Option Exercise Games and a Duopolistic q Theory of Investment*

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Abstract

Back and Paulsen (2009) examine Grenadier (2002)'s real-option exercise game under a closed-loop equilibrium, in which firms' dynamic state-contingent responses to rivals eliminate profits. They pose unresolved questions about how to define closed-loop equilibria and whether any with positive profits exist. We provide a definition of closed-loop equilibrium, develop a systematic method for constructing infinitely many such equilibria in closed form, and deploy these constructions to formulate a duopolistic q-theory of investment. Finally, we show that, across all closed-loop equilibria, investment proceeds more slowly than in the Back–Paulsen zero-profit closed-loop equilibrium but more rapidly than in the open-loop equilibrium.

Keywords: real-option games, closed-loop equilibria, irreversible investment, duopoly, marginal q, cross marginal q

JEL Codes: C61, C73, D43, E22, L13

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1 Introduction

How do firms exercise their growth options and accumulate capital under uncertainty in an oligopolistic competitive environment? Grenadier (2002) pioneers a real-option game framework in which firms undertake irreversible investments while the output price decreases with the aggregate industry capital stock. He obtains an explicit open-loop equilibrium solution. Back and Paulsen (2009) argue that a closed-loop equilibrium is more plausible than an open-loop equilibrium, as it allows firms to respond dynamically to their competitors' strategies, thereby yielding a Markov subgame perfect equilibrium.

Back and Paulsen (2009) also identify a closed-loop equilibrium in which all firms optimally follow the simple net present value (NPV) rule for investment. This equilibrium is Markov subgame perfect because adherence to the NPV rule constitutes a best response when competitors also employ it. Notably, this closed-loop equilibrium results in zero economic profits and a complete erosion of the value of industry growth options. Furthermore, Back and Paulsen (2009) highlight several open questions and avenues for future research. In their introduction, they note: *"There are difficulties in even defining the game in closed-loop form."* They further conjecture the existence of alternative equilibria—beyond the zero-profit outcome they analyzed—in which firms earn positive profits and preserve some option value of waiting. In their conclusion, they write: *"It is an open question whether the perfectly competitive boundary is the unique boundary such that closed-loop strategies ... are impervious to preemption."*

In this paper, we address these open questions by identifying novel closed-loop equilibria and exploring their implications for a duopolistic q theory of investment, which we subsequently employ for quantitative analysis. To set the stage, Section 2 presents Grenadier's duopoly real-option investment game setting with a linear (inverse) demand function.[1] We also summarize the monopoly benchmark and the open-loop equilibrium solution. Below we outline our main contributions.

First, in Section 3, we define closed-loop equilibria for the option exercise games studied by Grenadier (2002) and Back and Paulsen (2009). A key idea is to decompose a firm's capital accumulation process into a continuous component and a jump (lumpy) component, recog-

[1] In Appendix F, we extend the analysis to an isoelastic inverse demand function.

nizing that both components (weakly) increase if and only if the demand shock process $\{X_t\}$ exceeds its running maximum $\{M_t\}$, reflecting investment irreversibility. While this mechanism operates in monopoly and open-loop oligopoly settings, only in a closed-loop equilibrium does a firm's investment strategy respond dynamically to its competitor's strategies, consistent with Markov subgame perfection. Specifically, in our closed-loop equilibrium definition, the sensitivity of a firm's investment to the resetting of the demand shock's running maximum depends critically on its competitor's investment behavior.

Second, Section 4 develops a systematic solution method to identify and characterize symmetric closed-loop equilibria in closed form.[2] We conjecture and verify that there are infinitely many economically intuitive closed-loop equilibria in which firms earn positive profits. We derive closed-form expressions for investment strategies and firm values, revealing four distinct equilibrium regions: one region where both firms invest, two regions where only the smaller firm invests, and one region where neither firm invests.

Technically, we introduce an equilibrium trigger function $\mathcal{X}$ that depends on the pair of capital stocks (K_t^a, K_t^b) to delineate these regions. When industry demand $X_t < \mathcal{X}(K_t^a, K_t^b)$, neither firm invests, as the aggregate capital stock is sufficiently large to deter investment incentives. If $X_t > \mathcal{X}(K_t^a, K_t^b)$, the larger firm remains inactive while the smaller firm invests until the threshold $X_t = \mathcal{X}(K_t^a, K_t^b)$ is reached. When firms are equal in size, both invest simultaneously to maintain equality along the 45-degree line $K_t^a = K_t^b$. Importantly, once firms attain equal capital stocks, they evolve in lockstep, jointly responding to demand shocks along this line.

Third, in Section 5, we apply our solution method developed in Section 4 to obtain and then analyze all closed-loop equilibria in which a firm's investment policy is piecewise linear in both firms' capital stocks. To make an equilibrium closed loop so that it is Markov subgame perfect, we provide two economically motivated sufficient conditions. One is that a firm's marginal q must be weakly less than its marginal cost of investing; otherwise, the firm would expand capital to reduce its marginal q to the marginal cost of investing. The other condition is that a firm's *cross* marginal q—the sensitivity of its value to its competitor's capital stock–

[2]We focus on symmetric equilibria, where firms share the same investment trigger functional form but may differ in capital stock levels; asymmetric equilibria are discussed in Appendix D.

cannot be positive, reflecting the downward-sloping demand curve whereby an increase in one firm's capital depresses the other's value. In Appendix A, we present a verification theorem and its proof.

To fully characterize a closed-loop equilibrium, we also need to specify a firm's best response should its competitor deviate from the equilibrium strategy. If one firm over-invests relative to the equilibrium, the other firm adheres to its original equilibrium strategy. Conversely, if one firm under-invests, the other firm invests just enough to satisfy $X_t = \mathcal{X}(K_t^a, K_t^b)$. These best responses ensure that no firm benefits from unilateral deviation.

Fourth, in Section 6, we explore the q-theoretic implications of our closed-loop equilibria. We derive analytical and quantitative predictions for industry equilibrium outcomes across various closed-loop equilibria and contrast them with the open-loop benchmark. In all newly identified closed-loop equilibria, firms earn positive profits by preserving some option value of waiting, reflecting less aggressive competition than in the Back–Paulsen closed-loop equilibrium. Compared to the open-loop equilibrium, firms invest earlier, resulting in lower firm values across all closed-loop equilibria. For instance, in our numerical example, the best closed-loop equilibrium—yielding the highest firm value among all closed-loop equilibria—features investment rates 31% slower than in the Back–Paulsen equilibrium but 3% faster than in the open-loop equilibrium. This leads to significantly positive firm value, albeit 4% lower than in the open-loop equilibrium, as firms compete more aggressively under the Markov subgame perfect closed-loop equilibrium.

Fifth, in Section 7, we characterize two sets of closed-loop equilibria with *nonlinear* investment triggers. Compared to the closed-loop equilibria with linear investment triggers analyzed in Section 5, firms compete less aggressively when market demand is low but more fiercely as market demand increases due to the nonlinearity of the investment trigger. Economically, in these equilibria with nonlinear investment triggers, the larger firm's marginal q also equals the marginal cost of purchasing capital when the smaller firm invests. Finally, we conclude in Section 8 and provide supporting technical details in Appendices A-F and Internet Appendices IA.A-IA.B.

More broadly, our paper contributes to the literature that lies at the intersection between real options and dynamic games. Grenadier (1996) develops a duopoly real-option exercise

model, applying it to real estate development. Lambrecht (2001) develops a duopoly exit model in a real-option setting with two *ex ante* heterogeneous firms and studies pure-strategy equilibria.[3] Weeds (2002) studies irreversible investment in competing research projects with uncertain returns under a winner-takes-all patent system. Dai, Jiang, and Wang (2025) analyze a real-option game in which firms choose whether to enter as a leader or a follower, trading off first- versus second-mover advantages. In their setting, the second-mover advantage consideration gives rise to mixed entry strategies. Mathematically, these real-option game models build on a single firm's optimal-stopping formulation of the real-option exercising problem analyzed in McDonald and Siegel (1986) and Dixit and Pindyck (1994). In contrast, our duopolistic model builds on a singular control formulation of the real-option exercising problem analyzed in Abel and Eberly (1996). Kogan (2001, 2004) develop q-theoretic general-equilibrium asset-pricing models, using singular controls to model investment irreversibility.[4]

There is also a growing body of literature that integrates industrial organization considerations into asset pricing models. Dou, Ji, and Wu (2022) extend the standard Lucas-tree asset pricing model to incorporate endogenous strategic competition. Chen, Dou, Guo, and Ji (2024) investigate how strategic competition and financial distress interact dynamically.

2 Model

In this section, we first introduce a classic real-option game setting (Grenadier, 2002) in which firms compete by making recurrent investment decisions, then report the solution for a monopoly problem and the open-loop equilibrium solution for a duopoly problem. The solutions for the monopoly and open-loop equilibrium problems serve as benchmarks against which we compare our closed-loop equilibrium solutions.

[3]The heterogeneity arises from exogenously given different capital structures for two incumbents. Lambrecht and Perraudin (2003) introduce incomplete information into an equilibrium real-option exercising model. Anderson, Friedman, and Oprea (2010) generalize Lambrecht and Perraudin (2003) to settings with multiple firms.

[4]Harrison (1985) provides an introduction to singular control problems.

2.1 Setting

Consider a duopoly industry consisting of two firms: a and b. Let $\{K_t^i\}$ denote firm i's capital stock, where $i = a, b$. Each firm uses its capital stock to produce output. As in the literature, we assume that firm i's profit at time t is:

$$F^i(X_t, K_t^a, K_t^b) = \Pi(X_t, K_t) \cdot K_t^i - C(K_t^i)\,, \tag{1}$$

where $K_t = K_t^a + K_t^b$ is the industry's total capital stock at t, $\Pi(X_t, K_t)$ is downward sloping in K_t, and firm i's (total) production cost $C(\,\cdot\,)$ increases with its capital stock K_t^i. We refer to the exogenous $\{X_t\}$ process in (1) as the industry's demand shock, and firm i's profit increases with X_t. We assume that X_t follows a geometric Brownian motion (GBM):

$$dX_t = \mu X_t dt + \sigma X_t dW_t\,, \quad X_0 > 0\,, \tag{2}$$

where $\{W_t; t \geq 0\}$ is a one-dimensional standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with $W_0 = 0$, μ is the drift and $\sigma > 0$ is the volatility parameter.

As in Grenadier (2002) and Back and Paulsen (2009), we assume that corporate investment is irreversible so that $\{K_t^a\}$ and $\{K_t^b\}$ are nondecreasing over time. Investment irreversibility creates option value of waiting and thus it is helpful to introduce the following running maximum process for the market demand process $\{X_t\}$, as we show below:

$$M_t = \max_{s\in[0,t]} X_s. \tag{3}$$

Firm i's objective is

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\Big(F^i(X_t, K_t^a, K_t^b)dt - p\, dK_t^i\Big)\right], \tag{4}$$

where $r > 0$ is the constant discount rate and $p > 0$ is the constant marginal cost of purchasing a unit of capital. In the main body of our paper, we use the following specification:[5]

$$\Pi(x, k) = x - k \quad \text{and} \quad C(k) = ck, \tag{5}$$

where $c > 0$ is a constant. We assume $\mu < r$ to ensure the finiteness of value functions.

[5]Extending (5) to $\Pi(x, k) = x - \eta k$ where $\eta > 0$ measures the slope of the inverse demand function is straightforward. We report the solution for another specification of $\Pi(x, k)$ where demand elasticity is constant in Appendix F.

2.2 Monopoly Solution

A monopolist makes profits at the rate of $F(x,k) = (x-k)k - ck$ and chooses its capital process $\{K_t; t \geq 0\}$ to maximize its value $\mathbb{E}\left[\int_0^\infty e^{-rt}\Big([X_t - (K_t + c)]K_t dt - p\, dK_t\Big)\right]$. Next, we summarize the monopolist's investment strategy (Grenadier, 2002).

Proposition 1 *The monopoly's optimal capital accumulation process, $\{K_t^{\text{mon}}\}$, is given by*

$$K_t^{\text{mon}} = \max\left\{\frac{1}{2}\left(\frac{M_t}{\rho} - (rp + c)\right),\, K_0\right\}, \tag{6}$$

where M_t is the running maximum given in (3), $\rho > 0$ is a constant given by

$$\rho = \frac{\beta}{\beta - 1}\frac{r - \mu}{r}, \tag{7}$$

and $\beta > 2$ is a constant describing the optionality of the investment opportunity:[6]

$$\beta = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \tag{8}$$

We can describe the monopoly's strategy as follows. First, it does not invest until the running maximum M_t exceeds $\rho(2K_0 + rp + c)$.[7] Second, after M_t hits $\rho(2K_0 + rp + c)$, the monopoly invests whenever M_t increases, so that dK_t^{mon} is proportional to dM_t:

$$dK_t^{\text{mon}} = \frac{1}{2}\frac{dM_t}{\rho}. \tag{9}$$

For the running maximum M_t to increase over dt, both $X_t = M_t$ and $dX_t > 0$ over dt have to hold, so that $dM_t = dX_t > 0$. Therefore, we can equivalently summarize the monopoly's optimal capital process K_t^{mon} as follows: $K_t^{\text{mon}} = \inf\{k \geq K_0 \mid M_t \leq \mathcal{X}^{\text{mon}}(k)\}$ for $t > 0$, where $\mathcal{X}^{\text{mon}}(k)$ is the following threshold function for the demand shock $\{X_t\}$ process:

$$\mathcal{X}^{\text{mon}}(k) = \rho(2k + rp + c). \tag{10}$$

2.3 Duopoly: Open-loop Equilibrium Solution

In a duopoly setting, firm i makes profits at the rate of

$$F^i(x, k^a, k^b) = [x - (k^a + k^b)]k^i - ck^i. \tag{11}$$

[6] Here β is the positive root of the following (fundamental) quadratic equation: $\sigma^2 x(x-1)/2 + \mu x - r = 0$. For convergence, we require $\beta > 2$.

[7] If $X_0 > \rho(2K_0 + rp + c)$, the monopolist instantly increases its capital stock from K_0 to $\frac{1}{2}\left[\frac{X_0}{\rho} - (rp + c)\right]$.

At $t = 0$, firm i chooses its capital stock process $K^i \in \mathcal{A}(k_0^i)$ to maximize its market value given in (4), where $\mathcal{A}(\,\cdot\,)$ is the set of firm i's admissible open-loop strategies defined as:

$$\mathcal{A}(k_0^i) = \big\{\{K_t^i; t \geq 0\} \mid K_t^i \text{ is } \mathcal{F}_t\text{-adapted, nondecreasing, left-continuous}, K_0^i = k_0^i\big\}. \quad (12)$$

Firm i's open-loop strategy depends on the initial state vector $z_0 = (x_0, k_0^a, k_0^b)$ and the demand shock process $\{X_t; t \geq 0\}$. Importantly, its open-loop strategy does not depend on its competitor's strategy $\{K_t^{-i}; t > 0\}$. For a given pair of $K^a \in \mathcal{A}(k_0^a)$ and $K^b \in \mathcal{A}(k_0^b)$, let $\mathcal{J}^i(z_0; K^a, K^b)$ denote firm i's value given in (4).

Back and Paulsen (2009) define an open-loop equilibrium as follows.

Definition 1 $K^a \in \mathcal{A}(k_0^a)$ and $K^b \in \mathcal{A}(k_0^b)$ constitute an *open-loop equilibrium* if the following conditions hold: $\mathcal{J}^a(z_0; K^a, K^b) \geq \mathcal{J}^a(z_0; \widetilde{K}^a, K^b)$ for any $\widetilde{K}^a \in \mathcal{A}(k_0^a)$, and $\mathcal{J}^b(z_0; K^a, K^b) \geq \mathcal{J}^b(z_0; K^a, \widetilde{K}^b)$ for any $\widetilde{K}^b \in \mathcal{A}(k_0^b)$.

Proposition 2 *For $k_0^a = k_0^b$, there is an open-loop equilibrium such that $K_t^a = K_t^b$ for all $t > 0$, where*

$$K_t^i = \max\left\{\frac{1}{3}\left(\frac{M_t}{\rho} - (rp + c)\right),\, k_0^i\right\}, \quad (13)$$

with M_t denoting the running maximum as defined in (3), and $\rho > 0$ being the constant given by (7).

We can describe the open-loop strategy as follows. First, a firm does not invest until the running maximum M_t exceeds $\rho(3k_0^i + rp + c)$.[8] Second, after M_t hits $\rho(3k_0^i + rp + c)$, firm i invests whenever M_t increases, so that dK_t^i is proportional to dM_t:

$$dK_t^i = \frac{1}{3}\frac{dM_t}{\rho}\,. \quad (14)$$

This open-loop equilibrium can be equivalently described with the following threshold strategy: $K_t^i = \inf\big\{k \geq k_0^i : M_t \leq \mathcal{X}^{\mathrm{o}}(k, k)\big\}$, where $\mathcal{X}^{\mathrm{o}}(k^a, k^b)$ is the threshold function given by

$$\mathcal{X}^{\mathrm{o}}(k^a, k^b) := \begin{cases} \rho(2k^a + k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(k^a + 2k^b + rp + c), & \text{if } k^a \geq k^b. \end{cases} \quad (15)$$

Next we define closed-loop strategies and equilibria.

[8] If $X_0 > \rho(3k_0^i + rp + c)$, firm i instantly increases its capital stock from k_0^i to $\frac{1}{3}\big[\frac{X_0}{\rho} - (rp + c)\big]$.

3 Closed-loop Equilibrium Definition

Unlike in an open-loop equilibrium, at each time t a firm can respond to its competitor's strategy in a closed-loop equilibrium. Without loss of generality, we decompose firm i's investment over dt, dK_t^i, as follows:

$$dK_t^i = dK_t^{iC} + \Delta K_t^i\,, \quad t \geq 0\,, \tag{16}$$

where dK_t^{iC} denotes the continuous component of investment dK_t^i and $\Delta K_t^i := K_{t+}^i - K_t^i$ denotes the instantaneous increase (jump) of K^i. We consider a set of closed-loop Markovian (feedback) strategies.

First, we conjecture that the jump component of firm i's investment strategy, ΔK_t^i, depends on the state, $Z_t = (X_t, K_t^a, K_t^b)$, and the jump component of its competitor's strategy, ΔK_t^{-i}. That is, we can express the jump component of firm i's closed-loop strategy as follows:[9]

$$\Delta K_t^i = \boldsymbol{v}^i(Z_t, \Delta K_t^{-i})\,, \tag{17}$$

where $\boldsymbol{v}^i(z, v^{-i})$ is deterministic and measurable in $z = (x, k^a, k^b)$ and v^{-i}.

Second, we conjecture that the continuous part of the strategy can be written as:

$$dK_t^{iC} = u_t^i dM_t\,, \tag{18}$$

where M_t is the running maximum of the demand shock $\{X_s; s \leq t\}$ and u_t^i is an (adapted) Markov process chosen by firm i.[10] Equation (18) implies that a firm increases its capital stock when the running maximum of the demand shock, M_t, increases. Firm i's (Markovian) strategy, u_t^i, depends on the state, $Z_t = (X_t, K_t^a, K_t^b)$, and its competitor's strategy, u_t^{-i}. Thus we express the continuous component of firm i's closed-loop strategy as follows:

$$u_t^i = \boldsymbol{u}^i(Z_t, u_t^{-i}), \tag{19}$$

where $\boldsymbol{u}^i(z, u^{-i})$ is a deterministic and measurable function in $z = (x, k^a, k^b)$ and u^{-i}.

In summary, we have characterized firm i's investment using its closed-loop strategy $\boldsymbol{\varphi}^i = (\boldsymbol{u}^i, \boldsymbol{v}^i)$. By characterizing firm strategies this way, we can then define closed-loop

[9] Here $\boldsymbol{v}^i$ is allowed to depend on firm $-i$'s contemporaneous jump v^{-i}. See footnote 5 in Back and Paulsen (2009) for a related discussion and also Simon and Stinchcombe (1989) for a similar argument in deterministic continuous-time games.

[10] A more general formulation for the continuous component of firm i's strategy is given by $dK_t^{iC} = u_{1t}^i dt + u_{2t}^i dW_t + u_{3t}^i dL_t$, where L_t is a non-decreasing process in t, and $\{u_{1t}^i, u_{2t}^i, u_{3t}^i\}$ may depend on t, Z_t, and $\{u_{1t}^{-i}, u_{2t}^{-i}, u_{3t}^{-i}\}$. We can show that all results in this paper remain unchanged under this extended definition.

strategies and equilibria building on the dynamic-game methodology (Fudenberg and Tirole, 1991; Mailath and Samuelson, 2006).

Definition 2 A pair of closed-loop Markov strategies, $\boldsymbol{\varphi} := (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b)$ with $\boldsymbol{\varphi}^i = (\boldsymbol{u}^i, \boldsymbol{v}^i)$ for $i = a, b$, is feasible if the following conditions hold:

(i) For any $z = (x, k^a, k^b)$, there exists a unique pair $(v^a, v^b) \in \mathbb{R}^2_+$ satisfying

$$v^a = \boldsymbol{v}^a(z, v^b) \quad \text{and} \quad v^b = \boldsymbol{v}^b(z, v^a)\,, \tag{20}$$

and a unique pair $(u^a, u^b) \in \mathbb{R}^2_+$ satisfying

$$u^a = \boldsymbol{u}^a(z, u^b) \quad \text{and} \quad u^b = \boldsymbol{u}^b(z, u^a). \tag{21}$$

Moreover, for any initial value $(x, k^a, k^b) \in \mathbb{Z}$, there exists a unique pair $K^a \in \mathcal{A}(k^a)$ and $K^b \in \mathcal{A}(k^b)$ satisfying (16)-(17).[11]

(ii) For any $z \in \mathbb{Z}$, we have $\mathbb{E}^z\left[|K^i_T|^n\right] < \infty$ for any finite $T > 0$ and $n > 1$, and[12]

$$\mathbb{E}^z\left[\int_0^\infty e^{-rt}|F^i(X_t, K^a_t, K^b_t)|dt\right] < \infty. \tag{22}$$

Let $\mathcal{S}$ denote the set of all feasible (closed-loop) strategies.

For any $z \in \mathbb{Z}$ and any feasible (closed-loop) strategy pair $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b) \in \mathcal{S}$, we define

$$J^i(z; \boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b) = \mathbb{E}^z\left[\int_0^\infty e^{-rt}\Big(F^i(X_t, K^a_t, K^b_t)dt - pdK^i_t\Big)\right]. \tag{23}$$

Definition 3 A strategy pair $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ constitutes a *closed-loop equilibrium* if for any $z \in \mathbb{Z}$, the mutually best response conditions hold:

$$J^a(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \geq J^a(z; \boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b_*), \quad \forall\, (\boldsymbol{\varphi}^a, \boldsymbol{\varphi}^b_*) \in \mathcal{S}, \tag{24}$$

$$J^b(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \geq J^b(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b), \quad \forall\, (\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b) \in \mathcal{S}\,. \tag{25}$$

Let $V^a(z) = J^a(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ and $V^b(z) = J^b(z; \boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ denote the closed-loop equilibrium value functions for a and b, respectively. Note that the strategies are Markov subgame perfect.

Next, we characterize the closed-loop equilibrium solution.

[11] By definition, the continuous investment rate (u^a_t, u^b_t) is the unique solution to (21) with $z = (X_t, K^a_t, K^b_t)$, and the lumpy investment $(\Delta K^a_t, \Delta K^b_t)$ is the unique solution to (20) with $z = (X_t, K^a_t, K^b_t)$.

[12] We denote $\mathbb{E}^z_t[\cdot] = \mathbb{E}_t[\cdot|Z_t = z]$, where $Z_t = (X_t, K^a_t, K^b_t)$ takes the value of $z = (x, k^a, k^b)$. For $t = 0$, we simply write $\mathbb{E}^z_0[\cdot]$ as $\mathbb{E}^z[\cdot]$. Let $\mathbb{Z} = [0, \infty) \times [k^a_0, \infty) \times [k^b_0, \infty)$ denote the state space of the game, where $k^i_0 \geq 0$ is firm i's initial capital.

4 Closed-loop Equilibrium Solution Characterization

Four-region solution. Let $\mathbb{N}_{ab}$ and $\mathbb{I}_i$ denote the respective inaction region and firm i's investment region in the (x, k^a, k^b) space:

$$\mathbb{N}_{ab} = \Big\{(x, k^a, k^b) \in \mathbb{R}^3_+ : x < \mathcal{X}(k^a, k^b)\Big\}, \tag{26}$$

$$\mathbb{I}_i = \left\{(x, k^a, k^b) \in \mathbb{R}^3_+ : x \geq \mathcal{X}(k^a, k^b), x \geq \mathcal{X}(k^i, k^i)\right\}, \tag{27}$$

where $\mathcal{X}(k^a, k^b)$ is an endogenous investment trigger function to be determined in equilibrium. As we show below in equilibrium firm i undertakes lumpy investment only at time 0, when (X_0, K^a_0, K^b_0) lies in the interior of $\mathbb{I}_i$. At $t > 0$, firms either engage in continuous investment or take no action, so that the evolution of (X_t, K^a_t, K^b_t) always lies within the closure of $\mathbb{N}_{ab}$ for all $t > 0$.

