Climate Risk in Dynamic Spatial Equilibrium*

Preliminary and incomplete

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Abstract

We study the economic implications, over time and space, of the aggregate and local risks arising from climate change. We develop a dynamic spatial model of the U.S. economy disaggregated at the county level that features costly forward-looking migration and capital investment decisions, aggregate risk due to uncertain warming trajectories, and local risk due to uncertain exposure to extreme weather. We achieve tractability by leveraging the 'Master Equation' representation of the economy. We estimate structural elasticities and damage functions by matching reduced-form estimates linking population, income, and investment to coastal storms and heat waves. Counterfactual exercises show that local risk generates sizable welfare losses for capitalists in exposed coastal areas and in the aggregate, while raising capital through a precautionary savings mechanism. By contrast, the real option embedded in capital investments implies that aggregate risk leads to modest capital depletion in some southern coastal counties and negligible aggregate welfare losses.

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1 Introduction

Risk is a central feature of weather and climate patterns. Even with all the advances in weather and climate forecasting, there is substantial uncertainty about how many storms will hit Miami next year or about how extreme and frequent heat waves will be in Houston. This uncertainty is further aggravated by climate change and the uncertainty about the resulting global long-term temperature paths, due to geoscientific and policy uncertainties. The consequences of the local and aggregate risk embedded in the evolution of the world's climate are a central feature of the impact of climate change at the local, country-wide, and global levels. In this paper, we study the economic implications of climate risk in the United States over time and across space.

Faced with climate risk, households and businesses will not wait passively for it to resolve, but instead take preemptive actions to adapt as best they can. By moving out of high-risk locations and redeploying capital in low-risk locations, workers and investors may limit their exposure to climate change risk more than if they were confined to staying in place. How does climate change risk reshape economic activity when workers and capital can reallocate across space?

In this paper, we answer this question by developing a dynamic spatial model of the United States economy with local and aggregate climate risk. We use a fine spatial resolution at the 3,143-county level to capture climate change impacts and adaptation. Our framework incorporates costly forward-looking migration, capital investment decisions, and multiple dimensions of risk: local realizations of damaging extreme weather and aggregate warming paths are both uncertain. The analysis remains computationally tractable despite aggregate risk in warming trajectories and the high-dimensional state space by leveraging the 'Master Equation' representation of the economy. We estimate structural elasticities and damage functions by matching reduced-form estimates linking population, income, and investment to coastal storms and heat waves. Counterfactual exercises indicate that, in the U.S., changes in local risk due to climate change are an order of magnitude larger than the effect of aggregate risk, and substantially amplify the impact of climate change relative to a deterministic scenario.

Our starting point is a framework with forward-looking dynamic migration and in-

vestment decisions at a high level of spatial resolution as in Bilal and Rossi-Hansberg (2025). As in Caliendo et al. (2019), forward-looking risk-averse workers decide where to live and work. They earn an equilibrium wage in their location, which they use to consume and lease housing at equilibrium rents. They receive random preference shocks for locations that motivate them to migrate, subject to bilateral migration costs. We enrich this environment along several dimensions. As in Kleinman et al. (2023), risk-averse local capitalists own the capital stock in the location where they live and face a dynamic consumption-investment decision that accounts for the future expected path of the economy and its climate. Capitalists rent out their capital stock to developers who use it to produce housing and commercial structures. Local firms produce consumption goods using labor and commercial structures. The evolution of the local paths of capital depreciation rates, productivities, and amenities depends on climate change with heterogeneous local exposures: as global mean temperatures rise, the frequency of storms and heat waves changes over time.

We introduce aggregate and local risk in this environment. The path of global mean temperature is not known with certainty. Instead, it is subject to aggregate shocks. Thus, local capital depreciation rates, productivities, and amenities fluctuate in tandem with aggregate shocks. In addition, we allow local exposures of capital depreciation rates, productivities, and amenities themselves to be risky. This local risk reflects idiosyncratic realizations of rare weather extremes. Local risk in capital depreciation generates local risk premia relative to an economy in which capital is a safe asset.[1]

The resulting environment is a heterogeneous agent framework with a high-dimensional state space—population and capital in every county—rich interdependencies between decisions and prices in all locations, and aggregate risk. To make progress, we leverage the 'Master Equation' methodology from Bilal (2023). Specifically, we use the 'Second-order Approximation to the Master Equation' (SAME), which consists of an analytic second-order perturbation around a long-run deterministic steady-state. It allows us to solve efficiently for quantitative general equilibrium counterfactuals in heterogeneous agent models and capture the effects of aggregate risk, by solving a simple set of matrix and

[1] In future drafts, we plan to incorporate two additional components of local risk: scientific uncertainty in how storms and heat waves change, location by location, and estimation uncertainty in climate impacts.

three-dimensional tensor Sylvester equations.

Equipped with our solution method, which allows us to compute an equilibrium of the model quickly, we estimate the central parameters of our framework: the migration elasticity, capitalist risk-aversion, which controls the investment elasticity, and the damage functions that map global temperature changes into local changes in capital depreciation rates, productivity, and amenities. We use the storm and heat wave probability data for each county collected in Bilal and Rossi-Hansberg (2025). Specifically, we measure how their frequency changes with global mean temperature in each county. We also obtain reduced-form estimates of these weather extremes on population, income, and investment from Bilal and Rossi-Hansberg (2025), with the associated standard errors. We obtain expert elicitations of the distribution of global mean temperature paths for the next century from the Intergovernmental Panel on Climate Change (IPCC).

We target the reduced-form impulse response functions to quantify our framework. We jointly estimate the migration elasticity and capitalist risk aversion, along with the magnitude of shocks corresponding to a single extreme event realization. Given a guess for the migration elasticity and capitalist risk aversion, and a standard calibration of the rest of the model parameters, we can invert the model in steady state and obtain all the local characteristics, including local productivities, amenities, and the level of local capital investment costs.

We then leverage a key property of the SAME: impulse response functions are quadratic in the magnitude of shocks. Therefore, conditional on the two elasticities, it suffices to evaluate two impulse response functions for the target outcomes per extreme event—investment for storms and income per capita for heat waves—and compare them with the data. The magnitude of the underlying capital depreciation or productivity shock then solves a standard scalar quadratic equation. This observation sidesteps an additional numerical search over the magnitude of the shocks.

Next, we estimate the migration elasticity and capital risk-aversion with a method of simulated moments in an outer loop. We target the relative impulse response functions of employment and income to heat waves, and the relative impulse response function of investment to income to storms. We leverage the computational efficiency of our solution method to propose global identification guarantees.

Storms and heat waves have large effects on capital depreciation, productivity, and amenities. Through the lens of our model, storms generate a 12% capital depreciation shock, while heat waves generate a 2.5% productivity shock together with a 2.4% amenity shock.[2] The resulting structural damage functions imply that the southeastern Atlantic coast should expect a 1 to 3 p.p. increase in the annual capital depreciation rate for every 1°C increase in global mean temperatures because of rising storm activity. Southern Florida should expect a 5% reduction in productivity, and a 1.6% reduction in amenities, for every 1°C increase in global mean temperature.

We are left with converting our estimates of the impact of a single extreme event into a damage function that maps global mean temperature to productivity, amenity, and capital depreciation losses. We interact damages from a given event with changes in the empirical probability of that event for each location in response to changes in temperature to obtain the average effect of rising temperature. We combine the same objects to construct how local volatilities in capital depreciation, productivity, and amenities depend on global mean temperature.

We use the quantified model to evaluate the social cost of climate change risk in the U.S. economy. Our first exercise isolates the role of local and aggregate risk associated with climate change by removing level effects in temperatures. For now, we focus on capital depreciation risk, but we will incorporate productivity and amenity risk in the future.

Capital depreciation risk due to coastal storms reduces the value of capital on the southeastern Atlantic coast of the United States. A precautionary motive partly offsets this reduction in the value of capital, leading to rising capital stocks on the coast. Workers respond to higher capital stocks by moving in, thereby increasing the overall exposure of the economy to climate risk. On net, local capitalists in these regions lose from rising local risk. They lose up to 25% in consumption-equivalent welfare terms for changes in local risk corresponding to two degrees Celsius of long-run warming. Workers benefit by up to 0.3% from changes in local capital depreciation risk because of the precautionary effect and rising capital stocks.

[2]We obtain a lower capital depreciation effect than Bilal and Rossi-Hansberg (2025) because capitalists are risk-averse instead of risk-neutral but do not face adjustment costs.

Aggregate capital depreciation risk due to uncertainty in long-run warming has the opposite effect and is weaker in magnitude. To understand this result, recall that local risk from idiosyncratic extreme weather realizations surprises capitalists. Thus, they cannot react to it on impact, leading to their precautionary savings response. In contrast, aggregate risk due to long-run warming uncertainty leads to persistent deviations in capital returns around their mean. Capitalists can then intertemporally arbitrage these return differentials to their advantage, by investing more if warming is less severe than expected and consuming more otherwise. This leads to a convex return function that can yield benefits from aggregate risk. Thus, capitalists consume more, accumulate less capital, and benefit from aggregate risk relative to local risk.

Workers do not benefit similarly from aggregate warming risk for two reasons. First, they dislike lower capital stocks. Second, they cannot arbitrage intertemporally as easily as capitalists. Migration decisions are infrequent and subject to bilateral migration frictions. Thus, workers on the southeastern Atlantic coast lose up to 0.6% due to aggregate capital depreciation risk. In preliminary results, we find that these losses grow moderately larger once we incorporate productivity and amenity risk.

Our second exercise compares the magnitude of welfare losses from a median climate change scenario—a gradual increase of global mean temperature from 1°C to 4°C above pre-industrial levels by 2100, consistent with the IPCC's business-as-usual scenario—with the associated cost of risk.

We find that local risk substantially magnifies aggregate welfare losses for capitalists early in the transition. Deterministic warming creates winners and losers among capitalists by pushing workers out of the Southeast and into the rest of the U.S. These gains and losses partly offset initially. By contrast, local risk—uncertain extreme weather realizations—leads to less migration because of the precautionary savings effect, which retains workers on the coast and thus generates fewer winners among capitalists. Quantitatively, local risk unambiguously magnifies aggregate capitalist welfare losses from less than 1% to over 5%. By the end of the century, deterministic warming leads to aggregate capitalist losses of just under 20%, while local risk adds an additional 8%.

Perhaps surprisingly, aggregate climate change risk—uncertain warming trajectories—appears to have negligible aggregate welfare effects relative to local risk. This is partly

due to the real-option nature of capital returns. It is also due to the fact that aggregate climate change risk is intermediated by the expected capital depreciation rate due to warming, which is bounded by the relatively low baseline hazard of extreme weather. By contrast, local risk directly changes the hazard of extreme weather realizations. Therefore, aggregate risk is always quantitatively less important than local risk as long as the hazard of extreme weather remains low, as in our counterfactuals.

The methodology that allows us to reach these conclusions is related to the literature on mean-field games. Together, the elements of our framework constitute a heterogeneous-agent environment with aggregate shocks, in which the full distribution of workers and capital across counties matters for the economy's evolution. Standard solution methods are not well-suited for the dimensionality of this framework. To make progress, we leverage the 'Master Equation' representation of the economy (Bilal, 2023), itself introduced in the mathematics of mean field games literature (Cardaliaguet et al., 2019): the state space representation of our framework in which the distribution is a state variable. We take analytic Second-order Approximations of the Master Equation in aggregate temperature shocks and in the underlying distribution around the final deterministic steady-state to obtain low-dimensional Bellman equations for the directional derivatives of individual value functions with respect to the underlying distribution and aggregate shocks.[3]

Our paper is also related to the literature on dynamic spatial models (Desmet et al., 2018; Caliendo et al., 2019; Kleinman et al., 2023), an active part of which focuses on climate change (Desmet and Rossi-Hansberg, 2015; Desmet et al., 2021; Cruz and Rossi-Hansberg, 2023; Conte et al., 2022; Cruz, 2021; Balboni, 2021; Bilal and Rossi-Hansberg, 2025). Relative to these papers that focus on perfect foresight transitions or first-order perturbations that feature certainty equivalence, our methodology allows us to introduce aggregate and local risk and compute counterfactuals at a high level of spatial resolution. Our computation method also lets us estimate the two key elasticities, the climate damage

[3]Sequence-space first-order perturbation approaches as in Auclert et al. (2021) and Bilal and Goyal (2025) are not well-suited for our purposes: they stop at first order and thus abstract from risk. In addition, it can be a challenge to even construct and store the Jacobian in a spatial framework with complex interdependencies. A parallel and nascent literature uses neural networks to propose global solutions (Sun, 2025). While this approach sidesteps perturbations, it necessarily relies on functional form extrapolation and lacks convergence guarantees given the challenges associated with training the network on an infinite-dimensional state space.

functions, local and aggregate risk exposures, by precisely matching the model results with empirical reduced-form local responses.

Of course, the main contribution of our paper is to evaluate the economic losses from climate change risk at the aggregate and local levels, accounting for the economy's ability to adapt by reallocating activity across space. An active literature assesses the role of climate change risk in representative agent models (Weitzman, 2009; Weitzman, 2014; Cai and Lontzek, 2019; Barnett et al., 2020; Bremer and Ploeg, 2021; Olijslagers and Wijnbergen, 2024). By contrast, our methodology enables us to introduce a fine level of spatial disaggregation into our framework, thereby accounting for adaptation and connecting our damage functions and risk exposure to the data. Sun (2025) is perhaps the only paper that shares our focus on risk in a spatial economy, but does not study the relative importance of aggregate and local risk.

The rest of this paper is organized as follows. Section 2 outlines the framework. Section 3 describes our solution method. Section 4 details our quantification. Section 5 presents our results. Section 6 concludes.

2 A Dynamic Spatial Framework with Local and Aggregate Risk

This section presents the setup of our dynamic spatial model with aggregate and local risk. We model an economy with many locations and two types of agents: workers and capitalists. Workers can move subject to mobility costs and earn a wage, which they consume each period. Capitalists are fixed in a location, earn returns on their risky capital investments, and face a dynamic consumption-savings decision.

2.1 Agents and preferences

There is a unit mass of infinitely-lived workers who choose in which location $i \in \{1, ..., \mathcal{I}\}$ to live. Workers have Constant Relative Risk Aversion (CRRA) preferences $u(C_{it}) = \frac{C_{it}^{1-\gamma}-1}{1-\gamma}$. C_{it} denotes a Cobb-Douglas aggregator of amenities in the worker's current residential location, A_{it}, a freely traded final good, c_{it}, used as the numeraire, and housing,

h_{it}. Namely, $C_{it} = A_{it}\left(\frac{c_{it}}{1-\beta}\right)^{1-\beta}\left(\frac{h_{it}}{\beta}\right)^{\beta}$, where β denotes the share of housing in expenditure. Workers discount the future at rate ρ. Workers are allowed to move at rate μ, in which case they draw extreme-value distributed idiosyncratic preference shocks for potential destinations, with dispersion parameter ν. If they move, they pay a bilateral moving cost τ_{ij}. We define the long-run (A_i) and time-varying (a_{it}) components of amenities in location i as $A_{it} = A_i e^{a_{it}}$.

Each location is populated by a unit measure of capitalists. They are immobile and have CRRA preferences $u^{\text{cap}}(C) = \frac{C^{1-\zeta}-1}{1-\zeta}$ over the final consumption good, with a possibly different risk-aversion parameter ζ from workers. They do not consume housing.

2.2 Technology

Workers in location i are employed by a representative firm that produces the final good using commercial structures and labor and faces constant returns to scale. Locations are endowed with a local productivity Z_{it} and a local stock of land L_i. A representative firm is endowed with a production function for the final good given by $Y_{it} = Z_{it} S_{it}^{\alpha} (N_{it}^P)^{1-\alpha}$, where S_{it} denotes local commercial structures and N_{it}^P the local number of workers employed in final good production. We denote by N_{it} the total number of workers in location i at time t. We define the long-run (Z_i) and time-varying (χ_{it}) components of productivity as $Z_{it} \equiv Z_i e^{\chi_{it}}$.

In every location, there is an equilibrium capital stock K_{it}. Capital can be combined with land L_i and construction labor N_{it}^B to produce buildings B_{it} according to the production function $B_{it} = L_i^{\omega} (N_{it}^B)^{\varpi} K_{it}^{1-\omega-\varpi}$. Buildings B_{it} can be costlessly repurposed between residential housing and commercial structures, so that $B_{it} = S_{it} + H_{it}$. Denote by r_{it} the common rental rate of buildings to firms and workers. Land rents are aggregated into a national portfolio and rebated to workers in proportion to their wages.

Capitalists decide how much to invest in their location. Their only financial asset is their own capital stock, which is not mobile across locations. We denote by R_{it} the return on a unit of capital. The local capital stock depreciates by $d\overline{\Delta}_{it}$ over a time interval of

length dt, where:

$$d\overline{\Delta}_{it} = \Delta_{it}dt + \Sigma_{it}dW_{it}, \tag{1}$$

and dW_{it} is a standard Brownian motion. In our baseline, we assume it is i.i.d. across locations, but plan to incorporate spatial correlation in the future. We define the long-run (Δ_i) and time-varying (δ_{it}) components of the average capital depreciation rate by $\Delta_{it} \equiv \Delta_i + \delta_{it}$. While our Brownian motion specification remains agnostic at this stage, it also corresponds to the capital depreciation process that arises when infrequent storms occur at some hazard rate and destroy part of the capital stock. We detail this microfoundation in Appendix C, as it guides our estimation strategy.

2.3 Climate change

The path of global mean temperatures, $\overline{T}_t$, is exogenous. Going forward, the vast majority of carbon emissions is expected to stem from developing economies and is thus largely exogenous to U.S. economic activity. Global mean temperatures, measured relative to a baseline level T^B—for instance, the pre-industrial level—satisfy: $\overline{T}_t - T^B = \epsilon\big(\sigma_t z_t + T_t\big)$.

The path T_t is deterministic and corresponds to a central climate scenario. The stationary stochastic process z_t captures uncertainty around the central warming scenario. We denote by $\mathcal{A}(z)$ its infinitesimal generator, the continuous-time equivalent of its transition matrix. We later assume that z follows a continuous-time AR(1) process $dz_t = -\theta z_t + dW_t$. In this case, the operator $\mathcal{A}(z)$ is given by: $\mathcal{A}(z)[V] = -\theta z \frac{\partial V}{\partial z} + \frac{1}{2}\frac{\partial^2 V}{\partial z^2}$. The deterministic path σ_t scales the shock and encapsulates the amount of uncertainty in future carbon emissions. For instance, when σ_t is constant, the ex-ante variance of global temperature around its deterministic path is constant over time. When σ_t scales with T_t, the variance of global temperature increases proportionally with the level of temperature itself. ϵ is a scaling factor used to define the second-order approximation in our solution method.

Local productivity χ_{it}, amenities a_{it}, average capital depreciation δ_{it} and capital depreciation volatility Σ_{it} depend on time through the path of global mean temperatures. We

parameterize these shifters through the following functions:

$$\chi_{it} = \overline{\chi}_i \left(\overline{T}_t - T^B \right) \qquad a_{it} = \overline{a}_i \left(\overline{T}_t - T^B \right)$$
$$\delta_{it} = \overline{\delta}_i \left(\overline{T}_t - T^B \right) \qquad \Sigma_{it} = \overline{\Sigma}_i \left(\overline{T}_t - T^B \right), \tag{2}$$

where $\overline{\chi}_i(0) = \overline{a}_i(0) = \overline{\delta}_i(0)$. By contrast, $\overline{\Sigma}_i(0) > 0$ as we allow for volatility in our baseline equilibrium.

We specify these damage functions as follows and without loss of generality given our solution method:

$$\overline{\chi}_i(T) = \chi_i T + \frac{\chi_{i2} T^2}{2} \qquad \overline{a}_i(T) = a_i T + \frac{a_{i2} T^2}{2}$$
$$\overline{\delta}_i(T) = \delta_i T + \frac{\delta_{i2} T^2}{2} \qquad \overline{\Sigma}_i(T) = \epsilon \Sigma_i \sqrt{\vartheta_i + T}, \tag{3}$$

where the microfoundation for the capital depreciation process in Appendix C indicates that we must include the additional ϵ in the definition of $\overline{\Sigma}_i$ and that excess temperatures enter inside a square root. $\vartheta_i > 0$ denotes baseline volatility absent any warming. The restrictions from the microfoundation ensure that local risk is a second-order contribution.

2.4 Static equilibrium

We turn to the static equilibrium of our framework. The maximization problem of firms implies that the wage w_{it} in location i satisfies the firm's first-order condition $w_{it} = (1 - \alpha) Z_{it} (S_{it} / N_{it}^P)^{\alpha}$. Similarly, the rental rate for structures satisfies $r_{it} = \alpha Z_{it} (S_{it} / N_{it}^P)^{-(1-\alpha)}$, since depreciation is not covered by the users of buildings. The optimal choice of labor in building production implies $\varpi r_{it} B_{it} / N_{it}^B = w_{it}$. Local demand for housing by workers implies that $\beta w_{it} N_{it} = r_{it} H_{it}$.

We determine equilibrium prices and quantities as functions of the local capital stock K_{it} and local number of workers N_{it}. We show that labor is allocated to goods and buildings production in constant shares: $N_{it}^P = x N_{it}$ and $H_{it} = y B_{it}$, where x, y are combinations of parameters described in equation (19), Appendix A.1. Using these shares and the production function of buildings, the wage and the rental rate of buildings become:

$w_{it} = w_{i0} Z_{it} L_i^{\omega\alpha} (K_{it}^{1-\omega-\varpi} N_{it}^{\varpi-1})^{\alpha}$ and $r_{it} = r_{i0} Z_{it} L_i^{-\omega(1-\alpha)} (K_{it}^{1-\omega-\varpi} N_{it}^{\varpi-1})^{-(1-\alpha)}$, where w_{i0}, r_{i0} are combinations of parameters and local fundamentals expressed in Appendix A.1.

Profit maximization by developers implies that the rental rate of capital satisfies: $R_{it} = R_{0i} e^{\chi_{it}} K_{it}^{-\phi} N_{it}^{\psi} \equiv R_i(\chi_{it}, K_{it}, N_{it})$, where R_{0i} is a location-specific constant that depends on the permanent component of productivity and available land, and ϕ and ψ are combinations of parameters detailed in Appendix A.1.

We can now solve for consumption given prices. Consumption by workers in location i is equal to the real wage. It satisfies $c_{it} = w_{it}/r_{it}^{\beta} = c_{0i} e^{(1-\beta)\chi_{it}} \left(K_{it}^{1-\omega-\varpi} / N_{it}^{1-\varpi} \right)^{\xi}$, where c_{0i} is a location-specific constant, and ξ a combination of parameters detailed in Appendix A.1. Finally, flow utility of workers in location i at time t is $u\left(A_{it} w_{it}/r_{it}^{\beta}\right) = u\left(C_{0i} e^{a_{it} + (1-\beta)\chi_{it}} \left(K_{it}^{1-\omega-\varpi} / N_{it}^{1-\varpi} \right)^{\xi} \right) \equiv U_i(a_{it}, \chi_{it}, K_{it}, N_{it})$, where $C_{0i} \equiv A_i c_{0i}$.

2.5 Migration decisions

Workers solve a forward-looking dynamic migration decision problem. We denote by V_{it} the value of being located in i at time t. V_{it} satisfies the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\rho V_{it} = U_i(a_{it}, \chi_{it}, K_{it}, N_{it}) + \mu \left\{ \frac{1}{\nu} \log \left(\sum_j e^{\nu(V_{jt} - \tau_{ij})} \right) - V_{it} \right\} + \frac{\mathbb{E}[dV_{it}]}{dt}, \tag{4}$$

where we used standard discrete choice results. The second term on the left-hand side represents the continuation value from migration, which we denote by $\mathcal{M}_i[V]$. Namely,

$$\mathcal{M}_i[V] \equiv \mu \left\{ \frac{1}{\nu} \log \left(\sum_j e^{\nu(V_{jt} - \tau_{ij})} \right) - V_{it} \right\}.$$

The last term in the worker's HJB equation represents the continuation value from changes in the aggregate state, where the notation d denotes the time increment due to changes in aggregates only and not due to individual migration. Standard discrete choice results

then imply migration shares as a function of the vector of values V_t given by

$$m_{ij}(V_t) = \frac{e^{\nu(V_{jt}-\tau_{ij})}}{\sum_k e^{\nu(V_{kt}-\tau_{ik})}}. \tag{5}$$

The population distribution evolves according to the Kolmogorov Forward (KF) equation

$$\frac{dN_{it}}{dt} = \mu\left(\sum_k m_{ki}(V_t)N_{kt} - N_{it}\right) \equiv \mu\Big(\big(m^*(V_t) - \mathrm{Id}\big)N_t\Big)_i, \tag{6}$$

where $(\cdot)_i$ denotes the i^{th} entry of the vector inside the parenthesis. In equation (6), m denotes the matrix of migration shares $m_{ij}(V_t)$. m^* denotes the matrix transpose of the matrix m. We also denote by $M^*(V) = \mu\big(m^*(V) - \mathrm{Id}\big)$ a matrix that collects migration shares and the identity matrix, so that (6) becomes, in matrix notation, $\frac{dN_t}{dt} = M^*(V)N_t$. We denote $M(V) = \mu(m(V) - \mathrm{Id}))$ the transpose of $M^*(V)$ for future reference.

2.6 Investment decisions

Capitalists solve a forward-looking dynamic investment decision problem. We denote $\mathcal{P}_{it}(K)$ the value of being located in i at time t and holding K units of local capital. The level of capital evolves according to the law of motion:

$$dK_{it} = (I_{it} - \Delta_{it}K_{it})dt - \Sigma_{it}K_{it}dW_{it}, \tag{7}$$

where I_{it} denotes local investment. The capitalist budget constraint is $C_{it} + I_{it} = R_{it}K_{it}$. The capitalist HJB equation is then

$$\rho\mathcal{P}_{it}(K) = \max_C \frac{C^{1-\zeta}-1}{1-\zeta} + [(R_{it} - \Delta_{it})K - C]\frac{\partial\mathcal{P}_{it}}{\partial K} + \frac{\Sigma_{it}^2K^2}{2}\frac{\partial^2\mathcal{P}_{it}}{\partial K^2} + \mathbb{E}_t\left[\frac{d\mathcal{P}_{it}}{dt}\right],$$

The first term in the capitalist HJB is the flow utility from consumption. The second term is the continuation value from net investment. Net investment is given by returns on capital, minus expenditures on average capital depreciation and consumption. Net investment is valued at the marginal value $\frac{\partial\mathcal{P}_{it}}{\partial K}$. The third term represents the continuation value from capital depreciation *risk*. It depends on the local variance of capital deprecia-

tion $\Sigma_{it}^2 K^2$ and is valued through the concavity of the value function $\frac{\partial^2 \mathcal{P}_{it}}{\partial K^2}$. The final term in the capitalists' HJB is the continuation value from changes in the aggregate state.

We show in Appendix A.2 that $\mathcal{P}_{it}(K) = \frac{Q_{it} K^{1-\zeta} - 1/\rho}{1-\zeta}$, where Q_{it} satisfies

$$\rho Q_{it} = Q_{it}^{1-1/\zeta} + (1-\zeta)\left[R_{it} - \Delta_{it} - Q_{it}^{-1/\zeta} - \frac{\zeta \Sigma_{it}^2}{2}\right] Q_{it} + \mathbb{E}_t\left[\frac{dQ_{it}}{dt}\right]. \tag{8}$$

While not strictly necessary for our solution method, this expression simplifies exposition and allows us to work with the value of capital Q_{it} rather than with the entire value function $\mathcal{P}_{it}(K)$. Finally, investment decisions satisfy $I_{it} = \left(R_{it} - Q_{it}^{-1/\zeta}\right) K_{it}$.

For numerical accuracy, we will use log capital $\mathcal{K}_{it} = \log K_{it}$ in our solution method. The KF equation for log capital is given by

$$d\mathcal{K}_{it} = \left(R_{it} - Q_{it}^{-1/\zeta} - \Delta_{it} - \frac{\Sigma_{it}^2}{2}\right) dt - \Sigma_{it} dW_{it}. \tag{9}$$

We denote by $\mathbb{E}_t\left[\frac{d\mathcal{K}_{it}}{dt}\right] = R_{it} - Q_{it}^{-1/\zeta} - \Delta_{it} - \frac{\Sigma_{it}^2}{2}$ the drift of log capital for future reference.

2.7 Deterministic Steady-state

In the deterministic steady-state at baseline temperatures, $\epsilon = 0$. All time derivatives are equal to zero. The HJB equation of workers (4) and capitalist capitalists (8) become

$$\rho V_i^{ss} = U_i(0, 0, K_i^{ss}, N_i^{ss}) + \mathcal{M}_i[V^{ss}], \qquad \text{and} \qquad Q_i^{ss} = \rho^{-\zeta}. \tag{10}$$

The steady-state population distribution $N^{ss} = \{N_i^{ss}\}_i$ and capital distribution $K^{ss} = \{K_i^{ss}\}_i$ satisfy

$$0 = M^*(V^{ss}) N^{ss}, \qquad \text{and} \qquad R_i^{ss} = \rho + \Delta_i. \tag{11}$$

Appendix B describes a simple algorithm to solve for the steady-state that leverages equations (10)-(11).

3 Solution Method

3.1 Strategy

Our economy is a dynamic general equilibrium economy in which the distributions of workers $N_t = \{N_{it}\}_i$ and capital $K_t = \{K_{it}\}_i$ are aggregate state variables that determine local prices. The distribution of workers and capital evolves slowly according to the laws of motions (6) and (9). These laws of motion in turn depend on the values of workers given by the HJB equations (4) and (8). We solve this dynamic fixed point problem by leveraging the 'Master Equation' representation of the economy.

The Master Equation approach is structured in two steps. In the first step, we merge the individual decision problems with the laws of motion of the population and capital distributions into a single Bellman equation: the Master Equation. The Master Equation is a state-space representation of the equilibrium. The second step consists in taking a local perturbation of the Master Equation in the scale parameter ϵ around a deterministic steady-state of the economy.

3.2 The Master Equation

We start with the Master Equation for workers. We combine the HJB equation (4) with the laws of motion (6) and (9) and use a simple change of variables. We index value functions by the population and capital distributions instead of calendar time alone. The core idea is that the population and capital distributions are no more than large-dimensional state variables from the perspective of any given worker. Of course, we need to keep calendar time as a separate index because of climate change embedded in the deterministic path T_t.

Specifically, we change variables by writing $V_{it} = V_{it}(z_t, N_t, \mathcal{K}_t)$ and $Q_{it} = Q_{it}(z_t, N_t, \mathcal{K}_t)$, where now the time subscript t only captures dependence on deterministic temperature. Recall that $N_t = \{N_{it}\}_i$ denotes the vector of population across locations at time t, and $\mathcal{K}_t = \{\mathcal{K}_{it}\}_i$ similarly denotes the log capital distribution. Using this change of variables, we use the chain rule to express the continuation value from changes in aggregate states

as

$$\mathbb{E}_{it}\left[\frac{dV_{it}}{dt}\right] = \underbrace{\frac{\partial V_{it}}{\partial t}}_{\substack{\text{change} \\ \text{in } T_t}} + \underbrace{\mathcal{A}(z_t)[V_{it}]}_{\text{change in } z_t} + \underbrace{\sum_j \frac{\partial V_{it}}{\partial N_{jt}}\frac{dN_{jt}}{dt}}_{\substack{\text{change in } N_t\text{:} \\ \text{chain rule}}} + \underbrace{\sum_j \frac{\partial V_{it}}{\partial \mathcal{K}_j}\mathbb{E}_t\left[\frac{d\mathcal{K}_{jt}}{dt}\right]}_{\substack{\text{mean change in } \mathcal{K}_t\text{:} \\ \text{and chain rule}}} + \underbrace{\frac{1}{2}\sum_j \frac{\partial^2 V_{it}}{\partial \mathcal{K}_j}\Sigma^2_{jt}}_{\substack{\text{volatility in } \mathcal{K}_t\text{:} \\ \text{chain rule} \\ \text{+ Ito correction}}}, \quad (12)$$

The last three components of (12) encode how workers' values change with the population and capital distributions across locations. When workers reallocate across locations in equilibrium or when the distribution of capital shifts in equilibrium, the distribution of local wages, building rental rates, and capital rental rates changes. These changes in these three price distributions affect residents of location i.

Importantly, the continuation value from changes in the capital distribution contains two components. The first component captures how expected change in (log) capital $\mathcal{K}_{jt}$ across all locations j affect the value function of residents of location i, and therefore depends on the expected change in log capital $\mathbb{E}_t\left[\frac{d\mathcal{K}_{jt}}{dt}\right] = R_{it} - Q_{it}^{-1/\zeta} - \Delta_{it} - \frac{\Sigma^2_{it}}{2}$. The second component captures how residents of location i value capital depreciation volatility in all locations j. This component therefore depends on the local variance of the local log capital process Σ^2_{it}.

We then use the laws of motion of the population and capital distributions (6) and (9) to express the time change of population and capital in (12). After this substitution into the continuation value (12), we obtain the Master Equation for workers. We omit the full dependence of the value function and its derivatives on $(\mathcal{K}, N, t, \epsilon)$ on the right-hand-side

for brevity, and we also omit the direct dependence of $a_{it}, \chi_{it}, \Delta_{it}$ on $\epsilon(\sigma_t z + T_t)$, and so

$$\begin{aligned}\rho V_{it}(z_t, N_t, \mathcal{K}_t) &= \underbrace{U_i\Big(a_{it}, \chi_{it}, N_i, \mathcal{K}_i\Big)}_{\text{flow payoff}} + \underbrace{\mathcal{M}_i[V_t]}_{\substack{\text{continuation value} \\ \text{from own migration}}} \\ &+ \underbrace{\sum_j \frac{\partial V_{it}}{\partial N_j}\big(M^*(V_t)N_t\Big)_j + \mathcal{A}(z_t)[V_{it}] + \frac{\partial V_i}{\partial t}}_{\text{continuation values from aggregate changes in population, } z \text{ and } t} \\ &+ \underbrace{\sum_j \frac{\partial V_{it}}{\partial \mathcal{K}_j}\left\{R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right\} + \frac{1}{2}\sum_j \frac{\partial^2 V_{it}}{\partial \mathcal{K}_j^2}\Sigma_{jt}^2}_{\text{continuation values from aggregate changes in capital distribution}}.\end{aligned} \tag{13}$$

The Master Equation for workers depends on the value of capitalists Q because the latter affects investment, capital accumulation, prices, and, ultimately, worker welfare.

A similar logic delivers the Master Equation for capitalists, which is given by

$$\begin{aligned}\rho Q_{it}(z_t, N_t, \mathcal{K}_t) &= \underbrace{Q_{it}^{1-1/\zeta} + (1-\zeta)\left[R_{it} - \Delta_{it} - Q_{it}^{-1/\zeta} - \frac{\zeta\Sigma_{it}^2}{2}\right]Q_{it}}_{\text{flow payoff + continuation values from own investment}} \\ &+ \underbrace{\sum_j \frac{\partial Q_{it}}{\partial N_j}\left(M^*(V)N_t\right)_j + \mathcal{A}(z_t)[Q_{it}] + \frac{\partial Q_{it}}{\partial t}}_{\text{continuation values from aggregate changes in population, } z \text{ and } t} \\ &+ \underbrace{\sum_j \frac{\partial Q_{it}}{\partial \mathcal{K}_j}\left\{R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right\} + \frac{1}{2}\sum_j \frac{\partial^2 Q_{it}}{\partial \mathcal{K}_j^2}\Sigma_{jt}^2}_{\text{continuation values from aggregate changes in capital distribution}}.\end{aligned} \tag{14}$$

We omit the equilibrium definition, since it is standard. The main advantage of the Master Equations (13)–(14) is that they deliver a parsimonious characterization of the economy's general equilibrium. The derivations above establish our first result.

Proposition 1. *(Equilibrium)*
A solution (V, Q) to the Master Equations (13)-(14) together with a solution to the population and capital distributions $\{N_t, K_t\}_t$ to the KF equations (6) and (9) is an equilibrium of our economy.

3.3 Expansion of the Master Equation

Although the Master Equations (13)–(14) provide a parsimonious characterization of equilibrium, they are difficult to solve numerically because they depend on high-dimensional distributions. As a result, standard global methods quickly run into the curse of dimensionality. We instead follow Bilal (2023) and adopt an analytic second-order perturbation. In particular, we seek a solution to (13)–(14) when the scale parameter ϵ is sufficiently small. Namely, we define n as the vector of scaled deviations from steady-state for the population distribution, and k the vector of scaled deviations from steady-state for the log capital distribution, such that:

$$N = N^{ss} + \epsilon n, \qquad \text{and} \qquad \mathcal{K} = \mathcal{K}^{ss} + \epsilon k.$$

To state our strategy as compactly as possible, we define $\times_k$ as the tensor product with respect to index k. It takes as argument a tensor–a three-dimensional array—and a matrix or vector, and returns the summation along the k^{th} index of the tensor and the first index of the associated matrix. As such, it generalizes matrix products. For instance, for a tensor $\mathcal{V}_{ij\ell}$ and a vector n_j, the (i,ℓ) entry of $\mathcal{V} \times_2 n$ is $\sum_j \mathcal{V}_{ij\ell} n_j$. Similarly, the i^{th} entry of the term of $\mathcal{V} \times_2 n \times_3 k$ is $\sum_{j,\ell} \mathcal{V}_{ij\ell} n_j k_\ell$.

We then seek a solution of the form for workers, in vector notation:

$$\begin{aligned} V_t(z, N^{ss} + \epsilon n, \mathcal{K}^{ss} + \epsilon k) - V^{ss} &= \epsilon \left\{ \sum_{x \in \{n,k\}} V^x x + v_t^Z z + v_t^T \right\} \\ &+ \frac{\epsilon^2}{2} \left\{ \sum_{\{x,y\} \in \{n,k\}} \mathcal{V}^{X,Y} \times_2 x \times_3 y + \sum_{x \in \{n,k\}} \left(\mathrm{V}_t^{X,T} + z \mathrm{V}_t^{X,Z} \right) x + \overline{v}_t + v_t^{Z,Z} z^2 + v_t^{T,T} + 2 v_t^{Z,T} z \right\}. \end{aligned} \tag{15}$$

For completeness, we spell out the expanded expression in Appendix G.1. Similarly, we seek a solution for the value of capital:

$$\begin{aligned} Q_t(z, N^{ss} + \epsilon n, \mathcal{K}^{ss} + \epsilon k) - Q^{ss} &= \epsilon \left\{ \sum_{x \in \{n,k\}} Q^x x + q_t^Z z + q_t^T \right\} \\ &+ \frac{\epsilon^2}{2} \left\{ \sum_{\{x,y\} \in \{n,k\}} \mathcal{Q}^{X,Y} \times_2 x \times_3 y + \sum_{x \in \{n,k\}} \left(\mathrm{Q}_t^{X,T} + z \mathrm{Q}_t^{X,Z} \right) x + \overline{q}_t + q_t^{Z,Z} z^2 + q_t^{T,T} + 2 q_t^{Z,T} z \right\}. \end{aligned} \tag{16}$$

Rather than solving for the full equilibrium objects V and Q in the Master Equations, the perturbation approach reduces the problem to a finite set of coefficients that we collect into two groups. The first group consists of the first order directional derivatives V_{ij}^N, V_{ij}^K, v_{it}^Z, and v_{it}^T of the worker value function with respect to the population and capital distributions, temperature shocks, and temperature trends, and their counterparts for capitalists. These directional derivatives constitute the FAME.

