Option Exercise Games and a Duopolistic q Theory of Investment*

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Abstract

Back and Paulsen (2009) define the open-loop equilibrium for the real-option exercise game studied in Grenadier (2002), show that Grenadier's solution corresponds to the open-loop equilibrium, and identify a closed-loop equilibrium in which firms earn zero profits upon investment. They pose two open questions: how to define a closed-loop equilibrium and whether any such equilibrium with positive profits exists.

We address these questions by defining a subgame-perfect closed-loop equilibrium using a marginal q condition, with Back and Paulsen's zero-profit closed-loop equilibrium serving as a punishment equilibrium to support other equilibria with positive profits. We then characterize infinitely many such closed-loop equilibria in closed form and develop a duopolistic q-theory of investment in which, in addition to marginal q, cross marginal q—which measures how a firm's value responds to changes in its rival's capital stock—plays a central role in determining investment under imperfect competition. Across all closed-loop equilibria, investment proceeds more slowly than in the Back–Paulsen zero-profit equilibrium, but more rapidly than in the open-loop equilibrium.

Keywords: real-option games, closed-loop equilibria, irreversible investment, duopoly, marginal q, cross marginal q

JEL Codes: C61, C73, D43, E22, L13

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1 Introduction

How do firms exercise their growth options and accumulate capital under uncertainty in an oligopolistic environment? Grenadier (2002) develops a real-options game framework in which firms undertake irreversible investment in a setting where the output price decreases with industry capital stock. Back and Paulsen (2009) show that Grenadier's solution is an open-loop equilibrium. In this equilibrium, each firm chooses its investment strategy at time 0 and commits to it regardless of its competitor's actions, even though the dynamic game provides no mechanism to make such commitment credible. In particular, if one firm deviates from its prescribed strategy by investing more, its competitor is required to ignore the deviation and continue following its original investment plan. However, such behavior is inconsistent with subgame perfection.

In light of this observation, Back and Paulsen (2009) argue that a closed-loop equilibrium is a more appropriate solution concept, as it allows firms to condition their strategies at time t on the history of the game up to t and thus respond dynamically to competitors' actions over time, thereby yielding a subgame-perfect equilibrium. Back and Paulsen (2009) also identify a closed-loop equilibrium in which firms optimally follow the simple net present value (NPV) rule for investment. Notably, this closed-loop equilibrium yields zero economic profits and completely erodes the value of industry growth options.

Furthermore, Back and Paulsen (2009) highlight several open questions and avenues for future research. In their introduction, they note: *"There are difficulties in even defining the game in closed-loop form."* They further conjecture the existence of alternative equilibria in which firms earn positive profits and preserve some option value of waiting, but state that *"It is an open question whether the perfectly competitive boundary is the unique boundary such that closed-loop strategies ... are impervious to preemption."*

In this paper, we address these open questions by identifying closed-loop equilibria in which firms earn positive profits and exploring their implications for a duopolistic q-theory of investment, which we subsequently employ for a quantitative illustration. To set the stage, Section 2 presents Grenadier's duopoly real-options investment game with a linear inverse demand function and summarizes the monopoly benchmark (In Appendix E, we extend the analysis to an

isoelastic inverse demand function). In Section 3, we summarize Back and Paulsen's open-loop equilibrium definition and the corresponding solution derived by Grenadier (2002).

In Section 4, we define closed-loop equilibria in which firms earn positive profits for the option exercise games, inspired by Back and Paulsen's conjecture that *"there should be equilibria . . . , in which firms invest less than the perfectly competitive amount and threaten to punish any firm that deviates."* We begin by introducing Back and Paulsen's closed-loop equilibrium (with zero profits) using a sequential-game formulation in which the two firms are assigned *ex ante* the roles of leader and follower. This zero-profit closed-loop equilibrium strategy serves as the punishment strategy in a closed-loop equilibrium in which firms make positive profits.

Next, we define closed-loop strategies by decomposing a firm's capital accumulation process into a continuous component and a jump (lumpy) component. As investment is irreversible, both the continuous and jump components increase only when the demand process reaches a new high, that is, when the running maximum $\{M_t\}$ increases.

We then define a closed-loop equilibrium using the following two conditions. One condition requires that at any t a firm's capital stock must be at a level such that the firm cannot further increase its value by investing more. This corresponds to the firm's first-order necessary condition, which requires its marginal q not to exceed its capital purchase cost in equilibrium.

The other condition specifies the consequences when a firm undertakes a one-shot deviation from its equilibrium strategy. Fix a small time interval ε during which firm a deviates. Firm b observes the deviation at the end of this interval and responds by switching to the leader's strategy in Back and Paulsen's zero-profit closed-loop equilibrium. This grim-trigger strategy deters firm a from deviating. We show that these two conditions jointly characterize the set of admissible closed-loop equilibria.

It is worth pointing out that the use of Back and Paulsen's zero-profit equilibrium as a punishment device to sustain a positive-profit equilibrium is analogous to employing the static Nash equilibrium of a prisoner's dilemma to support cooperative outcomes in an infinitely repeated game (Fudenberg and Tirole, 1992). Recently, in their study of dynamic capital structure, Malenko and Tsoy (2025) adopt a similar approach: they use the Markov perfect equilibrium of DeMarzo and He (2021)—in which firms experience leverage ratcheting without benefits—to support a time-consistent equilibrium in which firms choose more prudent target

leverage and avoid such ratcheting.

Section 5 develops a solution method to identify and characterize symmetric closed-loop equilibria in closed form. We conjecture and verify that there exist infinitely many economically intuitive closed-loop equilibria in which firms earn positive profits. We derive closed-form expressions for investment strategies and firm values, revealing four distinct equilibrium regions: one region where both firms invest, two regions where only the smaller firm invests, and one region where neither firm invests.

We introduce an equilibrium trigger function $\mathcal{X}$ that depends on the pair of capital stocks (K_t^a, K_t^b) to delineate these regions. When industry demand $X_t < \mathcal{X}(K_t^a, K_t^b)$, neither firm invests, as the aggregate capital stock is sufficiently large that investing would not be profitable. If $X_t > \mathcal{X}(K_t^a, K_t^b)$, the larger firm remains inactive while the smaller firm invests until the threshold $X_t = \mathcal{X}(K_t^a, K_t^b)$ is reached. When firms are equal in size, both invest simultaneously to maintain equality along the 45° line $K_t^a = K_t^b$. Once firms attain equal capital stocks, they evolve in lockstep, jointly responding to demand shocks along this line.

In any subgame of a closed-loop equilibrium (as opposed to an open-loop equilibrium), the following two conditions must hold: (i) when a firm invests, its marginal q equals the capital purchase cost p; and (ii) when its competitor invests, the firm's value must be locally insensitive to the competitor's capital stock, so that its cross-marginal q equals zero. These two conditions imply the following boundary conditions on the endogenous investment boundary $\mathcal{X}(K_t^a, K_t^b)$ for firm a: (1) marginal q equals p on the segment of $\mathcal{X}(K_t^a, K_t^b)$ with $K_t^a \leq K_t^b$, where firm a invests; and (2) cross-marginal q equals zero on the segment of $\mathcal{X}(K_t^a, K_t^b)$ with $K_t^a \geq K_t^b$, where the competitor, firm b, invests.

Our analysis in Section 5 does not yet fully characterize a closed-loop equilibrium solution. In Section 6, we complete the equilibrium characterization by conjecturing and verifying the existence of a class of closed-loop equilibria in which the investment threshold $\mathcal{X}(K_t^a, K_t^b)$ is *piecewise linear* in the two firms' capital stocks.

In Section 7, we explore the q-theoretic implications of our closed-loop equilibria. We derive analytical and quantitative predictions for industry equilibrium outcomes across various closed-loop equilibria and contrast them with the open-loop benchmark. In all newly identified closed-loop equilibria, firms earn positive profits by preserving some option value of waiting,

reflecting less aggressive competition than in Back and Paulsen's closed-loop equilibrium. Compared to the open-loop equilibrium, firms invest earlier, resulting in lower firm values across all closed-loop equilibria. For instance, in our numerical example, the best closed-loop equilibrium—yielding the highest firm value among all closed-loop equilibria—features investment rates 26% slower than in Back and Paulsen's closed-loop equilibrium but 10% faster than in the open-loop equilibrium. This leads to significantly positive firm value, albeit lower than in the open-loop equilibrium, as firms compete more aggressively under the closed-loop equilibrium.

In Section 8, we characterize two sets of closed-loop equilibria with *nonlinear* investment triggers. Compared to the closed-loop equilibria with piecewise linear investment triggers analyzed in Section 6, firms compete less aggressively when market demand is low but more fiercely as market demand increases due to the nonlinearity of the investment trigger. Economically, in these equilibria with nonlinear investment triggers, the larger firm's marginal q also equals the marginal cost of purchasing capital when the smaller firm invests. We conclude in Section 9 and provide supporting technical details in Appendices A-E.

Finally, we emphasize that the smooth-pasting conditions, intuitively exposited by Dixit (1993) and widely used in irreversible investment models (see, e.g., Abel and Eberly, 1996), hold in neither Back and Paulsen's zero-profit closed-loop equilibrium nor any positive-profit closed-loop equilibrium because each firm dynamically responds to its competitor's strategy in every subgame.[1]

Related literature. The distinction between open-loop and closed-loop equilibria has been studied in deterministic oligopoly investment games; see, for example, Fershtman and Muller (1984), Reynolds (1987), Tirole (1988), Simon and Stinchcombe (1989), and Fudenberg and Tirole (1992). Introducing shocks into these models substantially complicates the analysis, making even the definition of a closed-loop equilibrium conceptually challenging, as emphasized by Back and Paulsen (2009), let alone the explicit characterization of closed-loop equilibria. A key contribution of our work is to define a closed-loop equilibrium concept and

[1]The smooth-pasting condition also fails to hold in Back and Paulsen's closed-loop equilibrium for the same reason. By contrast, Grenadier (2002) and Back and Paulsen (2009) use smooth-pasting conditions to characterize open-loop equilibria, which are defined under time-0 commitment and therefore do not require subgame perfection.

characterize a class of infinitely many such equilibria in closed form. Central to our analysis is the identification of the running maximum process of the demand shock, $\{M_t\}$, as an additional state variable. This state variable enables us to obtain a closed-form Markovian characterization of firms' feedback strategies in response to the evolution of $\{M_t\}$.

Our paper contributes to the game-theoretic real-options literature. Grenadier (1996) develops a duopoly real-options exercise model and applies it to real estate development. Lambrecht (2001) develops a duopoly real-options exit model with two *ex ante* heterogeneous firms. Dai, Jiang, and Wang (2025) analyze a real-options game in which firms choose whether to enter as a leader or a follower, trading off first- versus second-mover advantages. In their setting, second-mover advantages give rise to mixed entry strategies. All these real-options game models feature optimal-stopping problems, whose single-agent formulations have been analyzed by McDonald and Siegel (1986) and Dixit and Pindyck (1994). In contrast, our duopolistic model features recurrent irreversible investments over time.

Our work is also related to the literature on repeated games with imperfect monitoring. A key contribution is Sannikov (2007), who formulates and solves the problem in continuous time by characterizing the set of continuation values achievable in public perfect equilibria via differential equations derived from the martingale representation theorem.

Models featuring recurrent irreversible investment have been widely studied in the literature. Abel and Eberly (1996) analyze a monopolist's optimal investment policies. In addition to the work by Grenadier (2002) and Back and Paulsen (2009) discussed earlier, Baldursson (1998) uses the same Nash, i.e., open-loop, equilibrium concept as Grenadier (2002) to study oligopoly games.[2] Leahy (1993) and Baldursson and Karatzas (1996) study industry equilibrium in perfectly competitive settings. Kogan (2001, 2004) develop q-theoretic general-equilibrium asset-pricing models with irreversible investment.

There is also a growing body of literature that integrates industrial organization considerations into asset pricing models. Dou, Ji, and Wu (2022) extend the standard Lucas-tree asset pricing model to incorporate endogenous strategic competition. Chen, Dou, Guo, and Ji (2024) investigate how strategic competition and financial distress interact dynamically.

[2]Novy-Marx (2007) and Aguerrevere (2009) extend Grenadier's open-loop setting to analyze risk and return dynamics.

2 Model

In this section, we first introduce a classic real-option game-theoretic setting (Grenadier, 2002) in which firms compete by making recurrent irreversible investment decisions in subsection 2.1 and then report the monopoly solution, which serves a useful benchmark for our duopoly analysis, in subsection 2.2.

2.1 Setting

Consider a duopoly industry consisting of two firms: a and b. Let $\{K_t^i\}$ denote firm i's capital stock, where $i = a, b$. Each firm uses its capital stock to produce output. Let $\pi^i(X_t, K_t^a, K_t^b)$ denote firm i's profit at time t, where $\{X_t\}$ is an exogenous process representing the industry's demand shock. As in Grenadier (2002) and Back and Paulsen (2009), we assume that X_t follows a geometric Brownian motion (GBM):

$$dX_t = \mu X_t dt + \sigma X_t d\mathcal{B}_t\,, \quad X_0 > 0\,, \tag{1}$$

where $\{\mathcal{B}_t; t \geq 0\}$ is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $\mathcal{B}_0 = 0$, μ is the drift and $\sigma > 0$ is the volatility parameter.

As in the literature, investment is irreversible in that $\{K_t^a\}$ and $\{K_t^b\}$ are nondecreasing over time. Let $\{M_t\}$ denote the running-maximum process for market demand $\{X_t\}$:

$$M_t = \max_{s\in[0,t]} X_s. \tag{2}$$

As investment irreversibility induces an option value of waiting, this running-maximum process plays a crucial role in our analysis, as we show later.

Let $Z_t = (X_t, K_t^a, K_t^b)$ denote the state, and let $\mathbb{Z} = [0, \infty) \times [k_0^a, \infty) \times [k_0^b, \infty)$ denote the state space, where $k_0^i \geq 0$ is firm i's initial capital stock. For a given pair of capital stock processes $(\{K_t^a\}, \{K_t^b\})$, firm i's value is

$$\mathbb{E}^{z_0}\left[\int_0^\infty e^{-rt}\Big(\pi^i(Z_t)dt - p\, dK_t^i\Big)\right], \tag{3}$$

where $\mathbb{E}^{z_0}[\,\cdot\,] = \mathbb{E}[\,\cdot \mid Z_0 = z_0]$ is the $t = 0$ expectation operator, $r > 0$ is a constant, and $p > 0$ is the constant marginal cost of purchasing a unit of capital. In the main body of our

paper, we specify:[3]

$$\pi^i(x, k^a, k^b) = [x - (k^a + k^b)]k^i - ck^i, \tag{4}$$

where $c > 0$ is a constant. We assume $\mu < r$ to ensure the finiteness of value functions.

2.2 Monopoly Solution

If the industry is instead a monopoly, then the monopolist makes profits at the rate of $\pi(x, k) = (x - k)k - ck$ and chooses its capital process $\{K_t; t \geq 0\}$ to maximize its value $\mathbb{E}\left[\int_0^\infty e^{-rt}\Big([X_t - (K_t + c)]K_t dt - p\, dK_t\Big)\right]$. Next, we summarize the monopolist's investment strategy (Abel and Eberly, 1996; Grenadier, 2002). Mathematically, the monopolist solves a singular control problem (Harrison, 1985).

Proposition 1 *The monopoly's optimal capital accumulation process,* $\{K_t^{\text{mon}}\}$*, is given by*

$$K_t^{\text{mon}} = \max\left\{\frac{1}{2}\left(\frac{M_t}{\rho} - (rp + c)\right),\, k_0^{\text{mon}}\right\}, \text{ for all } t > 0, \tag{5}$$

where k_0^{mon} *is the initial capital stock of the monopolist,* M_t *is the running maximum given in (2),* $\rho > 0$ *is a constant given by*

$$\rho = \frac{\beta}{\beta - 1}\frac{r - \mu}{r}, \tag{6}$$

and $\beta > 2$ *is a constant describing the optionality of the investment opportunity:*[4]

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \tag{7}$$

We can describe the monopoly's strategy as follows. First, if the demand is high in that $X_0 > \rho(2k_0^{\text{mon}} + rp + c)$, the monopolist instantly increases its capital stock at $t = 0$ from its initial level, k_0^{mon}, to $\frac{1}{2}\big[\frac{X_0}{\rho} - (rp + c)\big]$. Then for all $t \geq 0$, the monopolist does not invest until the running maximum M_t exceeds $\rho(2K_t^{\text{mon}} + rp + c)$, where K_t^{mon} is given in (5). Finally, after M_t reaches $\rho(2K_t^{\text{mon}} + rp + c)$, the monopolist invests if and only if M_t increases and the investment rate, dK_t^{mon}, is proportional to dM_t:

$$dK_t^{\text{mon}} = \frac{1}{2}\frac{dM_t}{\rho}. \tag{8}$$

[3] In Appendix E, we report the solution for the isoelastic demand case, another widely studied case in the literature, see, e.g., Grenadier (2002).

[4] Here β is the positive root of the following (fundamental) quadratic equation: $\sigma^2 x(x-1)/2 + \mu x - r = 0$. For convergence, we require $\beta > 2$.

For a given time interval dt, the running maximum $\{M_t\}$ increases, when both $X_t = M_t$ and $dX_t > 0$ hold. Under this scenario, $dM_t = dX_t > 0$. We can equivalently summarize the monopolist's capital accumulation process as follows: $K_t^{\text{mon}} = \inf\{k \geq k_0^{\text{mon}} \mid M_t \leq \mathcal{X}^{\text{mon}}(k)\}$ for $t > 0$, where $\mathcal{X}^{\text{mon}}(k)$ is the investment threshold function given by

$$\mathcal{X}^{\text{mon}}(k) = \rho(2k + rp + c)\,. \tag{9}$$

To ease comparison with a closed-loop equilibrium solution, we report Back and Paulsen's open-loop equilibrium definition and the open-loop equilibrium solution.

3 Open-loop Equilibrium Definition and Solution

In an open-loop equilibrium, firms choose their strategies at $t = 0$. Specifically, at $t = 0$, firm i chooses its capital stock process $K^i \in \mathcal{S}^i_{\text{open}}(z_0)$ to maximize its market value given in (3), where $\mathcal{S}^i_{\text{open}}(z_0)$ is the set of firm i's admissible open-loop strategies defined as:

$$\mathcal{S}^i_{\text{open}}(z_0) = \big\{\{K_t^i; t \geq 0\} \mid K_t^i \text{ is } \mathcal{F}_t\text{-adapted, nondecreasing, left-continuous}, K_0^i = k_0^i\big\}. \tag{10}$$

Note that firm i's open-loop strategy only depends on the value of the initial state $Z_0 = z_0 := (x_0, k_0^a, k_0^b)$ and the demand-shock process $\{X_t; t \geq 0\}$, and importantly, does not depend on its competitor's strategy $\{K_t^{-i}; t > 0\}$. That is, firms compete by choosing their capital-stock paths at $t = 0$ and fully committing to their chosen paths. For a given pair, $K^a \in \mathcal{S}^a_{\text{open}}(z_0)$ and $K^b \in \mathcal{S}^b_{\text{open}}(z_0)$, let $\mathcal{J}^i(z_0; K^a, K^b)$ denote firm i's value given in (3).

Back and Paulsen (2009) define an open-loop equilibrium as follows.

Definition 1 $K^a_{\text{open}} \in \mathcal{S}^a_{\text{open}}(z_0)$ and $K^b_{\text{open}} \in \mathcal{S}^b_{\text{open}}(z_0)$ constitute an *open-loop equilibrium* if the following conditions hold:

$$\mathcal{J}^a(z_0; K^a_{\text{open}}, K^b_{\text{open}}) \geq \mathcal{J}^a(z_0; K^a, K^b_{\text{open}}), \quad \forall\, K^a \in \mathcal{S}^a_{\text{open}}(z_0), \tag{11}$$

$$\mathcal{J}^b(z_0; K^a_{\text{open}}, K^b_{\text{open}}) \geq \mathcal{J}^b(z_0; K^a_{\text{open}}, K^b), \quad \forall\, K^b \in \mathcal{S}^b_{\text{open}}(z_0). \tag{12}$$

An open-loop equilibrium is a Nash equilibrium over entire time paths chosen at $t = 0$, and firms cannot revise their strategies in the future. Inequalities (11)-(12) hold only at $t = 0$.

Proposition 2 *For $k_0^a = k_0^b$, there exists an open-loop equilibrium in which*

$$K^i_{\text{open},t} = \max\left\{\frac{1}{3}\left(\frac{M_t}{\rho} - (rp+c)\right), k_0^i\right\}, \text{ for all } t > 0, \tag{13}$$

where $\{M_t\}$ is the running-maximum process defined in (2) and $\rho > 0$ is given in (6).

Starting with the same level of the initial capital stock, firms adopt the same strategy so that their capital stocks are equal at all t. If the demand is high in that $X_0 > \rho(3k_0^i + rp + c)$, both firms instantly jump their capital stocks from k_0^i to $\frac{1}{3}\left[\frac{X_0}{\rho} - (rp+c)\right]$. Then for all $t \geq 0$, neither firm invests until the running maximum M_t exceeds $\rho(3K^i_{\text{open},t} + rp + c)$, where $K^i_{\text{open},t}$ is given in (13). After M_t reaches $\rho(3K^i_{\text{open},t} + rp + c)$, both firms invest whenever M_t increases. Finally, each firm invests at a rate proportional to dM_t:

$$dK^i_{\text{open},t} = \frac{1}{3}\frac{dM_t}{\rho}. \tag{14}$$

Therefore, the duopoly industry's investment rate equals $\frac{2}{3}\frac{dM_t}{\rho}$, which is faster than the monopoly's investment rate $\frac{1}{2}\frac{dM_t}{\rho}$ given in (8). This confirms the intuition that competition erodes the option value even in the open-loop equilibrium where firms compete by committing to their strategies at $t = 0$.

We can equivalently describe firm i's capital process in the open-loop equilibrium using: $K^i_{\text{open},t} = \inf\left\{k \geq k_0^i : M_t \leq \mathcal{X}^{\text{open}}(k,k)\right\}$, where $\mathcal{X}^{\text{open}}(k^a, k^b)$ is the threshold given by

$$\mathcal{X}^{\text{open}}(k^a, k^b) := \begin{cases} \rho(2k^a + k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(k^a + 2k^b + rp + c), & \text{if } k^a \geq k^b. \end{cases} \tag{15}$$

4 Closed-loop Equilibrium: Definition

In this section, we define closed-loop strategies and equilibria, and then provide a verification theorem. We proceed in three steps. First, in subsection 4.1, we define and characterize Back and Paulsen's closed-loop equilibrium in which firms make zero profits when investing. Then, in subsection 4.2, we define and characterize closed-loop equilibria in which firms make positive profits when investing, using Back and Paulsen's closed-loop equilibrium as a key building block. Finally, we provide a verification theorem for the closed-loop equilibria.

4.1 Back and Paulsen's Closed-loop Equilibrium with Zero Profits

Back and Paulsen (2009) identified a closed-loop equilibrium that features a perfectly competitive setting in which firms make zero profits when investing. In this equilibrium, which we refer to as the BP equilibrium, firms invest if and only if $X_t \geq \mathcal{X}^{\mathrm{BP}}(K_t^a, K_t^b)$ where $\mathcal{X}^{\mathrm{BP}}(k^a, k^b)$ is the investment threshold function given by:

$$\mathcal{X}^{\mathrm{BP}}(k^a, k^b) = \rho(k^a + k^b + rp + c)\,. \tag{16}$$

Back and Paulsen (2009) summarized this equilibrium using:

$$K_t^i = \inf\left\{k \geq K_0^i \mid \sup_{0\leq s\leq t}[X_s - \mathcal{X}^{\mathrm{BP}}(k, K_s^{-i})] \leq 0\right\}, \quad t > 0 \tag{17}$$

and pointed out that equation (17) "defines K^i as the smallest nondecreasing process such that $X_t \leq \mathcal{X}^{\mathrm{BP}}(K_t^i, K_t^{-i})$ for all t." Equation (17) implies that as long as (at least) one firm invests, the post-investment capital-stock pair, (K_t^a, K_t^b), must lie on the investment threshold: $X_t = \mathcal{X}^{\mathrm{BP}}(K_t^a, K_t^b) = \rho(K_t^a + K_t^b + rp + c)$ so that $K_t^a + K_t^b = \frac{X_t}{\rho} - (rp + c)$ holds.

We define the BP equilibrium using a sequential-game formulation in which the two firms are assigned *ex ante* the roles of leader and follower. In this formulation, the follower's closed-loop strategy at t is naturally a best response to the leader's strategy up to t. In Appendix A, we define this sequential game and characterize the leader's and the follower's closed-loop strategies. Below, we summarize the firms' capital-stock paths $(\{K_t^a\}, \{K_t^b\})$ and their value functions in the BP equilibrium. Without loss of generality, let firm a be the follower and firm b be the leader.

Theorem 1 *On the BP equilibrium path, the follower firm a's capital stock remains constant: $K_t^a = k_0^a$ for all $t > 0$, and the leader firm b's capital stock evolves as follows: $K_t^b = k_0^b + \left(\frac{M_t}{\rho} - (rp + c) - k_0^a - k_0^b\right)^+$ for all $t > 0$. Firm i's value $V_{\mathrm{BP}}^i(z)$ is given by*

$$V_{\mathrm{BP}}^i(z) = k^i p, \quad x \geq \mathcal{X}^{\mathrm{BP}}(k^a, k^b), \tag{18}$$

$$V_{\mathrm{BP}}^i(z) = k^i\left[p + \frac{x}{r-\mu} - \frac{\mathcal{X}^{\mathrm{BP}}(k^a, k^b)}{r\rho}\left(1 + \frac{1}{\beta - 1}\left(\frac{x}{\mathcal{X}^{\mathrm{BP}}(k^a, k^b)}\right)^{\beta}\right)\right], \quad x < \mathcal{X}^{\mathrm{BP}}(k^a, k^b), \tag{19}$$

where $z = (x, k^a, k^b)$, and $\mathcal{X}^{\mathrm{BP}}(k^a, k^b)$ is the investment threshold given by (16).

In this BP equilibrium, if the demand is high in that $X_0 > \rho(k_0^a + k_0^b + rp + c)$, as the

leader, firm b instantly increases its capital stock from k_0^b to $\frac{X_0}{\rho} - (rp + c) - k_0^a$. Then for all $t \geq 0$, the leader does not invest until the running maximum M_t exceeds $\rho(K_t^b + k_0^a + rp + c)$ and whenever M_t reaches a new record, the leader increases its capital stock as follows:

$$dK_t^b = \frac{dM_t}{\rho}\,.$$

By contrast, as the follower, firm a never invests so that $K_t^a = k_0^a$ at all $t > 0$ on the equilibrium path. As we show in the next subsection, this BP equilibrium solution is a key building block for our definition of a closed-loop equilibrium in which firms make positive profits.

4.2 Closed-loop Equilibria with Positive Profits

We first define feasible closed-loop strategies and then define closed-loop equilibria.

4.2.1 Closed-loop Strategies

We decompose firm i's investment over dt, dK_t^i, as follows:

$$dK_t^i = dK_t^{i,C} + \Delta K_t^i\,, \quad t \geq 0\,, \tag{20}$$

where $dK_t^{i,C}$ denotes the continuous component of investment dK_t^i and $\Delta K_t^i := K_{t+}^i - K_t^i$ denotes the instantaneous increase (jump) in K^i. We consider a set of closed-loop Markovian strategies. Assume that the jump component, ΔK_t^i, depends on the state, $Z_t = (X_t, K_t^a, K_t^b)$:

$$\Delta K_t^i = \boldsymbol{w}^i(Z_t)\,, \tag{21}$$

where $\boldsymbol{w}^i(\,\cdot\,) \geq 0$ is a function of Z_t.

Inspired by the monopoly's strategy (8) and the open-loop equilibrium strategy (14), both of which are driven by the running-maximum process $\{M_t\}$, we conjecture and later verify that the continuous part of the closed-loop strategy, $dK_t^{i,C}$, can be written as:

$$dK_t^{i,C} = \boldsymbol{u}^i(Z_t)dM_t\,, \tag{22}$$

where $\boldsymbol{u}^i(\,\cdot\,) \geq 0$ is a function of Z_t. That is, firm i increases its capital stock continuously only if the running maximum of the demand shock, M_t, increases.

Definition 2 Let $\boldsymbol{\varphi}^i(Z_t) = (\boldsymbol{u}^i(Z_t), \boldsymbol{w}^i(Z_t))$. A pair of closed-loop strategies, $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a(\,\cdot\,), \boldsymbol{\varphi}^b(\,\cdot\,))$, is feasible if the following conditions hold:

(i) For any given $z = (x, k^a, k^b) \in \mathbb{Z}$, there exists a unique pair of $\mathcal{F}_t$-adapted, nondecreasing, left-continuous processes, (K^a, K^b), satisfying (20)-(22);

(ii) For any $z \in \mathbb{Z}$, $\mathbb{E}^z\left[|K_T^i|^n\right] < \infty$ for any finite $T > 0$ and $n > 1$, and

$$\mathbb{E}^z\left[\int_0^\infty e^{-rt}|\pi^i(X_t, K_t^a, K_t^b)|dt\right] < \infty. \tag{23}$$

Let $\mathcal{S}$ denote the set of all feasible (closed-loop) strategies.

For any $z \in \mathbb{Z}$ and any feasible closed-loop strategy pair $\boldsymbol{\varphi} \in \mathcal{S}$, we define firm i's value as:

$$J^i(z; \boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b) = \mathbb{E}_t^z\left[\int_t^\infty e^{-r(s-t)}\Big(\pi^i(X_s, K_s^{a,\boldsymbol{\varphi}}, K_s^{b,\boldsymbol{\varphi}})ds - pdK_s^{i,\boldsymbol{\varphi}}\Big)\right], \tag{24}$$

where $\mathbb{E}_t^z[\,\cdot\,] = \mathbb{E}_t[\,\cdot\,|Z_t = z]$ and $(K^{a,\boldsymbol{\varphi}}, K^{b,\boldsymbol{\varphi}})$ is the associated capital-stock process pair.

Now we are ready to define closed-loop equilibria for our capital accumulation game that involves singular controls.

4.2.2 Closed-loop Equilibria

Definition 3 A strategy pair $\boldsymbol{\varphi}_* = (\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ constitutes a *closed-loop equilibrium* if for any $Z_t = z = (x, k^a, k^b) \in \mathbb{Z}$, the following conditions hold for $J^a(z; \boldsymbol{\varphi}_*)$:

(i) for any $\lambda^a > 0$, the following inequality holds:

$$J^a(x, k^a, k^b; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \geq J^a(x, k^a + \lambda^a, k^b; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) - p\lambda^a; \tag{25}$$

(ii) for any $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$, there exists an (infinitesimal) $\varepsilon > 0$ such that

$$J^a(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \geq \mathbb{E}_t^z\left[\int_t^{t+\varepsilon} e^{-r(s-t)}\Big(\pi^a(Z_s^{\boldsymbol{\varphi}})ds - p\,dK_s^{a,\boldsymbol{\varphi}}\Big) + e^{-r\varepsilon}V_{\mathrm{BP}}^a(Z_{t+\varepsilon}^{\boldsymbol{\varphi}})\right], \tag{26}$$

where $Z_s^{\boldsymbol{\varphi}} = (X_s, K_s^{a,\boldsymbol{\varphi}}, K_s^{b,\boldsymbol{\varphi}})$.

Note that symmetric conditions hold for $J^b(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$. Let $V^a(z) = J^a(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ and $V^b(z) = J^b(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ denote the closed-loop equilibrium value functions for a and b, respectively.

Inequality (25) is firm a's first-order necessary condition in a subgame. To see this, note that the right side of (25) represents firm a's net payoff if it instantaneously increases its capital stock by $\lambda^a > 0$. In equilibrium, firm a cannot increase its value by instantly adjusting its capital stock; hence (25) must hold for all $\lambda^a > 0$ for $J^a(x, k^a, k^b; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ to be firm a's equilibrium value.

Inequality (26) describes the equilibrium condition that there is no profitable one-shot deviation by firm a. To see this, fix a small ε and suppose that firm a deviates from its equilibrium strategy $\boldsymbol{\varphi}_*^a$ to another strategy $\boldsymbol{\varphi}^a$ over an infinitesimal period $[t, t+\varepsilon)$. Then, firm b penalizes firm a by switching from its equilibrium strategy $\boldsymbol{\varphi}_*^b$ to the leader's strategy in the BP equilibrium given in Theorem 1. Firm b's switching of its strategy is feasible and subgame perfect as it observes whether firm a has deviated from its equilibrium strategy $\boldsymbol{\varphi}_*^a$ or not in the $[t, t+\varepsilon)$ period. Immediately after firm b's punishment, firm a plays the follower's strategy in the BP equilibrium (again see Theorem 1). For other applications of the one-shot deviation principle in continuous time, see, for example, Sannikov (2007) for repeated games under imperfect monitoring and Malenko and Tsoy (2025) for dynamic capital structure analysis.

In Appendix A, we verify that the BP zero-profit closed-loop equilibrium, characterized in Theorem 1, satisfies inequalities (25) and (26), and is therefore subgame perfect.

Next, we state a verification theorem for a closed-loop equilibrium in which firms earn positive profits (see Appendix C for a proof.)

Theorem 2 *A strategy pair $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ constitutes a closed-loop equilibrium in which firms earn positive profits if firm value $V^i(z) = J^i(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ for $i = a, b$ is differentiable and satisfies*

$$\frac{\partial V^i(z)}{\partial k^i} \leq p, \quad \textit{for any } z \in \mathbb{Z} \tag{27}$$

and

$$V^i(z) > V^i_{\mathrm{BP}}(z), \quad \textit{for any } z \in \mathbb{Z}. \tag{28}$$

Inequality (27) is equivalent to condition (25), stating that, in a closed-loop equilibrium, firm i's marginal q cannot exceed the marginal cost of acquiring capital, p. In addition, inequality (28) implies condition (26), thereby ensuring the existence of a closed-loop equilibrium with positive profits.

Next, we characterize the closed-loop equilibrium solution in Sections 5 and 6.

5 Closed-loop Equilibrium Solution Characterization

In this section, we first conjecture that a closed-loop equilibrium solution features four regions (subsection 5.1) and then analyze general properties for the solution (subsection 5.2).

5.1 Four-region Solution

At each point (x, k^a, k^b) in the state space, a firm either invests or not. We therefore conjecture that the solution can be characterized by four mutually exclusive regions. First, consider the region where neither firm invests. Let $\mathbb{N}_{ab}$ denote this region:

$$\mathbb{N}_{ab} = \left\{(x, k^a, k^b) \in \mathbb{R}^3_+ : x < \mathcal{X}(k^a, k^b)\right\}, \tag{29}$$

where $\mathcal{X}(k^a, k^b)$ is an endogenously determined equilibrium investment boundary.[5] Suppose demand X_t is low relative to the industry capital stock in that $X_t < \mathcal{X}(K^a_t, K^b_t)$. Then, it is optimal for firms to wait and invest only when $\{X_s\}_{s>t}$ rises in the future, exceeding the threshold $\mathcal{X}(K^a_t, K^b_t)$. Now consider the region where firm i invests. Let $\mathbb{I}_i$ denote this region:

$$\mathbb{I}_i = \left\{(x, k^a, k^b) \in \mathbb{R}^3_+ : x \geq \mathcal{X}(k^a, k^b),\, x \geq \mathcal{X}(k^i, k^i)\right\}. \tag{30}$$

Note that $\mathbb{I}_i$ requires two conditions. The first condition $X_t \geq \mathcal{X}(K^a_t, K^b_t)$ implies that at least one firm invests, as market demand X_t is larger than the investment threshold $\mathcal{X}(K^a_t, K^b_t)$. The second condition $X_t \geq \mathcal{X}(K^i_t, K^i_t)$ implies that firm i must invest at t.[6]

In summary, we conjecture and later verify that the solution features four mutually exclusive regions: (1) $\mathbb{N}_{ab}$ where neither firm invests, (2) $\mathbb{I}_a \setminus \mathbb{I}_b$ where only firm a invests, (3) $\mathbb{I}_b \setminus \mathbb{I}_a$ where only firm b invests, and (4) $\mathbb{I}_a \cap \mathbb{I}_b$, the intersection of $\mathbb{I}_a$ and $\mathbb{I}_b$, where both firms invest.[7] Projecting the three-dimensional equilibrium-solution regions described above onto the $(K^a_t, K^b_t) = (k^a, k^b)$ plane for a given demand shock $X_t = x > 0$, we obtain the four-region solution described in Figure 1. Strictly above the investment boundary (the ACE curve) is

[5] We only consider the investment boundary $\mathcal{X}(k^a, k^b)$ that is continuous and increasing in both k^a and k^b, and symmetric, i.e., $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$.