Projecting three-dimensional equilibrium-solution regions characterized in (26)-(27) onto the plane of $(K^a_t, K^b_t) = (k^a, k^b)$ for a given demand shock $X_t = x > 0$ yields Figure 1. Strictly above the investment boundary (the ACE curve) where $\mathcal{X}(k^a, k^b) = x$ is the inaction region, which is the projection of $\mathbb{N}_{ab}$:

$$\mathbf{N}_{ab}(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : x < \mathcal{X}(k^a, k^b)\}, \tag{28}$$

where neither firm invests. Elsewhere in the first quadrant, at least one firm invests, which we further divide into three regions: a.) the ABC region where only firm a invests; b.) the CDE region where only firm b invests; and c.) the OBCD (square) region where both firms invest.[13] A key component of our conjecture is that if market demand is high enough so that the industry expands, it must be optimal for at least the smaller firm (and possible the larger one as well) to invest. The intuition is that the smaller firm benefits more from investing because of diminishing marginal returns. The option for a firm to condition its investment on its competitor's size makes a closed-loop equilibrium subgame perfect.

Characterizing firm value in the three investment regions. Let $V^i(z)$ denote firm i's value. When firm i invests, in equilibrium its marginal q must equal the marginal cost of

[13] ABC region is the projection of $\mathbb{I}_a \setminus \mathbb{I}_b$, CDE region is the projection of $\mathbb{I}_b \setminus \mathbb{I}_a$, and OBCD region is the projection of $\mathbb{I}_a \cap \mathbb{I}_b$.

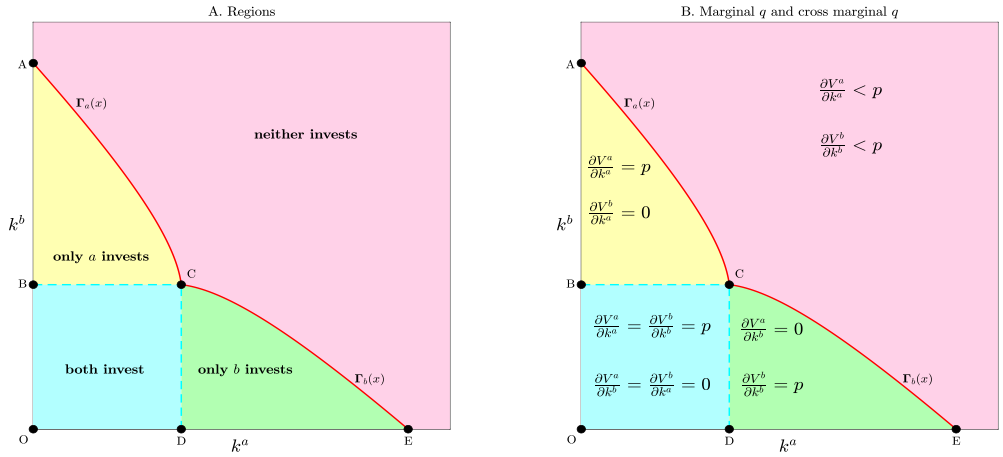


Figure 1: EQUILIBRIUM SOLUTION REGIONS ASSOCIATED WITH AN INVESTMENT BOUNDARY ACE. Inaction region $\mathbf{N}_{ab}(x)$ is given in (28) and firm i's investment boundary is $\mathbf{\Gamma}_i(x) := \{(k^a, k^b) \in \mathbb{R}^2_+ : \mathcal{X}(k^a, k^b) = x, k^i \leq k^{-i}\}$.

purchasing capital p and firm $-i$'s cross marginal q must be zero. In summary, we have:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = 0 \ \text{ in the ABC region,} \tag{29}$$

$$\frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^a(z)}{\partial k^b} = 0 \ \text{ in the CDE region,} \tag{30}$$

$$\frac{\partial V^a(z)}{\partial k^a} = \frac{\partial V^b(z)}{\partial k^b} = p \quad \text{and} \quad \frac{\partial V^b(z)}{\partial k^a} = \frac{\partial V^a(z)}{\partial k^b} = 0 \ \text{ in the OBCD region.} \tag{31}$$

Panel B of Figure 1 summarizes these properties for a firm's marginal q and cross marginal q.

Characterizing firm value in the inaction region. We proceed in three steps. First, we calculate a firm's value absent any future industry growth in that neither firm would ever invest from time t onward: $dK^a_s = dK^b_s \equiv 0$ for all $s \geq t$. Then, firm i's value equals the value of its assets in place, which is given by:

$$\Psi^i(z) = \mathbb{E}^z \left[\int_0^\infty e^{-rt} F^i(X_t, k^a, k^b) dt \right] = \left(\frac{x}{r-\mu} - \frac{k^a + k^b + c}{r} \right) k^i \,. \tag{32}$$

Second, we incorporate the growth option value under imperfect competition. In the inaction (waiting) region $\mathbf{N}_{ab}(x)$ where $x < \mathcal{X}(k^a, k^b)$, $V^i(z)$ satisfies the following HJB equation as in a standard real-option analysis:

$$rV^i(z) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V^i(z)}{\partial x^2} + \mu x \frac{\partial V^i(z)}{\partial x} + F^i(z), \tag{33}$$

where $F^i(z)$ is the profit function given in (11). The general solution for (33) takes the form:

$$V^i(z) = \Psi^i(z) + H^i(k^a, k^b)x^\beta\,, \tag{34}$$

where $\Psi^i(z)$ is given in (32) and $H^i(k^a, k^b)$ is a function that we determine below. Naturally we can interpret $H^i(k^a, k^b)$ as firm i's option value under imperfect competition.

Third, we solve $H^i(k^a, k^b)$. Without loss of generality, we analyze from firm a's perspective. Note that (29) and (30) imply the following conditions:

$$\frac{\partial V^a(z)}{\partial k^a} = p \quad \text{when} \;\; \mathcal{X}(k^a, k^b) = x \;\; \text{and} \;\; k^a \leq k^b\,, \tag{35}$$

$$\frac{\partial V^a(z)}{\partial k^b} = 0 \quad \text{when} \;\; \mathcal{X}(k^a, k^b) = x \;\; \text{and} \;\; k^a \geq k^b\,. \tag{36}$$

By substituting (34) into (35) and (36), we obtain

$$\frac{\partial H^a(k^a, k^b)}{\partial k^a} = \left(\frac{2k^a + k^b + rp + c}{r} - \frac{\mathcal{X}(k^a, k^b)}{r - \mu}\right)\mathcal{X}(k^a, k^b)^{-\beta} \tag{37}$$

for any $k^a \leq k^b$, and

$$\frac{\partial H^a(k^a, k^b)}{\partial k^b} = \frac{k^a}{r}\mathcal{X}(k^a, k^b)^{-\beta} \tag{38}$$

for any $k^a \geq k^b$. As the industry becomes sufficiently large, neither firm will ever invest due to decreasing returns to scale. Mathematically, $\lim_{k\to\infty} H^a(k, k) = 0$. We can then obtain the following closed-form solution for $H^a(k, k)$:[14]

$$H^a(k, k) = -\int_k^\infty \Big(\frac{4w + rp + c}{r} - \frac{\mathcal{X}(w, w)}{r - \mu}\Big)\mathcal{X}(w, w)^{-\beta}dw. \tag{39}$$

Then, integrating (37) and (38) and using (39) for $H^a(k, k)$, we obtain

$$\begin{aligned} H^a(k^a, k^b) &= H^a(k^b, k^b) - \int_{k^a}^{k^b} \left(\frac{2w + k^b + rp + c}{r} - \frac{\mathcal{X}(w, k^b)}{r - \mu}\right)\mathcal{X}(w, k^b)^{-\beta}dw, \quad k^a \leq k^b, \\ H^a(k^a, k^b) &= H^a(k^a, k^a) - \int_{k^b}^{k^a} \frac{k^a}{r}\mathcal{X}(k^a, w)^{-\beta}dw, \quad k^a \geq k^b\,. \end{aligned} \tag{40}$$

Having characterized $V^i(z)$ for a given threshold $\mathcal{X}(k^a, k^b)$ in the four regions, next we conjecture and later verify the following two inequality conditions for firm value.

Two inequalities for firm value. First, firm i's marginal value of capital, i.e., its marginal q, cannot exceed its marginal cost of acquiring capital: p. The intuition is as follows. If this were not true, firm i would benefit by increasing its capital stock, eventually bringing its

[14]Specifically, we integrate along the 45^o degree line in the (k^a, k^b) plane by using (37), (38), and $\lim_{k\to\infty} H^a(k, k) = 0$.

marginal q (weakly) below p. This leads us to conjecture:

$$\frac{\partial V^i(z)}{\partial k^i} \leq p, \quad \text{for any } z = (x, k^a, k^b) \in \mathbb{Z}\,. \tag{41}$$

Second, firm i must be (weakly) worse off if its competitor's capital stock, K^{-i}, increases. This is because an increase of K_t^{-i} causes the industry capital stock to increase, which lowers the unit price for the output $\Pi(X_t, K_t^i + K_t^{-i})$. This leads us to conjecture

$$\frac{\partial V^i(z)}{\partial k^{-i}} \leq 0, \quad \text{for any } z = (x, k^a, k^b) \in \mathbb{Z}. \tag{42}$$

Summary. A closed-loop equilibrium has the following properties. Firm i's value, $V^i(z)$, satisfies (34) subject to (35)-(36) in the inaction region, (29)-(31) in the three investment regions, and inequalities (41)-(42) in all four regions. Additionally, we need to determine the equilibrium investment boundary $\mathcal{X}(k^a, k^b)$. In Appendix A, we provide a (closed-loop equilibrium) verification theorem.

5 Closed-loop Equilibria with Piecewise Linear Triggers

In this section, we apply our solution methodology developed in Section 4 to obtain a set of closed-loop equilibria with piecewise linear boundaries in subsection 5.1 and then characterize these equilibria in subsection 5.2.

5.1 Conjecture and Verify Closed-loop Equilibria

Inspired by the piecewise linear boundary given in (15) for the open-loop equilibrium, we conjecture that there also exists a set of closed-loop equilibria characterized by the following piecewise linear boundary:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho(\theta_2 k^a + \theta_1 k^b + rp + c), & \text{if } k^a \leq k^b, \\ \rho(\theta_1 k^a + \theta_2 k^b + rp + c), & \text{if } k^a \geq k^b, \end{cases} \tag{43}$$

where (θ_1, θ_2) is a pair of positive real numbers from a set Θ to be determined. Note the symmetry between the $k^a \leq k^b$ and $k^b \leq k^a$ cases: θ_2 is the coefficient multiplying the capital stock of the smaller firm, and θ_1 is the coefficient multiplying the capital stock of the larger firm. We then solve the associated firm value, $V^i(z)$, in closed form by substituting the conjectured boundary (43) into (39)-(40).

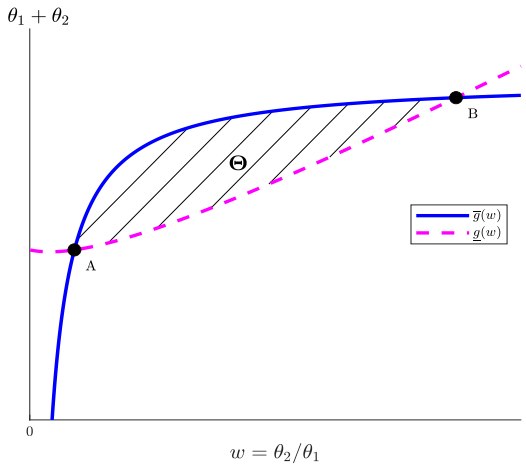


Figure 2: ADMISSIBLE (θ_1, θ_2) FOR CLOSED-LOOP EQUILIBRIA. The shaded area, bounded by the solid line $\overline{g}(\,\cdot\,)$ given in (45) and the dashed line $\underline{g}(\,\cdot\,)$ given in (46), defines a set Θ for (θ_1, θ_2) with $\theta_2 > \theta_1 > 0$. A pair $(\theta_1, \theta_2) \in \Theta$ (including the boundary) corresponds to a closed-loop equilibrium. The Back–Paulsen closed-loop equilibrium is the one with the lowest firm value, corresponding to point A where $\theta_1 = \theta_2 = 1$. The closed-loop equilibrium with the highest firm value corresponds to point B.

Recall that for our conjectured (symmetric) closed-loop equilibrium to exist, this candidate value function, $V^i(z)$, has to satisfy two conditions: (i) inequality (41), which states that firm i's (own) marginal q cannot exceed the capital purchase price; and (ii) inequality (42), which states that its cross marginal q cannot be positive. Inequalities (41) and (42) for $V^i(z)$ can be simplified to the right and left inequalities in (44) in the following Condition 1 for (θ_1, θ_2), respectively. See Corollary 1 in Appendix E.5 for more details.

Condition 1 *The pair, $\theta_2 > 0$ and $\theta_1 \in (0, \theta_2]$, satisfy the following inequality bounds:*

$$\underline{g}(\theta_2/\theta_1) \le \theta_1 + \theta_2 \le \overline{g}(\theta_2/\theta_1)\,, \tag{44}$$

where the upper and lower bound functions, $\overline{g}(\,\cdot\,)$ and $\underline{g}(\,\cdot\,)$, are respectively given by:

$$\overline{g}(w) = 3 - 1/w \quad \textit{and} \tag{45}$$

$$\underline{g}(w) = \left(1 - \frac{2}{\beta}\right)(1+w) + \frac{2}{\beta}\left(1 + \frac{1}{w}\right) + \left(2 - \frac{2}{\beta}\right)\frac{w-1}{(1+w)^{\beta-1} - 1}\,. \tag{46}$$

Let Θ denote the set of pair (θ_1, θ_2) satisfying (44)-(46).

In Figure 2, the shaded area, bounded by the solid line $\overline{g}(\,\cdot\,)$ given in (45) and the dashed line $\underline{g}(\,\cdot\,)$ given in (46), defines a closed set Θ for (θ_1, θ_2) with $\theta_2 \ge \theta_1 > 0$. Any pair $(\theta_1, \theta_2) \in \Theta$ corresponds to a closed-loop equilibrium. As we show later, $\theta_1 + \theta_2$ plays a key

role in determining a duopoly industry's investment rate in equilibrium. The right inequality (44) combined with $\theta_2/\theta_1 > 0$ implies $\theta_1 + \theta_2 < 3$. Point A where $\theta_1 = \theta_2 = 1$ corresponds to the closed-loop equilibrium identified by Back and Paulsen (2009) in which firms make zero profits. Among all closed-loop equilibria, this Back–Paulsen equilibrium is the one with the lowest firm value.

Point B corresponds to the **best closed-loop equilibrium:** the closed-loop equilibrium with the highest firm value. The unique pair (θ_1^*, θ_2^*) for the best closed-loop equilibrium is given by:

$$\theta_1^* = \frac{3 - 1/w^*}{1 + w^*} \text{ and } \theta_2^* = \frac{3w^* - 1}{1 + w^*}, \tag{47}$$

where w^* is the unique root of function $h(w) := [(1+w)^{\beta-1} - 1][\beta + 2 - (\beta - 2)w] - 2(\beta - 1)w$ on the $w > 1$ domain.[15]

As we explain later, the sum $\theta_1 + \theta_2$ dictates the rate at which the industry (with symmetric firms) invests. The larger the sum the slower the investment rate. As the pair (θ_1^*, θ_2^*) attains the highest value of $(\theta_1 + \theta_2)$ at point B in the closed set Θ, the equilibrium associated with this (θ_1^*, θ_2^*) pair is the best closed-loop equilibrium as it attains the highest firm value.

5.2 Closed-loop Equilibrium Solution

In this subsection, we analyze the solution for a closed-loop equilibrium associated with an admissible pair $(\theta_1, \theta_2) \in \Theta$.

Theorem 1 *Under Condition 1 for an investment boundary $\mathcal{X}(k^a, k^b)$ given in* (43)*, a strategy pair $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) = \{(\boldsymbol{u}^a, \boldsymbol{v}^a), (\boldsymbol{u}^b, \boldsymbol{v}^b)\} \in \mathcal{S}$ constitutes a closed-loop equilibrium if*

$$\boldsymbol{u}^i(z, u^{-i}) = \left[\frac{\mathbf{1}_{k^i < k^{-i}}}{\rho\theta_2} + \left(\frac{1}{\rho\theta_1} - \frac{\theta_2}{\theta_1}u^{-i}\right)\mathbf{1}_{k^i \geq k^{-i}}\right]\mathbf{1}_{x \geq \mathcal{X}(k^a, k^b)}, \tag{48}$$

$$\boldsymbol{v}^i(z, v^{-i}) = \inf\left\{\delta \geq 0 : \mathcal{X}\big(k^i + \delta, \max\{k^{-i}, \min\{k^{-i} + v^{-i}, \mathcal{K}(x)\}\}\big) \geq x\right\}, \tag{49}$$

for all $z = (x, k^a, k^b) \in \mathbb{Z}$, where $\mathbf{1}_{\{\cdot\}}$ is an indicator function and $\mathcal{K}(x)$ is given by

$$\mathcal{K}(x) = \frac{1}{\theta_1 + \theta_2}\left[\frac{x}{\rho} - (rp + c)\right]. \tag{50}$$

[15]The function $h(\cdot)$ is related to $\overline{g}(w) - \underline{g}(w)$ as follows: $h(w)(w-1) = (\overline{g}(w) - \underline{g}(w))\beta w[(1+w)^{\beta-1} - 1]$. It can be verified that $h(w)$ is concave for $w \geq 1$, $h(1) > 0$, and $h(w) < 0$ for sufficiently large w. Therefore, $h(w) = 0$, $w > 1$ has a unique solution $w^* > 1$.

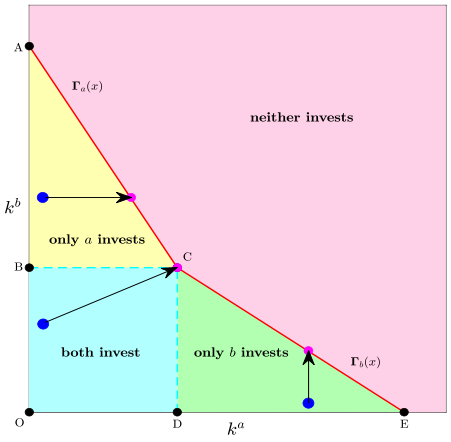


Figure 3: EQUILIBRIUM SOLUTION REGIONS WHEN THE INVESTMENT BOUNDARY IS PIECEWISE LINEAR.

Theorem 1 has two parts: equilibrium investment dynamics and a firm's response to its competitor's off-equilibrium deviation, which we analyze in subsection 5.2.1 and subsection 5.2.2, respectively.

5.2.1 Equilibrium investment and marginal q

In a closed-loop equilibrium, firm i's capital evolves as follows:

$$K_t^i = \max\left\{\min\left\{\widehat{K}_t^i, \mathcal{K}_t^M\right\},\, K_0^i\right\} \;, \text{ for } \; i = a, b, \tag{51}$$

where $\widehat{K}_t^a$ and $\widehat{K}_t^b$, uniquely solve

$$\mathcal{X}(\widehat{K}_t^a, K_0^b) = M_t \quad \text{and} \quad \mathcal{X}(K_0^a, \widehat{K}_t^b) = M_t, \tag{52}$$

respectively, and $\mathcal{K}_t^M$ is given by

$$\mathcal{K}_t^M := \mathcal{K}(M_t) = \frac{1}{\theta_1 + \theta_2}\left[\frac{M_t}{\rho} - (rp + c)\right]. \tag{53}$$

Equations (51)-(53) imply the following four scenarios depicted in Figure 3.

First, if starting from within the OBCD square region (including the boundaries), both firms invest so that $K_t^a = K_t^b = \mathcal{K}_t^M$, reaching point C in Figure 3. Afterwards, firms invest along the 45° degree by the same magnitude which is proportional to dM_t:

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)}\,. \tag{54}$$

Second, if starting from within the ABC triangle region, firm a increases its capital to $\widehat{K}_t^a$

given in (52) but firm b chooses to be inactive. As a result, the point $(\widehat{K}_t^a, K_t^b)$ on the AC boundary in Figure 3 is reached. Afterwards, firm a's investment is proportional to dM_t and firm b does not invest:

$$dK_t^a = \frac{dM_t}{\rho\theta_2} \text{ and } dK_t^b = 0. \tag{55}$$

This process continues until $\tau = \inf\{t : \widehat{K}_t^a = \mathcal{K}_t^M\}$, when point C in Figure 3 is reached. Thereafter for all $t > \tau$, firms invest along the 45° degree as described in the first scenario. Third, by symmetry the same analysis applies if firms start from within the CDE triangle region. In this case, we simply switch the strategies of firms a and b. Finally, recall that neither firm invests if the economy is in the region that is to the northeast of the ACE boundary (excluding the boundary). These four scenarios discussed above are endogenously connected over time by the stochastically evolving boundary ACE where $X_t = \mathcal{X}(K_t^a, K_t^b)$.

What are the implications on a firm's marginal q? Panel A of Figure 4 illustrates firms' marginal q for a closed-loop equilibrium in which firms make profits. In the inaction region (to the right of the ACE investment boundary), both firms' marginal q, q^a and q^b, are lower than the purchase cost of capital. In the OBCD square region, $q^a = q^b = p$ as both firms invest so that their first-order necessary conditions for investment hold with equality. In the ABC triangle region, the smaller firm (firm a) invests and the larger firm (firm b) is inactive, so that $q^a = p$ and $q^b < p$. By symmetry in the other triangle region CDE, $q^b = p$ and $q^a < p$.

Note that a firm's marginal q is always (weakly) lower than the purchase cost of capital p in a closed-loop equilibrium. This is because a firm can always respond to its competitor's investment strategy. For the special closed-loop equilibrium (the Back–Paulsen equilibrium) where firms compete most fiercely and thus earn zero profits, there are only two regions for marginal q. In the region where demand is high enough, i.e., $X_t \geq \mathcal{X}(K_t^a, K_t^b)$, at least one firm invests thus driving both firms' marginal q to the purchase cost of capital p: $q^a = q^b = p$ because of zero profits. When demand is not high enough so that $X_t < \mathcal{X}(K_t^a, K_t^b)$, neither firm invests and therefore both $q^a < p$ and $q^b < p$ hold. Panel B of Figure 4 summarizes these properties for firms' marginal q in the Back–Paulsen equilibrium.

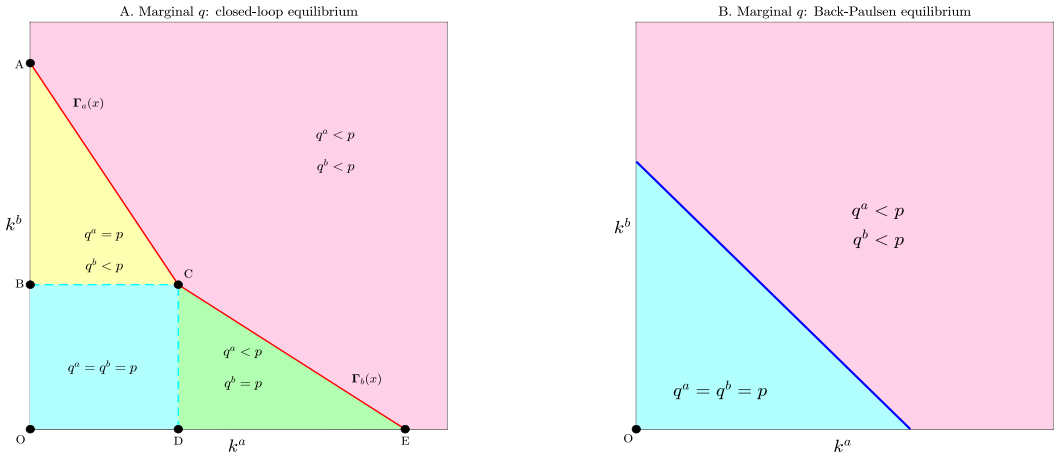


Figure 4: MARGINAL q IN EQUILIBRIA WITH LINEAR TRIGGER FUNCTIONS. This figure compares marginal q in a (typical) closed-loop equilibrium with positive profits (panel A) to marginal q in the Back–Paulsen zero-profit equilibrium (panel B).

5.2.2 Response to off-equilibrium deviation

Theorem 1 also describes how a firm responds to its competitor's potential equilibrium deviation: 1.) if firm i invests more than prescribed by its equilibrium strategy, firm $-i$ follows its own equilibrium strategy; 2.) if firm i invests less than prescribed, firm $-i$ invests just enough to bring the state to the boundary of the inaction region $\mathbb{N}_{ab}$. As a result, the state process always remains within the closure of the no-action region. This response ensures that no firm can benefit from deviating, thereby deterring deviations from equilibrium. Figure 5 depicts two examples to illustrate the mechanism.