The second group of coefficients contains the second-order directional derivatives—or Hessian—of the value function with respect to the population and capital distributions, temperature shocks, and temperature trends. These Hessians include three-dimensional arrays—or tensors—$\mathcal{V}_{ijs}^{N,N}$, $\mathcal{V}_{ijs}^{K,K}$, $\mathcal{V}_{ijs}^{N,K}$, and $\mathcal{V}_{ijs}^{K,N}$, which correspond to the second-order directional derivatives of the worker's value function with respect to the population and capital distributions, and include the capitalists' counterparts as well. These tensors or symmetric by construction. Next, the matrices $V_{ijt}^{N,Z}$, $V_{ijt}^{K,Z}$, $V_{ijt}^{N,T}$, and $V_{ijt}^{K,T}$ and the capitalist counterparts represent second-order cross directional derivatives of the value function with respect to the population distribution, capital distribution, temperature shocks, and temperature trends. Similarly, $v_{it}^{Z,Z}$ and $v_{it}^{T,T}$ and the capitalist counterparts are the second-order derivatives of values with respect to temperature shocks and temperature trends, with $v_{it}^{Z,T}$ denoting the cross derivative. Finally, $\overline{v}_{it}$ and its capitalist counterpart represent the ex-ante valuation of risk arising from the second-order continuation value of aggregate temperature shocks. These second-order directional derivatives constitute the SAME.

Of course, we still need obtain restrictions on the first- and second-order directional derivatives. To that end, we substitute the second-order Taylor expansions (15)-(16) into the Master Equations (13)–(14) and obtain a system of restrictions that determines the directional derivatives. The restrictions on the first-order derivatives are the FAME, and the restrictions on the second-order derivatives is the SAME.

3.4 The FAME

We first characterize the FAME as it is a necessary step before evaluating the effect of risk in the SAME. To that end, we collect the matrices V^N, V^K, V^N, V^K and time-dependent

vectors v_t^T, q_t^T, n_t, k_t into the matrix and vectors: $V^d = \begin{pmatrix} V^N & V^K \\ Q^N & Q^K \end{pmatrix}$, $v_t^T = \begin{pmatrix} v_t^T \\ q_t^T \end{pmatrix}$, $v_t^Z = \begin{pmatrix} v_t^Z \\ q_t^Z \end{pmatrix}$ and $p_t = \begin{pmatrix} n_t \\ k_t \end{pmatrix}$.

Proposition 2. *(FAME)*
V^d *satisfies the generalized Sylvester matrix equation and* v_t^T, p_t *satisfy the ordinary differential equation systems:*

$$\rho \mathrm{V}^d = \mathrm{D}^d + \mathrm{M}\mathrm{V}^d + \mathrm{V}^d\mathrm{M}^* + \mathrm{V}^d\mathrm{P}\mathrm{V}^d \qquad \rho \mathrm{v}_t^T = D_t^T + (\mathrm{M} + \mathrm{V}^d\mathrm{P})\mathrm{v}_t^T + \frac{\partial \mathrm{v}_t^T}{\partial t}$$
$$\rho \mathrm{v}_t^Z = D_t^Z + (\mathrm{M} + \mathrm{V}^d\mathrm{P})\mathrm{v}_t^Z + \frac{\partial \mathrm{v}_t^Z}{\partial t} \qquad \frac{dp_t}{dt} = D_t^p + \mathrm{P}(\mathrm{v}_t^T + \mathrm{v}_t^Z z_t) + (\mathrm{M}^* + \mathrm{P}\mathrm{V}^d)p_t,$$

where the matrices $\mathrm{D}^d, \mathrm{M}, \mathrm{P}$ *and vectors* D_t^T, D_t^Z, D_t^p *only depend on parameters, steady-state objects, and* T_t, σ_t *and are specified in Appendix G.*

Proof. See Appendix G. □

The first equation in Proposition 2 is the deterministic FAME and characterizes the directional derivatives V^N, V^K, V^N, V^K as the solution to a nonlinear Sylvester matrix equation. The right-hand side of the deterministic FAME features four terms, each mapping to a distinct economic force. First, the direct price impact matrix D^d captures how marginal changes in the population and capital distributions shift equilibrium prices and therefore flow payoffs for workers and capitalists.

The second term $\mathrm{M}\mathrm{V}^d$ encodes the partial equilibrium option value of migration and investment, as workers expect to migrate and capitalists expect to change their consumption and savings in response to changes in valuations across all locations. To first order, it suffices to evaluate these option values using steady-state decision rules embedded in the matrix M.

The third term $\mathrm{V}^d\mathrm{M}^*$ captures a first general equilibrium force. When assessing the effect of an additional worker in location j, agents in location i must account for the fact that this marginal entrant will migrate or invest over time, which affects future population and capital distributions and thus future payoffs. The FAME reveals that these expectations are summarized by the same steady-state migration and investment matrices collected in M^*.

The fourth term $\mathrm{V}^d\mathrm{P}\mathrm{V}^d$ captures a second general equilibrium force. It reflects how the marginal entrant changes the law of motion for the population and capital distributions because all other agents react to the resulting price changes.

The structure of the trend FAME mirrors the structure of the deterministic FAME. The direct impact component, D_t^T, which depends linearly on global mean temperature T_t^D and the damage coefficients a_i, χ_i, δ_i, and then propagates this impact through analogous partial and general equilibrium channels.

These first-order results imply three practical properties that come directly from the local perturbation. First, the deterministic FAME is finite-dimensional. Second, all inputs to the FAME can be constructed from steady-state objects. Third, the system is block-recursive: the deterministic FAME can be solved first, the trend and stochastic impulse values can then be obtained conditional on V^d, and finally the equilibrium path of allocations p_t can be computed by forward iteration given the impulse values. This structure delivers a straightforward computational algorithm, which follows Bilal (2023) and Bilal and Rossi-Hansberg (2025).

As a first order perturbation, the FAME features certainty equivalence and thus cannot capture the effect of local and aggregate risk. To incorporate risk into our analysis, we therefore proceed to a second-order approximation.

3.5 The SAME

The restrictions of the Hessian of the value functions constitute the SAME. As a second-order perturbation, it incorporates the effect of local and aggregate risk. We collect the tensors $\mathcal{V}^{N,N}, \mathcal{V}^{N,K}, \mathcal{V}^{K,N}, \mathcal{V}^{K,K}, \mathcal{Q}^{N,N}, \mathcal{Q}^{N,K}, \mathcal{Q}^{K,N}, \mathcal{Q}^{K,K}$ in the tensor $\mathcal{T} = (\mathcal{V}, \mathcal{Q})$. We collect the matrices $V^{N,T}, V^{K,T}, Q^{N,T}, Q^{K,T}$ in the matrix $\mathrm{V}_t^T = \begin{pmatrix} V^{N,T} & V^{K,T} \\ Q^{N,T} & Q^{K,T} \end{pmatrix}$, and similarly $\mathrm{V}_t^Z = \begin{pmatrix} V^{N,Z} & V^{K,Z} \\ Q^{N,Z} & Q^{K,Z} \end{pmatrix}$. For all pairs $X, Y \in \{Z, T\}$, we collect $\mathrm{v}_t^{X,Y} = \begin{pmatrix} v_t^{X,Y} \\ q_t^{X,Y} \end{pmatrix}$. Finally, we define $\widehat{v}_t = \begin{pmatrix} \overline{v}_t \\ \overline{q}_t \end{pmatrix}$.

Proposition 3. *(SAME)*
$\mathcal{T} = (\mathcal{V}, \mathcal{Q})$ satisfies the linear symmetric tensor Sylvester equation, V_t^T satisfies the linear Sylvester ordinary differential equation system, for all pairs $X \in \{Z, T\}, Y \in \{Z, T\}$ $\mathrm{v}_t^{X,X}, \widehat{v}_t$

satisfy the linear ordinary differential equation systems:

$$\rho\mathcal{T} = \mathbb{T} + \mathcal{T} \times_1 \Phi + \mathcal{T} \times_2 \Theta + \mathcal{T} \times_3 \Theta \qquad \rho \mathrm{V}_t^T = \widetilde{D}_t^T + \Phi \mathrm{V}_t^T + \mathrm{V}_t^T \Theta + \frac{\partial \mathrm{V}_t^T}{\partial t}$$

$$\rho \mathrm{v}_t^{X,Y} = \widetilde{D}_t^{X,Y} + \Phi \mathrm{v}_t^{X,Y} + \mathrm{v}_t^{X,Y} \Theta + \frac{\partial \mathrm{v}_t^{X,Y}}{\partial t} \qquad \rho \widehat{v}_t = \widehat{d}_t + \Phi \widehat{v}_t + \widehat{v}_t \Theta + \frac{\partial \widehat{v}_t}{\partial t},$$

where the tensor $\mathbb{T}$*, the matrices* $\widetilde{D}_t^T, \widetilde{D}_t^{X,Y}$, H, F *and the vector* $\widehat{d}_t$ *only depend on parameters, steady-state objects, and* T_t *and* σ_t *and are specified in Appendix H together with the law of motion for population and capital, and where* $\Phi = \mathrm{M} + \mathrm{V}^d \mathrm{P}$, $\Theta = \mathrm{H} + \mathrm{V}^d \mathrm{F}$.

Proof. See Appendix H. □

The SAME has a similar structure to the FAME, albeit in higher dimensions. The equation that determines the tensor $\mathcal{T}$ is the deterministic SAME. Its solution $\mathcal{T}$ encodes how value functions change through second-order effects when the distributions of population and capital move. The deterministic SAME is a linear tensor Sylvester equation, with four components.

The first component of the deterministic SAME is the impact represented by the tensor $\mathbb{T}$. As a second-order object, it encapsulates multiple forces. It contains the direct second-order derivatives of workers' and capitalists' flow payoffs with respect to capital and population. It also contains indirect second-order effects that arise as the product of first-order changes. For instance, it includes the square of first-order changes in the value of locating in particular destination due to shifts in population and capital distributions. This squared term enters because the option value of migration is itself nonlinear.

The second component of the deterministic SAME $\mathcal{T} \times_1 \Phi$ captures a similar force to the FAME. Workers expect to migrate, and capitalists expect to save. Therefore, the evaluate future second-order benefits from migration and savings using steady-state decision rules embedded in Φ. The third and fourth components of the SAME, $\mathcal{T} \times_2 \Theta + \mathcal{T} \times_3 \Theta$, reflect second-order counterparts of similar general equilibrium forces to the FAME. When contemplating an additional worker in a given location, agents expect this worker to migrate and affect all other agents' decisions, thereby changing the distribution of population and capital in the future. These components embed how workers and capitalists account for these general equilibrium effects. These two components have the

same structure because, by construction, the deterministic SAME is symmetric in its last two coordinates.

The matrices in $\mathrm{V}_t^T, \mathrm{V}_t^Z$ and the vectors in $\mathrm{v}_t^{X,Y}$ for $X, Y \in \{Z, T\}$ capture cross effects between changes in the distribution and aggregate shocks. As for the FAME, the SAME is block-recursive and the cross SAME depend on the solution $\mathcal{T}$ to the deterministic SAME. Crucially, $\mathrm{v}_t^{Z,Z}$ encodes the nonlinear valuation of aggregate shocks z_t, which is the central determinant of the valuation of aggregate risk. By contrast, $\mathrm{v}_t^{T,T}$ embeds the valuation of local risk through local volatilities.

The vectors in $\widehat{v}_t^T$ contain the ex-ante valuation of aggregate and local risk. This component is generically different from zero. Workers and capitalists expect baseline local volatility, even absent warming. They also expect aggregate shocks to occur, even when they turn out not to. The valuation of risk $\widehat{v}_t^T$ is unique to the SAME and is the focus of counterfactuals. To that end, we next turn to the quantification of the model.

4 Model Estimation

4.1 Data

We assemble three types of data. First, we compile economic information at the county-year level from multiple sources. Employment, wages, and related statistics are drawn from the Bureau of Economic Analysis, while investment data at five-year intervals come from the Census of Manufactures. These datasets cover the period from 1960 to 2019.

Second, we construct a detailed weather dataset. We use daily data on surface temperature, wind speed, and precipitation on a 0.5° by 0.5° grid beginning in 1901 from the Inter-Sectoral Impact Model Intercomparison Project (ISMIP). These reanalysis data combine historical station-level weather observations with climate model outputs to generate high-resolution weather series. We aggregate this information to build a dataset of annual weather extremes for each U.S. county and year.

Third, to quantify our model in a way that captures the economic impact of climate change, we draw on the reduced-form estimates in Bilal and Rossi-Hansberg (2025). Using a distributed lag specification that controls for a rich set of possible confounders in

the same data as we use for quantification, they estimate the causal effects of storms and heat waves on observables at multiple horizons: income, employment and investment. We use these reduced-form impulse response functions as targets in our estimation.

4.2 Baseline Parameters

We start by determining baseline preference and production parameters. Importantly, we estimate the migration elasticity of workers and the relative risk aversion of capitalists in Section 4.4. Throughout the rest of the paper, we interpret an interval $[t, t+1)$ as one year.

We first determine preference parameters. We use a pure rate of time preference $\rho = 0.02$, which is standard in the climate change literature (Rennert et al., 2022). We set the housing expenditure share for workers to the common value of $\beta = 0.3$ according to (Davis and Ortalo-Magne, 2011). For now, we set worker risk aversion parameter to $\gamma = 1$. We plan to relax this assumption in future iterations of this paper and estimate it internally. We set the rate of migration opportunities to $\mu = 2.30$ so that 90% of workers have the option to migrate within a year.

We calibrate the parameters in the production functions for goods and buildings to standard values. In the model, the share of commercial structures in goods production captures both capital and physical structures in the data. Standard estimates place these shares at 0.3 and 0.1, respectively, so we set $\alpha = 0.4$. To determine the labor share ϖ in building production, we use the labor share in the structures production sector, denoted by x, from BEA input-output tables. This yields $x = 0.95$, which in turn implies $\varpi = \frac{(1-\alpha)(1-x)}{\alpha x+(1-\alpha)\beta} = 0.05$. Finally, to calibrate the land share ω in building production, we rely on standard estimates from the real estate literature. For a representative urban property, land accounts for roughly 35 percent of total property value, with the remaining 65 percent corresponding to the building structure (Davis et al., 2021). We therefore set $\omega = 0.35$.

At baseline, we impose a common capital depreciation rate across counties, setting $\Delta_i = \Delta = 0.08$. In future work, we plan to use our investment data to estimate location-specific baseline depreciation rates. Of course, climate change affects depreciation heterogeneously across locations. These location-specific effects are precisely the local damage

functions we estimate below.

Finally, we set mean-reversion in global mean temperature shocks $\theta = 1$, so that the long-run variance of z_t is exactly 1.

4.3 Inversion of Fundamentals

Our second step is to estimate time-invariant fundamentals for each location, taking as given estimates of the migration elasticity ν and the investment elasticity ζ.

We use 2012 data for wages w_i, employment N_i, and migration flows $\mathbb{m}_{ij}$ for every pair of counties i, j. We interpret 2012 as the initial stead-state of the economy. We use a recursive scheme to recover the time-invariant fundamentals in every location, Z_i, A_i, and τ_{ij}. The following proposition shows that, given the observed data, these fundamentals can be uniquely recovered. We provide a formal proof in Appendix E.

Proposition 4. *(Inversion of fundamentals)*
Given data w_i, N_i, m_{ij}, there exists a unique set of vectors of fundamentals, (Z_i, A_i), and a unique symmetric migration cost matrix, τ_{ij}.

The proof of Proposition 4 is constructive and therefore delivers an algorithm to recover the fundamentals. It extends standard inversion results from static environments to our fully forward-looking dynamic setting. Because we invert migration costs non-parametrically, however, we can recover only their symmetric component.[4]

4.4 Migration Elasticity and Capitalist Preferences

Our inversion result is derived conditional on the migration elasticity ν and the capitalists' coefficient of relative risk aversion ζ. We estimate those internally using the reduced-form estimates of the impact of extreme events on economic activity of Bilal and Rossi-Hansberg (2025). We nest our inversion procedure in an outer loop in which we search over candidate values for ν and ζ.

To conduct our estimation exercise, we parameterize the effects of storms and heat waves in the model. Consistent with the reduced-form estimates, we assume that storms

[4]Without symmetry, migration costs are not separately identified from local amenities. If, instead, we imposed that migration costs depend on a distance metric, this dependence could be estimated flexibly.

generate only a capital depreciation shock in coastal counties, with scalar magnitude $\Gamma^{\text{cap dep}}$. Heat waves, by contrast, generate both a productivity shock and an amenity shock in warm counties. We denote the magnitude of the productivity shock by Γ^{prod} and the magnitude of the amenity shock by $\eta\Gamma^{\text{prod}}$.[5]

For now, we exploit a key property of the FAME to estimate the migration and investment elasticities without having to pin down the magnitude of the underlying shocks. By construction, the response of the economy is linear in shock size. As a result, relative impulse responses are invariant to the magnitude of a common shock. For example, the impulse response of employment relative to that of income per capita following a productivity shock does not depend on the size of the productivity shock itself. Therefore, once we target relative impulse responses, we can normalize the shock magnitude arbitrarily. This property allows us to estimate ν, ζ, and η independently of the magnitudes of the underlying shocks, $\Gamma^{\text{cap dep}}$ and Γ^{prod}.

In future iterations of this paper, we plan to use the SAME to estimate these elasticities. Importantly, under the SAME, the impulse responses of employment relative to income per capita following a productivity shock vary quadratically with the magnitude of the shock. Therefore, unlike in the FAME, relative impulse responses no longer isolate the elasticities from the size of the shocks. In principle, the three parameters and the shock magnitudes must therefore be estimated jointly. In practice, we exploit the quadratic property of the SAME. It implies that it suffices to evaluate two impulse response functions for the target outcomes per extreme event— investment for storms and income per capita for heat waves—and compare them with the data. The magnitude of the underlying capital depreciation or productivity shock then solves a standard scalar quadratic equation. This observation sidesteps an additional numerical search over the magnitude of the shocks. We do not expect the estimation using the SAME to materially differ from the FAME because the first-order approximation tends to be highly accurate for the realized shocks in sample.

We estimate the elasticities by indirect inference. Our procedure is conceptually close to a standard grid search, although we implement it differently to improve computational efficiency. In the first step, we generate candidate parameter vectors (ν, ζ, η) using

[5] We set the persistence of the productivity shock, ψ^{prod}, to match a half-life of 5 years.

a Sobol sequence. Relative to a standard tensor-product grid, this approach provides a more efficient coverage of the multivariate parameter space. We choose the bounds of the parameter space to be sufficiently broad to encompass a wide range of plausible values, including our point estimates.

In the second step, we invert fundamentals as described in Appendix E and solve for the steady state following Section 2.7 and Appendix B. We then compute the FAME of the economy around that steady state. Next, we simulate the dynamic response of the full economy to two shocks: a 1% transitory productivity shock accompanied by a η% amenity shock, and a 1% capital depreciation shock. A 1% normalization is sufficient because the FAME is linear and our estimation is based on relative responses, a point to which we return below. We use these simulated responses to construct model counterparts to the event-study estimates in Bilal and Rossi-Hansberg (2025). To incorporate the empirical spatial correlation of storms, we weight locations by their observed frequency of extreme events. Finally, we use these model-implied event-study estimates to build the target moments that identify (ν, ζ, η).

In a third step, we find the parameter vector (ν, ζ, η) among our set that best matches the data. Then, starting from this parameter vector, we use a standard minimum distance solver to adjust the parameters and close any remaining discrepancy relative to the data.

Although we estimate the three parameters (ν, ζ, η) jointly, we briefly discuss how we select each moment to provide intuition for identification. The gravity structure of migration implies that the migration elasticity ν is identified by the response of local employment relative to that of income per capita, holding amenities fixed. Accordingly, we use the response of employment relative to income per capita following a capital depreciation shock, which does not affect amenities.

To inform the amenity-productivity ratio η, we use the cumulative impulse response of employment relative to that of income per capita ten years after a productivity-amenity shock. We match this moment to its empirical counterpart following a heat wave in warm counties. Intuitively, if employment responds strongly relative to income per capita, holding the migration elasticity ν fixed, then amenities must constitute an important component of the overall effect of heat waves.

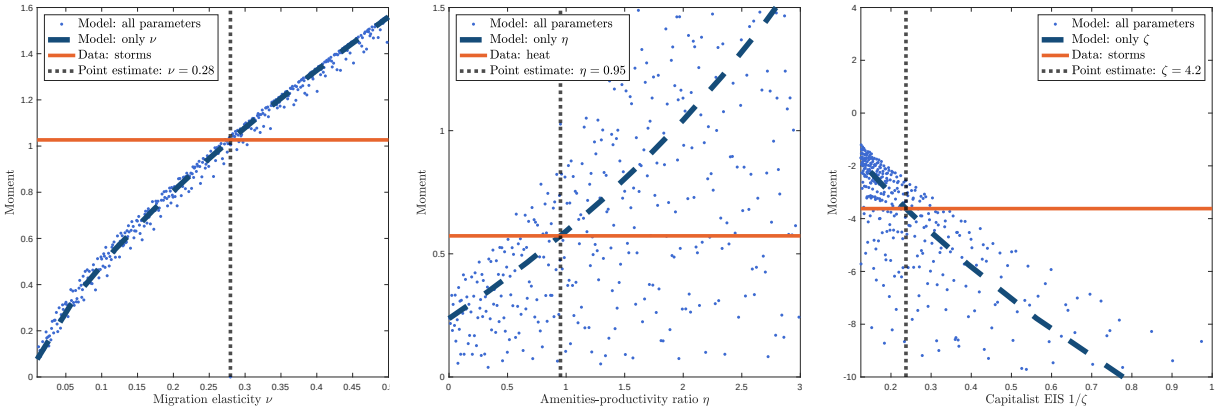
The capitalist preference parameter ζ is informed by the cumulative impulse response

Figure 1: Identification of migration elast., amenity-productivity ratio, and capitalist IES.

(a) Storms and capital depreciation: employment/income per capita CIR

(b) Heat waves and prod.-amenity: employment/ income per capita CIR

(c) Storms and capital depreciation: investment/employment CIR



Note: Blue dots: 5,000 model simulations sequencing parameter vectors (ν, η, ζ) using a Sobol sequence in the cube $[0.01, 0.5] \times [0, 3] \times [1, 10]$. Dashed dark blue line: moment in the model when only the parameter on the x-axis varies, holding other parameters at their point estimate. Horizontal solid orange line: moment in the data. Vertical dashed black line: parameter estimate.

of investment relative to employment ten years after a capital depreciation shock. Intuitively, the employment response captures the overall size of the shock, while the response of investment relative to employment reflects intertemporal substitution in capitalists' consumption and saving decisions. Hence, if investment responds strongly relative to employment, capitalists must be more willing to shift consumption across time, which corresponds to a lower value of ζ. We target the corresponding empirical moment using the response to storms in coastal counties.

We invert fundamentals and solve for the model's steady state, the FAME, and the associated impulse responses for 500 candidate triplets (ν, ζ, η). For each triplet, we compute the model-implied target moments and compare them with their empirical counterparts. Figure 1 summarizes the results of this exercise. After identifying the triplet among the 500 candidates that delivers the best fit, we refine the estimates using a Newton search to eliminate the remaining discrepancy between model and data moments.[6]

Panel 1(a) plots the cumulative response of employment relative to income per capita

[6]To cover the parameter space efficiently, we aggregate counties into 100 clusters for this exercise. We form these clusters using a k-means algorithm based on latitude, longitude, population, income, and coastal status. The results change only slightly when we increase the number of clusters.

following a capital depreciation shock. Each blue dot corresponds to the model-implied relative cumulative response for a given parameter triplet (ν, η, ζ). To highlight identification, we order the results on the x-axis by the migration elasticity ν. The scatterplot reveals a clear positive relationship: across all possible values of (η, ζ), higher values of ν are associated with a stronger response of employment relative to income per capita, which illustrates multivariate identification. The same pattern is present when we hold (η, ζ) fixed at their point estimates, in which case the moment continues to rise with ν, illustrating univariate identification. The orange horizontal line corresponds to the empirical moment for storms in coastal counties.

Panel 1(b) plots the cumulative response of employment relative to income per capita following a productivity-amenity shock against the productivity-amenity ratio η. While there is more residual dispersion that depends on other parameters, the model produces an upward-sloping relationship on average.

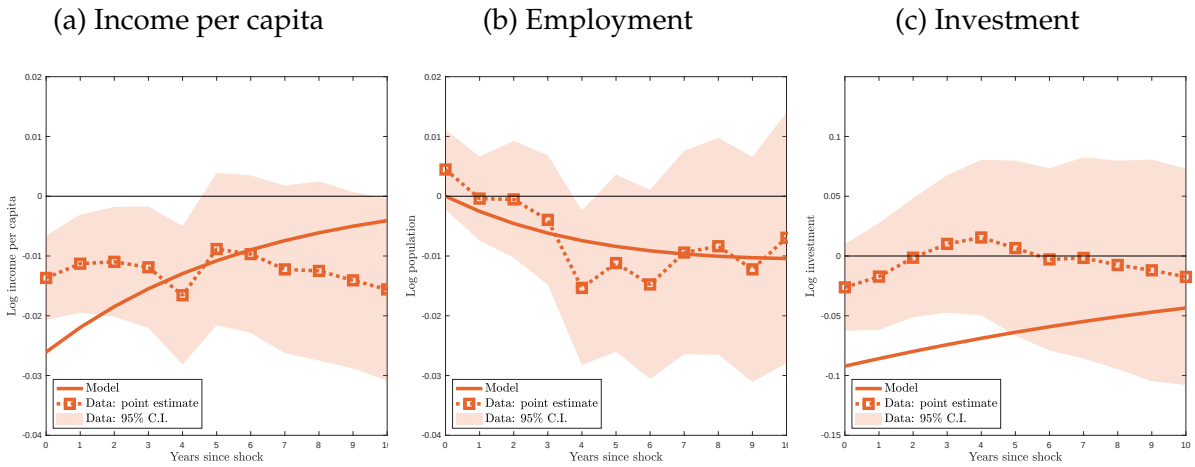
Finally, Panel 1(c) plots the cumulative response of investment relative to employment following a capital depreciation shock against the inverse of the capitalist coefficient of relative risk aversion. The figure highlights that this parameter is a central driver of the investment response relative to employment.

Our indirect inference approach implies that the migration elasticity is $\nu = 0.28$, which is similar to the value in the county-level specification in Bilal and Rossi-Hansberg (2025) and moderately higher than the value of 0.2 reported by Caliendo et al. (2019) in their state-level specification. We also find that amenities respond $\eta = 0.95$ times as much as productivity following a heat wave. This value exceeds the estimate reported by Cruz and Rossi-Hansberg (2023) for the world as a whole, suggesting that, relative to the rest of the world, production in the U.S. may be more resilient than amenities following such a shock. Finally, our estimate of the capitalist coefficient of relative risk aversion, $\zeta = 4.2$, lies within the relatively wide range of values commonly reported in the literature between 1 and 10.

4.5 Damage Functions and Local Risk Exposures

With the migration and investment elasticities in hand, we turn to estimating the damage functions for productivity, amenities, and capital depreciation χ_i, a_i, δ_i and local capital

Figure 2: Heat waves: impulse responses in model and in data.



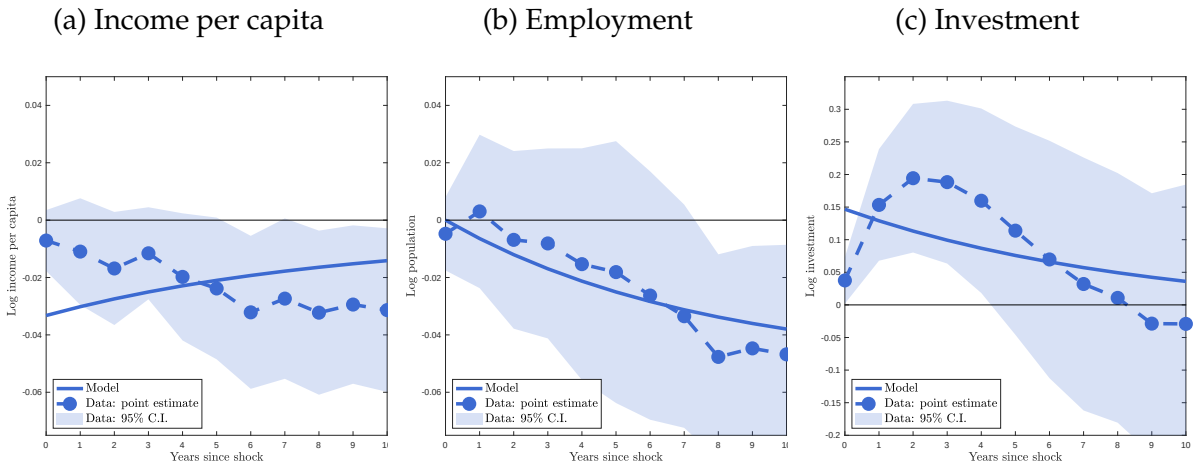
Note: Impulse responses for income per capita (a), employment (b), and investment (c) after a 1-in-20-years heat wave in a warm county. For (a) we use the response of wages, w_{it}, in the model and income per capita in the data. Solid line: model. Dashed line with circle markers: data point estimate. Shaded area: data 95% confidence band. The 10-year cumulative impulse responses for income per capita (a) and employment (b) are targeted. The impulse response for investment (c) is untargeted.

depreciation risk exposures Σ_i. We proceed in two steps. First, we estimate common economic damages from a given realization of storms $\Gamma^{\text{cap dep}}$ and heat waves Γ^{prod}. Second, we interact these damages from single events with local changes in the probability that these events occur as global mean temperatures rise.

In the first step, we estimate the economic damages associated with a given realization of storms, $\Gamma^{\text{cap dep}}$, and heat waves, Γ^{prod}, by matching the magnitude of cumulative impulse responses. This step exploits the linearity of the FAME. In particular, it implies that the size of shock s is given by $\Gamma^s = \frac{\text{CIR}_{10,v,s}}{\text{cir}_{10,v,s,\widehat{\tau}}}$, where $\text{CIR}_{10,v,s}$ denotes the observed 10-year cumulative response of variable v to shock s in the data, and $\text{cir}_{10,v,s,\widehat{\tau}}$ denotes the corresponding 10-year cumulative response in the model evaluated at the estimated parameter triplet $\widehat{\tau}$.

For heat waves in warm counties, we target the response of income per capita. Since the estimated model matches the response of employment relative to income per capita exactly in Figure 1, the model also matches exactly the cumulative responses of both employment and income per capita at the 10-year horizon. For storms in coastal counties, we target the cumulative effect on employment ten years after the shock. Because the estimated model matches the response of investment relative to employment exactly in Figure 1, the model also matches exactly the cumulative response of investment at that

Figure 3: Storms: impulse responses model and in data.



Note: Impulse responses for income per capita (a), employment (b), and investment (c) after a 1-in-50-years storm in a coastal county. For (c) we use the response of wages, w_{it}, in the model and income per capita in the data. Solid line: model. Dotted line with circle markers: data point estimate. Shaded area: data 95% confidence band. The 10-year cumulative impulse response for income per capita (a), employment (b), and investment (c) are targeted.

horizon.

Figure 2 compares the impulse responses of income per capita, employment, and investment following heat waves in warm counties. The model replicates the path of employment, and income per capita to heat waves. The response of investment, which is not directly targeted, is also close to its empirical counterpart. For all outcomes, the model-implied responses lie within the 95% confidence interval around the empirical estimates for nearly all horizons.

Our impulse-response matching exercise implies that a 1-in-20-year heat wave induces a negative productivity shock of $\Gamma^{\text{prod}} = 2.5\%$ in a warm county, along with a $\eta\Gamma^{\text{prod}} = 2.4\%$ decline in local amenities. While economically meaningful, this effect is smaller than the local impacts estimated for the world as a whole, including developing countries.[7]

Figure 3 compares the impulse responses of income per capita, employment, and investment following storms in coastal counties. As in the previous figure, the data are shown by circles and a dotted line, while the model is shown by the solid line. The figure shows that the model provides a reasonably good account of the full time path of employment and investment, remaining within the 95% confidence interval for most horizons.

[7]For instance, Cruz and Rossi-Hansberg (2023) find that, in the warmest locations, a 1°C increase in winter temperatures can lower productivity by more than 15%.

The fit is weaker for income per capita, as the model predicts a response that is too rapid relative to the data.

Our impulse-response matching exercise implies that a 1-in-50-year storm generates a capital depreciation shock of 10 percent, $\Gamma^{\text{cap dep}} = 12\%$, in a coastal county. While sizable, this estimate is smaller than the 30% value reported in Bilal and Rossi-Hansberg (2025). This discrepancy arises because of different structure of the capitalist problem. When capitalists are risk-averse, shifting consumption intertemporally is more costly. A smaller capital depreciation shock results in larger changes in investment, thereby rationalizing the same empirical impulse response function.

We now turn to the second step of the estimation strategy, which consists of constructing damage functions that vary with global mean temperature. This part of the analysis closely follows Bilal and Rossi-Hansberg (2025), but we include it here for completeness. We combine our estimated shock magnitudes, $(\Gamma^{\text{cap dep}}, \Gamma^{\text{prod}}, \eta\Gamma^{\text{prod}})$, with long-run changes in the probability of the corresponding extreme events.

To estimate how the probability of extreme events evolves across locations, we use the historical record starting in 1901. For each location i and event type $e \in$ storm, heat, we regress changes in the probability of the event, p_{it}^e, on changes in global mean temperature, T_t, over the period 1901 to 2019:

$$dp_{it}^e = \text{cst} + q_i^e \, dT_t + \epsilon_{it}. \tag{17}$$

Equation (17) delivers a statistical downscaling, a standard approach in the climate science and economics literature (Cruz and Rossi-Hansberg, 2023), which links local event probabilities to global mean temperature.

We then construct the damage functions as follows:

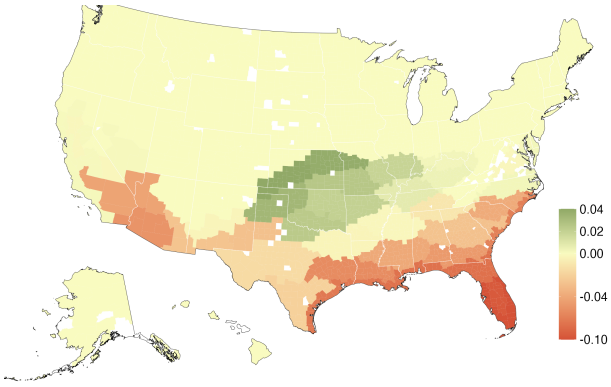
$$\delta_i = \mathbb{1}\{i \text{ coastal}\} q_i^{\text{storm}} \Gamma^{\text{cap dep}}, \quad \chi_i = \mathbb{1}\{i \text{ warm}\} q_i^{\text{heat}} \frac{\Gamma^{\text{prod}}}{\psi^{\text{prod}}}, \quad a_i = \mathbb{1}\{i \text{ warm}\} q_i^{\text{heat}} \frac{\eta\Gamma^{\text{prod}}}{\psi^{\text{prod}}},$$

where ψ^{prod} denotes the mean-reversion coefficient of productivity following heat waves.[8] Finally, we obtain local capital depreciation risk exposures by leveraging a central restric-

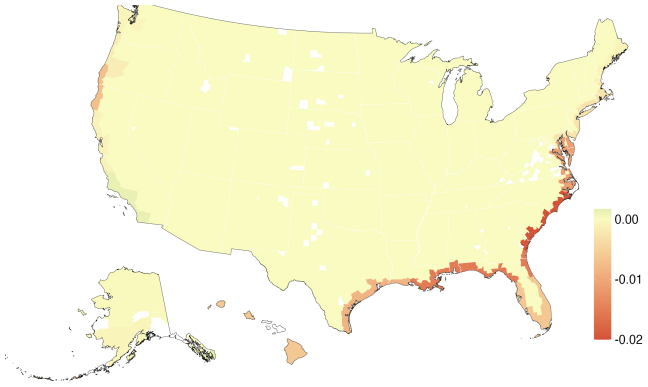
[8] We divide by ψ^{prod} in order to cumulate the lagged effects of heat waves and convert our estimate into the effect of a permanent increase in exposure to extreme heat.

Figure 4: Damage functions from heat and storms in the United States.

(a) Change in log productivity



(b) Change in annual capital depreciation



Note: Effect of a 1°C increase in global temperature on productivity and annual capital depreciation rates across counties in the U.S. Panel (a): coefficients χ_i such that $d \log Z_{it} = \chi_i dT_t$. Panel (b): negative of coefficients δ_i such that $d\Delta_{it} = \delta_i dT_t$.

tion from our microfoundation in Appendix C: $\Sigma_i^2 = \frac{\delta_i}{\Gamma^{\text{cap dep}}}$.

Figure 4 maps the estimated damage functions across U.S. counties. Panel 4(a) shows the effect of a 1°C increase in global mean temperature on productivity in each county. These effects are highly heterogeneous across space. We estimate productivity losses above 8 percent in southern Florida, where the frequency of heat waves rises particularly quickly. Similar losses are prevalent across much of the southern United States. In contrast, northern Texas and Oklahoma experience modest gains, consistent with a historical decline in heat wave exposure in those locations. By assumption, amenity changes are proportional to productivity changes, so we omit a separate map for amenities.

Panel 4(b) illustrates the impact of a 1°C increase in global mean temperatures on capital depreciation for every county in the U.S. The impact is highly heterogeneous across space and only imperfectly correlated with the one for heat waves. Storms increase capital depreciation rates substantially in the South-Eastern Atlantic coast. In Louisiana and North and South Carolina, a 1°C increase in global temperature increases the capital depreciation rate by more than 2 p.p. This effect amounts to more than a 6 p.p. increase for a warming scenario of 3°C , more than doubling the baseline capital depreciation rate. These capital depreciation effects are substantial, but affect a much smaller fraction of counties than heat waves. Local risk exposures Σ_i^2 and the baseline level of local risk ϑ_i are proportional to average capital depreciation exposures δ_i and baseline storm proba-

bilities, so that we omit a separate map. Through the lens of our structural model, we can compare the relative importance of these channels.

5 The Impact of Climate Risk in the United States

5.1 Setup

We evaluate the effects of aggregate and local risk on economic activity across the United States. We start the economy in the stochastic steady state in which there is background local risk ϑ_i. We take 2025 as the year in which agents learn the warming path T_t and the paths of aggregate and local risk, σ_t and Σ_{it}.

We consider a warming scenario in which T_t rises gradually by 2°C by 2100 and remains constant thereafter. We set $\sigma_t = T_t/4$ to impose a constant relative uncertainty around our central warming path. We choose the factor $1/4$ as it corresponds almost exactly to the IPCC's "high confidence" range of plausible additional 2100 warming realizations: 1°C to 3°C . Through the lens of our framework, the 95% confidence interval of 2100 global temperature is then $T_t \pm 1.96\sigma_t$, which corresponds to a 1.0-3.0°C range.

To highlight the role of capital depreciation, we focus only on the effects of storms, so we set the damage functions $\chi_i = a_i = 0$. We plan to incorporate the effect of productivity and amenities in future iterations of this paper.

5.2 Local and Aggregate Risk in the Long Run

We first study the long-run effects of local and aggregate risk. To do so, we set as baseline an economy already subject to a 2°C increase in temperature, but with no local or aggregate risk. That is, we set $\sigma_t = 0$ and $\Sigma_{it} = 0$ for all i and t. We then compute new stochastic steady states in which we first introduce local climate risk by setting $\Sigma_{it} > 0$, and then aggregate risk by setting $\sigma_t > 0$.

Figure 5 displays the effects of local risk on capitalists and the capital stock. Panel 5(a) reveals that local risk leads to substantial welfare losses for capitalists in southern coastal locations. In coastal South Carolina or Louisiana, capitalists lose up to 25% in consumption-equivalent terms from the increase in local risk. These losses are driven by

a stark reduction in the value of capital: as global mean temperatures rise, storms become more frequent. As they are a binomial stochastic process in our microfoundation, the local volatility of the capital depreciation process rises steeply. This volatility renders capital less valuable and thus hurts capitalists.