[6] This result follows from two equilibrium properties: (1) it is always the smaller firm that invests in equilibrium and (2) the investment boundary $\mathcal{X}(K^a_t, K^b_t)$ increases in a firm's capital stock.

[7] As we show below in equilibrium firm i undertakes lumpy investment only at time 0, when (x_0, k^a_0, k^b_0) lies in the interior of $\mathbb{I}_i$. At $t > 0$, firms either engage in continuous investment or take no action, so that the evolution of (X_t, K^a_t, K^b_t) always lies within the closure of $\mathbb{N}_{ab}$ for all $t > 0$.

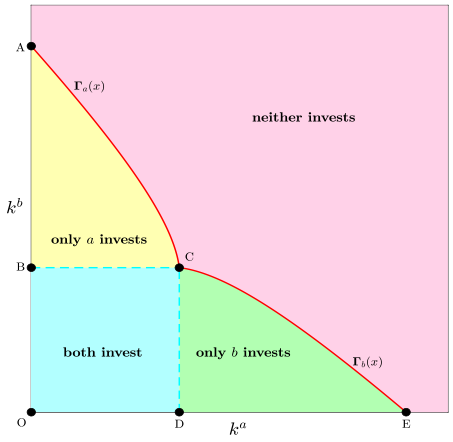


Figure 1: EQUILIBRIUM SOLUTION REGIONS ASSOCIATED WITH AN INVESTMENT BOUNDARY ACE. Inaction region $\mathbf{N}_{ab}(x)$ is given in (31) and firm i's investment boundary is $\mathbf{\Gamma}_i(x) := \{(k^a, k^b) \in \mathbb{R}^2_+ : \mathcal{X}(k^a, k^b) = x, k^i \leq k^{-i}\}$.

the inaction region, which is the projection of $\mathbb{N}_{ab}$, where neither firm invests:

$$\mathbf{N}_{ab}(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : x < \mathcal{X}(k^a, k^b)\}. \tag{31}$$

Elsewhere in the first quadrant, we have the other three regions: a.) the ABC region where only firm a invests; b.) the CDE region where only firm b invests; and c.) the OBCD (square) region where both firms invest.

Note that when market demand is high so that the industry expands, the smaller firm always invests. The intuition is that the smaller firm benefits more from investing due to the diminishing marginal return.[8] The option for a firm to condition its investment on its competitor's size makes a closed-loop equilibrium subgame perfect.

5.2 Characterizing General Properties of the Solution

Below, we first characterize firm i's marginal q and its competitor's cross marginal q in the region when firm i invests in subsection 5.2.1, and then solve the partial differential equation in closed form for firm i's value subject to a given boundary in subsection 5.2.2.

[8]It is only in the square region where both firms invest in equilibrium. This is because the marginal product is high enough even for the larger firm in this region.

5.2.1 Firm i's Marginal q and Firm $-i$'s Cross Marginal q when Firm i Invests

When firm i invests, its marginal q must equal the marginal cost of purchasing capital p. This is the standard first-order condition as in a classic q theory of investment for a single firm (Abel and Eberly, 1996). Also note that its competitor's value function cannot change when firm i invests. That is, the sensitivity of $V^{-i}(z)$ with respect to k^i, which we refer to as firm $-i$'s *cross marginal* q, must be zero when firm i invests. This is because firm i can instantly increase its capital stock when investing at t and firm $-i$'s value must satisfy the value-matching condition. In summary, we have:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = 0 \ \text{ in ABC where firm } a \text{ invests,} \tag{32}$$

$$\frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^a(z)}{\partial k^b} = 0 \ \text{ in CDE where firm } b \text{ invests,} \tag{33}$$

$$\frac{\partial V^a(z)}{\partial k^a} = \frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = \frac{\partial V^a(z)}{\partial k^b} = 0 \ \text{ in OBCD where both invest.} \tag{34}$$

5.2.2 Characterizing Firm i's Value, $V^i(z)$, for a Given Investment Boundary

Having characterized the general properties in the regions where at least one firm invests, we now turn to the region $\mathbf{N}_{ab}(x)$ where neither firm invests: $x < \mathcal{X}(k^a, k^b)$. Firm i's value, $V^i(z)$, satisfies the following HJB equation as in standard real-option settings:

$$rV^i(z) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V^i(z)}{\partial x^2} + \mu x \frac{\partial V^i(z)}{\partial x} + \pi^i(z), \tag{35}$$

where $\pi^i(z)$ is the profit function given in (4). The general solution for (35) takes the form:

$$V^i(z) = \Psi^i(z) + H^i(k^a, k^b) x^\beta \,, \tag{36}$$

where $\Psi^i(z)$ is the value of firm i's assets in place given by

$$\Psi^i(z) = \mathbb{E}_t^z \left[\int_t^\infty e^{-r(s-t)} \pi^i(X_s, k^a, k^b) ds \right] = \left(\frac{x}{r-\mu} - \frac{k^a + k^b + c}{r} \right) k^i \tag{37}$$

and $H^i(k^a, k^b)$ measures firm i's option value under imperfect competition, which we determine below. Note that $\Psi^i(z)$ is the special solution for (35) and would be firm i's value under the counterfactual that the industry has no growth from t onward: $dK_s^a = dK_s^b = 0$ for all $s \geq t$. That is, $\Psi^i(z)$ measures the value of firm i's assets in place. Next, we solve for $H^a(k^a, k^b)$.

Equations (32) and (33) imply the following boundary conditions:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{when} \ \ \mathcal{X}(k^a, k^b) = x \ \ \text{and} \ \ \ k^a \leq k^b \,, \tag{38}$$

$$\frac{\partial V^a(z)}{\partial k^b} = 0 \quad \text{when} \;\; \mathcal{X}(k^a, k^b) = x \;\; \text{and} \;\; k^a \geq k^b \,. \tag{39}$$

Substituting (36) into (38) and (39), and then simplifying the algebra, we obtain the following result. First, if firm a is the small one: $k^a \leq k^b$, we have

$$\frac{\partial H^a(k^a, k^b)}{\partial k^a} = \left(\frac{2k^a + k^b + rp + c}{r} - \frac{\mathcal{X}(k^a, k^b)}{r - \mu} \right) \mathcal{X}(k^a, k^b)^{-\beta}. \tag{40}$$

Second, if firm a is the larger one: $k^a \geq k^b$, we have

$$\frac{\partial H^a(k^a, k^b)}{\partial k^b} = \frac{k^a}{r} \mathcal{X}(k^a, k^b)^{-\beta} \,. \tag{41}$$

In addition, we have the boundary condition: $\lim_{k \to \infty} H^a(k, k) = 0$. This is because as the industry becomes very large, neither firm will ever invest and the option value of investing is zero. Using (40), (41), and $\lim_{k \to \infty} H^a(k, k) = 0$, and integrating along the 45° line in the (k^a, k^b) plane, we obtain the following closed-form solution on the 45° line:

$$H^a(k, k) = -\int_k^\infty \left(\frac{4\alpha + rp + c}{r} - \frac{\mathcal{X}(\alpha, \alpha)}{r - \mu} \right) \mathcal{X}(\alpha, \alpha)^{-\beta} d\alpha. \tag{42}$$

Finally, using (42) for $H^a(k, k)$ to integrate (40) and (41), we obtain the following expressions for $H^a(k^a, k^b)$, firm a's option value off the 45° line:

$$\begin{aligned} H^a(k^a, k^b) &= H^a(k^b, k^b) - \int_{k^a}^{k^b} \left(\frac{2\alpha + k^b + rp + c}{r} - \frac{\mathcal{X}(\alpha, k^b)}{r - \mu} \right) \mathcal{X}(\alpha, k^b)^{-\beta} d\alpha, \quad k^a \leq k^b, \\ H^a(k^a, k^b) &= H^a(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r} \mathcal{X}(k^a, \alpha)^{-\beta} d\alpha, \quad k^a \geq k^b \,. \end{aligned} \tag{43}$$

Summary. We have thus far characterized the following properties of a closed-loop equilibrium, given an investment boundary $\mathcal{X}(k^a, k^b)$. Firm i's value, $V^i(z)$, satisfies (36) subject to (38)-(39) in the inaction region where neither firm invests, (32)-(34) in the other three regions where at least one firm invests, and the inequalities (27)-(28) in all four regions. To fully characterize the closed-loop equilibrium, we still have to determine the equilibrium investment boundary $\mathcal{X}(k^a, k^b)$, to which we turn next.

6 Closed-loop Equilibria with Piecewise Linear Triggers

To complete our analysis in the preceding section and obtain a set of closed-loop equilibria, we first conjecture and verify that the investment boundary is piecewise linear in subsection 6.1, and then derive closed-form solutions for these closed-loop equilibria in subsection 6.2.

6.1 Closed-loop Equilibria with Piecewise Linear Thresholds

Inspired by the piecewise linear boundary given in (15) for the open-loop equilibrium (Grenadier, 2002; Back and Paulsen, 2009), we conjecture and then verify that there exists a set of closed-loop equilibria, also characterized by piecewise linear investment boundaries, but with slopes and intercepts different from those in the open-loop equilibrium. Specifically, we conjecture and verify that the boundary $\mathcal{X}(k^a, k^b)$ takes the following symmetric form:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho(\theta_2 k^a + \theta_1 k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(\theta_1 k^a + \theta_2 k^b + rp + c), & \text{if } k^a \geq k^b, \end{cases} \tag{44}$$

where (θ_1, θ_2) is a pair of positive real numbers from a set Θ to be determined. Note the symmetry between the $k^a \leq k^b$ and $k^b \leq k^a$ subspaces: θ_2 is the coefficient multiplying the capital stock of the smaller firm and θ_1 is the coefficient multiplying the capital stock of the larger firm. We solve the associated firm values, $V^a(z)$ and $V^b(z)$, in closed form by substituting the conjectured boundary (44) into (42)-(43).

Using (42)-(44) to simplify (27) and (28) for firm a (and by symmetry conducting the same exercise for firm b), we obtain the following conditions for (θ_1, θ_2) appearing in $\mathcal{X}(k^a, k^b)$.

Condition 1 *The pair (θ_1, θ_2), with $\theta_2 \geq \theta_1 > 0$, satisfies the following inequality bounds:*

$$\underline{g}(\theta_2/\theta_1) \leq \theta_1 + \theta_2 \leq \overline{g}(\theta_2/\theta_1)\,, \tag{45}$$

where the upper and lower bound functions, $\overline{g}(\,\cdot\,)$ and $\underline{g}(\,\cdot\,)$, are respectively given by:

$$\overline{g}(\eta) = 3 - 1/\eta \quad \textit{and} \tag{46}$$

$$\underline{g}(\eta) = g\left(1 + \frac{(1+\eta)^\beta - 1}{\eta}\right). \tag{47}$$

Here, $g(\,\cdot\,)$ is the inverse function of $\check{g}(s) = s^\beta + \frac{\beta}{\beta-2}(s-2)$ for $s \geq 2$.

Let Θ denote the set of pairs (θ_1, θ_2) satisfying **Condition 1.**

In Figure 2, the shaded area, bounded by the solid blue line $\overline{g}(\,\cdot\,)$ given in (46) and the dashed magenta line $\underline{g}(\,\cdot\,)$ given in (47), defines a closed set Θ for (θ_1, θ_2) with $\theta_2 \geq \theta_1 > 0$. Any pair $(\theta_1, \theta_2) \in \Theta$ corresponds to a closed-loop equilibrium. Next, we characterize the properties of the set of closed-loop equilibria with piecewise-linear triggers.

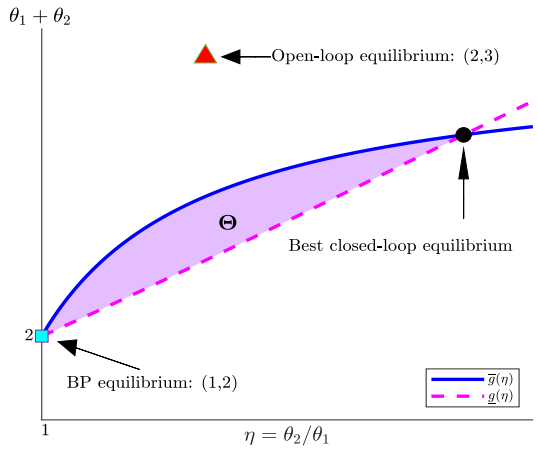


Figure 2: Characterizing the set of all closed-loop equilibria with piecewise linear triggers using the admissible set Θ for (θ_1,θ_2). The area, bounded by the solid blue line $\overline{g}(\,\cdot\,)$ given in (46) and the dashed magenta line $\underline{g}(\,\cdot\,)$ given in (47), defines a set Θ for (θ_1,θ_2) with $\theta_2\geq 1\geq\theta_1>0$. A pair $(\theta_1,\theta_2)\in\Theta$ (including those on the boundary) corresponds to a closed-loop equilibrium. The BP equilibrium is the one with the lowest firm value, corresponding to the cyan square: $\theta_1=\theta_2=1$. The closed-loop equilibrium with the highest firm value, referred to as the best equilibrium, corresponds to the black dot. The open-loop equilibrium corresponds to the red triangle, which is outside of the set Θ.

Lemma 1 *The equation $\overline{g}(\eta)-\underline{g}(\eta)=0$ has a unique root η^* in the domain $\eta>1$. Moreover, $\overline{g}(\eta)>\underline{g}(\eta)$ for any $\eta\in(1,\eta^*)$, and $\overline{g}(\eta)<\underline{g}(\eta)$ for any $\eta>\eta^*$.*

Under **Condition 1**, *$\theta_1+\theta_2$ attains its maximum if and only if $(\theta_1,\theta_2)=(\theta_1^*,\theta_2^*)$, and attains its minimum if and only if $(\theta_1,\theta_2)=(1,1)$, where*

$$\theta_1^*=\frac{3-1/\eta^*}{1+\eta^*}\qquad\text{and}\qquad\theta_2^*=\frac{3\eta^*-1}{1+\eta^*}. \tag{48}$$

Furthermore, for any $(\theta_1,\theta_2)\in\Theta\setminus\{(1,1)\}$, we have $\theta_2>1>\theta_1>0$ and $2<\theta_1+\theta_2<3$.

The equilibrium in the set Θ with the highest firm value, which we also refer to as the best closed-loop equilibrium (see the black dot in Figure 2), is uniquely characterized by (θ_1^*,θ_2^*) given in (48).

The BP equilibrium is the one with the lowest firm value, corresponding to the cyan square in Figure 2: $\theta_1=\theta_2=1$. Note that the open-loop equilibrium corresponds to the red triangle, which is outside of the set Θ. This is because the open-loop equilibrium is not subgame perfect, as there exists a set of (K_t^a,K_t^b) at time t where a firm's marginal q exceeds the purchase cost of capital p, violating Condition (i) in Definition 3 for a closed-loop equilibrium.

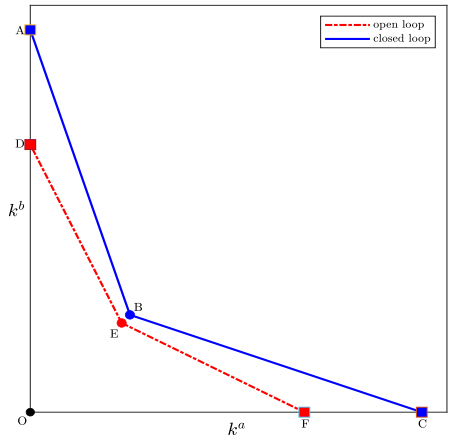


Figure 3: PIECEWISE LINEAR INVESTMENT THRESHOLDS IN DIFFERENT EQUILIBRIA. Fixing time t, $X_t = x$ and $(K_t^a, K_t^b) = (k^a, k^b)$, this figure plots investment thresholds at t in the open-loop equilibrium (DEF), and a (representative) closed-loop equilibrium (ABC) on the (k^a, k^b) plane.

In Figure 3, fixing time t, $X_t = x$, and $(K_t^a, K_t^b) = (k^a, k^b)$, we illustrate how the investment threshold in a closed-loop equilibrium differs from that in the open-loop equilibrium on the (k^a, k^b) plane. By symmetry, we focus on the upper half of the first quadrant, where firm a is smaller than firm b, i.e., $k^a \leq k^b$.

First, because competition is fiercer in a closed-loop equilibrium (due to subgame perfection) than in the open-loop equilibrium, the investment threshold in the former lies to the right of that in the latter.

Second, in a closed-loop equilibrium associated with (θ_1, θ_2), the investment threshold given in (44) is piecewise linear, with slope $-\frac{\theta_2}{\theta_1}$, whose absolute value exceeds one (the AB line segment). Using the expression for the investment threshold in a closed-loop equilibrium, $x = \mathcal{X}(k^a, k^b)$, where $\mathcal{X}(\cdot, \cdot)$ is given in (44), we obtain the coordinates of point A on the vertical axis: $(k^a, k^b) = \left(0, \frac{x/\rho - (rp+c)}{\theta_1}\right)$, and the coordinates of point B on the 45° line: $k^a = k^b = \frac{x/\rho - (rp+c)}{\theta_1 + \theta_2}$.

Third, in the open-loop equilibrium, the investment threshold (the DE segment) has slope -2, as $\theta_1 = 1$ and $\theta_2 = 2$. Using the expression for the open-loop investment threshold, $x = \mathcal{X}^{\text{open}}(k^a, k^b)$, yields the coordinates of point E on the 45° line: $k^a = k^b = \frac{x/\rho - (rp+c)}{3}$. Because $\theta_1 + \theta_2 < 3$ (see Condition 1), point E lies to the southwest of point B. This property again

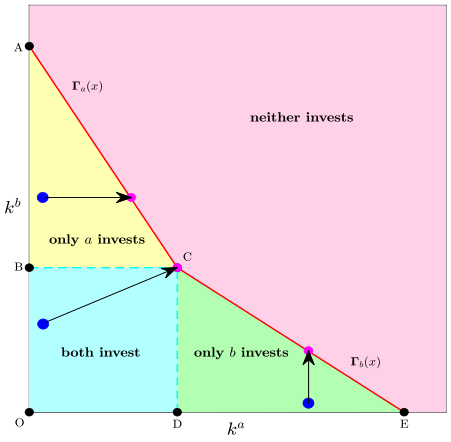


Figure 4: EQUILIBRIUM SOLUTION REGIONS WHEN THE INVESTMENT BOUNDARY IS PIECEWISE LINEAR.

reflects the prediction that firms compete more fiercely under the closed-loop equilibrium than under the open-loop equilibrium, as a lower value of $\theta_1 + \theta_2$ implies more intense competition.

6.2 Closed-loop Equilibrium Solution

Next, we summarize the closed-loop equilibrium strategy (see Appendix C for a proof.)

Theorem 3 *Under* **Condition 1**, *for the investment boundary* $\mathcal{X}(k^a, k^b)$ *given in* (44), *a strategy pair* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) = \{(\boldsymbol{u}^a_*, \boldsymbol{w}^a_*), (\boldsymbol{u}^b_*, \boldsymbol{w}^b_*)\} \in \mathcal{S}$ *constitutes a closed-loop equilibrium if*

$$\boldsymbol{u}^i_*(z) = \left[\frac{\mathbf{1}_{k^i<k^{-i}}}{\rho\theta_2} + \frac{\mathbf{1}_{k^i=k^{-i}}}{\rho(\theta_1+\theta_2)}\right]\mathbf{1}_{x\geq\mathcal{X}(k^a,k^b)}, \tag{49}$$

$$\boldsymbol{w}^i_*(z) = \inf\left\{\delta \geq 0 : \mathcal{X}\big(k^i + \delta, \max\{k^{-i}, \mathcal{K}(x)\}\big) \geq x\right\}, \tag{50}$$

for all $z = (x, k^a, k^b) \in \mathbb{Z}$, *where* $\mathbf{1}_{\{\cdot\}}$ *is an indicator function and* $\mathcal{K}(x)$ *is given by*

$$\mathcal{K}(x) = \frac{1}{\theta_1+\theta_2}\left[\frac{x}{\rho} - (rp + c)\right]. \tag{51}$$

Next, we use Figure 4 to illustrate the industry equilibrium dynamics in a given closed-loop equilibrium. Depending on where the state vector Z_t is, there are four scenarios starting at t.

First, if the economy is located in the region northeast of the ACE boundary (excluding the boundary itself), neither firm invests. Second, if the initial state lies within the OBCD square region (including its boundaries), both firms invest so that point C in Figure 4 is instantly reached: $K^a_t = K^b_t = \mathcal{K}(M_t)$. Thereafter, the firms invest along the 45° line, with

the same increments proportional to dM_t:

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)}\,. \tag{52}$$

The sum $\theta_1 + \theta_2$ dictates the rate at which firms invest. Equation (52) shows that the larger the sum the slower firms invest, and the more option value they preserve (see Section 7 for details). The pair (θ_1^*, θ_2^*), which attains the highest value of $\theta_1 + \theta_2$ in the set Θ, is associated with the closed-loop equilibrium with the highest firm value, referred to as the best closed-loop equilibrium (see the black dot in Figure 2).

Third, if the initial state lies within the ABC triangle region, firm a increases its capital to the AC boundary in Figure 4, while firm b remains inactive. Afterwards, firm a's investment is proportional to dM_t, whereas firm b does not invest:

$$dK_t^a = \frac{dM_t}{\rho\theta_2} \;\text{ and }\; dK_t^b = 0. \tag{53}$$

This process continues until point C in Figure 4 is reached. Once point C is reached, both firms invest along the 45° line as described in the above scenario. Finally, by symmetry, the same analysis applies if the initial state lies within the CDE triangle region. In this case, we simply switch the strategies of firms a and b.

These four scenarios are endogenously connected over time through the stochastically evolving ACE boundary: $X_t = \mathcal{X}(K_t^a, K_t^b)$.

Below, we report the associated equilibrium firm values in a given closed-loop equilibrium in Proposition 3 (see Appendix C for a proof.)

Proposition 3 *In an equilibrium characterized in Theorem 3, we calculate $V^i(z)$ for $i = a, b$ as follows:*

1. *in the region where $x \leq \mathcal{X}(k^a, k^b)$, $V^i(z)$ is given by* (36)*;*

2. *in the region where $k^a \leq \mathcal{K}(x)$ and $k^b \leq \mathcal{K}(x)$, $V^i(z) = V^i(x, \mathcal{K}(x), \mathcal{K}(x)) - p(\mathcal{K}(x) - k^i)$;*

3. *in the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^b > \mathcal{K}(x)$, $V^a(z) = V^a(x, \widehat{k}^a, k^b) - p(\widehat{k}^a - k^a)$ and $V^b(z) = V^b(x, \widehat{k}^a, k^b)$, where $V^i(x, \widehat{k}^a, k^b)$ is given in* (36) *and $\widehat{k}^a$ satisfies*

$$\mathcal{X}(\widehat{k}^a, k^b) = x. \tag{54}$$

By symmetry, we can calculate $V^i(z)$ in the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^a > \mathcal{K}(x)$ by switching the roles of firms a and b.

Next, we establish a key result for cross marginal q.

Proposition 4 *In the closed-loop equilibrium characterized in Theorem 3, a firm's cross marginal q is weakly negative. Specifically, for each firm $i = a, b$, the marginal effect of the rival firm's capital stock on firm i's value is nonpositive for all $z \in \mathbb{Z}$; that is,*

$$\frac{\partial V^i(z)}{\partial k^{-i}} \leq 0. \tag{55}$$

See Appendix C for a proof. This result is intuitive. A higher level of rival capital, K^{-i}, raises total industry capital and output, thereby lowering the output price and making firm i weakly worse off.

7 A Duopolistic q Theory of Investment

In this section, we develop a duopolistic q theory of investment by using the best closed-loop equilibrium solution for the real-option game analyzed earlier. We focus on a symmetric setting where the two firms have the same initial capital stock: $k_0^a = k_0^b$.

7.1 Industry Investment Dynamics and Valuation

Let $V^{\text{duo}}_{\text{closed}}(z) = V^a(z) + V^b(z)$ denote the value of a duopoly industry in a closed-loop equilibrium where $z := (x, k, k)$. The industry value in the Back–Paulsen solution, $V^{\text{duo}}_{\text{BP}}(z)$, is associated with the strategy pair $(\theta_1, \theta_2) = (1, 1)$. The industry value in the best closed-loop equilibrium, $V^{\text{duo}}_{\text{best}}(z)$, is associated with the strategy pair (θ_1^*, θ_2^*) given in (48) and is the highest among all closed-loop equilibria. As a reference, we also report the industry value in the open-loop equilibrium, denoted by $V^{\text{duo}}_{\text{open}}(z)$. Below we summarize the rank order of these industry values and industry investment rates.

Proposition 5 *For any $(\theta_1, \theta_2) \in \Theta$ satisfying Condition 1, the following inequalities hold:*

$$V^{\text{mon}}(x, 2k) > V^{\text{duo}}_{\text{open}}(x, k, k) > V^{\text{duo}}_{\text{best}}(x, k, k) \geq V^{\text{duo}}_{\text{closed}}(x, k, k) \geq V^{\text{duo}}_{\text{BP}}(x, k, k)\,, \tag{56}$$

where V^{mon} denotes the monopoly value given in Appendix B. Finally, the industry value in a closed-loop equilibrium, $V^{\text{duo}}_{\text{closed}}(x,k,k)$, is a strictly increasing function of $\theta_1+\theta_2$.

Consistent with our intuition, as competition is fiercest in the BP closed-loop equilibrium where $(\theta_1,\theta_2)=(1,1)$, the industry value $V^{\text{duo}}_{\text{BP}}(x,k,k)$ is the lowest. The industry value in the open-loop equilibrium is larger than in all closed-loop equilibria including the best closed-loop equilibrium associated with (θ_1^*,θ_2^*). This is because, by definition, unlike in a closed-loop equilibrium, a firm in the open-loop equilibrium has to commit to its strategy at $t=0$, which creates value *ex ante*. Specifically, a firm cannot condition its strategy on its competitor's future capital stock, which significantly mitigates value destruction arising from future competition. Obviously, a monopoly industry has the highest value (given the same total industry capital stock $2k$) as it faces no competition.

As discussed earlier, the industry value, $V^{\text{duo}}_{\text{closed}}(x,k,k)$, increases with $\theta_1+\theta_2$, which dictates the rate at which the industry invests: the larger this sum, the slower the industry invests.

Next, we compare a duopoly industry's investment rate to a monopoly's investment rate. Starting at $t=0$ with no capital, $k_0^a=k_0^b=0$, in a closed-loop equilibrium associated with a pair (θ_1,θ_2), a duopoly industry's investment rate is proportional to a monopoly's investment rate with a coefficient of $\frac{4}{\theta_1+\theta_2}$, which lies between 4/3 and 2 (recall that $\theta_1+\theta_2\in[2,3)$, as discussed when introducing Condition 1).

Duopolistic competition lowers the industry value at $t=0$. As a duopoly's investment rate decreases with the sum $\theta_1+\theta_2$, the industry value increases with this sum.[9] The closer $\theta_1+\theta_2$ is to the lower bound of 2, the faster the industry invests and the lower the industry value. In the BP closed-loop equilibrium where $\theta_1=\theta_2=1$, firms compete most fiercely. Indeed, the industry invests twice as fast as a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}}=2$, thus completely eliminating the option value of waiting so that $V^{\text{duo}}_{\text{BP}}(x,0,0)=0$. Next, we summarize these results.

Proposition 6 *As a fraction of the monopoly's investment rate, the duopoly industry investment rate in a closed-loop equilibrium is constant and decreases with $\theta_1+\theta_2$.*[10]

$$\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}}=\frac{4}{\theta_1+\theta_2}\in\left(\frac{4}{3},2\right], \tag{57}$$

[9] The coefficient in (58), $4\left[\frac{2}{\theta_1+\theta_2}-\left(\frac{2}{\theta_1+\theta_2}\right)^2\right]$, increases with $\theta_1+\theta_2$ as $\theta_1+\theta_2\in[2,3)$.

[10] This is conditional on investing, i.e., $M_t=X_t>\rho(rp+c)$ and $dM_t>0$ for $t>0$.

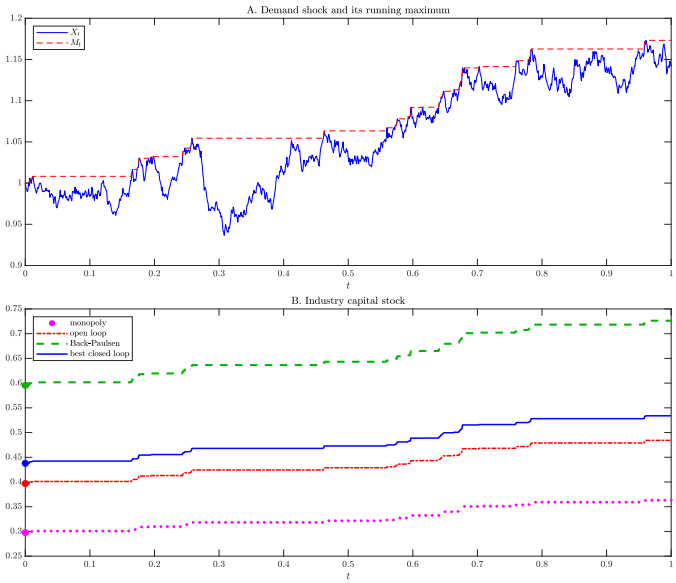


Figure 5: INDUSTRY INVESTMENT DYNAMICS. Panel A plots the running maximum M_t process (dashed red line) for a simulated demand shock path X_t (solid blue line). Panel B shows that at all t, the industry capital stock is larger under the best closed-loop equilibrium (solid blue line) than under the open-loop equilibrium (dash-dotted red line), attains the lowest level under monopoly (dotted magenta line) and the largest level in the BP equilibrium (dashed green line). Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

where $K_t^{\text{duo}} = K_t^a + K_t^b$. *As a fraction of the monopoly value, the duopoly industry value in a closed-loop equilibrium is constant and increases with* $\theta_1 + \theta_2$.[11]

$$\frac{V^{\text{duo}}_{\text{closed}}(x,0,0)}{V^{\text{mon}}(x,0)} = 4\left[\frac{2}{\theta_1+\theta_2} - \left(\frac{2}{\theta_1+\theta_2}\right)^2\right] < \frac{8}{9}\,. \tag{58}$$

Finally, we compare closed-loop equilibrium solutions with the open-loop equilibrium solution. In the open-loop equilibrium, a duopoly industry invests at a rate that is 1/3 faster than a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = \frac{4}{3}$, which causes the industry value to be 1/9 lower than that of a monopoly: $\frac{V^{\text{duo}}_{\text{open}}(x,0,0)}{V^{\text{mon}}(x,0)} = \frac{8}{9}$. Consistent with our intuition, because a firm can respond to its competitor's investment strategy in a closed-loop equilibrium, but cannot in the open-loop equilibrium, the industry invests faster and hence its value is lower in all closed-loop equilibria than in the open-loop equilibrium.

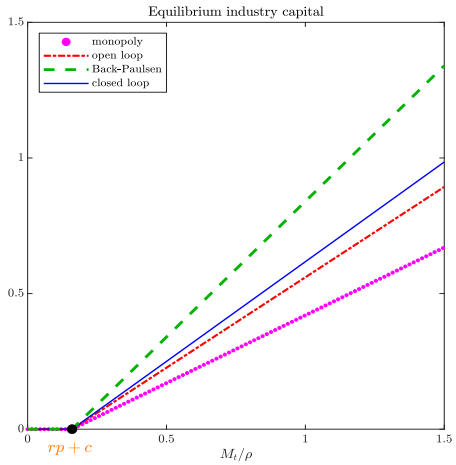


Figure 6: Comparing Equilibrium Industry Capital Processes. We plot industry capital stock as a function of M_t/ρ. In all cases, industry investment starts when M_t/ρ exceeds $rp + c$ (the black dot) but responds to M_t/ρ at different rates. Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

7.2 A Quantitative Illustration

In this subsection, we show that a duopoly industry's investment dynamics differ substantially across different equilibria. In panel A of Figure 5, we simulate a demand shock $\{X_t\}$ path and report the associated running maximum $\{M_t\}$, which is noticeably smoother than $\{X_t\}$. Panel B compares the industry capital stock processes in three equilibria: the BP equilibrium, the best closed-loop equilibrium, and the open-loop equilibrium.

Among these three equilibria, in the BP equilibrium, a duopoly industry starts with the highest level of capital stock and increases its K_t^{duo} at the fastest rate (see the dashed green line in panel B of Figure 5). This is because firms compete most fiercely in this equilibrium.

In the open-loop equilibrium, a duopoly industry starts with the lowest level of capital stock and increases its capital stock at the slowest rate (see the dash-dotted red line in panel B). This is because a firm does not respond to its competitor's investment strategy at any $t > 0$, thus preserving maximal option value in a duopoly setting.

In the best closed-loop equilibrium, where the industry attains the highest value among all closed-loop equilibria, a duopoly industry starts with a level of capital stock that lies between those in the BP and open-loop equilibria and increases its capital stock at a rate that also lies

[11] See Corollary 2 in Appendix C.6 for details.

between those in the BP and open-loop equilibria (see the solid blue line in panel B).

The intuition is as follows. On the one hand, firms preserve some option value of waiting by setting $\theta_1^* + \theta_2^* = 2.72$ in our numerical example, which is more than 36% larger than $\theta_1 + \theta_2 = 2$ in the BP equilibrium. On the other hand, firms must choose strategies that are subgame perfect in the best closed-loop equilibrium. This important equilibrium restriction induces more aggressive competition than in the open-loop equilibrium, thereby eroding some of the option value. (Recall that $\theta_1 + \theta_2 = 3$ in the open-loop equilibrium, and that a larger sum implies slower industry investment and greater preservation of option value.)

Finally, a monopolist starts with a lower level of capital stock and invests less than in the open-loop equilibrium, as the absence of competition allows it to preserve the maximal option value.

Figure 6 illustrates industry dynamics as a function of M_t/ρ. In all duopoly equilibria, the industry (starting with no capital) remains dormant until M_t/ρ reaches $rp + c = 0.16$ in our numerical example (the black dot), and the industry then grows at different rates (slopes) in different equilibria.

A duopoly industry invests at the fastest rate in the BP equilibrium with a slope of 1 in our numerical example. The investment rate in the open-loop equilibrium is the slowest among all equilibria – it is only 2/3 of the investment rate in the BP equilibrium. In the best closed-loop equilibrium, the industry invests at a rate of $2/(\theta_1^* + \theta_2^*) = 0.74$ in our numerical example, which is 26% lower than the investment rate in the BP equilibrium. That is, in the best closed-loop equilibrium, the industry invests much slower and attains a substantially higher value than in the BP equilibrium. Finally, a monopoly invests at a rate that is slower than that in any duopoly equilibria: it is only 1/2 of the investment rate in the BP equilibrium.