In the first example, firm a is smaller than firm b ($K_t^a < K_t^b$) so that the economy is in the ABC region. What if firm a deviates by investing less than the level prescribed by its equilibrium strategy? Firm b's best response would require it to invest just enough to reach the investment boundary AC in Panel A of Figure 5, so that the resulting outcome is described by the magenta solid dot on the AC boundary. But this outcome is suboptimal for firm a because it produces less and moreover faces a lower output price, compared with the outcome had it followed its equilibrium strategy.[16] In summary, in a triangle region, it is always optimal for the smaller firm to invest and its investment size should take it to reach

[16]Geometrically, this result follows from the property that the absolute value of the AC line slope is larger than one as implied by $\theta_2 \geq \theta_1 > 0$.

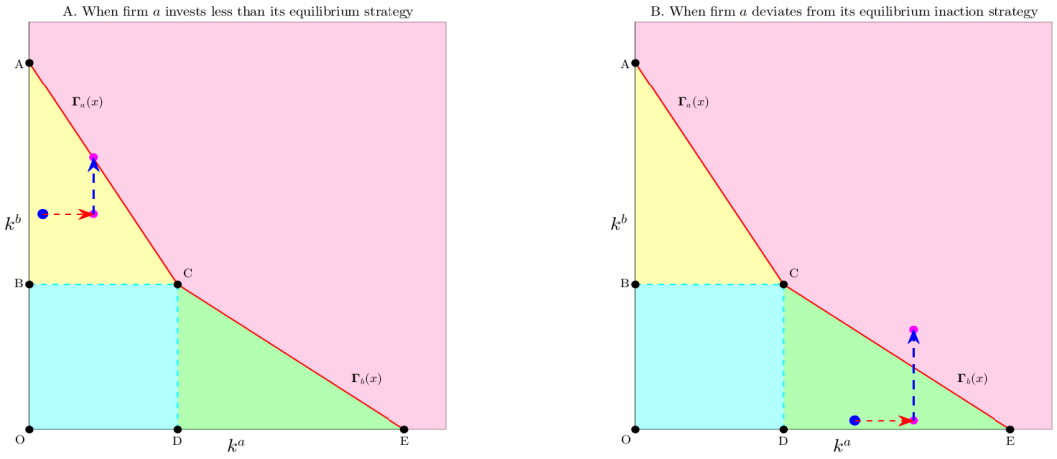


Figure 5: Firm b's response to firm a's deviation from its equilibrium strategy.

the investment boundary.

The second example considers a deviation by the larger firm. Let firm a be the larger one so that the economy is in the CDE region ($K_t^a > K_t^b$). Suppose that firm a invests by deviating from its equilibrium inaction strategy in this region. Then, firm b's response would require it to invest the same amount as that prescribed by the equilibrium strategy, so that the resulting outcome is described by the magenta solid dot in the inaction region which is above the ACE boundary in Panel B. But this outcome is suboptimal for firm a because the output price would be too low, destroying its value when compared with the alternative of sticking to its equilibrium strategy. In summary, in a triangle region CDE, it is always optimal for the larger firm to stay inactive.

6 Duopolistic q Theory of Investment

In this section, we analyze the q-theoretic implications of our model solution obtained in the preceding section, focusing on a symmetric duopoly setting where $K_0^a = K_0^b$. We report analytical results comparing different equilibrium solutions in subsection 6.1, and then conduct a quantitative analysis in subsection 6.2.

6.1 Analytical Results

Let $V^{\text{duo}}_{\text{closed}}(z) = V^a(z)+V^b(z)$ denote industry value in a closed-loop equilibrium where $z := (x, k, k)$. The industry value in the Back–Paulsen equilibrium, $V^{\text{duo}}_{\text{BP}}(z)$, is associated with the strategy pair $(\theta_1, \theta_2) = (1, 1)$. The industry value in the best closed-loop equilibrium, $V^{\text{duo}}_{\text{best}}(z)$, is associated with the strategy pair (θ^*_1, θ^*_2) given in (47) and is the highest among all closed-loop equilibria. Finally, let $V^{\text{duo}}_{\text{open}}(z)$ denote the industry value in the open-loop equilibrium. Below we summarize the rank order of these industry values and industry investment rates.

Proposition 3 *For any $(\theta_1, \theta_2) \in \Theta$ satisfying Condition 1, the following inequalities hold:*

$$V^{\text{mon}}(x, 2k) > V^{\text{duo}}_{\text{open}}(x, k, k) > V^{\text{duo}}_{\text{best}}(x, k, k) \geq V^{\text{duo}}_{\text{closed}}(x, k, k) \geq V^{\text{duo}}_{\text{BP}}(x, k, k)\,, \tag{56}$$

where V^{mon} denotes the monopoly value given in Appendix B. Finally, $V^{\text{duo}}_{\text{closed}}(x, k, k)$ is increasing in $\theta_1 + \theta_2$.

Proposition 4 *As a fraction of the monopoly's investment rate, the duopoly industry investment rate in a closed-loop equilibrium is constant and decreases with $\theta_1 + \theta_2$:*[17]

$$\frac{dK^{\text{duo}}_t}{dK^{\text{mon}}_t} = \frac{4}{\theta_1 + \theta_2} \in \Big(\frac{4}{3}, 2\Big]\,, \tag{57}$$

where $K^{\text{duo}}_t = K^a_t + K^b_t$. As a fraction of the monopoly value, the duopoly industry value in a closed-loop equilibrium is constant and increases with $\theta_1 + \theta_2$:[18]

$$\frac{V^{\text{duo}}_{\text{closed}}(x, 0, 0)}{V^{\text{mon}}(x, 0)} = 4\bigg[\frac{2}{\theta_1 + \theta_2} - \Big(\frac{2}{\theta_1 + \theta_2}\Big)^2\bigg] < \frac{8}{9}\,. \tag{58}$$

The intuition for (56) is as follows. First, as competition is the most fierce in the Back–Paulsen equilibrium where $(\theta_1, \theta_2) = (1, 1)$, the industry value $V^{\text{duo}}_{\text{BP}}(x, k, k)$ is the lowest. This is the case where the last inequality in (56) holds with equality. Second, at the other end of the spectrum is the best closed-loop equilibrium where $(\theta_1, \theta_2) = (\theta^*_1, \theta^*_2)$. This equilibrium attains the highest industry value among all closed-loop equilibria and is the only equilibrium in which the third inequality in (56) holds with equality. Third, the industry value in the open-loop equilibrium is larger than in all closed-loop equilibria. This is because, by definition, unlike in a closed-loop equilibrium, a firm in an open-loop equilibrium cannot choose its strategy based

[17] This is conditional on investing, i.e., $M_t = X_t > \rho(rp + c)$ for $t > 0$

[18] See Corollary 2 in Appendix E.5 for details.

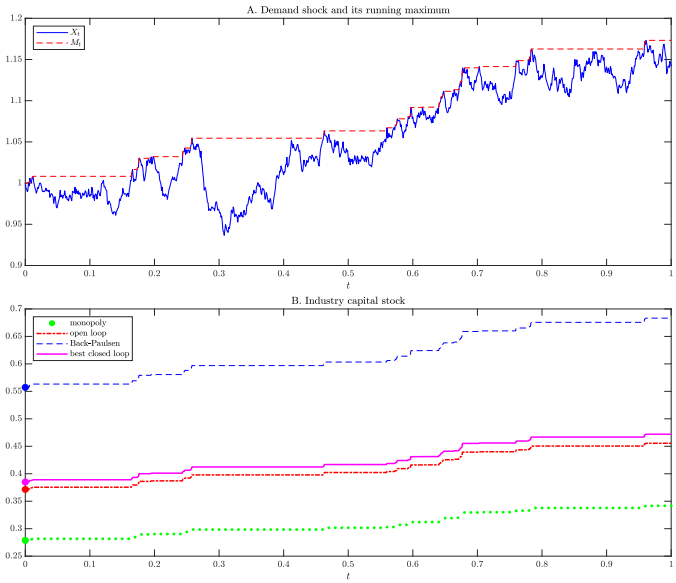


Figure 6: INDUSTRY INVESTMENT DYNAMICS. Panel A plots the running maximum M_t process (dashed red line) for a simulated demand shock path X_t (solid blue line). Panel B shows that at all t, the industry capital stock is larger under the best closed-loop equilibrium (solid magenta line) than under the open-loop equilibrium (dash-dotted red line), attains the lowest level under monopoly (solid dotted line) and the largest level in the Back–Paulsen equilibrium (dashed blue line). Parameter values: $r = 0.07$, $\mu = 0.015$, $\sigma = 0.15$, $c = 0.1$, and $p = 1$.

on its competitor's strategy at any $t > 0$. Obviously, a monopoly industry has the highest value as it faces no competition (see the first inequality in (56)).

In addition to the inequalities discussed above, Proposition 3 also states that $V^{\text{duo}}_{\text{closed}}(x, k, k)$ increases with $\theta_1 + \theta_2$. This is because $\theta_1 + \theta_2$ dictates the rate at which the industry invests, and the larger this sum, the slower the industry invests. Proposition 4 corroborates this result by stating two key findings for an industry starting from scratch with no capital, i.e., $K^a_0 = K^b_0 = 0$, in a closed-loop equilibrium associated with a pair (θ_1, θ_2). First, a duopoly industry's investment rate is proportional to a monopoly's investment rate with a coefficient of $\frac{4}{\theta_1+\theta_2}$, which lies between 4/3 and 2 (recall that $\theta_1 + \theta_2 \in [2, 3)$, as discussed when introducing Condition 1). Second, the corresponding duopoly industry value is proportional to the monopoly value, with a coefficient that increases with $\theta_1 + \theta_2$.[19]

Next, consider the open-loop equilibrium.[20] In this equilibrium, a duopoly industry invests

[19]The coefficient in (58), $4\left[\frac{2}{\theta_1+\theta_2} - \left(\frac{2}{\theta_1+\theta_2}\right)^2\right]$, increases with $\theta_1 + \theta_2$ as $\theta_1 + \theta_2 \in [2, 3)$.

[20]The open-loop equilibrium solution can be obtained as the limit of a closed-loop equilibrium solution by setting $\theta_1 + \theta_2 = 3$.

at a rate that is 1/3 faster than a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = \frac{4}{3}$, which causes the industry value to be 1/9 lower than that of a monopoly: $\frac{V_{\text{open}}^{\text{duo}}(x,0,0)}{V^{\text{mon}}(x,0)} = \frac{8}{9}$. Consistent with our intuition, because a firm can respond to its competitor's investment strategy in a closed-loop equilibrium, the industry invests faster and hence its value is lower in all closed-loop equilibria than in the open-loop equilibrium. The closer $\theta_1 + \theta_2$ is to the lower bound of 2, the faster the industry invests and the lower the industry value. In the limiting case where firms compete most fiercely, $\theta_1 = \theta_2 = 1$—that is, in the Back–Paulsen closed-loop equilibrium—the duopoly industry invests twice as fast as a monopoly: $\frac{dK_t^{\text{duo}}}{dK_t^{\text{mon}}} = 2$, completely eliminating the option value of waiting so that $V_{\text{BP}}^{\text{duo}}(x, 0, 0) = 0$.

6.2 Quantitative Illustration

Figure 6 shows that industry investment dynamics differ substantially across different equilibria. In Panel A, we simulate a demand shock $\{X_t\}$ path and report the associated running maximum $\{M_t\}$, which is noticeably smoother than $\{X_t\}$. Panel B compares the corresponding duopoly industry capital stock processes in three different equilibria.

The highest line in panel B (the dashed blue line) depicts the industry capital stock process in the Back–Paulsen equilibrium. The industry starts with the highest level of capital, $\frac{X_0}{\rho} - (rp + c)$, among all equilibria and, moreover, increases its K_t^{duo} at the fastest rate. This is because firms compete most fiercely in this equilibrium.

The second highest line (the magenta solid line) depicts the industry capital stock process in the best closed-loop equilibrium where the industry attains the highest value among all closed-loop equilibria. The industry starts with an initial capital stock of $\frac{2}{\theta_1^* + \theta_2^*}\left[\frac{X_0}{\rho} - (rp + c)\right]$, which is lower than in the Back–Paulsen equilibrium, and moreover, increases its capital stock at a much slower rate than in the Back–Paulsen equilibrium. This is because firms preserve some option value of waiting by setting $\theta_1^* + \theta_2^*$ to 2.9 in our numerical example, which is more than 40% larger than $\theta_1 + \theta_2 = 2$ in the Back–Paulsen equilibrium (Recall that the sum $\theta_1 + \theta_2$ dictates the investment rate and hence the industry value.)

Lower than the best closed-loop equilibrium $\{K_t^{\text{duo}}\}$ process is the open-loop equilibrium $\{K_t^{\text{duo}}\}$ process (the dash-dotted red line). In an open-loop equilibrium, a firm does not respond to its competitor's investment strategy at any $t > 0$, thus preserving even more

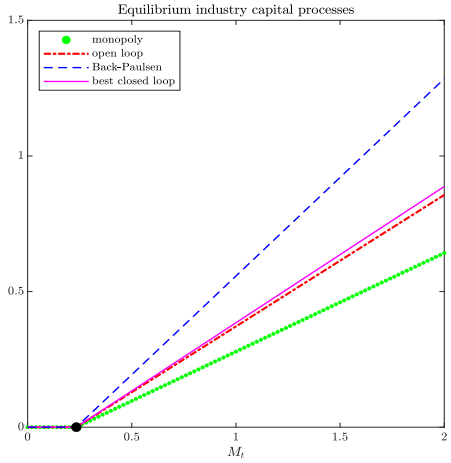


Figure 7: Comparing Equilibrium Industry Capital Processes. We plot industry capital stock as a function of M_t, the running maximum of X_t. In all cases, industry investment starts when M_t exceeds $\rho(rp + c)$ (the black dot) but responds to M_t at different rates. Parameter values: $r = 0.07$, $\mu = 0.015$, $\sigma = 0.15$, $c = 0.1$, and $p = 1$.

option value of waiting than in the best closed-loop equilibrium. This explains why the industry starts with an initial capital stock lower than in the best closed-loop equilibrium and increases at the slowest rate among all duopoly equilibria. Finally, a monopoly has the lowest capital stock at all time as there is no competition.

Figure 7 sheds additional insights on industry dynamics. It shows that in all duopoly equilibria, the industry remains dormant (with no capital) until the demand shock's running maximum, $\{M_t\}$, reaches $\rho(rp + c)$ (the black dot on the horizontal axis), and the industry then grows at very different rates in different equilibria (different slopes for the four lines.)

A duopoly industry invests at the fastest rate in the Back–Paulsen equilibrium with a slope of $1/\rho$, which is 0.727 in our numerical example. The investment rate in the open-loop equilibrium is the slowest among all equilibria – it is only 0.485, which is 2/3 of the investment rate in the Back–Paulsen equilibrium. In the best closed-loop equilibrium, the industry invests at a rate of $2/[(\theta_1^* + \theta_2^*)\rho]$. In our numerical example, it is 0.502, which is 31% lower than the investment rate in the Back–Paulsen equilibrium. That is, in the best closed-loop equilibrium, the industry invests much slower and attains a substantially higher value than in the Back–Paulsen equilibrium. Finally, a monopoly invests at a rate that is slower than in any duopoly equilibria: it is only 1/2 of the investment rate in the Back–Paulsen equilibrium.

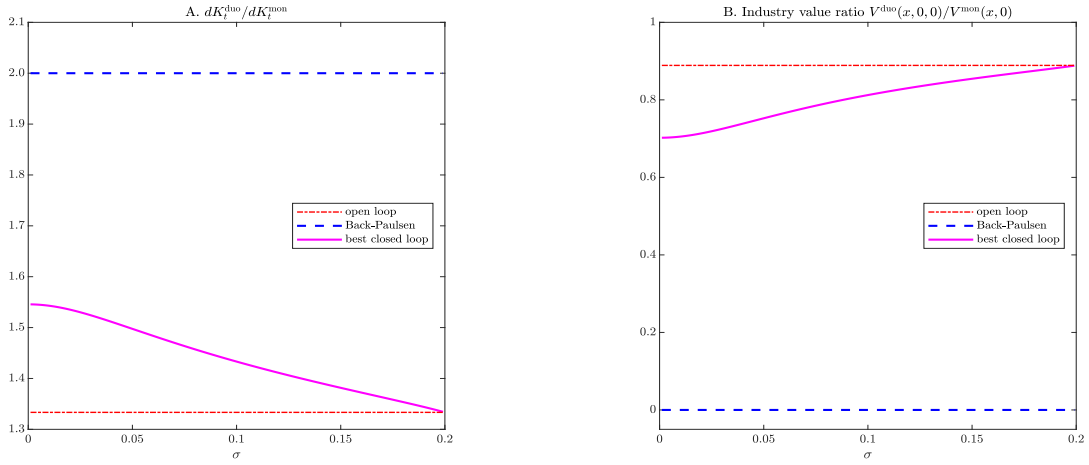


Figure 8: INDUSTRY INVESTMENT DYNAMICS AND INDUSTRY VALUE. Panel A plots a duopoly industry investment as a fraction of a monopoly's investment. Panel B plots the duopoly industry value as a fraction of the monopoly value. We report results for the open-loop equilibrium (dash-dotted red line), the Back–Paulsen closed-loop equilibrium (dashed blue line), and the best closed-loop equilibrium (solid magenta line). Parameter values: $r = 0.07$, $\mu = 0.015$, $c = 0.1$, and $p = 1$.

Finally, in Figure 8 we show that the effects of volatility σ on investment dynamics (panel A) and industry value (panel B) are quantitatively large, even after we control for the standard option value of waiting for a monopoly. To control for the standard volatility effect, we report duopoly industry investment rates as a fraction of the monopoly's investment rate: In the Back–Paulsen equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 2$ and in the open-loop equilibrium $dK_t^{\text{duo}}/dK_t^{\text{mon}} = 4/3$, both of which are independent of volatility σ. Figure 8 plots $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ (panel A) and $V^{\text{duo}}(x,0,0)/V^{\text{mon}}(x,0)$ (panel B) for the best closed-loop equilibrium together with these two special equilibria. As we increase volatility from around zero to 0.2, in our numerical example for the best closed-loop equilibrium, $dK_t^{\text{duo}}/dK_t^{\text{mon}}$ decreases by 14% from 1.55 to 1.34, causing the duopoly industry value as a fraction of the monopoly value, $V^{\text{duo}}(x,0,0)/V^{\text{mon}}(x,0)$, to increase by 27% from 0.70 to 0.89. This monotonicity is consistent with the intuition that the industry as a whole competes less aggressively when demand uncertainty is higher as the industry's option value of waiting is larger.

7 Closed-loop Equilibria with Nonlinear Boundaries

In this section, we demonstrate the existence of closed-loop equilibria with nonlinear investment thresholds. Specifically, we conjecture and verify that there exists a nonlinear investment threshold $\mathcal{X}(k^a, k^b)$ such that in the investment region $X_t \geq \mathcal{X}(k^a, k^b)$, firm i's marginal q always equals the marginal cost of purchasing, p, so that the following holds on $\mathcal{X}(k^a, k^b)$:[21]

$$\left.\frac{\partial V^i(x, k^a, k^b)}{\partial k^i}\right|_{x=\mathcal{X}(k^a,k^b)} = p. \tag{59}$$

In a symmetric equilibrium, we conjecture and verify that $\mathcal{X}(k^a, k^b)$ takes the form of:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho\big(\mathcal{G}(k^a, k^b) + rp + c\big), & \text{if } k^a \leq k^b, \\ \rho\big(\mathcal{G}(k^b, k^a) + rp + c\big), & \text{if } k^a \geq k^b\,, \end{cases} \tag{60}$$

where $\mathcal{G}(k^a, k^b)$ is a nonlinear function to be determined. Next we present a sufficient condition for $\mathcal{G}(k^a, k^b)$ so that the inequality conditions (41)-(42) and the equality condition (59) are satisfied. Given symmetry along the 45^o degree line, it is sufficient for us to characterize $\mathcal{G}(k^a, k^b)$ for the $k^b \geq k^a \geq 0$ domain in the first quadrant.

Condition 2 *In the $k^b \geq k^a$ region, there exists a lower bound, $\underline{k} > 0$, for k^a and k^b, such that the following conditions hold:*

(i.) $k^a + k^b < \mathcal{G}(k^a, k^b) < 2k^a + k^b$;

(ii.) $\mathcal{G}(k^a, k^b)$ is continuously differentiable ($\mathcal{C}^1$) and increasing in (k^a, k^b); and

(iii.) there exists a $\mathcal{C}^1$ function $\phi(z_1, z_2)$ such that at each (z_1, z_2), either $\frac{\partial \phi(z_1,z_2)}{\partial z_1} \neq 0$ or $\frac{\partial \phi(z_1,z_2)}{\partial z_2} \neq 0$ holds, and

$$\phi\big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\big) = 0. \tag{61}$$

Part (i) requires that the nonlinear investment boundary $\mathcal{X}(k^a, k^b)$ strictly lies above the linear boundary $\rho(k^a + k^b + rp + c)$ for the Back–Paulsen closed-loop equilibrium, where firms make zero profits, and below the linear boundary $\rho(2k^a + k^b + rp + c)$ for the open-loop equilibrium (analyzed in subsection 2.3). The intuition for why a nonlinear boundary $\mathcal{X}(k^a, k^b)$ has to lie between these two linear boundaries is as follows. The Back–Paulsen equilibrium is the closed-loop equilibrium where firms engage in the most aggressive competition, eroding

[21] Recall that in the Back–Paulsen equilibrium, the investment threshold is linear and both firms' marginal q, q^a and q^b, equal p in the investment region (see panel B of Figure 4).

all the option value of waiting. Intuitively, competition in other closed-loop equilibrium is less severe than in the Back–Paulsen equilibrium, firms invest at lower rates, suggesting that the associated investment boundary shall lie above the investment boundary for the Back–Paulsen equilibrium. On the other hand, intuitively a firm competes less in the open-loop equilibrium than in a closed-loop equilibrium as it effectively pre-commits at time 0 not to respond to its competitor's strategy at time $t > 0$. This reasoning suggests that the investment boundary for a closed-loop equilibrium lies above the investment boundary for the open-loop equilibrium. Note that the right and left equalities in part (i) are implied by (41) and (42), respectively.

As industry output price decreases in either firm's capital stock, for a fixed demand shock X_t, a firm invests less in an industry with a larger capital stock, *ceteris paribus.* Part (ii) captures this property. Both parts (i) and (ii) are needed to ensure that (41) and (42) hold.

Finally, part (iii) is implied by the first-order condition (59) for marginal q on the investment boundary $\mathcal{X}(k^a, k^b)$. To see this, first note that for any (k^a, k^b), (37) follows from (59). Second, using (37), (38), $\frac{\partial^2 H^a(k^a,k^b)}{\partial k^a \partial k^b} = \frac{\partial^2 H^a(k^a,k^b)}{\partial k^b \partial k^a}$, and the symmetry property of the investment boundary $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$, we obtain the following partial differential equation (PDE):

$$\rho k^b \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b} + \Big[\mathcal{X}(k^a, k^b) - \rho\big(2k^b + k^a + rp + c\big)\Big] \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} = 0, \quad k^a \leq k^b. \tag{62}$$

Solving the PDE (62) yields $\mathcal{X}(k^a, k^b) = \rho\big(\mathcal{G}(k^a, k^b) + rp + c\big)$ for $k^a \leq k^b$, where $\mathcal{G}(k^a, k^b)$ satisfies the condition in part (iii). See Appendix C for further details.

Having explained economic forces that give rise to Condition 2, we next characterize all (symmetric) closed-loop equilibria satisfying this condition.

Theorem 2 *Under Condition 2 for a nonlinear investment boundary $\mathcal{X}(k^a, k^b)$ given in* (60)*, a strategy pair $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) = \{(\boldsymbol{u}^a, \boldsymbol{v}^a), (\boldsymbol{u}^b, \boldsymbol{v}^b)\} \in \mathcal{S}$ constitutes a closed-loop equilibrium if $\min(k^a_0, k^b_0) \geq \underline{k}$ and*

$$\boldsymbol{u}^i(z, u^{-i}) = \left[\frac{\mathbf{1}_{k^i < k^{-i}}}{\vartheta_2(k^a, k^b)} + \Big(\frac{1}{\vartheta_1(k^a, k^b)} - \frac{\vartheta_2(k^a, k^b)}{\vartheta_1(k^a, k^b)} u^{-i}\Big)\mathbf{1}_{k^i \geq k^{-i}}\right]\mathbf{1}_{x \geq \mathcal{X}(k^a,k^b)}, \tag{63}$$

$$\boldsymbol{v}^i(z, v^{-i}) = \inf\Big\{\delta \geq 0 : \mathcal{X}\big(k^i + \delta, \max\{k^{-i}, \min\{k^{-i} + v^{-i}, \mathcal{K}(x)\}\}\big) \geq x\Big\}, \tag{64}$$

for any $z = (x, k^a, k^b) \in \mathbb{Z} := (0, \infty) \times [\underline{k}, \infty) \times [\underline{k}, \infty)$, with $\mathcal{K}(x)$ being given by

$$\mathcal{K}(x) := \inf\{k \geq \underline{k} : \mathcal{X}(k, k) \geq x\} \tag{65}$$

and $(\vartheta_1, \vartheta_2)$ *being given as follows:*

$$\vartheta_1(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a \le k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a \ge k^b, \end{cases} \quad \text{and} \quad \vartheta_2(k^a, k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } k^a \le k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } k^a \ge k^b. \end{cases} \tag{66}$$

There exist many closed-loop equilibria with nonlinear boundaries. A key step in identifying one such equilibrium is to specify $\phi(z_1, z_2)$ in (61). Below we characterize closed-loop equilibria in closed form for two linear $\phi(z_1, z_2)$ specifications: $\phi(z_1, z_2) = z_1 - \lambda$ and $\phi(z_1, z_2) = \lambda z_1 - z_2$. For both specifications, $\lambda > 0$ is a constant.