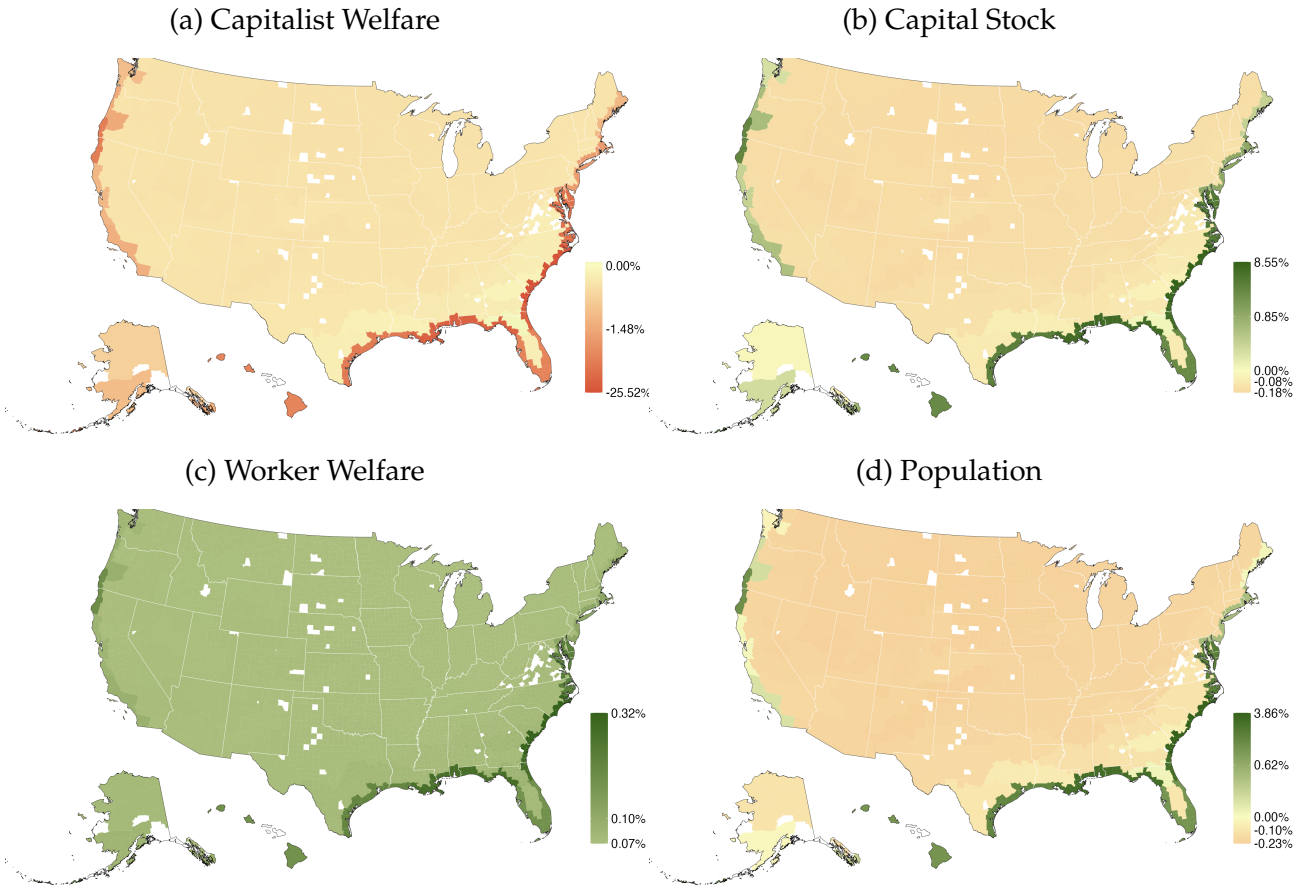
Panel 5(b) illustrates how capitalists adapt to local capital depreciation risk. Subject to more risk, they build precautionary savings that translate into more capital accumulation on exposed coasts.[9] Naturally, these precautionary savings offset only a small fraction of the losses associated with falling values of capital.

Panel 5(c) shows that workers benefit modestly from local capital depreciation risk throughout the United States, but particularly on the southeastern coast. In our current quantification, workers have logarithmic utility, which implies that they are risk-neutral with respect to *proportional* risk. Thus, they do not experience any direct disutility of *proportional* fluctuations in real wages and the capital stock. By contrast, they benefit from the additional capital driven by the precautionary motives of capitalists. Panel 5(d) shows that this force indeed pulls workers away from inland locations and into coastal regions. In future work we plan to consider settings in which the risk aversion of workers is different from unity, rendering them more sensitive to local risk.

Aggregate risk has opposite implications for welfare and allocations, with magnitudes that are more moderate. Figure 6 illustrates the effect of aggregate risk on the capitalist and worker welfare across U.S. counties, and capital stocks and population. Panel 6(a) shows that capitalists benefit from aggregate risk throughout the country. The reason is that, unlike local risk, aggregate climate risk leads to predictable average capital depreciation before making investment decisions. Thus, capitalists can respond to aggregate shocks and arbitrage return differentials over time. Namely, they can invest when the climate evolution is more benign, and consume when the negative climate trends are more harmful to their investments in their home locations. In effect, capital returns are convex in aggregate climate risk, implying that capitalists can gain from increases in this form of risk. Ultimately, capitalists more exposed to climate risk experience welfare gains of up to 3% in consumption-equivalent terms. Faced with effectively higher returns, capitalists

[9]Of course, if capitalists had other assets at their disposal to save, they would direct part of their precautionary savings there. Introducing a portfolio problem together with irreversible investment in physical capital is, however, beyond the scope of this paper.

Figure 5: Effect of Local Risk on Capitalists and Workers.



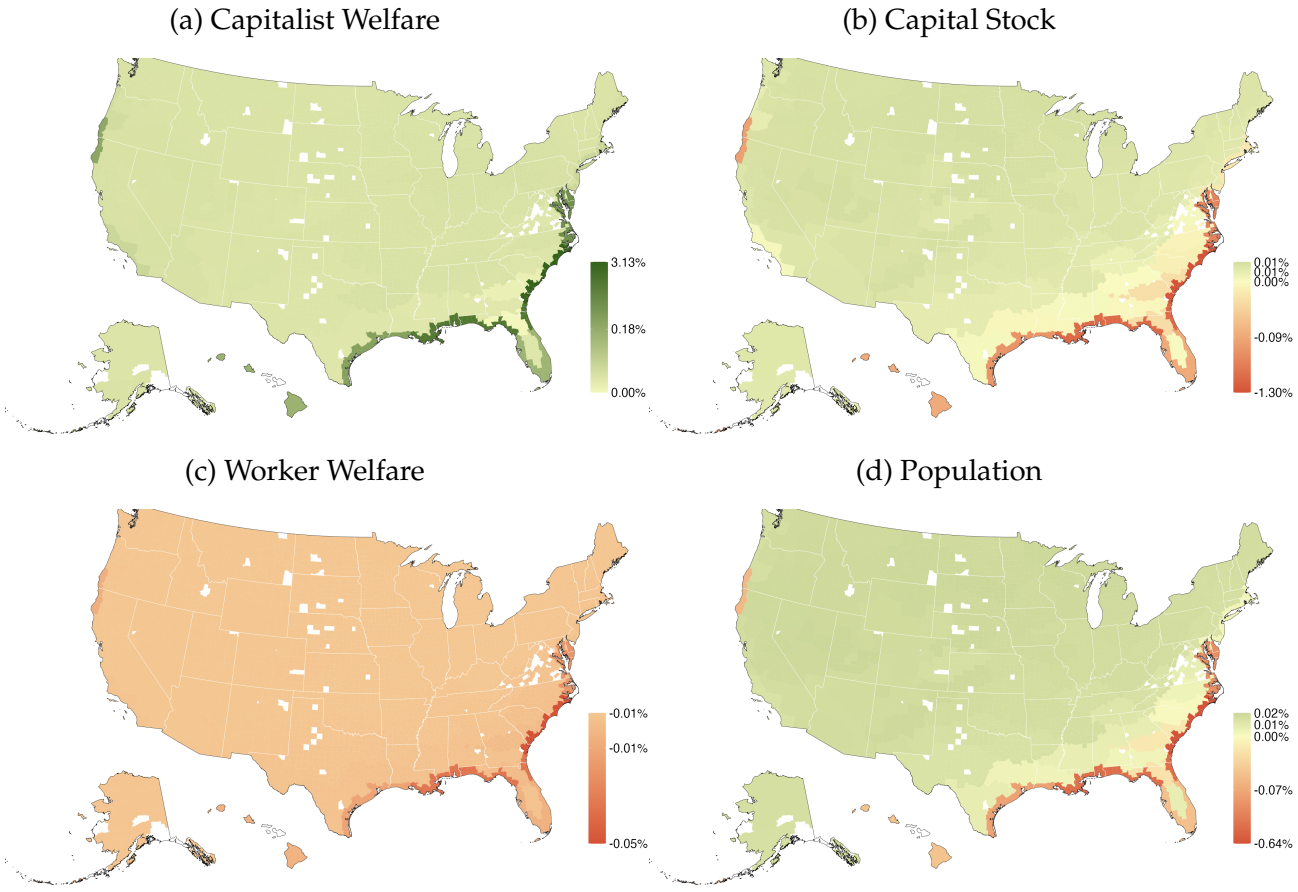
Note: Effect of local risk on capitalists in an economy already exposed to a 2°C increase in global temperature. Panel (a): change in capitalist welfare relative to the deterministic steady state. Panel (b): change in capital stock relative to the deterministic steady state. Panel (c): change in worker welfare relative to the deterministic steady state. Panel (d): change in population relative to the deterministic steady state.

can afford to hold less capital and consume more. Panel 6(b) shows that the capital stock decreases modestly in coastal counties in the southeast region.

Workers lose from the decline in capital stocks across the U.S. Panel 6(c) exhibits negative though small welfare declines. Panel 6(d) shows that workers move out from southeastern coastal locations that lose the most capital.

Quantitatively, the effects from aggregate climate risk are an order of magnitude smaller than the effects of local climate risk—regardless of their signs. This may appear surprising given the nontrivial uncertainty surrounding possible warming paths. Yet, aggregate climate change risk is intermediated by the expected capital depreciation rate due to warming, which is bounded by the relatively low baseline hazard of extreme weather: the

Figure 6: Effect of Aggregate Climate Risk on Capitalists and Workers.



Note: Effect of aggregate risk on capitalists in an economy already exposed to a 2°C increase in global temperature. Panel (a): change in capital stock relative to the deterministic steady state. Panel (b): change in capitalist welfare relative to the deterministic steady state.

local variance of capital depreciation due to aggregate climate risk is $(\delta_i \sigma_t)^2 = \delta_i^2 T_t^2/16$. By contrast, local risk directly changes the hazard of extreme weather realizations, with local variance $\Sigma_i^2 T_t^2 = \delta_i T_t^2/\Gamma^{\text{cap dep}}$. As the baseline damage function $\delta_i < 1$ is less than unity, damages associated with storms $\Gamma^{\text{cap dep}} < 1$ are also less than unity, and the likely range of warming represents a fraction of the central scenario, aggregate risk is always quantitatively less important than local risk.

5.3 Local and Aggregate Risk Over Time

We now compare the effects of local and aggregate risk to the deterministic impact of warming along a transition to a warmer world. We allow for the full impact of tempera-

ture change, including its effect on mean capital depreciation. We set 2025 as our baseline stochastic steady state with background local risk and study the transition to a new steady state associated with a 2°C increase in temperature by 2100. Throughout the transition, local and aggregate risk evolve over time as described in Section 5.1.

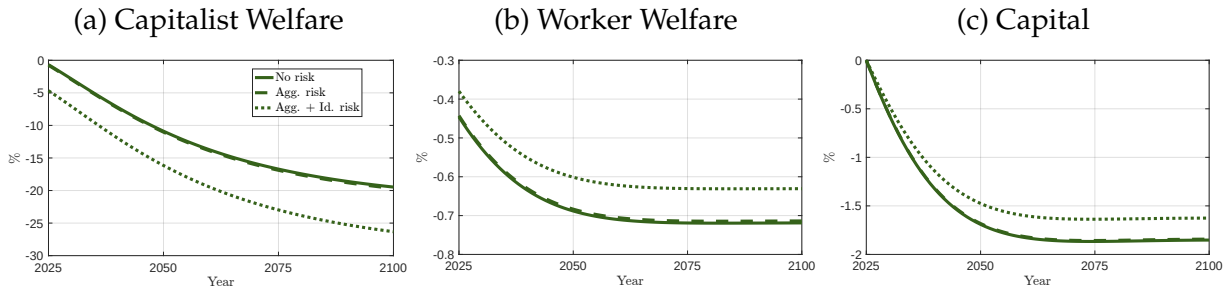
Figure 7 displays transitional dynamics under three configurations. The first configuration examines an increase in temperature but no risk at all. The second configuration assesses the effect of temperature increases with aggregate risk only. The third configuration considers temperature increases with both aggregate and local risk.

Panel 7(a) depict capitalist welfare. Deterministic warming creates winners and losers among capitalists by pushing workers out of the Southeast and into the rest of the U.S. These gains and losses partly offset initially, leading to capitalist welfare losses below 1%. Consistent with the results from Section 5.2, aggregate risk contributes little to capitalist welfare losses. By contrast, local risk—uncertain extreme weather realizations—leads to less migration because of the precautionary effect which retains workers on the coast, and thus yields fewer winners among capitalists. Quantitatively, local risk unambiguously magnifies aggregate capitalist welfare losses from less than 1% to over 5% on impact. By the end of the century, deterministic warming leads to aggregate capitalists losses just under 20%, while local risk contributes an additional 8% in losses.

Panel 7(b) displays worker welfare during the transition. Workers experience more moderate welfare losses of 0.7% in the long run in consumption-equivalent terms following a 2°C increase in temperature, because a smaller share of their income is dependent on capital. Introducing aggregate risk leaves the welfare effects largely unchanged. Local risk mitigates these welfare losses by 0.1 p.p. in the long run due to the excess capital accumulation from precautionary savings.

Figure 7(c) shows the path of the aggregate capital stock. Warming leads to capital depreciation and a lower capital stock at all horizons. By 2100, the U.S. economy experiences a decline in aggregate capital of slightly less than 2%. Aggregate risk has a negligible effect on the capital stock, consistent with Figure 6. By contrast, local risk has a larger effect, offsetting 0.3 p.p. of the decline in the long run.

Figure 7: Effect of Local and Aggregate Risk in a Warming World.



Note: Effect of aggregate local and aggregate risk during the transition towards a a 2°C increase in global temperature by 2100. Panel (a): change in aggregate capitalist welfare relative to the 2025 deterministic steady state. Panel (b): change in aggregate workers welfare relative to the 2025 deterministic steady state. Panel (c): change in capital stock relative to the 2025 deterministic steady state.

6 Conclusion

A distinctive feature of climate change is that it exacerbates aggregate and local risk. On the one hand, we are uncertain about its aggregate severity and, therefore, about how it will affect specific regions. On the other hand, global warming will increase the variability of local weather patterns. These two forms of climate-induced risk are distinct and affect agents and regions differentially. The former, which we refer to as aggregate risk, involves uncertainty about long-term climate paths, allowing local investors to react and invest conditionally on realizations. The latter, which we dubbed local risk, is realized ex post and implies capital destruction that unequivocally harms investors. This distinction is essential for understanding the magnitude of the effects of each type of risk, its effect on capitalists and workers, and its impact on the distribution of economic activity across counties in the U.S. Local risk concentrates economic activity in the locations that are more severely affected by climate change, as capitalists invest more for precautionary savings purposes, and the additional capital attracts workers. In contrast, aggregate risk has less harmful effects on capitalists and reduces capital in the worst-affected locations, driving workers out. Overall, we show that local risk leads to effects an order of magnitude larger, and thus its consequences should be more salient to policymakers. Of course, the impacts of climate risk are all in addition to the average negative consequences of climate change across locations.

Our paper is among the first to study climate risk in a dynamic spatial economy with labor mobility and forward-looking capital investments, given the methodological com-

plexity of such analysis. Of course, the model we propose can be extended in several ways. Particularly important is, perhaps, to expand the investment opportunities of local capitalists and to incorporate non-reversibilities in investment. Our framework forces local capitalists either to consume or to invest in the capital of their home location. They cannot invest in a national bond or in capital in other locations. The lack of alternative investment options likely accentuates the importance of the precautionary savings mechanism. Our modeling of capital as reversible implies that capitalists can always disinvest and consume if climate paths turn out to be more harmful than expected, accentuating the real option mechanism for aggregate risk. Incorporating these realistic features of the investment problem faced by local capitalists is, therefore, a promising avenue for future work. The current paper can serve as the foundation to do so.

APPENDIX

A Setup

A.1 Static Equilibrium

Final good production. The wage w_{it} in location i satisfies the firm's first-order condition $w_{it} = (1-\alpha)Z_{it}(S_{it}/N_{it}^P)^{\alpha}$. Neither renters nor firms cover depreciation, so that the rental rate for structures satisfies $r_{it} = \alpha Z_{it}(S_{it}/N_{it}^P)^{-(1-\alpha)}$.

Buildings production. The optimal choice of labor in building production implies $\varpi r_{it} B_{it}/N_{it}^B = w_{it}$.

Housing demand. Local demand for housing clears the housing market $\beta w_{it} N_{it} = r_{it} H_{it}$.

Equilibrium prices. We now solve for equilibrium prices and quantities as functions of the local capital stock K_{it} and local number of workers N_{it}. Combining labor demand in final good and building production, we obtain $\varpi\alpha\frac{B_{it}}{N_{it}^B} = (1-\alpha)\frac{S_{it}}{N_{it}^P}$. Housing demand rewrites $\beta(1-\alpha)N_{it}\frac{S_{it}}{N_{it}^P} = \alpha H_{it}$.

We look for shares x, y such that

$$N_{it}^P = xN_{it} \ , \ H_{it} = yB_{it} \tag{18}$$

and so $N_{it}^B = (1-x)N_{it}$ and $S_{it} = (1-y)B_{it}$. Substituting into the previous equations, we obtain

$$\frac{\varpi\alpha}{1-\alpha} = \frac{(1-y)(1-x)}{x} \ , \ \frac{\beta(1-\alpha)}{\alpha} = \frac{xy}{1-y}.$$

These two equations imply[10]

$$x = \frac{(1-\alpha)(1-\varpi\beta)}{\alpha\varpi + (1-\alpha)} \ , \ y = \beta\frac{(1-\alpha)+\alpha\varpi}{\alpha+(1-\alpha)\beta} \ , \ 1-y = \alpha\frac{1-\beta\varpi}{\alpha+(1-\alpha)\beta}. \tag{19}$$

Using these shares, we express the wage and the rental rate as

$$\begin{aligned} w_{it} &= (1-\alpha)\alpha^{\alpha}\Xi^{-\alpha}Z_{it}(B_{it}/N_{it})^{\alpha} \\ r_{it} &= \alpha^{\alpha}\Xi^{1-\alpha}Z_{it}(B_{it}/N_{it})^{-(1-\alpha)} \end{aligned} \tag{20}$$

where

$$\Xi = (1-\alpha)\frac{\alpha+(1-\alpha)\beta}{\alpha\varpi+(1-\alpha)}$$

Before proceeding, it is useful to substitute out buildings and express prices in terms of capital:

$$\begin{aligned} w_{it} &= (1-\alpha)\alpha^{\alpha}\Xi^{-\alpha}(1-x)^{\varpi\alpha}Z_{it}L_i^{\omega\alpha}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{\alpha} \\ r_{it} &= \alpha^{\alpha}\Xi^{1-\alpha}(1-x)^{-(1-\alpha)\varpi}Z_{it}L_i^{-\omega(1-\alpha)}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{-(1-\alpha)} \end{aligned} \tag{21}$$

Rental rate of capital. We also need the rental rate of capital:

$$\begin{aligned} R_{it} &= (1-\omega-\varpi)(1-x)^{\varpi}K_{it}^{-(\omega+\varpi)}L_i^{\omega}N_{it}^{\varpi}r_{it} \\ &= (1-\omega-\varpi)\alpha^{\alpha}\Xi^{1-\alpha}(1-x)^{\alpha\varpi}Z_{it}L_i^{\omega\alpha}K_{it}^{-\phi}N_{it}^{1-\alpha+\alpha\varpi} \end{aligned} \tag{22}$$

We obtain:

$$\begin{aligned} R_{it} &= \underbrace{(1-\omega-\varpi)\alpha^{\alpha}\Xi^{1-\alpha}(1-x)^{\alpha\varpi}Z_iL_i^{\omega\alpha}}_{\equiv R_{0i}}e^{\chi_{it}}K_{it}^{-\phi}N_{it}^{\psi} \\ &= R_{0i}e^{\chi_{it}}K_{it}^{-\phi}N_{it}^{\psi} \\ &\equiv R_i(\chi_{it},K_{it},N_{it}) \end{aligned} \tag{23}$$

[10]Multiplying both equations leads to $y(1-x) = \varpi\beta$ which can then be substituted into either of of the equations.

where $\phi = \omega + \varpi + (1-\omega-\varpi)(1-\alpha)$ and $\psi = 1 - \alpha + \alpha\varpi$.

Consumption and flow utility. We also express total consumption (real wage) in location i as

$$\begin{aligned}
c_{it} &= \frac{w_{it}}{r_{it}^{\beta}} \\
&= \frac{(1-\alpha)\alpha^{\alpha}\Xi^{-\alpha}(1-x)^{\varpi\alpha}Z_{it}L_i^{\omega\alpha}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{\alpha}}{\left[\alpha^{\alpha}\Xi^{1-\alpha}(1-x)^{-(1-\alpha)\varpi}Z_{it}L_i^{-\omega(1-\alpha)}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{-(1-\alpha)}\right]^{\beta}} \\
&= (1-\alpha)\alpha^{\alpha(1-\beta)}\Xi^{-\xi}(1-x)^{\varpi\xi}Z_{it}^{1-\beta}L_i^{\omega\xi}K_{it}^{(1-\omega-\varpi)\xi}N_{it}^{-\xi(1-\varpi)} \\
&= \underbrace{(1-\alpha)\alpha^{\alpha(1-\beta)}\Xi^{-\xi}(1-x)^{\varpi\xi}Z_i^{1-\beta}L_i^{\omega\xi}}_{\equiv C_{0i}} \cdot e^{(1-\beta)\chi_{it}}\left(\frac{K_{it}^{1-\omega-\varpi}}{N_{it}^{1-\varpi}}\right)^{\xi} \\
&\equiv c_{0i}e^{(1-\beta)\chi_{it}}\left(\frac{K_{it}^{1-\omega-\varpi}}{N_{it}^{1-\varpi}}\right)^{\xi} \qquad (24)
\end{aligned}$$

where $\xi = \alpha + \beta(1-\alpha)$. With consumption at hand, we express the flow utility in location i at time t:

$$u\left(A_{it}\frac{w_{it}}{r_{it}^{\beta}}\right) = u\left(C_{0i}e^{a_{it}+(1-\beta)\chi_{it}}\left(\frac{K_{it}^{1-\omega-\varpi}}{N_{it}^{1-\varpi}}\right)^{\xi}\right) \equiv U_i(a_{it},\chi_{it},K_{it},N_{it}), \qquad (25)$$

where $C_{0i} \equiv A_i C_{0i}$.

A.2 Capitalists

We guess and verify that:

$$\mathcal{Q}_{it}(K) = \frac{Q_{it}K^{1-\zeta}}{1-\zeta} + P_{it} \qquad C = c_{it}K \qquad (26)$$

Under this guess: $\mathcal{Q}_{it,K} = Q_{it}K^{-\zeta}$ and $\mathcal{Q}_{it,KK} = -\zeta Q_{it}K^{-\zeta-1}$. Substituting the guess into the HJB:

$$\rho\frac{Q_{it}K^{1-\zeta}}{1-\zeta} + \rho P_{it} = \max_c \frac{c^{1-\zeta}K^{1-\zeta}-1}{1-\zeta} + [(R_{it}-\Delta_{it})-c]Q_{it}K^{1-\zeta} - \frac{\zeta\Sigma_{it}^2}{2}Q_{it}K^{1-\zeta} + \frac{K^{1-\zeta}}{1-\zeta}\mathbb{E}_t\left[\frac{dQ_{it}}{dt}\right] + \mathbb{E}_t$$

Identifying components:

$$\rho Q_{it} = \max_c c^{1-\zeta} + (1-\zeta)\left[(R_{it}-\Delta_{it})-c\right]Q_{it} - \frac{\zeta(1-\zeta)\Sigma_{it}^2}{2}Q_{it} + \mathbb{E}_t\left[\frac{dQ_{it}}{dt}\right]$$
$$\rho P_{it} = -\frac{1}{1-\zeta} + \mathbb{E}_t\left[\frac{dP_{it}}{dt}\right] \quad \Rightarrow \quad P_{it} = -\frac{1}{\rho(1-\zeta)}$$

So we get the FOC:

$$c^{-\zeta} = Q_{it} \quad \Rightarrow \quad c = Q_{it}^{-1/\zeta} \quad \Rightarrow \quad \frac{I}{K} = R_{it} - c = R_{it} - Q_{it}^{-1/\zeta}$$

where we have used that $I = R_{it}K - C_{it}$.

Substitute back into the HJB:

$$\begin{aligned}\rho Q_{it} &= Q_{it}^{1-1/\zeta} + (1-\zeta)\left[R_{it}-\Delta_{it}-Q_{it}^{-1/\zeta}-\frac{\zeta\Sigma_{it}^2}{2}\right]Q_{it} + \mathbb{E}_t\left[\frac{dQ_{it}}{dt}\right] \\ &= \zeta Q_{it}^{1-1/\zeta} + (1-\zeta)\left[R_{it}-\Delta_{it}-\frac{\zeta\Sigma_{it}^2}{2}\right]Q_{it} + \mathbb{E}_t\left[\frac{dQ_{it}}{dt}\right].\end{aligned}$$

With log utility $\zeta = 1$, we obtain:

$$\rho Q_{it} = 1 \qquad\qquad \mathcal{Q}_{it}(K) = \frac{1}{\rho}\log K,$$

and so proportional risk Σ_{it} does not matter as income and substitution effects cancel out.

B Steady State Computation

Solving for steady-state values is relatively straightforward:

1. Start by using (10) to solve for Q_i^{ss} and the consumption rate:

$$(Q_i^{ss})^{-1/\zeta} = \rho = c^{ss}$$

2. Given Q_i^{ss}, then use (11) to solve for the equilibrium capital as a function of population:

$$K_i^{ss} = \left(\frac{R_{0i}(N_i^{ss})^{\psi}}{\rho + \Delta_i} \right)^{1/\phi} \equiv \overline{K}_i(N_i^{ss}) \tag{27}$$

3. Substitute (27) into (10) to obtain

$$\rho V_i^{ss} = U_i(\mathcal{K}_i(N_i^{ss}), N_i^{ss}) + \mathcal{M}_i[V^{ss}] \tag{28}$$

4. Alternate on (28) and (11) to jointly solve the equilibrium values V_i^{ss} and population N_i^{ss}:

   (a) Start from a uniform guess for N_i^{ss} (or any other guess)

   (b) Solve (28) for V_i^{ss}

   (c) Update N_i^{ss} by solving (11) given V_i^{ss}

   (d) Keep iterating on (b-c) until both N_i^{ss} and V_i^{ss} have converged.

C Microfoundation of the capital depreciation process

C.0.1 Microfoundation

We lay out a simple microfoundation starting from Poisson capital depreciation shocks for the diffusion process for capital depreciation. Suppose that there are infrequent depreciation shocks of magnitude $1 - D < 1$. They arise with some Poisson rate that depends on the global mean temperature relative to the pre-industrial level. D is the amount of capital that is destroyed by the shock. We denote excess temperature by $T = \overline{T} - T^B$. We omit i subscripts for now.

The law of motion of local capital follows a drift-compound Poisson process: $\frac{dK_t}{K_t} = \frac{I_t}{K_t}dt + d\overline{\Delta}_t$, where $d\overline{\Delta}_t = \Delta_t dt + DdJ_t$, where $J_t \in \{0,1\}$ indicates whether a Poisson shock hits at all. $\lambda(T)$ denotes the Poisson intensity of these Poisson shocks and depends on global mean excess temperature. D is a single scalar that denotes the size of the shock when it occurs. This law of motion implies that when a Poisson shock of magnitude D hits, capital jumps from K_t to $DK_t < K_t$. We are interested in the structure of the generator associated with the Poisson destruction part:

$$\mathcal{P}[V](K) = \lambda(T)(V((1-D)K) - V(K)) \tag{29}$$

We assume that D is small.[11] Using a second-order expansion in D: $V((1-D)K) - V(K) = -V'(K)KD + \frac{1}{2}V''(K)K^2D^2$. Substituting into (29):

$$\mathcal{D}[V](K) = -KV'(K)\lambda(T)D + \frac{K^2V''(K)}{2}\lambda(T)D^2$$

So, holding D fixed, the generator indeed has the same functional for than that of a Brownian motion. Consistent with the small D expansion, we posit that depreciation shocks are also small:

$$D = \sqrt{\epsilon} \times \overline{D},$$

where ϵ is the same scale parameter as the one that we use for the FAME/SAME. As will become clear below, scaling the shocks with $\sqrt{\epsilon}$ is necessary to keep volatility second order.

C.1 Drift and volatility formulas

We now revert to the full notation, where everything depends on $\overline{T} - T^P$. We posit that the arrival rate is:

$$\lambda_i(\overline{T} - T^P) = \overline{p}_{i0}(\epsilon)e^{\frac{p_{i1}}{\overline{p}_{i0}(\epsilon)}(\overline{T}-T^P)}.$$

[11] We work with the generator, which is the dual of the law of motion, so as to take an expansion in the space of functions rather than in a space of stochastic processes.

for a constants $p_{i1} > 0$ and a function $\overline{p}_{i0}(\epsilon) = p_{i0}\epsilon^{\eta}$ as $\epsilon \to 0$, for a constant $\eta < 1$. This formulation imposes that the baseline rate $\lambda_i(0) = \overline{p}_{i0}(\epsilon)$ is small too. It also ensures that $\lambda_i \geq 0$. This property allows us to use a small noise expansion at baseline. Then, to first order in ϵ because $\eta < 1$:

$$\lambda_i(\overline{T} - T^P) \approx \overline{p}_{i0}(\epsilon) + p_{i1}(\overline{T} - T^P).$$

We only need to consider the first order in ϵ, because the second order in ϵ will be multiplied by $\sqrt{\epsilon}$ below, and so is higher order than the SAME. Recall that $\overline{T} - T^P = \epsilon(T + \sigma z)$, so that:

$$\lambda_{it} \approx \overline{p}_{i0}(\epsilon) + p_{i1}\epsilon(T_t + \sigma_t z_t)\big).$$

Substituting into the drift and volatility equations, we obtain:

$$\begin{aligned}
\delta_{it} &= \epsilon^{1/2}\overline{D}\big(\overline{p}_{i0}(\epsilon) + p_{i1}\epsilon(T_t + \sigma_t z_t)\big) \equiv \epsilon^{1/2+\eta}\widetilde{\Delta}_i + \epsilon^{3/2}\delta_i(T_t + \sigma_t z_t)\Big) \\
\Sigma_{it}^2 &= \overline{D}^2\big(p_{i0}\epsilon^{1+\eta} + \epsilon^2 p_{i1}(T_t + \sigma_t z_t)\big) \equiv \Sigma_i^2\Big(\vartheta_i\epsilon^{1+\eta} + \epsilon^2(T_t + \sigma_t z_t)\Big),
\end{aligned}$$

where we defined $\widetilde{\Delta}_i = \overline{D}p_{i0}$, and

$$\delta_i = \overline{D}p_{i1} \qquad \Sigma_i = \overline{D}\sqrt{p_{i1}} \qquad \vartheta_i = \frac{p_{i0}}{p_{i1}}.$$

Since $\widetilde{\Delta}_i$ is isomorphic to a shift in the baseline local capital depreciation rate Δ_i we abstract from it from now on. We summarize the findings from the microfoundation as:

$$\boxed{\delta_{it} = \epsilon^{3/2}\delta_i(\sigma_t z_t + T_t) \qquad \Sigma_{it}^2 = \Sigma_i^2\Big(\vartheta_i\epsilon^{1+\eta} + \epsilon^2(\sigma_t z_t + T_t)\Big).} \tag{30}$$

We address two final observations related to the powers of ϵ. First, although the FAME is stated as a first-order perturbation in ϵ, the leading capital depreciation terms scale in $\epsilon^{3/2}$ in the microfoundation. In fact, this discrepancy is immaterial because the FAME is additive in productivity, amenity and capital depreciation damages. Therefore, it is entirely internally consistent to solve the FAME as stated, and incorporate the capital

depreciation components to leading order up to $\epsilon^{3/2}$. Therefore, to simplify notation and without loss of generality in the derivations below and in the main text we use the reduced-form diffusion specification in which δ_{it} scales with ϵ.

Second, and similarly, although the SAME is stated as a second-order perturbation in ϵ, baseline volatility $\vartheta_i \epsilon^{1+\eta}$ is technically of a somewhat lower order. This property is, again, immaterial. Because the SAME is additive, it is entirely internally consistent to solve the SAME as stated, and incorporate baseline volatility to leading order up to ϵ^2. Therefore, to simplify notation and without loss of generality in the derivations below and in the main text we use the reduced-form diffusion specification in which baseline volatility scales with ϵ^2 instead of $\epsilon^{1+\eta}$.

Finally, in the derivations below we introduce additional notation that allows us to separate idiosyncratic capital risk from aggregate risk associated with rising global temperatures. Specifically, we denote

$$\boxed{\Sigma_{it}^2 = \epsilon^2 \Sigma_i^2 \Big(\vartheta_i + \sigma_t z_t + \boldsymbol{\varsigma_t} \Big),} \tag{31}$$

where ς_t is a deterministic function of calendar time. Note that this specification nests the previous one as a special case when $\varsigma_t = T_t$.

D Data description

D.1 Economic data

Our economic dataset combines county-year information from several sources. Population data come from the BEA, which constructs annual county population estimates by interpolating between decennial Census counts and later replaces them with official intercensal figures as those become available. For income, we use BEA personal income per capita, a broad measure that includes labor income, transfers, employer contributions, interest income, and various imputed components. To capture labor earnings more directly, we also use BEA salary income. Employment is likewise taken from the BEA and reflects both full-time and part-time jobs.

To measure investment, we use county-level manufacturing capital expenditures from the Census of Manufacturers for 1958 to 1992 and from the Economic Censuses for 1997 to 2012. The earlier Census of Manufacturers reports are available through the ICPSR 2896 digitization project, while the later Economic Census data are released at five-year intervals. Because these investment data are not observed annually, we linearly interpolate values between Census years before estimating the event studies. Finally, we collect data on local public spending from the Government Finance Database, which harmonizes information from the Census of Governments. We use total local government expenditures as a control variable and interpolate these data between Census years as well.

D.2 ISIMIP3a data

Our climate data come from the Inter-Sectoral Impact Model Intercomparison Project (ISIMIP), which develops datasets designed to study the effects of climate change on both natural and socioeconomic systems. We use the high-frequency observed climate dataset from ISIMIP simulation round 3a, which covers 1901 to 2019 on a $0.5^\circ \times 0.5^\circ$ grid. This product combines two underlying datasets: the Global Soil Wetness Project Phase 3 (GSWP3) for 1901 to 1978 (Kim, 2017), and the WATCH Forcing Data methodology applied to the ERA5 reanalysis dataset (W5E5) for 1979 to 2019 (Lange, 2021a). Both datasets are constructed by blending numerical weather prediction models with station-based atmospheric observations (Mengel et al., 2021).

The historical GSWP3 component ultimately builds on the Twentieth Century Reanalysis Project (20CR), which spans 1871 to 2008 at a 2° spatial resolution and 6-hour frequency. In 20CR, atmospheric conditions are first generated using NOAA's NCEP Global Forecast System and then updated using observed and derived surface and sea-level pressure data through an Extended Kalman Filter assimilation procedure (Cucchi, 2020). The lower-boundary inputs for these simulations are based on sea surface temperature and sea-ice concentration from the UK Met Office HadISST1.1 dataset (Rayner, 2003), interpolated from monthly to daily frequency. The pressure observations used in the assimilation stage come from version 2 of the International Surface Pressure Databank (Cram, 2015), which compiles records from a wide range of meteorological agencies.

GSWP3 is obtained from 20CR through dynamic downscaling and bias correction of

selected variables. The original 2° fields are first downscaled to roughly 0.5° resolution for 1901 to 2010 at a 3-hour frequency using spectral nudging within a Global Spectral Model (Yoshimura and Kanamitsu, 2008). The resulting series are then bilinearly interpolated to a regular 0.5° grid and further bias-adjusted using observational reference datasets. In particular, precipitation is corrected using the Global Precipitation Climatology Centre Full Data Monthly Product Version 7 (Schneider, 2014), windspeed using the Climatic Research Unit CL2.0 dataset (New et al., 2002), and temperature, pressure, and humidity using the Climatic Research Unit TS3.23 dataset (Harris, 2014).

The more recent W5E5 component is based on the European Centre for Medium-Range Weather Forecasts reanalysis dataset ERA5 (Hersbach, 2020). ERA5 begins with numerical weather predictions generated by the ECMWF Integrated Forecasting System Cycle 41r2 and then assimilates observational data through a two-part procedure: an incremental 4D-Var method for atmospheric variables and ozone (Courtier et al., 1994), and a land data assimilation system that combines optimal interpolation methods for temperature, relative humidity, and snow with a Simplified Extended Kalman Filter for soil moisture. Across 1979 to 2019, this assimilation system incorporates roughly 96 billion observations from a wide range of conventional and satellite sources worldwide. The resulting ERA5 product has hourly frequency and a spatial resolution of about 31 kilometers.

To construct the forcing dataset used in ISIMIP, ERA5 is then bias-corrected and aggregated to a 0.5° grid following the WATCH framework for land-surface and global hydrological modeling. This produces the WFDE5 dataset (Cucchi, 2020). The bias adjustments are designed to align monthly moments with observed data, using the Climatic Research Unit TS4.03 dataset (Harris, 2020) for surface air temperature, shortwave radiation, rainfall, and snowfall, and the Global Precipitation Climatology Centre Full Data Monthly Product Version 2018 (Schneider, 2018) for rainfall and snowfall. W5E5, the ISIMIP input dataset, is then formed by combining WFDE5 over land with ERA5 over the ocean (Lange, 2021a).

Because the older 20CR-based data and the bias-correction procedures underlying GSWP3 are known to contain artifacts and spurious trends, and because ERA5 relies on a much larger observational base, the ISIMIP3a authors view W5E5 as the more realistic

dataset. To extend W5E5 back to 1901, the earlier GSWP3 series are homogenized to ensure a smooth transition at the 1978–1979 break. Specifically, over the common 1979 to 2004 reference period, GSWP3 is quantile-mapped to the corresponding W5E5 distributions (Lange, 2019; Lange, 2021b). The final GSWP3-W5E5 dataset therefore preserves the long-run trends present in GSWP3, coincides exactly with W5E5 from 1979 onward, and differs from the original GSWP3 series for 1901 to 1978.

D.3 Extreme event indicators

Storms—For comparison with our residualized percentiles, the 99$^{\text{th}}$ percentile of the un-residualized distribution of daily maximum windspeed for the years 1901-1910 is 10 meters per second (m/s). The largest windspeed is 18 m/s. Recall that these measures are daily averages, and thus allow for much faster gusts in small time windows. Similarly, the 99$^{\text{th}}$ percentile of the un-residualized distribution of daily maximum precipitation for the years 1901-1910 is 100 millimeters per day (mm/d). The maximum is above 220 mm/d.