Finally, in Figure 7 we show that the effects of volatility σ on investment dynamics (panel A) and industry value (panel B) are quantitatively large, even after we control for the standard option value of waiting for a monopoly. To control for the standard volatility effect, we report duopoly industry investment rates as a fraction of the monopoly's investment rate: In the BP closed-loop equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 2$ and in the open-loop equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 4/3$, both of which are independent of volatility σ. Figure 7 plots $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ (panel A) and $V^{\text{duo}}(x, 0, 0)/V^{\text{mon}}(x, 0)$ (panel B) for the best closed-loop equilibrium together with these two

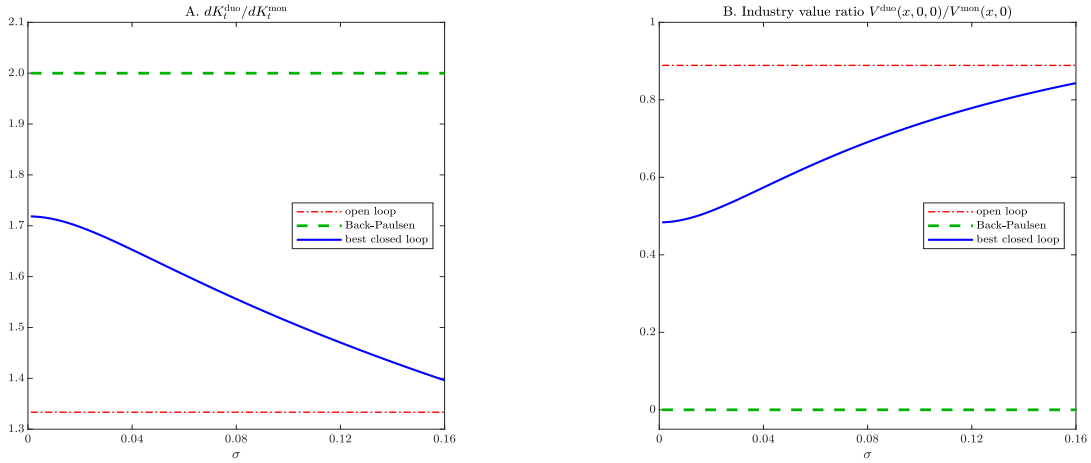


Figure 7: Industry investment dynamics and value. Panel A plots a duopoly industry investment as a fraction of a monopoly's investment and panel B plots the duopoly industry value as a fraction of the monopoly value for the open-loop equilibrium (dash-dotted red line), the BP closed-loop equilibrium (dashed green line), and the best closed-loop equilibrium (solid blue line). Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

special equilibria. As we increase volatility from around zero to 0.16, in our numerical example for the best closed-loop equilibrium, $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ decreases by 19% from 1.72 to 1.39, causing the duopoly industry value as a fraction of the monopoly value, $V^{\text{duo}}(x,0,0)/V^{\text{mon}}(x,0)$, to increase by 74% from 0.48 to 0.84. This monotonicity is consistent with the intuition that the industry as a whole competes less aggressively when demand uncertainty is higher as the industry's option value of waiting is larger.

8 Closed-loop Equilibria with Nonlinear Boundaries

So far, we have analyzed closed-loop equilibria with piecewise linear investment thresholds. In this section, we show that there also exist closed-loop equilibria with nonlinear investment thresholds. Unlike in a closed-loop equilibrium with a piecewise linear investment threshold, firm i's marginal q equals the marginal cost of purchasing capital, p, in that

$$\left.\frac{\partial V^i(x,k^a,k^b)}{\partial k^i}\right|_{x=\mathcal{X}(k^a,k^b)} = p \tag{59}$$

holds not only on the segment of the investment threshold $\mathcal{X}(k^a,k^b)$ where $k^i < k^{-i}$, but also on the other segment of the investment threshold $\mathcal{X}(k^a,k^b)$ where $k^i > k^{-i}$.

We conjecture and verify that the nonlinear investment boundary takes the form:

$$\mathcal{X}(k^a,k^b):=\begin{cases}\rho\big(\mathcal{G}(k^a,k^b)+rp+c\big), & \text{if } k^a\leq k^b,\\ \rho\big(\mathcal{G}(k^b,k^a)+rp+c\big), & \text{if } k^a\geq k^b\,,\end{cases} \tag{60}$$

where $\mathcal{G}(k^a,k^b)$ is a nonlinear function to be determined. Appendix D provides Condition A.3 under which the following four results hold in a closed-loop equilibrium with nonlinear thresholds: (i) a firm's marginal q cannot exceed the capital purchase cost p, as given in (27); (ii) firms earn positive profits so that (28) holds; (iii) a firm's cross marginal q is non-positive so that (55) holds; and (iv) a firm's marginal q equals p on the investment boundary $\mathcal{X}(k^a,k^b)$, as given in (59).

Theorem 4 *Under Condition A.3, with $\mathcal{X}(k^a,k^b)$ given in* (60)*, a strategy pair $(\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)=\{(\boldsymbol{u}^a_*,\boldsymbol{w}^a_*),(\boldsymbol{u}^b_*,\boldsymbol{w}^b_*)\}\in\mathcal{S}$ constitutes a closed-loop equilibrium if $\min(k^a_0,k^b_0)\geq\underline{k}$ and*

$$\boldsymbol{u}^i_*(z)=\left[\frac{\mathbf{1}_{k^i<k^{-i}}}{\vartheta_2(k^a,k^b)}+\frac{\mathbf{1}_{k^i=k^{-i}}}{\vartheta_1(k^a,k^b)+\vartheta_2(k^a,k^b)}\right]\mathbf{1}_{x\geq\mathcal{X}(k^a,k^b)}, \tag{61}$$

$$\boldsymbol{w}^i_*(z)=\inf\Big\{\delta\geq 0:\mathcal{X}\big(k^i+\delta,\max\{k^{-i},\mathcal{K}(x)\}\big)\geq x\Big\}, \tag{62}$$

for any $z=(x,k^a,k^b)\in\mathbb{Z}:=(0,\infty)\times[\underline{k},\infty)\times[\underline{k},\infty)$. In (61)-(62)*,*

$$\mathcal{K}(x):=\inf\{k\geq\underline{k}:\mathcal{X}(k,k)\geq x\} \tag{63}$$

and

$$\vartheta_1(k^a,k^b):=\begin{cases}\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a\leq k^b,\\ \frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a\geq k^b,\end{cases}\quad\text{and}\quad\vartheta_2(k^a,k^b):=\begin{cases}\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a\leq k^b,\\ \frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a\geq k^b.\end{cases} \tag{64}$$

There exist many closed-loop equilibria with nonlinear boundaries. Below we characterize closed-loop equilibria in closed form for two specifications.

Proposition 7 *For any $\lambda>0$, there exist closed-loop equilibria characterized by the following nonlinear investment boundaries $\mathcal{X}(k^a,k^b)$:*

(i) If $\min(k^a_0,k^b_0)>\sqrt{\lambda}$, then

$$\mathcal{X}(k^a,k^b)=\rho\left(k^a+k^b+\frac{\lambda}{\max\{k^a,k^b\}}+rp+c\right). \tag{65}$$

(ii) If $\min(k^a_0,k^b_0)>\frac{3}{\lambda}$, then

$$\mathcal{X}(k^a,k^b)=\rho\left((k^a+k^b)\frac{\lambda\max\{k^a,k^b\}}{\lambda\max\{k^a,k^b\}-1}+rp+c\right). \tag{66}$$

Nonlinear triggers in (65) and (66) are captured by the terms involving the max operators. To illustrate the industry capital stock dynamics in a closed-loop equilibrium, we plot K_t^{duo} (scaled by the monopoly's capital stock K_t^{mon} for the same $\{X_t\}$ path) as a function of the running maximum $\{M_t\}$ process in Figure 8 using the nonlinear trigger function given in (65).

Panel A shows that the industry investment dynamics are highly nonlinear, unlike the dynamics in the closed-loop equilibrium with linear triggers analyzed in Section 6. With a nonlinear trigger, K_t^{duo} is larger than in the best closed-loop equilibrium (with a linear trigger).

Panel B shows that the resulting industry value is lower in an equilibrium with a nonlinear trigger than in the best closed-loop equilibrium. The quantitative effect in our numerical example is large. For example, when $M_t = 5$, the duopoly industry capital stock is 62% larger than a monopoly's: $K_t^{\text{duo}}/K_t^{\text{mon}} = 1.62$, reducing the industry's net present value (NPV) to only 15% of a monopoly's NPV. By contrast, the industry's NPV in the best closed-loop equilibrium is 78% of a monopoly's when $M_t = 5$. That is, the industry's NPV in the equilibrium with a nonlinear trigger is only 19% of that in the best closed-loop equilibrium (with a linear trigger) analyzed in the preceding section. This large value destruction occurs because firms compete much more aggressively in the equilibrium with a nonlinear trigger than in the best closed-loop equilibrium, significantly eroding the option value of waiting.

In the limit as $M_t \to \infty$, the nonlinear term in (65) becomes increasingly small relative to the linear terms, and the ratio $K_t^{\text{duo}}/K_t^{\text{mon}}$ converges to two, approaching the BP closed-loop equilibrium solution. In summary, firms compete less aggressively when market demand is low but more fiercely as market demand increases in a closed-loop equilibrium with a nonlinear investment trigger.

9 Conclusion

Working with Grenadier (2002)'s classic real-option game setting, Back and Paulsen (2009) propose using the closed-loop equilibrium solution concept, which allows a firm to respond to its competitor's strategy over time, to characterize investment strategies. They further conjecture and verify a closed-loop equilibrium in which firms compete fiercely, yielding a perfectly competitive equilibrium outcome. Finally, they pose two important open questions:

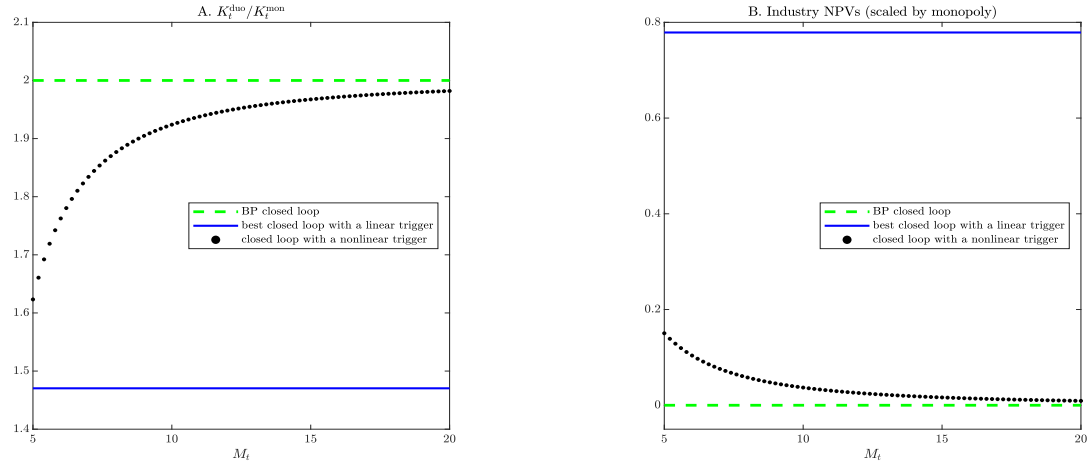


Figure 8: Equilibrium industry capital processes and industry net present values (NPVs). Panel A plots $K_t^{\text{duo}}/K_t^{\text{mon}}$ as a function of the demand shock's running maximum M_t in a closed-loop equilibrium with a nonlinear investment trigger, the BP equilibrium, and the best closed-loop equilibrium with a linear investment trigger. Panel B plots the duopoly industry's NPV (scaled by a monopoly's NPV) in the respective equilibria, where a duopoly industry's NPV is defined as $V^{\text{duo}}(M_t, K_t^{\text{duo}}/2, K_t^{\text{duo}}/2) - pK_t^{\text{duo}}$. Parameter values: $r = 0.06$, $\mu = 0.01$, $\sigma = 0.12$, $c = 0.1$, and $p = 1$.

(1) how to define a closed-loop equilibrium and (2) whether a closed-loop equilibrium, other than the one they identified, exists.

Responding to these open questions posed by Back and Paulsen, we first define a closed-loop equilibrium, identify infinitely many closed-loop equilibria in which firms earn positive profits, and then derive the corresponding value functions and investment policies in closed form.

We also contribute to the q theory of investment literature (Lucas and Prescott, 1971; Hayashi, 1982; Abel and Eberly, 1994, 1996) by extending it to an imperfectly competitive equilibrium setting. When analyzing the duopolistic equilibrium, we focus on the best closed-loop equilibrium, the one associated with the highest firm value among all closed-loop equilibria. Consistent with our intuition, compared with the BP closed-loop equilibrium in which firms earn zero profits, the option value of waiting in the best closed-loop equilibrium is quantitatively large. However, compared with the less economically plausible open-loop equilibrium, firms compete more fiercely, causing firm value to be lower.

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Appendices

A Sequential-game Formulation of the BP Equilibrium

In this appendix, we provide a sequential-game formulation of the BP closed-loop equilibrium. We decompose the leader's and follower's investment rates, dK_t^L and dK_t^F, over a small time interval dt into two components:

$$dK_t^L = dK_t^{L,C} + \Delta K_t^L, \tag{A.1}$$

$$dK_t^F = dK_t^{F,C} + \Delta K_t^F, \tag{A.2}$$

where $dK_t^{L,C}$ and $dK_t^{F,C}$ denote the continuous components of dK_t^L and dK_t^F, respectively, and $\Delta K_t^L := K_{t+}^L - K_t^L$ and $\Delta K_t^F := K_{t+}^F - K_t^F$ denote the instantaneous increases (jumps) of K^L and K^F, respectively. Assume that the jump and continuous components of the leader's investment strategy are given by

$$\Delta K_t^L = \boldsymbol{w}^L(Z_t)\,, \tag{A.3}$$

$$dK_t^{L,C} = u_t^L dM_t = \boldsymbol{u}^L(Z_t) dM_t, \tag{A.4}$$

where $\boldsymbol{w}^L(\,\cdot\,) \geq 0$ and $\boldsymbol{u}^L(\,\cdot\,) \geq 0$ are functions of Z_t.

By contrast, the jump component of the follower's investment strategy, ΔK_t^F, is allowed to depend not only on the state, $Z_t = (X_t, K_t^a, K_t^b)$, but also on the jump component of the leader's strategy, ΔK_t^L, so that we can express ΔK_t^F as:

$$\Delta K_t^F = \boldsymbol{w}^F(Z_t, \Delta K_t^L)\,, \tag{A.5}$$

where $\boldsymbol{w}^F(\,\cdot\,,\,\cdot\,) \geq 0$ is a function of Z_t and ΔK_t^L. Similarly, we can express the continuous component of the follower's investment strategy $dK_t^{F,C}$ as:

$$dK_t^{F,C} = \boldsymbol{u}^F(Z_t, u_t^L) dM_t\,, \tag{A.6}$$

where $\boldsymbol{u}^F(\,\cdot\,,\,\cdot\,) \geq 0$ is a function of Z_t and u_t^L.

We characterize the leader's and follower's investment strategies using their closed-loop strategies: $\boldsymbol{\varphi}^i = (\boldsymbol{u}^i, \boldsymbol{w}^i)$ for $i = L$ and F. Feasible closed-loop strategies and closed-loop equilibria are defined as in Definitions 2 and 3, respectively.

Next, we summarize the BP closed-loop equilibrium solution (see Appendix C.5 for a

proof.)

Theorem 5 *A strategy pair* $(\boldsymbol{\varphi}_*^L, \boldsymbol{\varphi}_*^F) = \{(\boldsymbol{u}_*^L, \boldsymbol{w}_*^L), (\boldsymbol{u}_*^F, \boldsymbol{w}_*^F)\} \in \mathcal{S}$ *constitutes a closed-loop equilibrium if*

$$\boldsymbol{u}_*^L(z) = \frac{1}{\rho}\mathbf{1}_{x \geq \mathcal{X}^{\mathrm{BP}}(k^a,k^b)}, \tag{A.7}$$

$$\boldsymbol{u}_*^F(z,u^L) = \left(\frac{1}{\rho} - u^L\right)^+ \mathbf{1}_{x \geq \mathcal{X}^{\mathrm{BP}}(k^a,k^b)}, \tag{A.8}$$

$$\boldsymbol{w}_*^L(z) = \left(\frac{x}{\rho} - (rp+c) - k^a - k^b\right)^+, \tag{A.9}$$

$$\boldsymbol{w}_*^F(z,w^L) = \left(\frac{x}{\rho} - (rp+c) - k^a - k^b - w^L\right)^+, \tag{A.10}$$

for all $z = (x, k^a, k^b) \in \mathbb{Z}$.

B Monopoly Solution

In this appendix, we report closed-form expressions for the monopolist's value function associated with the optimal strategy given in Proposition 1.

Proposition 8 *The monopoly value in the inaction region where* $x < \mathcal{X}^{\mathrm{mon}}(k)$ *is given by*

$$V^{\mathrm{mon}}(x,k) = \frac{k}{r-\mu}x - \frac{k^2+ck}{r} + H^{\mathrm{mon}}(k)x^{\beta}\,, \tag{B.1}$$

where

$$H^{\mathrm{mon}}(k) = \frac{1}{2\rho\beta(\beta-2)(r-\mu)}(\mathcal{X}^{\mathrm{mon}}(k))^{2-\beta}. \tag{B.2}$$

In the investment region where $x \geq \mathcal{X}^{\mathrm{mon}}(k)$*, the value function is given by*

$$V^{\mathrm{mon}}(x,k) = V^{\mathrm{mon}}(x, \mathcal{K}^{\mathrm{mon}}(x)) - p(\mathcal{K}^{\mathrm{mon}}(x) - k), \tag{B.3}$$

where $\mathcal{K}^{\mathrm{mon}}(x) = \frac{x/\rho-(rp+c)}{2}$ *is the inverse function of* $\mathcal{X}^{\mathrm{mon}}(k)$ *and*

$$V^{\mathrm{mon}}(x, \mathcal{K}^{\mathrm{mon}}(x)) = \frac{\mathcal{K}^{\mathrm{mon}}(x)}{r-\mu}x - \frac{(\mathcal{K}^{\mathrm{mon}}(x))^2 + c\mathcal{K}^{\mathrm{mon}}(x)}{r} + \frac{x^2}{2\rho\beta(\beta-2)(r-\mu)}.$$

The first two terms in (B.1) represent the value of the monopolist's existing assets in place, that is, the value when it has no future investment option and $K_t = k$ for all t. The last term in (B.1) is convex in x and measures the monopolist's option value of investing.

C Proofs

C.1 Regularity Conditions and Some Useful Technical Results

In this appendix, we first state regularity conditions and then establish some useful technical results under these regularity conditions (see Proposition 9.) Motivated by (32)–(34), we define the investment and inaction regions associated with the value functions, $V^a(z)$ and $V^b(z)$, as follows:

$$\mathbb{I}_i := \left\{ z \in \mathbb{Z} : \frac{\partial V^i(z)}{\partial k^i} = p \text{ and } \frac{\partial V^{-i}(z)}{\partial k^i} = 0 \right\}, \tag{C.1}$$

$$\mathbb{N}_i := \mathbb{Z} \setminus \mathbb{I}_i. \tag{C.2}$$

Let $\mathbb{N}_{ab}$ denote the intersection of $\mathbb{N}_a$ and $\mathbb{N}_b$:

$$\mathbb{N}_{ab} := \mathbb{N}_a \bigcap \mathbb{N}_b. \tag{C.3}$$

Let $\overline{\mathbb{N}}_{ab}$ denote the closure of $\mathbb{N}_{ab}$, which includes both its interior and boundary.

Condition A.1 *Assume $V^i \in \mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab}) \cap \mathcal{C}^{1,1,1}(\mathbb{Z})$ and equation (35) holds for any $z \in \mathbb{N}_{ab}$.*

Condition A.2 *For any initial state $(X_0, K_0^a, K_0^b) = (x, k^a, k^b) \in \mathbb{Z}$, the capital stock process pair (K^a, K^b) generated by $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ has the following properties:*

- *At the initial time $t = 0$, the value-matching condition holds (from $t = 0$ to $t = 0+$):*

$$V^i(x, K_{0+}^a, K_{0+}^b) - p(K_{0+}^i - k^i) = V^i(x, k^a, k^b); \tag{C.4}$$

- *For all $t > 0$, $(X_t, K_t^a, K_t^b) \in \overline{\mathbb{N}}_{ab}$ and*

$$K_t^i = K_{0+}^i + \int_{0+}^t \mathbf{1}_{\{Z_s \in \mathbb{I}_i\}} dK_s^i \tag{C.5}$$

  is continuous in $t > 0$, where the integrand is an indicator function.

- *Regularity conditions hold: for any $z \in \mathbb{Z}$, $\lim_{n\to\infty} \mathbb{E}^z \left[e^{-rt_n} V^i \left(X_{t_n}, K_{t_n}^a, K_{t_n}^b \right) \right] = 0$ for some sequence $\{t_n\} \to \infty$ and $\mathbb{E}^z \left[\int_0^t \left| e^{-rs} \frac{\partial V^i}{\partial x} \left(X_s, K_s^a, K_s^b \right) \sigma X_s \right|^2 ds \right] < \infty$, for all $t \geq 0$.*

Equation (C.4) is implied by the first-order condition for investment when firm i undertakes an initial lumpy investment. Using (C.5), we can show $dK_s^i = 0$ for $Z_s \notin \mathbb{I}_i$.

Proposition 9 *If a pair of functions,* $(V^a(z), V^b(z))$*, and a pair of strategies,* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$*, satisfy Conditions A.1-A.2,*[12]*, then* $V^a(z) = J^a(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ *and* $V^b(z) = J^b(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$*.*

Proof of Proposition 9 Recall the set $\mathbb{N}_{ab}$ as defined in (C.3), and let $\overline{\mathbb{N}}_{ab}$ denote its closure. Since $V^i_{k^a}(z)$ and $V^i_{k^b}(z)$ are continuous, the set $\mathbb{N}_{ab}$ is an open subset of $\mathbb{Z}$.

Recall that (K^a, K^b) denote the firms' capital stocks associated with the strategy pair $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$. Before applying Itô's formula, we note that Condition A.1 ensures that $V^a \in \mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab})$, while Condition A.2 guarantees that $(X_t, K^a_t, K^b_t) \in \overline{\mathbb{N}}_{ab}$ for any $t > 0$. Applying Itô's formula to $e^{-rs}V^a(X_s, K^a_s, K^b_s)$, $s \in (0, t]$, for $t > 0$ and noting that both K^a_s and K^b_s are continuous for $s > 0$ (as stated in Condition A.2), we obtain

$$\begin{aligned} V^a(x, K^a_{0+}, K^b_{0+}) = e^{-rt}V^a(X_t, K^a_t, K^b_t) - \int_0^t e^{-rs}\mathcal{L}V^a(X_s, K^a_s, K^b_s)ds \\ - \int_{0+}^t e^{-rs}\left[\frac{\partial V^a}{\partial k^a}(X_s, K^a_s, K^b_s)dK^a_s + \frac{\partial V^a}{\partial k^b}(X_s, K^a_s, K^b_s)dK^b_s\right] \\ + \int_0^t e^{-rs}\frac{\partial V^a}{\partial x}(X_s, K^a_s, K^b_s)\sigma X_s d\mathcal{B}_s, \end{aligned} \tag{C.6}$$

where $\mathcal{L}$ is the infinitesimal generator given by

$$\mathcal{L}V(z) := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(z)}{\partial x^2} + \mu x \frac{\partial V(z)}{\partial x} - rV(z). \tag{C.7}$$

Taking expectations on both sides of the above equation, and using the regularity condition $\mathbb{E}^z\left[\int_0^t \left|e^{-rs}\frac{\partial V^a}{\partial x}\left(X_s, K^a_s, K^b_s\right)\sigma X_s\right|^2 ds\right] < \infty$ from Condition A.2, we obtain

$$\begin{aligned} V^a(x, K^a_{0+}, K^b_{0+}) =& \mathbb{E}^z\Bigg[e^{-rt}V^a(X_t, K^a_t, K^b_t) - \int_0^t e^{-rs}\mathcal{L}V^a(X_s, K^a_s, K^b_s)ds \\ & - \int_{0+}^t e^{-rs}\Big[\frac{\partial V^a}{\partial k^a}(X_s, K^a_s, K^b_s)dK^a_s + \frac{\partial V^a}{\partial k^b}(X_s, K^a_s, K^b_s)dK^b_s\Big]\Bigg]. \end{aligned} \tag{C.8}$$

According to (C.1), we have $\frac{\partial V^a(z)}{\partial k^a} = p$ for any $z \in \mathbb{I}_a$, and $\frac{\partial V^a(z)}{\partial k^b} = 0$ for any $z \in \mathbb{I}_b$. Therefore,

$$\begin{aligned} & \int_{0+}^t e^{-rs}\left[\frac{\partial V^a}{\partial k^a}(X_s, K^a_s, K^b_s)dK^a_s + \frac{\partial V^a}{\partial k^b}(X_s, K^a_s, K^b_s)dK^b_s\right] \\ =& \int_{0+}^t e^{-rs}\left[\frac{\partial V^a}{\partial k^a}(X_s, K^a_s, K^b_s)\mathbf{1}_{(X_s, K^a_s, K^b_s)\in\mathbb{I}_a}dK^a_s + \frac{\partial V^a}{\partial k^b}(X_s, K^a_s, K^b_s)\mathbf{1}_{(X_s, K^a_s, K^b_s)\in\mathbb{I}_b}dK^b_s\right] \\ =& \int_{0+}^t e^{-rs}p dK^a_s, \end{aligned} \tag{C.9}$$

where the first equality uses (C.5).

[12]Under Conditions A.1 and A.2, we can apply Itô's lemma.

Moreover, Condition A.1 implies that equation (35) holds for any $z \in \mathbb{N}_{ab}$, and Condition A.2 ensures that $(X_t, K_t^a, K_t^b) \in \overline{\mathbb{N}}_{ab}$ for any $t > 0$. Hence, $\mathcal{L}V^a(X_s, K_s^a, K_s^b) + \pi^a(X_s, K_s^a, K_s^b) = 0$ for any $s > 0$. Substituting $-\mathcal{L}V^a(X_s, K_s^a, K_s^b) = \pi^a(X_s, K_s^a, K_s^b)$, $s > 0$ and (C.9) into (C.8), we obtain

$$\begin{aligned} & V^a(x, K_{0+}^a, K_{0+}^b) \\ =& \mathbb{E}^z\left[e^{-rt}V^a(X_t, K_t^a, K_t^b) + \int_0^t e^{-rs}\pi^a(X_s, K_s^a, K_s^b)ds - \int_{0+}^t e^{-rs}pdK_s^a\right]. \end{aligned} \tag{C.10}$$

Using the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n}, K_{t_n}^a, K_{t_n}^b)] = 0$ from Condition A.2, replacing t by t_n in (C.10), and letting $n \to \infty$, we obtain

$$V^a(x, K_{0+}^a, K_{0+}^b) = \mathbb{E}^z\left[\int_0^{+\infty} e^{-rs}\pi^a(X_s, K_s^a, K_s^b)ds - \int_{0+}^{+\infty} e^{-rs}pdK_s^a\right], \tag{C.11}$$

where the convergence of the first integral follows from the dominated convergence theorem. Finally, plugging (C.4) into (C.11), we have

$$\begin{aligned} V^a(z) &= \mathbb{E}^z\left[\int_0^{+\infty} e^{-rs}\pi^a(X_s, K_s^a, K_s^b)ds - \int_0^{+\infty} e^{-rs}pdK_s^a\right] \\ &= J^a(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b). \end{aligned}$$

Similarly, we can also show that $V^b(z) = J^b(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$. $\square$

C.2 Proof of Theorem 2

First, the inequality (27) together with $V^a(z) = J^a(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ implies (25).

Next, we prove (26) for any $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ and infinitesimal $\varepsilon > 0$. Recall that $(K^{a,\boldsymbol{\varphi}}, K^{b,\boldsymbol{\varphi}})$ denote the firms' capital stocks associated with the strategy pair $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}_*^b)$ and let $Z_t^{\boldsymbol{\varphi}} = (X_t, K_t^{a,\boldsymbol{\varphi}}, K_t^{b,\boldsymbol{\varphi}})$. By (23), the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^{\varepsilon} e^{-rs}\pi^a(Z_s^{\boldsymbol{\varphi}})ds\right] = 0. \tag{C.12}$$

Using (18)-(19), $|V_{\mathrm{BP}}^a(z)|$ is of polynomial growth in $z = (x, k^a, k^b)$: there exist constants $C_0 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_3 > 0$ such that

$$|V_{\mathrm{BP}}^a(z)| \le C_0(|x|^{\gamma_1} + |k^a|^{\gamma_2} + |k^b|^{\gamma_3}), \quad \forall z \in \mathbb{Z}. \tag{C.13}$$

Therefore, there exist constants $C_1 > 0$ and $\varepsilon_0 > 0$, such that for any $\varepsilon \in [0, \varepsilon_0]$,

$$|V_{\mathrm{BP}}^a(Z_\varepsilon^{\boldsymbol{\varphi}})| \le C_1(1 + |M_{\varepsilon_0}|^{\gamma_1} + |K_{\varepsilon_0}^{a,\boldsymbol{\varphi}}|^{\gamma_2} + |K_{\varepsilon_0}^{b,\boldsymbol{\varphi}}|^{\gamma_3}), \tag{C.14}$$

where M_t is the running maximum of the demand shock $\{X_s; s \le t\}$. Using (2.16) in Chapter

4 of Friedman (2012), and Definition 2, we have

$$\mathbb{E}^z\left[|M_{\varepsilon_0}|^{\gamma_1}+|K^{a,\boldsymbol{\varphi}}_{\varepsilon_0}|^{\gamma_2}+|K^{b,\boldsymbol{\varphi}}_{\varepsilon_0}|^{\gamma_3}\right]<\infty. \tag{C.15}$$

Hence, by (C.14), (C.15), and the dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0}\ \mathbb{E}^z\left[e^{-r\varepsilon}V^a_{\mathrm{BP}}(Z^{\boldsymbol{\varphi}}_\varepsilon)\right]=\mathbb{E}^z\left[V^a_{\mathrm{BP}}(Z^{\boldsymbol{\varphi}}_{0+})\right]. \tag{C.16}$$

Moreover, by the monotone convergence theorem, $\lim_{\varepsilon\to 0}\mathbb{E}^z\Big[\int_0^\varepsilon e^{-rs}dK^{a,\boldsymbol{\varphi}}_s\Big]=K^{a,\boldsymbol{\varphi}}_{0+}-K^{a,\boldsymbol{\varphi}}_0=\Delta K^{a,\boldsymbol{\varphi}}_0$. Combining this with (C.16), we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\ \mathbb{E}^z\left[e^{-r\varepsilon}V^a_{\mathrm{BP}}(Z^{\boldsymbol{\varphi}}_\varepsilon)-p\int_0^\varepsilon e^{-rs}dK^{a,\boldsymbol{\varphi}}_s\right]\\ =&\mathbb{E}^z\left[V^a_{\mathrm{BP}}(Z^{\boldsymbol{\varphi}}_{0+})-p\Delta K^{a,\boldsymbol{\varphi}}_0\right]\le V^a_{\mathrm{BP}}(z),\end{aligned} \tag{C.17}$$

where the inequality uses $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^a}\le p$ and $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^b}\le 0$ for all $z\in\mathbb{Z}$.

Combining (C.12) and (C.17), we obtain

$$V^a(z)>V^a_{\mathrm{BP}}(z)\ge\lim_{\varepsilon\to 0}\mathbb{E}^z\left[\int_0^\varepsilon e^{-rs}\Big(\pi^a(Z^{\boldsymbol{\varphi}}_s)ds-p\,dK^{a,\boldsymbol{\varphi}}_s\Big)+e^{-r\varepsilon}V^a_{\mathrm{BP}}(Z^{\boldsymbol{\varphi}}_\varepsilon)\right], \tag{C.18}$$

where the first inequality uses (28). Then, it follows from (C.18) that there exists $\varepsilon_0>0$, such that (26) holds for all $\varepsilon\in(0,\varepsilon_0]$.

By symmetry, we can also show that the same conditions hold for $J^b(z;\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)$. Then we have proved that $(\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)$ constitutes a closed-loop equilibrium strategy. □

C.3 Proofs of Theorem 3 and Propositions 3-4

In Appendix C.5, we will show that the BP solution constitutes a closed-loop equilibrium. Therefore, we only need to consider the case $(\theta_1,\theta_2)\in\Theta\setminus\{(1,1)\}$ and verify the conditions stated in Theorem 2 and Proposition 9.

We first present an auxiliary lemma used to verify that inequalities (27) and (55), as well as Condition A.1, hold.

Lemma 2 *Under Condition 1, the function $V^i(z)$ in Proposition 3 satisfies inequalities* (27) *and* (55)*, as well as Condition A.1.*

The proof of Lemma 2 is relegated to Internet Appendix IA.B.

Next, it remains to verify that $(\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)\in\mathcal{S}$, and that Condition A.2 is satisfied for the

functions $V^a(z)$ and $V^b(z)$ from Proposition 3, with the trigger function $\mathcal{X}(k^a, k^b)$ as defined in (44). The remainder of the proof is organized into three steps.

Step 1: Derive industry dynamics.

We first consider the region to the northeast of the ACE boundary in Figure 4, where $X_t \leq \mathcal{X}(K_t^a, K_t^b)$. Since $\mathcal{X}(k^a, k^b)$ increases in k^a and k^b, it follows from (50) that $\boldsymbol{w}_*^i(x, k^a, k^b) = 0$ for all $x \leq \mathcal{X}(k^a, k^b)$. Thus,

$$\Delta K_t^i = \boldsymbol{w}_*^i(X_t, K_t^a, K_t^b) = 0 \quad \text{for all } X_t \leq \mathcal{X}(K_t^a, K_t^b). \tag{C.19}$$

Then, no lumpy investment occurs whenever $X_t \leq \mathcal{X}(K_t^a, K_t^b)$.

Second, we consider the OBCD square region in Figure 4. From (50) and $\mathcal{X}(\mathcal{K}(x), \mathcal{K}(x)) = x$, we have[13]

$$\Delta K_t^i = \mathcal{K}(X_t) - K_t^i > 0 \quad \text{when } (K_t^a, K_t^b) \in [0, \mathcal{K}(X_t)) \times [0, \mathcal{K}(X_t)), \tag{C.20}$$

indicating that both firms invest lumpily to reach $(\mathcal{K}(X_t), \mathcal{K}(X_t))$.

Third, we consider the triangular region ABC in Figure 4. From (50) and the monotonicity of $\mathcal{X}(k^a, k^b)$ in k^a and k^b, we infer that

$$\Delta K_t^a = \inf\{\delta \geq 0 : \mathcal{X}\left(K_t^a + \delta, K_t^b\right) \geq X_t\} > 0 \text{ and } \Delta K_t^b = 0 \tag{C.21}$$

$$\text{when } X_t > \mathcal{X}(K_t^a, K_t^b) \text{ and } K_t^b \geq \mathcal{K}(X_t), \tag{C.22}$$

where we have used $K_t^a < \mathcal{K}(X_t)$. This indicates that firm b does not make a lumpy investment while firm a does at time t, moving instantaneously to $\boldsymbol{\Gamma}_a(X_t)$. By symmetry, the jump component of firm i's investment strategy in the triangular region CDE in Figure 4 can be derived analogously.

In summary, (C.19)-(C.22) imply the jump component of firm i's investment strategy is uniquely determined. Next, we solve the continuous component of firm i's investment strategy.

In equation (49), the indicator outside the square brackets implies that $\boldsymbol{u}_*^a(z) = \boldsymbol{u}_*^b(z) = 0$ in the inaction region where $x < \mathcal{X}(k^a, k^b)$. In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^i < k^{-i}$, (49) implies $\boldsymbol{u}_*^i(z) = \frac{1}{\rho\theta_2}$, $\boldsymbol{u}_*^{-i}(z) = 0$. That is, only the smaller firm invests:

$$dK_t^i = \frac{dM_t}{\rho\theta_2} \text{ and } dK_t^{-i} = 0 \quad \text{when } K_t^i < K_t^{-i} \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \tag{C.23}$$

Combining the above with (C.21), we conclude that starting from within the ABC and CDE

[13]For $k^a < \mathcal{K}(x)$ and $k^b < \mathcal{K}(x)$, (50) implies $\mathcal{X}\big(k^i + \boldsymbol{w}_*^i(z), \mathcal{K}(x)\big) = x$. Thus, we must have $\boldsymbol{w}_*^i(z) = \mathcal{K}(x) - k^i$, $i = a, b$.

triangular regions, the smaller firm i increases its capital to the boundary $\mathbf{\Gamma}_i(X_t)$, but the larger firm $-i$ chooses to be inactive. Subsequently, the smaller firm i invests continuously at a rate proportional to dM_t, as described in (C.23), and the larger firm $-i$ does not invest until the capital levels of both firms become equal.

In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^a = k^b$, (49) implies that

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)} \quad \text{when} \;\; K_t^a = K_t^b \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \tag{C.24}$$

Combining the above with (C.20), we conclude that starting from within the OBCD square region, both firms invest so that $K_t^a = K_t^b = \mathcal{K}(M_t)$, reaching point C in Figure 4. Subsequently, the two firms invest at the same rate at all t and thus are always of the same size. In summary, (C.23)-(C.24) imply the continuous component of firm i's investment strategy is uniquely determined.