Proposition 5 *There exist closed-loop equilibria with the following nonlinear investment boundaries* $\mathcal{X}(k^a, k^b)$*:*

(i) For $\min(k_0^a, k_0^b) > \sqrt{\lambda}$ *for any* $\lambda > 0$*,*

$$\mathcal{X}(k^a, k^b) = \rho\Big(k^a + k^b + \frac{\lambda}{\max\{k^a, k^b\}} + rp + c\Big); \tag{67}$$

(ii) For $\min(k_0^a, k_0^b) > \frac{3}{\lambda}$ *for any* $\lambda > 0$*,*

$$\mathcal{X}(k^a, k^b) = \rho\Big((k^a + k^b)\frac{\lambda \max\{k^a, k^b\}}{\lambda \max\{k^a, k^b\} - 1} + rp + c\Big). \tag{68}$$

Nonlinear triggers in (67) and (68) are captured by the terms involving max operators. To illustrate the industry capital stock dynamics in a closed-loop equilibrium with a nonlinear investment trigger, we plot K_t^{duo} (scaled by the monopoly's capital stock K_t^{mon}) as a function of the market demand's running maximum M_t in Figure 9 using the nonlinear trigger function given in (67).

Panel A shows that the industry investment dynamics is highly nonlinear unlike the dynamics in an equilibrium with linear triggers analyzed in Section 5. With a nonlinear trigger, K_t^{duo} is larger than K_t^{duo} in the best closed-loop equilibrium (with a linear trigger). Panel B shows that the resulting industry value is much lower in an equilibrium with a nonlinear trigger than in the best closed-loop equilibrium. The quantitative effect for our numerical example is very large. For example, when $M_t = 5$, the duopoly industry capital stock is 58% larger than a monopoly's: $K_t^{\text{duo}}/K_t^{\text{mon}} = 1.58$, causing the industry value so much that the industry's net present value (NPV) is only 8.8% of a monopoly's NPV. Recall that the industry's NPV in the best closed-loop equilibrium is 85.4% of a monopoly's when $M_t = 5$. That is, the industry's NPV in the equilibrium with a nonlinear trigger is only 10.3% of that

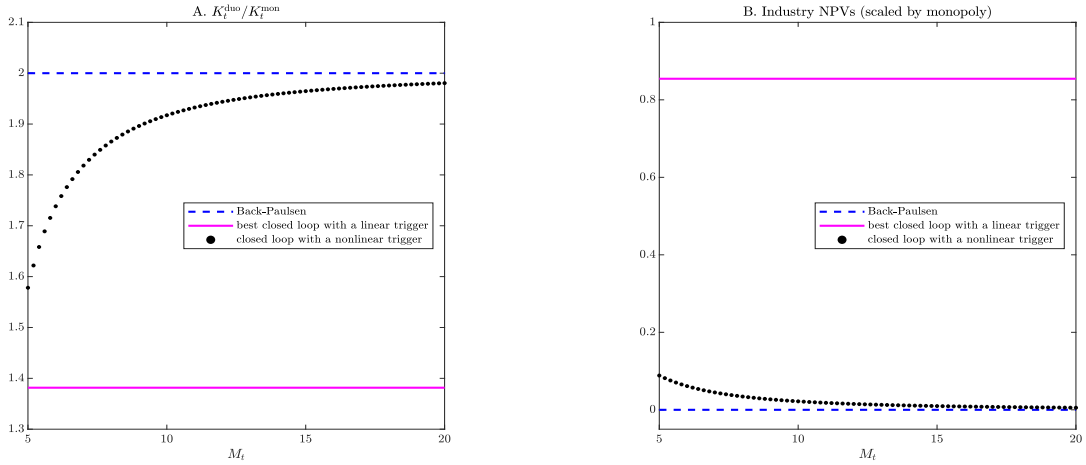


Figure 9: Equilibrium Industry Capital Processes and Industry Net Present Values (NPVs). Panel A plots $K_t^{\text{duo}}/K_t^{\text{mon}}$ as a function of the demand shock's running maximum M_t in a closed-loop equilibrium with a nonlinear investment trigger, the Back–Paulsen equilibrium, and the best closed-loop equilibrium with a linear investment trigger. Panel B plots the duopoly industry's NPV (scaled by a monopoly's NPV) in the respective equilibria, where a duopoly industry's NPV is defined as $V^{\text{duo}}(M_t, K_t^{\text{duo}}/2, K_t^{\text{duo}}/2) - pK_t^{\text{duo}}$. Parameter values: $r = 0.07$, $\mu = 0.015$, $c = 0.1$, and $p = 1$.

in the best closed-loop equilibrium (with a linear trigger) analyzed in the preceding section. This very large value destruction is because in our numerical example of the equilibrium with a nonlinear trigger, firms compete so much more aggressively than in the best closed-loop equilibrium, eroding some option value of waiting, but they still earn some profits, competing less aggressively than in the Back–Paulsen equilibrium in which firms make zero profits.

In the limit as $M_t \to \infty$, the nonlinear trigger term in (67) becomes increasingly smaller relative to the linear terms, the ratio $K_t^{\text{duo}}/K_t^{\text{mon}}$, converges to two, approaching the Back–Paulsen equilibrium solution. In summary, firms compete less aggressively when market demand is low and compete more fiercely as market demand increases in a closed-loop equilibrium with a nonlinear investment trigger.

8 Conclusion

Working with Grenadier (2002)'s classic real-option game setting, Back and Paulsen (2009) propose using the closed-loop equilibrium solution concept, which allows a firm to respond to its competitor's strategy over time, to characterize investment strategies. They further conjecture and verify a closed-loop equilibrium in which firms compete fiercely, yielding a perfectly competitive equilibrium outcome. Finally, they pose two important open questions: (1) how to define a closed-loop equilibrium and (2) whether a closed-loop equilibrium, other than the one they identified, exists.

We respond to Back and Paulsen's questions by first defining a closed-loop equilibrium and then identifying infinitely many closed-loop equilibria in which firms earn positive profits. Using equilibrium restrictions on a firm's marginal q and its *cross* marginal q, the sensitivity of the firm's value with respect to its competitor's capital stock, we obtain closed-form solutions for 1.) a firm's value, which also depends on its competitor's capital stock in addition to its own capital stock and the demand shock, and 2.) optimal investment policies, characterized by an explicitly solved investment boundary.

Using our newly proposed closed-loop equilibrium solution concept, we also contribute to the q theory of investment literature (Lucas and Prescott, 1971; Hayashi, 1982; Abel and Eberly, 1994, 1996) by extending these single-firm decision models to an imperfectly competitive equilibrium setting. When analyzing the duopolistic equilibrium, we focus on the best closed-loop equilibrium as firm value is the highest among all closed-form equilibria. Consistent with our intuition, compared with the Back–Paulsen closed-loop equilibrium in which firms make zero profits, the option value of waiting in the best closed-loop equilibrium is quantitatively large. However, compared with the less plausible open-loop equilibrium, firms compete more fiercely, causing firm value to be lower.

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Appendices

A Verification Theorem and Value Function

We present a verification theorem for closed-loop equilibria and defer its proof to Appendix E.3. The theorem provides key conditions on a pair of candidate value functions, $V^a(z)$ and $V^b(z)$, and the associated candidate equilibrium strategies $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$. We begin by imposing a condition on the candidate value functions.

Condition A.1 *Assume $V^i(z) \in \mathcal{C}^{0,1,1}(\mathbb{Z})$ for $i = a, b$, and inequalities* (41)–(42) *hold.*

Motivated by (29)–(31), we define the investment and no-action regions associated with the candidate value functions, adopting the same set of symbols as in the main body of the paper, where these regions are defined via trigger functions:

$$\mathbb{I}_i := \left\{ z \in \mathbb{Z} : \frac{\partial V^i(z)}{\partial k^i} = p \text{ and } \frac{\partial V^{-i}(z)}{\partial k^i} = 0 \right\}, \tag{A.1}$$

$$\mathbb{N}_i := \mathbb{Z} \setminus \mathbb{I}_i. \tag{A.2}$$

Let $\mathbb{N}_{ab}$ denote the intersection of $\mathbb{N}_a$ and $\mathbb{N}_b$:

$$\mathbb{N}_{ab} := \mathbb{N}_a \bigcap \mathbb{N}_b. \tag{A.3}$$

Let $\overline{\mathbb{N}}_{ab}$ denote the closure of $\mathbb{N}_{ab}$, which includes both its interior and boundary. These symbols are used throughout this appendix.

Next, we impose another condition on candidate equilibrium value functions.

Condition A.2 *Assume $V^i \in \mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab})$, and equation* (33) *holds for any $z \in \mathbb{N}_{ab}$.*

The following condition is imposed on candidate equilibrium strategies.

Condition A.3 *Consider a candidate equilibrium strategy pair $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ associated with the candidate value function pair (V^a, V^b) satisfying Condition A.1. For any given initial state $(X_0, K^a_0, K^b_0) = (x, k^a, k^b) \in \mathbb{Z}$, the capital stock process pair (K^a_t, K^b_t) generated by $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ has the following properties:*

- *At the initial time $t = 0$, the value-matching condition holds (from $t = 0$ to $t = 0+$):*

$$V^i(x, K^a_{0+}, K^b_{0+}) - p(K^i_{0+} - k^i) = V^i(x, k^a, k^b); \tag{A.4}$$

- *For all $t > 0$, $(X_t, K_t^a, K_t^b) \in \overline{\mathbb{N}}_{ab}$, and K_t^i is continuous in $t > 0$ with dynamics:*

$$K_t^i = K_{0+}^i + \int_{0+}^t \mathbf{1}_{\left\{(X_s, K_s^a, K_s^b) \in \mathbb{I}_i\right\}} dK_s^i, \tag{A.5}$$

where the integrand is an indicator function.

Equation (A.4) is implied by the first-order necessary condition for investment when firm i undertakes an initial lumpy investment.

Next, we impose a condition restricting firms' responses to competitors' deviations. For expositional clarity, we focus on firm a's deviation.

Condition A.4 *Let $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ be a candidate equilibrium strategy pair associated with (V^a, V^b) satisfying Condition A.1. Consider any feasible deviation strategy pair $(\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ where firm a deviates.*[22] *Under $(\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b)$, the state process satisfies:*

$$(X_t, K_t^{a,\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b}, K_t^{b,\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b}) \in \overline{\mathbb{N}}_{ab}, \quad \forall t > 0. \tag{A.6}$$

Condition A.4 restricts off-equilibrium state processes to the closure of the no-investment region associated with candidate value functions.

Theorem 3 *Assume there exist functions $V^a(z)$ and $V^b(z)$ satisfying Conditions A.1 and A.2, and a feasible strategy $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ associated with (V^a, V^b) that satisfies Conditions A.3 and A.4. Then $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ constitutes a closed-loop equilibrium strategy, with equilibrium value functions $V^a(z)$ and $V^b(z)$ for firms a and b under the regularity condition*[23]*:*

$$J^a(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) = V^a(z), \qquad J^b(z; \boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) = V^b(z). \tag{A.7}$$

B Firm Values

We first report firm value functions for monopoly in Appendix B.1 and then for duopoly settings in Appendix B.2.

[22] By symmetry, an analogous condition applies to deviations by firm b. For brevity, we omit details.

[23] For any initial state $z \in \mathbb{Z}$ and deviation strategy $\widetilde{\boldsymbol{\varphi}} \in \mathcal{S}$ (where $\widetilde{\boldsymbol{\varphi}} = (\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b)$ if $i = a$, or $(\boldsymbol{\varphi}_*^a, \widetilde{\boldsymbol{\varphi}}^b)$ if $i = b$), the capital processes $K_s^{a,\widetilde{\boldsymbol{\varphi}}}$ and $K_s^{b,\widetilde{\boldsymbol{\varphi}}}$ satisfy: $\lim_{n\to\infty} \mathbb{E}^z \left[e^{-rt_n} V^i \left(X_{t_n}, K_{t_n}^{a,\widetilde{\boldsymbol{\varphi}}}, K_{t_n}^{b,\widetilde{\boldsymbol{\varphi}}} \right) \right] = 0$ for some sequence $\{t_n\} \to \infty$, and $\mathbb{E}^z \left[\int_0^t \left| e^{-rs} \frac{\partial V^i}{\partial x} \left(X_s, K_s^{a,\widetilde{\boldsymbol{\varphi}}}, K_s^{b,\widetilde{\boldsymbol{\varphi}}} \right) \sigma(X_s) \right|^2 ds \right] < \infty$, $\forall t \geq 0$.

B.1 Monopoly Value

Proposition 6 *The monopoly value in the inaction region where $x < \mathcal{X}^{\text{mon}}(k)$ is given by*

$$V^{\text{mon}}(x,k) = \frac{k}{r-\mu}x - \frac{k^2+ck}{r} + H^{\text{mon}}(k)x^{\beta}\,, \tag{B.1}$$

where

$$H^{\text{mon}}(k) = \frac{1}{2\rho\beta(\beta-2)(r-\mu)}(\mathcal{X}^{\text{mon}}(k))^{2-\beta}. \tag{B.2}$$

In the investment region where $x \geq \mathcal{X}^{\text{mon}}(k)$, the value function is given by

$$V^{\text{mon}}(x,k) = V^{\text{mon}}(x,\mathcal{K}^{\text{mon}}(x)) - p(\mathcal{K}^{\text{mon}}(x)-k), \tag{B.3}$$

where $\mathcal{K}^{\text{mon}}(x) = \frac{x/\rho-(rp+c)}{2}$ is the inverse function of $\mathcal{X}^{\text{mon}}(k)$ and

$$V^{\text{mon}}(x,\mathcal{K}^{\text{mon}}(x)) = \frac{\mathcal{K}^{\text{mon}}(x)}{r-\mu}x - \frac{(\mathcal{K}^{\text{mon}}(x))^2+c\mathcal{K}^{\text{mon}}(x)}{r} + \frac{x^2}{2\rho\beta(\beta-2)(r-\mu)}.$$

The first two terms in (B.1) represent the monopolist's value if it has no future investment options at all, i.e., $K_t = k$ for all t. The $H^{\text{mon}}(k)x^{\beta}$ term is convex in x and measures the option value of investing.

Next, we report closed-form expressions for value functions associated with the equilibrium strategy given in Theorems 1 and 2.

B.2 Firm Value in Duopoly

Theorem 4 *The value function $V^i(z)$ associated with the equilibrium strategy given in Theorem 1 or 2, where $z = (x, k^a, k^b)$, is given below.*

1. *In the region that is to the northeast of the ACE boundary in Figure 1, i.e., in the region where $x \leq \mathcal{X}(k^a,k^b)$, $V^i(z)$ is given by* (34).

2. *In the square BCDO region in Figure 1, i.e., in the region where $k^a \leq \mathcal{K}(x)$ and $k^b \leq \mathcal{K}(x)$, both firms instantly invest to the point C and*

$$V^i(z) = V^i(x,\mathcal{K}(x),\mathcal{K}(x)) - p(\mathcal{K}(x)-k^i)\,. \tag{B.4}$$

3. *In the ABC region in Figure 1, i.e., in the region where $x \geq \mathcal{X}(k^a,k^b)$ and $k^b > \mathcal{K}(x)$, firm b does not invest and firm a invests so that its capital stock $\widehat{k}^a$ reaches the curve*

AC and satisfies the following equation:

$$\mathcal{X}(\widehat{k}^a, k^b) = x. \tag{B.5}$$

The equilibrium value functions for firms a and b are given by

$$V^a(z) = V^a(x, \widehat{k}^a, k^b) - p(\widehat{k}^a - k^a)\,, \tag{B.6}$$

and

$$V^b(z) = V^b(x, \widehat{k}^a, k^b)\,, \tag{B.7}$$

respectively, where $V^i(x, \widehat{k}^a, k^b)$ is given in (34)*.*

By symmetry, we can obtain the solution in the CDE region in Figure 1, i.e., in the region where $x \geq \mathcal{X}(k^a, k^b)$ and $k^a > \mathcal{K}(x)$.

C Solving the PDE (62)

Denote

$$\mathcal{G}(k^a, k^b) = \frac{\mathcal{X}(k^a, k^b)}{\rho} - \big(rp + c\big) \tag{C.1}$$

for $k^b \geq k^a > 0$. Then, (62) is equivalent to

$$k^b \mathcal{G}_{k^b}(k^a, k^b) + \Big(\mathcal{G}(k^a, k^b) - k^a - 2k^b\Big)\mathcal{G}_{k^a}(k^a, k^b) = 0. \tag{C.2}$$

Note that (C.2) is a quasilinear first-order PDE, and $k^b > 0$. The general (local) solution to (C.2) can be obtained by solving the following characteristic equations (see e.g., Chapter 1 in McOwen (2004)):

$$\frac{dk^b}{k^b} = \frac{dk^a}{\mathcal{G} - k^a - 2k^b}, \quad d\mathcal{G} = 0. \tag{C.3}$$

Solving $d\mathcal{G} = 0$, we have $\mathcal{G} = \lambda_1$ for some constant λ_1. Moreover, we have

$$\frac{d(\mathcal{G} - k^a - k^b)}{dk^b} = \frac{d(-k^a - k^b)}{dk^b} = \frac{-dk^a}{dk^b} - 1 = -\frac{\mathcal{G} - k^a - k^b}{k^b}, \tag{C.4}$$

where the first equality uses $d\mathcal{G} = 0$, and the last equality uses the first equation in (C.3). Solving the equation (C.4), we obtain $\mathcal{G} - k^a - k^b = \frac{\lambda_2}{k^b}$ for some constant λ_2. Hence, two independent solutions to (C.3) are given by

$$\mathcal{G} = \lambda_1, \quad (\mathcal{G} - k^a - k^b)k^b = \lambda_2, \tag{C.5}$$

where λ_1 and λ_2 are constants. Then, the general solution to the PDE (C.2) is characterized by (61). Hence, the PDE (62) admits the following general solution locally.

$$\mathcal{X}(k^a, k^b) = \rho\Big(\mathcal{G}(k^a, k^b) + rp + c\Big), \quad k^a \le k^b. \tag{C.6}$$

In addition, if $\mathcal{G}(k^a, k^b)$ satisfies (61) for any $k^b \ge k^a \ge \underline{k} > 0$, by taking partial derivatives with respect to k^a and k^b in (61), we obtain

$$\phi_{z_1}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)k^b(\mathcal{G}_{k^a} - 1) + \phi_{z_2}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)\mathcal{G}_{k^a} = 0, \tag{C.7}$$

$$\phi_{z_1}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)(\mathcal{G} - k^a - 2k^b + k^b\mathcal{G}_{k^b}) + \phi_{z_2}\Big((\mathcal{G} - k^a - k^b)k^b, \mathcal{G}\Big)\mathcal{G}_{k^b} = 0. \tag{C.8}$$

Since $\phi_{z_1} \neq 0$ or $\phi_{z_2} \neq 0$, it follows from (C.7) and (C.8) that

$$(\mathcal{G} - k^a - 2k^b + k^b\mathcal{G}_{k^b})\mathcal{G}_{k^a} = k^b\mathcal{G}_{k^b}(\mathcal{G}_{k^a} - 1). \tag{C.9}$$

Simplifying the above equation, we see that (C.2) holds for any $k^b \ge k^a \ge \underline{k}$.

D Asymmetric Equilibria

Consider a closed-loop equilibrium with the following asymmetric trigger function:

$$\mathcal{X}(k^a, k^b) := \begin{cases} \rho(\theta_2^a k^a + \theta_1^a k^b + rp + c), & \text{if } \xi k^a \le k^b, \\ \rho(\theta_1^b k^a + \theta_2^b k^b + rp + c), & \text{if } \xi k^a \ge k^b. \end{cases} \tag{D.1}$$

Let

$$\mathbf{I}_i(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : x \ge \mathcal{X}(k^a, k^b), k^i \le \mathcal{K}_i(x)\} \text{ and } \mathbf{N}_i(x) = \mathbb{R}^2_+ \setminus \mathbf{I}_i(x). \tag{D.2}$$

where

$$\mathcal{K}_a(x) := \inf\{k^a \ge 0 : \mathcal{X}(k^a, \xi k^a) \ge x\}, \quad \mathcal{K}_b(x) = \xi\mathcal{K}_b(x). \tag{D.3}$$

Let $\mathbf{I}_{ab}(x) = \mathbf{I}_a(x) \cap \mathbf{I}_b(x)$ and $\mathbf{N}_{ab}(x) = \mathbf{N}_a(x) \cap \mathbf{N}_b(x)$. These regions satisfy:

$$\mathbf{I}_{ab}(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : k^a \le \mathcal{K}_a(x), k^b \le \mathcal{K}_b(x)\}, \tag{D.4}$$

$$\mathbf{N}_{ab}(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : x < \mathcal{X}(k^a, k^b)\}. \tag{D.5}$$

As in Section 4, the value function in the region $\mathbf{N}_{ab}(x)$ is given by (34) with modified option value H^i, where (35) and (37) hold for $\xi k^a \le k^b$, while (36) and (38) hold for $\xi k^a \ge k^b$. We can derive the boundary solution:

$$H^a(k^a, \xi k^a) = -\int_{k^a}^{\infty} \left(\frac{(2 + 2\xi)w + rp + c}{r} - \frac{\mathcal{X}(w, \xi w)}{r - \mu}\right)\mathcal{X}(w, \xi w)^{-\beta} dw \tag{D.6}$$

by using (37), (38), and $\lim_{k^a\to\infty} H^a(k^a,\xi k^a)=0$. The general solution is:

$$H^a(k^a,k^b)=H^a(k^b/\xi,k^b)-\int_{k^a}^{k^b/\xi}\left(\frac{2w+k^b+rp+c}{r}-\frac{\mathcal{X}(w,k^b)}{r-\mu}\right)\mathcal{X}(w,k^b)^{-\beta}dw,\quad \xi k^a\le k^b \tag{D.7}$$

$$H^a(k^a,k^b)=H^a(k^a,\xi k^a)-\int_{k^b}^{\xi k^a}\frac{k^a}{r}\mathcal{X}(k^a,w)^{-\beta}dw,\quad \xi k^a\ge k^b \tag{D.8}$$

with $H^a(k,\xi k)$ given by (D.6).

Condition A.5 *The parameters $\xi>0$, (θ_1^a,θ_2^a), and (θ_1^b,θ_2^b) satisfy: (i) $\theta_2^i\ge\theta_1^i>0$ and $\theta_1^i\le 1$ for $i=a,b$; (ii) $\theta_1^a\xi+\theta_2^a=\theta_1^b+\theta_2^b\xi$; and (iii)*

$$\underline{g}\Big(\frac{\theta_2^i}{\theta_1^i\xi^i},\xi^i\Big)\le\theta_1^i\xi^i+\theta_2^i\le\overline{g}\Big(\frac{\theta_2^i}{\theta_1^i\xi^i},\xi^i\Big) \tag{D.9}$$

for $i=a,b$, where $\xi^a=\xi$, $\xi^b=1/\xi$, and the upper and lower bound functions, $\overline{g}(\,\cdot,\cdot\,)$ and $\underline{g}(\,\cdot,\cdot\,)$, are given by

$$\overline{g}(w,\zeta)=1+2\zeta-1/w, \tag{D.10}$$

$$\underline{g}(w,\zeta)=\left(1-\frac{2}{\beta}\right)\zeta(1+w)+\frac{2}{\beta}\Big(1+\frac{1}{w}\Big)+\left(2-\frac{2}{\beta}\right)\frac{\zeta w-1}{(1+w)^{\beta-1}-1}\,. \tag{D.11}$$

Next we summarize the solution for closed-loop equilibria associated with the asymmetric linear trigger functions given in (D.1).