E Steady-state inversion

We recover an estimate of the stock of buildings B_i and housing H_i from the constant shares rule (18)-(19) and the buildings production function as before:

$$B_i = L_i^{\omega}\Big((1-x)N_i\Big)^{\varpi}K_i^{1-\omega-\varpi} \;\;,\;\; H_i = yB_i$$

The wage equation (20) then delivers an estimate of fundamental productivity in county i as before:

$$Z_i = \frac{w_i}{(1-\alpha)\alpha^{\alpha}\Xi^{-\alpha}\left(B_i/N_i\right)^{\alpha}}.$$

From there we construct $C_{0i} = (1-\alpha)\alpha^{\alpha(1-\beta)}\Xi^{-\xi}(1-x)^{\varpi\xi}Z_i^{1-\beta}L_i^{\omega\xi}$ and $R_{0i} = (1-\omega-\varpi)\alpha^{\alpha}\Xi^{1-\alpha}(1-x)^{\alpha\varpi}Z_iL_i^{\omega\alpha}$ as before.

We recover migration costs as before. We construct migration shares as $m_{ij} = \frac{\mathtt{m}_{ij}}{\sum_k \mathtt{m}_{ik}}$. The equation for migration shares (5) implies that $X_{ij} \equiv \log m_{ij} - \log m_{ii} = \nu(V_j - V_i) -$

$\nu\tau_{ij} \equiv Y_j - Y_i + \epsilon_{ij}$ after imposing the normalization $\tau_{ii} = 0$. We exploit that $X_{ij} + X_{ji} = \epsilon_{ij} + \epsilon_{ji}$. We assume symmetric migration costs $\tau_{ij} = \tau_{ji}$, which we then recover as

$$\tau_{ij} = -\frac{X_{ij} + X_{ji}}{2\nu}.$$

We are left with estimating local amenities, which is now slightly different from before. We proceed in two steps. The first step consists in recovering local valuations from the steady-state population distribution. Defining $X_i = e^{\nu V_i}$ and $T_{ki} = e^{-\nu\tau_{ki}}$, equation (11) implies, together with a normalization of values:

$$X_i = \frac{N_i}{\sum_k \frac{N_k T_{ki}}{\sum_j T_{kj} X_j}} \ , \ \sum_i X_i = 1. \tag{32}$$

The system of equations (32) satisfies the gross substitutes property, and so has at most one solution. A simple iterative method delivers the solution to equation (32), and we recover $V_i = \frac{1}{\nu} \log\left(X_i\right)$.

With an estimate of local valuations V_i at hand, the second step consists in inverting the HJB (4) to recover amenities. In particular, we obtain flow utility as:

$$U_i = \rho V_i - \mathcal{M}_i[V].$$

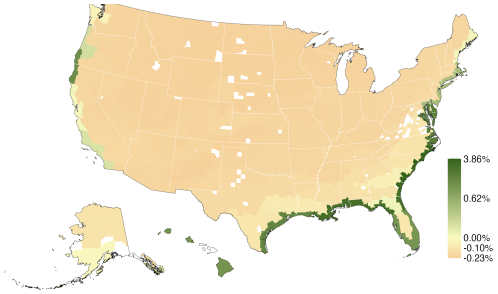
From there, we recover amenities as:

$$A_i = \frac{u^{-1}(U_i)}{c_{0i} K_i^{(1-\omega-\varpi)\xi} N_i^{-(1-\varpi)\xi}}.$$

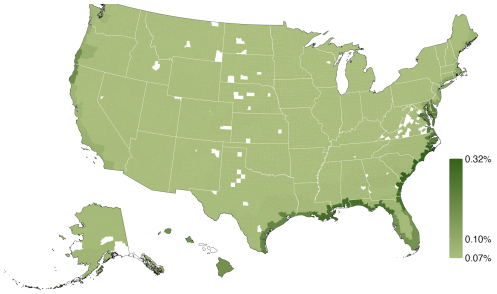
F Additional Figures

Figure 8: Effect of Local and Aggregate Risk on Workers

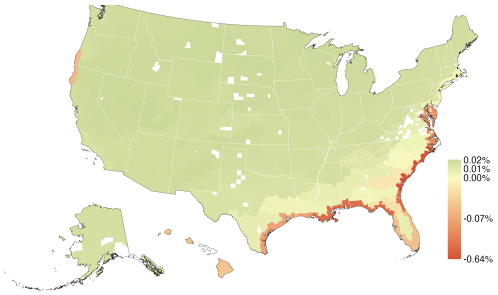
(a) Population



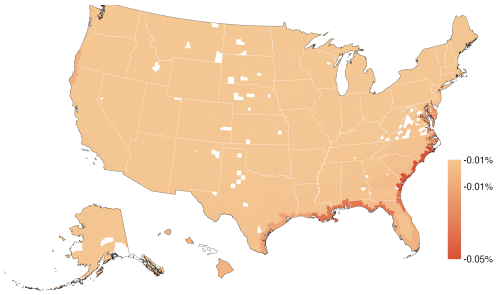
(b) Worker Welfare



(c) Population



(d) Worker Welfare



Note: Effect of local and aggregate risk on workers in an economy already exposed to a 2°C increase in global temperature. Panels (a) and (b) report the effects of local risk, while panels (c) and (d) report the effects of aggregate risk. Panel (a): change in population relative to the deterministic steady state. Panel (b): change in workers welfare relative to the deterministic steady state. Panel (c): change in population relative to the deterministic steady state. Panel (d): change in workers welfare relative to the deterministic steady state.

G Master Equation and FAME

G.1 Master Equation Expansion

The worker value expansion is:

$$
\begin{aligned}
&V_{it}(z, N^{ss} + \epsilon n, \mathcal{K}^{ss} + \epsilon k) - V_i^{ss} \\
&= \epsilon \left\{ \sum_j V_{ij}^N n_j + \sum_j V_{ij}^K k_j + v_{it}^Z z + v_{it}^T \right\} \\
&+ \frac{\epsilon^2}{2} \Bigg\{ \sum_j \sum_s \mathcal{V}_{ijs}^{N,N} n_j n_s + \sum_j \sum_s \mathcal{V}_{ijs}^{K,K} k_j k_s + \sum_j \sum_s \mathcal{V}_{ijs}^{N,K} n_j k_s + \sum_j \sum_s \mathcal{V}_{ijs}^{K,N} k_j n_s \\
&\qquad + 2 \sum_j V_{ijt}^{N,Z} n_j z + 2 \sum_j V_{ijt}^{K,Z} k_j z + 2 \sum_j V_{ijt}^{N,T} n_j + 2 \sum_j V_{ijt}^{K,T} k_j \\
&\qquad + \overline{v}_{it} + v_{it}^{Z,Z} z^2 + v_{it}^{T,T} + 2 v_{it}^{Z,T} z \Bigg\}.
\end{aligned}
\tag{33}
$$

The capital value expansion is:

$$
\begin{aligned}
&Q_{it}(z, N^{ss} + \epsilon n, \mathcal{K}^{ss} + \epsilon k) - V_i^{ss} \\
&= \epsilon \left\{ \sum_j Q_{ij}^N n_j + \sum_j Q_{ij}^K k_j + q_{it}^Z z + q_{it}^T \right\} \\
&+ \frac{\epsilon^2}{2} \Bigg\{ \sum_j \sum_s \mathcal{Q}_{ijs}^{N,N} n_j n_s + \sum_j \sum_s \mathcal{Q}_{ijs}^{K,K} k_j k_s + \sum_j \sum_s \mathcal{Q}_{ijs}^{N,K} n_j k_s + \sum_j \sum_s \mathcal{Q}_{ijs}^{K,N} k_j n_s \\
&\qquad + 2 \sum_j Q_{ijt}^{N,Z} n_j z + 2 \sum_j Q_{ijt}^{K,Z} k_j z + 2 \sum_j Q_{ijt}^{N,T} n_j + 2 \sum_j Q_{ijt}^{K,T} k_j \\
&\qquad + \overline{q}_{it} + q_{it}^{Z,Z} z^2 + q_{it}^{T,T} + 2 q_{it}^{Z,T} z \Bigg\}.
\end{aligned}
\tag{34}
$$

G.2 Flow payoffs

G.2.1 Workers

The flow payoffs become, to first order,

$$\frac{U_{it}-U_i^{ss}}{\epsilon} \approx u'(C_i^{ss})C_i^{ss}\cdot\left(\left(a_{i1}+(1-\beta)\chi_{i1}\right)\left(\sigma_t z+T_t\right)+\xi(1-\omega-\varpi)k_i-\xi(1-\varpi)\frac{n_i}{N_i^{ss}}\right)$$

with $u'(C)=C^{-\gamma}$. In vector notation,

$$\begin{aligned}\frac{U_t-U^{ss}}{\epsilon} &= \mathbf{diag}\Big(u'(C_i^{ss})C_i^{ss}\Big)\Bigg[\mathbf{vec}\left(a_{i1}+(1-\beta)\chi_{i1}\right)\left(\sigma_t z+T_t\right)\\ &\qquad +\xi(1-\omega-\varpi)k-\xi(1-\varpi)\mathbf{diag}\Big(1/N_i^{ss}\Big)n\Bigg]\\ &\equiv \mathrm{D}_{U\mathcal{K}}k+\mathrm{D}_{UN}n+\mathrm{d}_{UT}\left(\sigma_t z+T_t\right),\end{aligned} \tag{35}$$

where we have defined:

$$\begin{aligned}&\mathrm{d}_{UT}=\mathbf{vec}\Big(u'(C_i^{ss})C_i^{ss}\left(a_{i1}+(1-\beta)\chi_{i1}\right)\Big),\\ &\mathrm{D}_{U\mathcal{K}}=\xi(1-\omega-\varpi)\mathbf{diag}\Big(u'(C_i^{ss})C_i^{ss}\Big),\qquad \mathrm{D}_{UN}=-\xi(1-\varpi)\mathbf{diag}\Big(u'(C_i^{ss})C_i^{ss}/N_i^{ss}\Big).\end{aligned} \tag{36}$$

Alternatively, we could have written $\epsilon^{-1}(U_t-U^{ss}) = \mathrm{d}_{UZ}\sigma_t z+\mathrm{d}_{UT}T_t+\mathrm{D}_{U\mathcal{K}}k-\mathrm{D}_{UN}n$, with $\mathrm{d}_{UZ}=\mathrm{d}_{UT}$ for completion. Nevertheless, to avoid coding mistakes, we just keep the vector d_{UT} for the derivations.

G.2.2 Capitalists

We include the flow payoff from the return to capital and the continuation value from investment. To first order and in vector notation, we obtain

$$\begin{aligned}&\epsilon^{-1}\left[\left(\zeta Q_{it}^{1-1/\zeta}+(1-\zeta)\Big\{R_{it}-\Delta_{it}-\frac{\zeta\Sigma_{it}^2}{2}\Big\}Q_{it}\right)-\left(\zeta(Q_i^{ss})^{1-1/\gamma}+(1-\zeta)\Big\{\left(R_i^{ss}-\Delta_i-0\right\}Q_{it}\right)\right]\\ =\;&\mathrm{D}_{Q\mathcal{K}}k+\mathrm{D}_{QN}n+\mathrm{d}_{QT}\left(\sigma_t z+T_t\right)+\mathrm{D}_{QQ}\left[Q^N n+Q^K k+q_t^Z z+q_t^T\right]\end{aligned}$$

where

$$\mathrm{D}_{Q\mathcal{K}} = -(1-\zeta)\phi\mathbf{diag}\Big(R_i^{ss}Q_i^{ss}\Big), \qquad \mathrm{D}_{QN} = (1-\zeta)\psi\mathbf{diag}\Big(R_i^{ss}Q_i^{ss}/N_i^{ss}\Big),$$
$$\mathrm{D}_{QQ} = 0, \qquad \mathrm{d}_{QT} = (1-\zeta)\mathbf{vec}\Big((R_i^{SS}\chi_{i1} - \delta_{i1})Q_i^{ss}\Big). \tag{37}$$

where once again we ommit the vector d_{QZ}, which is equal to d_{RT}, to simplify the code.

Also note that now the matrices D_{QK}, D_{QN} do not correspond to the derivatives of the interest rate anymore. Note also that the term corresponding to the steady-state capital drift times the change in Q drops out because the capital drift is zero in steady-state.

Finally, there might have been a sign typo in how D_{RN} entered the FAME, and a typo in the sign of $D_{R\mathcal{K}}$. We need to check the code (it was present in the ACC code too).

We will also change the sign convention for how d_{RT} enters the FAME to avoid excess minus signs.

Other than these changes (and correcting the signs!), we can simply plug in these new equations into our existing code.

For use below, we also define the derivative of the interest rate alone:

$$\mathrm{D}_{R\mathcal{K}} = -\phi\mathbf{diag}\Big(R_i^{ss}\Big), \qquad \mathrm{D}_{RN} = \psi\mathbf{diag}\Big(R_i^{ss}/N_i^{ss}\Big) \qquad \mathrm{d}_{RT} = \mathbf{vec}\Big(R_i^{SS}\chi_{i1}\Big).$$

G.3 Continuation value from migration

The continuation value from migration becomes

$$\frac{\mathcal{M}[V] - \mathcal{M}[V^{ss}]}{\epsilon} \approx MV^N n + MV^K k + Mv_t^Z z + Mv_t^T, \tag{38}$$

where henceforth M denotes the steady-state matrix $M(V^{ss})$.

Continuation value from aggregate shocks. We impose z to follow an AR(1) process with mean-reversion parameter θ and unit variance. Therefore, the continuation values

become

$$\mathcal{A}(z)[V] = \mathcal{A}(z)[v_t^Z z] = -\epsilon\theta z v_t^Z,$$
$$\mathcal{A}(z)[Q] = \mathcal{A}(z)[q_t^Z z] = -\epsilon\theta z q_t^Z.$$

G.4 Population distribution

To linearize the law of motion of population, first note that

$$\frac{\partial m_{ji}}{\partial V_k} = \begin{cases} -\nu m_{ji} m_{jk} & \text{if} \quad i \neq k \\ \nu m_{ji}(1 - m_{ji}) & \text{if} \quad i = k \end{cases}$$

and so

$$\begin{aligned} M_{ij}^*(V + dV) &= M_{ij}^*(V) + \mu \sum_k \frac{\partial m_{ji}}{\partial V_k} dV_k \\ &= M_{ij}^*(V) + \mu\nu m_{ji}\Big\{dV_i - \big[m \cdot dV\big]_j\Big\} \end{aligned}$$

Then

$$\begin{aligned} M_i^*(V + dV)[N^{ss}] &= M_i^*(V)[N^{ss}] + \mu\nu \sum_j m_{ji}\Big\{dV_i - \big[m \cdot dV\big]_j\Big\} N_j^{ss} \\ &= M_i^*(V)[N^{ss}] + \nu\mu\Big[\Big(\mathbf{diag}(m^* N^{ss}) - m^* \mathbf{diag}(N^{ss}) m\Big) \cdot dV\Big]_i \\ &\equiv M_i^*(V)[N^{ss}] + (G \cdot dV)_i \end{aligned}$$

where

$$G = \nu\mu\Big(\mathbf{diag}(m^* N^{ss}) - m^* \mathbf{diag}(N^{ss}) m\Big) \tag{39}$$

G.5 Continuation value from changes in population distribution

We can then move on to the continuation value from changes in the population distribution. In vector notation,

$$\epsilon^{-1}\left(\sum_j \frac{\partial V_i}{\partial N_j}(M^*(V)N)_j\right)_{i=1}^{I} \approx V^N M^* n + V^N G\big(V^N n + V^K k + v_t^Z z + v_t^T\big) \quad (40)$$

Similarly,

$$\epsilon^{-1}\left(\sum_j \frac{\partial Q_i}{\partial N_j}(M^*(V)N)_j\right)_{i=1}^{I} \approx Q^N M^* n + Q^N G\big(V^N n + V^K k + v_t^Z z + v_t^T\big) \quad (41)$$

G.6 Capital distribution

The log capital LoM becomes to first order:

$$\frac{dk_t}{dt} = \mathrm{D}_{R\mathcal{K}} k_t + \mathrm{D}_{RN} n_t + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}\big[Q^N n_t + Q^K k_t + q_t^Z z + q_t^T\big],$$

where the Ito correction drops out to first order, and where:

$$\mathrm{d}_{\mathcal{I}T} = \mathbf{vec}\Big(R_i^{SS}\chi_{i1} - \delta_{i1}\Big) \qquad \mathrm{D}_{\mathcal{I}Q} = \mathbf{diag}\left(\frac{(Q_i^{ss})^{-1-1/\zeta}}{\zeta}\right)$$

There are two changes relative to what we had before: (1) the definition of $\mathrm{D}_{\mathcal{I}Q}, \mathrm{d}_{\mathcal{I}T}$ has changed (also note the change of sign in the definition/how they enter the LoM), and (2) two new terms enter, $\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n$, that we will need to keep track of in the continuation values.

G.7 Continuation value from changes in capital distribution

All the Ito correction terms are second order, so we can ignore them in the FAME. We obtain

$$\begin{aligned}
&\epsilon^{-1}\sum_j \frac{\partial V_i}{\partial \mathcal{K}_j}\left\{R_i(\chi_{it},N_i,\mathcal{K}_i) - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right\} \\
\approx\ & \sum_j V_{ij}^K\Big\{\mathrm{D}_{R\mathcal{K}}k_t + \mathrm{D}_{RN}n_t + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}\big[Q^N n_t + Q^K k_t + q_t^Z z + q_t^T\big]\Big\} \\
=\ & V^K\left\{\mathrm{D}_{\mathcal{I}Q}\Big[Q^N n + Q^K k + q_t^Z z + q_t^T\Big] + \mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \big(\sigma_t z + T_t\big)\mathrm{d}_{\mathcal{I}T}\right\} \qquad (42)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\epsilon^{-1}\sum_j \frac{\partial Q_i}{\partial \mathcal{K}_j}\left\{R_i(\chi_{it},N_i,\mathcal{K}_i) - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right\} \\
\approx\ & Q^K\left\{\mathrm{D}_{\mathcal{I}Q}\Big[Q^N n + Q^K k + q_t^Z z + q_t^T\Big] + \mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \big(\sigma_t z + T_t\big)\mathrm{d}_{\mathcal{I}T}\right\} \qquad (43)
\end{aligned}$$

G.8 Expansion of prices and investment

For estimation, it is also useful to collect the expansion of prices along any transition. We have, in level changes:

$$\begin{aligned}
\widehat{w}_t &= \mathrm{D}_{wN}n_t + \mathrm{D}_{w\mathcal{K}}k_t + \mathrm{d}_{wT}(\sigma_t z_t + T_t) \\
\widehat{R}_t &= \mathrm{D}_{RN}n_t + \mathrm{D}_{R\mathcal{K}}k_t + \mathrm{d}_{RT}(\sigma_t z_t + T_t) \\
\widehat{I}_t &= \mathrm{D}_{IN}n_t + \mathrm{D}_{I\mathcal{K}}k_t + \mathrm{d}_{IQ}\widehat{Q}_t + \mathrm{d}_{IT}(\sigma_t z_t + T_t)
\end{aligned}$$

where:

$$\begin{aligned}
&\mathrm{D}_{wN} = -\alpha(1-\varpi)\mathbf{diag}\big(w_i^{ss}/N_i^{ss}\big) \quad \mathrm{D}_{w\mathcal{K}} = \alpha(1-\omega-\varpi)\mathbf{diag}\big(w_i^{ss}\big) \quad \mathrm{d}_{wT} = \mathbf{vec}\big(\chi_i w_i^{ss}\big) \\
&\mathrm{D}_{IN} = \mathbf{diag}(K_i^{ss})\mathrm{D}_{RN} \quad \mathrm{D}_{I\mathcal{K}} = \mathbf{diag}(K_i^{ss})\mathrm{D}_{R\mathcal{K}} + \mathbf{diag}(I_i^{ss}) \quad \mathrm{D}_{IQ} = \mathbf{diag}(K_i^{ss})\mathrm{D}_{\mathcal{I}Q} \\
&\mathrm{d}_{IT} = \mathbf{diag}(K_i^{ss})\mathrm{d}_{RT},
\end{aligned}$$

and $\mathrm{D}_{RN}, \mathrm{D}_{R\mathcal{K}}, \mathrm{d}_{RT}$ are defined in Section G.2.2.

G.9 FAME in vector notation

G.9.1 Workers

Thus, the linearized master equation for the worker value becomes, in vector notation,

$$
\begin{aligned}
& \rho\big(V^N n + V^K k + v_t^Z z + v_t^T\big) - \frac{\partial v_t^Z}{\partial t} z - \frac{\partial v_t^T}{\partial t} \\
= \; & \overbrace{\mathrm{D}_{U\mathcal{K}} k + \mathrm{D}_{UN} n + (\sigma_t z + T_t)\,\mathrm{d}_{UT}}^{\text{flow payoff}} \\
+ \; & \overbrace{M\big(V^N n + V^K k + v_t^Z z + v_t^T\big)}^{\text{continuation value from migration}} \\
- \; & \overbrace{\theta z v_t^Z}^{\text{c.v. from climate shocks}} \\
+ \; & \underbrace{V^N M^* n + V^N G\big(V^N n + V^K k + v_t^Z z + v_t^T\big)}_{\text{continuation value from changes in the population distribution}} \\
+ \; & \underbrace{V^K\Big\{\mathrm{D}_{\mathcal{I}Q}\left[Q^N n + Q^K k + q_t^Z z + q_t^T\right] + \mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + (\sigma_t z + T_t)\mathrm{d}_{\mathcal{I}T}\Big\}}_{\text{continuation value from changes in the capital distribution}}
\end{aligned}
$$

Identifying coefficients and unknowns in bold notation,

$$
\begin{aligned}
\rho \boldsymbol{V^N} &= \mathrm{D}_{UN} + M\boldsymbol{V^N} + \boldsymbol{V^N} M^* + \boldsymbol{V^K}\mathrm{D}_{RN} + \boldsymbol{V^N} G \boldsymbol{V^N} + \boldsymbol{V^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{Q^N} \\
\rho \boldsymbol{V^K} &= \mathrm{D}_{U\mathcal{K}} + M\boldsymbol{V^K} \qquad\qquad + \boldsymbol{V^K}\mathrm{D}_{R\mathcal{K}} + \boldsymbol{V^N} G \boldsymbol{V^K} + \boldsymbol{V^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{Q^K} \\
(\rho+\theta)\boldsymbol{v_t^Z} - \frac{\partial \boldsymbol{v_t^Z}}{\partial t} &= \mathrm{d}_{UT}\sigma_t + M\boldsymbol{v_t^Z} + \boldsymbol{V^N} G \boldsymbol{v_t^Z} + \boldsymbol{V^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{q_t^Z} + \boldsymbol{V^K}\mathrm{d}_{\mathcal{I}T}\sigma_t \\
\rho \boldsymbol{v_t^T} - \frac{\partial \boldsymbol{v_t^T}}{\partial t} &= \mathrm{d}_{UT} T_t + M\boldsymbol{v_t^T} + \boldsymbol{V^N} G \boldsymbol{v_t^T} + \boldsymbol{V^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{q_t^T} + \boldsymbol{V^K}\mathrm{d}_{\mathcal{I}T} T_t
\end{aligned}
$$

G.9.2 Capitalists

Similarly, the linearized master equation for capitalists becomes, in vector notation,

$$
\begin{aligned}
& \rho\big(Q^N n + Q^K k + q_t^Z z + q_t^T\big) - \frac{\partial q_t^Z}{\partial t} - \frac{\partial q_t^T}{\partial t} \\
= \; & \overbrace{\mathrm{D}_{Q\mathcal{K}} k + \mathrm{D}_{QN} n + \mathrm{d}_{QT}\left(\sigma_t z + T_t\right) + \mathrm{D}_{QQ}\big[Q^N n + Q^K k + q_t^Z z + q_t^T\big]}^{\text{flow payoff + own continuation value}} \\
- \; & \overbrace{\theta z q_t^Z}^{\text{c.v. from climate shocks}} \\
+ \; & \underbrace{Q^N M^* n + Q^N G\big(V^N n + V^K k + v_t^Z z + v_t^T\big)}_{\text{continuation value from changes in the population distribution}} \\
+ \; & \underbrace{Q^K\Big\{\mathrm{D}_{\mathcal{I}Q}\left[Q^N n + Q^K k + q_t^Z z + q_t^T\right] + \mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + (\sigma_t z + T_t)\mathrm{d}_{\mathcal{I}T}\Big\}}_{\text{continuation value from changes in the capital distribution}}
\end{aligned}
$$

Identifying coefficients and with unknowns in bold notation,

$$
\begin{aligned}
\rho \boldsymbol{Q^N} &= \mathrm{D}_{QN} + \boldsymbol{Q^N} M^* + \boldsymbol{Q^K}\mathrm{D}_{RN} + D_{QQ}\boldsymbol{Q^N} + \boldsymbol{Q^N} G \boldsymbol{V^N} + \boldsymbol{Q^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{Q^N} \\
\rho \boldsymbol{Q^K} &= \mathrm{D}_{Q\mathcal{K}} + \boldsymbol{Q^K}\mathrm{D}_{R\mathcal{K}} + D_{QQ}\boldsymbol{Q^K} + \boldsymbol{Q^N} G \boldsymbol{V^K} + \boldsymbol{Q^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{Q^K} \\
(\rho+\theta)\boldsymbol{q_t^Z} - \frac{\partial \boldsymbol{q_t^Z}}{\partial t} &= \mathrm{d}_{QT}\sigma_t + \boldsymbol{Q^N} G \boldsymbol{v_t^Z} + \boldsymbol{Q^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{q_t^Z} + \boldsymbol{Q^K}\mathrm{d}_{\mathcal{I}T}\sigma_t \\
\rho \boldsymbol{q_t^T} - \frac{\partial \boldsymbol{q_t^T}}{\partial t} &= \mathrm{d}_{QT} T_t + \boldsymbol{Q^N} G \boldsymbol{v_t^T} + \boldsymbol{Q^K}\mathrm{D}_{\mathcal{I}Q}\boldsymbol{q_t^T} + \boldsymbol{Q^K}\mathrm{d}_{\mathcal{I}T} T_t
\end{aligned}
$$

We can collect terms and solve for a stacked system step by step.

G.10 Deterministic FAME

$$
\begin{aligned}
\rho \begin{pmatrix} \boldsymbol{V^N} & \boldsymbol{V^K} \\ \boldsymbol{Q^N} & \boldsymbol{Q^K} \end{pmatrix} &= \begin{pmatrix} \mathrm{D}_{UN} & \mathrm{D}_{U\mathcal{K}} \\ \mathrm{D}_{QN} & \mathrm{D}_{Q\mathcal{K}} \end{pmatrix} + \begin{pmatrix} M & 0 \\ 0 & D_{QQ} \end{pmatrix} \begin{pmatrix} \boldsymbol{V^N} & \boldsymbol{V^K} \\ \boldsymbol{Q^N} & \boldsymbol{Q^K} \end{pmatrix} + \begin{pmatrix} \boldsymbol{V^N} & \boldsymbol{V^K} \\ \boldsymbol{Q^N} & \boldsymbol{Q^K} \end{pmatrix} \begin{pmatrix} M^* & 0 \\ \mathrm{D}_{RN} & \mathrm{D}_{R\mathcal{K}} \end{pmatrix} \\
&+ \begin{pmatrix} \boldsymbol{V^N} & \boldsymbol{V^K} \\ \boldsymbol{Q^N} & \boldsymbol{Q^K} \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & \mathrm{D}_{\mathcal{I}Q} \end{pmatrix} \begin{pmatrix} \boldsymbol{V^N} & \boldsymbol{V^K} \\ \boldsymbol{Q^N} & \boldsymbol{Q^K} \end{pmatrix}
\end{aligned} \tag{44}
$$

We recognize a nonlinear Sylvester equation which we can easily solve by iterating on it.

G.11 Stochastic FAME: z

Once we have the solution $\boldsymbol{V^N}, \boldsymbol{V^K}, \boldsymbol{Q^N}, \boldsymbol{Q^K}$, we can solve for v_t^Z and q_t^Z. Define the vector Ψ and the matrix Φ as:

$$\Psi = \begin{pmatrix} \mathrm{d}_{UT} + V^K \mathrm{d}_{\mathcal{I}T} \\ \mathrm{d}_{QT} + Q^K \mathrm{d}_{\mathcal{I}T} \end{pmatrix}, \qquad \Phi = \begin{pmatrix} M + V^N G & V^K \mathrm{D}_{\mathcal{I}Q} \\ Q^N G & Q^K \mathrm{D}_{\mathcal{I}Q} \end{pmatrix} \tag{45}$$

Then, $\boldsymbol{v_t^Z}$ and $\boldsymbol{q_t^Z}$ satisfy a linear Sylvester equation:

$$(\rho + \theta) \begin{pmatrix} \boldsymbol{v_t^Z} \\ \boldsymbol{q_t^Z} \end{pmatrix} = \Psi \sigma_t + \Phi \begin{pmatrix} \boldsymbol{v_t^Z} \\ \boldsymbol{q_t^Z} \end{pmatrix} + \begin{pmatrix} \boldsymbol{v_t^Z} \\ \boldsymbol{q_t^Z} \end{pmatrix} \mathcal{D}_T^*, \tag{46}$$

where $\mathcal{D}_T$ is a matrix that discretizes the time derivative. We solve for $p = (v_t^Z; q_t^Z)$ by iterating back in time:

$$\begin{aligned} p_\infty &= ((\rho + \theta)\mathrm{Id} - \Phi)^{-1} \Psi \sigma_\infty \\ p_t &= \Big((1 + (\rho + \theta)dt) - dt\ \Phi\Big)^{-1} \Big(p_{t+dt} + \Psi \sigma_t\ dt\Big) \end{aligned}$$

G.12 Trend FAME: T_t

Finally, we solve for the time-dependent piece. v_t^T and q_t^T satisfy:

$$\rho \begin{pmatrix} \boldsymbol{v_t^T} \\ \boldsymbol{q_t^T} \end{pmatrix} = \Psi T_t + \Phi \begin{pmatrix} \boldsymbol{v_t^T} \\ \boldsymbol{q_t^T} \end{pmatrix} + \begin{pmatrix} \boldsymbol{v_t^T} \\ \boldsymbol{q_t^T} \end{pmatrix} \mathcal{D}_T^* \tag{47}$$

This is, again, a linear Sylvester equation that we solve iteratively for $p \equiv (v_t^T; q_t^T)$ by iterating backward in time:

$$\begin{aligned} p_\infty &= (\rho \mathrm{Id} - \Phi)^{-1} \Psi T_\infty^D \\ p_t &= \Big((1 + \rho\ dt) - dt\ \Phi\Big)^{-1} \Big(p_{t+dt} + \Psi T_t\ dt\Big) \end{aligned}$$

G.13 Linearized KFEs

To obtain the linearized laws of motions given T_t and a sequence z_t, we simply use the same terms as those in the "continuation value from changes in the population/capital distribution" from the FAMEs, since those correspond precisely to the linearization of the KFE. Let k_t be the vector of capital stocks across locations at time t, and n_t be the vector of populations across locations at time t. Then, in vector notation, the linearized KFE's are:

$$\begin{aligned}\frac{dn_t}{dt} &= M^* n_t + G\big(V^N n_t + V^K k_t + v_t^Z z_t + v_t^T\big) \\ \frac{dk_t}{dt} &= \mathrm{D}_{\mathcal{I}Q}\left[Q^N n_t + Q^K k_t + q_t^Z z_t + q_t^T\right] + \mathrm{D}_{RN} n_t + \mathrm{D}_{R\mathcal{K}} k_t + (\sigma_t z_t + T_t)\mathrm{d}_{\mathcal{I}T} \end{aligned} \tag{48}$$

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Written in matrix form:

$$\begin{aligned} v_1 &= V^N n + V^K k + v_t^Z z + v_t^T, \\ v_2 &= \mathcal{V}^{N,N} \times_3 n \times_2 n + \mathcal{V}^{N,K} \times_3 k \times_2 n + \mathcal{V}^{K,N} \times_3 n \times_2 k + \mathcal{V}^{K,K} \times_3 k \times_2 k \\ &\quad + 2V_t^{N,Z} nz + 2V_t^{K,Z} kz + 2V_t^{N,T} n + 2V_t^{K,T} k \\ &\quad + \overline{v}_t + v_t^{Z,Z} z^2 + v_t^{T,T} + 2v_t^{Z,T} z + v_t^{\Sigma}, \end{aligned} \tag{49}$$

and

$$\begin{aligned} q_1 &= Q^N n + Q^K k + q_t^Z + q_t^T \\ q_2 &= \mathcal{Q}^{N,N} \times_3 n \times_2 n + \mathcal{Q}^{N,K} \times_3 k \times_2 n + \mathcal{Q}^{K,N} \times_3 n \times_2 k + \mathcal{Q}^{K,K} \times_3 k \times_2 k \\ &\quad + 2Q_t^{N,Z} nz + 2Q_t^{K,Z} kz + 2Q_t^{N,T} n + 2Q_t^{K,T} k \\ &\quad + \overline{q}_t + q_t^{Z,Z} z^2 + q_t^{T,T} + 2q_t^{Z,T} z + q_t^{\Sigma}, \end{aligned} \tag{50}$$

where $\times_3$ and $\times_2$ are the 3-node and 2-node tensor products, respectively, which we defined earlier.

Importantly, from (**??**), we notice that the tensors $\mathcal{V}^{N,K}$ and $\mathcal{V}^{K,N}$ are symmetric in their last two coordinates. Similarly, the expression (**??**) implies that $\mathcal{Q}^{N,K}$ and $\mathcal{Q}^{K,N}$ are also

symmetric in their last two coordinates.

H.1 Flow payoffs

H.1.1 Workers

We start with the utility function. For the second-order effect of the distribution, we have:

$$\begin{aligned}
\left.\frac{\partial^2 U_{it}}{\partial N_{it}^2}\right|_{SS} &= \frac{\xi(1-\varpi)\widetilde{C}_i^{ss}\left[\xi(1-\varpi)u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}+(1+\xi(1-\varpi))u'\left(\widetilde{C}_i^{ss}\right)\right]}{(N_i^{ss})^2},\\
\left.\frac{\partial^2 U_{it}}{\partial \mathcal{K}_{it}\partial N_{it}}\right|_{SS} &= -\frac{\xi^2(1-\varpi)(1-\omega-\varpi)\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}+u'\left(\widetilde{C}_i^{ss}\right)\right]}{N_i^{ss}},\\
\left.\frac{\partial^2 U_{it}}{\partial \mathcal{K}_{it}^2}\right|_{SS} &= \left[\xi(1-\omega-\varpi)\right]^2\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}+u'\left(\widetilde{C}_i^{ss}\right)\right].
\end{aligned}$$

For the cross effect of the distribution and the aggregate shocks, we have:

$$\begin{aligned}
\left.\frac{\partial^2 U_{it}}{\partial(\sigma_t z_t)\partial N_{it}}\right|_{SS} &= \left.\frac{\partial^2 U_{it}}{\partial T_t\partial N_{it}}\right|_{SS} = -\frac{\epsilon\xi(1-\varpi)\left(a_{i1}+(1-\beta)\chi_{i1}\right)\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}+u'\left(\widetilde{C}_i^{ss}\right)\right]}{N_i^{ss}},\\
\left.\frac{\partial^2 U_{it}}{\partial(\sigma_t z_t)\partial \mathcal{K}_{it}}\right|_{SS} &= \left.\frac{\partial^2 U_{it}}{\partial T_t\partial \mathcal{K}_{it}}\right|_{SS} = \epsilon\xi(1-\omega-\varpi)\left(a_{i1}+(1-\beta)\chi_{i1}\right)\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}+u'\left(\widetilde{C}_i^{ss}\right)\right].
\end{aligned}$$

Finally, for the second-order effect of the aggregate shocks, we have:

$$\begin{aligned}
\left.\frac{\partial^2 U_{it}}{\partial(\sigma_t z_t)^2}\right|_{SS} &= \left.\frac{\partial^2 U_{it}}{\partial(T_t)^2}\right|_{SS} = \left.\frac{\partial^2 U_{it}}{\partial(\sigma_t z_t)\partial T_t}\right|_{SS}\\
&= \epsilon^2\Big\{u''\left(\widetilde{C}_i^{ss}\right)\left(\widetilde{C}_i^{ss}\right)^2\left(a_{i1}+(1-\beta)\chi_{i1}\right)^2+u'\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}\Big(\left(a_{i1}+(1-\beta)\chi_{i1}\right)^2+a_{i2}+(1-\beta)
\end{aligned}$$

Putting all the terms together, we have:

$$
\begin{aligned}
&U_t - U^{ss} \\
&\quad \approx \epsilon\left\{\mathrm{D}_{UN} n + \mathrm{D}_{U\mathcal{K}} k + \mathrm{d}_{UT}\left(\sigma_t z + T_t\right)\right\} \longleftarrow \text{ first-order term} \\
&\quad + \frac{\epsilon^2}{2}\left\{\mathbb{D}_{UNN} \times_3 n \times_2 n + \mathbb{D}_{UN\mathcal{K}} \times_3 k \times_2 n + \mathbb{D}_{U\mathcal{K}N} \times_3 n \times_2 k + \mathbb{D}_{U\mathcal{K}\mathcal{K}} \times_3 k \times_2 k\right\} \\
&\quad + \epsilon^2 \mathrm{D}_{U\mathcal{K}T} k\left(\sigma_t z + T_t\right) + \epsilon^2 \mathrm{D}_{UNT} n\left(\sigma_t z + T_t\right) \\
&\quad + \frac{\epsilon^2}{2} \mathrm{d}_{UTT}\left[(\sigma_t z)^2 + (T_t)^2 + 2(\sigma_t z) T_t\right],
\end{aligned}
$$

where $\mathbb{D}_{UNN}$, $\mathbb{D}_{U\mathcal{K}N}$, $\mathbb{D}_{UN\mathcal{K}}$, and $\mathbb{D}_{U\mathcal{K}\mathcal{K}}$ are diagonal tensors defined as:

$$
\begin{aligned}
\mathbb{D}_{UNN} &= \xi(1-\varpi)\mathbf{diag}_3\left(\frac{\widetilde{C}_i^{ss}\left[\xi(1-\varpi)u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss} + (1+\xi(1-\varpi))u'\left(\widetilde{C}_i^{ss}\right)\right]}{(N_i^{ss})^2}\right), \\
\mathbb{D}_{U\mathcal{K}N} &= \mathbb{D}_{UN\mathcal{K}} = -\xi^2(1-\varpi)(1-\omega-\varpi)\mathbf{diag}_3\left(\frac{\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss} + u'\left(\widetilde{C}_i^{ss}\right)\right]}{N_i^{ss}}\right), \\
\mathbb{D}_{U\mathcal{K}\mathcal{K}} &= \left[\xi(1-\omega-\varpi)\right]^2 \mathbf{diag}_3\left(\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss} + u'\left(\widetilde{C}_i^{ss}\right)\right]\right),
\end{aligned}
$$

where $\mathbf{diag}_3$ is the 3-dimensional diagonal operator; and D_{UN}, $\mathrm{D}_{U\mathcal{K}}$, D_{UZ}, and d_{UT} are defined as in section 4. The remaining objects are defined as follows:

$$
\begin{aligned}
\mathrm{D}_{UNT} &= -\xi(1-\varpi)\mathbf{diag}\left(\frac{(a_{i1} + (1-\beta)\chi_{i1})\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss} + u'\left(\widetilde{C}_i^{ss}\right)\right]}{N_i^{ss}}\right), \\
\mathrm{D}_{U\mathcal{K}T} &= \xi(1-\omega-\varpi)\mathbf{diag}\left(\left(a_{i1} + (1-\beta)\chi_{i1}\right)\widetilde{C}_i^{ss}\left[u''\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss} + u'\left(\widetilde{C}_i^{ss}\right)\right]\right), \\
\mathrm{d}_{UTT} &= \mathbf{vec}\left(u''\left(\widetilde{C}_i^{ss}\right)\left(\widetilde{C}_i^{ss}\right)^2 (a_{i1} + (1-\beta)\chi_{i1})^2\right) + \mathbf{vec}\left(u'\left(\widetilde{C}_i^{ss}\right)\widetilde{C}_i^{ss}\left(\left(a_{i1} + (1-\beta)\chi_{i1}\right)^2 + a_{i2} + (1 -\right.\right.
\end{aligned}
$$