Combining (C.19)-(C.24), we can obtain the industry dynamics:

$$K_t^i = \max\left\{\min\left\{\widehat{K}_t^i, \mathcal{K}_t^M\right\}, k_0^i\right\} \;\text{, for } \; i = a, b, \tag{C.25}$$

where $\widehat{K}_t^a$ and $\widehat{K}_t^b$, uniquely solve

$$\mathcal{X}(\widehat{K}_t^a, k_0^b) = M_t \quad \text{and} \quad \mathcal{X}(k_0^a, \widehat{K}_t^b) = M_t, \tag{C.26}$$

respectively, and $\mathcal{K}_t^M$ is given by

$$\mathcal{K}_t^M := \mathcal{K}(M_t) = \frac{1}{\theta_1 + \theta_2}\left[\frac{M_t}{\rho} - (rp + c)\right]. \tag{C.27}$$

Step 2: Verify $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$.

According to the results proved in Step 1, we conclude that the strategy $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ satisfies the conditions in Definition 2-(i). Next, we proceed to verify the conditions in Definition 2-(ii).

By (C.25), we have $K_t^i \leq k_0^i \vee \mathcal{K}(M_t)$, $t \geq 0$. Since $\mathcal{K}(x) = \frac{1}{\theta_1+\theta_2}\left[\frac{x}{\rho} - (rp + c)\right]$ is linear, there exists a constant $C_1 > 0$ such that $K_t^i \in [k_0^i, C_1(1 + M_t)]$, $\forall\, t \geq 0$, $i = a, b$. Then, for any finite $T > 0$ and $n > 1$, a sufficient condition for $\mathbb{E}^z\left[|K_T^i|^n\right] < \infty$ is

$$\mathbb{E}^z\left[|M_T|^n\right] < \infty. \tag{C.28}$$

Since $\pi^i(x, k^a, k^b) = (x - (k^a + k^b) - c)k^i$, and since $K_t^i \in [k_0^i, C_1(1 + M_t)]$ and $X_t \in [0, M_t]$, $\forall\, t \geq 0$, there exists a constant $C_2 > 0$ such that $|\pi^i(X_t, K_t^a, K_t^b)| \leq C_2(1 + |M_t|^2)$, $\forall\, t \geq 0$.

Thus, a sufficient condition for (23) is

$$\int_0^{+\infty} \mathbb{E}^z[e^{-rt}|M_t|^2]dt < +\infty. \tag{C.29}$$

By (2.16) in Chapter 4 of Friedman (2012), (C.28) holds for any $T > 0$ and $n > 1$. For any initial value z, there exists a constant $C_3 > 0$ such that

$$\begin{aligned}\mathbb{E}^z[M_t^2] &\leq 3X_0^2 + 3\mathbb{E}^z\left[\Big(\int_0^t |\mu|X_s ds\Big)^2\right] + 3\mathbb{E}^z\left[\sup_{\tau\in[0,t]}\Big(\int_0^\tau \sigma X_s d\mathcal{B}_s\Big)^2\right] \\ &\leq 3X_0^2 + (3\mu^2 t + C_3\sigma^2)x_0^2\int_0^t e^{(\sigma^2+2\mu)s}ds, \end{aligned}\tag{C.30}$$

where the first inequality uses $X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s d\mathcal{B}_s$ and $|x_1 + x_2 + x_3|^2 \leq 3(|x_1|^2 + |x_2|^2 + |x_3|^2)$, and the second inequality follows from the Cauchy inequality and the Burkholder-Davis-Gundy Inequality. Since $\beta > 2$ implies $r > \sigma^2 + 2\mu$, we conclude from (C.30) that (C.29) holds. Therefore, the conditions in Definition 2-(ii) are satisfied, and $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$.

Step 3: Verify Condition A.2.

By (C.25), the definition of $V^i(z)$ in Proposition 3, and symmetry, it follows that (C.4) holds.

By (C.25), we have $(X_t, K_t^a, K_t^b) \in \overline{\mathbb{N}}_{ab}$ for all $t > 0$, and K_t^i is continuous in $t > 0$ and satisfies (C.5).

It remains to verify the regularity conditions in Condition A.2. By symmetry, we focus on firm a. Denote

$$\begin{aligned}\mathcal{H}(k) &:= |H^a(k,k)| + k\Big(\frac{3k + pr + c}{r} + \frac{\mathcal{X}(k,k)}{r-\mu}\Big)\mathcal{X}(0,k)^{-\beta}, \\ \widetilde{\mathcal{H}}(k) &:= |H^a(k,k)| + \frac{k^2}{r}\mathcal{X}(k,0)^{-\beta},\end{aligned}$$

where $H^a(k^a, k^b)$ is given by (42)-(43). We first show that $\lim_{k\to+\infty}\mathcal{H}(k) = 0$, $\lim_{k\to+\infty}\widetilde{\mathcal{H}}(k) = 0$, and

$$|H^a(k^a,k^b)| \leq \mathcal{H}(k^b)\mathbf{1}_{k^a\leq k^b} + \widetilde{\mathcal{H}}(k^a)\mathbf{1}_{k^a>k^b}, \quad \forall (k^a,k^b) \in \mathbb{R}_+^2. \tag{C.31}$$

Since $\lim_{k\to+\infty} H^a(k,k) = 0$ and $\beta > 2$, and $\mathcal{X}(k,k)$ and $\mathcal{X}(0,k)$ are positive and linearly increasing in k, we have $\lim_{k\to+\infty}\mathcal{H}(k) = 0$. By (43), for $k^b \geq k^a \geq 0$, we have

$$|H^a(k^a,k^b)| = \Big|H^a(k^b,k^b) - \int_{k^a}^{k^b}\Big(\frac{2\alpha + k^b + rp + c}{r} - \frac{\mathcal{X}(\alpha,k^b)}{r-\mu}\Big)\mathcal{X}(\alpha,k^b)^{-\beta}d\alpha\Big| \leq \mathcal{H}(k^b),$$

where the inequality follows from the monotonicity of $\mathcal{X}(k^a,k^b)$ in $k^a \in [0,k^b]$. Similarly, using

(43), we can obtain

$$|H^a(k^a,k^b)| = \left|H^a(k^a,k^a) - \int_{k^b}^{k^a} \frac{k^a}{r}\mathcal{X}(k^a,\alpha)^{-\beta}d\alpha\right| \le \widetilde{\mathcal{H}}(k^a)$$

for $k^a \ge k^b \ge 0$. Thus, we can see (C.31) holds.

Fix any $(k^a,k^b) \in \overline{\mathbf{N}}_{ab}(x)$, i.e., $x \le \mathcal{X}(k^a,k^b)$. If $k^a \le k^b$, then $\mathcal{X}(k^b,k^b) \ge \mathcal{X}(k^a,k^b) \ge x$, which implies that $k^b \ge \mathcal{K}(x)$. If $k^a \ge k^b$, then $\mathcal{X}(k^a,k^a) \ge \mathcal{X}(k^a,k^b) \ge x$, which implies that $k^a \ge \mathcal{K}(x)$. Hence, we have

$$k^b\mathbf{1}_{k^a\le k^b} + k^a\mathbf{1}_{k^a>k^b} \ge \mathcal{K}(x). \tag{C.32}$$

Since $\lim_{k\to\infty}\mathcal{H}(k) = 0$, $\lim_{k\to\infty}\widetilde{\mathcal{H}}(k) = 0$, and $\lim_{x\to+\infty}\mathcal{K}(x) = \infty$, for any $\delta > 0$, there exists $x_\delta > 0$ such that $|\mathcal{H}(k^b)| \le \delta$ and $|\widetilde{\mathcal{H}}(k^a)| \le \delta$ for any $k^a \ge \mathcal{K}(x_\delta)$ and $k^b \ge \mathcal{K}(x_\delta)$. Combining (C.31) and (C.32), we see that $k^b\mathbf{1}_{k^a\le k^b} + k^a\mathbf{1}_{k^a>k^b} \ge \mathcal{K}(x_\delta)$ and $|H^a(k^a,k^b)| \le \delta$ when $\mathcal{X}(k^a,k^b) \ge x_\delta$. Denote $\tau_\delta := \inf\{t \ge 0 : X_t \ge x_\delta\}$. Then for any $t > 0$,

$$|\mathbb{E}^z[e^{-rt}H^a(K_t^a,K_t^b)X_t^\beta\mathbf{1}_{t\ge\tau_\delta}]| \le \delta|\mathbb{E}^z[e^{-rt}X_t^\beta]| = \delta x^\beta, \tag{C.33}$$

where the inequality follows from $\mathcal{X}(K_t^a,K_t^b) \ge \mathcal{X}(K_{\tau_\delta}^a,K_{\tau_\delta}^b) \ge X_{\tau_\delta} = x_\delta$, $\forall\ t \ge \tau_\delta$ and $|H^a(k^a,k^b)| \le \delta$ for $\mathcal{X}(k^a,k^b) \ge x_\delta$, and the equality holds as $e^{-rt}X_t^\beta$ is a martingale. The continuity of $H^a(k^a,k^b)$ in $(k^a,k^b) \in \mathbb{R}_+^2$ implies that $H^a(k^a,k^b)$ is bounded in the compact set $\{(k^a,k^b) \in \mathbb{R}_+^2 : \mathcal{X}(k^a,k^b) \le x_\delta\}$. Owing to $|H^a(k^a,k^b)| \le \delta$ for $\mathcal{X}(k^a,k^b) \ge x_\delta$, $H^a(k^a,k^b)$ is bounded in $(k^a,k^b) \in \mathbb{R}_+^2$. Recalling that $r > 0$ and $X_t \ge 0$, $\forall t \ge 0$, we infer that $\lim_{t\to+\infty}\mathbb{E}^z[e^{-rt}H^a(K_t^a,K_t^b)X_t^\beta\mathbf{1}_{t<\tau_\delta}] = 0$. Combining this with (C.33), we have $\lim_{t\to+\infty}\mathbb{E}^z[e^{-rt}H^a(K_t^a,K_t^b)X_t^\beta] \le \delta x^\beta$. Since $\delta > 0$ is arbitrary, we obtain

$$\lim_{t\to\infty}\mathbb{E}^z[e^{-rt}H^a(K_t^a,K_t^b)X_t^\beta] = 0. \tag{C.34}$$

According to (C.30), $r > 0$ and $r > \sigma^2 + 2\mu$, we have

$$\lim_{t\to\infty}\mathbb{E}^z[e^{-rt}M_t^2] = 0. \tag{C.35}$$

We derive from (23) and Tonelli's Theorem that

$$\int_0^\infty \mathbb{E}^z[e^{-rt}|\pi^i(X_t,K_t^a,K_t^b)|]dt < \infty.$$

It follows that there exists a sequence $\{t_n\}$ of real numbers such that $\lim_{n\to\infty}t_n = \infty$ and

$$\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}\pi^i(X_{t_n},K_{t_n}^a,K_{t_n}^b)] = 0. \tag{C.36}$$

Since $\pi^i(x,k^a,k^b) = k^a(x-\frac{1}{4}k^a) - k^a(\frac{3}{4}k^a + k^b + c) \le k^a(x-\frac{1}{4}k^a)$ for any $(x,k^a,k^b) \in \mathbb{R}_+^3$, we

derive from (C.36) that

$$\liminf_{n\to\infty} \mathbb{E}^z\Big[e^{-rt_n}K^a_{t_n}\Big(X_{t_n}-\frac{1}{4}K^a_{t_n}\Big)\Big]\geq 0. \tag{C.37}$$

In addition, we have

$$\begin{aligned}0\geq&\limsup_{n\to\infty}\mathbb{E}^z\left[e^{-rt_n}\Big(-X^2_{t_n}+K^a_{t_n}(X_{t_n}-\frac{1}{4}K^a_{t_n})\Big)\right]\\=&\limsup_{n\to\infty}\mathbb{E}^z\left[e^{-rt_n}K^a_{t_n}\Big(X_{t_n}-\frac{1}{4}K^a_{t_n}\Big)\right],\end{aligned} \tag{C.38}$$

where the first inequality follows from $-X^2_{t_n}+K^a_{t_n}(X_{t_n}-\frac{1}{4}K^a_{t_n})=-(X_{t_n}-\frac{1}{2}K^a_{t_n})^2$, and the equality uses (C.35). Hence, we derive from (C.37) and (C.38) that

$$\lim_{n\to\infty}\mathbb{E}^z\left[e^{-rt_n}K^a_{t_n}\Big(X_{t_n}-\frac{1}{4}K^a_{t_n}\Big)\right]=0. \tag{C.39}$$

Combining (C.39), (C.36), and $\pi^i(x,k^a,k^b)=k^a(x-\frac{1}{4}k^a)-k^a(\frac{3}{4}k^a+k^b+c)$, we conclude that $\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}K^a_{t_n}(\frac{3}{4}K^a_{t_n}+K^b_{t_n}+c)]=0$. Noticing $K^a\geq 0$, $K^b\geq 0$, and $c\geq 0$, we have

$$\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}(K^a_{t_n})^2]=0, \tag{C.40}$$

$$\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}K^a_{t_n}K^b_{t_n}]=0. \tag{C.41}$$

Combining (C.40) with (C.35) and using the Cauchy-Schwarz inequality, we obtain

$$\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}K^a_{t_n}X_{t_n}]=0. \tag{C.42}$$

Since we have shown $X_t\ \leq\mathcal{X}(K^a_t,K^b_t)$ for any $t>0$, it follows from (C.34) and (C.40)-(C.42) that $\lim_{n\to\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n},K^a_{t_n},K^b_{t_n})]=0$, where $V^a(X_t,K^a_t,K^b_t)$ is given by (36).

Using (36), we have $V^a_x(x,k^a,k^b)=\frac{k^a}{r-\mu}+\beta H^a(k^a,k^b)x^{\beta-1}$ for any $(k^a,k^b)\in\overline{\mathbf{N}}_{ab}(x)$. Thus, there exists a constant $C_1>0$ such that for any $(k^a,k^b)\in\overline{\mathbf{N}}_{ab}(x)$,

$$\begin{aligned}|V^a_x(x,k^a,k^b)x|^2\leq&2\Big(\frac{k^ax}{r-\mu}\Big)^2+2\beta^2|H^a(k^a,k^b)|^2x^{2\beta}\\\leq&C_1((k^a)^4+x^4+x^{2\beta})\leq C_1((k^a)^4+1+2x^{2\beta}),\end{aligned} \tag{C.43}$$

where the second inequality follows from the boundedness of $H^a(k^a,k^b)$ in $\mathbb{R}^2_+$, and the last inequality uses $\beta>2$ and $x^4<1+x^{2\beta}$. Since $K^a_t\geq 0$ is nondecreasing in t, and since Step 2 establishes that $\mathbb{E}^z[|K^a_t|^n]<+\infty$ for any $t>0$, $n>1$, it follows that $\mathbb{E}^z[\int_0^t|K^a_s|^4ds]\leq t\mathbb{E}^z[|K^a_t|^4]<+\infty$, $\forall t>0$. Additionally, direct computation yields $\mathbb{E}^z[X^{2\beta}_t]=x^{2\beta}e^{\beta t(2\mu+(2\beta-1)\sigma^2)}$, implying $\mathbb{E}^z[\int_0^t X^{2\beta}_s ds]<\infty$, $\forall t>0$. We then conclude from $X_t\ \leq\mathcal{X}(K^a_t,K^b_t)$, $t>0$, and (C.43) that $\mathbb{E}^z\left[\int_0^t|e^{-rs}V^a_x(X_s,K^a_s,K^b_s)\sigma X_s|^2ds\right]<+\infty,\ \forall\ t\geq 0$. $\square$

Next, we compare the firms' value functions.

Lemma 3 *Let $V^i(z)$ denote firm i's value associated with the symmetric investment threshold $\mathcal{X}(k^a,k^b)$. Suppose that $\frac{\partial V^i(x,k^a,k^b)}{\partial k^i}\leq p$, $\frac{\partial V^i(x,k^a,k^b)}{\partial k^{-i}}\leq 0$, and that $\mathcal{X}^{\text{open}}(k^a,k^b)\geq\mathcal{X}(k^a,k^b)>\mathcal{X}^{\text{BP}}(k^a,k^b)$ for any $x>0$ and $(k^a,k^b)\in\mathbb{D}$, where $\mathcal{X}^{\text{open}}(k^a,k^b)$ is defined in (15), $\mathcal{X}^{\text{BP}}(k^a,k^b)$ is defined in (16) and $\mathbb{D}=\{(k^a,k^b)\in[0,\infty)\times[0,\infty):k^i>l(k^{-i}-\underline{k})+\underline{k},\ i=a,b\}$ for some $\underline{k}\geq 0$ and $l\in[0,1)$. Then $V^i(x,k^a,k^b)>V^i_{\text{BP}}(x,k^a,k^b)$ for all $x>0$ and $(k^a,k^b)\in\overline{\mathbb{D}}=\{(k^a,k^b)\in[0,\infty)\times[0,\infty):k^i\geq l(k^{-i}-\underline{k})+\underline{k},\ i=a,b\}$.*

Proof of Lemma 3 By symmetry, it suffices to prove the result for firm a. According to (36),

$$V^a(z)=\Psi^a(z)+H^a(k^a,k^b)x^\beta,\quad x\leq\mathcal{X}(k^a,k^b), \tag{C.44}$$

$$V^a_{\text{BP}}(z)=\Psi^a(z)+H^a_{\text{BP}}(k^a,k^b)x^\beta,\quad x\leq\mathcal{X}^{\text{BP}}(k^a,k^b), \tag{C.45}$$

where $H^a(k^a,k^b)$ is given by (42)-(43), and $H^a_{\text{BP}}(k^a,k^b)$ is defined similarly, except that $\mathcal{X}(k^a,k^b)$ is replaced by $\mathcal{X}^{\text{BP}}(k^a,k^b)$.

We first show that $H^a(k^a,k^b)>H^a_{\text{BP}}(k^a,k^b)$ for any $(k^a,k^b)\in\overline{\mathbb{D}}$. To this end, define $f(x,y)=-\Big(\frac{y}{r}-\frac{x}{r-\mu}\Big)x^{-\beta}$. Then

$$\begin{aligned}\frac{\partial f(x,y)}{\partial x}=&\frac{x^{-\beta}}{r-\mu}-\beta\Big(\frac{x}{r-\mu}-\frac{y}{r}\Big)x^{-\beta-1}\\ =&x^{-\beta-1}\frac{\beta-1}{r-\mu}(\rho y-x)>0,\quad\text{when }\ x<\rho y.\end{aligned} \tag{C.46}$$

According to (42), for any $(k,k)\in\overline{\mathbb{D}}$, we have

$$\begin{aligned}H^a(k,k)=&\int_k^\infty f(\mathcal{X}(\alpha,\alpha),4\alpha+rp+c)d\alpha\\ >&\int_k^\infty f(\mathcal{X}^{\text{BP}}(\alpha,\alpha),4\alpha+rp+c)d\alpha=H^a_{\text{BP}}(k,k),\end{aligned} \tag{C.47}$$

where the inequality uses (C.46), and $\rho(4\alpha+rp+c)>\mathcal{X}^{\text{open}}(\alpha,\alpha)\geq\mathcal{X}(\alpha,\alpha)>\mathcal{X}^{\text{BP}}(\alpha,\alpha)$ for $\alpha>k$.

Next, consider any $(k^a,k^b)\in\overline{\mathbb{D}}$ such that $k^b\geq k^a$. By (43),

$$\begin{aligned}H^a(k^a,k^b)=&H^a(k^b,k^b)+\int_{k^a}^{k^b}f(\mathcal{X}(\alpha,k^b),2\alpha+k^b+rp+c)d\alpha\\ >&H^a_{\text{BP}}(k^b,k^b)+\int_{k^a}^{k^b}f(\mathcal{X}^{\text{BP}}(\alpha,k^b),2\alpha+k^b+rp+c)d\alpha=H^a_{\text{BP}}(k^a,k^b),\end{aligned} \tag{C.48}$$

where the inequality uses (C.47), (C.46), and $\rho(2\alpha+k^b+rp+c)=\mathcal{X}^{\text{open}}(\alpha,k^b)\geq\mathcal{X}(\alpha,k^b)\geq\mathcal{X}^{\text{BP}}(\alpha,k^b)$.

Finally, consider any $(k^a, k^b) \in \overline{\mathbb{D}}$ such that $k^a \geq k^b$. Again by (43),

$$\begin{aligned} H^a(k^a, k^b) &= H^a(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r} \mathcal{X}(k^a, \alpha)^{-\beta} d\alpha \\ &> H^a_{\text{BP}}(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r} \mathcal{X}^{\text{BP}}(k^a, \alpha)^{-\beta} d\alpha = H^a_{\text{BP}}(k^a, k^b), \end{aligned} \tag{C.49}$$

where the inequality uses (C.47), and $\mathcal{X}(k^a, \alpha) \geq \mathcal{X}^{\text{BP}}(k^a, \alpha)$.

We have therefore shown that $H^a(k^a, k^b) > H^a_{\text{BP}}(k^a, k^b)$ for any $(k^a, k^b) \in \overline{\mathbb{D}}$. It then follows from (C.44)-(C.45) that

$$V^a(z) > V^a_{\text{BP}}(z), \quad x \leq \mathcal{X}^{\text{BP}}(k^a, k^b), \ (k^a, k^b) \in \overline{\mathbb{D}}. \tag{C.50}$$

Now consider any $x > \mathcal{X}^{\text{BP}}(k^a, k^b)$ and $(k^a, k^b) \in \overline{\mathbb{D}}$. Let $\Delta k > 0$ satisfy $x = \mathcal{X}^{\text{BP}}(k^a + \Delta k, k^b + \Delta k)$. For any $\delta \in [0, \Delta k]$, we have $x \geq \mathcal{X}^{\text{BP}}(k^a + \delta, k^b + \delta)$, and

$$\begin{aligned} &\frac{\partial}{\partial \delta}[V^a(x, k^a + \delta, k^b + \delta) - V^a_{\text{BP}}(x, k^a + \delta, k^b + \delta)] \\ =&\frac{\partial V^a(x, k^a + \delta, k^b + \delta)}{\partial k^a} - \frac{\partial V^a_{\text{BP}}(x, k^a + \delta, k^b + \delta)}{\partial k^a} \\ &+ \frac{\partial V^a(x, k^a + \delta, k^b + \delta)}{\partial k^b} - \frac{\partial V^a_{\text{BP}}(x, k^a + \delta, k^b + \delta)}{\partial k^b} \leq 0, \end{aligned} \tag{C.51}$$

where the inequality uses $\frac{\partial V^a(z)}{\partial k^a} \leq p$, $\frac{\partial V^a_{\text{BP}}(z)}{\partial k^a} = p$, $\frac{\partial V^a(z)}{\partial k^b} \leq 0$, $\frac{\partial V^a_{\text{BP}}(z)}{\partial k^b} = 0$ for all $x \geq \mathcal{X}^{\text{BP}}(k^a, k^b)$. Using (C.50), we have $V^a(x, k^a + \Delta k, k^b + \Delta k) > V^a_{\text{BP}}(x, k^a + \Delta k, k^b + \Delta k)$. Combining it with (C.51), we have $V^a(x, k^a, k^b) > V^a_{\text{BP}}(x, k^a, k^b)$. $\square$

The following lemma shows that the equilibrium value $V^i(z)$ given in Proposition 3 is strictly greater than $V^i_{\text{BP}}(z)$ for all $z \in \mathbb{Z}$.

Lemma 4 *Let $V^i(z)$ denote firm i's value associated with the investment threshold $\mathcal{X}(k^a, k^b)$ given in (44), where $(\theta_1, \theta_2) \in \Theta \setminus \{(1,1)\}$. Then, $V^i(z) > V^i_{\text{BP}}(z)$ for all $z \in \mathbb{Z}$.*

Proof of Lemma 4 Using Lemma 1, we have $\theta_2 > 1 > \theta_1$ and $\theta_1 + \theta_2 > 2$, implying that $\frac{\theta_2 - 1}{1 - \theta_1} > 1$. When $k^a \leq k^b$, $\mathcal{X}(k^a, k^b) > \mathcal{X}^{\text{BP}}(k^a, k^b)$ if and only if $\theta_2 k^a + \theta_1 k^b > k^a + k^b$, which is equivalent to $k^b < \frac{\theta_2 - 1}{1 - \theta_1} k^a$. By symmetry, it follows that $\mathcal{X}(k^a, k^b) > \mathcal{X}^{\text{BP}}(k^a, k^b)$ if and only if $(k^a, k^b) \in \mathbb{D}$, where

$$\begin{aligned} \mathbb{D} =&\Big\{(k^a, k^b) \in [0, \infty) \times [0, \infty) : k^i > \frac{1 - \theta_1}{\theta_2 - 1} k^{-i}\Big\} \\ =&\Big\{(k^a, k^b) \in [0, \infty) \times [0, \infty) : \frac{1 - \theta_1}{\theta_2 - 1} k^a < k^b < \frac{\theta_2 - 1}{1 - \theta_1} k^a\Big\}. \end{aligned} \tag{C.52}$$

Since Lemma 2 shows that $V^i(z)$ satisfies inequalities (27) and (55), Lemma 3 implies that $V^i(x,k^a,k^b) > V^i_{\rm BP}(x,k^a,k^b)$ for any $x>0$ and $(k^a,k^b)\in\overline{\mathbb{D}} = \{(k^a,k^b)\in[0,\infty)\times[0,\infty) : \frac{1-\theta_1}{\theta_2-1}k^a \le k^b \le \frac{\theta_2-1}{1-\theta_1}k^a\}$.

The remainder of the proof is divided into two steps.

Step 1: We prove $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ for any $x>0$ and $0\le\frac{\theta_2-1}{1-\theta_1}k^a < k^b$.

Using (36) and the fact that $\Psi^a(x,0,k^b)=0$, we obtain

$$\lim_{k^b\to\infty} V^a(x,0,k^b) = \lim_{k^b\to\infty} H^a(0,k^b)x^\beta = 0.$$

Using (IA.B.26), we have $V^a_{k^b}(x,0,k^b)<0$ for any $x<\mathcal{X}(0,k^b)$. Combining it with (55), we have $V^a(x,0,k^b)>0$ for any $x>0$ and $k^b\ge 0$. Using (19), we have $V^a_{\rm BP}(x,0,k^b)=0$ for any $x>0$ and $k^b\ge 0$.

Using (IA.B.8), we have $\frac{\partial V^a(z)}{\partial k^a} = p$ for any $x\ge\mathcal{X}(k^a,k^b)$, $0\le k^a\le k^b$. Then for any $x\ge\mathcal{X}(k^a,k^b)$, $0\le k^a\le k^b$,

$$\begin{aligned} V^a(x,k^a,k^b) - V^a_{\rm BP}(x,k^a,k^b) &= V^a(x,0,k^b) - V^a_{\rm BP}(x,0,k^b) \\ &\quad + \int_0^{k^a}\left(\frac{\partial V^a(x,\alpha,k^b)}{\partial\alpha} - \frac{\partial V^a_{\rm BP}(x,\alpha,k^b)}{\partial\alpha}\right)d\alpha \\ &\ge V^a(x,0,k^b) - V^a_{\rm BP}(x,0,k^b) > 0, \end{aligned} \tag{C.53}$$

where the first inequality uses $\frac{\partial V^a_{\rm BP}(x,\alpha,k^b)}{\partial\alpha} \le p = \frac{\partial V^a(x,\alpha,k^b)}{\partial\alpha}$, the second inequality uses $V^a_{\rm BP}(x,0,k^b)=0$ and $V^a(x,0,k^b)>0$.

Let $x=\mathcal{X}(k^a,k^b)$. Since $\mathcal{X}(k^a,k^b)\le\mathcal{X}^{\rm BP}(k^a,k^b)$ for $0\le\frac{\theta_2-1}{1-\theta_1}k^a<k^b$, it follows from $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ and (C.44)-(C.45) that $H^a(k^a,k^b) > H^a_{\rm BP}(k^a,k^b)$. Using this inequality and applying (C.44)–(C.45) once more, we further conclude that $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ for any $x\le\mathcal{X}(k^a,k^b)\le\mathcal{X}^{\rm BP}(k^a,k^b)$ and $0\le\frac{\theta_2-1}{1-\theta_1}k^a<k^b$. Combining it with (C.53), we have $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ for any $x>0$ and $0\le\frac{\theta_2-1}{1-\theta_1}k^a<k^b$.

Step 2: We prove $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ for any $x>0$ and $0\le\frac{\theta_2-1}{1-\theta_1}k^b<k^a$.

We begin by showing that

$$\frac{\partial}{\partial k^a}H^a_{\rm BP}(k^a,0) - \frac{\partial}{\partial k^a}H^a(k^a,0) > 0, \quad k^a>0. \tag{C.54}$$

Using (IA.B.32), we obtain

$$\frac{\partial}{\partial k^a}H^a_{\rm BP}(k^a,0) = \frac{1}{\beta-1}\frac{(\beta-2)k^a-(rp+c)}{\mathcal{X}^{\rm BP}(k^a,0)^\beta}, \tag{C.55}$$

and

$$\frac{\partial}{\partial k^a}H^a(k^a,0)=\frac{1}{(\beta-1)\theta_2}\Big(\frac{\varpi k^a-(\theta_2-1)(rp+c)}{\mathcal{X}(k^a,k^a)^\beta}+\frac{(\beta-2)\theta_1 k^a-(rp+c)}{\mathcal{X}(k^a,0)^\beta}\Big), \tag{C.56}$$

where $\varpi=\theta_2(3\beta-2-\beta(\theta_1+\theta_2))-(\beta-2)\theta_1$. Denote $\theta=\theta_1+\theta_2$ and

$$\mathcal{E}(y)=\theta_2((\beta-2)y-1)\Big(\frac{1+\theta y}{1+y}\Big)^\beta-((\beta-2)\theta_1 y-1)\Big(\frac{1+\theta y}{1+\theta_1 y}\Big)^\beta-\varpi y+\theta_2-1. \tag{C.57}$$

A direct calculation shows that for $y=\frac{k^a}{rp+c}$, we have

$$\mathcal{E}(y)=(rp+c)^{\beta-1}\theta_2(1+\theta y)^\beta(\beta-1)\bigg(\frac{\partial}{\partial k^a}H^a_{\mathrm{BP}}(k^a,0)-\frac{\partial}{\partial k^a}H^a(k^a,0)\bigg). \tag{C.58}$$

Hence, it suffices to show that $\mathcal{E}(y)>0$ for all $y>0$.

Using $\varpi=\theta_2(3\beta-2-\beta\theta)-(\beta-2)\theta_1$, we have $\mathcal{E}(0)=\mathcal{E}'(0)=0$. Moreover,

$$\mathcal{E}''(y)=\beta(\beta-1)\theta_2(1+\theta y)^{\beta-2}\mathcal{H}(y), \tag{C.59}$$

where

$$\mathcal{H}(y)=(\theta-1)\frac{\alpha_1+\alpha_2 y}{(1+y)^{\beta+2}}-\frac{\alpha_3+\alpha_4 y}{(1+\theta_1 y)^{\beta+2}}, \tag{C.60}$$

and $\alpha_1=3-\theta$, $\alpha_2=\beta(\theta-1)+2$, $\alpha_3=2\theta_1-\theta_2$, $\alpha_4=\theta_1(2\theta_1+\beta\theta_2)$. By Lemma 1, we have $\theta_1<1<\theta_2$ and $2<\theta<3$. Then we have

$$\begin{aligned}\mathcal{H}(0)=&(\theta-1)\alpha_1-\alpha_3=(\theta-1)(3-\theta)+\theta-3\theta_1\\ >&(\theta-1)(3-\theta)+\theta-3=(\theta-2)(3-\theta)>0,\end{aligned} \tag{C.61}$$

where the first inequality uses $\theta_1<1$, and the second inequality uses $2<\theta<3$.

Next, we prove that $\mathcal{H}(y)$ has at most one root for $y>0$. Since Lemma 1 implies that $2<\theta<3$, we have $\alpha_1>0$ and $\alpha_2>0$. Thus, $\mathcal{H}(y)>0$ when $\alpha_3+\alpha_4 y\le 0$. Hence, to show that $\mathcal{H}(y)$ has at most one positive root, it suffices to consider the case $\alpha_3+\alpha_4 y>0$. We have $\mathcal{H}(y)=(\theta-1)\frac{\alpha_1+\alpha_2 y}{(1+y)^{\beta+2}}(1-\mathcal{R}(y))$, where $\mathcal{R}(y)$ is given by

$$\mathcal{R}(y)=\frac{\alpha_3+\alpha_4 y}{(\theta-1)(\alpha_1+\alpha_2 y)}\frac{(1+y)^{\beta+2}}{(1+\theta_1 y)^{\beta+2}}. \tag{C.62}$$

Since $\alpha_1>0$, $\alpha_2>0$ and $\alpha_3+\alpha_4 y>0$, we have $\mathcal{R}(y)>0$ for any $y>0$. We have

$$\frac{\mathcal{R}'(y)}{\mathcal{R}(y)}=\frac{\alpha_1\alpha_4-\alpha_2\alpha_3}{(\alpha_1+\alpha_2 y)(\alpha_3+\alpha_4 y)}+(\beta+2)\frac{1-\theta_1}{(1+y)(1+\theta_1 y)}, \tag{C.63}$$

Denote $\varrho=\theta_1/\theta_2$. We have

$$\alpha_1\alpha_4-\alpha_2\alpha_3=\frac{\theta}{(1+\varrho)^2}[\varrho\theta(\beta+2\varrho)(3-\theta)+(1+\varrho)(1-2\varrho)(\beta(\theta-1)+2)]. \tag{C.64}$$

When $\varrho \in (0, 1/2]$, the right-hand side of (C.64) is strictly positive. When $\varrho \in (1/2, 1)$,

$$\begin{aligned}&\frac{\partial}{\partial\theta}[\varrho\theta(\beta+2\varrho)(3-\theta)+(1+\varrho)(1-2\varrho)(\beta(\theta-1)+2)]\\=&\varrho(\beta+2\varrho)(3-2\theta)+(1+\varrho)(1-2\varrho)\beta<0.\end{aligned}\tag{C.65}$$

The right inequality in (45) implies that $\theta \leq 3-\varrho$. Using (C.65), we have

$$\begin{aligned}&\varrho\theta(\beta+2\varrho)(3-\theta)+(1+\varrho)(1-2\varrho)(\beta(\theta-1)+2)\\\geq&\varrho(3-\varrho)(\beta+2\varrho)\varrho+(1+\varrho)(1-2\varrho)(\beta(2-\varrho)+2)\\=&(1-\varrho)^2[\beta(\varrho+2)+2(1+\varrho-\varrho^2)]>0,\end{aligned}\tag{C.66}$$

where the inequality uses $\varrho \in (1/2, 1)$. Therefore, $\alpha_1\alpha_4 - \alpha_2\alpha_3 > 0$ for any $\varrho \in (0, 1)$. Combining this with (C.63), we conclude that $\mathcal{R}'(y) > 0$ for any $y > 0$. Thus, $\mathcal{H}(y) = (\theta-1)\frac{\alpha_1+\alpha_2 y}{(1+y)^{\beta+2}}(1-\mathcal{R}(y))$ has at most one root in the region $y > 0$. Since $\mathcal{H}(0) > 0$, it follows from (C.59) that $\mathcal{E}''(0) > 0$, and $\mathcal{E}''(y)$ has at most one root in the region $y > 0$. That is, $\mathcal{E}''(y) > 0$ for any $y > 0$, or there exists $\hat{y} > 0$, such that $\mathcal{E}''(y) > 0$ for $y \in (0, \hat{y})$, and $\mathcal{E}''(y) < 0$ in $y > \hat{y}$.

Therefore, either $\mathcal{E}'(y)$ is increasing for all $y > 0$, or there exists $\hat{y} > 0$ such that $\mathcal{E}'(y)$ is increasing on $y \in (0, \hat{y})$ and decreasing for $y > \hat{y}$. Note that

$$\lim_{y\to\infty}\mathcal{E}'(y)= \quad \theta_2(\beta-2)\left[(\theta_1+\theta_2)^\beta+\frac{\beta}{\beta-2}(\theta_1+\theta_2-2)-1+\frac{\theta_1}{\theta_2}-\frac{\theta_1}{\theta_2}\left(1+\frac{\theta_2}{\theta_1}\right)^\beta\right]\geq 0,\tag{C.67}$$

where the inequality uses the left inequality in (45). Thus, $\mathcal{E}'(y) > 0$ for any $y > 0$. Since $\mathcal{E}(0) = 0$, we have $\mathcal{E}(y) > 0$ for any $y > 0$, which implies (C.54).