Theorem 5 *Under Condition A.5 for an investment boundary $\mathcal{X}(k^a,k^b)$ given in* (D.1)*, a strategy pair $(\boldsymbol{\varphi}_*^a,\boldsymbol{\varphi}_*^b)=\{(\boldsymbol{u}^a,\boldsymbol{v}^a),(\boldsymbol{u}^b,\boldsymbol{v}^b)\}\in\mathcal{S}$ constitutes a closed-loop equilibrium if*

$$\boldsymbol{u}^i(z,u^{-i})=\Big[\mathbf{1}_{\xi^ik^i<k^{-i}}\frac{}{\rho\theta_2^i}+\Big(\frac{1}{\rho\theta_1^{-i}}-\frac{\theta_2^{-i}}{\theta_1^{-i}}u^{-i}\Big)\mathbf{1}_{\xi^ik^i\ge k^{-i}}\Big]\mathbf{1}_{x\ge\mathcal{X}(k^a,k^b)}, \tag{D.12}$$

$$\boldsymbol{v}^i(z,v^{-i})=\inf\Big\{\delta\ge0:\mathcal{X}\big(k^i+\delta,\max\{k^{-i},\min\{k^{-i}+v^{-i},\mathcal{K}_{-i}(x)\}\}\big)\ge x\Big\}, \tag{D.13}$$

for any $z=(x,k^a,k^b)\in\mathbb{Z}$, with $\mathcal{K}_i(x)$ defined in (D.3)*.*

Theorem 6 *The value function $V^i(z)$ under the equilibrium in Theorem 5 mirrors that in Theorem 4 except $V^i(z)=V^i(x,\mathcal{K}_a(x),\mathcal{K}_b(x))-p(\mathcal{K}_i(x)-k^i)$ in region where $k^a\le\mathcal{K}_a(x)$ and $k^b\le\mathcal{K}_b(x)$.*

E Proofs

E.1 Proof of Theorem 1

We only need to verify the conditions stated in Theorem 3. To begin, we establish a supporting lemma that demonstrates Conditions A.1 and A.2 are satisfied.

Lemma 1 *Under Condition 1, the functions $V^a(z)$ and $V^b(z)$ from Theorem 4 with trigger function $\mathcal{X}(k^a, k^b)$ in (43) satisfy Conditions A.1 and A.2. For the special case where $\theta_1 = \theta_2 = 1$, we have*

$$V^i(x, k^a, k^b) = pk^i, \qquad x \geq \mathcal{X}(k^a, k^b), \quad (k^a, k^b) \in \mathbb{R}^2_+. \tag{E.1}$$

Lemma 1 follows from Lemma 4 in Internet Appendix IA.A, with proof relegated to Internet Appendix IA.B.

Next, it remains to verify that $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$, and that Conditions A.3-A.4 as well as Footnote 23 are satisfied for the functions $V^a(z)$ and $V^b(z)$ from Theorem 4, with the trigger function $\mathcal{X}(k^a, k^b)$ as defined in (43). The remainder of the proof is organized into five steps.

Step 1: Derive industry dynamics.

We first consider the region to the northeast of the ACE boundary in Figure 3, where $X_t \leq \mathcal{X}(K^a_t, K^b_t)$. Since $\mathcal{X}(k^a, k^b)$ increases in k^a and k^b, it follows from (49) that $\boldsymbol{v}^i(x, k^a, k^b; v^{-i}) = 0$ for all $v^{-i} \geq 0$ and $x \leq \mathcal{X}(k^a, k^b)$. Thus,

$$\Delta K^i_t = \boldsymbol{v}^i(X_t, K^a_t, K^b_t; \Delta K^{-i}_t) = 0 \quad \text{for all } X_t \leq \mathcal{X}(K^a_t, K^b_t). \tag{E.2}$$

Then, no lumpy investment occurs whenever $X_t \leq \mathcal{X}(K^a_t, K^b_t)$.

Second, we consider the OBCD square region in Figure 3. From (49) and $\mathcal{X}(\mathcal{K}(x), \mathcal{K}(x)) = x$, we have[24]

$$\Delta K^i_t = \mathcal{K}(X_t) - K^i_t > 0 \quad \text{when } (K^a_t, K^b_t) \in [0, \mathcal{K}(X_t)) \times [0, \mathcal{K}(X_t)), \tag{E.3}$$

indicating that both firms invest lumpily to reach $(\mathcal{K}(X_t), \mathcal{K}(X_t))$.

Third, we consider two triangular regions ABC and CDE in Figure 3. From (49) and the

[24]For $k^a < \mathcal{K}(x)$ and $k^b < \mathcal{K}(x)$, (49) implies $k^i + \boldsymbol{v}^i(z; v^{-i}) \geq \mathcal{K}(x)$ if $k^{-i} + v^{-i} < \mathcal{K}(x)$, and $k^i + \boldsymbol{v}^i(z; v^{-i}) = \mathcal{K}(x)$ if $k^{-i} + v^{-i} \geq \mathcal{K}(x)$. Thus, the unique solution pair (v^a, v^b) to the system of equations $v^a = \boldsymbol{v}^a(z, v^b)$ and $v^b = \boldsymbol{v}^b(z, v^a)$ is $(v^a, v^b) = (\mathcal{K}(x) - k^a, \mathcal{K}(x) - k^b)$.

monotonicity of $\mathcal{X}(k^a, k^b)$ in k^a and k^b, we infer that[25]

$$\Delta K_t^i = \inf\{\delta \geq 0 : \mathcal{X}\left(K_t^i + \delta, K_t^{-i}\right) \geq X_t\} > 0 \text{ and } \Delta K_t^{-i} = 0 \tag{E.4}$$

$$\text{when } X_t > \mathcal{X}(K_t^a, K_t^b) \text{ and } K_t^{-i} \geq \mathcal{K}(X_t), \tag{E.5}$$

where we have used $K_t^i < \mathcal{K}(X_t)$. This indicates that firm $-i$ does not make a lumpy investment while firm i does at time t, moving instantaneously to $\mathbf{\Gamma}_i(X_t)$.

In summary, (E.2)-(E.5) imply for any $z = (x, k^a, k^b)$, there exists a unique pair $(v^a, v^b) \in \mathbb{R}_+^2$ satisfying (20), and the jump component of firm i's investment strategy is uniquely determined. Next, we solve the continuous component of firm i's investment strategy.

Equation (48) describes the continuous component of firm i's equilibrium strategy using three indicator functions. In equation (48), the indicator outside of the square-bracket pair states that $\boldsymbol{u}^a(z, u^b) = \boldsymbol{u}^b(z, u^a) = 0$ in the inaction region where $x < \mathcal{X}(k^a, k^b)$.

In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^i < k^{-i}$, (48) implies $\boldsymbol{u}^i(z, u^{-i}) = \frac{1}{\rho\theta_2}$, $\boldsymbol{u}^{-i}(z, u^i) = 0$. That is, only the smaller firm invests:

$$dK_t^i = \frac{dM_t}{\rho\theta_2} \text{ and } dK_t^{-i} = 0 \quad \text{when } K_t^i < K_t^{-i} \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \tag{E.6}$$

Combining the above with (E.4), we conclude that starting from within the ABC and CDE triangular regions, the smaller firm i increases its capital to the boundary $\mathbf{\Gamma}_i(X_t)$, but the larger firm $-i$ chooses to be inactive. Subsequently, the smaller firm i's investment is proportional to dM_t, as described in (E.6), and the larger firm $-i$ does not invest until the capital levels of both firms become equal.

In the action region $x \geq \mathcal{X}(k^a, k^b)$ with $k^a = k^b$, (48) implies $\boldsymbol{u}^i(z, u^{-i}) = \frac{1}{\rho\theta_1} - \frac{\theta_2}{\theta_1}u^{-i}$. Jointly solving $u_t^a = \frac{1}{\rho\theta_1} - \frac{\theta_2}{\theta_1}u_t^b$ and $u_t^b = \frac{1}{\rho\theta_1} - \frac{\theta_2}{\theta_1}u_t^a$, we obtain

$$dK_t^a = dK_t^b = \frac{dM_t}{\rho(\theta_1 + \theta_2)} \quad \text{when } K_t^a = K_t^b \text{ and } \mathcal{X}(K_t^i, K_t^{-i}) = X_t. \tag{E.7}$$

Combining the above with (E.3), we conclude that starting from within the OBCD square region, both firms invest so that $K_t^a = K_t^b = \mathcal{K}_t^M$, reaching point C in Figure 3. Subsequently, the two firms invest at the same rate at all t and thus are always of the same size. In summary, (E.6)-(E.7) imply for any $z = (x, k^a, k^b)$, there exists a unique pair $(u^a, u^b) \in \mathbb{R}_+^2$ satisfying (21), and the continuous component of firm i's investment strategy is uniquely determined.

[25] In the case $x > \mathcal{X}(k^a, k^b)$ and $k^{-i} \geq \mathcal{K}(x)$, we have $\boldsymbol{v}^i(z; v^{-i}) = \inf\{\delta \geq 0 : \mathcal{X}\left(k^i + \delta, k^{-i}\right) \geq x\}$ and $\boldsymbol{v}^{-i}(z; v^i) = \inf\{\delta \geq 0 : \mathcal{X}\left(k^{-i} + \delta, k^i + v^i\right) \geq x\}$ for any $v^i \geq 0$ and $v^{-i} \geq 0$.

Combining (E.2)-(E.7), we can obtain the industry dynamics (51).

Step 2: Verify $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$.

According to the results proved in Step 1, we conclude that the strategy $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b)$ satisfies the conditions in Definition 2-(i). Next, we proceed to verify the conditions in Definition 2-(ii).

By (51), we have $K_t^i \leq k_0^i \vee \mathcal{K}(M_t)$, $t \geq 0$. Since $\mathcal{K}(x)$ in (50) is linear, there exists a constant $C_1 > 0$ such that $K_t^i \in [k_0^i, C_1(1+M_t)]$, $\forall\ t \geq 0$, $i = a, b$. Then, for any finite $T > 0$ and $n > 1$, a sufficient condition for $\mathbb{E}^z\left[|K_T^i|^n\right] < \infty$ is

$$\mathbb{E}^z\left[|M_T|^n\right] < \infty. \tag{E.8}$$

Using $F^i(x, k^a, k^b) = (x - (k^a + k^b) - c)k^i$ with $K_t^i \in [k_0^i, C_1(1+M_t)]$ and $X_t \in [0, M_t]$, $\forall\ t \geq 0$, there exists a constant $C_2 > 0$ such that $|F^i(X_t, K_t^a, K_t^b)| \leq C_2(1 + |M_t|^2)$, $\forall\ t \geq 0$. Thus, a sufficient condition for (22) is

$$\int_0^{+\infty} \mathbb{E}^z[e^{-rt}|M_t|^2]dt < +\infty. \tag{E.9}$$

By (2.16) in Chapter 4 of Friedman (2012), (E.8) holds for any $T > 0$ and $n > 1$. For any initial value $z = (x_0, k_0^a, k_0^b)$ and $X_t = x_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s$, there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
\mathbb{E}^z[M_t^2] &\leq 3x_0^2 + 3\mathbb{E}^z\left[\Big(\int_0^t |\mu| X_s ds\Big)^2\right] + 3\mathbb{E}^z\left[\sup_{\tau\in[0,t]}\Big(\int_0^\tau \sigma X_s dW_s\Big)^2\right] \\
&\leq 3x_0^2 + (3\mu^2 t + C_3\sigma^2)x_0^2 \int_0^t e^{(\sigma^2+2\mu)s} ds,
\end{aligned} \tag{E.10}$$

where the first inequality uses $|x_1 + x_2 + x_3|^2 \leq 3(|x_1|^2 + |x_2|^2 + |x_3|^2)$, and the second follows from the Cauchy inequality and the Burkholder-Davis-Gundy Inequality. Since $\beta > 2$ implies $r > \sigma^2 + 2\mu$, we conclude from (E.10) that (E.9) holds. Therefore, the conditions in Definition 2-(ii) are satisfied, and $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$.

Step 3: Verify Condition A.3.

Using (51), the definition of $V^i(z)$ given in Theorem 4, and symmetry, we confirm that (A.4) holds. By (51), we have $(X_t, K_t^a, K_t^b) \in \overline{\mathbb{N}}_{ab}$ for all $t > 0$, and K_t^i is continuous in $t > 0$ and satisfies (A.5).

Step 4: Verify Condition A.4.

Next, we present the dynamics of capital stocks driven by the off-equilibrium strategy. For ease of exposition, we focus on the deviation of firm a. Denote $(\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ as the strategy

where firm a deviates from time 0, and denote resulting capital processes as:

$$\widetilde{K}^a := K^{a,\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*}, \quad \widetilde{K}^b := K^{b,\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*}. \tag{E.11}$$

Using (49), one can deduce $\Delta\widetilde{K}^b_t$ in the following three cases.

(i) When $\widetilde{K}^a_{t+} < \mathcal{K}(X_t)$, we have

$$\Delta\widetilde{K}^b_t = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\widetilde{K}^a_{t+}, \widetilde{K}^b_t + \delta\right) \geq X_t\}. \tag{E.12}$$

(ii) When $\widetilde{K}^a_{t+} \geq \mathcal{K}(X_t) \geq \widetilde{K}^a_t$, we have

$$\Delta\widetilde{K}^b_t = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\mathcal{K}(X_t), \widetilde{K}^b_t + \delta\right) \geq X_t\} = \left(\mathcal{K}(X_t) - \widetilde{K}^b_t\right)^+. \tag{E.13}$$

(iii) When $\widetilde{K}^a_t > \mathcal{K}(X_t)$, we have

$$\Delta\widetilde{K}^b_t = \inf\{\delta \geq 0 : \ \mathcal{X}\left(\widetilde{K}^a_t, \widetilde{K}^b_t + \delta\right) \geq X_t\}. \tag{E.14}$$

By (E.12)-(E.14), firm b's strategy ensures $X_t \ \leq \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t)$ for almost all $t > 0$. To extend this to all $t > 0$, suppose for contradiction that there exists a $t > 0$ such that $X_t > \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t)$. Define

$$\Delta t := \max\{\delta \in [0, t] : X_s > \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t), \quad \forall s \in [t - \delta, t]\}. \tag{E.15}$$

By continuity of X_s, we have $\Delta t > 0$ almost surely. It follows that for any $s \in (t - \Delta t, t]$,

$$X_s > \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t) \geq \mathcal{X}(\widetilde{K}^a_s, \widetilde{K}^b_s), \tag{E.16}$$

where the second inequality follows from the monotonicity of $\mathcal{X}(k^a, k^b)$ in $k^a \geq 0$ and $k^b \geq 0$ and the nondecreasing paths of $\widetilde{K}^a$ and $\widetilde{K}^b$. This contradicts $X_s \leq \mathcal{X}(\widetilde{K}^a_s, \widetilde{K}^b_s)$ holding almost everywhere.

Thus, $X_t \ \leq \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t)$ for all $t > 0$, i.e., (A.6) holds. By (49), $\Delta\widetilde{K}^b_t = 0$ when $X_t \ \leq \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t)$, so $\widetilde{K}^b_t$ is continuous for all $t > 0$.

Step 5: Verify the regularity conditions in Footnote 23.

By symmetry, we focus on V^a. Consider any deviation strategy $\widetilde{\boldsymbol{\varphi}}^a$ such that $(\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ and recall the notation (E.11). Denote

$$\mathcal{H}(k) := |H^a(k,k)| + k\Big(\frac{3k + pr + c}{r} + \frac{\mathcal{X}(k,k)}{r - \mu}\Big)\mathcal{X}(0,k)^{-\beta},$$
$$\widetilde{\mathcal{H}}(k) := |H^a(k,k)| + \frac{k^2}{r}\mathcal{X}(k,0)^{-\beta},$$

where $H^a(k^a,k^b)$ is given by (39)-(40). We first show that $\lim_{k\to+\infty}\mathcal{H}(k)=0$, $\lim_{k\to+\infty}\widetilde{\mathcal{H}}(k)=0$, and

$$|H^a(k^a,k^b)|\le \mathcal{H}(k^b)\mathbf{1}_{k^a\le k^b}+\widetilde{\mathcal{H}}(k^a)\mathbf{1}_{k^a>k^b},\quad \forall (k^a,k^b)\in\mathbb{R}^2_+. \tag{E.17}$$

Since $\lim_{k\to+\infty}H^a(k,k)=0$ and $\beta>2$ with $\mathcal{X}(k,k)$ and $\mathcal{X}(0,k)$ being positive and linearly increasing in k, we have $\lim_{k\to+\infty}\mathcal{H}(k)=0$. By (40), for $k^b\ge k^a\ge 0$, we have

$$|H^a(k^a,k^b)|=|H^a(k^b,k^b)-\int_{k^a}^{k^b}\Big(\frac{2w+k^b+rp+c}{r}-\frac{\mathcal{X}(w,k^b)}{r-\mu}\Big)\mathcal{X}(w,k^b)^{-\beta}dw|\le\mathcal{H}(k^b),$$

where the inequality follows from the monotonicity of $\mathcal{X}(k^a,k^b)$ in $k^a\in[0,k^b]$. Similarly, using (40), we can obtain

$$|H^a(k^a,k^b)|=|H^a(k^a,k^a)-\int_{k^b}^{k^a}\frac{k^a}{r}\mathcal{X}(k^a,w)^{-\beta}dw|\le\widetilde{\mathcal{H}}(k^a)$$

for $k^a\ge k^b\ge 0$. Thus, we can see (E.17) holds.

Fix any $(k^a,k^b)\in\overline{\mathbf{N}}_{ab}(x)$, i.e., $x\le\mathcal{X}(k^a,k^b)$. If $k^a\le k^b$, then $\mathcal{X}(k^b,k^b)\ge\mathcal{X}(k^a,k^b)\ge x$, which implying $k^b\ge\mathcal{K}(x)$. If $k^a\ge k^b$, then $\mathcal{X}(k^a,k^a)\ge\mathcal{X}(k^a,k^b)\ge x$, implying $k^a\ge\mathcal{K}(x)$. Hence, we have

$$k^b\mathbf{1}_{k^a\le k^b}+k^a\mathbf{1}_{k^a>k^b}\ge\mathcal{K}(x). \tag{E.18}$$

Since $\lim_{k\to\infty}\mathcal{H}(k)=0$, $\lim_{k\to\infty}\widetilde{\mathcal{H}}(k)=0$, and $\lim_{x\to+\infty}\mathcal{K}(x)=\infty$, for any $\delta>0$, there exists $x_\delta>0$ such that $|\mathcal{H}(k^b)|\le\delta$ and $|\widetilde{\mathcal{H}}(k^a)|\le\delta$ for any $k^a\ge\mathcal{K}(x_\delta)$ and $k^b\ge\mathcal{K}(x_\delta)$. Combining (E.17) and (E.18), we see that $k^b\mathbf{1}_{k^a\le k^b}+k^a\mathbf{1}_{k^a>k^b}\ge\mathcal{K}(x_\delta)$ and $|H^a(k^a,k^b)|\le\delta$ when $\mathcal{X}(k^a,k^b)\ge x_\delta$. Denote $\tau_\delta:=\inf\{t\ge0:X_t\ge x_\delta\}$. Then for any $t>0$,

$$|\mathbb{E}^z[e^{-rt}H^a(\widetilde{K}^a_t,\widetilde{K}^b_t)X^\beta_t\mathbf{1}_{t\ge\tau_\delta}]|\le\delta|\mathbb{E}^z[e^{-rt}X^\beta_t]|=\delta x^\beta, \tag{E.19}$$

where the inequality follows from $\mathcal{X}(\widetilde{K}^a_t,\widetilde{K}^b_t)\ge\mathcal{X}(\widetilde{K}^a_{\tau_\delta},\widetilde{K}^b_{\tau_\delta})\ge X_{\tau_\delta}=x_\delta$, $\forall\ t\ge\tau_\delta$ (as stated by (A.6)) and $|H^a(k^a,k^b)|\le\delta$ for $\mathcal{X}(k^a,k^b)\ge x_\delta$, and the equality holds as $e^{-rt}X^\beta_t$ is a martingale. The continuity of $H^a(k^a,k^b)$ in $(k^a,k^b)\in\mathbb{R}^2_+$ suggests that $H^a(k^a,k^b)$ is bounded in the compact set $\{(k^a,k^b)\in\mathbb{R}^2_+:\mathcal{X}(k^a,k^b)\le x_\delta\}$. Owing to $|H^a(k^a,k^b)|\le\delta$ for $\mathcal{X}(k^a,k^b)\ge x_\delta$, $H^a(k^a,k^b)$ is bounded in $(k^a,k^b)\in\mathbb{R}^2_+$. Recalling that $r>0$ and $X_t\ge0$, $\forall t\ge0$, we infer that $\lim_{t\to+\infty}\mathbb{E}^z[e^{-rt}H^a(\widetilde{K}^a_t,\widetilde{K}^b_t)X^\beta_t\mathbf{1}_{t<\tau_\delta}]=0$. Combining this with (E.19), we have $\lim_{t\to+\infty}\mathbb{E}^z[e^{-rt}H^a(\widetilde{K}^a_t,\widetilde{K}^b_t)X^\beta_t]\le\delta x^\beta$. Since $\delta>0$ is arbitrary, we obtain

$$\lim_{t\to\infty}\mathbb{E}^z[e^{-rt}H^a(\widetilde{K}^a_t,\widetilde{K}^b_t)X^\beta_t]=0. \tag{E.20}$$

According to (E.10), $r > 0$ and $r > \sigma^2 + 2\mu$, we have

$$\lim_{t\to\infty} \mathbb{E}^z[e^{-rt} M_t^2] = 0. \tag{E.21}$$

For $(\widetilde{\boldsymbol{\varphi}}^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$, we derive from (22) and Tonelli's Theorem that

$$\int_0^\infty \mathbb{E}^z[e^{-rt}|F^i(X_t, \widetilde{K}_t^a, \widetilde{K}_t^b)|]dt < \infty.$$

It follows that there exists a sequence $\{t_n\}$ of real numbers such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} F^i(X_{t_n}, \widetilde{K}_{t_n}^a, \widetilde{K}_{t_n}^b)] = 0. \tag{E.22}$$

Since $F^i(x, k^a, k^b) = k^a(x - \frac{1}{4}k^a) - k^a(\frac{3}{4}k^a + k^b + c) \le k^a(x - \frac{1}{4}k^a)$ for any $(x, k^a, k^b) \in \mathbb{R}_+^3$, we derive from (E.22) that

$$\liminf_{n\to\infty} \mathbb{E}^z\Big[e^{-rt_n}\widetilde{K}_{t_n}^a\Big(X_{t_n} - \frac{1}{4}\widetilde{K}_{t_n}^a\Big)\Big] \ge 0. \tag{E.23}$$

In addition, we have

$$\begin{aligned} 0 \ge &\limsup_{n\to\infty} \mathbb{E}^z\left[e^{-rt_n}\big(-X_{t_n}^2 + \widetilde{K}_{t_n}^a(X_{t_n} - \frac{1}{4}\widetilde{K}_{t_n}^a)\big)\right] \\ = &\limsup_{n\to\infty} \mathbb{E}^z\left[e^{-rt_n}\widetilde{K}_{t_n}^a\big(X_{t_n} - \frac{1}{4}\widetilde{K}_{t_n}^a\big)\right], \end{aligned} \tag{E.24}$$

where the first inequality follows from $-X_{t_n}^2 + \widetilde{K}_{t_n}^a(X_{t_n} - \frac{1}{4}\widetilde{K}_{t_n}^a) = -[X_{t_n} - \frac{1}{2}\widetilde{K}_{t_n}^a]^2$, and the equality uses (E.21). Hence, we derive from (E.23) and (E.24) that

$$\lim_{n\to\infty} \mathbb{E}^z\left[e^{-rt_n}\widetilde{K}_{t_n}^a\big(X_{t_n} - \frac{1}{4}\widetilde{K}_{t_n}^a\big)\right] = 0. \tag{E.25}$$

Combining (E.25), (E.22), and $F^i(x, k^a, k^b) = k^a(x - \frac{1}{4}k^a) - k^a(\frac{3}{4}k^a + k^b + c)$, we conclude that $\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n}\widetilde{K}_{t_n}^a(\frac{3}{4}\widetilde{K}_{t_n}^a + \widetilde{K}_{t_n}^b + c)] = 0$. Noticing $\widetilde{K}^a \ge 0$, $\widetilde{K}^b \ge 0$, and $c \ge 0$, we have

$$\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n}(\widetilde{K}_{t_n}^a)^2] = 0, \tag{E.26}$$

$$\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n}\widetilde{K}_{t_n}^a\widetilde{K}_{t_n}^b] = 0. \tag{E.27}$$

Combining (E.26) with (E.21) and using the Cauchy inequality, we obtain

$$\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n}\widetilde{K}_{t_n}^a X_{t_n}] = 0. \tag{E.28}$$

Since we have shown $X_t \le \mathcal{X}(\widetilde{K}_t^a, \widetilde{K}_t^b)$ for any $t > 0$, it follows from (E.20) and (E.26)-(E.28) that $\lim_{n\to\infty} \mathbb{E}^z[e^{-rt_n} V^a(X_{t_n}, \widetilde{K}_{t_n}^a, \widetilde{K}_{t_n}^b)] = 0$, where $V^a(X_t, \widetilde{K}_t^a, \widetilde{K}_t^b)$ is given by (34).