H.1.2 Capitalists

We let $\overline{Q}_{it} \equiv \zeta Q_{it}^{1-1/\zeta} + (1-\zeta)\Big\{R_{it} - \Delta_{it} - \frac{\zeta\Sigma_{it}^2}{2}\Big\}Q_{it}$, to be the flow pay-off for the capitalists. First, for the second-order effect of the distribution, we have:

$$\begin{aligned}
\left.\frac{\partial^2\overline{Q}_{it}}{\partial N_{it}^2}\right|_{SS} &= \frac{\psi(\psi-1)(1-\zeta)Q_i^{ss}R_i^{ss}}{(N_i^{ss})^2},\\
\left.\frac{\partial^2\overline{Q}_{it}}{\partial \mathcal{K}_{it}\partial N_{it}}\right|_{SS} &= -\frac{\psi\phi(1-\zeta)Q_i^{ss}R_i^{ss}}{N_i^{ss}},\\
\left.\frac{\partial^2\overline{Q}_{it}}{\partial \mathcal{K}_{it}^2}\right|_{SS} &= \phi^2(1-\zeta)Q_i^{ss}R_i^{ss}.
\end{aligned}$$

Second, for the cross-effect of the distribution and the aggregate shocks, we have:

$$\begin{aligned}
\left.\frac{\partial^2\overline{Q}_{it}}{\partial(\sigma_t z)\partial N_{it}}\right|_{SS} &= \left.\frac{\partial^2\overline{Q}_{it}}{\partial T_t\partial N_{it}}\right|_{SS} = \epsilon\psi(1-\zeta)Q_i^{ss}\chi_{i1}\frac{R_i^{ss}}{N_i^{ss}},\\
\left.\frac{\partial^2\overline{Q}_{it}}{\partial(\sigma_t z)\partial \mathcal{K}_{it}}\right|_{SS} &= \left.\frac{\partial^2\overline{Q}_{it}}{\partial T_t\partial \mathcal{K}_{it}}\right|_{SS} = -\epsilon\phi(1-\zeta)Q_i^{ss}\chi_{i1}R_i^{ss}.
\end{aligned}$$

Third, the coefficients of the second-order effect of the aggregate shocks are given by:

$$\left.\frac{\partial^2\overline{Q}_{it}}{\partial(\sigma_t z)^2}\right|_{SS} = \left.\frac{\partial^2\overline{Q}_{it}}{\partial(T_t)^2}\right|_{SS} = \left.\frac{\partial^2\overline{Q}_{it}}{\partial(\sigma_t z)\partial T_t}\right|_{SS} = \epsilon^2(1-\zeta)Q_i^{ss}\left[R_i^{ss}\left(\chi_{i1}^2+\chi_{i2}\right)-\delta_{i2}\right].$$

Forth, coefficients for the capital depreciation coming from the Brownian motion:

$$\left.\frac{\partial\overline{Q}_{it}}{\partial\Sigma_{it}^2}\right|_{SS} = -\epsilon^2\frac{(1-\zeta)\zeta}{2}Q_i^{ss}$$

Finally, the terms involving Q_{it}:

$$\begin{aligned}
\left.\frac{\partial^2 \overline{Q}_{it}}{\partial Q_{it}^2}\right|_{SS} &= \frac{1-\zeta}{\zeta}\left(Q_i^{ss}\right)^{-\frac{1+\zeta}{\zeta}} \\
\left.\frac{\partial^2 \overline{Q}_{it}}{\partial \mathcal{K}_{it}\partial Q_{it}}\right|_{SS} &= -\phi(1-\zeta)R_i^{ss} \\
\left.\frac{\partial^2 \overline{Q}_{it}}{\partial N_{it}\partial Q_{it}}\right|_{SS} &= \psi(1-\zeta)\frac{R_i^{ss}}{N_i^{ss}} \\
\left.\frac{\partial^2 \overline{Q}_{it}}{\partial(\sigma_t z)\partial Q_{it}}\right|_{SS} &= \left.\frac{\partial^2 \overline{Q}_{it}}{\partial T_t \partial Q_{it}}\right|_{SS} = \epsilon(1-\zeta)[R_i^{ss}\chi_{i1} - \delta_{i1}].
\end{aligned}$$

Putting all the terms together, we have that:

$$\begin{aligned}
\overline{Q}_{it} - \overline{Q}_i^{ss} \approx\ & \overline{Q}_{i,N}\left(\epsilon n_i\right) + \overline{Q}_{i,\mathcal{K}}\left(\epsilon k_i\right) + \overline{Q}_{i,Q}\left(\epsilon q_{1i} + \frac{\epsilon^2}{2}q_{2i}\right) + \overline{Q}_{i,T}z + \overline{Q}_{i,T}T_t \\
& + \frac{1}{2}\overline{Q}_{i,NN}\left(\epsilon n_i\right)^2 + \frac{1}{2}\overline{Q}_{i,\mathcal{K}\mathcal{K}}\left(\epsilon k_i\right)^2 + \overline{Q}_{i,N\mathcal{K}}\left(\epsilon n_i\right)\left(\epsilon k_i\right) \\
& + \frac{1}{2}\overline{Q}_{i,QQ}\left(\epsilon q_{1i} + \frac{\epsilon^2}{2}q_{2i}\right)^2 \\
& + \overline{Q}_{i,NT}\left(\epsilon n_i\right)\sigma_t z + \overline{Q}_{i,NT}\left(\epsilon n_i\right)T_t \\
& + \overline{Q}_{i,\mathcal{K}T}\left(\epsilon k_i\right)\sigma_t z + \overline{Q}_{i,\mathcal{K}T}\left(\epsilon k_i\right)T_t \\
& + \overline{Q}_{i,QN}\left(\epsilon q_{1i} + \frac{\epsilon^2}{2}q_{2i}\right)n_i + \overline{Q}_{i,Q\mathcal{K}}\left(\epsilon q_{1i} + \frac{\epsilon^2}{2}q_{2i}\right)k_i + \overline{Q}_{i,QT}\left(\epsilon q_{1i} + \frac{\epsilon^2}{2}q_{2i}\right)\left(\sigma_t z + T_t\right) \\
& + \frac{1}{2}\overline{Q}_{i,TT}(\sigma_t z)^2 + \frac{1}{2}\overline{Q}_{i,TT}(T_t)^2 + \overline{Q}_{i,TT}(\sigma_t z)T_t + \overline{Q}_{i,\Sigma}\Sigma_i^2\varsigma_t,
\end{aligned}$$

where $\overline{Q}_{i,XY}$ denote the cross partial derivative of $\overline{Q}_i$ with respect to X and Y, evaluated at steady state. Now, before writing all terms in matrix notation, we work on the quadratic term q_{1i}^2:

$$
\begin{aligned}
q_{1i}^2 = & \sum_j \sum_s Q_{ij}^N Q_{is}^N n_j n_s + \sum_j \sum_s Q_{ij}^K Q_{is}^K k_j k_s + \sum_j \sum_s Q_{ij}^N Q_{is}^K n_j k_s + \sum_j \sum_s Q_{ij}^K Q_{is}^N k_j n_s \\
& + \left(q_{it}^Z z\right)^2 + \left(q_{it}^T\right)^2 + 2 q_{it}^Z z q_{it}^T \\
& + 2 \sum_j Q_{ij}^N n_j q_{it}^Z z + 2 \sum_j Q_{ij}^N n_j q_{it}^T \\
& + 2 \sum_j Q_{ij}^K k_j q_{it}^Z z + 2 \sum_j Q_{ij}^K k_j q_{it}^T.
\end{aligned}
$$

First, note that:

$$
\begin{aligned}
\sum_j \sum_s Q_{ij}^N Q_{is}^N n_j n_s &= \left[\mathbb{Q}^{N,N} \times_3 n \times_2 n\right]_i, \qquad \text{with } \mathbb{Q}_{ijs}^{N,N} = Q_{ij}^N Q_{is}^N \\
\sum_j \sum_s Q_{ij}^N Q_{is}^K n_j k_s &= \left[\mathbb{Q}^{N,K} \times_3 k \times_2 n\right]_i, \qquad \text{with } \mathbb{Q}_{ijs}^{N,K} = Q_{ij}^N Q_{is}^K, \\
\sum_j \sum_s Q_{ij}^K Q_{is}^N k_j n_s &= \left[\mathbb{Q}^{K,N} \times_3 n \times_2 k\right]_i, \qquad \text{with } \mathbb{Q}_{ijs}^{K,N} = Q_{ij}^K Q_{is}^N \\
\sum_j \sum_s Q_{ij}^K Q_{is}^K k_j k_s &= \left[\mathbb{Q}^{K,K} \times_3 k \times_2 k\right]_i, \qquad \text{with } \mathbb{Q}_{ijs}^{K,K} = Q_{ij}^K Q_{is}^K
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\sum_j Q_{ij}^N q_{it}^Z z n_j &= \left[\mathbf{diag}(q_t^Z) Q^N n z\right], \qquad \sum_j Q_{ij}^N q_{it}^T n_j = \left[\mathbf{diag}(q_t^T) Q^N n\right]_i, \\
\sum_j Q_{ij}^K q_{it}^Z z k_j &= \left[\mathbf{diag}(q_t^Z) Q^K k z\right]_i, \qquad \sum_j Q_{ij}^K q_{it}^T k_j = \left[\mathbf{diag}(q_t^T) Q^K k\right]_i,
\end{aligned}
$$

where **diag** is the diagonal operator. Putting all terms together, we get:

$$
\begin{aligned}
q_1 \cdot q_1 = & \ \mathbb{Q}^{N,N} \times_3 n \times_2 n \ + \ \mathbb{Q}^{N,K} \times_3 k \times_2 n \ + \ \mathbb{Q}^{K,N} \times_3 n \times_2 k \ + \ \mathbb{Q}^{K,K} \times_3 k \times_2 k \\
& + (q_t^Z \cdot q_t^Z) z^2 \ + \ q_t^T \cdot q_t^T + 2 (q_t^Z \cdot q_t^T) z \\
& + 2 \mathbf{diag}(q_t^Z) Q^N n z \ + \ 2 \mathbf{diag}(q_t^T) Q^N n \\
& + 2 \mathbf{diag}(q_t^Z) Q^K n z \ + \ 2 \mathbf{diag}(q_t^T) Q^K n.
\end{aligned}
$$

where the operator $\cdot$ denotes the inner product between two vectors. Then, to the leading order, we get:

$$
\begin{aligned}
\overline{Q}_t - \overline{Q}^{ss} \approx\ & \epsilon \left\{ \mathrm{D}_{QN} n + \mathrm{D}_{Q\mathcal{K}} k + \mathrm{d}_{QT} \left(\sigma_t z + T_t \right) \right\} \\
& + \frac{\epsilon^2}{2} \mathrm{D}_{QQ} \Big\{ \mathcal{Q}^{N,N} \times_3 n \times_2 n + \mathcal{Q}^{N,K} \times_3 k \times_2 n + \mathcal{Q}^{K,N} \times_3 n \times_2 k + \mathcal{Q}^{K,K} \times_3 k \times_2 k \\
& \qquad + 2Q_t^{N,Z} n z + 2Q_t^{K,Z} k z + 2Q_t^{N,T} n + 2Q_t^{K,T} k \\
& \qquad + \overline{q}_t + q_t^{Z,Z} z^2 + q_t^{T,T} + 2 q_t^{Z,T} z + q_t^{\Sigma} \Big\} \\
& + \frac{\epsilon^2}{2} \left\{ \mathbb{D}_{QNN} \times_3 n \times_2 n + \mathbb{D}_{Q\mathcal{K}\mathcal{K}} \times_3 k \times_2 k + \mathbb{D}_{QN\mathcal{K}} \times_3 n \times_2 k + \mathbb{D}_{Q\mathcal{K}N} \times_3 k \times_2 n \right\} \\
& + \frac{\epsilon^2}{2} \mathrm{D}_{QQQ} \Big\{ \mathbb{Q}^{N,N} \times_3 n \times_2 n \ + \ \mathbb{Q}^{N,K} \times_3 k \times_2 n \ + \ \mathbb{Q}^{K,N} \times_3 n \times_2 k \ + \ \mathbb{Q}^{K,K} \times_3 k \times_2 k \\
& \qquad + (q_t^Z \cdot q_t^Z) z^2 \ + \ q_t^T \cdot q_t^T + 2 (q_t^Z \cdot q_t^T) z \\
& \qquad + 2\mathbf{diag}(q_t^Z) Q^N n z \ + \ 2\mathbf{diag}(q_t^T) Q^N n \\
& \qquad + 2\mathbf{diag}(q_t^Z) Q^K k z \ + \ 2\mathbf{diag}(q_t^T) Q^K k \Big\} \\
& + \epsilon^2 \Big\{ \mathrm{D}_{QNT} n \left[\sigma_t z + T_t \right] + \mathrm{D}_{Q\mathcal{K}T} k \left[\sigma_t z + T_t \right] \Big\} \\
& + \epsilon^2 \Big\{ \left(\mathbb{D}_{QQN} \times_1 Q^N \right) \times_2 n \times_3 n + \left(\mathbb{D}_{QQN} \times_1 Q^K \right) \times_2 k \times_3 n + (\mathbb{D}_{QQN} \times_1 q_t^Z) n z + (\mathbb{D}_{QQN} \times_1 \\
& + \epsilon^2 \Big\{ \left(\mathbb{D}_{QQ\mathcal{K}} \times_1 Q^N \right) \times_2 n \times_3 k + \left(\mathbb{D}_{QQ\mathcal{K}} \times_1 Q^K \right) \times_2 k \times_3 k + (\mathbb{D}_{QQ\mathcal{K}} \times_1 q_t^Z) k z + (\mathbb{D}_{QQ\mathcal{K}} \times_1 \\
& + \epsilon^2 \mathrm{D}_{QQT} \left[Q^N n + Q^K k + q_t^Z z + q_t^T \right] \left[\sigma_t z + T_t \right] \\
& + \frac{\epsilon^2}{2} \mathrm{d}_{QTT} \left[(\sigma_t z)^2 + (T_t)^2 + 2 (\sigma_t z) T_t \right] + \frac{\epsilon^2}{2} \Sigma^2 \mathrm{d}_{Q\Sigma T} \varsigma_t
\end{aligned}
$$

where $\mathbb{D}_{QNN}$, $\mathbb{D}_{Q\mathcal{K}N}$, $\mathbb{D}_{QN\mathcal{K}}$, $\mathbb{D}_{Q\mathcal{K}\mathcal{K}}$, $\mathbb{D}_{QQN}$ and $\mathbb{D}_{QQ\mathcal{K}}$ are diagonal tensors defined as:

$$
\begin{aligned}
& \mathbb{D}_{QNN} = \psi(\psi - 1)(1 - \zeta) \mathbf{diag}_3 \left(\frac{Q_i^{ss} R_i^{ss}}{(N_i^{ss})^2} \right), && \mathbb{D}_{Q\mathcal{K}\mathcal{K}} = \phi^2 (1 - \zeta) \mathbf{diag}_3 \left(Q_i^{ss} R_i^{ss} \right), \\
& \mathbb{D}_{QN\mathcal{K}} = \mathbb{D}_{Q\mathcal{K}N} = -\psi\phi(1 - \zeta) \mathbf{diag}_3 \left(\frac{Q_i^{ss} R_i^{ss}}{N_i^{ss}} \right), && \\
& \mathbb{D}_{QQN} = (1 - \zeta) \psi \mathbf{diag}_3 \left(\frac{R_i^{ss}}{N_i^{ss}} \right) && \mathbb{D}_{QQ\mathcal{K}} = -(1 - \zeta) \phi \mathbf{diag}_3 \left(R_i^{ss} \right)
\end{aligned}
$$

Moreover, Σ^2 is a diagonal matrix with components Σ_i^2, and $\mathrm{D}_{Q\mathcal{K}}$, D_{QN}, D_{QQ}, and d_{QT}

are defined as in section 4. The remaining matrices are and vectors:

$$\mathrm{D}_{QQQ} = \frac{1-\zeta}{\zeta}\mathbf{diag}\left((Q_i^{ss})^{-1-\frac{1}{\zeta}}\right), \qquad \mathrm{D}_{QQT} = (1-\zeta)\mathbf{diag}\left(R_i^{ss}\chi_{i1} - \delta_{i1}\right)$$

$$\mathrm{D}_{QNT} = \psi(1-\zeta)\mathbf{diag}\left(\chi_{i1}\frac{Q_i^{ss}R_i^{ss}}{N_i^{ss}}\right), \qquad \mathrm{D}_{Q\mathcal{K}T} = -\phi(1-\zeta)\mathbf{diag}\left(\chi_{i1}Q_i^{ss}R_i^{ss}\right),$$

$$\mathrm{d}_{QTT} = (1-\zeta)\mathbf{vec}\left(Q_i^{ss}\left(R_i^{ss}\left(\chi_{i1}^2+\chi_{i2}\right)-\delta_{i2}\right)\right), \quad \mathrm{d}_{Q\Sigma T} = -\frac{(1-\zeta)\zeta}{2}\mathbf{vec}\left(Q_i^{ss}\right).$$

Grouping terms according to their order and using $\mathrm{D}_{QQ} = 0$ gives the final expression:

$$\begin{aligned}
\overline{Q}_t - \overline{Q}_i^{ss} \approx\ & \epsilon\left\{\mathrm{D}_{QK}k + \mathrm{D}_{QN}n + \mathrm{d}_{QT}\left(\sigma_t z + T_t\right)\right\} \longleftarrow \text{ first-order term} \\
& + \frac{\epsilon^2}{2}\left\{\mathbb{D}_{QNN} + \mathbb{Q}^{N,N}\times_1\mathrm{D}_{QQQ} + \mathbb{D}_{QQN}\times_1 Q^N\right\}\times_3 n\times_2 n \\
& + \frac{\epsilon^2}{2}\left\{\mathbb{D}_{Q\mathcal{K}N} + \mathbb{Q}^{N,K}\times_1\mathrm{D}_{QQQ} + \mathbb{D}_{QQN}\times_1 Q^K\right\}\times_3 k\times_2 n \\
& + \frac{\epsilon^2}{2}\left\{\mathbb{D}_{QN\mathcal{K}} + \mathbb{Q}^{K,N}\times_1\mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}}\times_1 Q^N\right\}\times_3 n\times_2 k \\
& + \frac{\epsilon^2}{2}\left\{\mathbb{D}_{Q\mathcal{K}\mathcal{K}} + \mathbb{Q}^{K,K}\times_1\mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}}\times_1 Q^K\right\}\times_3 k\times_2 k \\
& + \epsilon^2\left[\mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT}Q^N\right)\sigma_t + \mathbb{D}_{QQN}\times_1 q_t^Z\right]nz \\
& + \epsilon^2\left[\mathrm{D}_{QQQ}\mathbf{diag}(q_t^T)Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT}Q^N\right)T_t + \mathbb{D}_{QQN}\times_1 q_t^T\right]n \\
& + \epsilon^2\left[\mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^K + (\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT}Q^K)\sigma_t + \mathbb{D}_{QQ\mathcal{K}}\times_1 q_t^Z\right]kz \\
& + \epsilon^2\left[\mathrm{D}_{QQQ}\mathbf{diag}(q_t^T)Q^K + \left(\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT}Q^K\right)T_t + \mathbb{D}_{QQ\mathcal{K}}\times_1 q_t^T\right]k \\
& + \frac{\epsilon^2}{2}\left[\mathrm{D}_{QQQ}(q_t^Z\cdot q_t^Z) + 2\mathrm{D}_{QQT}q_t^Z\sigma_t + \mathrm{d}_{QTT}\sigma_t^2\right]z^2 \\
& + \frac{\epsilon^2}{2}\left[\mathrm{D}_{QQQ}(q_t^T\cdot q_t^T) + 2\mathrm{D}_{QQT}q_t^T T_t + \mathrm{d}_{QTT}(T_t)^2\right] \\
& + \epsilon^2\left[\mathrm{D}_{QQQ}q_t^Z q_t^T + \mathrm{D}_{QQT}(q_t^Z T_t + q_t^T\sigma_t) + \mathrm{d}_{QTT}\sigma_t T_t\right]z. \\
& + \frac{\epsilon^2}{2}\Sigma^2\mathrm{d}_{Q\Sigma T}\varsigma_t.
\end{aligned}$$

H.2 Continuation Value from Migration

We have that:

$$\mathcal{M}_i(V_t) - \mathcal{M}_i(V^{ss}) \approx \nabla[\mathcal{M}_i](V^{ss})(V - V^{ss}) + \frac{1}{2}(V - V^{ss})^T H\left[\mathcal{M}_i\right](V^{ss})(V - V^{ss}),$$

where $\nabla[\mathcal{M}_i](\cdot)$ and $H\left[\mathcal{M}_i\right](\cdot)$ are the gradient and the Hessian of $\mathcal{M}_i$, respectively. We have that, up to a second order, $V - V^{ss} = \epsilon v_1 + \frac{\epsilon^2}{2} v_2$, with v_1 and v_2 defined in (**??**) and (**??**). Therefore, omitting the dependence on V^{ss}, we can write the expression above as:

$$\mathcal{M}_i(V_t) - \mathcal{M}_i(V^{ss}) \approx \epsilon \left[M v_1\right]_i + \frac{\epsilon^2}{2}\left[M v_2\right]_i + \frac{1}{2}(\epsilon v_1 + \frac{\epsilon^2}{2} v_2)^T H\left[\mathcal{M}_i\right](V^{ss})(\epsilon v_1 + \frac{\epsilon^2}{2} v_2).$$

In matrix form:

$$\begin{aligned}
M v_1 &= M V^N n + M V^K k + M v_t^Z z + M v_t^T, \\
M v_2 &= M\Big[\mathcal{V}^{N,N} \times_3 n \times_2 n + \mathcal{V}^{N,K} \times_3 k \times_2 n + \mathcal{V}^{K,N} \times_3 n \times_2 k + \mathcal{V}^{K,K} \times_3 k \times_2 k \\
&\quad + 2V_t^{N,Z} n z + 2V_t^{K,Z} k z + 2V_t^{N,T} n + 2V_t^{K,T} k + \overline{v}_t + v_t^{Z,Z} z^2 + v_t^{T,T} + 2v_t^{Z,T} z + v_t^{\Sigma,Z} z + v_t^{\Sigma,T}\Big] \\
&= \mathcal{V}^{N,N} \times_1 M \times_3 n \times_2 n + \mathcal{V}^{N,K} \times_1 M \times_3 k \times_2 n + \mathcal{V}^{K,N} \times_1 M \times_3 n \times_2 k + \mathcal{V}^{K,K} \times_1 M \times_3 k \times_2 k \\
&\quad + 2MV_t^{N,Z} n z + 2MV_t^{K,Z} k z + 2MV_t^{N,T} n + 2MV_t^{K,T} k + M\overline{v}_t + M v_t^{Z,Z} z^2 + M v_t^{T,T} + 2M v_t^{Z,T} z \\
&\quad + M v_t^{\Sigma,Z} z + M v_t^{\Sigma,T}.
\end{aligned}$$

On the other hand, the Hessian matrix is given by:

$$H\left[\mathcal{M}_i\right](V^{ss}) = \left[\frac{\partial m_{ik}(V^{ss})}{\partial V_h}\right]_{hk}.$$

To ease notation, we omit the dependence of the migration probabilities on the steady-state vector V^{ss}. Then, we have that:

$$(\epsilon v_1 + \frac{\epsilon^2}{2} v_2)^T H\left[\mathcal{M}_i\right](V^{ss})(\epsilon v_1 + \frac{\epsilon^2}{2} v_2) = \mu\nu\left[\sum_k m_{ik}(\epsilon v_{1k} + \frac{\epsilon^2}{2} v_{2k})^2 - \sum_h \sum_k m_{ih} m_{ik}(\epsilon v_{1h} + \frac{\epsilon^2}{2} v_{2h})(\epsilon v_{1k}\right.$$

Note that the second-order terms in v_2 will drop out of the approximation when raised to

a higher power or if multiplied by $v1$. Thus, we have that:

$$(\epsilon v_1 + \frac{\epsilon^2}{2} v_2)^T H\left[\mathcal{M}_i\right](V^{ss})(\epsilon v_1 + \frac{\epsilon^2}{2} v_2) = \epsilon^2 \mu\nu \left[\sum_k m_{ik} v_{1k}^2 - \sum_h \sum_k m_{ih} m_{ik} v_{1h} v_{1k}\right],$$

which in matrix notation is:

$$(\epsilon v_1 + \frac{\epsilon^2}{2} v_2)^T H\left[\mathcal{M}\right](V^{ss})(\epsilon v_1 + \frac{\epsilon^2}{2} v_2) = \epsilon^2 \mu\nu m (v_1 \cdot v_1) - \epsilon^2 \mu\nu \mathbb{M} \times_3 (v_1) \times_2 (v_1),$$

where $\mathbb{M}$ is a tensor defined as $\mathbb{M}_{ikh} = -m_{ik} m_{ih}$. We can use the expression for $q_1 \cdot q_1$ in the previous section to expand the term $m(v_1 \cdot v_1)$:

$$\begin{aligned} m(v_1 \cdot v_1) =& \mathbb{V}^{N,N} \times_1 m \times_3 n \times_2 n + \mathbb{V}^{N,K} \times_1 m \times_3 k \times_2 n + \mathbb{V}^{K,N} \times_1 m \times_3 n \times_2 k + \mathbb{V}^{K,K} \times_1 m \times_3 k \times_2 k \\ &+ m\left(v_t^Z \cdot v_t^Z\right) z^2 + m\left(v_t^T \cdot v_t^T\right) + 2m\left(v_t^Z \cdot v_t^T\right) z \\ &+ 2m\mathbf{diag}(v_t^Z) V^N n z \ + \ 2m\mathbf{diag}(v_t^T) V^N n \\ &+ 2m\mathbf{diag}(v_t^Z) V^K n z \ + \ 2m\mathbf{diag}(v_t^T) V^K n, \end{aligned}$$

where the tensors $\mathbb{V}^{N,N}$, $\mathbb{V}^{N,K}$, $\mathbb{V}^{K,N}$, and $\mathbb{V}^{K,K}$ are defined as:

$$\mathbb{V}_{ijs}^{N,N} = V_{ij}^N V_{is}^N \qquad \mathbb{V}_{ijs}^{N,K} = V_{ij}^N V_{is}^K \qquad \mathbb{V}_{ijs}^{K,N} = V_{ij}^K V_{is}^N \qquad \mathbb{V}_{ijs}^{K,K} = V_{ij}^K V_{is}^K$$

We can use Lemmas 1 and 3 in the appendix jointly with the fact that $\mathbb{M}$ is symmetric in its last two coordinates to expand the term $\mathbb{M} \times_3 v_1 \times_2 v_1$ as:

$$\begin{aligned} \mathbb{M} \times_3 (v_1) \times_2 (v_1) =& \left(\mathbb{M} \times_3 V^N \times_2 V^N\right) \times_3 n \times_2 n + \left(\mathbb{M} \times_3 V^K \times_2 V^N\right) \times_3 k \times_2 n \\ &+ \left(\mathbb{M} \times_3 V^N \times_2 V^K\right) \times_3 n \times_2 k + \left(\mathbb{M} \times_3 V^K \times_2 V^K\right) \times_3 k \times_2 k \\ &+ 2\left(\mathbb{M} \times_2 V^N \times_3 v_t^Z\right) n z + 2\left(\mathbb{M} \times_2 V^N \times_3 v_t^T\right) n \\ &+ 2\left(\mathbb{M} \times_2 V^K \times_3 v_t^Z\right) k z + 2\left(\mathbb{M} \times_2 V^K \times_3 v_t^T\right) k \\ &+ \mathbb{M} \times_3 v_t^Z \times_2 v_t^Z z^2 + \mathbb{M} \times_3 v_t^T \times_2 v_t^T + 2\mathbb{M} \times_3 v_t^Z \times_2 v_t^T z. \end{aligned}$$

Gathering all the terms gives:

$$
\begin{aligned}
\mathcal{M}(V_t) - \mathcal{M}(V^{ss}) = \epsilon & \left(MV^N n + MV^K k + Mv_t^Z z + Mv_t^T + M \right) \\
& + \frac{\epsilon^2}{2} \Big[\mathcal{V}^{N,N} \times_1 M \times_3 n \times_2 n + \mathcal{V}^{N,K} \times_1 M \times_3 k \times_2 n + \mathcal{V}^{K,N} \times_1 M \times_3 n \times_2 k + \mathcal{V}^{K,K} \times_1 M \times_3 k \times_2 k \\
& \qquad + 2MV_t^{N,Z} nz + 2MV_t^{K,Z} kz + 2MV_t^{N,T} n + 2MV_t^{K,T} k + M\overline{v}_t + Mv_t^{Z,Z} z^2 + Mv_t^{T,T} + 2Mv_t^{Z,T} z \\
& \qquad + Mv_t^{\Sigma,Z} z + Mv_t^{\Sigma,T} \Big] \\
& + \frac{\epsilon^2 \mu\nu}{2} \Big[\mathbb{V}^{N,N} \times_1 m \times_3 n \times_2 n + \mathbb{V}^{N,K} \times_1 m \times_3 k \times_2 n + \mathbb{V}^{K,N} \times_1 m \times_3 n \times_2 k + \mathbb{V}^{K,K} \times_1 m \times_3 k \times_2 k \\
& \qquad + m \left(v_t^Z \cdot v_t^Z \right) z^2 + m \left(v_t^T \cdot v_t^T \right) + 2m \left(v_t^Z \cdot v_t^T \right) z \\
& \qquad + 2m \mathbf{diag}(v_t^Z) V^N nz + 2m \mathbf{diag}(v_t^T) V^N n + 2m \mathbf{diag}(v_t^Z) V^K nz + 2m \mathbf{diag}(v_t^T) V^K n \Big] \\
& + \frac{\epsilon^2 \mu\nu}{2} \Big[\left(\mathbb{M} \times_3 V^N \times_2 V^N \right) \times_3 n \times_2 n + \left(\mathbb{M} \times_3 V^K \times_2 V^N \right) \times_3 k \times_2 n \\
& \qquad + \left(\mathbb{M} \times_3 V^N \times_2 V^K \right) \times_3 n \times_2 k + \left(\mathbb{M} \times_3 V^K \times_2 V^K \right) \times_3 k \times_2 k \\
& \qquad + 2 \left(\mathbb{M} \times_2 V^N \times_3 v_t^Z \right) nz + 2 \left(\mathbb{M} \times_2 V^N \times_3 v_t^T \right) n + 2 \left(\mathbb{M} \times_2 V^K \times_3 v_t^Z \right) kz + 2 \left(\mathbb{M} \times_2 V^K \times \right. \\
& \qquad + \mathbb{M} \times_3 v_t^Z \times_2 v_t^Z z^2 + \mathbb{M} \times_3 v_t^T \times_2 v_t^T + 2\mathbb{M} \times_3 v_t^Z \times_2 v_t^T z \Big]
\end{aligned}
$$

We obtain the final expression grouping terms according to their order:

$$
\begin{aligned}
\mathcal{M}(V_t) - \mathcal{M}(V^{ss}) = \epsilon &\left(MV^N n + MV^K k + Mv_t^Z z + Mv_t^T \right) \longleftarrow \text{ first-order term} \\
&+ \frac{\epsilon^2}{2}\left[\mathcal{V}^{N,N} \times_1 M + \mu\nu \left(\mathbb{V}^{N,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^N \right) \right] \times_3 n \times_2 n \\
&+ \frac{\epsilon^2}{2}\left[\mathcal{V}^{N,K} \times_1 M + \mu\nu \left(\mathbb{V}^{N,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^N \right) \right] \times_3 k \times_2 n \\
&+ \frac{\epsilon^2}{2}\left[\mathcal{V}^{K,N} \times_1 M + \mu\nu \left(\mathbb{V}^{K,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^K \right) \right] \times_3 n \times_2 k \\
&+ \frac{\epsilon^2}{2}\left[\mathcal{V}^{K,K} \times_1 M + \mu\nu \left(\mathbb{V}^{K,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^K \right) \right] \times_3 k \times_2 k \\
&+ \epsilon^2 \left[MV_t^{N,Z} + \mu\nu \left(m\, \mathbf{diag}(v_t^Z) V^N + \mathbb{M} \times_2 V^N \times_3 v_t^Z \right) \right] nz \\
&+ \epsilon^2 \left[MV_t^{N,T} + \mu\nu \left(m\, \mathbf{diag}(v_t^T) V^N + \mathbb{M} \times_2 V^N \times_3 v_t^T \right) \right] n \\
&+ \epsilon^2 \left[MV_t^{K,Z} + \mu\nu \left(m\, \mathbf{diag}(v_t^Z) V^K + \mathbb{M} \times_2 V^K \times_3 v_t^Z \right) \right] kz \\
&+ \epsilon^2 \left[MV_t^{N,T} + \mu\nu \left(m\, \mathbf{diag}(v_t^T) V^K + \mathbb{M} \times_2 V^K \times_3 v_t^T \right) \right] k \\
&+ \frac{\epsilon^2}{2}\left[Mv_t^{Z,Z} + \mu\nu \left(m(v_t^Z \cdot v_t^Z) + \mathbb{M} \times_3 v_t^Z \times_2 v_t^Z \right) \right] z^2 \\
&+ \frac{\epsilon^2}{2}\left[Mv_t^{T,T} + \mu\nu \left(m\,(v_t^T \cdot v_t^T) + \mathbb{M} \times_3 v_t^T \times_2 v_t^T \right) \right] \\
&+ \epsilon^2 \left[Mv_t^{Z,T} + \mu\nu \left(m\,(v_t^Z \cdot v_t^T) + \mathbb{M} \times_3 v_t^Z \times_2 v_t^T \right) \right] z \\
&+ \frac{\epsilon^2}{2} Mv_t^{\Sigma,Z} z + \frac{\epsilon^2}{2} Mv_t^{\Sigma,T} \\
&+ \frac{\epsilon^2}{2} M\overline{v}_t
\end{aligned}
$$

H.3 Continuation Value from Changes in Population Distribution

H.3.1 Workers

The continuation value for workers from changes in the population distribution is given by:

$$\mu \sum_j \frac{\partial V_i}{\partial N_j} \left(\sum_k m_{kj}(V) N_k - N_j \right)$$

We will approximate individual terms and then take the product. First, applying the

chain rule to (**??**), (**??**) and (**??**), we get that:

$$\frac{\partial V_i}{\partial N_j} = V_{ij}^N + \frac{\epsilon}{2}\left(\sum_s \mathcal{V}_{ijs}^{N,N} n_s + \sum_j \mathcal{V}_{ijs}^{N,N} n_j + \sum_s \mathcal{V}_{ijs}^{N,K} k_s + \sum_j \mathcal{V}_{ijs}^{K,N} k_j + 2V_{ijt}^{N,Z} z + 2V_{ijt}^{N,T}\right).$$

Now, fixing j and k, we work on the term $m_{kj}N_k - N_j$. First, we start approximating m_{kj}:

$$M_{jk}^*(V_{SS} + dv) = M_{jk}(V_{SS})^* + \nabla\left[M_{jk}^*\right](V_{SS})dv + \frac{1}{2}dv^* H\left[M_{jk}^*\right](V_{SS})dv$$

Recall that we have already derived the gradient component in the previous section, which we now define as a tensor:

$$\frac{\partial m_{kj}}{\partial V_h} = \begin{cases} -\nu m_{kj} m_{kh} & \text{if } j \neq h \\ \nu m_{kj}(1 - m_{kj}) & \text{if } j = h \end{cases}$$

Therefore,

$$\frac{\partial M_{jk}^*}{\partial V_h} = \mu\nu \begin{cases} -m_{kj} m_{kh} & \text{if } j \neq h \\ m_{kj}(1 - m_{kj}) & \text{if } j = h \end{cases} \equiv \mathbb{G}_{jkh}$$

This implies that,

$$\left(\nabla\left[M_{jk}^*\right](V_{SS})dv\right)_{jk} = \mathbb{G} \times_3 dv$$

On the other hand, for the Hessian, we have that $H\left[M_{jk}^*\right] = \mu\frac{\partial^2 m_{kj}}{\partial V_h \partial V_s}$. Now,

$$\begin{aligned}
\frac{\partial^2 m_{kj}}{\partial^2 V_h} &= \nu^2 m_{kj} m_{kh}(2m_{kh} - 1) && \text{for } h \neq j \\
\frac{\partial^2 m_{kj}}{\partial V_h \partial V_s} &= \nu^2 2 m_{kj} m_{kh} m_{ks} && \text{for } h \neq s \text{ and } h, s \neq j \\
\frac{\partial^2 m_{kj}}{\partial V_h \partial V_j} &= \nu^2 m_{kj} m_{kh}(2m_{kj} - 1) && \text{for } h \neq j \\
\frac{\partial^2 m_{kj}}{\partial^2 V_j} &= \nu^2 m_{kj}\left(1 - 3m_{kj} + 2m_{kj}^2\right)
\end{aligned}$$

Hence, define:

$$\mathbb{H}_{kjhs} = \mu \frac{\partial^2 m_{jk}}{\partial V_h \partial V_s},$$

a four-dimensional tensor. Using these expressions, one obtains:

$$\left(dv^* H \left[M^*_{jk} \right] dv \right)_{jk} = \mathbb{H} \times_4 dv \times_3 dv,$$

where the tensor product with respect to the fourth coordinate, $\times_4$, is defined as analogously to $\times_3$ in Section **??**. Then, to the leading order (where the 0th order term drops out in steady-state):

$$\begin{aligned}
(M(V)^* N)_j &= \sum_k \left(M^*_{jk} + \mathbb{G}_{jk\bullet} \times_3 dv + \frac{1}{2} \mathbb{H}_{jk\bullet\bullet} \times_4 dv \times_3 dv \right) (N_k + \epsilon n_k), \\
&= \left(\mathbb{G}_{j\bullet\bullet} \times_2 N \right) dv + \epsilon (M^* n)_j \\
&+ \epsilon \mathbb{G}_{j\bullet\bullet} \times_3 dv \times_2 n + \frac{1}{2} \left(\mathbb{H}_{j\bullet\bullet\bullet} \times_2 N \right) \times_3 dv \times_2 dv, \\
&= \epsilon \left\{ \left(\mathbb{G}_{j\bullet\bullet} \times_2 N \right) v_1 + (M^* n)_j \right\} \\
&+ \epsilon^2 \left\{ \frac{1}{2} \left(\mathbb{G}_{j\bullet\bullet} \times_2 N \right) v_2 + \mathbb{G}_{j\bullet\bullet} \times_3 v_1 \times_2 n + \frac{1}{2} \left(\mathbb{H}_{j\bullet\bullet\bullet} \times_2 N \right) \times_3 v_1 \times_2 v_1 \right\} \\
&\equiv \epsilon \mathcal{M}_j + \epsilon^2 \mathcal{N}_j, \qquad (51)
\end{aligned}$$

where we drop 3rd-order terms to get to the second and third lines. Note that:

$$\begin{aligned}
\mathcal{M} &= M^* n + (\mathbb{G} \times_2 N) v_1, \\
&= M^* n + (\mathbb{G} \times_2 N) \left(V^N n + V^K k + v^Z_t z + v^T_t \right), \\
&= F^N n + F^K k + f^Z_t z + f^T_t,
\end{aligned}$$

with

$$\begin{aligned}
F^N &= M^* + (\mathbb{G} \times_2 N) V^N, & f^Z_t &= (\mathbb{G} \times_2 N) v^Z_t, \\
F^K &= (\mathbb{G} \times_2 N) V^K, & f^T_t &= (\mathbb{G} \times_2 N) v^T_t.
\end{aligned}$$

Thus, again, to the leading order:

$$\begin{aligned}
&\sum_j \frac{\partial V_i}{\partial N_j} \left(M^*(V)N\right)_j = \\
&\sum_j \left\{ V_{ij}^N + \frac{\epsilon}{2} \left(\sum_s \mathcal{V}_{ijs}^{N,N} n_s + \sum_s \mathcal{V}_{isj}^{N,N} n_s + \sum_s \mathcal{V}_{ijs}^{N,K} k_s + \sum_s \mathcal{V}_{isj}^{K,N} k_s + 2V_{ijt}^{N,Z} z + 2V_{ijt}^{N,T} \right) \right\} \\
&\quad \times \left\{ \epsilon \mathcal{M}_j + \epsilon^2 \mathcal{N}_j \right\}, \\
&= \sum_j \left\{ V_{ij}^N + \frac{\epsilon}{2} \partial \mathcal{V}_j \right\} \left\{ \epsilon \mathcal{M}_j + \epsilon^2 \mathcal{N}_j \right\}
\end{aligned}$$

Focusing on the second-order terms, we start by expanding the term involving $\partial \mathcal{V}_j$ and $\mathcal{M}_j$:

$$\frac{\epsilon^2}{2} \sum_j \partial \mathcal{V}_j \times \mathcal{M}_j = \left(\mathcal{V}^{N,N} \times_3 n + \mathcal{V}^{N,N} \times_2 n + \mathcal{V}^{N,K} \times_3 k + \mathcal{V}^{K,N} \times_2 k + 2V_t^{N,Z} z + 2V_t^{N,T} \right) \mathcal{M}$$

Expanding term by term, we get:

$$\begin{aligned}
\left(\mathcal{V}^{N,N} \times_3 n \right) \mathcal{M} = \mathcal{V}^{N,N} \times_2 F^N \times_3 n \times_2 n + \mathcal{V}^{N,N} \times_2 F^K \times_3 k \times_2 n \\
+ \left(\mathcal{V}^{N,N} \times_2 f_t^Z \right) nz + \left(\mathcal{V}^{N,N} \times_2 f_t^T \right) n.
\end{aligned}$$

$$\begin{aligned}
\left(\mathcal{V}^{N,N} \times_2 n \right) \mathcal{M} = \mathcal{V}^{N,N} \times_3 F^N \times_3 n \times_2 n + \mathcal{V}^{N,N} \times_3 F^K \times_3 k \times_2 n \\
+ \left(\mathcal{V}^{N,N} \times_3 f_t^Z \right) nz + \left(\mathcal{V}^{N,N} \times_3 f_t^T \right) n.
\end{aligned}$$

$$\begin{aligned}
\left(\mathcal{V}^{N,K} \times_3 k \right) \mathcal{M} = \mathcal{V}^{N,K} \times_2 F^N \times_3 k \times_2 n + \mathcal{V}^{N,K} \times_2 F^K \times_3 k \times_2 k \\
+ \left(\mathcal{V}^{N,K} \times_2 f_t^Z \right) kz + \left(\mathcal{V}^{N,K} \times_2 f_t^T \right) k.
\end{aligned}$$

$$\begin{aligned}
\left(\mathcal{V}^{K,N} \times_2 k \right) \mathcal{M} = \mathcal{V}^{K,N} \times_3 F^N \times_3 n \times_2 k + \mathcal{V}^{K,N} \times_3 F^K \times_3 k \times_2 k \\
+ \left(\mathcal{V}^{K,N} \times_3 f_t^Z \right) kz + \left(\mathcal{V}^{K,N} \times_3 f_t^T \right) k.
\end{aligned}$$

$$2V_t^{N,Z} z\mathcal{M} = 2\left(V_t^{N,Z} F^N n z + V_t^{N,Z} F^K k z + V_t^{N,Z} f_t^Z z^2 + V_t^{N,Z} f_t^T z\right).$$

$$2V_t^{N,T} \mathcal{M} = 2\left(V_t^{N,T} F^N n + V_t^{N,T} F^K k + V_t^{N,T} f_t^Z z + V_t^{N,T} f_t^T\right).$$

Now we work on the term involving V^N and $\mathcal{N}$:

$$\begin{aligned}\sum_j V_{ij}^N \epsilon^2 \mathcal{N}_j &= \epsilon^2 \left\{\frac{1}{2}\left(\mathbb{G} \times_1 V^N \times_2 N\right) v_2 + \mathbb{G} \times_1 V^N \times_3 v_1 \times_2 n + \frac{1}{2}\left(\mathbb{H} \times_1 V^N \times_2 N\right) \times_3 v_1 \times_2 v_1\right\}, \\ &= \epsilon^2 \left\{\frac{1}{2} X v_2 + \mathbb{Y} \times_3 v_1 \times_2 n + \frac{1}{2}\mathbb{Z} \times_3 v_1 \times_2 v_1\right\},\end{aligned}$$

with the matrix X and the tensors $\mathbb{Y}$ and $\mathbb{Z}$ defined as

$$X \equiv \mathbb{G} \times_1 V^N \times_2 N, \qquad \mathbb{Y} \equiv \mathbb{G} \times_1 V^N, \qquad \mathbb{Z} = \mathbb{H} \times_1 V^N \times_2 N,$$

and we denote $\mathbb{Y}^*$ as the tensor resulting when transposing the last two coordinates of $\mathbb{Y}$.