Recall that, in the region $0 \leq \frac{\theta_2-1}{1-\theta_1}k^b \leq k^a$, we have $\mathcal{X}(k^a, k^b) \leq \mathcal{X}^{\mathrm{BP}}(k^a, k^b)$. Thus, for $0 < x \leq \mathcal{X}(k^a, 0) \leq \mathcal{X}^{\mathrm{BP}}(k^a, 0)$, we derive from (C.44)-(C.45) that

$$\begin{aligned}&\frac{\partial V^a(x,k^a,0)}{\partial k^a}-\frac{\partial V^a_{\mathrm{BP}}(x,k^a,0)}{\partial k^a}\\=&x^\beta\Big[\frac{\partial}{\partial k^a}H^a(k^a,0)-\frac{\partial}{\partial k^a}H^a_{\mathrm{BP}}(k^a,0)\Big]<0,\end{aligned}\tag{C.68}$$

where the inequality uses (C.54).

For $k^a > k^b \geq 0$, (IA.B.30) implies that

$$H^a_{k^bk^a}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta-\frac{1}{r}=\frac{-\beta k^a\rho\theta_1\mathcal{X}(k^a,k^b)^{-1}}{r}<0.\tag{C.69}$$

Thus, for any $0 < x \le \mathcal{X}(k^a, k^b)$ and $k^a > k^b \ge 0$,

$$H^a_{k^b k^a}(k^a, k^b)x^\beta - \frac{1}{r} < 0. \tag{C.70}$$

For $\mathcal{X}^{\rm BP}(k^a, 0) \ge x \ge \mathcal{X}(k^a, 0)$, we have

$$\begin{aligned}
&\frac{\partial V^a(x, k^a, 0)}{\partial k^a} = \frac{\partial V^a(x, k^a, \widehat{k}^b)}{\partial k^a} \\
=&\frac{x}{r-\mu} - \frac{2k^a + \widehat{k}^b + c}{r} + \frac{\partial}{\partial k^a} H^a(k^a, \widehat{k}^b)x^\beta \\
\le&\frac{x}{r-\mu} - \frac{2k^a + c}{r} + \frac{\partial}{\partial k^a} H^a(k^a, 0)x^\beta \\
<&\frac{x}{r-\mu} - \frac{2k^a + c}{r} + \frac{\partial}{\partial k^a} H^a_{\rm BP}(k^a, 0)x^\beta = \frac{\partial V^a_{\rm BP}(x, k^a, 0)}{\partial k^a},
\end{aligned} \tag{C.71}$$

where $\widehat{k}^b \ge 0$ satisfies $x = \mathcal{X}(k^a, \widehat{k}^b)$, the first inequality uses (C.70), the second inequality uses $\frac{\partial}{\partial k^a} H^a_{\rm BP}(k^a, 0) - \frac{\partial}{\partial k^a} H^a(k^a, 0) > 0$, and the second equality and the last equality use (C.44).

For $x \ge \mathcal{X}^{\rm BP}(k^a, 0)$, we have

$$\frac{\partial V^a(x, k^a, 0)}{\partial k^a} \le p = \frac{\partial V^a_{\rm BP}(x, k^a, 0)}{\partial k^a}. \tag{C.72}$$

Combining (C.68), (C.71), (C.72), we have

$$\frac{\partial V^a(x, k^a, 0)}{\partial k^a} - \frac{\partial V^a_{\rm BP}(x, k^a, 0)}{\partial k^a} < 0, \quad k^a > \frac{x}{\rho} - (rp + c), \tag{C.73}$$

$$\frac{\partial V^a(x, k^a, 0)}{\partial k^a} - \frac{\partial V^a_{\rm BP}(x, k^a, 0)}{\partial k^a} \le 0, \quad 0 \le k^a \le \frac{x}{\rho} - (rp + c). \tag{C.74}$$

Combining (C.73)-(C.74) with

$$\lim_{k^a \to \infty} [V^a(x, k^a, 0) - V^a_{\rm BP}(x, k^a, 0)] = \lim_{k^a \to \infty} [H^a(k^a, 0) - H^a_{\rm BP}(k^a, 0)]x^\beta = 0,$$

we conclude that $V^a(x, k^a, 0) > V^a_{\rm BP}(x, k^a, 0)$ for any $x > 0$ and $k^a > 0$.

For any $x \ge \mathcal{X}(k^a, k^b)$, $0 \le \frac{\theta_2 - 1}{1 - \theta_1} k^b < k^a$, we have $\frac{\partial V^a(x, k^a, k^b)}{\partial k^b} = 0 \ge \frac{\partial V^a_{\rm BP}(x, k^a, k^b)}{\partial k^b}$, and thus

$$\begin{aligned}
V^a(x, k^a, k^b) - V^a_{\rm BP}(x, k^a, k^b) &= V^a(x, k^a, 0) - V^a_{\rm BP}(x, k^a, 0) \\
&\quad + \int_0^{k^b} \left(\frac{\partial V^a(x, k^a, \alpha)}{\partial \alpha} - \frac{\partial V^a_{\rm BP}(x, k^a, \alpha)}{\partial \alpha} \right) d\alpha \\
&\ge V^a(x, k^a, 0) - V^a_{\rm BP}(x, k^a, 0) > 0.
\end{aligned} \tag{C.75}$$

Let $x = \mathcal{X}(k^a, k^b)$. Since $\mathcal{X}(k^a, k^b) \le \mathcal{X}^{\rm BP}(k^a, k^b)$ for $0 \le \frac{\theta_2 - 1}{1 - \theta_1} k^b < k^a$, it follows from $V^a(x, k^a, k^b) > V^a_{\rm BP}(x, k^a, k^b)$ and (C.44)-(C.45) that $H^a(k^a, k^b) > H^a_{\rm BP}(k^a, k^b)$. Using this inequality and applying (C.44)–(C.45) once more, we further conclude that $V^a(x, k^a, k^b) > V^a_{\rm BP}(x, k^a, k^b)$ for any $x \le \mathcal{X}(k^a, k^b) \le \mathcal{X}^{\rm BP}(k^a, k^b)$ and $0 \le \frac{\theta_2 - 1}{1 - \theta_1} k^b < k^a$. Combining it with

(C.75), we have $V^a(x,k^a,k^b) > V^a_{\rm BP}(x,k^a,k^b)$ for any $x>0$ and $0 \le \frac{\theta_2-1}{1-\theta_1}k^b < k^a$. $\square$

C.4 Proof of Lemma 1

By the definition of $\underline{g}(\eta)$, we have

$$\underline{g}(\eta)^\beta + \frac{\beta}{\beta-2}(\underline{g}(\eta)-2) = 1 + \frac{(1+\eta)^\beta - 1}{\eta}. \tag{C.76}$$

Denote $y = 1/\eta$. Using $\overline{g}(\eta) = 3 - 1/\eta$, we obtain

$$\begin{aligned} &1 + \frac{(1+\eta)^\beta - 1}{\eta} - [\overline{g}(\eta)^\beta + \frac{\beta}{\beta-2}(\overline{g}(\eta)-2)] \\ =&(1+y)^\beta y^{1-\beta} - (3-y)^\beta + \frac{2(y-1)}{\beta-2}. \end{aligned} \tag{C.77}$$

Define

$$G(y) = (1+y)^\beta y^{1-\beta} - (3-y)^\beta + \frac{2(y-1)}{\beta-2}.$$

Then $G(1)=0$, $G'(1) = 2^\beta + \frac{2}{\beta-2} > 0$, and $G(y) \to \infty$ as $y \to 0+$. Moreover, we have

$$\begin{aligned} G''(y) =& \beta(\beta-1)\Big((1+y)^{\beta-2}y^{-\beta-1} - (3-y)^{\beta-2}\Big) \\ =&\beta(\beta-1)y^{2-\beta}\Big((1+y)^{\beta-2}y^{-2} - y^{\beta-2}(3-y)^{\beta-2}\Big) \\ >&\beta(\beta-1)y^{2-\beta}\Big((1+y)^{\beta-2} - y^{\beta-2}(3-y)^{\beta-2}\Big) > 0, \end{aligned} \tag{C.78}$$

where the first inequality uses $y \in (0,1)$, and the second inequality uses $(1+y) - y(3-y) = (y-1)^2 > 0$. Thus, $G(y)$ is strictly convex on $y \in (0,1)$. Together with $G(1)=0$, $G'(1)>0$, and $G(y) \to \infty$ as $y \to 0+$, the convexity of $G(y)$ implies that $G(y)$ has a unique root y^* in the region $y \in (0,1)$, and that $G(y) > 0$ for all $y \in (0, y^*)$ and $G(y) < 0$ for all $y \in (y^*, 1)$. Combining this with (C.77), we obtain that, for $\eta^* = 1/y^*$,

$$\overline{g}(\eta^*)^\beta + \frac{\beta}{\beta-2}(\overline{g}(\eta^*)-2) = 1 + \frac{(1+\eta^*)^\beta - 1}{\eta^*}, \tag{C.79}$$

$$\overline{g}(\eta)^\beta + \frac{\beta}{\beta-2}(\overline{g}(\eta)-2) > 1 + \frac{(1+\eta)^\beta - 1}{\eta}, \quad \forall \eta \in (1,\eta^*), \tag{C.80}$$

$$\overline{g}(\eta)^\beta + \frac{\beta}{\beta-2}(\overline{g}(\eta)-2) < 1 + \frac{(1+\eta)^\beta - 1}{\eta}, \quad \forall \eta > \eta^*. \tag{C.81}$$

Since the function $\breve{g}(s) = s^\beta + \frac{\beta}{\beta-2}(s-2)$ is strictly increasing for $s>0$, (C.76) and (C.79)-(C.81) imply that $\overline{g}(\eta) - \underline{g}(\eta) = 0$ has a unique root η^* on the domain $\eta > 1$. Moreover, $\overline{g}(\eta) > \underline{g}(\eta)$ for any $\eta \in (1,\eta^*)$, and $\overline{g}(\eta) < \underline{g}(\eta)$ for any $\eta > \eta^*$.

Therefore, for any $(\theta_1,\theta_2) \in \Theta$, the maximum value of θ_2/θ_1 is η^*. Since $\overline{g}(\eta)$ is increasing

in $\eta > 0$, the sum $\theta_1 + \theta_2$ attains its maximum over $(\theta_1, \theta_2) \in \Theta$ if and only if $(\theta_1, \theta_2) = (\theta_1^*, \theta_2^*)$, where (θ_1^*, θ_2^*) is defined in (48).

Moreover, by Lemma 5, we have $\theta_2 > 1 > \theta_1 > 0$ and $2 < \theta_1 + \theta_2 < 3$ for any $(\theta_1, \theta_2) \in \Theta \setminus \{(1,1)\}$. $\square$

C.5 Proofs of Theorems 1 and 5

Recall that firm a is the follower and firm b is the leader. Let K^{F*} and K^{L*} denote, respectively, the capital stocks of the follower and the leader under the BP closed-loop strategy $(\boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L)$. Using (A.7) and (A.9), we obtain

$$K_t^{L*} = k_0^b + \left(\frac{M_t}{\rho} - (rp + c) - k_0^a - k_0^b \right)^+, \quad t > 0. \tag{C.82}$$

Combining (A.8), (A.10) and (C.82), we obtain

$$\boldsymbol{u}_*^F(Z_t^*, \boldsymbol{u}_*^L(Z_t^*)) = 0, \qquad \boldsymbol{w}_*^F(Z_t^*, \Delta K_t^{L*}) = 0, \tag{C.83}$$

where $Z_t^* = (X_t, K_t^{F*}, K_t^{L*})$. Therefore, (C.83) implies that $K_t^{F*} = K_0^{F*} = k_0^a$ for any $t \geq 0$.

Following the arguments in Appendix C.3, we can show that $V_{\rm BP}^i(z)$ satisfies inequalities (27) and (55), as well as Conditions A.1-A.2. To verify that the BP closed-loop strategy $(\boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L)$ satisfies inequalities (25)-(26), it suffices to prove that

$$V_{\rm BP}^a(z) = J^a(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L) \geq J^a(z; \boldsymbol{\varphi}^F, \boldsymbol{\varphi}_*^L), \tag{C.84}$$

$$V_{\rm BP}^b(z) = J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L) \geq J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}^L). \tag{C.85}$$

We proceed in three steps.

Step 1: We prove $V_{\rm BP}^a(z) \geq J^a(z; \boldsymbol{\varphi}^F, \boldsymbol{\varphi}_*^L)$, where $(\boldsymbol{\varphi}^F, \boldsymbol{\varphi}_*^L) \in \mathcal{S}$.

Let K^F and K^L denote the capital stocks of the follower and the leader, respectively, associated with the strategy pair $(\boldsymbol{\varphi}^F, \boldsymbol{\varphi}_*^L)$. Using (A.7) and (A.9), we conclude that $X_t \leq \mathcal{X}^{\rm BP}(K_t^F, K_t^L)$ for any $t > 0$.

Denote $\overline{\mathbb{N}}_{ab} = \{(x, k^a, k^b) \in \mathbb{Z} : x \leq \mathcal{X}^{\rm BP}(k^a, k^b)\}$. Then $(X_t, K_t^F, K_t^L) \in \overline{\mathbb{N}}_{ab}$ for all $t > 0$. Since $V_{\rm BP}^a \in \mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab})$, applying Itô formula to $e^{-rs} V_{\rm BP}^a(X_s, K_s^F, K_s^L)$, $s \in (0, t]$, for $t > 0$ and taking expectation, we have

$$V_{\rm BP}^a(x, K_{0+}^F, K_{0+}^L) = \mathbb{E}^z \left[e^{-rt} V_{\rm BP}^a(X_t, K_t^F, K_t^L) - \int_0^t e^{-rs} \mathcal{L} V_{\rm BP}^a(X_s, K_s^F, K_s^L) ds \right.$$

$$- \int_0^t e^{-rs} \left[\frac{\partial V^a_{\mathrm{BP}}}{\partial k^a}(X_s, K^F_s, K^L_s) dK^{F,C}_s + \frac{\partial V^a_{\mathrm{BP}}}{\partial k^b}(X_s, K^F_s, K^L_s) dK^{L,C}_s \right]$$
$$- \sum_{s \in (0,t]} e^{-rs} [V^a_{\mathrm{BP}}(X_s, K^F_{s+}, K^L_{s+}) - V^a_{\mathrm{BP}}(X_s, K^F_s, K^L_s)] \Bigg], \tag{C.86}$$

where $\mathcal{L}$ is the infinitesimal generator given by

$$\mathcal{L}V(z) := \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V(z)}{\partial x^2} + \mu x \frac{\partial V(z)}{\partial x} - rV(z), \tag{C.87}$$

$K^{i,C}$ denotes the continuous part of K^i. The expectation of the local martingale term in (C.86) vanishes by the regularity condition $\mathbb{E}^z \left[\int_0^t |e^{-rs} \frac{\partial V^a_{\mathrm{BP}}}{\partial x}(X_s, K^F_s, K^L_s) \sigma(X_s)|^2 ds \right] < +\infty, \ \forall\, t \geq 0$, as stated in Condition A.2.

Using $(X_s, K^F_s, K^L_s) \in \overline{\mathbb{N}}_{ab}$ for all $s > 0$, we can show that

$$-\mathcal{L}V^a_{\mathrm{BP}}(X_s, K^F_s, K^L_s) = \pi^a(X_s, K^F_s, K^L_s) \ \text{ for any } s > 0.$$

Substituting this into (C.86) and using $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^a} \leq p$, $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^b} \leq 0$, we obtain

$$\begin{aligned} &V^a_{\mathrm{BP}}(x, K^F_{0+}, K^L_{0+}) \\ \geq &\mathbb{E}^z \Big[e^{-rt} V^a_{\mathrm{BP}}(X_t, K^F_t, K^L_t) + \int_0^t e^{-rs} \pi^a(X_s, K^F_s, K^L_s) ds - \int_{0+}^t e^{-rs} p dK^F_s \Big]. \end{aligned} \tag{C.88}$$

Next, applying the condition $\lim_{n \to +\infty} \mathbb{E}^z [e^{-rt_n} V^a_{\mathrm{BP}}(X_{t_n}, K^F_{t_n}, K^L_{t_n})] = 0$ from Condition A.2, replacing t with t_n in (C.88), and letting $n \to \infty$, we obtain

$$\begin{aligned} &V^a_{\mathrm{BP}}(x, K^F_{0+}, K^L_{0+}) \\ \geq &\lim_{n \to +\infty} \mathbb{E}^z \Big[\int_0^{t_n} e^{-rs} \pi^a(X_s, K^F_s, K^L_s) ds - \int_{0+}^{t_n} e^{-rs} p dK^F_s \Big] \\ = &\mathbb{E}^z \Big[\int_0^{+\infty} e^{-rs} \pi^a(X_s, K^F_s, K^L_s) ds - \int_{0+}^{+\infty} e^{-rs} p dK^F_s \Big], \end{aligned} \tag{C.89}$$

where the convergence of the first integral follows from the dominated convergence theorem together with (23), and the second integral is well-defined by the monotone convergence theorem.

Finally, using $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^a} \leq p$, $\frac{\partial V^a_{\mathrm{BP}}(z)}{\partial k^b} \leq 0$, $K^L_{0+} \geq k^b$, and $K^F_{0+} \geq k^a$, we derive

$$V^a_{\mathrm{BP}}(x, k^a, k^b) \geq V^a_{\mathrm{BP}}(x, k^a, K^L_{0+}) \geq V^a_{\mathrm{BP}}(x, K^F_{0+}, K^L_{0+}) - p(K^F_{0+} - k^a). \tag{C.90}$$

Combining (C.90) with (C.89), we obtain

$$V^a_{\mathrm{BP}}(z) \geq \mathbb{E}^z \Big[\int_0^{+\infty} e^{-rs} \pi^a(X_s, K^F_s, K^L_s) ds - \int_0^{+\infty} e^{-rs} p dK^F_s \Big] = J^a(z; \boldsymbol{\varphi}^F, \boldsymbol{\varphi}^L_*).$$

Step 2: We prove $V^a_{\mathrm{BP}}(z) = J^a(z; \boldsymbol{\varphi}^F_*, \boldsymbol{\varphi}^L_*)$.

Replacing K^L and K^F in (C.86) with K^{L*} and K^{F*}, respectively, and noting that K_s^{L*} is continuous in $s > 0$ and $K_s^{F*} = K_0^{F*}$ for all $s \geq 0$, we obtain

$$V_{\rm BP}^a(x, K_{0+}^{F*}, K_{0+}^{L*}) = \mathbb{E}^z\Big[e^{-rt}V_{\rm BP}^a(X_t, K_t^{F*}, K_t^{L*}) - \int_0^t e^{-rs}\mathcal{L}V_{\rm BP}^a(X_s, K_s^{F*}, K_s^{L*})ds - \int_{0+}^t e^{-rs}\frac{\partial V_{\rm BP}^a}{\partial k^b}(X_s, K_s^{F*}, K_s^{L*})dK_s^{L*}\Big]. \tag{C.91}$$

Since $\frac{\partial V_{\rm BP}^a}{\partial k^b}(z) = 0$ for any $x \geq \mathcal{X}^{\rm BP}(k^a, k^b)$ and $dK_s^{L*} = 0$ for $X_s < \mathcal{X}^{\rm BP}(K_s^{F*}, K_s^{L*})$, we have

$$\int_{0+}^t e^{-rs}\frac{\partial V_{\rm BP}^a}{\partial k^b}(X_s, K_s^{F*}, K_s^{L*})dK_s^{L*} = 0. \tag{C.92}$$

Substituting $-\mathcal{L}V_{\rm BP}^a(X_s, K_s^{F*}, K_s^{L*}) = \pi^a(X_s, K_s^{F*}, K_s^{L*})$, $s > 0$ and (C.92) into (C.91), we obtain

$$V_{\rm BP}^a(x, K_{0+}^{F*}, K_{0+}^{L*}) = \mathbb{E}^z\Big[e^{-rt}V_{\rm BP}^a(X_t, K_t^{F*}, K_t^{L*}) + \int_0^t e^{-rs}\pi^a(X_s, K_s^{F*}, K_s^{L*})ds\Big]. \tag{C.93}$$

Using the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V_{\rm BP}^a(X_{t_n}, K_{t_n}^{F*}, K_{t_n}^{L*})] = 0$ from Condition A.2, replacing t with t_n in (C.93), and letting n approach infinity, we obtain

$$V_{\rm BP}^a(x, K_0^{F*}, K_{0+}^{L*}) = V_{\rm BP}^a(x, K_{0+}^{F*}, K_{0+}^{L*}) = \mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}\pi^a(X_s, K_s^{F*}, K_s^{L*})ds\Big]. \tag{C.94}$$

Note that $K_{0+}^{L*} > K_0^{L*}$ if and only if $X_0 > \mathcal{X}^{\rm BP}(K_0^{F*}, K_0^{L*})$. Since $V_{\rm BP}^a(x, k^a, k^b) = pk^a$ for $x \geq \mathcal{X}^{\rm BP}(k^a, k^b)$, and since $X_0 = \mathcal{X}^{\rm BP}(K_0^{F*}, K_{0+}^{L*})$ when $X_0 > \mathcal{X}^{\rm BP}(K_0^{F*}, K_0^{L*})$, we have $V_{\rm BP}^a(x, K_0^{F*}, K_{0+}^{L*}) = V_{\rm BP}^a(x, K_0^{F*}, K_0^{L*}) = V_{\rm BP}^a(z)$. Finally, plugging $V_{\rm BP}^a(x, K_0^{F*}, K_{0+}^{L*}) = V_{\rm BP}^a(z)$ into (C.94), we have

$$V_{\rm BP}^a(z) = \mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}\pi^a(X_s, K_s^{F*}, K_s^{L*})ds\Big] = J^a(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L),$$

where the second equality uses $K_s^{F*} = K_0^{F*}$ for all $s \geq 0$.

In summary, combining the results in Steps 1 and 2, we obtain (C.84).

Step 3: We prove $V_{\rm BP}^b(z) = J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L) \geq J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}^L)$, where $(\boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}^L) \in \mathcal{S}$.

Let K^F and K^L denote the capital stocks of the follower and the leader, respectively, associated with the strategy pair $(\boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}^L)$. Using (A.8) and (A.10), we conclude that $X_t \leq \mathcal{X}^{\rm BP}(K_t^F, K_t^L)$ for any $t > 0$.

Using $\frac{\partial V_{\rm BP}^b(z)}{\partial k^b} \leq p$, $\frac{\partial V_{\rm BP}^b(z)}{\partial k^a} \leq 0$, together with the fact that $X_t \leq \mathcal{X}^{\rm BP}(K_t^F, K_t^L)$ for any $t > 0$, and proceeding as in Step 1, we obtain $V_{\rm BP}^b(z) \geq J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}^L)$.

Since $\frac{\partial V_{\rm BP}^b(z)}{\partial k^b} = p$ for any $x \geq \mathcal{X}^{\rm BP}(k^a, k^b)$ and $dK_s^{L*} = 0$ for $X_s < \mathcal{X}^{\rm BP}(K_s^{F*}, K_s^{L*})$, it

follows that

$$\int_{0+}^{t} e^{-rs} \frac{\partial V_{\rm BP}^b}{\partial k^b}(X_s, K_s^{F*}, K_s^{L*}) dK_s^{L*} = \int_{0+}^{t} e^{-rs} p dK_s^{L*}. \tag{C.95}$$

Using (C.95) and $V_{\rm BP}^b(x, K_{0+}^{F*}, K_{0+}^{L*}) - p(K_{0+}^{L*} - k_0^b) = V_{\rm BP}^b(z)$, and repeating the argument in Step 2, we obtain $V_{\rm BP}^b(z) = J^b(z; \boldsymbol{\varphi}_*^F, \boldsymbol{\varphi}_*^L)$. $\square$

C.6 Other Technical Results

We present two corollaries that are used in the paper, and relegate their proofs to Internet Appendix.

Corollary 1 *Let $\mathcal{X}(k^a, k^b)$ be as defined in (44), where $\theta_2 \geq \theta_1 > 0$. Let $V^a(z)$ and $V^b(z)$ be the functions given in Proposition 3. Then the right inequality in (45) is a necessary condition for (27), and the left inequality in (45) is a necessary condition for $V^i(z) \geq V_{\rm BP}^i(z)$, $\forall\ z \in \mathbb{Z}$.*

Corollary 2 *For the equilibrium strategies in Theorem 3 and the special case $k_0^a = k_0^b = 0$, firm value at time 0 is given as follows.*

(i) In the inaction region where $x < \underline{x} = \rho(rp + c)$, for any $(\theta_1, \theta_2) \in \Theta$,

$$V^i(x, 0, 0) = \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2}\right)^2\right] \frac{\underline{x}^2}{\rho\beta(\beta - 2)(r - \mu)} \left(\frac{x}{\underline{x}}\right)^{\beta}. \tag{C.96}$$

(ii) In the investment region where $x \geq \underline{x} = \rho(rp + c)$, for any $(\theta_1, \theta_2) \in \Theta$,

$$V^i(x, 0, 0) = \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2}\right)^2\right] \left[\frac{\beta}{\beta - 2} \frac{(x - \underline{x})^2}{2r\rho^2} + \frac{(rp + c)(x - \underline{x}) + \frac{\underline{x}^2}{\rho\beta}}{(\beta - 2)(r - \mu)}\right]. \tag{C.97}$$

D Technical Details for Section 8

In this appendix, we provide technical details for the closed-loop equilibria with nonlinear triggers in Section 8. Given the symmetry of $\mathcal{X}(k^a, k^b)$ given in (60) along the 45° line, it is sufficient for us to characterize $\mathcal{G}(k^a, k^b)$ for the $k^b \geq k^a \geq 0$ domain in the first quadrant.

Condition A.3 *In the $k^b \geq k^a$ region, there exists a lower bound, $\underline{k} > 0$, for k^a and k^b, such that the following conditions hold:*

(i) $k^a + k^b < \mathcal{G}(k^a, k^b) < 2k^a + k^b$;

(ii) $\mathcal{G}(k^a, k^b)$ *is continuously differentiable* ($\mathcal{C}^1$) *and satisfies* $\frac{\partial \mathcal{G}(k^a,k^b)}{\partial k^a} > 0$, $\frac{\partial \mathcal{G}(k^a,k^b)}{\partial k^b} > 0$*; and*

(iii) there exists a $\mathcal{C}^1$ *function* $\phi(z_1, z_2)$ *such that at each* (z_1, z_2)*, either* $\frac{\partial \phi(z_1,z_2)}{\partial z_1} \neq 0$ *or* $\frac{\partial \phi(z_1,z_2)}{\partial z_2} \neq 0$ *holds, and*

$$\phi\big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\big) = 0. \tag{D.1}$$

Part (i) requires that the nonlinear investment boundary $\mathcal{X}(k^a, k^b)$ lie strictly above the linear boundary $\rho(k^a + k^b + rp + c)$ for the BP closed-loop equilibrium: $k^a + k^b < \mathcal{G}(k^a, k^b)$, and strictly below the linear boundary $\rho(2k^a + k^b + rp + c)$ for the open-loop equilibrium: $\mathcal{G}(k^a, k^b) < 2k^a + k^b$.

As competition in any other closed-loop equilibrium is less severe than in the BP zero-profit closed-loop equilibrium, firms must invest at slower rates than in the BP equilibrium. This explains why $\mathcal{X}(k^a, k^b)$ must lie above the investment boundary for the BP equilibrium so that $k^a + k^b < \mathcal{G}(k^a, k^b)$ holds.

On the other hand, as firms compete less in the open-loop equilibrium than in all closed-loop equilibria, firms must invest at faster rates in a closed-loop equilibrium than in the open-loop equilibrium. This explains why the nonlinear trigger $\mathcal{X}(k^a, k^b)$ must lie below the investment boundary for the open-loop equilibrium so that $\mathcal{G}(k^a, k^b) < 2k^a + k^b$ holds.

As the industry output price decreases in either firm's capital stock, for a fixed demand shock X_t, a firm invests less in an industry with a larger capital stock, *ceteris paribus.* Part (ii) captures this property. Both parts (i) and (ii) are needed to ensure that (27), (28) and (55) hold.

Finally, part (iii) is implied by the first-order condition (59) for marginal q on the investment boundary $\mathcal{X}(k^a, k^b)$. To see this, first note that for any (k^a, k^b), (40) follows from (59). Second, using (40), (41), $\frac{\partial^2 H^a(k^a,k^b)}{\partial k^a \partial k^b} = \frac{\partial^2 H^a(k^a,k^b)}{\partial k^b \partial k^a}$, and the symmetry property of the investment boundary $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$, we obtain the following partial differential equation (PDE):

$$\rho k^b \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b} + \Big[\mathcal{X}(k^a, k^b) - \rho\big(2k^b + k^a + rp + c\big)\Big]\frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} = 0, \quad k^a \leq k^b. \tag{D.2}$$

Next, we solve PDE (D.2) and show that $\mathcal{X}(k^a, k^b) = \rho\big(\mathcal{G}(k^a, k^b) + rp + c\big)$ for $k^a \leq k^b$,

where $\mathcal{G}(k^a, k^b)$ satisfies the condition in part (iii). To this end, we denote

$$\mathcal{G}(k^a, k^b) = \frac{\mathcal{X}(k^a, k^b)}{\rho} - \big(rp + c\big) \tag{D.3}$$

for $k^b \geq k^a > 0$. Then, (D.2) is equivalent to

$$k^b \mathcal{G}_{k^b}(k^a, k^b) + \Big(\mathcal{G}(k^a, k^b) - k^a - 2k^b\Big)\mathcal{G}_{k^a}(k^a, k^b) = 0. \tag{D.4}$$

Note that (D.4) is a quasilinear first-order PDE and $k^b > 0$. We obtain the general (local) solution for (D.4) by solving the following characteristic equations (McOwen, 2004):

$$\frac{dk^b}{k^b} = \frac{dk^a}{\mathcal{G} - k^a - 2k^b}, \quad d\mathcal{G} = 0. \tag{D.5}$$

Solving $d\mathcal{G} = 0$ yields $\mathcal{G} = \lambda_1$ for some constant λ_1. Also, we have

$$\frac{d(\mathcal{G} - k^a - k^b)}{dk^b} = \frac{d(-k^a - k^b)}{dk^b} = \frac{-dk^a}{dk^b} - 1 = -\frac{\mathcal{G} - k^a - k^b}{k^b}, \tag{D.6}$$

where the first equality uses $d\mathcal{G} = 0$ and the last equality uses the first equation in (D.5). Solving (D.6), we obtain $\mathcal{G} - k^a - k^b = \frac{\lambda_2}{k^b}$ for some constant λ_2. Hence, two independent solutions to (D.5) are

$$\mathcal{G} = \lambda_1 \quad \text{and} \quad (\mathcal{G} - k^a - k^b)k^b = \lambda_2, \tag{D.7}$$

where λ_1 and λ_2 are constants. The general solution to (D.4) is thus characterized by (D.1) and (D.2) admits the following general solution locally:

$$\mathcal{X}(k^a, k^b) = \rho\Big(\mathcal{G}(k^a, k^b) + rp + c\Big), \quad k^a \leq k^b. \tag{D.8}$$

In addition, if $\mathcal{G}(k^a, k^b)$ satisfies (D.1) for any $k^b \geq k^a \geq \underline{k} > 0$, taking partial derivatives with respect to k^a and k^b in (D.1), we obtain

$$\phi_{z_1}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)k^b(\mathcal{G}_{k^a} - 1) + \phi_{z_2}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)\mathcal{G}_{k^a} = 0, \tag{D.9}$$

$$\phi_{z_1}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)(\mathcal{G} - k^a - 2k^b + k^b\mathcal{G}_{k^b}) + \phi_{z_2}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)\mathcal{G}_{k^b} = 0. \tag{D.10}$$

Since $\phi_{z_1} \neq 0$ or $\phi_{z_2} \neq 0$, (D.9) and (D.10) imply:

$$(\mathcal{G} - k^a - 2k^b + k^b\mathcal{G}_{k^b})\mathcal{G}_{k^a} = k^b\mathcal{G}_{k^b}(\mathcal{G}_{k^a} - 1). \tag{D.11}$$

Simplifying the above equation, we can show that (D.4) holds for any $k^b \geq k^a \geq \underline{k}$.

D.1 Proof of Theorem 4

We only need to verify the conditions stated in Theorem 2 and Proposition 9 with $\mathbb{Z} = (0,\infty)\times[\underline{k},\infty)\times[\underline{k},\infty)$. By applying the same arguments as in the proof of Theorem 3, we can verify that Conditions A.1-A.2 are satisfied for functions $V^a(z)$ and $V^b(z)$ given in Proposition 3, with the trigger function $\mathcal{X}(k^a,k^b)$ as defined in (60). If (27) and (55) hold, then Lemma 3 and Condition A.3-(i) imply that $V^i(z) > V^i_{\rm BP}(z)$ for all $z\in\mathbb{Z}$. It therefore remains to prove (27) and (55).

For convenience, we denote

$$\mathbf{I}_i(x) = \{(k^a,k^b)\in[\underline{k},\infty)\times[\underline{k},\infty) : x \geq \mathcal{X}(k^a,k^b), k^i \leq \mathcal{K}(x)\}, \tag{D.12}$$

$$\mathbf{N}_i(x) = \{(k^a,k^b)\in[\underline{k},\infty)\times[\underline{k},\infty) : (k^a,k^b)\notin \mathbf{I}_i(x)\}, \tag{D.13}$$

$\mathbf{I}_{ab}(x) = \mathbf{I}_a(x)\cap\mathbf{I}_b(x)$, and $\mathbf{N}_{ab}(x) = \mathbf{N}_a(x)\cap\mathbf{N}_b(x)$.

We prove the case for $V^a(z)$ only, as the argument for $V^b(z)$ is analogous. It is straightforward to verify that $V^a(z)$ is continuously differentiable, $V^a_{k^a}(x,k^a,k^b) = p$ for $(k^a,k^b)\in\mathbf{I}_a(x)$, $V^a_{k^b}(x,k^a,k^b) = 0$ for $(k^a,k^b)\in\mathbf{I}_b(x)$, and (IA.B.5)-(IA.B.7) hold.

We complete the proof in three steps.

Step 1: We prove that

$$V^a_{k^a}(x,k^a,k^b) = p, \qquad x\geq\mathcal{X}(k^a,k^b),\ k^a\geq\underline{k},\ k^b\geq\underline{k}. \tag{D.14}$$

In Appendix D, we have shown that $\mathcal{X}(k^a,k^b)$ as given by (D.8) satisfies (D.2) for any $k^a\geq k^b\geq\underline{k}>0$. By the symmetry of $\mathcal{X}(k^a,k^b)$, it follows from (D.2) that

$$\rho k^a\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a} + \Big[\mathcal{X}(k^a,k^b) - \rho\big(2k^a+k^b+rp+c\big)\Big]\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b} = 0, \quad k^a\geq k^b. \tag{D.15}$$

Let $\mathcal{Y}(k^a,k^b) = \left(\frac{2k^a+k^b+rp+c}{r} - \frac{\mathcal{X}(k^a,k^b)}{r-\mu}\right)\mathcal{X}(k^a,k^b)^{-\beta}$. A direct calculation shows that (D.15) is equivalent to the following equation:

$$\frac{\partial\mathcal{Y}(k^a,k^b)}{\partial k^b} = \frac{\partial\big[\frac{k^a}{r}\mathcal{X}(k^a,k^b)^{-\beta}\big]}{\partial k^a}, \quad k^a>k^b\geq\underline{k}. \tag{D.16}$$

For $k^a\geq k^b$, substituting (41) into (D.16) yields $\frac{\partial\mathcal{Y}(k^a,k^b)}{\partial k^b} = \frac{\partial}{\partial k^a}[\frac{\partial H^a(k^a,k^b)}{\partial k^b}] = \frac{\partial}{\partial k^b}[\frac{\partial H^a(k^a,k^b)}{\partial k^a}]$. Hence, $\frac{\partial H^a(k^a,k^b)}{\partial k^a} - \mathcal{Y}(k^a,k^b)$ is independent of k^b:

$$\frac{\partial H^a(k^a,k^b)}{\partial k^a} - \mathcal{Y}(k^a,k^b) = C(k^a), \quad k^a\geq k^b\geq\underline{k}, \tag{D.17}$$

for some function $C(k^a)$.