Using (34), we have $V_x^a(x, k^a, k^b) = \frac{k^a}{r-\mu} + \beta H^a(k^a, k^b)x^{\beta-1}$ for any $(k^a, k^b) \in \overline{\mathbf{N}}_{ab}(x)$. Thus, there exists a constant $C_1 > 0$ such that for any $(k^a, k^b) \in \overline{\mathbf{N}}_{ab}(x)$,

$$|V_x^a(x, k^a, k^b)x|^2 \le 2\Big(\frac{k^a x}{r-\mu}\Big)^2 + 2\beta^2|H^a(k^a, k^b)|^2 x^{2\beta}$$

$$\leq C_1((k^a)^4 + x^4 + x^{2\beta}) \leq C_1((k^a)^4 + 1 + 2x^{2\beta}), \tag{E.29}$$

where the second inequality follows from the boundedness of $H^a(k^a, k^b)$ in $\mathbb{R}^2_+$, and the last inequality uses $\beta > 2$ and $x^4 < 1 + x^{2\beta}$. Since $\widetilde{K}^a_t \geq 0$ is nondecreasing in t, and $\mathbb{E}^z[|\widetilde{K}^a_t|^n] < +\infty$ for any $t > 0$, $n > 1$ by Definition 2-(i), we have $\mathbb{E}^z[\int_0^t |\widetilde{K}^a_s|^4 ds] \leq t\mathbb{E}^z[|\widetilde{K}^a_t|^4] < +\infty$, $\forall t > 0$. Additionally, direct computation yields $\mathbb{E}^z[X^{2\beta}_t] = e^{\beta t(2\mu+(2\beta-1)\sigma^2)}$, implying $\mathbb{E}^z[\int_0^t X^{2\beta}_s ds] < \infty$, $\forall t > 0$. We then conclude from $X_t \leq \mathcal{X}(\widetilde{K}^a_t, \widetilde{K}^b_t)$, $t > 0$, and (E.29) that $\mathbb{E}^z\left[\int_0^t |e^{-rs} V^a_x(X_s, \widetilde{K}^a_s, \widetilde{K}^b_s)\sigma(X_s)|^2 ds\right] < +\infty, \ \forall\, t \geq 0$. $\square$

E.2 Proof of Theorem 2

We only need to verify the conditions stated in Theorem 3 with $\mathbb{Z} = (0, \infty) \times [\underline{k}, \infty) \times [\underline{k}, \infty)$. By applying the same arguments as in the proof of Theorem 1, we can verify that Conditions A.2-A.4 and the regularity condition in Footnote 23 are satisfied for functions $V^a(z)$ and $V^b(z)$ given in Theorem 4, with the trigger function $\mathcal{X}(k^a, k^b)$ as defined in (60). Therefore, it remains to show that Condition A.1 is satisfied.

For convenience, we denote

$$\mathbf{I}_i(x) = \{(k^a, k^b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : x \geq \mathcal{X}(k^a, k^b), k^i \leq \mathcal{K}(x)\}, \tag{E.30}$$

$$\mathbf{N}_i(x) = \{(k^a, k^b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : (k^a, k^b) \notin \mathbf{I}_i(x)\}, \tag{E.31}$$

$\mathbf{I}_{ab}(x) = \mathbf{I}_a(x) \cap \mathbf{I}_b(x)$, and $\mathbf{N}_{ab}(x) = \mathbf{N}_a(x) \cap \mathbf{N}_b(x)$.

We prove the case for $V^a(z)$ only, as the argument for $V^b(z)$ is analogous. It is straightforward to verify that $V^a \in \mathcal{C}^{0,1,1}(\mathbb{Z})$, and $V^a_{k^a}(z) = p$ in the $\mathbf{I}_a(x)$ region, $V^a_{k^b}(z) = 0$ in the $\mathbf{I}_b(x)$ region. Hence, it remains to check that the two inequalities in Condition A.1.

We complete our proof in three steps.

Step 1: We prove that

$$\frac{\partial V^i(x, k^a, k^b)}{\partial k^i} = p, \qquad x \geq \mathcal{X}(k^a, k^b), \ k^a \geq \underline{k}, \ k^b \geq \underline{k}. \tag{E.32}$$

In Appendix C, we have shown that $\mathcal{X}(k^a, k^b)$ as given by (C.6) satisfies (62) for any $k^a \geq k^b \geq \underline{k} > 0$. By the symmetry of $\mathcal{X}(k^a, k^b)$, we derive from (62) that

$$\rho k^a \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} + \Big[\mathcal{X}(k^a, k^b) - \rho\big(2k^a + k^b + rp + c\big)\Big] \frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b} = 0, \quad k^a \geq k^b. \tag{E.33}$$

Denote $\mathcal{Y}(k^a, k^b) = \left(\frac{2k^a + k^b + rp + c}{r} - \frac{\mathcal{X}(k^a, k^b)}{r - \mu}\right) \mathcal{X}(k^a, k^b)^{-\beta}$. A direct calculation shows that

(E.33) is equivalent to the following equation:

$$\frac{\partial \mathcal{Y}(k^a,k^b)}{\partial k^b}=\frac{\partial\big[\frac{k^a}{r}\mathcal{X}(k^a,k^b)^{-\beta}\big]}{\partial k^a},\quad k^a>k^b\geq \underline{k}. \tag{E.34}$$

For $k^a\geq k^b$, substituting (38) into (E.34) yields $\frac{\partial \mathcal{Y}(k^a,k^b)}{\partial k^b}=\frac{\partial}{\partial k^a}[\frac{\partial H^a(k^a,k^b)}{\partial k^b}]=\frac{\partial}{\partial k^b}[\frac{\partial H^a(k^a,k^b)}{\partial k^a}]$. It follows that

$$\frac{\partial H^a(k^a,k^b)}{\partial k^a}=\mathcal{Y}(k^a,k^b)+C(k^a),\quad k^a\geq k^b\geq \underline{k}, \tag{E.35}$$

for some function $C(k^a)$.

Since (37) holds for $k^a\leq k^b$, we have $\frac{\partial H^a(k^a,k^b)}{\partial k^a}=\mathcal{Y}(k^a,k^b)$ for $k^a\leq k^b$, which, together with (E.35) and the continuity of $\frac{\partial H^a(k^a,k^b)}{\partial k^a}$ and $\mathcal{Y}(k^a,k^b)$, implies that $C(k^a)=0$. Hence, (37) also holds for $k^a>k^b\geq \underline{k}$. Then, it follows from (34) that

$$\frac{\partial V^a(x,k^a,k^b)}{\partial k^a}\mid_{x=\mathcal{X}(k^a,k^b)}=p\ \text{ for } k^a>k^b\geq \underline{k}. \tag{E.36}$$

Therefore, $\frac{\partial V^a(x,k^a,k^b)}{\partial k^a}=\frac{\partial V^a(x,k^a,\widehat{k}^b)}{\partial k^a}=p$ for any $(k^a,k^b)\in \mathbf{I}_b(x)\setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b$ is uniquely determined by $\mathcal{X}(k^a,\widehat{k}^b)=x$. Since $\frac{\partial V^a(x,k^a,k^b)}{\partial k^a}=p$ for any $(k^a,k^b)\in \mathbf{I}_a(x)$, we have proved (E.32).

Step 2: We prove that

$$V^a_{k^a}(x,k^a,k^b)<p,\quad (k^a,k^b)\in \mathbf{N}_{ab}(x),\ x>0. \tag{E.37}$$

For $k^b\geq k^a\geq \underline{k}$, we have

$$\mathcal{X}(k^a,k^b)-\rho(2k^a+k^b+pr+c)=\rho\Big(\mathcal{G}(k^a,k^b)-2k^a-k^b\Big)<0, \tag{E.38}$$

where the inequality follows from the second inequality in part (i) of Condition 2. Using (E.38) and the symmetry of $\mathcal{X}(k^a,k^b)$, for $k^a\geq k^b\geq \underline{k}$, we have

$$\mathcal{X}(k^a,k^b)<\rho(2k^b+k^a+pr+c)\leq \rho(2k^a+k^b+pr+c), \tag{E.39}$$

the second inequality uses $k^a\geq k^b$. Combining (E.38) and (E.39), we infer

$$\mathcal{X}(k^a,k^b)\leq \rho(2k^a+k^b+pr+c) \tag{E.40}$$

for any $k^a\geq \underline{k}$, $k^b\geq \underline{k}$.

By employing similar arguments as in Step 4 of the proof of Lemma 4, we can show that (E.37) holds by using (E.40) and $V^a_{k^a}(x,k^a,k^b)\mid_{x=\mathcal{X}(k^a,k^b)}=p$.

Step 3: We prove that

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)<0,\quad k^b>k^a\geq \underline{k}. \tag{E.41}$$

Using (34), we have $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = -\frac{k^a}{r} + H^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta$. Thus (E.41) is equivalent to $H^a_{k^b}(k^a,k^b) < \mathcal{H}(k^a,k^b)$, where $\mathcal{H}(k^a,k^b) := \frac{k^a}{r}\mathcal{X}(k^a,k^b)^{-\beta}$. Additionally, by using (38), we have $V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = 0$ for $k^a \geq k^b \geq \underline{k}$ and thus $H^a_{k^b}(k,k) = \mathcal{H}(k,k)$. As a result, (E.41) is equivalent to

$$\int_{k^a}^{k^b} \frac{\partial H^a_{k^b}(k,k^b)}{\partial k^a} dk > \int_{k^a}^{k^b} \frac{\partial \mathcal{H}(k,k^b)}{\partial k^a} dk, \quad k^b > k^a \geq \underline{k}. \tag{E.42}$$

Then, we only need to show that $H^a_{k^ak^b}(k^a,k^b) > \frac{\partial \mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b > k^a \geq \underline{k}$. Using (37), we have

$$\begin{aligned} &H^a_{k^ak^b}(k^a,k^b) - \frac{\partial \mathcal{H}(k^a,k^b)}{\partial k^a} \\ =&\frac{(\beta-1)\Big[\mathcal{X}_{k^b}(k^a,k^b)\big(\mathcal{X}(k^a,k^b) - \rho(2k^a+k^b+pr+c)\big) + \rho\mathcal{X}_{k^a}(k^a,k^b)k^a\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}}. \end{aligned} \tag{E.43}$$

According to part (ii) of Condition 2, we have $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} \geq 0$ for any $k^b \geq k^a \geq \underline{k}$. By combining (62) and (60), we find that $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} / \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} = \frac{k^a+2k^b-\mathcal{G}(k^a,k^b)}{k^b}$ for $k^b \geq k^a \geq \underline{k}$. We infer from part (i) of Condition 2 that $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} / \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} \in (0,1)$ for any $k^b \geq k^a \geq \underline{k}$, which implies that $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} > \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} > 0$ for any $k^b \geq k^a \geq \underline{k}$. By the symmetry of $\mathcal{X}(k^a,k^b)$, we conclude $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} > 0$, $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} > 0$ and

$$\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)} \in (0,1), \quad \forall k^a \geq \underline{k},\ k^b \geq \underline{k}. \tag{E.44}$$

For $k^b > k^a$, we have

$$\begin{aligned} &\mathcal{X}(k^a,k^b) - \rho(2k^a+k^b+pr+c) + \rho\frac{\mathcal{X}_{k^a}(k^a,k^b)}{\mathcal{X}_{k^b}(k^a,k^b)}k^a \\ =&\rho\left([2-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}]k^b + k^a\right) - \rho\left([2-\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}]k^a + k^b\right) \\ =&\rho\left([1-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}]k^b - [1-\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}]k^a\right) > 0, \end{aligned} \tag{E.45}$$

where the first equality uses $\mathcal{X}(k^a,k^b) = \rho\left([2-\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}]k^b + k^a + rp + c\right)$ for $k^b > k^a$, and the inequality uses (E.44).

Substituting (E.45) into (E.43) gives $H^a_{k^ak^b}(k^a,k^b) > \frac{\partial \mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b > k^a \geq \underline{k}$. Hence, (E.41) holds.

Finally, by applying the same arguments as in Step 2 of the proof of Lemma 4 in Internet Appendix IA.B, we can prove that $V^a_{k^b}(x,k^a,k^b) < 0$ for any $(k^a,k^b) \in \mathbf{N}_b(x)$, and $x > 0$. $\square$

E.3 Proof of Theorem 3

We employ the following abbreviated notation: $V^i_{k^j}(x,k^a,k^b) := \frac{\partial V^i(x,k^a,k^b)}{\partial k^j}$, where $j=i$ or $-i$. Recall the set $\mathbb{N}_{ab}$ as defined in (A.3), and let $\overline{\mathbb{N}}_{ab}$ denote its closure. Since Condition A.1 implies that $V^i_{k^a}(x,k^a,k^b)$ and $V^i_{k^b}(x,k^a,k^b)$ are continuous, the set $\mathbb{N}_{ab}$ is an open subset of $\mathbb{Z}$. In the following, we fix an initial state $z=(x,k^a,k^b)\in\mathbb{Z}$.

Step 1: We prove $V^a(x,k^a,k^b)\geq J^a(x,k^a,k^b;\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*)$, where $(\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*)\in\mathcal{S}$.

To simplify notation, we let $\widetilde{K}^a$ and $\widetilde{K}^b$ denote the firms' capital stocks under the off-equilibrium deviation $(\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*)$. By Condition A.4, it follows that $(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)\in\overline{\mathbb{N}}_{ab}$ for any $s>0$. As Condition A.2 implies $V^a\in\mathcal{C}^{2,1,1}(\overline{\mathbb{N}}_{ab})$, applying Itô formula to $e^{-rs}V^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)$, $s\in(0,t]$, for $t>0$ and taking expectation, we have

$$\begin{aligned}V^a(x,\widetilde{K}^a_{0+},\widetilde{K}^b_{0+}) =&\mathbb{E}^z\Big[e^{-rt}V^a(X_t,\widetilde{K}^a_t,\widetilde{K}^b_t)-\int_0^t e^{-rs}\mathcal{L}V^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)ds\\&-\int_0^t e^{-rs}[V^a_{k^a}(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)d\widetilde{K}^{a,C}_s+V^a_{k^b}(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)d\widetilde{K}^{b,C}_s]\\&-\sum_{s\in(0,t]}e^{-rs}[V^a(X_s,\widetilde{K}^a_{s+},\widetilde{K}^b_{s+})-V^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)]\Big],\end{aligned}\tag{E.46}$$

where $\mathcal{L}V^i$ is the infinitesimal generator given by

$$\mathcal{L}V^i(z):=\frac{1}{2}\sigma(x)^2\frac{\partial^2V^i(z)}{\partial x^2}+\mu(x)\frac{\partial V^i(z)}{\partial x}-rV^i(z),\tag{E.47}$$

$\widetilde{K}^{i,C}$ denotes the continuous part of $\widetilde{K}^i$, and the expectation of the local martingale term vanishes due to the regularity condition $\mathbb{E}^z\left[\int_0^t|e^{-rs}V^a_x(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)\sigma(X_s)|^2ds\right]<+\infty,\ \forall\, t\geq 0$, as stated in Footnote 23. According to Condition A.2 and (A.6), we have

$$-\mathcal{L}V^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)=F^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)\ \text{ for any } s>0.$$

Substituting this into (E.46) and using $V^a_{k^a}\leq p$, $V^a_{k^b}\leq 0$ (Condition A.1), we obtain

$$\begin{aligned}&V^a(x,\widetilde{K}^a_{0+},\widetilde{K}^b_{0+})\\\geq&\mathbb{E}^z\Big[e^{-rt}V^a(X_t,\widetilde{K}^a_t,\widetilde{K}^b_t)+\int_0^t e^{-rs}F^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)ds-\int_{0+}^t e^{-rs}pd\widetilde{K}^a_s\Big].\end{aligned}\tag{E.48}$$

By applying the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n},\widetilde{K}^a_{t_n},\widetilde{K}^b_{t_n})]=0$ from Footnote 23, substituting t with t_n in (E.48), and letting n approach infinity, we derive the following result:

$$\begin{aligned}&V^a(x,\widetilde{K}^a_{0+},\widetilde{K}^b_{0+})\\\geq&\lim_{n\to+\infty}\mathbb{E}^z\Big[\int_0^{t_n}e^{-rs}F^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)ds-\int_{0+}^{t_n}e^{-rs}pd\widetilde{K}^a_s\Big]\end{aligned}$$

$$=\mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)ds-\int_{0+}^{+\infty} e^{-rs}pd\widetilde{K}^a_s\Big], \tag{E.49}$$

where the convergence of the first integral is due to the dominated convergence theorem and (22), and the second integral is well-defined by the monotone convergence theorem. Using $V^a_{k^a}\leq p$, $V^a_{k^b}\leq 0$ (Condition A.1), $\widetilde{K}^a_{0+}\geq k^a$, and $\widetilde{K}^b_{0+}\geq k^b$, we derive

$$V^a(x,k^a,k^b)\geq V^a(x,k^a,\widetilde{K}^b_{0+})\geq V^a(x,\widetilde{K}^a_{0+},\widetilde{K}^b_{0+})-p(\widetilde{K}^a_{0+}-k^a). \tag{E.50}$$

Combining (E.50) with (E.49), we obtain

$$V^a(x,k^a,k^b)\geq \mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F^a(X_s,\widetilde{K}^a_s,\widetilde{K}^b_s)ds-\int_0^{+\infty} e^{-rs}pd\widetilde{K}^a_s\Big]=J^a(x,k^a,k^b;\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}^b_*).$$

Step 2: We prove $V^a(x,k^a,k^b)=J^a(x,k^a,k^b;\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)$.

Recall that (K^a,K^b) denote the firms' capital stocks under the strategy $(\boldsymbol{\varphi}^a_*,\boldsymbol{\varphi}^b_*)$. By replacing $\widetilde{K}^a$ and $\widetilde{K}^b$ in (E.46) with K^a and K^b, respectively, and noting that K^a_s and K^b_s are continuous in $s>0$ (as stated in Condition A.3), we obtain

$$\begin{aligned}V^a(x,K^a_{0+},K^b_{0+})=\mathbb{E}^z\Big[&e^{-rt}V^a(X_t,K^a_t,K^b_t)-\int_0^t e^{-rs}\mathcal{L}V^a(X_s,K^a_s,K^b_s)ds\\&-\int_{0+}^t e^{-rs}[V^a_{k^a}(X_s,K^a_s,K^b_s)dK^a_s+V^a_{k^b}(X_s,K^a_s,K^b_s)dK^b_s]\Big].\end{aligned} \tag{E.51}$$

According to (A.1), we have $V^a_{k^a}(z)=p$ for any $z\in\mathbb{I}_a$, and $V^a_{k^b}(z)=0$ for any $z\in\mathbb{I}_b$. Therefore,

$$\begin{aligned}&\int_{0+}^t e^{-rs}[V^a_{k^a}(X_s,K^a_s,K^b_s)dK^a_s+V^a_{k^b}(X_s,K^a_s,K^b_s)dK^b_s]\\=&\int_{0+}^t e^{-rs}[V^a_{k^a}(X_s,K^a_s,K^b_s)\mathbf{1}_{(X_s,K^a_s,K^b_s)\in\mathbb{I}_a}dK^a_s+V^a_{k^b}(X_s,K^a_s,K^b_s)\mathbf{1}_{(X_s,K^a_s,K^b_s)\in\mathbb{I}_b}dK^b_s]\\=&\int_{0+}^t e^{-rs}pdK^a_s,\end{aligned} \tag{E.52}$$

where the first equality uses (A.5). Substituting $-\mathcal{L}V^a(X_s,K^a_s,K^b_s)=F^a(X_s,K^a_s,K^b_s)$, $s>0$ and (E.52) into (E.51), we obtain

$$\begin{aligned}&V^a(x,K^a_{0+},K^b_{0+})\\=&\mathbb{E}^z\Big[e^{-rt}V^a(X_t,K^a_t,K^b_t)+\int_0^t e^{-rs}F^a(X_s,K^a_s,K^b_s)ds-\int_{0+}^t e^{-rs}pdK^a_s\Big].\end{aligned} \tag{E.53}$$

Using the condition $\lim_{n\to+\infty}\mathbb{E}^z[e^{-rt_n}V^a(X_{t_n},K^a_{t_n},K^b_{t_n})]=0$ from Footnote 23, replacing t with t_n in (E.53), and letting n approach infinity, we derive the following result:

$$V^a(x,K^a_{0+},K^b_{0+})=\mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F^a(X_s,K^a_s,K^b_s)ds-\int_{0+}^{+\infty} e^{-rs}pdK^a_s\Big], \tag{E.54}$$

where the convergence of the first integral follows from the dominated convergence theorem. Finally, plugging (A.4) into (E.54), we have

$$\begin{aligned} V^a(x,k^a,k^b) =& \mathbb{E}^z\Big[\int_0^{+\infty} e^{-rs}F^a(X_s,K_s^a,K_s^b)ds - \int_0^{+\infty} e^{-rs}pdK_s^a\Big] \\ =& J^a(x,k^a,k^b;\boldsymbol{\varphi}_*^a,\boldsymbol{\varphi}_*^b). \end{aligned}$$

In summary, combining the results in Steps 1 and 2, we obtain $V^a(z) = J^a(z;\boldsymbol{\varphi}_*^a,\boldsymbol{\varphi}_*^b) \geq J^a(z;\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}_*^b)$ for any $z = (x,k^a,k^b) \in \mathbb{Z}$ and $(\widetilde{\boldsymbol{\varphi}}^a,\boldsymbol{\varphi}_*^b) \in \mathcal{S}$. By symmetry, we also have $V^b(z) = J^b(z;\boldsymbol{\varphi}_*^a,\boldsymbol{\varphi}_*^b) \geq J^b(z;\boldsymbol{\varphi}_*^a,\widetilde{\boldsymbol{\varphi}}^b)$ for any $z = (x,k^a,k^b) \in \mathbb{Z}$ and $(\boldsymbol{\varphi}_*^a,\widetilde{\boldsymbol{\varphi}}^b) \in \mathcal{S}$. □

E.4 Proof of Proposition 5

We only need to verify that Condition 2 holds. First, consider the case $\phi(z_1,z_2) = z_1 - \lambda$ for a constant $\lambda > 0$. In this case, $\mathcal{G} = \frac{\lambda}{k^b} + k^a + k^b$ solves the equation (61). For any $\underline{k} > \sqrt{\lambda}$, we have $\lambda < k^a k^b$ for all $k^a \geq \underline{k}$ and $k^b \geq \underline{k}$. Hence, part (i) holds, and part (ii) is immediate.

Next, consider the case $\phi(z_1,z_2) = \lambda z_1 - z_2$ for a constant $\lambda > 0$. In this case, $\mathcal{G}(k^a,k^b) = (k^a+k^b)\frac{\lambda k^b}{\lambda k^b - 1}$ solves equation (61). For any $\underline{k} > \frac{3}{\lambda}$, we have $\lambda k^b - 1 > 0$ for all $k^b \geq \underline{k}$. Then, for any $k^b \geq k^a \geq \underline{k}$, we have $\mathcal{G}(k^a,k^b) - k^a - k^b = \frac{k^a+k^b}{\lambda k^b - 1} > 0$ and

$$\frac{\mathcal{G}(k^a,k^b) - k^a - k^b}{k^a} = \frac{k^a+k^b}{(\lambda k^b - 1)k^a} \leq \frac{\underline{k}+k^b}{(\lambda k^b - 1)\underline{k}} \leq \frac{2\underline{k}}{(\lambda \underline{k} - 1)\underline{k}} = \frac{2}{\lambda \underline{k} - 1} < 1, \tag{E.55}$$

where the first inequality follows from the fact that $\frac{k^a+k^b}{(\lambda k^b-1)k^a}$ is decreasing in $k^a \in [\underline{k},k^b]$, the second inequality follows from the fact that $\frac{\underline{k}+k^b}{(\lambda k^b-1)\underline{k}}$ is decreasing in $k^b \geq \underline{k}$, and the last inequality uses $\underline{k} > \frac{3}{\lambda}$. Hence, part (i) holds, and part (ii) is immediate. Therefore, Condition 2 holds.

□

E.5 Other Technical Results

We present two corollaries that are used in the paper, and relegate their proofs to Internet Appendix IA.B.