Expanding each individual term, we get:

$$\begin{aligned}X v_2 = &\left(\mathcal{V}^{N,N} \times_1 X\right) \times_3 n \times_2 n + \left(\mathcal{V}^{N,K} \times_1 X\right) \times_3 k \times_2 n \\ &+ \left(\mathcal{V}^{K,N} \times_1 X\right) \times_3 n \times_2 k + \left(\mathcal{V}^{K,K} \times_1 X\right) \times_3 k \times_2 k \\ &+ 2X V_t^{N,Z} n z + 2X V_t^{K,Z} k z + 2X V_t^{N,T} n + 2X V_t^{K,T} k \\ &+ X\overline{v}_t + X v_t^{Z,Z} z^2 + X v_t^{T,T} + 2X v_t^{Z,T} z + X v_t^{\Sigma,Z} z + X v_t^{\Sigma,T}.\end{aligned}$$

For the second term, we get:

$$\begin{aligned}\mathbb{Y} \times_3 v_1 \times_2 n &= \mathbb{Y} \times_3 \left(V^N n + V^K k + v_t^Z z + v_t^T\right) \times_2 n \\ &= \left(\mathbb{Y} \times_3 V^N\right) \times_3 n \times_2 n + \left(\mathbb{Y} \times_3 V^K\right) \times_3 k \times_2 n + \left(\mathbb{Y} \times_3 v_t^Z\right) n z + \left(\mathbb{Y} \times_3 v_t^T\right) n\end{aligned}$$

For the last term, we use the results from the previous section to write:

$$\begin{aligned}\mathbb{Z} \times_3 v_1 \times_2 v_1 = {} & \mathbb{Z} \times_3 V^N \times_2 V^N \times_3 n \times_2 n + \mathbb{Z} \times_3 V^K \times_2 V^N \times_3 k \times_2 n \\ & + \mathbb{Z} \times_3 V^N \times_2 V^K \times_3 n \times_2 k + \mathbb{Z} \times_3 V^K \times_2 V^K \times_3 k \times_2 k \\ & + 2\left(\mathbb{Z} \times_2 V^N \times_3 v_t^Z\right) nz + 2\left(\mathbb{Z} \times_2 V^N \times_3 v_t^T\right) n \\ & + 2\left(\mathbb{Z} \times_2 V^K \times_3 v_t^Z\right) kz + 2\left(\mathbb{Z} \times_2 V^K \times_3 v_t^T\right) k \\ & + \mathbb{Z} \times_3 v_t^Z \times_2 v_t^Z z^2 + \mathbb{Z} \times_3 v_t^T \times_2 v_t^T + 2\mathbb{Z} \times_3 v_t^Z \times_2 v_t^T z\end{aligned}$$

Collecting all terms, we get:

$$\begin{aligned}&\frac{1}{\epsilon^2}\sum_j \frac{\partial V_i}{\partial N_j}\left(M^*(V)N\right)_j \\ &= \frac{1}{2}\left\{\mathcal{V}^{N,N} \times_2 F^N + \mathcal{V}^{N,N} \times_3 F^N + \mathcal{V}^{N,N} \times_1 X + \mathbb{Y}^* \times_2 V^N + \mathbb{Y} \times_3 V^N + \mathbb{Z} \times_3 V^N \times_2 V^N\right\} \times_3 n \times_2 n \\ &+ \frac{1}{2}\left\{\mathcal{V}^{N,N} \times_3 F^K + \mathcal{V}^{N,K} \times_2 F^N + \mathcal{V}^{N,K} \times_1 X + \mathbb{Y} \times_3 V^K + \mathbb{Z} \times_3 V^K \times_2 V^N\right\} \times_3 k \times_2 n \\ &+ \frac{1}{2}\left\{\mathcal{V}^{N,N} \times_2 F^K + \mathcal{V}^{K,N} \times_3 F^N + \mathcal{V}^{K,N} \times_1 X + \mathbb{Y}^* \times_2 V^K + \mathbb{Z} \times_3 V^N \times_2 V^K\right\} \times_3 n \times_2 k \\ &+ \frac{1}{2}\left\{\mathcal{V}^{N,K} \times_2 F^K + \mathcal{V}^{K,N} \times_3 F^K + \mathcal{V}^{K,K} \times_1 X + \mathbb{Z} \times_3 V^K \times_2 V^K\right\} \times_3 k \times_2 k \\ &+ \left[\frac{1}{2}\mathcal{V}^{N,N} \times_2 f_t^Z + \frac{1}{2}\mathcal{V}^{N,N} \times_3 f_t^Z + V_t^{N,Z} F^N + X V_t^{N,Z} + \mathbb{Y} \times_3 v_t^Z + \mathbb{Z} \times_2 V^N \times_3 v_t^Z\right] nz \\ &+ \left[\frac{1}{2}\mathcal{V}^{N,N} \times_2 f_t^T + \frac{1}{2}\mathcal{V}^{N,N} \times_3 f_t^T + V_t^{N,T} F^N + X V_t^{N,T} + \mathbb{Y} \times_3 v_t^T + \mathbb{Z} \times_2 V^N \times_3 v_t^T\right] n \\ &+ \left[\frac{1}{2}\mathcal{V}^{N,K} \times_2 f_t^Z + \frac{1}{2}\mathcal{V}^{K,N} \times_3 f_t^Z + V_t^{N,Z} F^K + X V_t^{K,Z} + \mathbb{Z} \times_2 V^K \times_3 v_t^Z\right] kz \\ &+ \left[\frac{1}{2}\mathcal{V}^{N,K} \times_2 f_t^T + \frac{1}{2}\mathcal{V}^{K,N} \times_3 f_t^T + V_t^{N,T} F^K + X V_t^{K,T} + \mathbb{Z} \times_2 V^K \times_3 v_t^T\right] k \\ &+ \frac{1}{2}\left[2V_t^{N,Z} f_t^Z + X v_t^{Z,Z} + \mathbb{Z} \times_3 v_t^Z \times_2 v_t^Z\right] z^2 \\ &+ \frac{1}{2}\left[2V_t^{N,T} f_t^T + X v_t^{T,T} + \mathbb{Z} \times_3 v_t^T \times_2 v_t^T\right] \\ &+ \left[V_t^{N,Z} f_t^T + V_t^{N,T} f_t^Z + X v_t^{Z,T} + \mathbb{Z} \times_3 v_t^Z \times_2 v_t^T\right] z \\ &+ \frac{1}{2} X v_t^{\Sigma,Z} z + \frac{1}{2} X v_t^{\Sigma,T} \\ &+ \left[\frac{1}{2} X \overline{v}_t\right]\end{aligned}$$

H.3.2 Capitalists

For capitalists, the continuation value from changes in the population distribution is given by:

$$\mu \sum_j \frac{\partial Q_i}{\partial N_j} \left(\sum_k m_{kj}(V) N_k - N_j \right)$$

Similarly to the workers case, (**??**), (**??**), and (**??**) imply that:

$$\frac{\partial Q_i}{\partial N_j} = Q_{ij}^N + \frac{\epsilon}{2} \left(\sum_s \mathcal{Q}_{ijs}^{N,N} n_s + \sum_j \mathcal{Q}_{ijs}^{N,N} n_j + \sum_s \mathcal{Q}_{ijs}^{N,K} k_s + \sum_j \mathcal{Q}_{ijs}^{K,N} k_j + 2Q_{ijt}^{N,Z} z + 2Q_{ijt}^{N,T} \right)$$

Following the derivations for the worker's case, we get that to the leading order:

$$\begin{aligned}
&\sum_j \frac{\partial Q_i}{\partial N_j} \left(M^*(V) N \right)_j = \\
&\sum_j \left\{ Q_{ij}^N + \frac{\epsilon}{2} \left(\sum_s \mathcal{Q}_{ijs}^{N,N} n_s + \sum_j \mathcal{Q}_{ijs}^{N,N} n_j + \sum_s \mathcal{Q}_{ijs}^{N,K} k_s + \sum_j \mathcal{Q}_{ijs}^{K,N} k_j + 2Q_{ijt}^{N,Z} z + 2Q_{ijt}^{N,T} \right) \right\} \\
&\quad \times \left\{ \epsilon \mathcal{M}_j + \epsilon^2 \mathcal{N}_j \right\}, \\
&= \sum_j \left\{ Q_{ij}^N + \frac{\epsilon}{2} \partial \mathcal{Q}_j \right\} \left\{ \epsilon \mathcal{M}_j + \epsilon^2 \mathcal{N}_j \right\}
\end{aligned}$$

First, the term $\partial \mathcal{Q}_j \times \mathcal{M}_j$ is analogous to the worker's case. Second, we get that:

$$\begin{aligned}
\sum_j Q_{ij}^N \epsilon^2 \mathcal{N}_j &= \epsilon^2 \left\{ \frac{1}{2} \left(\mathbb{G} \times_1 Q^N \times_2 N \right) v_2 + \mathbb{G} \times_1 Q^N \times_3 v_1 \times_2 n + \frac{1}{2} \left(\mathbb{H} \times_1 Q^N \times_2 N \right) \times_3 v_1 \times_2 v_1 \right\}, \\
&= \epsilon^2 \left\{ \frac{1}{2} S v_2 + \mathbb{P} \times_3 v_1 \times_2 n + \frac{1}{2} \mathbb{R} \times_3 v_1 \times_2 v_1 \right\},
\end{aligned}$$

with the matrix S and the tensors $\mathbb{P}$ and $\mathbb{R}$ defined as

$$S \equiv \mathbb{G} \times_1 Q^N \times_2 N, \qquad \mathbb{P} \equiv \mathbb{G} \times_1 Q^N, \qquad \mathbb{R} = \mathbb{H} \times_1 Q^N \times_2 N.$$

We denote by $\mathbb{P}^*$ the tensor one obtains by transposing the last two coordinates of $\mathbb{P}$. Collecting all terms, the second-order term of the capitalist's continuation value from

changes in the population distribution is given by:

$$
\begin{aligned}
&\frac{1}{\epsilon^2}\sum_j \frac{\partial Q_i}{\partial N_j}\left(M^*(V)N\right)_j \\
&= \frac{1}{2}\left\{\mathcal{Q}^{N,N}\times_2 F^N + \mathcal{Q}^{N,N}\times_3 F^N + \mathcal{V}^{N,N}\times_1 S + \mathbb{P}^*\times_2 V^N + \mathbb{P}\times_3 V^N + \mathbb{R}\times_3 V^N\times_2 V^N\right\}\times_3 n\times_2 n \\
&+ \frac{1}{2}\left\{\mathcal{Q}^{N,N}\times_3 F^K + \mathcal{Q}^{N,K}\times_2 F^N + \mathcal{V}^{N,K}\times_1 S + \mathbb{P}\times_3 V^K + \mathbb{R}\times_3 V^K\times_2 V^N\right\}\times_3 k\times_2 n \\
&+ \frac{1}{2}\left\{\mathcal{Q}^{N,N}\times_2 F^K + \mathcal{Q}^{K,N}\times_3 F^N + \mathcal{V}^{K,N}\times_1 S + \mathbb{P}^*\times_2 V^K + \mathbb{R}\times_3 V^N\times_2 V^K\right\}\times_3 n\times_2 k \\
&+ \frac{1}{2}\left\{\mathcal{Q}^{N,K}\times_2 F^K + \mathcal{Q}^{K,N}\times_3 F^K + \mathcal{V}^{K,K}\times_1 S + \mathbb{R}\times_3 V^K\times_2 V^K\right\}\times_3 k\times_2 k \\
&+ \left[\frac{1}{2}\mathcal{Q}^{N,N}\times_2 f_t^Z + \frac{1}{2}\mathcal{Q}^{N,N}\times_3 f_t^Z + Q_t^{N,Z}F^N + SV_t^{N,Z} + \mathbb{P}\times_3 v_t^Z + \mathbb{R}\times_2 V^N\times_3 v_t^Z\right] nz \\
&+ \left[\frac{1}{2}\mathcal{Q}^{N,N}\times_2 f_t^T + \frac{1}{2}\mathcal{Q}^{N,N}\times_3 f_t^T + Q_t^{N,T}F^N + SV_t^{N,T} + \mathbb{P}\times_3 v_t^T + \mathbb{R}\times_2 V^N\times_3 v_t^T\right] n \\
&+ \left[\frac{1}{2}\mathcal{Q}^{N,K}\times_2 f_t^Z + \frac{1}{2}\mathcal{Q}^{K,N}\times_3 f_t^Z + Q_t^{N,Z}F^K + SV_t^{K,Z} + \mathbb{R}\times_2 V^K\times_3 v_t^Z\right] kz \\
&+ \left[\frac{1}{2}\mathcal{Q}^{N,K}\times_2 f_t^T + \frac{1}{2}\mathcal{Q}^{K,N}\times_3 f_t^T + Q_t^{N,T}F^K + SV_t^{K,T} + \mathbb{R}\times_2 V^K\times_3 v_t^T\right] k \\
&+ \frac{1}{2}\left[2Q_t^{N,Z}f_t^Z + Sv_t^{Z,Z} + \mathbb{R}\times_3 v_t^Z\times_2 v_t^Z\right] z^2 \\
&+ \frac{1}{2}\left[2Q_t^{N,T}f_t^T + Sv_t^{T,T} + \mathbb{R}\times_3 v_t^T\times_2 v_t^T\right] \\
&+ \left[Q_t^{N,Z}f_t^T + Q_t^{N,T}f_t^Z + Sv_t^{Z,T} + \mathbb{R}\times_3 v_t^Z\times_2 v_t^T\right] z \\
&+ \frac{1}{2}Sv_t^{\Sigma,Z}z + \frac{1}{2}Sv_t^{\Sigma,T} \\
&+ \frac{1}{2}\left[S\overline{v}_t\right]
\end{aligned}
$$

H.4 Continuation Value from Changes in the Capital Distribution

H.4.1 Workers

Worker's continuation value from changes in the capital distribution is given by:

$$\sum_j \frac{\partial V_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right)$$

We first work on approximating the law of motion for capital. Up to second-order we

get:

$$\begin{aligned}\frac{dk_t}{dt} &= \mathrm{D}_{R\mathcal{K}}k_t + \mathrm{D}_{RN}n_t + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_{1,t}, \\ &+ \frac{\epsilon}{2}\Big\{\mathbb{D}_{RNN}\times_3 n_t \times_2 n_t + \mathbb{D}_{RN\mathcal{K}}\times_3 k_t \times_2 n_t + \mathbb{D}_{R\mathcal{K}N}\times_3 n_t \times_2 n_k + \mathbb{D}_{R\mathcal{K}\mathcal{K}}\times_3 k_t \times_2 k_t \\ &\quad + 2\mathrm{D}_{RNT}n_t(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T}k_t(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2\mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t \\ &\quad + \mathrm{D}_{IQ}q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t}\cdot q_{1,t})\Big\},\end{aligned} \tag{52}$$

where:

$$\mathbb{D}_{RNN} = \psi(\psi-1)\mathbf{diag}_3\left(\frac{R_i^{ss}}{(N_i^{ss})^2}\right), \quad \mathbb{D}_{R\mathcal{K}\mathcal{K}} = \phi^2\mathbf{diag}_3\left(R_i^{ss}\right), \qquad \mathbb{D}_{RN\mathcal{K}} = \mathbb{D}_{R\mathcal{K}N} = -\phi\psi\mathbf{diag}_3\left(\frac{R_i^{ss}}{N_i^{ss}}\right)$$
$$\mathrm{D}_{RNT} = \psi\mathbf{diag}\left(\chi_{i1}\frac{R_i^{ss}}{N_i^{ss}}\right), \qquad \mathrm{D}_{R\mathcal{K}T} = -\phi\mathbf{diag}\left(\chi_{i1}R_i^{ss}\right), \quad \mathrm{D}_{\mathcal{I}QQ} = -\frac{1+\zeta}{\zeta}\mathbf{diag}\left((Q_i^{ss})^{-\frac{1}{\zeta}-2}\right),$$
$$\mathrm{d}_{\mathcal{I}TT} = \mathbf{vec}\left((\chi_{i1}^2+\chi_{i2})R_i^{ss} - \delta_{i2}\right), \quad \mathrm{d}_{\mathcal{I}\Sigma T} = -\frac{1}{2}\mathbb{1},$$

where $\mathbb{1}$ is a vector with ones in all of its entries. Now, turning to the value, we we have that:

$$\frac{\partial V_i}{\partial \mathcal{K}_j} = V_{ij}^K + \frac{\epsilon}{2}\underbrace{\left(\sum_s \mathcal{V}_{ijs}^{K,K}k_s + \sum_s \mathcal{V}_{isj}^{K,K}k_s + \sum_s \mathcal{V}_{isj}^{N,K}n_s + \sum_s \mathcal{V}_{ijs}^{K,N}n_s + 2V_{ijt}^{K,Z}z + 2V_{ijt}^{K,T}\right)}_{\equiv \partial_j V_i}.$$

Hence, we get:

$$
\begin{aligned}
&\frac{1}{\epsilon}\sum_j \frac{\partial V_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right) \\
&\quad = \sum_j \left(V_{ij}^K + \frac{\epsilon}{2}\partial_j V_i\right)\left[\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_1\right]_j \\
&\quad + \sum_j \left(V_{ij}^K + \frac{\epsilon}{2}\partial_j V_i\right)\left(\frac{\epsilon}{2}\Big[\mathbb{D}_{RNN}\times_3 n_t \times_2 n_t + \mathbb{D}_{RN\mathcal{K}}\times_3 k_t \times_2 n_t + \mathbb{D}_{R\mathcal{K}N}\times_3 n_t \times_2 n_k + \mathbb{D}_{R\mathcal{K}\mathcal{K}}\times_3 k_t\right. \\
&\qquad\qquad + 2\mathrm{D}_{RNT}n_t(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T}k_t(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2\mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t \\
&\qquad\qquad \left. + \mathrm{D}_{IQ}q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t}\cdot q_{1,t})\Big]_j\right) \\
&\quad = \sum_j V_{ij}^K\left[\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_1\right]_j \\
&\quad + \frac{\epsilon}{2}\sum_j \partial_j V_i\left[\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_1\right]_j \\
&\quad + \frac{\epsilon}{2}\sum_j V_{ij}^K\Big[\mathbb{D}_{RNN}\times_3 n_t \times_2 n_t + \mathbb{D}_{RN\mathcal{K}}\times_3 k_t \times_2 n_t + \mathbb{D}_{R\mathcal{K}N}\times_3 n_t \times_2 n_k + \mathbb{D}_{R\mathcal{K}\mathcal{K}}\times_3 k_t \times_2 k_t \\
&\qquad + 2\mathrm{D}_{RNT}n_t(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T}k_t(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2\mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t \\
&\qquad + \mathrm{D}_{IQ}q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t}\cdot q_{1,t})\Big]_j,
\end{aligned}
$$

where we drop the terms with an order higher than two. The first-order term is given by the first line:

$$
\sum_j V_{ij}^K\left[\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_1\right]_i = V^K\left\{\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_1\right\}
$$

Working on the second line, we get that

$$
\begin{aligned}
&\sum_j \partial_j V_i \left[\mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q} q_1\right]_j \\
&\quad = \sum_j \left(\sum_s \mathcal{V}^{K,K}_{ijs} k_s + \sum_s \mathcal{V}^{K,K}_{isj} k_s + \sum_s \mathcal{V}^{N,K}_{isj} n_s + \sum_s \mathcal{V}^{K,N}_{ijs} n_s + 2V^{K,Z}_{ijt} z + 2V^{K,T}_{ijt} \right) \\
&\quad \times \left(\mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q} q_1\right)_j \\
&\quad = \left(\mathcal{V}^{K,K} \times_2 \mathrm{D}_{R\mathcal{K}}\right) \times_3 k \times_2 k + \left(\mathcal{V}^{K,K} \times_3 \mathrm{D}_{R\mathcal{K}}\right) \times_3 k \times_2 k + \left(\mathcal{V}^{N,K} \times_3 \mathrm{D}_{R\mathcal{K}}\right) \times_3 k \times_2 n \\
&\qquad + \left(\mathcal{V}^{K,N} \times_2 \mathrm{D}_{R\mathcal{K}}\right) \times_3 n \times_2 k + 2V^{K,Z}_t \mathrm{D}_{R\mathcal{K}} k z + 2V^{K,T}_t \mathrm{D}_{R\mathcal{K}} k \\
&\quad + \left(\mathcal{V}^{K,K} \times_2 \mathrm{D}_{RN}\right) \times_3 k \times_2 n + \left(\mathcal{V}^{K,K} \times_3 \mathrm{D}_{RN}\right) \times_3 n \times_2 k + \left(\mathcal{V}^{N,K} \times_3 \mathrm{D}_{RN}\right) \times_3 n \times_2 n \\
&\qquad + \left(\mathcal{V}^{K,N} \times_2 \mathrm{D}_{RN}\right) \times_3 n \times_2 n + 2V^{K,Z}_t \mathrm{D}_{RN} n z + 2V^{K,T}_t \mathrm{D}_{RN} n \\
&\quad + \left(\mathcal{V}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T}\right) k(\sigma_t z + T_t) + \left(\mathcal{V}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T}\right) k(\sigma_t z + T_t) + \left(\mathcal{V}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T}\right) n(\sigma_t z + T_t) \\
&\qquad + \left(\mathcal{V}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T}\right) n(\sigma_t z + T_t) + 2V^{K,Z}_t \mathrm{d}_{\mathcal{I}T} z(\sigma_t z + T_t) + 2V^{K,T}_t \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) \\
&\quad + \left(\mathcal{V}^{K,K} \times_2 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 k \times_2 q_1 + \left(\mathcal{V}^{K,K} \times_3 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 q_1 \times_2 k + \left(\mathcal{V}^{N,K} \times_3 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 q_1 \times_2 n \\
&\qquad + \left(\mathcal{V}^{K,N} \times_2 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 n \times_2 q_1 + 2V^{K,Z}_t \mathrm{D}_{\mathcal{I}Q} q_1 z + 2V^{K,T}_t \mathrm{D}_{\mathcal{I}Q} q_1.
\end{aligned}
$$

We work on expanding the last two lines which involve the term q_1. Define the matrices:

$$
D^N \equiv \mathrm{D}_{\mathcal{I}Q} Q^N, \qquad D^K \equiv \mathrm{D}_{\mathcal{I}Q} Q^K, \qquad D^Z_t \equiv \mathrm{D}_{\mathcal{I}Q} q^Z_t, \qquad D^T_t \equiv \mathrm{D}_{\mathcal{I}Q} q^T_t.
$$

Using this notation, expanding each of the sub-terms gives:

$$\begin{aligned}
\left(\mathcal{V}^{K,K} \times_2 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 k \times_2 q_1 &= \left(\mathcal{V}^{K,K} \times_2 D^N\right) \times_3 k \times_2 n + \left(\mathcal{V}^{K,K} \times_2 D^K\right) \times_3 k \times_2 k \\
&\quad + \left(\mathcal{V}^{K,K} \times_2 D_t^Z\right) kz + \left(\mathcal{V}^{K,K} \times_2 D_t^T\right) k. \\
\left(\mathcal{V}^{K,K} \times_3 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 q_1 \times_2 k &= \left(\mathcal{V}^{K,K} \times_3 D^N\right) \times_3 n \times_2 k + \left(\mathcal{V}^{K,K} \times_3 D^K\right) \times_3 k \times_2 k \\
&\quad + \left(\mathcal{V}^{K,K} \times_3 D_t^Z\right) kz + \left(\mathcal{V}^{K,K} \times_3 D_t^T\right) k. \\
\left(\mathcal{V}^{N,K} \times_3 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 q_1 \times_2 n &= \left(\mathcal{V}^{N,K} \times_3 D^N\right) \times_3 n \times_2 n + \left(\mathcal{V}^{N,K} \times_3 D^K\right) \times_3 k \times_2 n \\
&\quad + \left(\mathcal{V}^{N,K} \times_3 D_t^Z\right) nz + \left(\mathcal{V}^{N,K} \times_3 D_t^T\right) n. \\
\left(\mathcal{V}^{K,N} \times_2 \mathrm{D}_{\mathcal{I}Q}\right) \times_3 n \times_2 q_1 &= \left(\mathcal{V}^{K,N} \times_2 D^N\right) \times_3 n \times_2 n + \left(\mathcal{V}^{K,N} \times_2 D^K\right) \times_3 n \times_2 k \\
&\quad + \left(\mathcal{V}^{K,N} \times_2 D_t^Z\right) nz + \left(\mathcal{V}^{K,N} \times_2 D_t^T\right) n. \\
2V_t^{K,Z} \mathrm{D}_{\mathcal{I}Q} q_1 z &= 2\left(V_t^{K,Z} D^N nz + V_t^{K,Z} D^K kz + V_t^{K,Z} D_t^Z z^2 + V_t^{K,Z} D_t^T z\right) \\
2V_t^{K,T} \mathrm{D}_{\mathcal{I}Q} q_1 &= 2\left(V_t^{K,T} D^N n + V_t^{K,T} D^K k + V_t^{K,T} D_t^Z z + V_t^{K,T} D_t^T\right)
\end{aligned}$$

Finally, we work on the last line of the initial expansion.

$$\begin{aligned}
&\sum_j V_{ij}^K \Big[\mathbb{D}_{RNN} \times_3 n \times_2 n + \mathbb{D}_{RN\mathcal{K}} \times_3 k \times_2 n + \mathbb{D}_{R\mathcal{K}N} \times_3 n \times_2 k + \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_3 k \times_2 k \\
&\qquad + 2\mathrm{D}_{RNT} n(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T} k(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T} \varsigma_t \\
&\qquad + \mathrm{D}_{IQ} q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t} \cdot q_{1,t}) \Big]_j, \\
&= (\mathbb{D}_{RNN} \times_1 V^K) \times_3 n \times_2 n + (\mathbb{D}_{RN\mathcal{K}} \times_1 V^K) \times_3 k \times_2 n + (\mathbb{D}_{R\mathcal{K}N} \times_1 V^K) \times_3 n \times_2 k \\
&\qquad + (\mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 V^K) \times_3 k \times_2 k + 2V^K \mathrm{D}_{RNT} n_t(\sigma_t z + T_t) + 2V^K \mathrm{D}_{R\mathcal{K}T} k_t(\sigma_t z + T_t) \\
&\qquad + V^K \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + V^K \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T} \varsigma_t + V^K \mathrm{D}_{IQ} q_{2,t} + V^K \mathrm{D}_{IQQ}(q_{1,t} \cdot q_{1,t})
\end{aligned}$$

Then, the term involving second-order shocks can be expressed as:

$$V^K \mathrm{d}_{\mathcal{I}TT}(\sigma_t z_t + T_t)^2 = V^K \mathrm{d}_{\mathcal{I}TT}(\sigma_t z)^2 + V^K \mathrm{d}_{\mathcal{I}TT}(T_t)^2 + 2V^K \mathrm{d}_{\mathcal{I}TT} \sigma_t z T_t.$$

Define the matrix $A \equiv V^K \mathrm{D}_{\mathcal{I}Q}$. By expanding the term involving q_2 we obtain:

$$\begin{aligned} Aq_2 = &\left(\mathcal{Q}^{N,N} \times_1 A\right) \times_3 n \times_2 n + \left(\mathcal{Q}^{N,K} \times_1 A\right) \times_3 k \times_2 n \\ &+ \left(\mathcal{Q}^{K,N} \times_1 A\right) \times_3 n \times_2 k + \left(\mathcal{Q}^{K,K} \times_1 A\right) \times_3 k \times_2 k \\ &+ 2AQ_t^{N,Z} nz + 2AQ_t^{N,T} n \\ &+ 2AQ_t^{K,Z} kz + 2AQ_t^{K,T} k \\ &+ A\overline{q}_t + Aq_t^{Z,Z} z^2 + Aq_t^{T,T} + 2Aq_t^{Z,T} z + Aq_t^{\Sigma}. \end{aligned}$$

Now, define the matrix $B \equiv V^K \mathrm{D}_{\mathcal{I}QQ}$. We expand the term involving $q_1 \cdot q_1$ as:

$$\begin{aligned} B\left(q_1 \cdot q_1\right) = &\left(\mathbb{Q}^{N,N} \times_1 B\right) \times_3 n \times_2 n + \left(\mathbb{Q}^{N,K} \times_1 B\right) \times_3 k \times_2 n \\ &+ \left(\mathbb{Q}^{K,N} \times_1 B\right) \times_3 n \times_2 k + \left(\mathbb{Q}^{K,K} \times_1 B\right) \times_3 k \times_2 k \\ &+ B\left(q_t^Z \cdot q_t^Z\right) z^2 + B\left(q_t^T \cdot q_t^T\right) + 2B\left(q_t^Z \cdot q_t^T\right) z \\ &+ 2B\mathbf{diag}(q_t^Z) Q^N nz + 2B\mathbf{diag}(q_t^T) Q^N n \\ &+ 2B\mathbf{diag}(q_t^Z) Q^K kz + 2B\mathbf{diag}(q_t^T) Q^K k. \end{aligned}$$

By collecting all terms, we construct the second-order approximation of the worker's continuation value from changes in capital distribution. Given that $\mathcal{V}^{K,K}$ is symmetric in the last two coordinates, we obtain:

$$
\begin{aligned}
&\sum_j \frac{\partial V_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right) = \epsilon V^K \left\{ \mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}\left[Q^N n + Q^K k + q_t^Z z + q_t^T\right] \right\} \\
&\quad = \frac{1}{2}\left\{ \mathcal{V}^{N,K} \times_3 \mathrm{D}_{RN} + \mathcal{V}^{K,N} \times_2 \mathrm{D}_{RN} + \mathcal{V}^{N,K} \times_3 D^N + \mathcal{V}^{K,N} \times_2 D^N + \mathbb{D}_{RNN} \times_1 V^K + \mathcal{Q}^{N,N} \times_1 A + \mathbb{Q}^{N,N} \times_1 B \right\} \times_3 n \times_2 n \\
&\quad + \frac{1}{2}\left\{ \mathcal{V}^{N,K} \times_3 \mathrm{D}_{R\mathcal{K}} + \mathcal{V}^{K,K} \times_2 \mathrm{D}_{RN} + \mathcal{V}^{K,K} \times_2 D^N + \mathcal{V}^{N,K} \times_3 D^K + \mathbb{D}_{RN\mathcal{K}} \times_1 V^K + \mathcal{Q}^{N,K} \times_1 A + \mathbb{Q}^{N,K} \times_1 B \right\} \times_3 k \times_2 n \\
&\quad + \frac{1}{2}\left\{ \mathcal{V}^{K,N} \times_2 \mathrm{D}_{R\mathcal{K}} + \mathcal{V}^{K,K} \times_3 \mathrm{D}_{RN} + \mathcal{V}^{K,K} \times_3 D^N + \mathcal{V}^{K,N} \times_2 D^K + \mathbb{D}_{R\mathcal{K}N} \times_1 V^K + \mathcal{Q}^{K,N} \times_1 A + \mathbb{Q}^{K,N} \times_1 B \right\} \times_3 n \times_2 k \\
&\quad + \frac{1}{2}\left\{ \mathcal{V}^{K,K} \times_2 \mathrm{D}_{R\mathcal{K}} + \mathcal{V}^{K,K} \times_3 \mathrm{D}_{R\mathcal{K}} + \mathcal{V}^{K,K} \times_2 D^K + \mathcal{V}^{K,K} \times_3 D^K + \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 V^K + \mathcal{Q}^{K,K} \times_1 A + \mathbb{Q}^{K,K} \times_1 B \right\} \times_3 k \times_2 k \\
&\quad + \left[V_t^{K,Z} \mathrm{D}_{RN} + \frac{1}{2}\left[\left(\mathcal{V}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \mathcal{V}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T} \right) \sigma_t + \mathcal{V}^{N,K} \times_3 D_t^Z + \mathcal{V}^{K,N} \times_2 D_t^Z \right] + V_t^{K,Z} D^N + V^K \mathrm{D}_{RNT} \sigma_t + A Q_t^{N,Z} + B \mathbf{diag}(q_t^Z \right. \\
&\quad + \left[V_t^{K,T} \mathrm{D}_{RN} + \frac{1}{2}\left[\left(\mathcal{V}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \mathcal{V}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T} \right) T_t + \mathcal{V}^{N,K} \times_3 D_t^T + \mathcal{V}^{K,N} \times_2 D_t^T \right] + V_t^{K,T} D^N + V^K \mathrm{D}_{RNT} T_t + A Q_t^{N,T} + B \mathbf{diag}(q_t^T \right. \\
&\quad + \left[V_t^{K,Z} \mathrm{D}_{R\mathcal{K}} + \frac{1}{2}\left[\left(\mathcal{V}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \mathcal{V}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T} \right) \sigma_t + \mathcal{V}^{K,K} \times_2 D_t^Z + \mathcal{V}^{K,K} \times_3 D_t^Z \right] + V_t^{K,Z} D^K + V^K \mathrm{D}_{R\mathcal{K}T} \sigma_t + A Q_t^{K,Z} + B \mathbf{diag}(q_t^Z) \mathcal{Q} \right. \\
&\quad + \left[V_t^{K,T} \mathrm{D}_{R\mathcal{K}} + \frac{1}{2}\left[\left(\mathcal{V}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \mathcal{V}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T} \right) T_t + \mathcal{V}^{K,K} \times_2 D_t^T + \mathcal{V}^{K,K} \times_3 D_t^T \right] + V_t^{K,T} D^K + V^K \mathrm{D}_{R\mathcal{K}T} T_t + A Q_t^{K,T} + B \mathbf{diag}(q_t^T) \mathcal{Q} \right. \\
&\quad + \frac{1}{2}\left[2 V_t^{K,Z} \mathrm{d}_{\mathcal{I}T} \sigma_t + 2 V_t^{K,Z} D_t^Z + V^K \mathrm{d}_{\mathcal{I}TT} \sigma_t^2 + A q_t^{Z,Z} + B\left(q_t^Z \cdot q_t^Z \right) \right] z^2 \\
&\quad + \frac{1}{2}\left[2 V_t^{K,T} \mathrm{d}_{\mathcal{I}T} T_t + 2 V_t^{K,T} D_t^T + V^K \mathrm{d}_{\mathcal{I}TT} (T_t)^2 + A q_t^{T,T} + B\left(q_t^T \cdot q_t^T \right) \right] \\
&\quad + \left[V_t^{K,Z} \mathrm{d}_{\mathcal{I}T} T_t + V_t^{K,T} \mathrm{d}_{\mathcal{I}T} \sigma_t + V_t^{K,Z} D_t^T + V_t^{K,T} D_t^Z + V^K \mathrm{d}_{\mathcal{I}TT} \sigma_t T_t + A q_t^{Z,T} + B\left(q_t^Z \cdot q_t^T \right) \right] z \\
&\quad + \frac{1}{2}\left[V^K \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T} \varsigma_t + A q_t^{\Sigma} \right] \\
&\quad + \frac{1}{2}\left[A \bar{q}_t \right]
\end{aligned}
$$