Since (40) holds for $k^a \leq k^b$, we have $\frac{\partial H^a(k^a,k^b)}{\partial k^a} = \mathcal{Y}(k^a,k^b)$ for $k^a \leq k^b$, which, together with (D.17) and the continuity of $\frac{\partial H^a(k^a,k^b)}{\partial k^a}$ and $\mathcal{Y}(k^a,k^b)$, implies that $C(k^a) = 0$. Hence, (40) also holds for $k^a > k^b \geq \underline{k}$. It then follows from (36) that

$$V^a_{k^a}(x,k^a,k^b) \mid_{x=\mathcal{X}(k^a,k^b)} = p \quad \text{for } k^a > k^b \geq \underline{k}. \tag{D.18}$$

Therefore, $V^a_{k^a}(x,k^a,k^b) = V^a_{k^a}(x,k^a,\widehat{k}^b) = p$ for any $(k^a,k^b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b$ is uniquely determined by $\mathcal{X}(k^a,\widehat{k}^b) = x$. Since $V^a_{k^a}(x,k^a,k^b) = p$ for any $(k^a,k^b) \in \mathbf{I}_a(x)$, we have proved (D.14).

Step 2: We prove that

$$V^a_{k^a}(x,k^a,k^b) < p, \quad (k^a,k^b) \in \mathbf{N}_{ab}(x),\ x > 0. \tag{D.19}$$

For $k^b \geq k^a \geq \underline{k}$, we have

$$\mathcal{X}(k^a,k^b) - \rho(2k^a + k^b + pr + c) = \rho\Big(\mathcal{G}(k^a,k^b) - 2k^a - k^b\Big) < 0, \tag{D.20}$$

where the inequality follows from the second inequality in part (i) of Condition A.3. Using (D.20) and the symmetry of $\mathcal{X}(k^a,k^b)$, for $k^a \geq k^b \geq \underline{k}$, we obtain

$$\mathcal{X}(k^a,k^b) < \rho(2k^b + k^a + pr + c) \leq \rho(2k^a + k^b + pr + c), \tag{D.21}$$

where the second inequality uses $k^a \geq k^b$. Combining (D.20) and (D.21), we infer

$$\mathcal{X}(k^a,k^b) \leq \rho(2k^a + k^b + pr + c), \quad k^a \geq \underline{k},\ k^b \geq \underline{k}. \tag{D.22}$$

By employing similar arguments to those in Step 4 of the proof of Lemma 2, we can show that (D.19) holds by using (D.22) and (D.14).

Step 3: We prove that

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) < 0, \quad k^b > k^a \geq \underline{k}. \tag{D.23}$$

Using (36), we have $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = -\frac{k^a}{r} + H^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta$. Thus (D.23) is equivalent to $H^a_{k^b}(k^a,k^b) < \mathcal{H}(k^a,k^b)$, where $\mathcal{H}(k^a,k^b) = \frac{k^a}{r}\mathcal{X}(k^a,k^b)^{-\beta}$. Additionally, by using (41), we have $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = 0$ for $k^a \geq k^b \geq \underline{k}$ and thus $H^a_{k^b}(k,k) = \mathcal{H}(k,k)$. As a result, $H^a_{k^b}(k^a,k^b) < \mathcal{H}(k^a,k^b)$ is equivalent to

$$\int_{k^a}^{k^b} \frac{\partial H^a_{k^b}(\alpha,k^b)}{\partial \alpha} d\alpha > \int_{k^a}^{k^b} \frac{\partial \mathcal{H}(\alpha,k^b)}{\partial \alpha} d\alpha, \quad k^b > k^a \geq \underline{k}. \tag{D.24}$$

Then, it suffices to show that $H^a_{k^a k^b}(k^a,k^b) > \frac{\partial \mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b > k^a \geq \underline{k}$. Using (40), we

have

$$\begin{aligned}&H^a_{k^ak^b}(k^a,k^b)-\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}\\=&\frac{(\beta-1)\Big[\mathcal{X}_{k^b}(k^a,k^b)\big(\mathcal{X}(k^a,k^b)-\rho(2k^a+k^b+pr+c)\big)+\rho\mathcal{X}_{k^a}(k^a,k^b)k^a\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}}.\end{aligned}\tag{D.25}$$

According to part (ii) of Condition A.3, we have $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}>0$, $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}>0$ for any $k^b\geq k^a\geq\underline{k}$. By combining (D.2) and (60), we find that

$$\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}\Big/\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}=\frac{k^a+2k^b-\mathcal{G}(k^a,k^b)}{k^b},\quad k^b\geq k^a\geq\underline{k}.\tag{D.26}$$

We infer from (D.26) and part (i) of Condition A.3 that $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}\Big/\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}\in(0,1)$ for any $k^b\geq k^a\geq\underline{k}$. By the symmetry of $\mathcal{X}(k^a,k^b)$, we conclude that

$$\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}\in(0,1),\quad\forall k^a\geq\underline{k},\ k^b\geq\underline{k}.\tag{D.27}$$

For $k^b>k^a$, we have

$$\begin{aligned}&\mathcal{X}(k^a,k^b)-\rho(2k^a+k^b+pr+c)+\rho\frac{\mathcal{X}_{k^a}(k^a,k^b)}{\mathcal{X}_{k^b}(k^a,k^b)}k^a\\=&\rho\left[\left(2-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}\right)k^b+k^a\right]-\rho\left[\left(2-\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}\right)k^a+k^b\right]\\=&\rho\left[\left(1-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}\right)k^b-\left(1-\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}\right)k^a\right]>0,\end{aligned}\tag{D.28}$$

where the first equality uses $\mathcal{X}(k^a,k^b)=\rho\left([2-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}]k^b+k^a+rp+c\right)$ for $k^b>k^a$, and the inequality follows from (D.27).

Substituting (D.28) into (D.25) gives $H^a_{k^ak^b}(k^a,k^b)>\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b>k^a\geq\underline{k}$. Hence, (D.23) holds.

Finally, by employing similar arguments to those in Step 2 of the proof of Lemma 2, we can prove that $V^a_{k^b}(x,k^a,k^b)<0$ for any $(k^a,k^b)\in\mathbf{N}_b(x)$ and $x>0$. $\square$

D.2 Proof of Proposition 7

We only need to verify that Condition A.3 holds. First, consider the case $\phi(z_1,z_2)=z_1-\lambda$ for a constant $\lambda>0$. In this case, $\mathcal{G}=\frac{\lambda}{k^b}+k^a+k^b$ solves equation (D.1). For any $\underline{k}>\sqrt{\lambda}$, we have $\lambda<k^ak^b$ for all $k^a\geq\underline{k}$ and $k^b\geq\underline{k}$. Hence,

$$0<\frac{\mathcal{G}-k^a-k^b}{k^a}=\frac{\lambda}{k^ak^b}<1.$$

Then, part (i) of Condition A.3 holds. Moreover,

$$\frac{\partial}{\partial k^a}\Big(\frac{\lambda}{k^b}+k^a+k^b\Big)=1>0,\quad \text{and}\quad \frac{\partial}{\partial k^b}\Big(\frac{\lambda}{k^b}+k^a+k^b\Big)=1-\frac{\lambda}{(k^b)^2}>0, \tag{D.29}$$

where the right inequality follows from $k^b\geq \underline{k}>\sqrt{\lambda}$. Thus, part (ii) of Condition A.3 holds.

Next, consider the case $\phi(z_1,z_2)=\lambda z_1-z_2$ for a constant $\lambda>0$. In this case, $\mathcal{G}(k^a,k^b)=(k^a+k^b)\frac{\lambda k^b}{\lambda k^b-1}$ solves equation (D.1). For any $\underline{k}>\frac{3}{\lambda}$, we have $\lambda k^b-1>0$ for all $k^b\geq\underline{k}$. Then, for any $k^b\geq k^a\geq\underline{k}$, we have $\mathcal{G}(k^a,k^b)-k^a-k^b=\frac{k^a+k^b}{\lambda k^b-1}>0$ and

$$\frac{\mathcal{G}(k^a,k^b)-k^a-k^b}{k^a}=\frac{k^a+k^b}{(\lambda k^b-1)k^a}\leq\frac{\underline{k}+k^b}{(\lambda k^b-1)\underline{k}}\leq\frac{2\underline{k}}{(\lambda\underline{k}-1)\underline{k}}=\frac{2}{\lambda\underline{k}-1}<1, \tag{D.30}$$

where the first inequality follows from the fact that $\frac{k^a+k^b}{(\lambda k^b-1)k^a}$ is decreasing in $k^a\in[\underline{k},k^b]$, the second inequality follows from the fact that $\frac{\underline{k}+k^b}{(\lambda k^b-1)\underline{k}}$ is decreasing in $k^b\geq\underline{k}$, and the last inequality uses $\underline{k}>\frac{3}{\lambda}$. Hence, part (i) of Condition A.3 holds. Moreover,

$$\frac{\partial}{\partial k^a}\Big(\frac{(k^a+k^b)\lambda k^b}{\lambda k^b-1}\Big)=\frac{\lambda k^b}{\lambda k^b-1}>0, \tag{D.31}$$

and

$$\frac{\partial}{\partial k^b}\Big(\frac{(k^a+k^b)\lambda k^b}{\lambda k^b-1}\Big)=\frac{(\lambda k^b-1)^2-1-\lambda k^a}{(\lambda k^b-1)^2}\geq\frac{(\lambda k^b-1)^2-1-\lambda k^b}{(\lambda k^b-1)^2}=\frac{\lambda k^b(\lambda k^b-3)}{(\lambda k^b-1)^2}>0, \tag{D.32}$$

where the first inequality uses $k^b\geq k^a$, the second inequality uses $k^b\geq\underline{k}>\frac{3}{\lambda}$. Thus, part (ii) of Condition A.3 holds. □

E Constant Elasticity Model

In this appendix, we extend our analysis to a setting where firm i's profit is given by

$$\pi^i(x,k^a,k^b)=x(k^a+k^b)^{-1/\gamma}k^i\,, \tag{E.1}$$

where $\gamma>1$ is a constant. Consider the counterfactual setting in which neither firm ever invests, i.e., when $dK_t^a=dK_t^b\equiv 0$ for all t. Then, firm i's value is solely derived from its existing assets in place and is given by the special solution to the valuation equation (35):

$$\Psi^i(z)=\mathbb{E}^z\left[\int_0^\infty e^{-rt}\pi^i(X_t,k^a,k^b)dt\right]=x\frac{(k^a+k^b)^{-1/\gamma}k^i}{r-\mu} \tag{E.2}$$

for $z=(x,k^a,k^b)$. We denote $\psi^i(k^a,k^b)=\Psi^i(x,k^a,k^b)/x=\frac{(k^a+k^b)^{-1/\gamma}k^i}{r-\mu}$.

E.1 Monopoly Solution

First, consider a monopoly setting where the monopolist's profit function is given by $\pi(x,k) = xk^{1-1/\gamma}$, where k is the monopolist's capital stock.

The monopolist optimally invests when and only when $X_t \geq \mathcal{X}^{\text{mon}}(K_t)$, where $\mathcal{X}^{\text{mon}}(\,\cdot\,)$ is the trigger function given by

$$\mathcal{X}^{\text{mon}}(k) = \frac{\rho\gamma}{\gamma - 1} k^{\frac{1}{\gamma}}\,. \tag{E.3}$$

In (E.3),

$$\rho = \frac{p\beta}{\beta - 1}(r - \mu), \tag{E.4}$$

and β is given in (7). For convergence, we require $\beta > \gamma$ and $r > \mu$.

E.2 Duopoly Solution

For brevity, we focus on symmetric closed-loop equilibria with nonlinear trigger functions, as we have done for the linear demand case in Section 8.

Given a symmetric trigger function $\mathcal{X}(k^a, k^b)$ under certain regularity conditions,[14] we define firm i's investment region $\mathbb{I}_i$ and the inaction region $\mathbb{N}_{ab}$ as (30) and (29), respectively.

We can show that the corresponding value function for $x \leq \mathcal{X}(k^a, k^b)$ is given by (36), where $\Psi^i(z)$ is defined in (E.2). By substituting (36) into (38) and (39), we obtain

$$\frac{\partial H^a(k^a,k^b)}{\partial k^a} = \Big(p - \mathcal{X}(k^a,k^b)\psi^a_{k^a}(k^a,k^b)\Big)\mathcal{X}(k^a,k^b)^{-\beta}, \quad k^a \leq k^b\,, \tag{E.5}$$

$$\frac{\partial H^a(k^a,k^b)}{\partial k^b} = -\psi^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^{1-\beta}, \quad k^a \geq k^b\,, \tag{E.6}$$

where $\psi^a_{k^a}(k^a,k^b) = (k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}$ and $\psi^a_{k^b}(k^a,k^b) = (k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$.

With the constraint (59), we can endogenously determine the following nonlinear trigger

[14] We only consider the trigger function $\mathcal{X}(k^a,k^b)$ that is continuously differentiable on $\{(k^a,k^b) \in [\underline{k},\infty) \times [\underline{k},\infty) : k^a \neq k^b\}$ with $\mathcal{X}_{k^a}(k^a,k^b) > 0$ and $\mathcal{X}_{k^b}(k^a,k^b) > 0$, where $\underline{k} > 0$ is a constant. Without loss of generality, we assume that $\mathcal{X}(k^a,k^b)$ has an upper bound given by the monopolist's trigger $\rho\frac{\gamma}{\gamma-1}(k^a+k^b)^{\frac{1}{\gamma}}$ and a lower bound given by the trigger of perfectly competitive equilibrium $\rho(k^a+k^b)^{\frac{1}{\gamma}}$ (Back and Paulsen, 2009). These bounds guarantee the well-posedness of the firms' value functions under the trigger function $\mathcal{X}(k^a,k^b)$.

functions (see Appendix E.3):

$$\mathcal{X}(k^a,k^b) := \begin{cases} \Big(\rho+\mathcal{G}(k^a,k^b)\Big)(k^a+k^b)^{\frac{1}{\gamma}}, & \text{if } k^a \le k^b, \\ \Big(\rho+\mathcal{G}(k^b,k^a)\Big)(k^a+k^b)^{\frac{1}{\gamma}}, & \text{if } k^a \ge k^b, \end{cases} \tag{E.7}$$

where ρ is given in (E.4), and $\mathcal{G}(k^a,k^b)$ is to be determined. Next we state conditions for $\mathcal{G}(k^a,k^b)$ under which there exist closed-loop equilibria satisfying (59) as well as (27) and (55). Given the symmetry along the 45° line, it is sufficient for us to characterize $\mathcal{G}(k^a,k^b)$ for the $k^b \ge k^a \ge 0$ region in the first quadrant.

Condition A.4 *There exists a constant $\underline{k} > 0$ such that in $k^b \ge k^a \ge \underline{k}$ region,*

(i) $0 < \mathcal{G}(k^a,k^b) < \frac{\rho k^a}{\gamma(k^a+k^b)-k^a}$*;*

(ii) $\mathcal{G}(k^a,k^b)$ *is* $\mathcal{C}^1$*, and the trigger function* $\mathcal{X}(k^a,k^b) = \big(\rho+\mathcal{G}(k^a,k^b)\big)(k^a+k^b)^{\frac{1}{\gamma}}$ *satisfies* $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} > 0$, $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} > 0$*; and*

(iii) there exists a $\mathcal{C}^1$ *function* $\phi(z_1,z_2)$ *such that at each* (z_1,z_2)*, either* $\frac{\partial \phi(z_1,z_2)}{\partial z_1} \neq 0$ *or* $\frac{\partial \phi(z_1,z_2)}{\partial z_2} \neq 0$ *holds, and*

$$\phi\left(k^b\mathcal{G}, \frac{1}{k^b\mathcal{G}} \ln \frac{1}{(\mathcal{G}+\rho)(k^a+k^b)^{\frac{1}{\gamma}}}\right) = 0. \tag{E.8}$$

The left inequality in part (i) of Condition A.4 ensures that $\mathcal{X}(k^a,k^b)$ given in (E.7) is higher than $\rho(k^a+k^b)^{\frac{1}{\gamma}}$, the trigger function of the perfect competition equilibrium. The right inequality in part (i) of Condition A.4 ensures that $\mathcal{X}(k^a,k^b)$ given in (E.7) is lower than the trigger function of the open-loop equilibrium, i.e.,

$$\mathcal{X}(k^a,k^b) < \frac{\rho}{1-\frac{\min\{k^a,k^b\}}{\gamma(k^a+k^b)}}(k^a+k^b)^{\frac{1}{\gamma}}. \tag{E.9}$$

By choosing different ϕ to obtain $\mathcal{G}$ satisfying Condition A.4, we can derive various endogenous nonlinear trigger functions. Below we present two sets of trigger functions, where $\phi(z_1,z_2) = z_1 - \lambda$ and $\phi(z_1,z_2) = \frac{\ln(\lambda/z_1)}{z_1} + z_2$, respectively, with $\lambda > 0$ being a constant.

Proposition 10 *For any $\lambda > 0$, there exist closed-loop equilibria characterized by the following nonlinear investment boundaries $\mathcal{X}(k^a,k^b)$:*

(i) If $\min(k^a,k^b) > \frac{\lambda(2\gamma-1)}{\rho}$*, then*

$$\mathcal{X}(k^a,k^b) = \Big(\rho + \frac{\lambda}{\max\{k^a,k^b\}}\Big)(k^a+k^b)^{\frac{1}{\gamma}}. \tag{E.10}$$

(ii) If $\min(k^a, k^b) > \frac{1}{2}\Big(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\Big)^{\frac{\gamma}{\gamma+1}}$, *then*

$$\mathcal{X}(k^a, k^b) = \rho(k^a + k^b)^{\frac{1}{\gamma}} + \frac{\lambda}{\sqrt{(\frac{\rho\max\{k^a,k^b\}}{2})^2 + \lambda\max\{k^a, k^b\}(k^a + k^b)^{-\frac{1}{\gamma}}} + \frac{\rho\max\{k^a,k^b\}}{2}}. \tag{E.11}$$

Next, we present symmetric closed-loop equilibria associated with the nonlinear trigger functions $\mathcal{X}(k^a, k^b)$ as given in equation (E.7). We define $\mathcal{K}(x)$ and $\vartheta_1(k^a, k^b)$, $\vartheta_2(k^a, k^b)$ as specified by (63) and (64), respectively, with the trigger functions $\mathcal{X}(k^a, k^b)$ as given in equation (E.7).

Theorem 6 *Let* $\boldsymbol{\varphi}^i_* := (\boldsymbol{u}^i_*, \boldsymbol{w}^i_*)$ *be firm* i*'s closed-loop strategy given by* (61)-(62)*, with* $\mathcal{X}(k^a, k^b)$*,* $\mathcal{K}(x)$*, and* $\vartheta_1(k^a, k^b)$*,* $\vartheta_2(k^a, k^b)$ *given by* (E.7)*,* (63)*, and* (64)*, respectively. Under Condition A.4 with* $\min(k^a_0, k^b_0) \geq \underline{k}$*, the closed-loop strategy pair* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ *constitutes a closed-loop equilibrium.*

We can verify that the capital processes under the strategy given in Theorem 6 take the explicit form as in (C.25), but with $\mathcal{X}(k^a, k^b)$ given by (E.7). Then, for the equilibrium strategy given in Theorem 6, firm i's equilibrium value $V^i(z)$ takes the same form as in Proposition 3, but with $\Psi^i(z)$ and $\mathcal{X}(k^a, k^b)$ as defined in (E.2) and (E.7), respectively.

E.3 Determining Endogenous Nonlinear Trigger Functions in Duopoly

By imposing the constraint (59), we infer that (38) holds for any (k^a, k^b), as does (E.5) for any (k^a, k^b). Combining (E.5) and (E.6) and using $\frac{\partial^2 H^a(k^a,k^b)}{\partial k^a \partial k^b} = \frac{\partial^2 H^a(k^a,k^b)}{\partial k^b \partial k^a}$ for $k^a \geq k^b$, and $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$, we obtain

$$\frac{k^b}{k^a + k^b}\mathcal{X}(k^a, k^b)\frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b} + \left[\Big(\gamma - \frac{k^b}{k^a + k^b}\Big)\mathcal{X}(k^a, k^b) - \gamma\rho(k^a + k^b)^{\frac{1}{\gamma}}\right]\frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} = 0 \tag{E.12}$$

for any $k^a \leq k^b$. Lemma 6 in Internet Appendix shows that the PDE (E.12) admits the following general solution:

$$\mathcal{X}(k^a, k^b) = \Big(\rho + \mathcal{G}(k^a, k^b)\Big)(k^a + k^b)^{\frac{1}{\gamma}}, \tag{E.13}$$

where $\mathcal{G}(k^a, k^b)$ satisfies (E.8). By using (E.13) and symmetry, we obtain the nonlinear trigger functions as given in (E.7).

Internet Appendices to

"Option Exercise Games and the q Theory of Investment"

IA.A Additional Technical Results

Lemma 5 *Assume the pair (θ_1, θ_2) satisfies Condition 1.*

(i) When $\theta_1 = \theta_2$, we have $\theta_1 = \theta_2 = 1$.

(ii) When $\theta_2 > \theta_1$, we have $\theta_2 > 1 > \theta_1$, and

$$2 - \frac{\theta_2}{\theta_1} - \theta_2 < 2(\theta_1 - 1). \tag{IA.A.1}$$

(iii) The largest value of $\theta_1 + \theta_2$ is reached only at (θ_1^*, θ_2^*) defined in (48)*.*

(iv) The smallest value of $\theta_1 + \theta_2$ is reached only at $\theta_2 = \theta_1 = 1$.

(v) When $\theta_2 > \theta_1$,

$$\underline{g}(\theta_2/\theta_1) > \breve{g}(\theta_2/\theta_1), \tag{IA.A.2}$$

where $\breve{g}(\,\cdot\,)$ is given by

$$\breve{g}(\eta) = \left(1 - \frac{2}{\beta}\right)(1+\eta) + \frac{2}{\beta}\left(1 + \frac{1}{\eta}\right) + \left(2 - \frac{2}{\beta}\right)\frac{\eta - 1}{(1+\eta)^{\beta-1} - 1}. \tag{IA.A.3}$$

Moreover, (IA.A.2) *together with $\theta_1 + \theta_2 \geq \underline{g}(\theta_2/\theta_1)$ implies that*

$$2 + (\beta - 2)\frac{\theta_2}{\theta_1} - \beta\theta_2 < \frac{\beta\frac{\theta_2}{\theta_1}(2 - \frac{\theta_2}{\theta_1} - \theta_2)}{(1 + \frac{\theta_2}{\theta_1})^{\beta} - 1}. \tag{IA.A.4}$$

Lemma 6 *The PDE* (E.12) *admits a general solution as given by* (E.13)*. Under Condition A.4, $\mathcal{X}(k^a, k^b)$ given by* (E.13) *satisfies* (E.12) *for any $k^b \geq k^a \geq \underline{k}$.*

Lemma 7 *Let the function $V^i(z)$ be as given in Proposition 3 but with $\Psi^i(z)$ and $\mathcal{X}(k^a, k^b)$ as defined in* (E.2) *and* (E.7)*, respectively. Under Condition A.4, the functions $V^a(z)$ and $V^b(z)$ satisfy* (D.14)*, the inequalities* (27)-(28) *and* (55)*, and Condition A.1 with $\mathbb{Z} = \{(x, k^a, k^b) : x > 0, k^a \geq \underline{k}, k^b \geq \underline{k}\}$.*

Lemma 8 *Assume Condition A.4 holds. Let the strategy* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ *be as defined in Theorem 6, and let the function* $V^i(z)$ *be as given in Proposition 3 but with* $\Psi^i(z)$ *and* $\mathcal{X}(k^a, k^b)$ *as defined in* (E.2) *and* (E.7)*, respectively. Then, the strategy* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ *and* (V^a, V^b) *satisfy Condition A.2 with* $\mathbb{Z} = \{(x, k^a, k^b) : x > 0, k^a \geq \underline{k}, k^b \geq \underline{k}\}$.

IA.B Additional Proofs

Proof of Lemma 2 We will prove the result for $V^a(z)$ only, as the case for $V^b(z)$ can be handled in a similar manner. To simplify notation, for any function $f \in \{V^a, H^a\}$, we denote the partial derivatives as follows: $f_{k^a}(x, k^a, k^b) = \frac{\partial f(x,k^a,k^b)}{\partial k^a}$, $f_{k^b}(x, k^a, k^b) = \frac{\partial f(x,k^a,k^b)}{\partial k^b}$, $f_{k^a k^a}(x, k^a, k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial (k^a)^2}$, $f_{k^a k^b}(x, k^a, k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial k^a \partial k^b}$, $f_{k^b k^a}(x, k^a, k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial k^b \partial k^a}$, and $f_{k^b k^b}(x, k^a, k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial (k^b)^2}$, where $H^a(k^a, k^b)$ is a function describing firm a's option value, as given by (42)-(43). Define

$$\vartheta_1(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a \leq k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a > k^b, \end{cases} \quad \text{and} \quad \vartheta_2(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a \leq k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a > k^b. \end{cases} \tag{IA.B.1}$$

Substituting $\mathcal{X}(k^a, k^b)$ as defined in (44) into (IA.B.1), we have

$$\vartheta_1(k^a, k^b) = \rho\theta_1 \quad \text{and} \quad \vartheta_2(k^a, k^b) = \rho\theta_2. \tag{IA.B.2}$$

In the following, we will simplify the notation by writing $\vartheta_1(k^a, k^b)$ and $\vartheta_2(k^a, k^b)$ as ϑ_1 and ϑ_2, respectively.

Given that $\beta > 2$, $r > 0$, and $c \geq 0$, it follows that $\mathcal{X}(k^a, k^b) > 0$ for any $(k^a, k^b) \in \mathbb{R}^2_+$. According to (42)-(43), we have $H^a \in \mathcal{C}^{1,1}(\mathbb{R}^2_+) \bigcap \mathcal{C}^{\infty,\infty}(\{(k^a, k^b) \in \mathbb{R}^2_+ : k^a \neq k^b\})$. Therefore, for $k^a \neq k^b$, we have $H^a_{k^a k^b}(k^a, k^b) = H^a_{k^b k^a}(k^a, k^b)$. By (36), we see that V^a belongs to $\mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab})$ and satisfies (35), where $\overline{\mathbb{N}}_{ab}$ is the closure of $\mathbb{N}_{ab}$ given by (29). Using (36), (42)-(43), we obtain that

$$V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p, \quad k^b \geq k^a \geq 0, \tag{IA.B.3}$$

$$V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) = 0, \quad k^a \geq k^b \geq 0. \tag{IA.B.4}$$

Using Proposition 3 and (IA.B.3)-(IA.B.4), we can obtain the following derivatives. In the region where $k^a \leq \mathcal{K}(x)$ and $k^b \leq \mathcal{K}(x)$ (the square BCDO region in Figure 4),

$$\frac{\partial V^a(x, k^a, k^b)}{\partial k^a} = p, \quad \frac{\partial V^a(x, k^a, k^b)}{\partial k^b} = 0, \quad \frac{\partial V^a(x, k^a, k^b)}{\partial x} = V^a_x(x, \mathcal{K}(x), \mathcal{K}(x)). \tag{IA.B.5}$$

In the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^b > \mathcal{K}(x)$ (the ABC region in Figure 4),

$$\frac{\partial V^a(x,k^a,k^b)}{\partial k^a} = p, \quad \frac{\partial V^a(x,k^a,k^b)}{\partial k^b} = V^a_{k_b}(x,\widehat{k}^a,k^b), \quad \frac{\partial V^a(x,k^a,k^b)}{\partial x} = V^a_x(x,\widehat{k}^a,k^b), \tag{IA.B.6}$$

where $\widehat{k}^a$ satisfies $\mathcal{X}(\widehat{k}^a, k^b) = x$. In the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^a > \mathcal{K}(x)$ (the CDE region in Figure 4),

$$\frac{\partial V^a(x,k^a,k^b)}{\partial k^a} = V^a_{k_a}(x,k^a,\widehat{k}^b), \quad \frac{\partial V^a(x,k^a,k^b)}{\partial k^b} = 0, \quad \frac{\partial V^a(x,k^a,k^b)}{\partial x} = V^a_x(x,k^a,\widehat{k}^b), \tag{IA.B.7}$$

where $\widehat{k}^b$ satisfies $\mathcal{X}(k^a, \widehat{k}^b) = x$.

By combining (IA.B.5)-(IA.B.7), we observe that $V^a_{k^a}(z)$, $V^a_{k^b}(z)$ and $V^a_x(z)$ are continuous in $z = (x, k^a, k^b) \in \mathbb{Z}$, particularly across the boundary $x = \mathcal{X}(k^a, k^b)$, and

$$V^a_{k^a}(z) = p, \quad z \in \mathbb{I}_a, \tag{IA.B.8}$$

$$V^a_{k^b}(z) = 0, \quad z \in \mathbb{I}_b, \tag{IA.B.9}$$

where $\mathbb{I}_i$ is defined by (30). Consequently, $V^a \in \mathcal{C}^{1,1,1}(\mathbb{Z})$ and Condition A.1 is satisfied. It remains to prove that inequalities (27) and (55) hold. We first consider the case $\theta_2 > \theta_1$. Lemma 5 implies that $\theta_2 > 1 > \theta_1$. We will complete our proof in four steps.

Step 1: We prove that

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) < 0, \quad k^b > k^a \geq 0. \tag{IA.B.10}$$

Using (36), for any $k^b \geq k^a \geq 0$, we have

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = -\frac{k^a}{r} + H^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta. \tag{IA.B.11}$$

Taking the partial derivative with respect to k^b in (40) and using the equality $H^a_{k^ak^b}(k^a,k^b) = H^a_{k^bk^a}(k^a,k^b)$ for $k^b > k^a \geq 0$, we obtain

$$\begin{aligned} &H^a_{k^ak^b}(k^a,k^b) - \frac{\vartheta_1}{\vartheta_2} H^a_{k^ak^a}(k^a,k^b) = \frac{1}{r}\Big(1 - 2\frac{\vartheta_1}{\vartheta_2}\Big)\mathcal{X}(k^a,k^b)^{-\beta} \\ =&\frac{1}{r}\Big(1 - 2\frac{\vartheta_1}{\vartheta_2}\Big)\frac{1}{(1-\beta)\vartheta_2}\frac{\partial(\mathcal{X}(k^a,k^b)^{1-\beta})}{\partial k^a}. \end{aligned}$$

Taking the integral with respect to k^a in both sides of the above equation and using the

relation $\frac{\vartheta_1}{\vartheta_2} = \frac{\theta_1}{\theta_2}$, we obtain

$$\int H^a_{k^a k^b}(k^a, k^b)dk^a = \frac{\theta_1}{\theta_2} H^a_{k^a}(k^a, k^b) + \frac{1}{r}\Big(1 - 2\frac{\theta_1}{\theta_2}\Big)\frac{\mathcal{X}(k^a, k^b)^{1-\beta}}{(1-\beta)\vartheta_2}. \tag{IA.B.12}$$

For $k^b \geq k^a \geq 0$, we derive from (IA.B.12) that

$$\begin{aligned} H^a_{k^b}(k^a, k^b) =& H^a_{k^b}(k^b, k^b) - \frac{\theta_1}{\theta_2}[H^a_{k^a}(k^b, k^b) - H^a_{k^a}(k^a, k^b)] \\ &- \frac{1}{r}\Big(1 - 2\frac{\theta_1}{\theta_2}\Big)\frac{\mathcal{X}(k^b, k^b)^{1-\beta} - \mathcal{X}(k^a, k^b)^{1-\beta}}{(1-\beta)\vartheta_2}. \end{aligned} \tag{IA.B.13}$$

Denote

$$\mathcal{G}(k^a, k^b) = \Big(2 - \frac{\theta_2}{\theta_1}\Big)k^a + k^b + rp + c - (\theta_2 k^a + \theta_1 k^b + pr + c)\frac{1}{\beta - 1}(\beta + \frac{1}{\theta_1} - \frac{2}{\theta_2}). \tag{IA.B.14}$$

Substituting (40)-(41) into (IA.B.13), and using (IA.B.11), we obtain

$$V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) = \frac{\theta_1}{r\theta_2}\left[\mathcal{G}(k^a, k^b) - \Big(\frac{\mathcal{X}(k^a, k^b)}{\mathcal{X}(k^b, k^b)}\Big)^{\beta}\mathcal{G}(k^b, k^b)\right]. \tag{IA.B.15}$$

To prove (IA.B.10), we first prove that

$$V^a_{k^b}(\mathcal{X}(0, k^b), 0, k^b) < 0, \quad k^b \geq 0. \tag{IA.B.16}$$

To achieve this, we define

$$\mathcal{H}(k^b) := \Big(\frac{\mathcal{X}(k^b, k^b)}{\mathcal{X}(0, k^b)}\Big)^{\beta}\mathcal{G}(0, k^b) - \mathcal{G}(k^b, k^b), \tag{IA.B.17}$$

and conclude that (IA.B.16) is equivalent to $\mathcal{H}(k^b) < 0$ for any $k^b \geq 0$, using (IA.B.15). Now, we prove that $\mathcal{H}(k^b) < 0$, $\forall\ k^b > 0$. Let $\mathcal{Y}(k^b) := \frac{\mathcal{X}(k^b, k^b)}{\mathcal{X}(0, k^b)}$. Then $\mathcal{H}(k^b) = \mathcal{Y}(k^b)^{\beta}\mathcal{G}(0, k^b) - \mathcal{G}(k^b, k^b)$. Given that $\theta_2/\theta_1 > 1$, $\theta_2 \geq 1$, and (45), we have

$$\begin{aligned} \mathcal{G}_{k^a} &:= \frac{\partial \mathcal{G}(k^a, k^b)}{\partial k^a} = \Big(2 - \frac{\theta_2}{\theta_1} - \theta_2\Big)\frac{\beta}{\beta - 1} < 0, \\ \mathcal{G}_{k^b} &:= \frac{\partial \mathcal{G}(k^a, k^b)}{\partial k^b} = (2 + (\beta - 2)\frac{\theta_2}{\theta_1} - \beta\theta_2)\frac{1}{\beta - 1}\frac{\theta_1}{\theta_2} < 0, \\ \mathcal{G}(0, 0) &= \frac{pr + c}{(\beta - 1)\theta_2}\Big(2 - \frac{\theta_2}{\theta_1} - \theta_2\Big) < 0. \end{aligned} \tag{IA.B.18}$$

It follows that

$$\mathcal{G}(k^a, k^b) < 0, \quad (k^a, k^b) \in \mathbb{R}^2_+. \tag{IA.B.19}$$

Using $\mathcal{Y}(k) = 1 + \frac{\theta_2}{\theta_1} - \frac{\theta_2}{\theta_1}\frac{\mathcal{X}(0,0)/(\rho\theta_1)}{k+\mathcal{X}(0,0)/(\rho\theta_1)}$, $\mathcal{Y}(0) = 1$, and $\mathcal{Y}'(0) = \frac{\theta_2}{\theta_1}\frac{1}{\mathcal{X}(0,0)/(\rho\theta_1)} = \frac{\theta_2}{pr+c}$, we infer

$$\mathcal{H}'(0) = \beta\mathcal{Y}'(0)\mathcal{G}(0,0) - \mathcal{G}_{k^a} = \beta\frac{\theta_2}{\xi}\frac{1}{pr+c}\mathcal{G}(0,0) - \mathcal{G}_{k^a} = 0 \tag{IA.B.20}$$

and

$$\begin{aligned}\mathcal{H}''(k) =& \beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[\Big(\beta - 1 + \frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\Big)\mathcal{G}(0,k) + 2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)}\mathcal{G}_{k^b}\right]\\ =& \beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[(\beta-1)\mathcal{G}(0,k) + \frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\mathcal{G}\Big(0, -\frac{\mathcal{X}(0,0)}{\vartheta_1}\Big)\right],\end{aligned} \tag{IA.B.21}$$

where the second equality uses $\mathcal{G}(0,k) = \mathcal{G}\big(0, -\frac{\mathcal{X}(0,0)}{\rho\theta_1}\big) + (k + \frac{\mathcal{X}(0,0)}{\rho\theta_1})\mathcal{G}_{k^b}$ and $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}(k + \frac{\mathcal{X}(0,0)}{\vartheta_1}) + 2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)} = 0$. A direct calculation yields that $\frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2} = -2\frac{\theta_1}{\theta_2}$, and

$$\begin{aligned}&(\beta-1)\mathcal{G}(0,0) + \frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2}\mathcal{G}\Big(0, -\frac{\mathcal{X}(0,0)}{\rho\theta_1}\Big)\\ =&\Big(\beta - 1 - 2\frac{\theta_1}{\theta_2}\Big)\mathcal{G}(0,0) + 2\frac{\theta_1}{\theta_2}\frac{\mathcal{X}(0,0)}{\rho\theta_1}\mathcal{G}_{k^b}\\ =&\frac{pr+c}{\theta_2}\Big[2 - \frac{\theta_2}{\theta_1} - \theta_2 + 2(1-\theta_1)\Big] < 0,\end{aligned} \tag{IA.B.22}$$

where the inequality utilizes (IA.A.1). By combining (IA.B.21), (IA.B.22), $\mathcal{Y}(0) = 1$, and $\mathcal{Y}'(0) > 0$, we can conclude that $\mathcal{H}''(0) < 0$.