Corollary 1 *Let $\mathcal{X}(k^a,k^b)$ be as defined in* (43)*, where $\theta_2 \geq \theta_1 > 0$. Let $V^a(z)$ and $V^b(z)$ be the functions given in Theorem 4. Then the right inequality in* (44) *is a necessary condition for* (41)*, and the left inequality in* (44) *is a necessary condition for* (42)*.*

Corollary 2 *For the equilibrium strategies in Theorem 1 and the special case $k_0^a = k_0^b = 0$, firm value at time 0 is given as follows.*

(i) In the inaction region where $x < \underline{x} = \rho(rp + c)$, for any $(\theta_1, \theta_2) \in \Theta$,

$$V^i(x, 0, 0) = \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2}\right)^2\right] \frac{\underline{x}^2}{\rho\beta(\beta - 2)(r - \mu)} \left(\frac{x}{\underline{x}}\right)^\beta. \tag{E.56}$$

(ii) In the investment region where $x \geq \underline{x} = \rho(rp + c)$, for any $(\theta_1, \theta_2) \in \Theta$,

$$V^i(x, 0, 0) = \left[\frac{2}{\theta_1 + \theta_2} - \left(\frac{2}{\theta_1 + \theta_2}\right)^2\right] \left[\frac{\beta}{\beta - 2} \frac{(x - \underline{x})^2}{2r\rho^2} + \frac{(rp + c)(x - \underline{x}) + \frac{\underline{x}^2}{\rho\beta}}{(\beta - 2)(r - \mu)}\right]. \tag{E.57}$$

F Constant Elasticity Model

In this appendix, we extend our results to the constant elasticity model, where firm i's profit is given by

$$F^i(x, k^a, k^b) = x(k^a + k^b)^{-1/\gamma} k^i \tag{F.1}$$

with $\gamma > 1$ being a constant. When neither firm ever invests, i.e., when $dK_t^a = dK_t^b \equiv 0$ for all t, firm i's value equals

$$\Psi^i(z) = \mathbb{E}^z \left[\int_0^\infty e^{-rt} F^i(X_t, k^a, k^b) dt\right] = x \frac{(k^a + k^b)^{-1/\gamma} k^i}{r - \mu} \tag{F.2}$$

for $z = (x, k^a, k^b)$. We denote $\psi^i(k^a, k^b) = \Psi^i(x, k^a, k^b)/x = \frac{(k^a + k^b)^{-1/\gamma} k^i}{r - \mu}$.

F.1 Monopoly Solution

First, we summarize the solution for the monopoly case in the constant elasticity model. The monopolist's profit flow function is given by $F(x, k) = xk^{1-1/\gamma}$, where k is the monopolist's capital stock.

The optimal strategy for the monopolist is to invest only when $X_t \geq \mathcal{X}^{\text{mon}}(K_t)$, where K_t is the monopolist's capital process. Here, $\mathcal{X}^{\text{mon}}(\cdot)$ is the trigger function for the monopolist and is given by

$$\mathcal{X}^{\text{mon}}(k) = \frac{\rho\gamma}{\gamma - 1} k^{\frac{1}{\gamma}}, \tag{F.3}$$

where ρ is a constant defined as

$$\rho = \frac{p\beta}{\beta - 1}(r - \mu), \tag{F.4}$$

with β being defined in (8). For convergence, we always require $\beta > \gamma$ and $r > \mu$ in the constant elasticity model.

F.2 Duopoly Solution

Next, we examine the symmetric closed-loop equilibria in a duopoly. For illustration, we focus on nonlinear trigger functions and follow the analysis in Section 7.

Given a symmetric trigger function $\mathcal{X}(k^a, k^b)$ under certain regularity conditions,[26] we define firm i's investment region $\mathbf{I}_i(x)$ and no-action regions $\mathbf{N}_i(x)$ as follows:

$$\mathbf{I}_i(x) = \{(k^a, k^b) \in \mathbb{R}^2_+ : x \geq \mathcal{X}(k^a, k^b), k^i \leq \mathcal{K}(x)\} \quad \text{and} \quad \mathbf{N}_i(x) = \mathbb{R}^2_+ \setminus \mathbf{I}_i(x). \tag{F.5}$$

We can show that the corresponding value function for $x \leq \mathcal{X}(k^a, k^b)$ is given by (34), where $\Psi^i(z)$ is defined in (F.2). By substituting (34) into (35) and (36), we obtain

$$\frac{\partial H^a(k^a, k^b)}{\partial k^a} = \Big(p - \mathcal{X}(k^a, k^b)\psi^a_{k^a}(k^a, k^b)\Big)\mathcal{X}(k^a, k^b)^{-\beta}, \quad k^a \leq k^b\,, \tag{F.6}$$

$$\frac{\partial H^a(k^a, k^b)}{\partial k^b} = -\psi^a_{k^b}(k^a, k^b)\mathcal{X}(k^a, k^b)^{1-\beta}, \quad k^a \geq k^b\,, \tag{F.7}$$

where $\psi^a_{k^a}(k^a, k^b) = (k^a + k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a + k^b}{r-\mu}$ and $\psi^a_{k^b}(k^a, k^b) = (k^a + k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$.

With the constraint (59), we can endogenously determine the following nonlinear trigger functions (see Appendix F.3):

$$\mathcal{X}(k^a, k^b) := \begin{cases} \Big(\rho + \mathcal{G}(k^a, k^b)\Big)(k^a + k^b)^{\frac{1}{\gamma}}, & \text{if } k^a \leq k^b, \\ \Big(\rho + \mathcal{G}(k^b, k^a)\Big)(k^a + k^b)^{\frac{1}{\gamma}}, & \text{if } k^a \geq k^b, \end{cases} \tag{F.8}$$

where $\mathcal{G}(k^a, k^b)$ is to be determined. Next we state conditions for $\mathcal{G}(k^a, k^b)$ under which there exist closed-loop equilibria satisfying (59) as well as (41) and (42). Given symmetry along the 45^o degree line, it is sufficient for us to characterize $\mathcal{G}(k^a, k^b)$ for the $k^b \geq k^a \geq 0$ region in the first quadrant.

Condition A.6 *There exists a constant $\underline{k} > 0$ such that in $k^b \geq k^a \geq \underline{k}$ region,*

(i) $0 < \mathcal{G}(k^a, k^b) < \frac{\rho k^a}{\gamma(k^a + k^b) - k^a}$;

[26] We only consider the trigger function $\mathcal{X}(k^a, k^b)$ that is continuously differentiable on $\{(k^a, k^b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : k^a \neq k^b\}$ with $\mathcal{X}_{k^a}(k^a, k^b) > 0$ and $\mathcal{X}_{k^b}(k^a, k^b) > 0$, where $\underline{k} > 0$ is a constant. Without loss of generality, we assume that $\mathcal{X}(k^a, k^b)$ has a upper bound given by the monopolist's trigger $\rho\frac{\gamma}{\gamma-1}(k^a + k^b)^{\frac{1}{\gamma}}$ and a lower bound given by the trigger of perfect competition equilibrium $\rho(k^a + k^b)^{\frac{1}{\gamma}}$ (Back and Paulsen, 2009). These bounds guarantee the well-posedness of the firms' value functions under the trigger function $\mathcal{X}(k^a, k^b)$.

(ii) $\mathcal{G}(k^a, k^b)$ *is* $\mathcal{C}^1$, $\big(\rho + \mathcal{G}(k^a, k^b)\big)(k^a + k^b)^{\frac{1}{\gamma}}$ *is increasing in* k^a, k^b*; and*

(iii) there exists a $\mathcal{C}^1$ *function* $\phi(z_1, z_2)$ *such that at each* (z_1, z_2)*, either* $\frac{\partial \phi(z_1,z_2)}{\partial z_1} \neq 0$ *or* $\frac{\partial \phi(z_1,z_2)}{\partial z_2} \neq 0$ *holds, and*

$$\phi\left(k^b \mathcal{G}, \frac{1}{k^b \mathcal{G}} \ln \frac{1}{(\mathcal{G} + \rho)(k^a + k^b)^{\frac{1}{\gamma}}}\right) = 0. \tag{F.9}$$

The left inequality in part (i) of Condition A.6 ensures that $\mathcal{X}(k^a, k^b)$ given in (F.8) is higher than $\rho(k^a + k^b)^{\frac{1}{\gamma}}$, the trigger function of the perfect competition equilibrium. The right inequality in part (i) of Condition A.6 ensures that $\mathcal{X}(k^a, k^b)$ given in (F.8) is lower than the trigger function of the open-loop equilibrium, i.e.,

$$\mathcal{X}(k^a, k^b) < \frac{\rho}{1 - \frac{\min\{k^a, k^b\}}{\gamma(k^a + k^b)}} (k^a + k^b)^{\frac{1}{\gamma}}. \tag{F.10}$$

By choosing different function ϕ to obtain $\mathcal{G}$ satisfying Condition A.6, we can derive various endogenous nonlinear trigger functions. Below we present two sets of trigger functions, where $\phi(z_1, z_2) = z_1 - \lambda$ and $\phi(z_1, z_2) = \frac{\ln(\lambda / z_1)}{z_1} + z_2$, respectively, with $\lambda > 0$ being a constant.

Proposition 7 *Let* $\lambda > 0$ *be a constant. We have the following nonlinear trigger functions:*

(i) For $\min(k^a, k^b) > \frac{\lambda(2\gamma - 1)}{\rho}$,

$$\mathcal{X}(k^a, k^b) = \Big(\rho + \frac{\lambda}{\max\{k^a, k^b\}}\Big)(k^a + k^b)^{\frac{1}{\gamma}}. \tag{F.11}$$

(ii) For $\min(k^a, k^b) > \frac{1}{2}\Big(\frac{(2\gamma - 1)^2 \lambda}{\gamma \rho^2}\Big)^{\frac{\gamma}{\gamma + 1}}$,

$$\mathcal{X}(k^a, k^b) = \rho(k^a + k^b)^{\frac{1}{\gamma}} + \frac{\lambda}{\sqrt{(\frac{\rho \max\{k^a, k^b\}}{2})^2 + \lambda \max\{k^a, k^b\}(k^a + k^b)^{-\frac{1}{\gamma}}} + \frac{\rho \max\{k^a, k^b\}}{2}}. \tag{F.12}$$

Next, we present symmetric closed-loop equilibria associated with the nonlinear trigger functions $\mathcal{X}(k^a, k^b)$ as given in equation (F.8). We define $\mathcal{K}(x)$ and $\vartheta_1(k^a, k^b)$, $\vartheta_2(k^a, k^b)$ as specified by (65) and (66), respectively, with the trigger functions $\mathcal{X}(k^a, k^b)$ as given in equation (F.8).

Theorem 7 *Let* $\boldsymbol{\varphi}^i_* := (\boldsymbol{u}^i, \boldsymbol{v}^i)$ *be firm* i*'s closed-loop strategy given by* (63)-(64)*, with* $\mathcal{X}(k^a, k^b)$*,* $\mathcal{K}(x)$*, and* $\vartheta_1(k^a, k^b)$*,* $\vartheta_2(k^a, k^b)$ *given by* (F.8)*,* (65)*, and* (66)*, respectively. Under Condition*

A.6 with $\min(K_0^a, K_0^b) \geq \underline{k}$*, the closed-loop strategy pair* $(\boldsymbol{\varphi}_*^a, \boldsymbol{\varphi}_*^b) \in \mathcal{S}$ *is a closed-loop equilibrium.*

We can verify that the capital processes under the strategy given in Theorem 7 take the explicit form as in (51), but with $\mathcal{X}(k^a, k^b)$ given by (F.8). Then, for the equilibrium strategy given in Theorem 7, firm i's equilibrium value $V^i(z)$ takes the same form as in Theorem 4, but with $\Psi^i(z)$ and $\mathcal{X}(k^a, k^b)$ as defined in (F.2) and (F.8), respectively.

F.3 Determining Endogenous Nonlinear Trigger Functions in Duopoly

By imposing the constraint (59), we infer that (35) holds for any (k^a, k^b), as does (F.6) for any (k^a, k^b). Combining (F.6) and (F.7) and using $\frac{\partial^2 H^a(k^a,k^b)}{\partial k^a \partial k^b} = \frac{\partial^2 H^a(k^a,k^b)}{\partial k^b \partial k^a}$ for $k^a \geq k^b$,and $\mathcal{X}(k^a, k^b) = \mathcal{X}(k^b, k^a)$, we obtain

$$\frac{k^b}{k^a + k^b}\mathcal{X}(k^a, k^b)\frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^b} + \left[\Big(\gamma - \frac{k^b}{k^a + k^b}\Big)\mathcal{X}(k^a, k^b) - \gamma\rho(k^a + k^b)^{\frac{1}{\gamma}}\right]\frac{\partial \mathcal{X}(k^a, k^b)}{\partial k^a} = 0 \tag{F.13}$$

for any $k^a \leq k^b$.

Lemma 5 in Internet Appendix IA.A indicates that the PDE (F.13) admits the following general solution:

$$\mathcal{X}(k^a, k^b) = \Big(\rho + \mathcal{G}(k^a, k^b)\Big)(k^a + k^b)^{\frac{1}{\gamma}}, \tag{F.14}$$

where $\mathcal{G}(k^a, k^b)$ satisfies (F.9). By using (F.14) and symmetry, we obtain the nonlinear trigger functions as given in (F.8).

Internet Appendices to

"Option Exercise Games and the q Theory of Investment"

IA.A Additional Technical Results

Lemma 2 *Assume the pair (θ_1, θ_2) satisfies Condition 1.*

(i) When $\theta_1 = \theta_2$, we have $\theta_1 = \theta_2 = 1$.

(ii) When $\theta_2 > \theta_1$, we have $\theta_2 > 1 > \theta_1$.

(iii) The largest value of $\theta_1 + \theta_2$ is reached only at (θ_1^*, θ_2^*) defined in (47).

(iv) The smallest value of $\theta_1 + \theta_2$ is reached only at $\theta_2 = \theta_1 = 1$.

Lemma 3 *Assume $\xi > 0$, the pair (θ_1^a, θ_2^a) and the pair (θ_1^b, θ_2^b) satisfy Condition A.5.*

(i) When $\theta_1^a = \theta_2^a$ or $\theta_1^b = \theta_2^b$, we have $\theta_1^a = \theta_2^a = \theta_1^b = \theta_2^b = 1$.

(ii) When $\theta_2^a > \theta_1^a$ or $\theta_2^b > \theta_1^b$, we have $\theta_2^i > 1 \geq \theta_1^i$, and

$$2 - \frac{\theta_2^i}{\theta_1^i} - \theta_2^i < 2\xi^i(\theta_1^i - 1). \tag{IA.A.1}$$

(iii) Moreover, the left inequality in (D.9) *is equivalent to*

$$2 + (\beta - 2)\frac{\theta_2^i}{\theta_1^i} - \beta\theta_2^i \leq \frac{\beta\frac{\theta_2^i}{\theta_1^i\xi^i}(2 - \frac{\theta_2^i}{\theta_1^i} - \theta_2^i)}{(1 + \frac{\theta_2^i}{\theta_1^i\xi^i})^\beta - 1}. \tag{IA.A.2}$$

Lemma 4 *Assume Condition A.5 holds, and recall the function $V^i(z)$ defined in Theorem 6 with $\mathcal{X}(k^a, k^b)$ as defined in* (D.1). *Then, the functions $V^a(z)$ and $V^b(z)$ satisfy Conditions A.1 and A.2.*

Lemma 5 *The PDE* (F.13) *admits a general solution as given by* (F.14). *Under Condition A.6, $\mathcal{X}(k^a, k^b)$ given by* (F.14) *satisfies* (F.13) *for any $k^b \geq k^a \geq \underline{k}$.*

Lemma 6 *Recall the function $V^i(z)$ defined in Theorem 4 with $\Psi^i(z)$ and $\mathcal{X}(k^a, k^b)$, as defined in* (F.2) *and* (F.8), *respectively. Under Condition A.6, the functions $V^a(z)$ and $V^b(z)$ satisfy Conditions A.1 and A.2 with $\mathbb{Z} = \{(x, k^a, k^b) : x > 0, k^a \geq \underline{k}, k^b \geq \underline{k}\}$, and* (E.32) *holds.*

Lemma 7 *Assume Condition A.6 holds. Let the strategy* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*)$ *be as defined in Theorem 7, and let the function* $V^i(z)$ *be as given in Theorem 4 but with* $\Psi^i(z)$ *and* $\mathcal{X}(k^a, k^b)$ *as defined in* (F.2) *and* (F.8)*, respectively. Then, the strategy* $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$ *and* (V^a, V^b) *satisfy Conditions A.3-A.4 and Footnote 23 with* $\mathbb{Z} = \{(x, k^a, k^b) : x > 0, k^a \geq \underline{k}, k^b \geq \underline{k}\}$.

IA.B Additional Proofs

Proof of Lemma 2 If $\theta_1 = \theta_2$, then by using $\overline{g}(1) = \underline{g}(1) = 1$, we observe that (44) simplifies to $2 \leq \theta_2 + \theta_1 \leq 2$. This implies that $\theta_2 = \theta_1 = 1$.

Denote

$$\widehat{g}(w) = \frac{[(1+w)^\beta - 1][(1-\frac{2}{\beta})w + \frac{2}{\beta}] + (w-2)w}{(1+w)^\beta - (1+w)}. \tag{IA.B.1}$$

Because $\theta_1 + \theta_2 = \theta_2(1 + \frac{\theta_1}{\theta_2})$ and $\underline{g}(w)/(1+\frac{1}{w}) = \widehat{g}(w)$, the left inequality in (44) is equivalent to

$$\widehat{g}\Big(\frac{\theta_2}{\theta_1}\Big) \leq \theta_2. \tag{IA.B.2}$$

Since $\beta > 2$, we have

$$\widehat{g}(w) > \frac{[(1+w)^\beta - 1][(1-\frac{2}{\beta}) + \frac{2}{\beta}] - w}{(1+w)^\beta - (1+w)} = 1, \quad w > 1. \tag{IA.B.3}$$

Thus, for $\theta_2 > \theta_1$, we derive from (IA.B.2) that $\theta_2 > 1$. The right inequality in (44) implies that

$$\theta_1 \leq \frac{3 - \theta_2}{1 + 1/\theta_2}. \tag{IA.B.4}$$

We can show that $f(x) := \frac{3-x}{1+1/x} = \frac{x(3-x)}{x+1}$ is decreasing in $x \geq 1$ by noticing that $f'(x) = \frac{4-(x+1)^2}{(x+1)^2} < 0$ for $x > 1$. Since $\theta_2 > 1$, we have $\frac{3-\theta_2}{1+1/\theta_2} < 1$, which, together with (IA.B.4), implies that $\theta_1 < 1$.

Recall the function h as defined below (47). A straightforward calculation shows that

$$h(w)(w-1) = (\overline{g}(w) - \underline{g}(w))\beta w[(1+w)^{\beta-1} - 1].$$

It can be verified that $h''(w) < 0$ for $w \geq 1$, $h(1) > 0$, and $h(w) < 0$ for sufficiently large w. Therefore, the equation $h(w) = 0$ in $w > 1$ has a unique solution $w^* > 1$. For $w > w^*$, we

have $\underline{g}(w) > \overline{g}(w)$. Thus, we have $\theta_2/\theta_1 \le w^*$. Since $\overline{g}(w) = 3 - 1/w$ is increasing for $w \ge 1$, it follows that $\overline{g}(\theta_2/\theta_1) \le 3 - 1/w^*$. From (44), we conclude that $\theta_1 + \theta_2 \le 3 - 1/w^*$, with equality if and only if $\theta_2/\theta_1 = w^*$, which implies $\theta_2 = \frac{3w^*-1}{1+w^*}$ and $\theta_1 = \frac{3-1/w^*}{1+w^*}$.

We can show that $\underline{g}(w) > 2$ for $w > 1$ (see (IA.B.10) with $\zeta = 1$). Hence, $\theta_1 + \theta_2 > \underline{g}(\theta_2/\theta_1) > 2$ when $\theta_2 > \theta_1 > 0$. Then, the smallest value of $\theta_1 + \theta_2$ is achieved only when $\theta_1 = \theta_2 = 1$.

□

Proof of Lemma 3 If $\theta_1^i = \theta_2^i$, then by using $\overline{g}(\frac{1}{\xi^i}, \xi^i) = \underline{g}(\frac{1}{\xi^i}, \xi^i) = 1 + \xi^i$, we observe that (D.9) simplifies to $1 + \xi^i \le \theta_1^i(1 + \xi^i) \le 1 + \xi^i$. This implies that $\theta_1^i = \theta_2^i = 1$.

Denote

$$\widehat{g}(w, \zeta) = \frac{[(1+w)^\beta - 1][(1 - \frac{2}{\beta})\zeta w + \frac{2}{\beta}] + (\zeta w - 2)w}{(1+w)^\beta - (1+w)}. \tag{IA.B.5}$$

Since $\widehat{g}(w,\zeta) = \frac{[(1+w)^\beta-(1+w)][(1-\frac{2}{\beta})\zeta w+\frac{2}{\beta}]+((1-\frac{2}{\beta})\zeta w+\frac{2}{\beta}+\zeta w-2)w}{(1+w)^\beta-(1+w)} = (1-\frac{2}{\beta})\zeta w + \frac{2}{\beta} + \frac{(2-\frac{2}{\beta})(\zeta w-1)w}{(1+w)^\beta-(1+w)}$, we have $\underline{g}(w,\zeta) = \widehat{g}(w,\zeta)(1 + \frac{1}{w})$. Combining it with $\theta_1^i\xi^i + \theta_2^i = \theta_2^i(\frac{\theta_1^i\xi^i}{\theta_2^i} + 1)$, one can see that the left inequality in (D.9) is equivalent to

$$\widehat{g}\Big(\frac{\theta_2^i}{\theta_1^i\xi^i}, \xi^i\Big) \le \theta_2^i. \tag{IA.B.6}$$

We can further verify that (IA.B.6) is equivalent to (IA.A.2).

Since $\beta > 2$, we have

$$\widehat{g}(w,\zeta) = (1 - \frac{2}{\beta})\zeta w + \frac{2}{\beta} + \frac{(2-\frac{2}{\beta})(\zeta w - 1)w}{(1+w)^\beta - (1+w)} > 1, \quad \text{if} \quad \zeta w > 1. \tag{IA.B.7}$$

Thus, we derive from (IA.B.6) that $\theta_2^i > 1$ when $\theta_2^i > \theta_1^i$.

Note that $\widehat{g}(w,\zeta) = \zeta w - \frac{w}{1+w}(\zeta w - 1) + (\zeta w - 1)\Big[\frac{(2-\frac{2}{\beta})w}{(1+w)^\beta-(1+w)} - \frac{2}{\beta} + \frac{w}{1+w}\Big]$. Next, we prove that

$$\Big[\frac{(2-\frac{2}{\beta})w}{(1+w)^\beta - (1+w)} - \frac{2}{\beta} + \frac{w}{1+w}\Big] > 0, \quad w > 0. \tag{IA.B.8}$$

We have

$$\frac{(2-\frac{2}{\beta})w}{(1+w)^\beta - (1+w)} - \frac{2}{\beta} + \frac{w}{1+w} = \frac{[(1+w)^{\beta-1} + 1 - \frac{2}{\beta}]w - \frac{2}{\beta}[(1+w)^\beta - (1+w)]}{(1+w)^\beta - (1+w)}. \tag{IA.B.9}$$

Let $y = 1+w$, then we only need to show that $h(y) = [y^{\beta-1}+1-\frac{2}{\beta}](y-1)-\frac{2}{\beta}[y^{\beta}-y] > 0$ for any $y > 1$. We have $h'(y) = (\beta-2)y^{\beta-1}-(\beta-1)y^{\beta-2}+1$, and $h''(y) = (\beta-2)(\beta-1)y^{\beta-3}(y-1) > 0$ for any $y > 1$. Since $h'(1) = 0$, we have $h'(y) > 0$ for any $y > 1$. Since $h(1) = 0$, we have $h(y) > 0$ for any $y > 1$. Hence, (IA.B.8) holds and $\widehat{g}(w,\zeta) > \zeta w - \frac{w}{1+w}(\zeta w - 1) = (1+\zeta)\frac{w}{1+w}$ when $\zeta w > 1$. It follows that

$$\underline{g}(w,\zeta) = \widehat{g}(w,\zeta)(1+\frac{1}{w}) > 1+\zeta, \quad \text{if} \quad \zeta w > 1. \tag{IA.B.10}$$

If $\frac{\theta_2^i}{\theta_1^i} > 1$, we derive from (IA.B.10) that $\underline{g}\Big(\frac{\theta_2^i}{\theta_1^i\xi^i}, \xi^i\Big) > 1+\xi^i$. Combining it with the left inequality in (D.9), we have

$$1+\xi^i < \theta_1^i\xi^i + \theta_2^i, \quad \text{if} \quad \theta_2^i > \theta_1^i. \tag{IA.B.11}$$

If $\theta_1^a = \theta_2^a$, as we shown above, we have $\theta_1^a = \theta_2^a = 1$, and thus $\theta_1^b + \theta_2^b\xi = 1+\xi$. If $\theta_2^b > \theta_1^b$, then we derive from (IA.B.11) that $1 + 1/\xi < \theta_1^b/\xi + \theta_2^b$. Then we arrive at a contradiction as $\theta_1^b + \theta_2^b\xi = 1+\xi$. Hence, we must have $\theta_2^b = \theta_1^b$ when $\theta_1^a = \theta_2^a$. Similarly, we can show that $\theta_2^a = \theta_1^a$ when $\theta_1^b = \theta_2^b$. Therefore, when $\theta_1^a = \theta_2^a$ or $\theta_1^b = \theta_2^b$, we have $\theta_1^a = \theta_2^a = \theta_1^b = \theta_2^b = 1$.