H.4.2 Capitalists

Capitalists continuation value from changes in the capital distribution is:

$$\sum_j \frac{\partial Q_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right).$$

The derivative of the capitalists' value with respect to capital is given by

$$\frac{\partial Q_i}{\partial \mathcal{K}_j} = Q_{ij}^K + \frac{\epsilon}{2}\underbrace{\left(\sum_s \mathcal{Q}_{ijs}^{K,K} k_s + \sum_s \mathcal{Q}_{isj}^{K,K} k_s + \sum_s \mathcal{Q}_{isj}^{N,K} n_s + \sum_s \mathcal{Q}_{ijs}^{K,N} n_s + 2Q_{ijt}^{K,Z} z + 2Q_{ijt}^{K,T}\right)}_{\equiv \partial_j Q_i}$$

Following the derivations for the worker's case, up to second order, we get that

$$\begin{aligned}
&\frac{1}{\epsilon}\sum_j \frac{\partial Q_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right) \\
&\quad = \sum_j Q_{ij}^K \left[\mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q} q_1\right]_j \\
&\quad + \frac{\epsilon}{2}\sum_j \partial_j Q_i \left[\mathrm{D}_{R\mathcal{K}} k + \mathrm{D}_{RN} n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q} q_1\right]_j \\
&\quad + \frac{\epsilon}{2}\sum_j Q_{ij}^K \Big[\mathbb{D}_{RNN} \times_3 n_t \times_2 n_t + \mathbb{D}_{RN\mathcal{K}} \times_3 k_t \times_2 n_t + \mathbb{D}_{R\mathcal{K}N} \times_3 n_t \times_2 n_k + \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_3 k_t \times_2 k_t \\
&\qquad + 2\mathrm{D}_{RNT} n_t(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T} k_t(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t \\
&\qquad + \mathrm{D}_{IQ} q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t} \cdot q_{1,t})\Big]_j.
\end{aligned}$$

Defining the matrices

$$D \equiv Q^K \mathrm{D}_{IQ}, \qquad E \equiv Q^K \mathrm{D}_{IQQ},$$

we can make the analogous derivations as in the worker's case to get that, up to the second order, the capitalist's continuation value from changes in the capital distribution is given by:

$$
\begin{aligned}
&\sum_j \frac{\partial Q_i}{\partial \mathcal{K}_j}\left(R_{jt} - Q_{jt}^{-1/\zeta} - \Delta_{jt} - \frac{\Sigma_{jt}^2}{2}\right) = \epsilon Q^K\left\{\mathrm{D}_{R\mathcal{K}}k + \mathrm{D}_{RN}n + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}\left[Q^N n + Q^K k + q_t^Z z + q_t^T\right]\right\} \\
&\quad = \frac{1}{2}\left\{\mathcal{Q}^{N,K}\times_3 \mathrm{D}_{RN} + \mathcal{Q}^{K,N}\times_2 \mathrm{D}_{RN} + \mathcal{Q}^{N,K}\times_3 D^N + \mathcal{Q}^{K,N}\times_2 D^N + \mathbb{D}_{RNN}\times_1 Q^K + \mathcal{Q}^{N,N}\times_1 D + \mathbb{Q}^{N,N}\times_1 E\right\}\times_3 n\times_2 n \\
&\quad + \frac{1}{2}\left\{\mathcal{Q}^{N,K}\times_3 \mathrm{D}_{R\mathcal{K}} + \mathcal{Q}^{K,K}\times_2 \mathrm{D}_{RN} + \mathcal{Q}^{K,K}\times_2 D^N + \mathcal{Q}^{N,K}\times_3 D^K + \mathbb{D}_{RN\mathcal{K}}\times_1 Q^K + \mathcal{Q}^{N,K}\times_1 D + \mathbb{Q}^{N,K}\times_1 E\right\}\times_3 k\times_2 n \\
&\quad + \frac{1}{2}\left\{\mathcal{Q}^{K,N}\times_2 \mathrm{D}_{R\mathcal{K}} + \mathcal{Q}^{K,K}\times_3 \mathrm{D}_{RN} + \mathcal{Q}^{K,K}\times_3 D^N + \mathcal{Q}^{K,N}\times_2 D^K + \mathbb{D}_{R\mathcal{K}N}\times_1 Q^K + \mathcal{Q}^{K,N}\times_1 D + \mathbb{Q}^{K,N}\times_1 E\right\}\times_3 n\times_2 k \\
&\quad + \frac{1}{2}\left\{\mathcal{Q}^{K,K}\times_2 \mathrm{D}_{R\mathcal{K}} + \mathcal{Q}^{K,K}\times_3 \mathrm{D}_{R\mathcal{K}} + \mathcal{Q}^{K,K}\times_2 D^K + \mathcal{Q}^{K,K}\times_3 D^K + \mathbb{D}_{R\mathcal{K}\mathcal{K}}\times_1 Q^K + \mathcal{Q}^{K,K}\times_1 D + \mathbb{Q}^{K,K}\times_1 E\right\}\times_3 k\times_2 k \\
&\quad + \left[Q_t^{K,Z}\mathrm{D}_{RN} + \frac{1}{2}\left[\left(\mathcal{Q}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \mathcal{Q}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\right)\sigma_t + \mathcal{Q}^{N,K}\times_3 D_t^Z + \mathcal{Q}^{K,N}\times_2 D_t^Z\right] + Q_t^{K,Z}D^N + Q^K\mathrm{D}_{RNT}\sigma_t + DQ_t^{N,Z} + E\mathbf{diag}(q_t^Z)\right] \\
&\quad + \left[Q_t^{K,T}\mathrm{D}_{RN} + \frac{1}{2}\left[\left(\mathcal{Q}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \mathcal{Q}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\right)T_t + \mathcal{Q}^{N,K}\times_3 D_t^T + \mathcal{Q}^{K,N}\times_2 D_t^T\right] + Q_t^{K,T}D^N + Q^K\mathrm{D}_{RNT}T_t + DQ_t^{N,T} + E\mathbf{diag}(q_t^T)\right] \\
&\quad + \left[Q_t^{K,Z}\mathrm{D}_{R\mathcal{K}} + \frac{1}{2}\left[\left(\mathcal{Q}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \mathcal{Q}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\right)\sigma_t + \mathcal{Q}^{K,K}\times_2 D_t^Z + \mathcal{Q}^{K,K}\times_3 D_t^Z\right] + Q_t^{K,Z}D^K + Q^K\mathrm{D}_{R\mathcal{K}T}\sigma_t + DQ_t^{K,Z} + E\mathbf{diag}(q_t^Z)\right] \\
&\quad + \left[Q_t^{K,T}\mathrm{D}_{R\mathcal{K}} + \frac{1}{2}\left[\left(\mathcal{Q}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \mathcal{Q}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\right)T_t + \mathcal{Q}^{K,K}\times_2 D_t^T + \mathcal{Q}^{K,K}\times_3 D_t^T\right] + Q_t^{K,T}D^K + Q^K\mathrm{D}_{R\mathcal{K}T}T_t + DQ_t^{K,T} + E\mathbf{diag}(q_t^T)\right] \\
&\quad + \frac{1}{2}\left[2Q_t^{K,Z}\mathrm{d}_{\mathcal{I}T}\sigma_t + 2Q_t^{K,Z}D_t^Z + Q^K\mathrm{d}_{\mathcal{I}TT}\sigma_t^2 + Dq_t^{Z,Z} + E\left(q_t^Z\cdot q_t^Z\right)\right]z^2 \\
&\quad + \frac{1}{2}\left[2Q_t^{K,T}\mathrm{d}_{\mathcal{I}T}T_t + 2Q_t^{K,T}D_t^T + Q^K\mathrm{d}_{\mathcal{I}TT}(T_t)^2 + Dq_t^{T,T} + E\left(q_t^T\cdot q_t^T\right)\right] \\
&\quad + \left[Q_t^{K,Z}\mathrm{d}_{\mathcal{I}T}T_t + Q_t^{K,T}\mathrm{d}_{\mathcal{I}T}\sigma_t + Q_t^{K,Z}D_t^T + Q_t^{K,T}D_t^Z + Q^K\mathrm{d}_{\mathcal{I}TT}\sigma_t T_t + Dq_t^{Z,T} + E\left(q_t^Z\cdot q_t^T\right)\right]z \\
&\quad + \frac{1}{2}\left[Q^K\Sigma^2\mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t + Dq_t^{\Sigma}\right] \\
&\quad + \frac{1}{2}\left[D\overline{q}_t\right]
\end{aligned}
$$

I Continuation Value from Climate Shocks

Recall that we impose z to follow an AR(1) process with mean-reversion parameter θ and unit variance.

I.1 Workers

Omitting the first-order terms, we have the following:

$$\begin{aligned}
\mathcal{A}(z)[V] &= \frac{\epsilon^2}{2}\mathcal{A}(z)[v_2] \\
&= \frac{\epsilon^2}{2}\left\{\mathcal{A}(z)[2V_t^{N,Z}nz] + \mathcal{A}(z)[2V_t^{K,Z}kz] + \mathcal{A}(z)[v_t^{Z,Z}z^2] + \mathcal{A}(z)[2v_t^{Z,T}z]\right\} \\
&= -\epsilon^2\theta\left\{V_t^{N,Z}nz + V_t^{K,Z}kz + v_t^{Z,Z}z^2 + v_t^{Z,T}z\right\} + \frac{\epsilon^2}{2}v_t^{Z,Z}.
\end{aligned}$$

I.2 Capitalists

Similarly, ignoring the first-order terms, we have that:

$$\begin{aligned}
\mathcal{A}(z)[Q] &= \frac{\epsilon^2}{2}\mathcal{A}(z)[q_2] \\
&= \frac{\epsilon^2}{2}\left\{\mathcal{A}(z)[2Q_t^{N,Z}nz] + \mathcal{A}(z)[2Q_t^{K,Z}kz] + \mathcal{A}(z)[q_t^{Z,Z}z^2] + \mathcal{A}(z)[2q_t^{Z,T}z]\right\} \\
&= -\epsilon^2\theta\left\{Q_t^{N,Z}nz + Q_t^{K,Z}kz + q_t^{Z,Z}z^2 + q_t^{Z,T}z\right\} + \frac{\epsilon^2}{2}q_t^{Z,Z}.
\end{aligned}$$

J SAME

J.1 SAME in matrix notation

Due to the length of the equations, we omit the full expression of the linearized master equation. Applying the method of identifying coefficients, we get the following equations for the second-order terms:

J.1.1 Workers

Block 1: N & K

$$\begin{aligned}\rho\mathcal{V}^{N,N} &= \mathbb{D}_{UNN} + \mu\nu\left(\mathbb{V}^{N,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^N\right) + \mathbb{Y}^* \times_2 V^N + \mathbb{Y} \times_3 V^N + \mathbb{Z} \times_3 V^N \times_2 V^N \\ &+ \mathbb{D}_{RNN} \times_1 V^K + \mathbb{Q}^{N,N} \times_1 B \\ &+ \mathcal{V}^{N,N} \times_1 (M + X) + \mathcal{V}^{N,N} \times_2 F^N + \mathcal{V}^{N,N} \times_3 F^N + \mathcal{V}^{N,K} \times_3 (\mathrm{D}_{RN} + D^N) + \mathcal{V}^{K,N} \times_2 (\mathrm{D}_{RN} + \\ &+ \mathcal{Q}^{N,N} \times_1 A\end{aligned}$$

$$\begin{aligned}\rho\mathcal{V}^{N,K} &= \mathbb{D}_{UN\mathcal{K}} + \mu\nu\left(\mathbb{V}^{N,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^N\right) + \mathbb{Y} \times_3 V^K + \mathbb{Z} \times_3 V^K \times_2 V^N \\ &+ \mathbb{D}_{RN\mathcal{K}} \times_1 V^K + \mathbb{Q}^{N,K} \times_1 B \\ &+ \mathcal{V}^{N,N} \times_3 F^K + \mathcal{V}^{N,K} \times_1 (M + X) + \mathcal{V}^{N,K} \times_2 F^N + \mathcal{V}^{N,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{K,K} \times_2 (\mathrm{D}_{RN} + D \\ &+ \mathcal{Q}^{N,K} \times_1 A\end{aligned}$$

$$\begin{aligned}\rho\mathcal{V}^{K,N} &= \mathbb{D}_{U\mathcal{K}N} + \mu\nu\left(\mathbb{V}^{K,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^K\right) + \mathbb{Y}^* \times_2 V^K + \mathbb{Z} \times_3 V^N \times_2 V^K \\ &+ \mathbb{D}_{R\mathcal{K}N} \times_1 V^K + \mathbb{Q}^{K,N} \times_1 B \\ &+ \mathcal{V}^{N,N} \times_2 F^K + \mathcal{V}^{K,N} \times_1 (M + X) + \mathcal{V}^{K,N} \times_3 F^N + \mathcal{V}^{K,N} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{K,K} \times_3 (\mathrm{D}_{RN} + D \\ &+ \mathcal{Q}^{K,N} \times_1 A\end{aligned}$$

$$\begin{aligned}\rho\mathcal{V}^{K,K} &= \mathbb{D}_{U\mathcal{K}\mathcal{K}} + \mu\nu\left(\mathbb{V}^{K,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^K\right) + \mathbb{Z} \times_3 V^K \times_2 V^K \\ &+ \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 V^K + \mathbb{Q}^{K,K} \times_1 B \\ &+ \mathcal{V}^{K,K} \times_1 (M + X) + \mathcal{V}^{K,K} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{K,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{N,K} \times_2 F^K + \mathcal{V}^{K,N} \times_3 F \\ &+ \mathcal{Q}^{K,K} \times_1 A\end{aligned}$$

Block 2: N & $\sigma_t z$, and T_t

$$
\begin{aligned}
&(\rho+\theta)\boldsymbol{V}_t^{N,Z} - \frac{\boldsymbol{V}_t^{N,Z}}{\partial t} \\
&\quad = \mathrm{D}_{UNT}\sigma_t + \mu\nu\left(m\,\mathbf{diag}(v_t^Z)V^N + \mathbb{M}\times_2 V^N \times_3 v_t^Z\right) + \mathbb{Y}\times_3 v_t^Z + \mathbb{Z}\times_2 V^N\times_3 v_t^Z \\
&\quad + V^K\mathrm{D}_{RNT}\sigma_t + B\mathbf{diag}(q_t^Z)Q^N \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,N}\times_2 f_t^Z + \boldsymbol{\mathcal{V}}^{N,N}\times_3 f_t^Z + \boldsymbol{\mathcal{V}}^{N,K}\times_3 D_t^Z + \boldsymbol{\mathcal{V}}^{K,N}\times_2 D_t^Z + \Big(\boldsymbol{\mathcal{V}}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\Big) \\
&\quad + (M+X)\,\boldsymbol{V}_t^{N,Z} + \boldsymbol{V}_t^{N,Z}F^N + A\boldsymbol{Q}_t^{N,Z} + \boldsymbol{V}_t^{K,Z}(\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

$$
\begin{aligned}
&\rho\boldsymbol{V}_t^{N,T} - \frac{\boldsymbol{V}_t^{N,T}}{\partial t} \\
&\quad = \mathrm{D}_{UNT}T_t + \mu\nu\left(m\,\mathbf{diag}(v_t^T)V^N + \mathbb{M}\times_2 V^N \times_3 v_t^T\right) + \mathbb{Y}\times_3 v_t^T + \mathbb{Z}\times_2 V^N\times_3 v_t^T \\
&\quad + V^K\mathrm{D}_{RNT}T_t + B\mathbf{diag}(q_t^T)Q^N \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,N}\times_2 f_t^T + \boldsymbol{\mathcal{V}}^{N,N}\times_3 f_t^T + \boldsymbol{\mathcal{V}}^{N,K}\times_3 D_t^T + \boldsymbol{\mathcal{V}}^{K,N}\times_2 D_t^T + \Big(\boldsymbol{\mathcal{V}}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\Big) \\
&\quad + (M+X)\,\boldsymbol{V}_t^{N,T} + \boldsymbol{V}_t^{N,T}F^N + A\boldsymbol{Q}_t^{N,T} + \boldsymbol{V}_t^{K,T}(\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

Block 3: K & $\sigma_t z$, and T_t

$$
\begin{aligned}
&(\rho+\theta)\boldsymbol{V}_t^{K,Z} - \frac{\partial\boldsymbol{V}_t^{K,Z}}{\partial t} \\
&\quad = \mathrm{D}_{UK\mathcal{T}}\sigma_t + \mu\nu\left(m\,\mathbf{diag}(v_t^Z)V^K + \mathbb{M}\times_2 V^K \times_3 v_t^Z\right) + \mathbb{Z}\times_2 V^K\times_3 v_t^Z + V^K\mathrm{D}_{R\mathcal{K}T}\sigma_t + B\mathbf{diag}(q_t^Z)Q^K \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,K}\times_2 f_t^Z + \boldsymbol{\mathcal{V}}^{K,N}\times_3 f_t^Z + \boldsymbol{\mathcal{V}}^{K,K}\times_2 D_t^Z + \boldsymbol{\mathcal{V}}^{K,K}\times_3 D_t^Z + \Big(\boldsymbol{\mathcal{V}}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\Big)\sigma_t\Big] \\
&\quad + (M+X)\,\boldsymbol{V}_t^{K,Z} + \boldsymbol{V}_t^{N,Z}F^K + A\boldsymbol{Q}_t^{K,Z} + \boldsymbol{V}_t^{K,Z}(\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
$$

$$
\begin{aligned}
&\rho\boldsymbol{V}_t^{K,T} - \frac{\partial\boldsymbol{V}_t^{K,T}}{\partial t} \\
&\quad = \mathrm{D}_{UK\mathcal{T}}T_t + \mu\nu\left(m\,\mathbf{diag}(v_t^T)V^K + \mathbb{M}\times_2 V^K \times_3 v_t^T\right) + \mathbb{Z}\times_2 V^K\times_3 v_t^T + V^K\mathrm{D}_{R\mathcal{K}T}T_t + B\mathbf{diag}(q_t^T)Q^K \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,K}\times_2 f_t^T + \boldsymbol{\mathcal{V}}^{K,N}\times_3 f_t^T + \boldsymbol{\mathcal{V}}^{K,K}\times_2 D_t^T + \boldsymbol{\mathcal{V}}^{K,K}\times_3 D_t^T + \Big(\boldsymbol{\mathcal{V}}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\Big)T_t\Big] \\
&\quad + (M+X)\,\boldsymbol{V}_t^{N,T} + \boldsymbol{V}_t^{N,T}F^K + A\boldsymbol{Q}_t^{K,T} + \boldsymbol{V}_t^{K,T}(\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
$$

Block 4: $(\sigma_t z)^2$**,** $(T_t)^2$**, and** $(\sigma_t z)T_t$

$$
\begin{aligned}
&(\rho+2\theta)v_t^{Z,Z} - \frac{\partial v_t^{Z,Z}}{\partial t} \\
&\qquad = \mathrm{d}_{UTT}\sigma_t^2 + \mu\nu\left(m(v_t^Z \cdot v_t^Z) + \mathbb{M}\times_3 v_t^Z \times_2 v_t^Z\right) + \mathbb{Z}\times_3 v_t^Z \times_2 v_t^Z + B\left(q_t^Z \cdot q_t^Z\right) + V^K \mathrm{d}_{\mathcal{I}TT}\sigma_t^2 \\
&\qquad + 2\left[V_t^{N,Z} f_t^Z + V_t^{K,Z}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right)\right] \\
&\qquad + (M+X)\, v_t^{Z,Z} + A q_t^{Z,Z}
\end{aligned}
$$

$$
\begin{aligned}
&\rho v_t^{T,T} - \frac{\partial v_t^{T,T}}{\partial t} \\
&\qquad = \mathrm{d}_{UTT}(T_t)^2 + \mu\nu\left(m\left(v_t^T \cdot v_t^T\right) + \mathbb{M}\times_3 v_t^T \times_2 v_t^T\right) + \mathbb{Z}\times_3 v_t^T \times_2 v_t^T + B\left(q_t^T \cdot q_t^T\right) + V^K \mathrm{d}_{\mathcal{I}TT}(T_t)^2 \\
&\qquad + 2\left[V_t^{N,T} f_t^T + V_t^{K,T}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right)\right] \\
&\qquad + (M+X)\, v_t^{T,T} + A q_t^{T,T}
\end{aligned}
$$

$$
\begin{aligned}
&(\rho+\theta)v_t^{Z,T} - \frac{\partial v_t^{Z,T}}{\partial t} \\
&\qquad = \mathrm{d}_{UTT}\sigma_t T_t + \mu\nu\left(m\left(v_t^Z \cdot v_t^T\right) + \mathbb{M}\times_3 v_t^Z \times_2 v_t^T\right) + \mathbb{Z}\times_3 v_t^Z \times_2 v_t^T + B\left(q_t^Z \cdot q_t^T\right) + V^K \mathrm{d}_{\mathcal{I}TT}\sigma \\
&\qquad + V_t^{N,Z} f_t^T + V_t^{N,T} f_t^Z + V_t^{K,Z}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right) + V_t^{K,T}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right) \\
&\qquad + (M+X)\, v_t^{Z,T} + A q_t^{Z,T}
\end{aligned}
$$

Block 6: $\Sigma_i^2 \varsigma_t$

$$
\rho v_t^{\Sigma} - \frac{\partial v_t^{\Sigma}}{\partial t} = V^K \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t + (M+X)v_t^{\Sigma} + A q_t^{\Sigma}
$$

Block 7: Constant

$$
\rho \overline{v}_t - \frac{\partial \overline{v}_t}{\partial t} = v_t^{Z,Z} + (M+X)\,\overline{v}_t + A\overline{q}_t
$$

J.1.2 Capitalists

Block 1: N & K

$$\begin{aligned}\rho\mathcal{Q}^{N,N} &= \mathbb{D}_{QNN} + \mathbb{Q}^{N,N} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQN} \times_1 Q^N + \mathbb{P}^* \times_2 V^N + \mathbb{P} \times_3 V^N + \mathbb{R} \times_3 V^N \times_2 V^N \\ &+ \mathbb{D}_{RNN} \times_1 Q^K + \mathbb{Q}^{N,N} \times_1 E \\ &+ \mathcal{Q}^{N,N} \times_1 D + \mathcal{Q}^{N,N} \times_2 F^N + \mathcal{Q}^{N,N} \times_3 F^N + \mathcal{Q}^{N,K} \times_3 (\mathrm{D}_{RN} + D^N) + \mathcal{Q}^{K,N} \times_2 (\mathrm{D}_{RN} + D^N) \\ &+ \mathcal{V}^{N,N} \times_1 S\end{aligned}$$

$$\begin{aligned}\rho\mathcal{Q}^{N,K} &= \mathbb{D}_{Q\mathcal{K}N} + \mathbb{Q}^{N,K} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQN} \times_1 Q^K + \mathbb{P} \times_3 V^K + \mathbb{R} \times_3 V^K \times_2 V^N \\ &+ \mathbb{D}_{RN\mathcal{K}} \times_1 Q^K + \mathbb{Q}^{N,K} \times_1 E \\ &+ \mathcal{Q}^{N,N} \times_3 F^K + \mathcal{Q}^{N,K} \times_1 D + \mathcal{Q}^{N,K} \times_2 F^N + \mathcal{Q}^{N,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{K,K} \times_2 (\mathrm{D}_{RN} + D^N) \\ &+ \mathcal{V}^{N,K} \times_1 S\end{aligned}$$

$$\begin{aligned}\rho\mathcal{Q}^{K,N} &= \mathbb{D}_{QN\mathcal{K}} + \mathbb{Q}^{K,N} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}} \times_1 Q^N + \mathbb{P}^* \times_2 V^K + \mathbb{R} \times_3 V^N \times_2 V^K \\ &+ \mathbb{D}_{R\mathcal{K}N} \times_1 Q^K + \mathbb{Q}^{K,N} \times_1 E \\ &+ \mathcal{Q}^{N,N} \times_2 F^K + \mathcal{Q}^{K,N} \times_1 D + \mathcal{Q}^{K,N} \times_3 F^N + \mathcal{Q}^{K,N} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{K,K} \times_3 (\mathrm{D}_{RN} + D^N) \\ &+ \mathcal{V}^{K,N} \times_1 S\end{aligned}$$

$$\begin{aligned}\rho\mathcal{Q}^{K,K} &= \mathbb{D}_{Q\mathcal{K}\mathcal{K}} + \mathbb{Q}^{K,K} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}} \times_1 Q^K + \mathbb{R} \times_3 V^K \times_2 V^K \\ &+ \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 Q^K + \mathbb{Q}^{K,K} \times_1 E \\ &+ \mathcal{Q}^{K,K} \times_1 D + \mathcal{Q}^{K,K} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{K,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{N,K} \times_2 F^K + \mathcal{Q}^{K,N} \times_3 F^K \\ &+ \mathcal{V}^{K,K} \times_1 S\end{aligned}$$

Block 2: N & $\sigma_t z$, and T_t

$$
\begin{aligned}
&(\rho+\theta)\boldsymbol{Q}_t^{N,Z} - \frac{\boldsymbol{Q}_t^{N,Z}}{\partial t} \\
&\quad = \mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT}Q^N\right)\sigma_t + \mathbb{D}_{QQN} \times_1 q_t^Z + \mathbb{P} \times_3 v_t^Z + \mathbb{R} \times_2 V^N \times_3 v_t^Z \\
&\quad + Q^K \mathrm{D}_{RNT}\sigma_t + E\mathbf{diag}(q_t^Z)Q^N \\
&\quad + \frac{1}{2}\left[\boldsymbol{\mathcal{Q}}^{N,N} \times_2 f_t^Z + \boldsymbol{\mathcal{Q}}^{N,N} \times_3 f_t^Z + \boldsymbol{\mathcal{Q}}^{N,K} \times_3 D_t^Z + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 D_t^Z + \left(\boldsymbol{\mathcal{Q}}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T}\right)\sigma_t\right] \\
&\quad + D\boldsymbol{Q}_t^{N,Z} + \boldsymbol{Q}_t^{N,Z}F^N + S\boldsymbol{V}_t^{N,Z} + \boldsymbol{Q}_t^{K,Z}(\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

$$
\begin{aligned}
&\rho\boldsymbol{Q}_t^{N,T} - \frac{\boldsymbol{Q}_t^{N,T}}{\partial t} \\
&\quad = \mathrm{D}_{QQQ}\mathbf{diag}(q_t^T)Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT}Q^N\right)T_t + \mathbb{D}_{QQN} \times_1 q_t^T + \mathbb{P} \times_3 v_t^T + \mathbb{R} \times_2 V^N \times_3 v_t^T \\
&\quad + Q^K \mathrm{D}_{RNT}T_t + E\mathbf{diag}(q_t^T)Q^N \\
&\quad + \frac{1}{2}\left[\boldsymbol{\mathcal{Q}}^{N,N} \times_2 f_t^T + \boldsymbol{\mathcal{Q}}^{N,N} \times_3 f_t^T + \boldsymbol{\mathcal{Q}}^{N,K} \times_3 D_t^T + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 D_t^T + \left(\boldsymbol{\mathcal{Q}}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T}\right)T_t\right] \\
&\quad + D\boldsymbol{Q}_t^{N,T} + \boldsymbol{Q}_t^{N,T}F^N + S\boldsymbol{V}_t^{N,T} + \boldsymbol{Q}_t^{K,T}(\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

Block 3: K & $\sigma_t z$, and T_t

$$
\begin{aligned}
&(\rho+\theta)\boldsymbol{Q}_t^{K,Z} - \frac{\partial\boldsymbol{Q}_t^{K,Z}}{\partial t} \\
&\quad = \mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^K + (\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT}Q^K)\sigma_t + \mathbb{D}_{QQ\mathcal{K}} \times_1 q_t^Z + \mathbb{R} \times_2 V^K \times_3 v_t^Z \\
&\quad + Q^K \mathrm{D}_{R\mathcal{K}T}\sigma_t + E\mathbf{diag}(q_t^Z)Q^K \\
&\quad + \frac{1}{2}\left[\boldsymbol{\mathcal{Q}}^{N,K} \times_2 f_t^Z + \boldsymbol{\mathcal{Q}}^{K,N} \times_3 f_t^Z + \boldsymbol{\mathcal{Q}}^{K,K} \times_2 D_t^Z + \boldsymbol{\mathcal{Q}}^{K,K} \times_3 D_t^Z + \left(\boldsymbol{\mathcal{Q}}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T}\right)\sigma_t\right] \\
&\quad + D\boldsymbol{Q}_t^{K,Z} + \boldsymbol{Q}_t^{N,Z}F^K + S\boldsymbol{V}_t^{K,Z} + \boldsymbol{Q}_t^{K,Z}(\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
$$

$$
\begin{aligned}
&\rho Q_t^{K,T} - \frac{\partial Q_t^{K,T}}{\partial t} \\
&= \mathrm{D}_{QQQ}\mathbf{diag}(q_t^T)Q^K + \left(\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT}Q^K\right) T_t + \mathbb{D}_{QQ\mathcal{K}} \times_1 q_t^T + \mathbb{R} \times_2 V^K \times_3 v_t^T \\
&+ Q^K \mathrm{D}_{R\mathcal{K}T} T_t + E\mathbf{diag}(q_t^T)Q^K \\
&+ \frac{1}{2}\left[\mathcal{Q}^{N,K} \times_2 f_t^T + \mathcal{Q}^{K,N} \times_3 f_t^T + \mathcal{Q}^{K,K} \times_2 D_t^T + \mathcal{Q}^{K,K} \times_3 D_t^T + \left(\mathcal{Q}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \mathcal{Q}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T}\right) T_t\right] \\
&+ DQ_t^{K,T} + Q_t^{N,T} F^K + SV_t^{K,T} + Q_t^{K,T}(\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
$$

Block 4: $(\sigma_t z)^2$**,** $(T_t)^2$**, and** $(\sigma_t z)T_t$

$$
\begin{aligned}
&(\rho + 2\theta) q_t^{Z,Z} - \frac{\partial q_t^{T,T}}{\partial t} \\
&\qquad = \mathrm{D}_{QQQ}(q_t^Z \cdot q_t^Z) + 2\mathrm{D}_{QQT} q_t^Z \sigma_t + \mathrm{d}_{QTT}\sigma_t^2 + \mathbb{R} \times_3 v_t^Z \times_2 v_t^Z + E\left(q_t^Z \cdot q_t^Z\right) + Q^K \mathrm{d}_{\mathcal{I}TT}\sigma_t^2 \\
&\qquad + 2\left[Q_t^{N,Z} f_t^Z + Q_t^{K,Z}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right)\right] \\
&\qquad + Dq_t^{Z,Z} + Sv_t^{Z,Z}
\end{aligned}
$$

$$
\begin{aligned}
&\rho q_t^{T,T} - \frac{\partial q_t^{T,T}}{\partial t} \\
&\qquad = \mathrm{D}_{QQQ}(q_t^T \cdot q_t^T) + 2\mathrm{D}_{QQT} q_t^T T_t + \mathrm{d}_{QTT}(T_t)^2 + \mathbb{R} \times_3 v_t^T \times_2 v_t^T + E\left(q_t^T \cdot q_t^T\right) + Q^K \mathrm{d}_{\mathcal{I}TT}(T_t)^2 \\
&\qquad + 2\left[Q_t^{N,T} f_t^T + Q_t^{K,T}\left(D_t^T + \mathrm{d}_{\mathcal{I}T} T_t\right)\right] \\
&\qquad + Dq_t^{T,T} + Sv_t^{T,T}
\end{aligned}
$$

$$
\begin{aligned}
&(\rho + \theta) q_t^{Z,T} - \frac{\partial q_t^{Z,T}}{\partial t} \\
&\qquad = \mathrm{D}_{QQQ} q_t^Z q_t^T + \mathrm{D}_{QQT}(q_t^Z T_t + q_t^T \sigma_t) + \mathrm{d}_{QTT}\sigma_t T_t + \mathbb{R} \times_3 v_t^Z \times_2 v_t^T + E\left(q_t^Z \cdot q_t^T\right) + Q^K \mathrm{d}_{\mathcal{I}TT}\sigma_t \\
&\qquad + Q_t^{N,Z} f_t^T + Q_t^{N,T} f_t^Z + Q_t^{K,Z}\left(D_t^T + \mathrm{d}_{\mathcal{I}T} T_t\right) + Q_t^{K,T}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right) \\
&\qquad + Dq_t^{Z,T} + Sv_t^{Z,T}
\end{aligned}
$$

Block 6: $\Sigma_i^2 \varsigma_t$

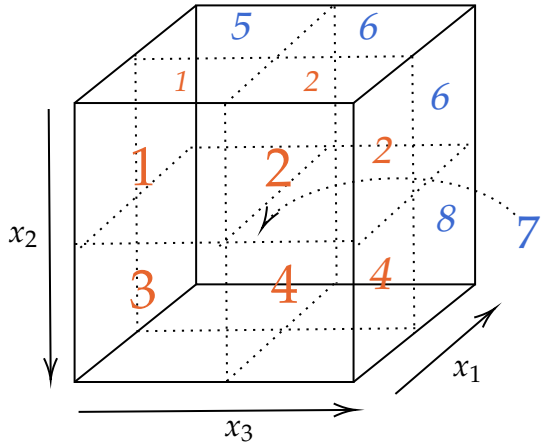
$$\rho q_t^{\Sigma} - \frac{\partial q_t^{\Sigma}}{\partial t} = \Sigma^2 \mathrm{d}_{Q\Sigma T} \varsigma_t + Q^K \Sigma^2 \mathrm{d}_{I\Sigma T} \varsigma_t + D q_t^{\Sigma} + S v_t^{\Sigma}$$

Block 7: Constant

$$\rho \overline{q}_t - \frac{\partial \overline{q}_t}{\partial t} = q_t^{Z,Z} + D \overline{q}_t + S \overline{v}_t$$

J.2 Deterministic SAME

We stack the deterministic tensors as follows:



$$\mathcal{V}^{N,N} = 1 \quad \mathcal{V}^{N,K} = 2, \quad \mathcal{V}^{K,N} = 3, \quad \mathcal{V}^{K,K} = 4$$
$$\mathcal{Q}^{N,N} = 5 \quad \mathcal{Q}^{N,K} = 6, \quad \mathcal{Q}^{K,N} = 7, \quad \mathcal{Q}^{K,K} = 8,$$

which we will represent as:

$$\mathcal{T} \equiv \underbrace{\begin{bmatrix} \mathcal{V}^{N,N} & \mathcal{V}^{N,K} \\ \mathcal{V}^{K,N} & \mathcal{V}^{K,K} \end{bmatrix}}_{\text{“front panel”}} \odot \underbrace{\begin{bmatrix} \mathcal{Q}^{N,N} & \mathcal{Q}^{N,K} \\ \mathcal{Q}^{K,N} & \mathcal{Q}^{K,K} \end{bmatrix}}_{\text{“back panel”}}$$

Define the tensor $\mathbb{T}_0$ as:

$$\mathbb{T}_0 \equiv \begin{bmatrix} \mathbb{V}_0^{N,N} & \mathbb{V}_0^{N,K} \\ \mathbb{V}_0^{K,N} & \mathbb{V}_0^{K,K} \end{bmatrix} \odot \begin{bmatrix} \mathbb{Q}_0^{N,N} & \mathbb{Q}_0^{N,K} \\ \mathbb{Q}_0^{K,N} & \mathbb{Q}_0^{K,K} \end{bmatrix},$$

with $\mathbb{V}_0^{N,N}$, $\mathbb{V}_0^{N,K}$, $\mathbb{V}_0^{K,N}$, $\mathbb{V}_0^{K,K}$, $\mathbb{Q}_0^{N,N}$, $\mathbb{Q}_0^{N,K}$, $\mathbb{Q}_0^{K,N}$ and $\mathbb{Q}_0^{K,K}$ defined as:

$$\mathbb{V}_0^{N,N} \equiv \mathbb{D}_{UNN} + \mu\nu\left(\mathbb{V}^{N,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^N\right) + \mathbb{Y}^* \times_2 V^N + \mathbb{Y} \times_3 V^N + \mathbb{Z} \times_3 V^N \times_2 V^N + \mathbb{D}_{RNN} \times_1 V^K + \mathbb{Q}^{N,N} \times_1 B$$

$$\mathbb{V}_0^{N,K} \equiv \mathbb{D}_{UN\mathcal{K}} + \mu\nu\left(\mathbb{V}^{N,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^N\right) + \mathbb{Y} \times_3 V^K + \mathbb{Z} \times_3 V^K \times_2 V^N + \mathbb{D}_{RN\mathcal{K}} \times_1 V^K + \mathbb{Q}^{N,K} \times_1 B$$

$$\mathbb{V}_0^{K,N} \equiv \mathbb{D}_{U\mathcal{K}N} + \mu\nu\left(\mathbb{V}^{K,N} \times_1 m + \mathbb{M} \times_3 V^N \times_2 V^K\right) + \mathbb{Y}^* \times_2 V^K + \mathbb{Z} \times_3 V^N \times_2 V^K + \mathbb{D}_{R\mathcal{K}N} \times_1 V^K + \mathbb{Q}^{K,N} \times_1 B$$

$$\mathbb{V}_0^{K,K} \equiv \mathbb{D}_{U\mathcal{K}\mathcal{K}} + \mu\nu\left(\mathbb{V}^{K,K} \times_1 m + \mathbb{M} \times_3 V^K \times_2 V^K\right) + \mathbb{Z} \times_3 V^K \times_2 V^K + \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 V^K + \mathbb{Q}^{K,K} \times_1 B$$

$$^{N,N}_0 \equiv \mathbb{D}_{QNN} + \mathbb{Q}^{N,N} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQN} \times_1 Q^N + \mathbb{P}^* \times_2 V^N + \mathbb{P} \times_3 V^N + \mathbb{R} \times_3 V^N \times_2 V^N + \mathbb{D}_{RNN} \times_1 Q^K + \mathbb{Q}^{N,N} \times_1$$

$$^{N,K}_0 \equiv \mathbb{D}_{QN\mathcal{K}} + \mathbb{Q}^{N,K} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQN} \times_1 Q^K + \mathbb{P} \times_3 V^K + \mathbb{R} \times_3 V^K \times_2 V^N + \mathbb{D}_{RN\mathcal{K}} \times_1 Q^K + \mathbb{Q}^{N,K} \times_1 E$$

$$^{K,N}_0 \equiv \mathbb{D}_{Q\mathcal{K}N} + \mathbb{Q}^{K,N} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}} \times_1 Q^N + \mathbb{P}^* \times_2 V^K + \mathbb{R} \times_3 V^N \times_2 V^K + \mathbb{D}_{R\mathcal{K}N} \times_1 Q^K + \mathbb{Q}^{K,N} \times_1 E$$

$$^{K,K}_0 \equiv \mathbb{D}_{Q\mathcal{K}\mathcal{K}} + \mathbb{Q}^{K,K} \times_1 \mathrm{D}_{QQQ} + \mathbb{D}_{QQ\mathcal{K}} \times_1 Q^K + \mathbb{R} \times_3 V^K \times_2 V^K + \mathbb{D}_{R\mathcal{K}\mathcal{K}} \times_1 Q^K + \mathbb{Q}^{K,K} \times_1 E$$

Recall from the FAME notation that we define the matrix Φ as:

$$\Phi \equiv \begin{pmatrix} M + X & A \\ S & D \end{pmatrix}.$$

Moreover, define the matrix Θ as:

$$\Theta \equiv \begin{pmatrix} F^N & F^K \\ \mathrm{D}_{RN} + D^N & \mathrm{D}_{R\mathcal{K}} + D^K \end{pmatrix}.$$