A direct calculation shows that $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}$ and $\mathcal{G}(0,k)$ are affine functions of k. Consequently, the expression in the bracket in (IA.B.21) is linear in k. Then we conclude from $\mathcal{H}''(0) < 0$, (IA.B.21), $\mathcal{Y}'(k)^2 > 0$, and $\mathcal{Y}(k) > 0$, $\forall k \geq 0$ that either $\mathcal{H}''(k) < 0$, $\forall\ k \geq 0$, or there exists $\widetilde{k} > 0$ such that $\mathcal{H}''(k) < 0$, $\forall k \in [0, \widetilde{k})$ and $\mathcal{H}''(k) > 0$, $\forall\ k > \widetilde{k}$. As a result, $\lim_{k\to\infty}\mathcal{H}'(k) \in [-\infty, \infty]$ exists, and

$$\lim_{k\to+\infty}\frac{\mathcal{H}(k)}{k} = \lim_{k\to\infty}\mathcal{H}'(k). \tag{IA.B.23}$$

Moreover, since we have shown $\mathcal{H}'(0) = 0$ in (IA.B.20), then $\mathcal{H}'(k) < 0$, $\forall\ k > 0$, or there exists $\widehat{k} \geq 0$ such that $\mathcal{H}'(k) < 0$, $\forall k \in (0, \widehat{k})$ and $\mathcal{H}'(k) > 0$, $\forall\ k > \widehat{k}$. Note that $\lim_{k\to+\infty}\mathcal{Y}(k) =$

$1+\frac{\theta_2}{\theta_1}$ and

$$\begin{aligned}&\lim_{k\to+\infty}\frac{\mathcal{H}(k)}{k}=\Big(1+\frac{\theta_2}{\theta_1}\Big)^{\beta}\mathcal{G}_{k^b}-(\mathcal{G}_{k^a}+\mathcal{G}_{k^b})\\=&\frac{\theta_1}{\theta_2}\frac{1}{\beta-1}\left[\Big(\Big(1+\frac{\theta_2}{\theta_1}\Big)^{\beta}-1\Big)\Big(2+(\beta-2)\frac{\theta_2}{\theta_1}-\beta\theta_2\Big)-\beta\frac{\theta_2}{\theta_1}\Big(2-\frac{\theta_2}{\theta_1}-\theta_2\Big)\right]\leq 0,\end{aligned}\tag{IA.B.24}$$

where the inequality utilizes (IA.A.4) and the condition $\theta_2\geq\theta_1>0$. Combining (IA.B.23) and (IA.B.24), we obtain $\lim_{k\to\infty}\mathcal{H}'(k)\leq 0$. Since we have shown that $\mathcal{H}'(k)<0$, $\forall\ k>0$, or there exists $\widehat{k}\geq 0$ such that $\mathcal{H}'(k)<0$, $\forall k\in(0,\widehat{k})$ and $\mathcal{H}'(k)>0$, $\forall\ k>\widehat{k}$, we must have $\mathcal{H}'(k)<0$ for all $k>0$. Since $\mathcal{H}(0)=0$, it follows that $\mathcal{H}(k)<0$, $\forall\ k>0$. Then we have completed the proof of (IA.B.16).

Given that (IA.B.18) implies that $\frac{\partial^2}{\partial(k^a)^2}\mathcal{G}(k^a,k^b)=0$, we derive from (IA.B.15) that

$$\frac{\partial^2}{\partial(k^a)^2}V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)=-\frac{\theta_1}{r\theta_2}\beta(\beta-1)\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^b,k^b)}\Big)^{\beta}\Big(\frac{\vartheta_2}{\mathcal{X}(k^a,k^b)}\Big)^2\mathcal{G}(k^b,k^b)>0,$$

for any $k^b\geq k^a\geq 0$, where the inequality uses (IA.B.19). The convexity of $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)$ with respect to k^a implies that

$$\begin{aligned}&V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)<\max\Big\{V^a_{k^b}(\mathcal{X}(0,k^b),0,k^b),V^a_{k^b}(\mathcal{X}(k^b,k^b),k^b,k^b)\Big\}\\=&\max\Big\{V^a_{k^b}(\mathcal{X}(0,k^b),0,k^b),0\Big\}=0,\qquad\forall k^a\in(0,k^b),\end{aligned}\tag{IA.B.25}$$

where the first equality uses $V^a_{k^b}(\mathcal{X}(k^b,k^b),k^b,k^b)=0$ and the second equality uses (IA.B.16). By combining (IA.B.16) and (IA.B.25), we can see that (IA.B.10) holds.

Step 2: We prove that

$$V^a_{k^b}(x,k^a,k^b)<0,\quad(k^a,k^b)\in\mathbf{N}_b(x),\ x>0.\tag{IA.B.26}$$

Recall that for any $x\in(0,\mathcal{X}(k^a,k^b)]$ and $(k^a,k^b)\in\mathbb{R}^2_+$, $V^a(x,k^a,k^b)$ is given by (36). Consequently, $V^a_{k^b}(x,k^a,k^b)=-\frac{k^a}{r}+H^a_{k^b}(k^a,k^b)x^{\beta}$, which is linear in $x^{\beta}>0$.

For $k^b>k^a\geq 0$, since $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)<0$ by (IA.B.10), and $\lim_{x\to 0+}V^a_{k^b}(x,k^a,k^b)=-\frac{k^a}{r}\leq 0$, it follows that $V^a_{k^b}(x,k^a,k^b)<0$ for any $x\in(0,\mathcal{X}(k^a,k^b))$. For $k^a\geq k^b\geq 0$ with $k^a>0$, since $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)=0$ by (IA.B.4), and $\lim_{x\to 0+}V^a_{k^b}(x,k^a,k^b)=-\frac{k^a}{r}<0$, it follows that $V^a_{k^b}(x,k^a,k^b)<0$ for any $x\in(0,\mathcal{X}(k^a,k^b))$. Thus, we conclude that $V^a_{k^b}(x,k^a,k^b)<0$ for any $(k^a,k^b)\in\mathbf{N}_{ab}(x)$, $x>0$.

In addition, for any $(k^a,k^b)\in\mathbf{I}_a(x)\setminus\mathbf{I}_{ab}(x)$, we have

$$V^a_{k^b}(x,k^a,k^b)=V^a_{k^b}(x,\widehat{k}^a,k^b)<0,\tag{IA.B.27}$$

where $\widehat{k}^a = \frac{1}{\theta_2}\left(\frac{x}{\rho} - (rp+c)\right) - \frac{\theta_1}{\theta_2}k^b$ is determined by (54), the equality is derived from (IA.B.3), and the inequality follows from (IA.B.10). Since $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)) \supseteq \mathbf{N}_b(x)$, we infer (IA.B.26).

Step 3: We prove that

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) < p, \quad k^a > k^b \geq 0. \tag{IA.B.28}$$

Using (36), for $k^a \geq k^b \geq 0$, we have

$$\begin{aligned} &V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) - p \\ =&\frac{\mathcal{X}(k^a,k^b)}{r-\mu} + H^a_{k^a}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta - \frac{2k^a+k^b+pr+c}{r}. \end{aligned} \tag{IA.B.29}$$

Taking the partial derivative with respect to k^a in (41), we obtain the following result for $k^a > k^b \geq 0$:

$$\begin{aligned} H^a_{k^bk^a}(k^a,k^b) =&\frac{1}{r}[\mathcal{X}(k^a,k^b)^{-\beta} - \beta k^a\vartheta_1\mathcal{X}(k^a,k^b)^{-1-\beta}] \\ =&\frac{1}{r}\Big[\frac{1}{(1-\beta)\vartheta_2}\frac{\partial(\mathcal{X}(k^a,k^b)^{1-\beta})}{\partial k^b} + k^a\frac{\vartheta_1}{\vartheta_2}\frac{\partial(\mathcal{X}(k^a,k^b)^{-\beta})}{\partial k^b}\Big]. \end{aligned} \tag{IA.B.30}$$

Integrating the above equation with respect to k^b and using the relation $\frac{\vartheta_1}{\vartheta_2} = \frac{\theta_1}{\theta_2}$, we have

$$\int H^a_{k^ak^b}(k^a,k^b)dk^b = \frac{1}{r}\left[\frac{\mathcal{X}(k^a,k^b)^{1-\beta}}{(1-\beta)\vartheta_2} + k^a\frac{\theta_1}{\theta_2}\mathcal{X}(k^a,k^b)^{-\beta}\right]. \tag{IA.B.31}$$

For $k^a \geq k^b \geq 0$, we derive from (40) and (IA.B.31) that

$$\begin{aligned} H^a_{k^a}(k^a,k^b) &= H^a_{k^a}(k^a,k^a) - \int_{k^b}^{k^a} H^a_{k^ak^b}(k^a,k)dk \\ &= \Big(\frac{2k^a+k^a+pr+c}{r} - \frac{\mathcal{X}(k^a,k^a)}{r-\mu}\Big)\mathcal{X}(k^a,k^a)^{-\beta} \\ &\quad - \frac{1}{r}\left[\frac{\mathcal{X}(k^a,k^a)^{1-\beta} - \mathcal{X}(k^a,k^b)^{1-\beta}}{(1-\beta)\rho\theta_2} + k^a\frac{\theta_1}{\theta_2}\big(\mathcal{X}(k^a,k^a)^{-\beta} - \mathcal{X}(k^a,k^b)^{-\beta}\big)\right]. \end{aligned} \tag{IA.B.32}$$

Plugging (IA.B.32) into (IA.B.29), we have

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) - p$$

$$=\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,k^a)}\Big)^{\beta}\left(\frac{(2-\frac{\theta_1}{\theta_2})k^a+k^a+pr+c}{r}-\mathcal{X}(k^a,k^a)\Big(\frac{1}{r-\mu}+\frac{1}{r(1-\beta)\rho\theta_2}\Big)\right)$$
$$-\left(\frac{\Big(2-\frac{\theta_1}{\theta_2}\Big)k^a+k^b+pr+c}{r}-\mathcal{X}(k^a,k^b)\Big(\frac{1}{r-\mu}+\frac{1}{r(1-\beta)\rho\theta_2}\Big)\right). \tag{IA.B.33}$$

Denote

$$\mathcal{B}(k^a,k^b)=\Big(2-\frac{\theta_1}{\theta_2}\Big)k^a+k^b+pr+c-(\theta_2k^b+\theta_1k^a+pr+c)\frac{1}{\beta-1}\Big(\beta-\frac{1}{\theta_2}\Big). \tag{IA.B.34}$$

Then for $k^a\geq k^b\geq 0$, (IA.B.33) can be simplified as

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b)-p=\frac{1}{r}\left[\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,k^a)}\Big)^{\beta}\mathcal{B}(k^a,k^a)-\mathcal{B}(k^a,k^b)\right]=\frac{\mathcal{R}(k^a,k^b)}{r}, \tag{IA.B.35}$$

where $\mathcal{R}(k^a,k^b)$ is given by

$$\mathcal{R}(k^a,k^b)=\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,k^a)}\Big)^{\beta}\mathcal{B}(k^a,k^a)-\mathcal{B}(k^a,k^b). \tag{IA.B.36}$$

We observe that (IA.B.28) is equivalent to

$$\mathcal{R}(k^a,k^b)<0,\quad k^a>k^b\geq 0. \tag{IA.B.37}$$

Denote $\psi(k^a)=[(2-\frac{\theta_1}{\theta_2})k^a+pr+c-(\theta_1k^a+pr+c)\frac{1}{\beta-1}(\beta-\frac{1}{\theta_2})]\frac{\beta-1}{\beta}\frac{1}{\theta_2-1}$. We have $\mathcal{B}(k^a,\psi(k^a))=0$. Given that $\theta_2>1$, we find $\mathcal{B}_{k^b}=\frac{\beta}{\beta-1}(1-\theta_2)<0$, and $\mathcal{B}(k^a,-\frac{\theta_1k^a+pr+c}{\theta_2})=2(1-\frac{\theta_1}{\theta_2})k^a+(pr+c)(1-\frac{1}{\theta_2})>0$. From $\mathcal{B}_{k^b}<0$, $\mathcal{B}(k^a,\psi(k^a))=0$, and $\mathcal{B}(k^a,-\frac{\theta_1k^a+pr+c}{\theta_2})>0$, we conclude that $-\frac{\theta_1k^a+pr+c}{\theta_2}<\psi(k^a)$. Since $\mathcal{X}(k^a,-\frac{\theta_1k^a+pr+c}{\theta_2})=0$, it follows that $\mathcal{X}(k^a,k^b)>0$ for $k^b\geq\psi(k^a)$ and $k^a\geq 0$. Additionally, from $rp+c>0$, we deduce that $\mathcal{X}(k^a,k^b)>0$ for $k^b\geq 0$ and $k^a\geq 0$. Therefore, $\mathcal{R}(k^a,k^b)$, as given by (IA.B.36), is well-defined for any $k^b\in[0\wedge\psi(k^a),k^a)$ with $k^a>0$.

For any $k^b\in[0\wedge\psi(k^a),k^a)$, $k^a>0$, we have

$$\begin{aligned}\mathcal{R}_{k^b}(k^a,k^a)=&\beta\frac{\rho\theta_2}{\mathcal{X}(k^a,k^a)}\mathcal{B}(k^a,k^a)-\frac{\beta}{\beta-1}(1-\theta_2)\\ =&\frac{\rho\theta_2}{\mathcal{X}(k^a,k^a)}\beta k^a\Big[2-\frac{\theta_1}{\theta_2}-\theta_1+(1-\theta_2)\Big]\geq 0,\end{aligned} \tag{IA.B.38}$$

where the inequality follows from the right inequality in (45). Additionally, we have

$$\mathcal{R}_{k^bk^b}(k^a,k^b)=\beta(\beta-1)\Big(\frac{\rho\theta_2}{\mathcal{X}(k^a,k^a)}\Big)^2\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,k^a)}\Big)^{\beta-2}\mathcal{B}(k^a,k^a). \tag{IA.B.39}$$

Fix $k^a\geq 0$. If $\mathcal{B}(k^a,k^a)=0$, then from $\mathcal{B}_{k^b}<0$, we derive that $\mathcal{B}(k^a,k^b)>0$ for $k^a>k^b$, and thus $\mathcal{R}(k^a,k^b)=-\mathcal{B}(k^a,k^b)<0$. If $\mathcal{B}(k^a,k^a)<0$, then we derive from (IA.B.39) that $\mathcal{R}_{k^bk^b}(k^a,k^b)<0$, which, together with (IA.B.38), implies that $\mathcal{R}_{k^b}(k^a,k^b)>\mathcal{R}_{k^b}(k^a,k^a)\geq 0$ for $k^b\in[0\wedge\psi(k^a),k^a)$. Hence, we have $\mathcal{R}(k^a,k^b)<\mathcal{R}(k^a,k^a)=0$ for any $k^b\in[0\wedge\psi(k^a),k^a)$.

To prove (IA.B.37), it suffices to consider any given $k^a\geq 0$ such that $\mathcal{B}(k^a,k^a)>0$. From (IA.B.39), we deduce that $\mathcal{R}(k^a,k^b)$ is convex in k^b, which implies that

$$\mathcal{R}(k^a,k^b)<\mathcal{R}(k^a,k^a)\vee\mathcal{R}(k^a,0\wedge\psi(k^a))=0\vee\mathcal{R}(k^a,0\wedge\psi(k^a)), \tag{IA.B.40}$$

for any $k^b\in(0\wedge\psi(k^a),k^a)$. Setting $k^b=0\wedge\psi(k^a)$ in (IA.B.36), we obtain

$$\begin{aligned}&\mathcal{R}(k^a,0\wedge\psi(k^a))=\Big(\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,k^a)}\Big)^{\beta}\mathcal{B}(k^a,k^a)-\mathcal{B}(k^a,0\wedge\psi(k^a))\\ \leq&\Big(\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,k^a)}\Big)^{\beta}[\mathcal{B}(k^a,k^a)-\mathcal{B}(k^a,0\wedge\psi(k^a))]<0,\quad k^a>0,\end{aligned} \tag{IA.B.41}$$

where the first inequality uses $\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,k^a)}\in(0,1]$ and $\mathcal{B}(k^a,0\wedge\psi(k^a))\geq\mathcal{B}(k^a,\psi(k^a))=0$, and the second inequality follows from $\mathcal{B}_{k^b}<0$ and $k^a>0$. Combining (IA.B.40) with (IA.B.41), we conclude that $\mathcal{R}(k^a,k^b)<0$ for any $k^b\in[0\wedge\psi(k^a),k^a)$ and $k^a>0$.

Therefore, (IA.B.37) holds. Referring to (IA.B.35), we see that (IA.B.28) holds.

Step 4: We prove that

$$V^a_{k^a}(x,k^a,k^b)<p,\quad (k^a,k^b)\in\mathbf{N}_a(x),\ x>0. \tag{IA.B.42}$$

For $k^a\geq k^b\geq 0$, we have

$$\begin{aligned}&\mathcal{X}(k^a,k^b)-\rho[2k^a+k^b+pr+c]=\rho[(\theta_2-1)k^b+(\theta_1-2)k^a]\\ \leq&\rho[(\theta_2-1)k^a+(\theta_1-2)k^a]=\rho k^a(\theta_1+\theta_2-3)\leq 0,\end{aligned} \tag{IA.B.43}$$

where the first inequality holds due to $\theta_2>1$, and the second inequality follows from $\theta_1+\theta_2\leq 3-\frac{\theta_1}{\theta_2}<3$ as given in the right inequality in (45) with $i=b$. For $k^b\geq k^a\geq 0$, we have

$$\begin{aligned}&\mathcal{X}(k^a,k^b)-\rho[2k^a+k^b+pr+c]=\rho[(\theta_2-2)k^a+(\theta_1-1)k^b]\\ \leq&\rho[(\theta_2-2)k^a+(\theta_1-1)k^a]=\rho k^a(\theta_1+\theta_2-3)\leq 0,\end{aligned} \tag{IA.B.44}$$

where the first inequality uses $\theta_1 \leq 1$, and the second inequality uses $\theta_1 + \theta_2 \leq 3 - \frac{\theta_1}{\theta_2}$ as given in the right inequality in (45) with $i = b$. Combining (IA.B.43), and (IA.B.44), we have

$$\mathcal{X}(k^a, k^b) \leq \rho[2k^a + k^b + pr + c], \quad (k^a, k^b) \in \mathbb{R}^2_+. \tag{IA.B.45}$$

Next, for each fixed $(k^a, k^b) \in \mathbb{R}^2_+$, we prove that $V^a_{k^a}(x, k^a, k^b) < p$ for all $x \in [0, \mathcal{X}(k^a, k^b))$. Using (36), we obtain

$$V^a_{k^a}(x, k^a, k^b) = \frac{x}{r-\mu} - \frac{2k^a + k^b + c}{r} + H^a_{k^a}(k^a, k^b)x^\beta, \quad x \in [0, \mathcal{X}(k^a, k^b)). \tag{IA.B.46}$$

If $H^a_{k^a}(k^a, k^b) \geq 0$, then using $r > \mu$, we see that $V^a_{k^a}(x, k^a, k^b)$ is increasing in $x \in (0, \mathcal{X}(k^a, k^b))$ and

$$V^a_{k^a}(x, k^a, k^b) < V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) \leq p, \tag{IA.B.47}$$

for any $x \in (0, \mathcal{X}(k^a, k^b))$, where the second inequality uses (IA.B.3) and (IA.B.28).

If $H^a_{k^a}(k^a, k^b) < 0$, then $\frac{x}{r-\mu} + H^a_{k^a}(k^a, k^b)x^\beta$ is first increasing and then decreasing in the region $x > 0$. Denote $\widehat{x} = \arg\max_{x \in [0, \mathcal{X}(k^a, k^b)]} \frac{x}{r-\mu} + H^a_{k^a}(k^a, k^b)x^\beta$. We have $\widehat{x} > 0$. If $\widehat{x} = \mathcal{X}(k^a, k^b)$, then $\frac{x}{r-\mu} + H^a_{k^a}(k^a, k^b)x^\beta$ is increasing in $x \in [0, \mathcal{X}(k^a, k^b)]$. Hence, we conclude from (IA.B.46) that $V^a_{k^a}(x, k^a, k^b) < V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) \leq p$, $\forall\ x \in (0, \mathcal{X}(k^a, k^b))$. If $\widehat{x} < \mathcal{X}(k^a, k^b)$, then we have $\widehat{x} \in (0, \mathcal{X}(k^a, k^b))$ and $\frac{\partial V^a_{k^a}(x, k^a, k^b)}{\partial x} \mid_{x=\widehat{x}} = 0$, which implies that $\frac{1}{r-\mu} + \beta H^a_{k^a}(k^a, k^b)\widehat{x}^{\beta-1} = 0$. It follows that

$$\frac{\widehat{x}}{r-\mu} + H^a_{k^a}(k^a, k^b)\widehat{x}^\beta = \frac{\beta-1}{\beta}\frac{\widehat{x}}{r-\mu} < \frac{\beta-1}{\beta}\frac{\mathcal{X}(k^a, k^b)}{r-\mu} \leq \frac{2k^a + k^b + pr + c}{r}, \tag{IA.B.48}$$

where the first inequality follows from $r - \mu > 0$ and $\beta > 2$, and the second inequality uses (IA.B.45). Combining (IA.B.48) with (IA.B.46), we have $V^a_{k^a}(x, k^a, k^b) \leq V^a_{k^a}(\widehat{x}, k^a, k^b) < p$ for any $x \in (0, \mathcal{X}(k^a, k^b))$.

Therefore, we have shown $V^a_{k^a}(x, k^a, k^b) < p$, for all $x \in (0, \mathcal{X}(k^a, k^b))$ and $(k^a, k^b) \in \mathbb{R}^2_+$, i.e., for any $(k^a, k^b) \in \mathbf{N}_{ab}(x)$ with $x \geq 0$. Using (IA.B.28), we have $V^a_{k^a}(x, k^a, k^b) = V^a_{k^a}(x, k^a, \widehat{k}^b) < p$ for any $(k^a, k^b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b$ is uniquely determined by the following linear equation:

$$\mathcal{X}(k^a, \widehat{k}^b) = x. \tag{IA.B.49}$$

Note that $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)) = \mathbf{N}_a(x)$. Then (IA.B.42) holds.

In summary, by combining our analyses in Steps 1-4 with (IA.B.8)-(IA.B.9), we have verified (27) and (55) hold.

Finally, it remains to consider the case $\theta_1 = \theta_2 = 1$. Note that (IA.B.8)-(IA.B.9), and (IA.B.11)-(IA.B.15) also hold for the case $\theta_1 = \theta_2 = 1$. In particular, for $\mathcal{G}(k^a, k^b)$ as defined in (IA.B.14), we have $\mathcal{G}(k^a, k^b) = 0$ for all k^a and k^b. Substituting it into (IA.B.15), we have $V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) = 0$. Thus, for any $(k^a, k^b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, $V^a_{k^b}(x, k^a, k^b) = V^a_{k^b}(x, \widehat{k}^a, k^b) = 0$, where $\widehat{k}^a = \frac{x}{\rho} - (rp + c) - k^b$. Combining this with (IA.B.9), we have

$$V^a_{k^b}(x, k^a, k^b) = 0, \quad x \geq \mathcal{X}(k^a, k^b) \tag{IA.B.50}$$

for any $(k^a, k^b) \in \mathbb{R}^2_+$.

Observe that (IA.B.29)-(IA.B.36) also apply when $\theta_1 = \theta_2 = 1$. In particular, for $\mathcal{B}(k^a, k^b)$ as defined in (IA.B.34), we have $\mathcal{B}(k^a, k^b) \equiv 0$. Substituting this into (IA.B.35), we find that $V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p$ for any $k^a \geq k^b \geq 0$. Thus $V^a_{k^a}(x, k^a, k^b) = V^a_{k^a}(x, k^a, \widehat{k}^b) = p$ for any $(k^a, k^b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b = \frac{x}{\rho} - (rp + c) - k^a$. Combining this with (IA.B.8), we have

$$V^a_{k^a}(x, k^a, k^b) = p, \quad x \geq \mathcal{X}(k^a, k^b) \tag{IA.B.51}$$

for any $(k^a, k^b) \in \mathbb{R}^2_+$.

Substituting $\theta_1 = \theta_2 = 1$ into (42), we have

$$H^a(k, k) = -\frac{\mathcal{X}(k, k)^{1-\beta}}{\beta(r - \mu)} k. \tag{IA.B.52}$$

Substituting (IA.B.52) and $\mathcal{X}(k^a, k^b) = \rho(k^a + k^b + rp + c)$ into (36) with $k^a = k^b \geq 0$, $x = \mathcal{X}(k^a, k^a)$, we have $V^a(\mathcal{X}(k^a, k^a), k^a, k^a) = pk^a$. Combining this with (IA.B.50) and (IA.B.51), we obtain $V^a(x, k^a, k^b) = pk^a$ for any $x \geq \mathcal{X}(k^a, k^b)$ and $(k^a, k^b) \in \mathbb{R}^2_+$.

Finally, by following similar procedures in Step 2 and Step 4, we can demonstrate that (27) and (55) hold for the case $\theta_1 = \theta_2 = 1$.

□

Proof of Lemma 5 Using (46) and (IA.A.3), we have

$$h(\eta)(\eta - 1) = (\overline{g}(\eta) - \breve{g}(\eta))\beta\eta[(1 + \eta)^{\beta-1} - 1], \tag{IA.B.53}$$

where $h(\eta) = [(1 + \eta)^{\beta-1} - 1][\beta + 2 - (\beta - 2)\eta] - 2(\beta - 1)\eta$. It can be verified that $h''(\eta) < 0$ for $\eta > 1$, $h(1) > 0$, and $h(\eta) < 0$ for sufficiently large η. Therefore, there exists a unique root, $\breve{\eta} > 1$, for $h(\eta) = 0$ in the $\eta > 1$ domain. Using (IA.B.53), we have

$$\overline{g}(\eta) > \breve{g}(\eta), \quad \eta \in (1, \breve{\eta}), \tag{IA.B.54}$$

$$\overline{g}(\eta) < \breve{g}(\eta), \quad \eta > \breve{\eta}. \tag{IA.B.55}$$

We first show that the two inequalities in (45) imply that (IA.A.2) holds for $\theta_2 > \theta_1 > 0$. It is sufficient to prove that $\underline{g}(\eta) > \breve{g}(\eta)$ for any $\eta \in (1, \breve{\eta}]$. According to Lemma 1, $\eta^* > 1$ denotes the unique root of $\overline{g}(\eta) = \underline{g}(\eta)$ in the region $\eta > 1$. We derive from (IA.B.54)-(IA.B.55), and $\underline{g}(\eta) > \breve{g}(\eta)$ for any $\eta \in (1, \breve{\eta}]$ that $\eta^* \in (1, \breve{\eta})$. Using $\overline{g}(\breve{\eta}) = \breve{g}(\breve{\eta}) < \underline{g}(\breve{\eta})$, $\overline{g}(\eta^*) = \underline{g}(\eta^*)$, and the fact that $\overline{g}(\eta) = \underline{g}(\eta)$ has only one root in the region $\eta > 1$, we conclude that $\overline{g}(\eta) < \underline{g}(\eta)$ for any $\eta \geq \breve{\eta}$. Since (45) implies that $\overline{g}(\theta_2/\theta_1) \geq \underline{g}(\theta_2/\theta_1)$, we have $\theta_2/\theta_1 < \breve{\eta}$. Then we derive from $\underline{g}(\eta) > \breve{g}(\eta)$, $\eta \in (1, \breve{\eta}]$ that (IA.A.2) holds.

Because $0 = h(\breve{\eta}) = [(1+\breve{\eta})^{\beta-1} - 1][\beta + 2 - (\beta-2)\breve{\eta}] - 2(\beta-1)\breve{\eta}$, we have $\beta + 2 - (\beta-2)\breve{\eta} > 0$, which implies that $\breve{\eta} < \frac{\beta+2}{\beta-2}$. Thus, we only need to prove that $\underline{g}(\eta) > \breve{g}(\eta)$ for any $\eta \in (1, \frac{\beta+2}{\beta-2})$. By the definition of $\underline{g}(\eta)$, we have

$$\underline{g}(\eta)^\beta + \frac{\beta}{\beta-2}(\underline{g}(\eta) - 2) = 1 + \frac{(1+\eta)^\beta - 1}{\eta}. \tag{IA.B.56}$$

Taking derivatives on both sides of the above equation, we obtain

$$\underline{g}'(\eta)\Big[\beta \underline{g}(\eta)^{\beta-1} + \frac{\beta}{\beta-2}\Big] = \frac{(1+\eta)^{\beta-1}(\beta\eta - \eta - 1) + 1}{\eta^2}. \tag{IA.B.57}$$

Substituting $\underline{g}(1) = 2$ into the above equation, we obtain $\underline{g}'(1) = 1 - \frac{2}{\beta}$. Denote $l(\eta) = \underline{g}(1) + \underline{g}'(1)(\eta - 1)$. We have $l(\eta) = 1 + \frac{2}{\beta} + (1 - \frac{2}{\beta})\eta$. As a result, we have

$$\begin{aligned}
& l(\eta) - \breve{g}(\eta) \\
=& \frac{2}{\beta} - \frac{2}{\beta\eta} - \left(2 - \frac{2}{\beta}\right) \frac{\eta - 1}{(1+\eta)^{\beta-1} - 1} \\
=& \frac{2(\eta-1)}{\beta}\left[\frac{1}{\eta} - \frac{\beta - 1}{(1+\eta)^{\beta-1} - 1}\right] \\
=& \frac{2(\eta-1)}{\beta} \frac{(1+\eta)^{\beta-1} - 1 - (\beta-1)\eta}{\eta[(1+\eta)^{\beta-1} - 1]} > 0,
\end{aligned} \tag{IA.B.58}$$

where the inequality uses $(1+\eta)^{\beta-1} > 1 + (\beta-1)\eta$ for any $\eta > 1$, $\beta > 2$.

Thus, using (IA.B.58), in order to prove that $\underline{g}(\eta) > \breve{g}(\eta)$ for any $\eta \in (1, \frac{\beta+2}{\beta-2})$, we only need to show that $\underline{g}(\eta) \geq l(\eta)$. Using (IA.B.56), we can see that $\underline{g}(\eta) \geq l(\eta)$ is equivalent to

$$\begin{aligned}
& l(\eta)^\beta + \frac{\beta}{\beta-2}(l(\eta) - 2) \leq 1 + \frac{(1+\eta)^\beta - 1}{\eta} \\
\iff & l(\eta)^\beta + \eta - 1 \leq 1 + \frac{(1+\eta)^\beta - 1}{\eta} \\
\iff & (1+\eta)^\beta - \eta l(\eta)^\beta \geq (\eta - 1)^2
\end{aligned}$$

$$\iff 1-\eta\Big(\frac{l(\eta)}{1+\eta}\Big)^{\beta}\geq\frac{(\eta-1)^2}{(1+\eta)^{\beta}}. \tag{IA.B.59}$$

Denote $y=\frac{\eta-1}{\eta+1}$. Then $\eta\in(1,\frac{\beta+2}{\beta-2})$ is equivalent to $y\in(0,\frac{2}{\beta})$. Moreover, using $1+\eta=\frac{2}{1-y}$, $\eta=\frac{1+y}{1-y}$, $\eta-1=\frac{2y}{1-y}$, $l(\eta)=\frac{2}{1-y}(1-\frac{2y}{\beta})=(1+\eta)(1-\frac{2y}{\beta})$, we can see that (IA.B.59) is equivalent to

$$1-\frac{1+y}{1-y}\Big(1-\frac{2y}{\beta}\Big)^{\beta}\geq 2^{2-\beta}y^2(1-y)^{\beta-2}. \tag{IA.B.60}$$

Denote $\mathcal{G}(y)=1-\frac{1+y}{1-y}\Big(1-\frac{2y}{\beta}\Big)^{\beta}-2^{2-\beta}y^2(1-y)^{\beta-2}$. We have

$$\begin{aligned}\mathcal{G}'(y)=&\frac{2y(\frac{2}{\beta}-y)}{(1-y)^2}\Big(1-\frac{2y}{\beta}\Big)^{\beta-1}-2^{2-\beta}\beta y\Big(\frac{2}{\beta}-y\Big)(1-y)^{\beta-3}\\ >&\frac{2y(\frac{2}{\beta}-y)}{(1-y)^2}\Big(1-y\Big)^{\beta-1}-2^{2-\beta}\beta y\Big(\frac{2}{\beta}-y\Big)(1-y)^{\beta-3}\\ =&2y\Big(\frac{2}{\beta}-y\Big)(1-y)^{\beta-3}[1-\beta 2^{1-\beta}]>0,\end{aligned} \tag{IA.B.61}$$

where the first inequality uses $\beta>2$ and $y\in(0,\frac{2}{\beta})$, and the second inequality uses $2^{\beta-1}=(1+1)^{\beta-1}>1+\beta-1=\beta$ for $\beta>2$. Since $\mathcal{G}(0)=0$, we have $\mathcal{G}(y)>0$ for any $y\in(0,\frac{2}{\beta})$. Therefore, (IA.B.59)-(IA.B.60) hold, and thus $\underline{g}(\eta)>\breve{g}(\eta)$ for any $\eta\in(1,\frac{\beta+2}{\beta-2})$. Then, (IA.A.2) holds for all $\theta_2>\theta_1>0$ satisfying Condition 1.

Moreover, (IA.A.2) together with $\theta_1+\theta_2\geq\underline{g}(\theta_2/\theta_1)$ implies that

$$\theta_1+\theta_2>\breve{g}(\theta_2/\theta_1). \tag{IA.B.62}$$

Denote

$$\widehat{g}(\eta)=\frac{[(1+\eta)^{\beta}-1][(1-\frac{2}{\beta})\eta+\frac{2}{\beta}]+(\eta-2)\eta}{(1+\eta)^{\beta}-(1+\eta)}. \tag{IA.B.63}$$

Because $\theta_1+\theta_2=\theta_2(1+\frac{\theta_1}{\theta_2})$ and $\breve{g}(\eta)/(1+\frac{1}{\eta})=\widehat{g}(\eta)$, (IA.B.62) is equivalent to

$$\widehat{g}\Big(\frac{\theta_2}{\theta_1}\Big)<\theta_2. \tag{IA.B.64}$$

We can further verify that (IA.B.64) is equivalent to (IA.A.4).

If $\theta_1=\theta_2$, then by using $\overline{g}(1)=\underline{g}(1)=1$, we observe that (45) simplifies to $2\leq\theta_2+\theta_1\leq 2$. This implies that $\theta_2=\theta_1=1$.