With this notation, the deterministic SAME equations take the form:

$$
\begin{aligned}
\rho \mathcal{V}^{N,N} &= \mathbb{V}_0^{N,N} \\
&+ \mathcal{V}^{N,N} \times_1 (M + X) + \mathcal{Q}^{N,N} \times_1 A \\
&+ \mathcal{V}^{N,N} \times_2 F^N + \mathcal{V}^{K,N} \times_2 (\mathrm{D}_{RN} + D^N) \\
&+ \mathcal{V}^{N,N} \times_3 F^N + \mathcal{V}^{N,K} \times_3 (\mathrm{D}_{RN} + D^N)
\end{aligned}
\qquad
\begin{aligned}
\rho \mathcal{Q}^{N,N} &= \mathbb{Q}_0^{N,N} \\
&+ \mathcal{Q}^{N,N} \times_1 D + \mathcal{V}^{N,N} \times_1 S \\
&+ \mathcal{Q}^{N,N} \times_2 F^N + \mathcal{Q}^{K,N} \times_2 (\mathrm{D}_{RN} + D^N) \\
&+ \mathcal{Q}^{N,N} \times_3 F^N + \mathcal{Q}^{N,K} \times_3 (\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

$$
\begin{aligned}
\rho \mathcal{V}^{N,K} &= \mathbb{V}_0^{N,K} \\
&+ \mathcal{V}^{N,K} \times_1 (M + X) + \mathcal{Q}^{N,K} \times_1 A \\
&+ \mathcal{V}^{N,K} \times_2 F^N + \mathcal{V}^{K,K} \times_2 (\mathrm{D}_{RN} + D^N) \\
&+ \mathcal{V}^{N,N} \times_3 F^K + \mathcal{V}^{N,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
\qquad
\begin{aligned}
\rho \mathcal{Q}^{N,K} &= \mathbb{V}_0^{N,K} \\
&+ \mathcal{Q}^{N,K} \times_1 D + \mathcal{V}^{N,K} \times_1 S \\
&+ \mathcal{Q}^{N,K} \times_2 F^N + \mathcal{Q}^{K,K} \times_2 (\mathrm{D}_{RN} + D^N) \\
&+ \mathcal{Q}^{N,N} \times_3 F^K + \mathcal{Q}^{N,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K)
\end{aligned}
$$

$$
\begin{aligned}
\rho \mathcal{V}^{K,N} &= \mathbb{V}_0^{K,N} \\
&+ \mathcal{V}^{K,N} \times_1 (M + X) + \mathcal{Q}^{K,N} \times_1 A \\
&+ \mathcal{V}^{N,N} \times_2 F^K + \mathcal{V}^{K,N} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) \\
&+ \mathcal{V}^{K,N} \times_3 F^N + \mathcal{V}^{K,K} \times_3 (\mathrm{D}_{RN} + D^N)
\end{aligned}
\qquad
\begin{aligned}
\rho \mathcal{Q}^{K,N} &= \mathbb{V}_0^{K,N} \\
&+ \mathcal{Q}^{K,N} \times_1 D + \mathcal{V}^{K,N} \times_1 S \\
&+ \mathcal{Q}^{N,N} \times_2 F^K + \mathcal{Q}^{K,N} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) \\
&+ \mathcal{Q}^{K,N} \times_3 F^N + \mathcal{Q}^{K,K} \times_3 (\mathrm{D}_{RN} + D^N)
\end{aligned}
$$

$$
\begin{aligned}
\rho \mathcal{V}^{K,K} &= \mathbb{V}_0^{K,K} \\
&+ \mathcal{V}^{K,K} \times_1 (M + X) + \mathcal{Q}^{K,K} \times_1 A \\
&+ \mathcal{V}^{K,K} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{N,K} \times_2 F^K \\
&+ \mathcal{V}^{K,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{V}^{K,N} \times_3 F^K
\end{aligned}
\qquad
\begin{aligned}
\rho \mathcal{Q}^{K,K} &= \mathbb{V}_0^{K,K} \\
&+ \mathcal{Q}^{K,K} \times_1 D + \mathcal{V}^{K,K} \times_1 S \\
&+ \mathcal{Q}^{K,K} \times_2 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{N,K} \times_2 F^K \\
&+ \mathcal{Q}^{K,K} \times_3 (\mathrm{D}_{R\mathcal{K}} + D^K) + \mathcal{Q}^{K,N} \times_3 F^K
\end{aligned}
$$

which can be written compactly as a tensor equation for $\boldsymbol{\mathcal{T}}$ as:

$$
\rho \boldsymbol{\mathcal{T}} = \mathbb{T}_0 + \boldsymbol{\mathcal{T}} \times_1 \Phi + \boldsymbol{\mathcal{T}} \times_2 \Theta + \boldsymbol{\mathcal{T}} \times_3 \Theta \tag{53}
$$

J.3 Cross SAME: $\sigma_t z$, T_t, N & K

Define the matrices:

$$
\begin{aligned}
\mathbf{V}_{0t}^{N,Z} &= \mathrm{D}_{UNT}\sigma_t + \mu\nu\left(m\,\mathbf{diag}(v_t^Z)V^N + \mathbb{M}\times_2 V^N \times_3 v_t^Z\right) + \mathbb{Y}\times_3 v_t^Z + \mathbb{Z}\times_2 V^N \times_3 v_t^Z \\
&\quad + V^K\mathrm{D}_{RNT}\sigma_t + B\mathbf{diag}(q_t^Z)Q^N \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,N}\times_2 f_t^Z + \boldsymbol{\mathcal{V}}^{N,N}\times_3 f_t^Z + \boldsymbol{\mathcal{V}}^{N,K}\times_3 D_t^Z + \boldsymbol{\mathcal{V}}^{K,N}\times_2 D_t^Z + \left(\boldsymbol{\mathcal{V}}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\right) \\
\mathbf{V}_{0t}^{K,Z} &= \mathrm{D}_{U\mathcal{K}T}\sigma_t + \mu\nu\left(m\,\mathbf{diag}(v_t^Z)V^K + \mathbb{M}\times_2 V^K \times_3 v_t^Z\right) + \mathbb{Z}\times_2 V^K \times_3 v_t^Z + V^K\mathrm{D}_{R\mathcal{K}T}\sigma_t + B\mathbf{diag}(q_t^Z)Q \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{V}}^{N,K}\times_2 f_t^Z + \boldsymbol{\mathcal{V}}^{K,N}\times_3 f_t^Z + \boldsymbol{\mathcal{V}}^{K,K}\times_2 D_t^Z + \boldsymbol{\mathcal{V}}^{K,K}\times_3 D_t^Z + \left(\boldsymbol{\mathcal{V}}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\right)\sigma_t \\
\mathbf{Q}_{0t}^{N,Z} &= \mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT}Q^N\right)\sigma_t + \mathbb{D}_{QQN}\times_1 q_t^Z + \mathbb{P}\times_3 v_t^Z + \mathbb{R}\times_2 V^N \times_3 v_t^Z \\
&\quad + Q^K\mathrm{D}_{RNT}\sigma_t + E\mathbf{diag}(q_t^Z)Q^N \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{Q}}^{N,N}\times_2 f_t^Z + \boldsymbol{\mathcal{Q}}^{N,N}\times_3 f_t^Z + \boldsymbol{\mathcal{Q}}^{N,K}\times_3 D_t^Z + \boldsymbol{\mathcal{Q}}^{K,N}\times_2 D_t^Z + \left(\boldsymbol{\mathcal{Q}}^{N,K}\times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,N}\times_2 \mathrm{d}_{\mathcal{I}T}\right) \\
\mathbf{Q}_{0t}^{K,Z} &= \mathrm{D}_{QQQ}\mathbf{diag}(q_t^Z)Q^K + (\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT}Q^K)\sigma_t + \mathbb{D}_{QQ\mathcal{K}}\times_1 q_t^Z + \mathbb{R}\times_2 V^K \times_3 v_t^Z \\
&\quad + Q^K\mathrm{D}_{R\mathcal{K}T}\sigma_t + E\mathbf{diag}(q_t^Z)Q^K \\
&\quad + \frac{1}{2}\Big[\boldsymbol{\mathcal{Q}}^{N,K}\times_2 f_t^Z + \boldsymbol{\mathcal{Q}}^{K,N}\times_3 f_t^Z + \boldsymbol{\mathcal{Q}}^{K,K}\times_2 D_t^Z + \boldsymbol{\mathcal{Q}}^{K,K}\times_3 D_t^Z + \left(\boldsymbol{\mathcal{Q}}^{K,K}\times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,K}\times_3 \mathrm{d}_{\mathcal{I}T}\right)
\end{aligned}
$$

Similarly, define the matrices:

$$
\begin{aligned}
\mathbf{V}_{0t}^{N,T} &= \mathrm{D}_{UNT} T_t + \mu\nu \left(m\, \mathbf{diag}(v_t^T) V^N + \mathbb{M} \times_2 V^N \times_3 v_t^T \right) + \mathbb{Y} \times_3 v_t^T + \mathbb{Z} \times_2 V^N \times_3 v_t^T \\
&\quad + V^K \mathrm{D}_{RNT} T_t + B\mathbf{diag}(q_t^T) Q^N \\
&\quad + \frac{1}{2} \left[\boldsymbol{\mathcal{V}}^{N,N} \times_2 f_t^T + \boldsymbol{\mathcal{V}}^{N,N} \times_3 f_t^T + \boldsymbol{\mathcal{V}}^{N,K} \times_3 D_t^T + \boldsymbol{\mathcal{V}}^{K,N} \times_2 D_t^T + \left(\boldsymbol{\mathcal{V}}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T} \right) T \right. \\
\mathbf{V}_{0t}^{K,T} &= \mathrm{D}_{U\mathcal{K}T} T_t + \mu\nu \left(m\, \mathbf{diag}(v_t^T) V^K + \mathbb{M} \times_2 V^K \times_3 v_t^T \right) + \mathbb{Z} \times_2 V^K \times_3 v_t^T + V^K \mathrm{D}_{R\mathcal{K}T} T_t + B\mathbf{diag}(q_t^T) Q \\
&\quad + \frac{1}{2} \left[\boldsymbol{\mathcal{V}}^{N,K} \times_2 f_t^T + \boldsymbol{\mathcal{V}}^{K,N} \times_3 f_t^T + \boldsymbol{\mathcal{V}}^{K,K} \times_2 D_t^T + \boldsymbol{\mathcal{V}}^{K,K} \times_3 D_t^T + \left(\boldsymbol{\mathcal{V}}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{V}}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T} \right) T_t \right. \\
\mathbf{Q}_{0t}^{N,T} &= \mathrm{D}_{QQQ} \mathbf{diag}(q_t^T) Q^N + \left(\mathrm{D}_{QNT} + \mathrm{D}_{QQT} Q^N \right) T_t + \mathbb{D}_{QQN} \times_1 q_t^T + \mathbb{P} \times_3 v_t^T + \mathbb{R} \times_2 V^N \times_3 v_t^T \\
&\quad + Q^K \mathrm{D}_{RNT} T_t + E\mathbf{diag}(q_t^T) Q^N \\
&\quad + \frac{1}{2} \left[\boldsymbol{\mathcal{Q}}^{N,N} \times_2 f_t^T + \boldsymbol{\mathcal{Q}}^{N,N} \times_3 f_t^T + \boldsymbol{\mathcal{Q}}^{N,K} \times_3 D_t^T + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 D_t^T + \left(\boldsymbol{\mathcal{Q}}^{N,K} \times_3 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,N} \times_2 \mathrm{d}_{\mathcal{I}T} \right) \right. \\
\mathbf{Q}_{0t}^{K,T} &= \mathrm{D}_{QQQ} \mathbf{diag}(q_t^T) Q^K + \left(\mathrm{D}_{Q\mathcal{K}T} + \mathrm{D}_{QQT} Q^K \right) T_t + \mathbb{D}_{QQ\mathcal{K}} \times_1 q_t^T + \mathbb{R} \times_2 V^K \times_3 v_t^T \\
&\quad + Q^K \mathrm{D}_{R\mathcal{K}T} T_t + E\mathbf{diag}(q_t^T) Q^K \\
&\quad + \frac{1}{2} \left[\boldsymbol{\mathcal{Q}}^{N,K} \times_2 f_t^T + \boldsymbol{\mathcal{Q}}^{K,N} \times_3 f_t^T + \boldsymbol{\mathcal{Q}}^{K,K} \times_2 D_t^T + \boldsymbol{\mathcal{Q}}^{K,K} \times_3 D_t^T + \left(\boldsymbol{\mathcal{Q}}^{K,K} \times_2 \mathrm{d}_{\mathcal{I}T} + \boldsymbol{\mathcal{Q}}^{K,K} \times_3 \mathrm{d}_{\mathcal{I}T} \right) \right.
\end{aligned}
$$

Then, $V_t^{N,Z}$, $V_t^{K,Z}$, $Q_t^{N,Z}$, $Q_t^{K,Z}$, $V_t^{N,T}$, $V_t^{K,T}$, $Q_t^{N,T}$, and $Q_t^{K,T}$ satisfy:

$$
\begin{aligned}
(\rho+\theta) \begin{pmatrix} V_t^{N,Z} & V_t^{K,Z} \\ Q_t^{N,Z} & Q_t^{K,Z} \end{pmatrix} &= \begin{pmatrix} \mathbf{V}_{0t}^{N,Z} & \mathbf{V}_{0t}^{K,Z} \\ \mathbf{Q}_{0t}^{N,Z} & \mathbf{Q}_{0t}^{K,Z} \end{pmatrix} + \Phi \begin{pmatrix} V_t^{N,Z} & V_t^{K,Z} \\ Q_t^{N,Z} & Q_t^{K,Z} \end{pmatrix} + \begin{pmatrix} V_t^{N,Z} & V_t^{K,Z} \\ Q_t^{N,Z} & Q_t^{K,Z} \end{pmatrix} \Theta \\
&\quad + \begin{pmatrix} V_t^{N,Z} & V_t^{K,Z} \\ Q_t^{N,Z} & Q_t^{K,Z} \end{pmatrix} \mathcal{D}_T^* \qquad (54) \\
\rho \begin{pmatrix} V_t^{N,T} & V_t^{K,T} \\ Q_t^{N,T} & Q_t^{K,T} \end{pmatrix} &= \begin{pmatrix} \mathbf{V}_{0t}^{N,T} & \mathbf{V}_{0t}^{K,T} \\ \mathbf{Q}_{0t}^{N,T} & \mathbf{Q}_{0t}^{K,T} \end{pmatrix} + \Phi \begin{pmatrix} V_t^{N,T} & V_t^{K,T} \\ Q_t^{N,T} & Q_t^{K,T} \end{pmatrix} + \begin{pmatrix} V_t^{N,T} & V_t^{K,T} \\ Q_t^{N,T} & Q_t^{K,T} \end{pmatrix} \Theta \\
&\quad + \begin{pmatrix} V_t^{N,T} & V_t^{K,T} \\ Q_t^{N,T} & Q_t^{K,T} \end{pmatrix} \mathcal{D}_T^*
\end{aligned}
$$

where $\mathcal{D}_T^*$ is the matrix that discretizes the time derivative.

J.4 Stochastic SAME: $(\sigma_t z)^2$, T_t^2, $(\sigma_t z)T_t$

Define the vectors

$$
\begin{aligned}
\mathbf{v}_{0t}^{Z,Z} &= \mathrm{d}_{UTT}\sigma_t^2 + \mu\nu\left(m(v_t^Z \cdot v_t^Z) + \mathbb{M} \times_3 v_t^Z \times_2 v_t^Z\right) + \mathbb{Z} \times_3 v_t^Z \times_2 v_t^Z + B\left(q_t^Z \cdot q_t^Z\right) + V^K \mathrm{d}_{\mathcal{I}TT}\sigma_t^2 \\
&\quad + 2\left[\boldsymbol{V_t^{N,Z}} f_t^Z + \boldsymbol{V_t^{K,Z}}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right)\right] \\
\mathbf{v}_{0t}^{T,T} &= \mathrm{d}_{UTT}(T_t)^2 + \mu\nu\left(m\left(v_t^T \cdot v_t^T\right) + \mathbb{M} \times_3 v_t^T \times_2 v_t^T\right) + \mathbb{Z} \times_3 v_t^T \times_2 v_t^T + B\left(q_t^T \cdot q_t^T\right) + V^K \mathrm{d}_{\mathcal{I}TT}(T_t)^2 \\
&\quad + 2\left[\boldsymbol{V_t^{N,T}} f_t^T + \boldsymbol{V_t^{K,T}}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right)\right] \\
\mathbf{v}_{0t}^{Z,T} &= \mathrm{d}_{UTT}\sigma_t T_t + \mu\nu\left(m\left(v_t^Z \cdot v_t^T\right) + \mathbb{M} \times_3 v_t^Z \times_2 v_t^T\right) + \mathbb{Z} \times_3 v_t^Z \times_2 v_t^T + B\left(q_t^Z \cdot q_t^T\right) + V^K \mathrm{d}_{\mathcal{I}TT}\sigma_t T_t \\
&\quad + \boldsymbol{V_t^{N,Z}} f_t^T + \boldsymbol{V_t^{N,T}} f_t^Z + \boldsymbol{V_t^{K,Z}}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right) + \boldsymbol{V_t^{K,T}}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right) \\
\mathbf{v}_{0t}^{\Sigma} &= V^K \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T}\varsigma_t, \\
\mathbf{q}_{0t}^{Z,Z} &= \mathrm{D}_{QQQ}\left(q_t^Z \cdot q_t^Z\right) + 2\mathrm{D}_{QQT} q_t^Z \sigma_t + \mathrm{d}_{QTT}\sigma_t^2 + \mathbb{R} \times_3 v_t^Z \times_2 v_t^Z + E\left(q_t^Z \cdot q_t^Z\right) + Q^K \mathrm{d}_{\mathcal{I}TT}\sigma_t^2 \\
&\quad + 2\left[\boldsymbol{Q_t^{N,Z}} f_t^Z + \boldsymbol{Q_t^{K,Z}}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right)\right] \\
\mathbf{q}_{0t}^{T,T} &= \mathrm{D}_{QQQ}(q_t^T \cdot q_t^T) + 2\mathrm{D}_{QQT} q_t^T T_t + \mathrm{d}_{QTT}(T_t)^2 + \mathbb{R} \times_3 v_t^T \times_2 v_t^T + E\left(q_t^T \cdot q_t^T\right) + Q^K \mathrm{d}_{\mathcal{I}TT}(T_t)^2 \\
&\quad + 2\left[\boldsymbol{Q_t^{N,T}} f_t^T + \boldsymbol{Q_t^{K,T}}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right)\right] \\
\mathbf{q}_{0t}^{Z,T} &= \mathrm{D}_{QQQ}(q_t^Z \cdot q_t^T) + \mathrm{D}_{QQT}(q_t^Z T_t + q_t^T \sigma_t) + \mathrm{d}_{QTT}\sigma_t T_t + \mathbb{R} \times_3 v_t^Z \times_2 v_t^T + E\left(q_t^Z \cdot q_t^T\right) + Q^K \mathrm{d}_{\mathcal{I}TT}\sigma_t T_t \\
&\quad + \boldsymbol{Q_t^{N,Z}} f_t^T + \boldsymbol{Q_t^{N,T}} f_t^Z + \boldsymbol{Q_t^{K,Z}}\left(D_t^T + \mathrm{d}_{\mathcal{I}T}T_t\right) + \boldsymbol{Q_t^{K,T}}\left(D_t^Z + \mathrm{d}_{\mathcal{I}T}\sigma_t\right) \\
\mathbf{q}_{0t}^{\Sigma} &= (\Sigma^2 \mathrm{d}_{Q\Sigma T} + Q^K \Sigma^2 \mathrm{d}_{\mathcal{I}\Sigma T})\varsigma_t.
\end{aligned}
$$

Then, $v_t^{Z,Z}$, $q_t^{Z,Z}$, $v_t^{T,T}$, $q_t^{T,T}$, $v_t^{Z,T}$, $q_t^{Z,T}$, $v_t^{\Sigma,Z}$, $v_t^{\Sigma,T}$, $q_t^{\Sigma,Z}$ and $v_t^{\Sigma,T}$ satisfy:

$$
\begin{aligned}
(\rho + 2\theta) \begin{pmatrix} v_t^{Z,Z} \\ q_t^{Z,Z} \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_{0t}^{Z,Z} \\ \mathbf{q}_{0t}^{Z,Z} \end{pmatrix} + \Phi \begin{pmatrix} v_t^{Z,Z} \\ q_t^{Z,Z} \end{pmatrix} + \begin{pmatrix} v_t^{Z,Z} \\ q_t^{Z,Z} \end{pmatrix} \mathcal{D}_T^* \\
\rho \begin{pmatrix} v_t^{T,T} \\ q_t^{T,T} \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_{0t}^{T,T} \\ \mathbf{q}_{0t}^{T,T} \end{pmatrix} + \Phi \begin{pmatrix} v_t^{T,T} \\ q_t^{T,T} \end{pmatrix} + \begin{pmatrix} v_t^{T,T} \\ q_t^{T,T} \end{pmatrix} \mathcal{D}_T^* \\
(\rho + \theta) \begin{pmatrix} v_t^{Z,T} \\ q_t^{Z,T} \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_{0t}^{Z,T} \\ \mathbf{q}_{0t}^{Z,T} \end{pmatrix} + \Phi \begin{pmatrix} v_t^{Z,T} \\ q_t^{Z,T} \end{pmatrix} + \begin{pmatrix} v_t^{Z,T} \\ q_t^{Z,T} \end{pmatrix} \mathcal{D}_T^* \\
\rho \begin{pmatrix} v_t^{\Sigma} \\ q_t^{\Sigma} \end{pmatrix} &= \begin{pmatrix} \mathbf{v}_{0t}^{\Sigma} \\ \mathbf{q}_{0t}^{\Sigma} \end{pmatrix} + \Phi \begin{pmatrix} v_t^{\Sigma} \\ q_t^{\Sigma} \end{pmatrix} + \begin{pmatrix} v_t^{\Sigma} \\ q_t^{\Sigma} \end{pmatrix} \mathcal{D}_T^*
\end{aligned}
$$

J.5 Constant SAME

Given $v_t^{Z,Z}$ and $q_t^{Z,Z}$, the vectors $\overline{v}_t$ and $\overline{q}_t$ satisfy:

$$
\rho \begin{pmatrix} \overline{v}_t \\ \overline{q}_t \end{pmatrix} = \begin{pmatrix} v_t^{Z,Z} \\ q_t^{Z,Z} \end{pmatrix} + \Phi \begin{pmatrix} \overline{v}_t \\ \overline{q}_t \end{pmatrix} + \begin{pmatrix} \overline{v}_t \\ \overline{q}_t \end{pmatrix} \mathcal{D}_T^*
$$

J.6 Linearized KFEs

On the one hand, the law of motion for the population is given by (51). Noticing that $\frac{dn_{it}}{dt} = \frac{1}{\epsilon}\frac{dN_{it}}{dt}$, $\mathbb{G} \times_2 N = G$, and defining the tensor $\mathbb{I} \equiv \mathbb{H} \times_2 N$, we obtain the following expression:

$$
\frac{dn_t}{dt} = M^* n_t + G v_{1,t} + \frac{\epsilon}{2}\left\{G v_{2,t} + 2\mathbb{G} \times_3 v_{1,t} \times_2 n_t + \mathbb{I} \times_3 v_{1,t} \times_2 v_{1,t}\right\} \tag{55}
$$

On the other hand, the law of motion for capital is given by (52). Just for convenience, we will write the same expression here:

$$\begin{aligned}\frac{dk_t}{dt} &= \mathrm{D}_{R\mathcal{K}}k_t + \mathrm{D}_{RN}n_t + \mathrm{d}_{\mathcal{I}T}(\sigma_t z + T_t) + \mathrm{D}_{\mathcal{I}Q}q_{1,t},\\ &+ \frac{\epsilon}{2}\Big\{\mathbb{D}_{RNN}\times_3 n_t \times_2 n_t + \mathbb{D}_{RN\mathcal{K}}\times_3 k_t \times_2 n_t + \mathbb{D}_{R\mathcal{K}N}\times_3 n_t \times_2 n_k + \mathbb{D}_{R\mathcal{K}\mathcal{K}}\times_3 k_t \times_2 k_t\\ &\quad + 2\mathrm{D}_{RNT}n_t(\sigma_t z + T_t) + 2\mathrm{D}_{R\mathcal{K}T}k_t(\sigma_t z + T_t) + \mathrm{d}_{\mathcal{I}TT}(\sigma_t z + T_t)^2 + \Sigma^2\mathrm{d}_{\mathcal{I}\Sigma T}(\sigma_t z + T_t)\\ &\quad + \mathrm{D}_{IQ}q_{2,t} + \mathrm{D}_{IQQ}(q_{1,t}\cdot q_{1,t})\Big\},\end{aligned}$$

J.7 Approximation of Prices

J.7.1 Wages

From equation (21) we have that:

$$\begin{aligned}w_{it} &= (1-\alpha)\alpha^{\alpha}\Xi^{-\alpha}(1-x)^{\varpi\alpha}Z_i L_i^{\omega\alpha}e^{\chi_{it}}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{\alpha},\\ &= w_{i,0}e^{\chi_{it}}(K_{it}^{1-\omega-\varpi}N_{it}^{\varpi-1})^{\alpha}.\end{aligned}$$

Therefore, up to second order:

$$\begin{aligned}\frac{w_t - w^{ss}}{\epsilon} &= \mathrm{D}_{wN}n + \mathrm{D}_{wK}k + \mathrm{d}_{wT}\left(\sigma_t z + T_t\right)\\ &+ \frac{\epsilon}{2}\left\{\mathbb{D}_{wNN}\times_3 n \times_2 n + \mathbb{D}_{wNK}\times_3 k \times_2 n + \mathbb{D}_{wKN}\times_3 n \times_2 k + \mathbb{D}_{wKK}\times_3 k \times_2 k\right\}\\ &+ \epsilon\left\{\mathrm{D}_{wNT}n\left(\sigma_t z + T_t\right) + \mathrm{D}_{wKT}k\left(\sigma_t z + T_t\right)\right\}\\ &+ \frac{\epsilon}{2}\mathrm{d}_{wTT}\left(\sigma_t z + T_t\right)^2.\end{aligned}$$

where $\mathbb{D}_{wNN}$, $\mathbb{D}_{wKK}$, $\mathbb{D}_{wNK}$, $\mathbb{D}_{wKN}$, D_{wNT}, D_{wKT}, and d_{WTT} are given by

$$\begin{aligned}\mathbb{D}_{wNN} &= \alpha(1-\varpi)\left[\alpha(1-\varpi)+1\right]\mathbf{diag}_3\left(\frac{w_i^{ss}}{(N_i^{ss})^2}\right) & \mathrm{D}_{WNT} &= -\alpha(1-\varpi)\mathbf{diag}\left(\chi_{i1}\frac{w_i^{ss}}{N_i^{ss}}\right)\\ \mathbb{D}_{wKK} &= \alpha(1-\varpi-\omega)\left[\alpha(1-\varpi-\omega)-1\right]\mathbf{diag}_3\left(w_i^{ss}\right) & \mathrm{D}_{WKT} &= \alpha(1-\varpi-\omega)\mathbf{diag}\left(\chi_{i1}w_i^{ss}\right)\\ \mathbb{D}_{wNK} &= \mathbb{D}_{WKN} = -\alpha^2(1-\varpi)(1-\omega-\varpi)\mathbf{diag}_3\left(\frac{w_i^{ss}}{N_i^{ss}}\right) & \mathrm{d}_{WTT} &= \mathbf{vec}\left(w_i^{ss}(\chi_{i1}^2+\chi_{i2})\right).\end{aligned}$$

The matrices D_{wN}, D_{wK} and the vector d_{wT} are defined as in Section G.8.

J.7.2 Capital's Rental Rate

The return to capital is given by equation (23):

$$R_{it} = R_{0i} e^{\chi_{it}} K_{it}^{-\phi} N_{it}^{\psi}$$

Hence, up to a second-order:

$$\begin{aligned}\frac{R_t - R^{ss}}{\epsilon} &= \mathrm{D}_{RN} n + \mathrm{D}_{RK} k + \mathrm{d}_{RT}\left(\sigma_t z + T_t\right) \\ &+ \frac{\epsilon}{2}\left\{\mathbb{D}_{RNN} \times_3 n \times_2 n + \mathbb{D}_{RNK} \times_3 k \times_2 n + \mathbb{D}_{RKN} \times_3 n \times_2 k + \mathbb{D}_{RKK} \times_3 k \times_2 k\right\} \\ &+ \epsilon\left\{\mathrm{D}_{RNT} n\left(\sigma_t z + T_t\right) + \mathrm{D}_{RKT} k\left(\sigma_t z + T_t\right)\right\} + \frac{\epsilon}{2}\mathrm{d}_{RTT}\left(\sigma_t z + T_t\right)^2,\end{aligned}$$

where D_{RNT}, D_{RKT} and d_{RT} are defined in Section G.2.2. Moreover, $\mathbb{D}_{RNN}$, $\mathbb{D}_{RNK}$, $\mathbb{D}_{RKN}$, $\mathbb{D}_{RKK}$, D_{RNT} and D_{RKT} are defined as in Section H.1.2. Finally, define the vector d_{RTT} as:

$$\mathrm{d}_{RTT} = \mathbf{vec}\left(R_i^{ss}(\chi_{i1}^2 + \chi_{i2})\right).$$

J.7.3 Investment

Investment is given by

$$I_{it} = \left(R_{it} - Q_{it}^{-1/\zeta}\right) K_{it}.$$

Hence, up to a second-order:

$$\begin{aligned}\frac{I_t - I^{ss}}{\epsilon} &= \mathrm{D}_{IN} n + \mathrm{D}_{I\mathcal{K}} k + \mathrm{D}_{IQ} q_1 + \mathrm{d}_{IT}(\sigma_t z + T_t) \\ &+ \frac{\epsilon}{2}\left\{\mathbb{D}_{INN} \times_3 n \times_2 n + \mathbb{D}_{IN\mathcal{K}} \times_3 k \times_2 n + \mathbb{D}_{I\mathcal{K}N} \times_3 n \times_2 k + \mathbb{D}_{I\mathcal{K}\mathcal{K}} \times_3 k \times_2 k\right\}, \\ &+ \epsilon\left\{\mathrm{D}_{INT} n(\sigma_t z + T_t) + \mathrm{D}_{I\mathcal{K}T} k(\sigma_t z + T_t)\right\}, \\ &+ \frac{\epsilon}{2}\left\{\mathbb{D}_{I\mathcal{K}Q} \times_3 q_1 \times_2 k + \mathbb{D}_{IQ\mathcal{K}} \times_3 k \times_2 q_1 + \mathrm{D}_{IQ} q_2 + \mathbb{D}_{IQQ} \times_3 q_1 \times_2 q_1 + \mathrm{d}_{ITT}(\sigma_t z + T_t)^2\right\}\end{aligned}$$

where:

$$\mathbb{D}_{INN} = \psi(\psi-1)\mathbf{diag}_3\left(K_i^{ss}R_i^{ss}/(N_i^{ss})^2\right), \qquad \mathbb{D}_{I\mathcal{K}N} = \mathbb{D}_{IN\mathcal{K}} = \psi(1-\phi)\mathbf{diag}_3\left(R_i^{ss}K_i^{ss}/N_i^{ss}\right)$$

$$\mathbb{D}_{I\mathcal{K}\mathcal{K}} = \phi(\phi-2)\mathbf{diag}_3\left(R_i^{ss}K_i^{ss}\right) + \mathbf{diag}_3\left(I_i^{ss}\right) \quad \mathbb{D}_{I\mathcal{K}Q} = \mathbb{D}_{IQ\mathcal{K}} = \frac{1}{\zeta}\mathbf{diag}_3\left((Q_i^{ss})^{-\frac{1}{\zeta}-1}K_i^{ss}\right)$$

$$\mathbb{D}_{IQQ} = -\left[\frac{1}{\zeta}+1\right]\frac{1}{\zeta}\mathbf{diag}_3\left((Q_i^{ss})^{-\frac{1}{\zeta}-2}K_i^{ss}\right) \qquad \mathrm{d}_{ITT} = \mathbf{vec}\left((\chi_{i1}^2+\chi_{i2})R_i^{ss}K_i^{ss}\right)$$

$$\mathrm{D}_{INT} = \psi\mathbf{diag}\left(\chi_{i1}K_i^{ss}R_i^{ss}/N_i^{ss}\right), \qquad \mathrm{D}_{I\mathcal{K}T} = (1-\phi)\mathbf{diag}\left(\chi_{i1}K_i^{ss}R_i^{ss}\right)$$

K Consumption equivalent welfare

K.1 Workers

For any period τ, we compute consumption equivalent welfare change between the steady-state SS and any counterfactual IRF path as $\omega_{i\tau}$, such that

$$\mathbb{E}_\tau\int_\tau^\infty e^{-\rho(t-\tau)}\left\{u\left((1+\omega_{i\tau})A_{i(t)}^{ss}C_{i(t)}^{ss}\right)+\epsilon_{i(t)t}^{ss}\right\}dt = \mathbb{E}_\tau\int_\tau^\infty e^{-\rho(t-\tau)}\left\{u(A_{i(t),t}^{IRF}C_{i(t),t}^{IRF})+\epsilon_{i(t)t}^{IRF}\right\}dt$$

where ϵ_{it} denotes the preference shock for the current (chosen) location. We use general CRRA utility $u(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$. We denote by:

$$\mathcal{E}_{i\tau}^{ss} = \mathbb{E}_\tau\int_\tau^\infty e^{-\rho(t-\tau)}\epsilon_{i(t)t}^{ss}dt$$

the present value of preference shocks.

Then:

$$\begin{aligned}
&\mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} \left\{ u\Big((1+\omega_{i\tau}) A^{ss}_{i(t)} C^{ss}_{i(t)} \Big) + \epsilon^{ss}_{i(t)t} \right\} dt \\
&= \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} \left\{ \frac{\Big((1+\omega_{i\tau}) A^{ss}_{i(t)} C^{ss}_{i(t)} \Big)^{1-\gamma} - 1}{1-\gamma} + \epsilon^{ss}_{i(t)t} \right\} dt \\
&= (1+\omega_{i\tau})^{1-\gamma} \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} \frac{\Big(A^{ss}_{i(t)} C^{ss}_{i(t)} \Big)^{1-\gamma}}{1-\gamma} dt + \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} \left\{ -\frac{1}{1-\gamma} + \epsilon^{ss}_{i(t)t} \right\} dt \\
&= (1+\omega_{i\tau})^{1-\gamma} \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} \left\{ \frac{\Big(A^{ss}_{i(t)} C^{ss}_{i(t)} \Big)^{1-\gamma} - 1}{1-\gamma} dt + \epsilon^{ss}_{i(t)t} + \frac{1}{1-\gamma} - \epsilon^{ss}_{i(t)t} \right\} dt - \frac{1}{\rho(1-\gamma)} + \mathcal{E}^{ss}_{i\tau} \\
&= (1+\omega_{i\tau})^{1-\gamma} \left\{ V^{ss}_{i\tau} + \frac{1}{\rho(1-\gamma)} - \mathcal{E}^{ss}_{i\tau} \right\} - \frac{1}{\rho(1-\gamma)} + \mathcal{E}^{ss}_{i\tau}
\end{aligned}$$

Then:

$$(1+\omega_{i\tau})^{1-\gamma} = \frac{V^{IRF}_{i\tau} + \frac{1}{\rho(1-\gamma)} - \mathcal{E}^{ss}_{i\tau}}{V^{ss}_{i\tau} + \frac{1}{\rho(1-\gamma)} - \mathcal{E}^{ss}_{i\tau}} = 1 + \frac{dV^{IRF}_{i\tau}}{V^{ss}_{i\tau} + \frac{1}{\rho(1-\gamma)} - \mathcal{E}^{ss}_{i\tau}},$$

To compute $\mathcal{E}^{ss}_{i,\tau}$, we use the identify:

$$V^{ss}_{i\tau} = V^{C,SS}_{i\tau} + \mathcal{E}^{ss}_{i,\tau},$$

where $V^{C,SS}_{i\tau}$ is the value excluding taste shocks and is defined by:

$$V^{C,SS}_{i\tau} = \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} u\Big(A^{ss}_{i(t)} C^{ss}_{i(t)} \Big) dt.$$

$V^{C,SS}_{i\tau}$ satisfies the Bellman equation:

$$\rho V^{C,SS}_{it} = u\Big(A^{ss}_i C^{ss}_i \Big) + \mu \sum_j m^{ss}_{ij} V^{C,SS}_{jt}$$

Hence, we can easily calculate $V^{C,SS}_{it}$ (as we were doing previously). Subtituting into the

welfare formula:

$$(1+\omega_{i\tau})^{1-\gamma} = 1 + \frac{dV_{i\tau}^{IRF}}{V_{i\tau}^{ss} + \frac{1}{\rho(1-\gamma)} + V_{i\tau}^{C,SS} - V_{i\tau}^{ss}}$$
$$= 1 + \frac{dV_{i\tau}^{IRF}}{V_{i\tau}^{C,SS} + \frac{1}{\rho(1-\gamma)}}$$

which implies:

$$\omega_{i\tau} = \left(1 + \frac{dV_{i\tau}^{IRF}}{V_{i\tau}^{C,SS} + \frac{1}{\rho(1-\gamma)}}\right)^{\frac{1}{1-\gamma}} - 1$$

When shocks are small, to a first order

$$\omega_{i\tau} \approx \frac{1}{1-\gamma} \frac{dV_{i\tau}}{V_{i\tau}^{C,SS} + \frac{1}{\rho(1-\gamma)}} = \frac{\rho\, dV_{i\tau}}{1+\rho(1-\gamma)V_{i\tau}^{C,SS}}. \tag{56}$$

Crucially, we need V^C in the denominator.

K.2 Capitalists

We define the capitalist's consumption equivalent welfare analogously as in the workers case. For any period τ, we compute consumption equivalent welfare change between the steady-state SS and any counterfactual IRF path as $\omega_{i\tau}^{cap}$, such that

$$\mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} u^{cap}((1+\omega_{i\tau}^{cap})C_{i(t)}^{ss})dt = \mathbb{E}_\tau \int_\tau^\infty e^{-\rho(t-\tau)} u^{cap}(C_{i(t),t}^{IRF})dt.$$

Using $u^{cap}(c) = \frac{c^{1-\zeta}-1}{1-\zeta}$, we obtain:

$$\omega_{i\tau}^{cap} = \left[\frac{\rho(1-\zeta)\mathcal{P}_{i\tau}^{IRF}+1}{\rho(1-\zeta)\mathcal{P}_{i\tau}^{SS}+1}\right]^{\frac{1}{1-\zeta}} - 1,$$

where

$$\mathcal{P}_{i\tau}^{SS} \equiv \int_{\tau}^{\infty} e^{-\rho(t-\tau)} u^{cap}(C_{i(t)}^{SS}) dt,$$
$$\mathcal{P}_{i\tau}^{IRF} \equiv \int_{\tau}^{\infty} e^{-\rho(t-\tau)} u^{cap}(C_{i(t),t}^{IRF}) dt.$$

Hence, for small shocks, we get:

$$\omega_{i\tau}^{cap} \approx \frac{\rho d\mathcal{P}_{i\tau}^{IRF}}{\rho(1-\zeta)\mathcal{P}_{i\tau}^{SS} + 1}.$$

Using the definition of $\mathcal{P}_{it}$ from (26) we get that:

$$\omega_{i\tau}^{cap} \approx \frac{1}{1-\zeta} \frac{dQ_{i\tau}^{IRF}}{Q_{i\tau}^{SS}} + \frac{dK_{i\tau}}{K_{i\tau}^{SS}}.$$