Since $\beta > 2$, we have

$$\widehat{g}(\eta) > \frac{[(1+\eta)^{\beta}-1][(1-\frac{2}{\beta})+\frac{2}{\beta}]-\eta}{(1+\eta)^{\beta}-(1+\eta)} = 1, \quad \eta > 1. \tag{IA.B.65}$$

Thus, for $\theta_2 > \theta_1$, we derive from (IA.B.64) that $\theta_2 > 1$. The right inequality in (45) implies that

$$\theta_1 \leq \frac{3-\theta_2}{1+1/\theta_2}. \tag{IA.B.66}$$

We can show that $f(x) := \frac{3-x}{1+1/x} = \frac{x(3-x)}{x+1}$ is decreasing in $x \geq 1$ by noticing that $f'(x) = \frac{4-(x+1)^2}{(x+1)^2} < 0$ for $x > 1$. Since $\theta_2 > 1$, we have $\frac{3-\theta_2}{1+1/\theta_2} < 1$, which, together with (IA.B.66), implies that $\theta_1 < 1$.

Define $\phi(\eta) := ((1+\eta)^{\beta}-1)((\beta-1)\eta-2)+2\beta\eta$. We observe that $\phi(\eta) > 0$, $\forall \eta > 0$, using the fact that $\phi''(\eta) = \beta(\beta-1)(1+\eta)^{\beta-2}(\beta+1)\eta > 0$, $\forall \eta > 0$, along with $\phi'(0) = 0$ and $\phi(0) = 0$. Recalling $\widehat{g}$ as defined in (IA.B.63), we have

$$\widehat{g}(\eta) - \frac{4-\eta}{1+\frac{2}{\eta}} = \frac{1}{(1+\eta)^{\beta}-(1+\eta)}\frac{\eta-1}{\eta+2}\frac{2}{\beta}\phi(\eta) > 0, \quad \forall \eta > 1. \tag{IA.B.67}$$

Setting $\eta = \frac{\theta_2}{\theta_1}$ in (IA.B.67) and combining it with (IA.B.64), we derive the following inequality

$$\theta_2 > \frac{4-\frac{\theta_2}{\theta_1}}{1+\frac{2\theta_1}{\theta_2}}, \tag{IA.B.68}$$

when $\theta_2/\theta_1 > 1$. It can then be seen that (IA.B.68) is equivalent to (IA.A.1).

Using (IA.B.57), we have $\underline{g}'(\eta) > 0$ for $\eta > 1$. Hence, $\theta_1 + \theta_2 \geq \underline{g}(\theta_2/\theta_1) > 2$ when $\theta_2 > \theta_1 > 0$. Then, the smallest value of $\theta_1 + \theta_2$ is achieved only when $\theta_1 = \theta_2 = 1$.

□

Proof of Lemma 6 Denote

$$\mathcal{G}(k^a, k^b) = \mathcal{X}(k^a, k^b)(k^a + k^b)^{-1/\gamma} - \rho. \tag{IA.B.69}$$

It can be shown that (E.12) is equivalent to

$$\begin{aligned}&\mathcal{G}_{k^b}(k^a, k^b)k^b\big[\mathcal{G}(k^a, k^b)+\rho\big] + \mathcal{G}_{k^a}(k^a, k^b)\Big[\mathcal{G}(k^a, k^b)\big(\gamma(k^a+k^b)-k^b\big)-\rho k^b\Big]\\ &+\mathcal{G}(k^a, k^b)(\mathcal{G}(k^a, k^b)+\rho) = 0.\end{aligned} \tag{IA.B.70}$$

Let $g(k^b,k)=\mathcal{G}(k^a,k^b)k^b$, where $k=k^a+k^b$. Then (IA.B.70) is equivalent to

$$g_{k^b}(k^b,k)(g(k^b,k)k^b+\rho(k^b)^2)+g_k(k^b,k)g(k^b,k)\gamma k=0. \tag{IA.B.71}$$

Note that (IA.B.71) is a quasilinear first-order PDE, and the coefficients of $g_{k^b}(k^b,k)$ and $g_k(k^b,k)$ cannot be zero simultaneously given $k\geq k^b>0$ and $\gamma>1$. The general (local) solution of (IA.B.71) can be obtained by solving the following characteristic equations (see e.g., Chapter 1 of McOwen (2004)):

$$\begin{cases} \frac{dk^b}{gk^b+\rho(k^b)^2}=\frac{dk}{g\gamma k}, \\ dg=0. \end{cases} \tag{IA.B.72}$$

Solving (IA.B.72) yields the following two independent first integrals of the characteristic system:

$$\frac{1}{g}\ln\frac{k^b}{(g+\rho k^b)k^{\frac{1}{\gamma}}}=\lambda_1, \quad g=\lambda_2, \tag{IA.B.73}$$

where λ_1, λ_2 are constants. Then, the general solution to the PDE (IA.B.70) is characterized by (E.8). Hence, the PDE (E.12) admits a general solution as given by (E.13).

In addition, if $\mathcal{G}(k^a,k^b)$ satisfies (E.8) for any $k^b\geq k^a\geq \underline{k}>0$, then for $k=k^a+k^b$, $g=g(k^b,k)=\mathcal{G}(k^a,k^b)k^b$ satisfies

$$\phi\Big(g,\frac{1}{g}\ln\frac{k^b}{(g+\rho k^b)k^{\frac{1}{\gamma}}}\Big)=0. \tag{IA.B.74}$$

Denote $U(k^b,k,z)=\frac{1}{z}\ln\frac{k^b}{(z+\rho k^b)k^{\frac{1}{\gamma}}}$. Taking partial derivatives with respect to k^b and k in (IA.B.74), we have

$$\phi_{z_1}\Big(g,U(k^b,k,g)\Big)g_{k^b}+\phi_{z_2}\Big(g,U(k^b,k,g)\Big)\big(U_{k^b}+U_zg_{k^b}\big)=0, \tag{IA.B.75}$$

$$\phi_{z_1}\Big(g,U(k^b,k,g)\Big)g_{k}+\phi_{z_2}\Big(g,U(k^b,k,g)\Big)\big(U_{k}+U_zg_{k}\big)=0, \tag{IA.B.76}$$

where $g=g(k^b,k)$. Since $\phi_{z_1}\neq 0$ or $\phi_{z_2}\neq 0$, we derive from (IA.B.75) and (IA.B.76) that $\big(U_k+U_zg_k\big)g_{k^b}=\big(U_{k^b}+U_zg_{k^b}\big)g_k$, which is equivalent to

$$g_{k^b}U_k=g_kU_{k^b}. \tag{IA.B.77}$$

Substituting $U_{k^b}(k^b,k,g)=\frac{1}{g}(\frac{1}{k^b}-\frac{\rho}{g+\rho k^b})$ and $U_k(k^b,k,g)=\frac{-1}{g\gamma k}$ into (IA.B.77), we obtain

(IA.B.71) and thus (IA.B.70) holds for any $k^b \geq k^a \geq \underline{k}$. $\square$

Proof of Proposition 5 To highlight the dependence on (θ_1, θ_2), we let $\boldsymbol{\varphi}^{\theta_1,\theta_2}$ and $V^i(z; \boldsymbol{\varphi}^{\theta_1,\theta_2})$ be respectively the strategy and firm i's value function associated with the linear trigger function given in (44). Due to symmetry, it suffices to prove that $V^a(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2})$ is increasing in $\theta_1 + \theta_2 \in [2, 4]$.

For any $\theta_2 \geq \theta_1 > 0$ and strategy $\boldsymbol{\varphi}^{\theta_1,\theta_2}$, the corresponding firm value is characterized by Proposition 3. In particular, the option value $H^i(k^a, k^b)$ is given by (42)-(43).

We can see that $H^a(k, k)$ given by (42) depends on (θ_1, θ_2) only through $\theta_1 + \theta_2$. We denote it as $H^a(k, k; \theta_1 + \theta_2)$. First, we show that $H^a(k, k; \theta_1 + \theta_2)$ is increasing in $\theta_1 + \theta_2 \in (0, 4]$. To this end, we define $f(\theta) := -\frac{(\theta k + rp + c)^{1-\beta}}{\theta}\left[(4(\beta - 1) - \beta\theta)k + (rp + c)(\frac{4}{\theta} - 2)\right]$. We can see that

$$H^a(k, k; \theta) = (\rho)^{1-\beta}\frac{f(\theta)}{\beta(\beta - 2)(r - \mu)}. \tag{IA.B.78}$$

A direct calculation shows that

$$f'(\theta) = \frac{(\theta k + rp + c)^{-\beta}}{\theta^3}(4 - \theta)\left[(\theta k)^2(\beta - 1)\beta + 2\theta k\beta(rp + c) + 2\Big(rp + c\Big)^2\right].$$

Thanks to $\beta > 2$ and $rp + c > 0$, we see that $f'(\theta) > 0$ for any $\theta \in [0, 4)$, and $f'(\theta) < 0$ for any $\theta > 4$. Hence, $H^a(k, k; \theta)$ is increasing in $\theta \in (0, 4]$.

Second, we can see $\mathcal{K}(x) = \frac{1}{\theta_1 + \theta_2}\left[\frac{x}{\rho} - (rp + c)\right]$ is decreasing in $\theta_1 + \theta_2$ for any $x \geq \rho(rp + c)$. To highlight the dependence of $\mathcal{K}(x)$ on $\theta_1 + \theta_2$, we denote it as $\mathcal{K}(x; \theta_1 + \theta_2)$.

Next, we fix $\theta_2 \geq \theta_1 > 0$ and $\widetilde{\theta}_2 \geq \widetilde{\theta}_1 > 0$ with $4 \geq \theta := \theta_1 + \theta_2 > \widetilde{\theta} := \widetilde{\theta}_1 + \widetilde{\theta}_2$. For any $k \geq \mathcal{K}(x; \widetilde{\theta})$, we conclude from (36) that

$$V^a(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x, k, k; \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = \Big[H^a(k, k; \theta) - H^a(k, k; \widetilde{\theta})\Big]x^\beta > 0. \tag{IA.B.79}$$

For $k \in [\mathcal{K}(x; \theta), \mathcal{K}(x; \widetilde{\theta})]$, we conclude from Proposition 3 that

$$\begin{aligned} &V^a(x, \mathcal{K}(x; \widetilde{\theta}), \mathcal{K}(x; \widetilde{\theta}); \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) - V^a(x, k, k; \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p[\mathcal{K}(x; \widetilde{\theta}) - k] \\ \geq &V^a(x, \mathcal{K}(x; \widetilde{\theta}), \mathcal{K}(x; \widetilde{\theta}); \boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2}), \end{aligned} \tag{IA.B.80}$$

where the equality follows from $V^a_{k^a}(x, k, k; \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p$ and $V^a_{k^b}(x, k, k; \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = 0$ for $k \in [\mathcal{K}(x; \theta), \mathcal{K}(x; \widetilde{\theta})]$, and the inequality follows from $V^a_{k^a}(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2}) \leq p$, $V^a_{k^b}(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2}) \leq$

0. Then we derive from (IA.B.80) that for $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$,

$$\begin{aligned} & V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) \\ \geq & V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x,\mathcal{K}(x;\widetilde{\theta}),\mathcal{K}(x;\widetilde{\theta});\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) > 0, \end{aligned} \tag{IA.B.81}$$

where the second inequality uses (IA.B.79). Since $V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) = V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p$, and $V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) = V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = 0$ for any $k \in [0,\mathcal{K}(x;\theta)]$, we derive from (IA.B.81) that $V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) > V^a(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2})$ for any $k \in [0,\mathcal{K}(x;\theta)]$.

As a result, we have shown that $V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2})$ is increasing in $\theta_1+\theta_2 \in [2,4]$. For any (θ_1,θ_2) satisfying Condition 1, $(\theta_1,\theta_2) \neq (1,1)$, and $(\theta_1,\theta_2) \neq (\theta_1^*,\theta_2^*)$, we have

$$3 > \theta_1^* + \theta_2^* > \theta_1 + \theta_2 > 2,$$

where the first inequality uses $\theta_1^* + \theta_2^* < 3$, the second inequality uses Lemma 5-(iii), and the last inequality uses Lemma 5-(iv).

Using $V^{\text{mon}}(x,2k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{2,2})$, $V^{\text{duo}}_{\text{open}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{1,2})$, $V^{\text{duo}}_{\text{best}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1^*,\theta_2^*})$, $V^{\text{duo}}_{\text{closed}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2})$, $V^{\text{duo}}_{\text{BP}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{1,1})$, we conclude that (56) holds. □

Proof of Proposition 6 (57) can be derived from (8), (C.24) and $dK^{\text{duo}}_t = dK^a_t + dK^b_t$.

Using Corollary 2, we can obtain (58). □

Proof of Corollary 1 Note that (IA.B.29)-(IA.B.35) hold for any $\theta_2 \geq \theta_1 > 0$. Recall the function $\mathcal{R}(k^a,k^b)$ defined in (IA.B.36) and the equalities in (IA.B.38). If the right inequality in (45) is reversed, then we derive from (IA.B.38) that $\mathcal{R}_{k^b}(k^a,k^a) < 0$. Since (IA.B.36) implies that $\mathcal{R}(k^a,k^a) = 0$, there exists $\epsilon \in (0,k^a)$ such that $\mathcal{R}(k^a,k^b) > 0$ for any $k^b \in (k^a-\epsilon,k^a)$. Using (IA.B.35), we have

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) > p \tag{IA.B.82}$$

for any $k^b \in (k^a-\epsilon,k^a)$. Therefore, the right inequality in (45) is a necessary condition for (27).

Note that (C.55)-(C.60) hold for any $\theta_2 \geq \theta_1 > 0$. Using (C.57), we have

$$\lim_{y\to\infty} \frac{\mathcal{E}(y)}{y} = \theta_2(\beta-2)\left[(\theta_1+\theta_2)^\beta + \frac{\beta}{\beta-2}(\theta_1+\theta_2-2) - 1 + \frac{\theta_1}{\theta_2} - \frac{\theta_1}{\theta_2}\left(1+\frac{\theta_2}{\theta_1}\right)^\beta\right]. \tag{IA.B.83}$$

If the left inequality in (45) is reversed, then (IA.B.83) is strictly negative. Combining this

with (C.58), we obtain $\frac{\partial}{\partial k^a}H^a_{\rm BP}(k^a,0)-\frac{\partial}{\partial k^a}H^a(k^a,0)<0$ for any sufficiently large $k^a>0$. Since $\lim_{k^a\to\infty}H^a_{\rm BP}(k^a,0)=\lim_{k^a\to\infty}H^a(k^a,0)=0$, we obtain $H^a_{\rm BP}(k^a,0)>H^a(k^a,0)$ for any sufficiently large $k^a>0$. For any fixed $x>0$ and any sufficiently large $k^a>0$, we have $x<\mathcal{X}(k^a,0)$ and $x<\mathcal{X}^{\rm BP}(k^a,0)$. Hence, by (C.44)-(C.45), we derive that $V^a_{\rm BP}(x,k^a,0)>V^a(x,k^a,0)$.

Therefore, the left inequality in (45) is a necessary condition for $V^a(z)\geq V^a_{\rm BP}(z)$, $\forall\ z\in\mathbb{Z}$.

□

Proof of Corollary 2 According to Proposition 3, for $x<\mathcal{X}(0,0)=\rho(rp+c)$, we have $V^i(x,0,0)=\Psi^i(x,0,0)+H^i(0,0)x^\beta=H^i(0,0)x^\beta$. By (IA.B.78), we obtain

$$H^i(0,0)=(\rho(rp+c))^{1-\beta}\frac{-\frac{1}{\theta}\left[(rp+c)(\frac{4}{\theta}-2)\right]}{\beta(\beta-2)(r-\mu)},$$

where $\theta=\theta_1+\theta_2$. Then we derive (C.96).

According to Proposition 3, for $x\geq\mathcal{X}(0,0)=\rho(rp+c)$, we have

$$V^i(x,0,0)=V^i(x,\mathcal{K}(x),\mathcal{K}(x))-p\mathcal{K}(x).$$

Using $\mathcal{K}(x)=\frac{1}{\theta_1+\theta_2}\left[\frac{x}{\rho}-(rp+c)\right]$, we can derive (C.97). □

Proof of Lemma 7 For convenience, we denote

$$\mathbf{I}_i(x)=\{(k^a,k^b)\in[\underline{k},\infty)\times[\underline{k},\infty):x\geq\mathcal{X}(k^a,k^b),k^i\leq\mathcal{K}(x)\},\tag{IA.B.84}$$

$$\mathbf{N}_i(x)=\{(k^a,k^b)\in[\underline{k},\infty)\times[\underline{k},\infty):(k^a,k^b)\notin\mathbf{I}_i(x)\},\tag{IA.B.85}$$

$\mathbf{I}_{ab}(x)=\mathbf{I}_a(x)\cap\mathbf{I}_b(x)$, and $\mathbf{N}_{ab}(x)=\mathbf{N}_a(x)\cap\mathbf{N}_b(x)$.

We only prove the case for $V^a(z)$ as the case for $V^b(z)$ can be treated similarly. It is straightforward to verify that $V^a\in\mathcal{C}^{1,1,1}(\mathbb{Z})$, $V^a_{k^a}(x,k^a,k^b)=p$ for $(k^a,k^b)\in\mathbf{I}_a(x)$, $V^a_{k^b}(x,k^a,k^b)=0$ for $(k^a,k^b)\in\mathbf{I}_b(x)$, and that Condition A.1 holds. The proof of (D.14) is similar to that in the proof of Theorem 4. If (27) and (55) hold, then Lemma 3 and Condition A.4-(i) imply that $V^i(z)>V^i_{\rm BP}(z)$ for all $z\in\mathbb{Z}$. It therefore remains to prove (27) and (55).

Next, we complete our proof in three steps.

Step 1: We prove that (D.19) *holds.*

Using (36) and (E.2), we obtain

$$V^a_{k^a}(x,k^a,k^b)=\psi^a_{k^a}(k^a,k^b)x+H^a_{k^a}(k^a,k^b)x^\beta,\quad x\in(0,\mathcal{X}(k^a,k^b)],\tag{IA.B.86}$$

where $\psi^a_{k^a}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}$. Using $r>\mu$ and $\gamma>1$, we have $\psi^a_{k^a}(k^a,k^b)>0$. If $H^a_{k^a}(k^a,k^b)\geq 0$, then $V^a_{k^a}(x,k^a,k^b)$ is increasing in $x\in[0,\mathcal{X}(k^a,k^b)]$. Hence, for any $x\in(0,\mathcal{X}(k^a,k^b))$,

$$V^a_{k^a}(x,k^a,k^b)<V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b)=p, \tag{IA.B.87}$$

where the equality uses (D.14).

If $H^a_{k^a}(k^a,k^b)<0$, then $\psi^a_{k^a}(k^a,k^b)x+H^a_{k^a}(k^a,k^b)x^\beta$ is first increasing and then decreasing in the region $x\geq 0$. Denote $\hat{x}:=\arg\max_{x\in[0,\mathcal{X}(k^a,k^b)]}[\psi^a_{k^a}(k^a,k^b)x+H^a_{k^a}(k^a,k^b)x^\beta]$. We have $\hat{x}>0$. If $\hat{x}=\mathcal{X}(k^a,k^b)$, then $\psi^a_{k^a}(k^a,k^b)x+H^a_{k^a}(k^a,k^b)x^\beta$ is increasing in $x\in[0,\mathcal{X}(k^a,k^b)]$. Hence, we conclude from (IA.B.86) that $V^a_{k^a}(x,k^a,k^b)<V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b)=p$ for all $x\in(0,\mathcal{X}(k^a,k^b))$. If $\hat{x}<\mathcal{X}(k^a,k^b)$, then we have $\hat{x}\in(0,\mathcal{X}(k^a,k^b))$ and $\frac{\partial V^a_{k^a}(x,k^a,k^b)}{\partial x}\mid_{x=\hat{x}}=0$, which implies that $\psi^a_{k^a}(k^a,k^b)+\beta H^a_{k^a}(k^a,k^b)\hat{x}^{\beta-1}=0$. It follows that

$$\begin{aligned}&\psi^a_{k^a}(k^a,k^b)\hat{x}+H^a_{k^a}(k^a,k^b)\hat{x}^\beta=\frac{\beta-1}{\beta}\psi^a_{k^a}(k^a,k^b)\hat{x}\\<&\frac{\beta-1}{\beta}(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}\mathcal{X}(k^a,k^b)\leq p,\end{aligned} \tag{IA.B.88}$$

where the first inequality uses $\psi^a_{k^a}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}>0$ and $\beta>1$, and the second inequality uses (E.9) and (E.4). Combining (IA.B.88) with (IA.B.86), we have $V^a_{k^a}(x,k^a,k^b)\leq V^a_{k^a}(\hat{x},k^a,k^b)<p$ for any $x\in(0,\mathcal{X}(k^a,k^b))$. Therefore, (D.19) holds.

Step 2: We prove that (D.23) *holds.*

Using (36) and (E.2), we have

$$V^a_{k^b}(x,k^a,k^b)=\psi^a_{k^b}(k^a,k^b)x+H^a_{k^b}(k^a,k^b)x^\beta,\quad x\in(0,\mathcal{X}(k^a,k^b)], \tag{IA.B.89}$$

where $\psi^a_{k^b}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$. Thus, (D.23) is equivalent to $H^a_{k^b}(k^a,k^b)<\mathcal{H}(k^a,k^b)$ for $k^b>k^a\geq\underline{k}$, where $\mathcal{H}(k^a,k^b):=-\psi^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^{1-\beta}$. Additionally, we can derive from (E.6) that $H^a_{k^b}(k^a,k^b)=\mathcal{H}(k^a,k^b)$ for $k^a=k^b$. As a result, (D.23) is equivalent to (D.24).

Then, we only need to show that $H^a_{k^ak^b}(k^a,k^b)>\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b>k^a\geq\underline{k}$. Using (E.5), we have

$$\begin{aligned}&H^a_{k^ak^b}(k^a,k^b)-\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}\\=&\frac{(\beta-1)\Big[\mathcal{X}(k^a,k^b)\big[\mathcal{X}_{k^b}(k^a,k^b)\psi^a_{k^a}(k^a,k^b)-\mathcal{X}_{k^a}(k^a,k^b)\psi^a_{k^b}(k^a,k^b)\big]-p\frac{\beta}{\beta-1}\mathcal{X}_{k^b}(k^a,k^b)\Big]}{\mathcal{X}(k^a,k^b)^{\beta+1}}\end{aligned}$$

$$
\begin{aligned}
=&\frac{(\beta-1)\Big[\frac{\mathcal{X}(k^a,k^b)}{(k^a+k^b)^{\frac{1}{\gamma}}}\Big[\mathcal{X}_{k^b}(k^a,k^b)+(\mathcal{X}_{k^a}(k^a,k^b)-\mathcal{X}_{k^b}(k^a,k^b))\frac{k^a}{\gamma(k^a+k^b)}\Big]-\rho\mathcal{X}_{k^b}(k^a,k^b)\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}}\\
=&\frac{(\beta-1)\mathcal{X}_{k^b}(k^a,k^b)\Big[\frac{\mathcal{X}(k^a,k^b)}{(k^a+k^b)^{\frac{1}{\gamma}}}\Big[1+\big(\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}-1\big)\frac{k^a}{\gamma(k^a+k^b)}\Big]-\rho\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}},
\end{aligned}
\tag{IA.B.90}
$$

where the second equality uses $\psi^a_{k^a}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}$ and $\psi^a_{k^b}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$, and the last equality uses $\vartheta_2(k^a,k^b)=\mathcal{X}_{k^a}(k^a,k^b)$, $\vartheta_1(k^a,k^b)=\mathcal{X}_{k^b}(k^a,k^b)$ for $k^b>k^a$.

According to part (ii) of Condition A.4, we have $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}>0$ and $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}>0$ for any $k^b\geq k^a\geq\underline{k}$. By combining (E.12) with (E.7), we obtain

$$
\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}=1-\frac{\gamma(k^a+k^b)\mathcal{G}(k^a,k^b)}{k^b(\rho+\mathcal{G}(k^a,k^b))},\quad k^b\geq k^a\geq\underline{k}. \tag{IA.B.91}
$$

Since $\mathcal{G}(k^a,k^b)>0$, we have $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}<1$ and $\mathcal{X}(k^a,k^b)>\rho(k^a+k^b)^{\frac{1}{\gamma}}$ for any $k^b\geq k^a\geq\underline{k}$. By the symmetry of $\mathcal{X}(k^a,k^b)$, we conclude $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}>0$, $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}>0$ and

$$
\vartheta_2(k^a,k^b)>\vartheta_1(k^a,k^b)>0,\quad\forall k^a\geq\underline{k},\quad k^b\geq\underline{k}. \tag{IA.B.92}
$$

Plugging $\mathcal{X}(k^a,k^b)>\rho(k^a+k^b)^{\frac{1}{\gamma}}$ and $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}\in(0,1)$ into (IA.B.90), we have $H^a_{k^ak^b}(k^a,k^b)>\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b>k^a\geq\underline{k}$. Hence, (D.23) holds.

Step 3: We prove that

$$
V^a_{k^b}(x,k^a,k^b)<0,\quad(k^a,k^b)\in\mathbf{N}_b(x),\ k^a\geq\underline{k},\ k^b\geq\underline{k},\ x>0. \tag{IA.B.93}
$$

We can see from (IA.B.89) that for $x\in(0,\mathcal{X}(k^a,k^b)]$,

$$
\frac{V^a_{k^b}(x,k^a,k^b)}{x}=\psi^a_{k^b}(k^a,k^b)+H^a_{k^b}(k^a,k^b)x^{\beta-1}
$$

is linear in $x^{\beta-1}>0$ and satisfies $\lim_{x\to0+}\frac{V^a_{k^b}(x,k^a,k^b)}{x}=\psi^a_{k^b}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}<0$. Using (39) and (D.23), we have $\lim_{x\to\mathcal{X}(k^a,k^b)-}\frac{V^a_{k^b}(x,k^a,k^b)}{x}\leq0$. Hence, we have $V^a_{k^b}(x,k^a,k^b)<0$ for any $x\in(0,\mathcal{X}(k^a,k^b))$.

In addition, for any $(k^a,k^b)\in\mathbf{I}_a(x)\setminus\mathbf{I}_{ab}(x)$ with $k^a\geq\underline{k}$, $k^b\geq\underline{k}$,

$$
V^a_{k^b}(x,k^a,k^b)=V^a_{k^b}(x,\widehat{k}^a,k^b)<0, \tag{IA.B.94}
$$

where $\widehat{k}^a$ is determined by $\mathcal{X}(\widehat{k}^a, k^b) = x$, the equality is derived from $V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p$, and the inequality follows from (D.23). Since $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)) \supseteq \mathbf{N}_b(x)$, we infer (IA.B.93). $\square$

Proof of Lemma 8 We only prove that $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$. The proof of Condition A.2 is similar to that of Theorem 3. Similarly, we can verify that Definition 2-(i) holds.

By (C.25), we have $K^i_t \leq K^i_0 \vee \mathcal{K}(M_t)$, $t \geq 0$. Recall that Condition A.4 implies $\mathcal{X}(k^a, k^b) \geq \rho(k^a + k^b)^{\frac{1}{\gamma}}$, and $\mathcal{K}(x)$ is defined in (63). It follows that there exists a constant $C_1 > 0$ such that $K^i_t \in [K^i_0, C_1(1 + |M_t|^\gamma)]$, $\forall\ t \geq 0$, $i = a, b$. Using $\pi^i(x, k^a, k^b) = x(k^a + k^b)^{-1/\gamma} k^i$, $K^i_t \in [K^i_0, C_1(1 + |M_t|^\gamma)]$ and $X_t \in [0, M_t]$, $\forall\ t \geq 0$, we can see there exists a constant $C_2 > 0$ such that $|\pi^i(X_t, K^a_t, K^b_t)| \leq C_2(1 + |M_t|^{\gamma+1})$, $\forall\ t \geq 0$. Recalling (C.28), (C.29), and (C.30), we can see the conditions in Definition 2-(ii) are satisfied, and $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$. $\square$

Proof of Theorem 6 We can apply Theorem 2 and Proposition 9 on the domain $\mathbb{Z} = (0, \infty) \times [\underline{k}, \infty) \times [\underline{k}, \infty)$. Hence, using Theorem 2, Proposition 9, and Lemmas 7-8, we can see that Theorem 6 holds. $\square$

Proof of Proposition 10 We only need to verify that Condition A.4 holds.

First, we consider the case $\phi(z_1, z_2) = z_1 - \lambda$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \frac{\lambda}{k^b}$ solves the equation (E.8). Since $\frac{\rho k^a k^b}{\gamma(k^a + k^b) - k^a} = \frac{\rho}{\gamma/k^a + (\gamma-1)/k^b}$ is increasing in $k^a \geq \underline{k}$ and $k^b \geq \underline{k}$, we have

$$\frac{\rho k^a k^b}{\gamma(k^a + k^b) - k^a} \geq \frac{\rho \underline{k}^2}{2\gamma \underline{k} - \underline{k}} = \frac{\rho \underline{k}}{2\gamma - 1} > \lambda, \tag{IA.B.95}$$

where the second inequality uses $\underline{k} > \frac{\lambda(2\gamma-1)}{\rho}$. It follows that $\frac{\rho k^a}{\gamma(k^a+k^b)-k^a} > \frac{\lambda}{k^b}$. Hence, Condition A.4-(i) holds. Using $\gamma > 1$ and $\rho > 0$, we can see that $\big(\rho + \mathcal{G}(k^a, k^b)\big)(k^a + k^b)^{\frac{1}{\gamma}} = \big(\rho + \frac{\lambda}{k^b}\big)(k^a + k^b)^{\frac{1}{\gamma}}$ is increasing in k^a and thus $\vartheta_2(k^a, k^b) = \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} > 0$ for $k^b \geq k^a \geq \underline{k}$.

Since Condition A.4-(i) has been verified, we have $\mathcal{G}(k^a, k^b) < \frac{\rho k^a}{\gamma(k^a+k^b)-k^a} \leq \frac{\rho k^b}{\gamma(k^a+k^b)-k^b}$ for $k^b \geq k^a \geq \underline{k}$. Hence, the right-hand side of (IA.B.91) is strictly positive. Since we have also shown that $\vartheta_2(k^a, k^b) > 0$, it follows that $\vartheta_1(k^a, k^b) > 0$. Then Condition A.4-(ii) holds.

Next, we consider the case $\phi(z_1, z_2) = \frac{\ln(\lambda/z_1)}{z_1} + z_2$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \sqrt{(\frac{\rho}{2})^2 + \frac{\lambda}{k^b(k^a+k^b)^{\frac{1}{\gamma}}}} - \frac{\rho}{2}$ solves the equation (E.8). Since $\lambda > 0$, we have $\mathcal{G} > 0$ for any $k^a \geq \underline{k}$, $k^b \geq \underline{k}$. Next, we prove the second inequality in Condition A.4-(i), which is equivalent to

$$\sqrt{\Big(\frac{\rho}{2}\Big)^2 + \frac{\lambda}{k^b(k^a + k^b)^{\frac{1}{\gamma}}}} < \frac{\rho k^a}{\gamma(k^a + k^b) - k^a} + \frac{\rho}{2}. \tag{IA.B.96}$$

Using $\frac{\rho k^a}{\gamma(k^a+k^b)-k^a}+\frac{\rho}{2}=\frac{\rho}{2}\frac{\gamma(k^a+k^b)+k^a}{\gamma(k^a+k^b)-k^a}$, one can see that (IA.B.96) is equivalent to

$$\Big(\frac{\rho}{2}\Big)^2+\frac{\lambda}{k^b(k^a+k^b)^{\frac{1}{\gamma}}}<\Big(\frac{\rho}{2}\Big)^2\Big(\frac{\gamma(k^a+k^b)+k^a}{\gamma(k^a+k^b)-k^a}\Big)^2. \tag{IA.B.97}$$

Using $\left(\frac{\gamma(k^a+k^b)+k^a}{\gamma(k^a+k^b)-k^a}\right)^2-1=\frac{4\gamma(k^a+k^b)k^a}{(\gamma(k^a+k^b)-k^a)^2}$, one can see that (IA.B.97) is equivalent to

$$\lambda<\gamma\rho^2\frac{k^ak^b(k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}. \tag{IA.B.98}$$

We prove that $\frac{k^ak^b(k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}$ is increasing in k^a and k^b in the region $k^b\geq k^a\geq\underline{k}$. Denote $f(k^a,k^b):=\ln\left(\frac{k^ak^b(k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}\right)=\ln k^a+\ln k^b+(1+\frac{1}{\gamma})\ln(k^a+k^b)-2\ln(\gamma(k^a+k^b)-k^a)$. We have

$$\frac{\partial f(k^a,k^b)}{\partial k^a}=\frac{1}{k^a}+\Big(1+\frac{1}{\gamma}\Big)\frac{1}{k^a+k^b}-\frac{2(\gamma-1)}{\gamma(k^a+k^b)-k^a}, \tag{IA.B.99}$$

$$\frac{\partial f(k^a,k^b)}{\partial k^b}=\frac{1}{k^b}+\Big(1+\frac{1}{\gamma}\Big)\frac{1}{k^a+k^b}-\frac{2\gamma}{\gamma(k^a+k^b)-k^a}. \tag{IA.B.100}$$

We can rewrite $\frac{\partial f(k^a,k^b)}{\partial k^b}$ as follows:

$$\frac{\partial f(k^a,k^b)}{\partial k^b}=\frac{(k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b}{k^b(k^a+k^b)(\gamma(k^a+k^b)-k^a)}. \tag{IA.B.101}$$

For any $k^b\geq k^a\geq\underline{k}$, we have $\frac{\partial\left((k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b\right)}{\partial k^b}=2k^b+(\gamma-1-\frac{1}{\gamma})k^a\geq 2k^a+(\gamma-1-\frac{1}{\gamma})k^a>0$, and $((k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b)\mid_{k^a=k^b}=(2\gamma-1-\frac{1}{\gamma})(k^a)^2>0$. Then, it follows from (IA.B.101) that $\frac{\partial f(k^a,k^b)}{\partial k^b}>0$. Comparing (IA.B.99) with (IA.B.100), we have $\frac{\partial f(k^a,k^b)}{\partial k^a}\geq\frac{\partial f(k^a,k^b)}{\partial k^b}>0$ for any $k^b\geq k^a\geq\underline{k}$.

Hence, $f(k^a,k^b)$, and consequently $\frac{k^ak^b(k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}$, are increasing in k^a and k^b in the region $k^b\geq k^a\geq\underline{k}$. It follows that (IA.B.98) holds for any $k^b\geq k^a\geq\underline{k}$ if and only if (IA.B.98) holds for $k^a=k^b=\underline{k}$, which is equivalent to $\underline{k}>\frac{1}{2}\Big(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\Big)^{\frac{\gamma}{\gamma+1}}$. Hence, Condition A.4-(i) holds for any $\underline{k}>\frac{1}{2}\Big(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\Big)^{\frac{\gamma}{\gamma+1}}$.

Since $\mathcal{G}(k^a,k^b)=\frac{\lambda(k^a+k^b)^{\frac{-1}{\gamma}}}{\sqrt{\left(\frac{\rho k^b}{2}\right)^2+\lambda k^b(k^a+k^b)^{\frac{-1}{\gamma}}}+\frac{\rho k^b}{2}}$, we have

$$\mathcal{G}(k^a,k^b)(k^a+k^b)^{\frac{1}{\gamma}}=\frac{\lambda}{\sqrt{\left(\frac{\rho k^b}{2}\right)^2+\lambda k^b(k^a+k^b)^{\frac{-1}{\gamma}}}+\frac{\rho k^b}{2}}, \tag{IA.B.102}$$

which is increasing in k^a. It follows that $\mathcal{X}(k^a, k^b) = \Big(\rho + \mathcal{G}(k^a, k^b)\Big)(k^a + k^b)^{\frac{1}{\gamma}}$ is increasing in k^a and $\vartheta_2(k^a, k^b) = \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} > 0$ for $k^b \geq k^a \geq \underline{k}$.

Since Condition A.4-(i) has been verified, we have $\mathcal{G}(k^a, k^b) < \frac{\rho k^a}{\gamma(k^a+k^b)-k^a} \leq \frac{\rho k^b}{\gamma(k^a+k^b)-k^b}$ for $k^b \geq k^a \geq \underline{k}$. Hence, the right-hand side of (IA.B.91) is strictly positive. Since we have also shown that $\vartheta_2(k^a, k^b) > 0$, it follows that $\vartheta_1(k^a, k^b) > 0$. Then Condition A.4-(ii) holds. $\square$