Define $\phi(z) := ((1+z)^{\beta}-1)((\beta-1)z-2) + 2\beta z$. We observe that $\phi(w) > 0$, $\forall w > 0$, using the fact that $\phi''(w) = \beta(\beta-1)(1+w)^{\beta-2}(\beta+1)w > 0$, $\forall w > 0$, along with $\phi'(0) = 0$ and $\phi(0) = 0$. Recalling $\widehat{g}$ as defined in (IA.B.5), we have

$$\widehat{g}(w,\zeta) - \frac{2+2\zeta-\zeta w}{1+\frac{2}{w}} = \frac{1}{(1+w)^{\beta}-(1+w)}\frac{\zeta w-1}{w+2}\frac{2}{\beta}\phi(w) > 0, \quad \forall \zeta w > 1. \tag{IA.B.12}$$

Setting $w = \frac{\theta_2^i}{\theta_1^i\xi^i}$ and $\zeta = \xi^i$ in (IA.B.12) and combining it with (IA.B.6), we derive the following inequality

$$\theta_2^i > \frac{2+2\xi^i-\frac{\theta_2^i}{\theta_1^i}}{1+\frac{2\theta_1^i\xi^i}{\theta_2^i}}, \tag{IA.B.13}$$

when $\theta_2^i/\theta_1^i > 1$. It can then be seen that (IA.B.13) is equivalent to (IA.A.1).

□

Proof of Lemma 4 We will prove the result for $V^a(z)$ only, as the case for $V^b(z)$ can be handled in a similar manner. To simplify notation, for any function $f \in \{V^a, H^a\}$, we denote the partial derivatives as follows: $f_{k^a}(x,k^a,k^b) = \frac{\partial f(x,k^a,k^b)}{\partial k^a}$, $f_{k^b}(x,k^a,k^b) = \frac{\partial f(x,k^a,k^b)}{\partial k^b}$, $f_{k^ak^a}(x,k^a,k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial (k^a)^2}$, $f_{k^ak^b}(x,k^a,k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial k^a\partial k^b}$, $f_{k^bk^a}(x,k^a,k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial k^b\partial k^a}$, and $f_{k^bk^b}(x,k^a,k^b) = \frac{\partial^2 f(x,k^a,k^b)}{\partial (k^b)^2}$, where $H^a(k^a,k^b)$ is a function describing firm a's option value, as

given by (D.6)-(D.8). Define

$$\vartheta_1(k^a,k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } \xi k^a \le k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } \xi k^a > k^b, \end{cases} \quad \text{and} \quad \vartheta_2(k^a,k^b) := \begin{cases} \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a}, & \text{if } \xi k^a \le k^b, \\ \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b}, & \text{if } \xi k^a > k^b. \end{cases} \tag{IA.B.14}$$

Substituting $\mathcal{X}(k^a,k^b)$ as defined in (D.1) into (IA.B.14), we have

$$\vartheta_1(k^a,k^b) := \begin{cases} \rho\theta_1^a, & \text{if } \xi k^a \le k^b, \\ \rho\theta_1^b, & \text{if } \xi k^a > k^b, \end{cases} \quad \text{and} \quad \vartheta_2(k^a,k^b) := \begin{cases} \rho\theta_2^a, & \text{if } \xi k^a \le k^b, \\ \rho\theta_2^b, & \text{if } \xi k^a > k^b. \end{cases} \tag{IA.B.15}$$

In the following, we will simplify the notation by writing $\vartheta_1(k^a,k^b)$ and $\vartheta_2(k^a,k^b)$ as ϑ_1 and ϑ_2, respectively. This is because they are constants in both regions of interest: $\xi k^a < k^b$ and $\xi k^a > k^b$.

Given that $\beta > 2$, $r > 0$, and $c \ge 0$, it follows that $\mathcal{X}(k^a,k^b) > 0$ for any $(k^a,k^b) \in \mathbb{R}^2_+$. According to (D.6)-(D.8), we have $H^a \in \mathcal{C}^{1,1}(\mathbb{R}^2_+) \bigcap \mathcal{C}^{\infty,\infty}(\{(k^a,k^b) \in \mathbb{R}^2_+ : \xi k^a \ne k^b\})$. Therefore, for $\xi k^a \ne k^b$, we have $H^a_{k^a k^b}(k^a,k^b) = H^a_{k^b k^a}(k^a,k^b)$. By (34), we see that V^a belongs to $\mathcal{C}^{2,1,1}(\{(x,k^a,k^b) \in \mathbb{R}^3_+ : (k^a,k^b) \in \overline{\mathbf{N}}_{ab}(x)\})$ and satisfies (33), where $\mathbf{N}_{ab}(x)$ is given by (D.5). Using (34), (D.6)-(D.8), we obtain that

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) = p, \qquad k^b \ge \xi k^a \ge 0, \tag{IA.B.16}$$

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b) = 0, \qquad \xi k^a \ge k^b \ge 0. \tag{IA.B.17}$$

By combining the above with the definition of $V^a(x,k^a,k^b)$ given in Theorem 4, we observe that $V^a_{k^a}(x,k^a,k^b)$ and $V^a_{k^b}(x,k^a,k^b)$ are continuous in $(x,k^a,k^b) \in \mathbb{Z}$, particularly across the boundary $x = \mathcal{X}(k^a,k^b)$, and

$$V^a_{k^a}(x,k^a,k^b) = p, \qquad (k^a,k^b) \in \mathbf{I}_a(x), \tag{IA.B.18}$$

$$V^a_{k^b}(x,k^a,k^b) = 0, \qquad (k^a,k^b) \in \mathbf{I}_b(x), \tag{IA.B.19}$$

where $\mathbf{I}_i(x)$ is defined by (D.2). Consequently, $V^a \in \mathcal{C}^{0,1,1}(\mathbb{Z})$ and Condition A.2 is satisfied. It remains to prove that the two inequalities in Condition A.1 hold.

We will complete our proof in four steps. Lemma 3 implies that when $\theta_1^a = \theta_2^a$ or $\theta_1^b = \theta_2^b$, we have $\theta_1^a = \theta_2^a = \theta_1^b = \theta_2^b = 1$, which is proved in Lemma 1. Thus, we focus on the case $\theta_2^a > \theta_1^a$, and $\theta_2^b > \theta_1^b$. Then, Lemma 3 implies that $\theta_2^a > 1 \ge \theta_1^a$, $\theta_2^b > 1 \ge \theta_1^b$.

Step 1: We prove that

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)<0,\quad k^b>\xi k^a\geq 0. \tag{IA.B.20}$$

Using (34), for any $k^b\geq\xi k^a\geq 0$, we have

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)=-\frac{k^a}{r}+H^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^\beta. \tag{IA.B.21}$$

Taking the partial derivative with respect to k^b in (37) and using the equality $H^a_{k^ak^b}(k^a,k^b)=H^a_{k^bk^a}(k^a,k^b)$ for $k^b>\xi k^a\geq 0$, we obtain

$$\begin{aligned}&H^a_{k^ak^b}(k^a,k^b)-\frac{\vartheta_1}{\vartheta_2}H^a_{k^ak^a}(k^a,k^b)=\frac{1}{r}\Big(1-2\frac{\vartheta_1}{\vartheta_2}\Big)\mathcal{X}(k^a,k^b)^{-\beta}\\ =&\frac{1}{r}\Big(1-2\frac{\vartheta_1}{\vartheta_2}\Big)\frac{1}{(1-\beta)\vartheta_2}\frac{\partial(\mathcal{X}(k^a,k^b)^{1-\beta})}{\partial k^a}.\end{aligned}$$

Taking the integral with respect to k^a in both sides of the above equation and using the relation $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}=\frac{\theta^a_1}{\theta^a_2}$ for $k^b>\xi k^a\geq 0$, we obtain

$$\int H^a_{k^ak^b}(k^a,k^b)dk^a=\frac{\theta^a_1}{\theta^a_2}H^a_{k^a}(k^a,k^b)+\frac{1}{r}\Big(1-2\frac{\theta^a_1}{\theta^a_2}\Big)\frac{\mathcal{X}(k^a,k^b)^{1-\beta}}{(1-\beta)\vartheta_2}. \tag{IA.B.22}$$

For $k^b\geq\xi k^a\geq 0$, we derive from (IA.B.22) that

$$\begin{aligned}H^a_{k^b}(k^a,k^b)=&H^a_{k^b}(k^b/\xi,k^b)-\frac{\theta^a_1}{\theta^a_2}[H^a_{k^a}(k^b/\xi,k^b)-H^a_{k^a}(k^a,k^b)]\\&-\frac{1}{r}\Big(1-2\frac{\theta^a_1}{\theta^a_2}\Big)\frac{\mathcal{X}(k^b/\xi,k^b)^{1-\beta}-\mathcal{X}(k^a,k^b)^{1-\beta}}{(1-\beta)\vartheta_2}.\end{aligned} \tag{IA.B.23}$$

Denote

$$\mathcal{G}(k^a,k^b)=\Big(2-\frac{\theta^a_2}{\theta^a_1}\Big)k^a+k^b+rp+c-(\theta^a_2k^a+\theta^a_1k^b+pr+c)\frac{1}{\beta-1}(\beta+\frac{1}{\theta^a_1}-\frac{2}{\theta^a_2}). \tag{IA.B.24}$$

Substituting (37)-(38) into (IA.B.23), and using (IA.B.21), we obtain

$$V^a_{k^b}(\mathcal{X}(k^a,k^b),k^a,k^b)=\frac{\theta^a_1}{r\theta^a_2}\left[\mathcal{G}(k^a,k^b)-\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^b/\xi,k^b)}\Big)^\beta\mathcal{G}(k^b/\xi,k^b)\right]. \tag{IA.B.25}$$

To prove (IA.B.20), we first prove that

$$V^a_{k^b}(\mathcal{X}(0,k^b),0,k^b)<0,\quad k^b\geq 0. \tag{IA.B.26}$$

To achieve this, we define

$$\mathcal{H}(k^b):=\Big(\frac{\mathcal{X}(k^b/\xi,k^b)}{\mathcal{X}(0,k^b)}\Big)^{\beta}\mathcal{G}(0,k^b)-\mathcal{G}(k^b/\xi,k^b), \tag{IA.B.27}$$

and conclude that (IA.B.26) is equivalent to $\mathcal{H}(k^b)<0$ for any $k^b\geq 0$, using (IA.B.25). Now, we prove that $\mathcal{H}(k^b)<0$, $\forall\ k^b>0$. Let $\mathcal{Y}(k^b):=\frac{\mathcal{X}(k^b/\xi,k^b)}{\mathcal{X}(0,k^b)}$. Then $\mathcal{H}(k^b)=\mathcal{Y}(k^b)^{\beta}\mathcal{G}(0,k^b)-\mathcal{G}(k^b/\xi,k^b)$. Given that $\theta^a_2/\theta^a_1>1$, $\theta^a_2\geq 1$, and (IA.A.2), we have

$$\begin{aligned}
&\mathcal{G}_{k^a}:=\frac{\partial\mathcal{G}(k^a,k^b)}{\partial k^a}=\Big(2-\frac{\theta^a_2}{\theta^a_1}-\theta^a_2\Big)\frac{\beta}{\beta-1}<0, \qquad \text{(IA.B.28)}\\
&\mathcal{G}_{k^b}:=\frac{\partial\mathcal{G}(k^a,k^b)}{\partial k^b}=(2+(\beta-2)\frac{\theta^a_2}{\theta^a_1}-\beta\theta^a_2)\frac{1}{\beta-1}\frac{\theta^a_1}{\theta^a_2}<0,\\
&\mathcal{G}(0,0)=\frac{pr+c}{(\beta-1)\theta^a_2}\Big(2-\frac{\theta^a_2}{\theta^a_1}-\theta^a_2\Big)<0.
\end{aligned}$$

It follows that

$$\mathcal{G}(k^a,k^b)<0,\quad (k^a,k^b)\in\mathbb{R}^2_+. \tag{IA.B.29}$$

Using $\mathcal{Y}(k)=1+\frac{\theta^a_2}{\theta^a_1\xi}-\frac{\theta^a_2}{\theta^a_1\xi}\frac{\mathcal{X}(0,0)/(\rho\theta^a_1)}{k+\mathcal{X}(0,0)/(\rho\theta^a_1)}$, $\mathcal{Y}(0)=1$, and $\mathcal{Y}'(0)=\frac{\theta^a_2}{\theta^a_1\xi}\frac{1}{\mathcal{X}(0,0)/(\rho\theta^a_1)}$, we infer

$$\mathcal{H}'(0)=\beta\mathcal{Y}'(0)\mathcal{G}(0,0)-\mathcal{G}_{k^a}/\xi=\beta\frac{\theta^a_2}{\xi}\frac{1}{pr+c}\mathcal{G}(0,0)-\mathcal{G}_{k^a}/\xi=0 \tag{IA.B.30}$$

and

$$\begin{aligned}
\mathcal{H}''(k)=&\beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[\Big(\beta-1+\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\Big)\mathcal{G}(0,k)+2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)}\mathcal{G}_{k^b}\right]\\
=&\beta\mathcal{Y}(k)^{\beta-2}\mathcal{Y}'(k)^2\left[(\beta-1)\mathcal{G}(0,k)+\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}\mathcal{G}\Big(0,-\frac{\mathcal{X}(0,0)}{\vartheta_1}\Big)\right], \qquad \text{(IA.B.31)}
\end{aligned}$$

where the second equality uses $\mathcal{G}(0,k)=\mathcal{G}\big(0,-\frac{\mathcal{X}(0,0)}{\rho\theta^a_1}\big)+(k+\frac{\mathcal{X}(0,0)}{\rho\theta^a_1})\mathcal{G}_{k^b}$ and $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}(k+\frac{\mathcal{X}(0,0)}{\vartheta_1})+2\frac{\mathcal{Y}(k)}{\mathcal{Y}'(k)}=0$. A direct calculation yields that $\frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2}=-2\frac{\xi\theta^a_1}{\theta^a_2}$, and

$$(\beta-1)\mathcal{G}(0,0)+\frac{\mathcal{Y}(0)\mathcal{Y}''(0)}{\mathcal{Y}'(0)^2}\mathcal{G}\Big(0,-\frac{\mathcal{X}(0,0)}{\rho\theta^a_1}\Big)$$

$$=\Big(\beta-1-2\frac{\xi\theta_1^a}{\theta_2^a}\Big)\mathcal{G}(0,0)+2\frac{\xi\theta_1^a}{\theta_2^a}\frac{\mathcal{X}(0,0)}{\rho\theta_1^a}\mathcal{G}_{k^b}$$
$$=\frac{pr+c}{\theta_2^a}\Big[2-\frac{\theta_2^a}{\theta_1^a}-\theta_2^a+2\xi(1-\theta_1^a)\Big]<0, \tag{IA.B.32}$$

where the inequality utilizes (IA.A.1). By combining (IA.B.31), (IA.B.32), $\mathcal{Y}(0)=1$, and $\mathcal{Y}'(0)>0$, we can conclude that $\mathcal{H}''(0)<0$.

A direct calculation shows that $\frac{\mathcal{Y}(k)\mathcal{Y}''(k)}{\mathcal{Y}'(k)^2}$ and $\mathcal{G}(0,k)$ are affine functions of k. Consequently, the expression in the bracket in (IA.B.31) is linear in k. Then we conclude from $\mathcal{H}''(0)<0$, (IA.B.31), $\mathcal{Y}'(k)^2>0$, and $\mathcal{Y}(k)>0$, $\forall k\geq 0$ that either $\mathcal{H}''(k)<0$, $\forall\ k\geq 0$, or there exists $\widetilde{k}>0$ such that $\mathcal{H}''(k)<0$, $\forall k\in[0,\widetilde{k})$ and $\mathcal{H}''(k)>0$, $\forall\ k>\widetilde{k}$. Recall that we have shown $\mathcal{H}'(0)=0$ in (IA.B.30). Then $\mathcal{H}'(k)<0$, $\forall\ k>0$, or there exists $\widehat{k}\geq 0$ such that $\mathcal{H}'(k)<0$, $\forall k\in(0,\widehat{k})$ and $\mathcal{H}'(k)>0$, $\forall\ k>\widehat{k}$. Thus, $\mathcal{H}(k)<\max\big\{\mathcal{H}(0),\lim_{w\to+\infty}\mathcal{H}(w)\big\}$ for any $k>0$. Note that $\lim_{k\to+\infty}\mathcal{Y}(k)=1+\frac{\theta_2^a}{\theta_1^a\xi}$ and

$$\lim_{k\to+\infty}\frac{\mathcal{H}(k)}{k}=\Big(1+\frac{\theta_2^a}{\theta_1^a\xi}\Big)^{\beta}\mathcal{G}_{k^b}-(\mathcal{G}_{k^a}/\xi+\mathcal{G}_{k^b})$$
$$=\frac{\theta_1^a}{\theta_2^a}\frac{1}{\beta-1}\left[\Big(\Big(1+\frac{\theta_2^a}{\theta_1^a\xi}\Big)^{\beta}-1\Big)\Big(2+(\beta-2)\frac{\theta_2^a}{\theta_1^a}-\beta\theta_2^a\Big)-\beta\frac{\theta_2^a}{\theta_1^a\xi}\Big(2-\frac{\theta_2^a}{\theta_1^a}-\theta_2^a\Big)\right]\leq 0, \tag{IA.B.33}$$

where the inequality utilizes (IA.A.2) and the condition $\theta_2^b\geq\theta_1^b>0$. Since $\mathcal{H}(0)=0$, it follows that $\mathcal{H}(k)<0$, $\forall\ k>0$. Therefore, we have completed the proof of (IA.B.26).

Given that (IA.B.28) implies that $\frac{\partial^2}{\partial(k^a)^2}\mathcal{G}(k^a,k^b)=0$, we derive from (IA.B.25) that

$$\frac{\partial^2}{\partial(k^a)^2}V_{k^b}^a(\mathcal{X}(k^a,k^b),k^a,k^b)=-\frac{\theta_1^a}{r\theta_2^a}\beta(\beta-1)\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^b/\xi,k^b)}\Big)^{\beta}\Big(\frac{\vartheta_2}{\mathcal{X}(k^a,k^b)}\Big)^2\mathcal{G}(k^b/\xi,k^b)>0,$$

for any $k^b\geq\xi k^a\geq 0$, where the inequality uses (IA.B.29). The convexity of $V_{k^b}^a(\mathcal{X}(k^a,k^b),k^a,k^b)$ with respect to k^a implies that

$$V_{k^b}^a(\mathcal{X}(k^a,k^b),k^a,k^b)<\max\Big\{V_{k^b}^a(\mathcal{X}(0,k^b),0,k^b),V_{k^b}^a(\mathcal{X}(k^b/\xi,k^b),k^b/\xi,k^b)\Big\}$$
$$=\max\Big\{V_{k^b}^a(\mathcal{X}(0,k^b),0,k^b),0\Big\}=0,\qquad\forall k^a\in(0,k^b/\xi), \tag{IA.B.34}$$

where the first equality uses $V_{k^b}^a(\mathcal{X}(k^b/\xi,k^b),k^b/\xi,k^b)=0$ and the second equality uses (IA.B.26). By combining (IA.B.26) and (IA.B.34), we can see that (IA.B.20) holds.

Step 2: We prove that

$$V_{k^b}^a(x,k^a,k^b)<0,\quad(k^a,k^b)\in\mathbf{N}_b(x)\setminus\{(0,0)\},\ x>0. \tag{IA.B.35}$$

Recall that for any $x \in (0, \mathcal{X}(k^a, k^b)]$ and $(k^a, k^b) \in \mathbb{R}^2_+$, $V^a(x, k^a, k^b)$ is given by (34). Consequently, $V^a_{k^b}(x, k^a, k^b) = -\frac{k^a}{r} + H^a_{k^b}(k^a, k^b)x^\beta$, which is linear in $x^\beta > 0$.

For $k^b > \xi k^a \geq 0$, since $V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) < 0$ by (IA.B.20), and $\lim_{x\to 0+} V^a_{k^b}(x, k^a, k^b) = -\frac{k^a}{r} \leq 0$, it follows that $V^a_{k^b}(x, k^a, k^b) < 0$ for any $x \in (0, \mathcal{X}(k^a, k^b))$. For $\xi k^a \geq k^b \geq 0$ with $k^a > 0$, since $V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) = 0$ by (IA.B.17), and $\lim_{x\to 0+} V^a_{k^b}(x, k^a, k^b) = -\frac{k^a}{r} < 0$, it follows that $V^a_{k^b}(x, k^a, k^b) < 0$ for any $x \in (0, \mathcal{X}(k^a, k^b))$. Thus, we conclude that $V^a_{k^b}(x, k^a, k^b) < 0$ for any $(k^a, k^b) \in \mathbf{N}_{ab}(x) \setminus \{(0, 0)\}$, $x > 0$.

In addition, for any $(k^a, k^b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, it follows from equation (B.6) that

$$V^a_{k^b}(x, k^a, k^b) = V^a_{k^b}(x, \widehat{k}^a, k^b) < 0, \tag{IA.B.36}$$

where $\widehat{k}^a = \frac{1}{\theta^a_2}\left(\frac{x}{\rho} - (rp + c)\right) - \frac{\theta^a_1}{\theta^a_2}k^b$ is determined by (B.5), the equality is derived from (IA.B.16), and the inequality follows from (IA.B.20). Since $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)) \supseteq \mathbf{N}_b(x)$, we infer (IA.B.35).

Step 3: We prove that

$$V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) < p, \quad \xi k^a > k^b \geq 0. \tag{IA.B.37}$$

Using (34), for $\xi k^a \geq k^b \geq 0$, we have

$$\begin{aligned} &V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) - p \\ =&\frac{\mathcal{X}(k^a, k^b)}{r - \mu} + H^a_{k^a}(k^a, k^b)\mathcal{X}(k^a, k^b)^\beta - \frac{2k^a + k^b + pr + c}{r}. \end{aligned} \tag{IA.B.38}$$

Taking the partial derivative with respect to k^a in (38), we obtain the following result for $\xi k^a > k^b \geq 0$:

$$\begin{aligned} H^a_{k^b k^a}(k^a, k^b) =&\frac{1}{r}[\mathcal{X}(k^a, k^b)^{-\beta} - \beta k^a \vartheta_1 \mathcal{X}(k^a, k^b)^{-1-\beta}] \\ =&\frac{1}{r}\Big[\frac{1}{(1-\beta)\vartheta_2}\frac{\partial(\mathcal{X}(k^a, k^b)^{1-\beta})}{\partial k^b} + k^a\frac{\vartheta_1}{\vartheta_2}\frac{\partial(\mathcal{X}(k^a, k^b)^{-\beta})}{\partial k^b}\Big]. \end{aligned}$$

Integrating the above equation with respect to k^b and using the relation $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)} = \frac{\theta^b_1}{\theta^b_2}$ for $\xi k^a > k^b \geq 0$, we have

$$\int H^a_{k^a k^b}(k^a, k^b)dk^b = \frac{1}{r}\left[\frac{\mathcal{X}(k^a, k^b)^{1-\beta}}{(1-\beta)\vartheta_2} + k^a\frac{\theta^b_1}{\theta^b_2}\mathcal{X}(k^a, k^b)^{-\beta}\right]. \tag{IA.B.39}$$

For $\xi k^a \geq k^b \geq 0$, we derive from (37) and (IA.B.39) that

$$\begin{aligned} H^a_{k^a}(k^a,k^b) &= H^a_{k^a}(k^a,\xi k^a) - \int_{k^b}^{\xi k^a} H^a_{k^a k^b}(k^a,k)dk \\ &= \Big(\frac{2k^a+\xi k^a+pr+c}{r} - \frac{\mathcal{X}(k^a,\xi k^a)}{r-\mu}\Big)\mathcal{X}(k^a,\xi k^a)^{-\beta} \\ &\quad - \frac{1}{r}\left[\frac{\mathcal{X}(k^a,\xi k^a)^{1-\beta}-\mathcal{X}(k^a,k^b)^{1-\beta}}{(1-\beta)\vartheta_2} + k^a\frac{\theta_1^b}{\theta_2^b}\big(\mathcal{X}(k^a,\xi k^a)^{-\beta}-\mathcal{X}(k^a,k^b)^{-\beta}\big)\right]. \end{aligned} \tag{IA.B.40}$$

Plugging (IA.B.40) into (IA.B.38), we have

$$\begin{aligned} &V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) - p \\ =&\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,\xi k^a)}\Big)^\beta\left(\frac{(2-\frac{\theta_1^b}{\theta_2^b})k^a+\xi k^a+pr+c}{r} - \mathcal{X}(k^a,\xi k^a)\Big(\frac{1}{r-\mu}+\frac{1}{r(1-\beta)\vartheta_2}\Big)\right) \\ &-\left(\frac{\Big(2-\frac{\theta_1^b}{\theta_2^b}\Big)k^a+k^b+pr+c}{r} - \mathcal{X}(k^a,k^b)\Big(\frac{1}{r-\mu}+\frac{1}{r(1-\beta)\vartheta_2}\Big)\right). \end{aligned} \tag{IA.B.41}$$

Denote

$$\mathcal{B}(k^a,k^b) = \Big(2-\frac{\theta_1^b}{\theta_2^b}\Big)k^a + k^b + pr + c - (\theta_2^b k^b + \theta_1^b k^a + pr + c)\frac{1}{\beta-1}\Big(\beta - \frac{1}{\theta_2^b}\Big). \tag{IA.B.42}$$

Then for $\xi k^a \geq k^b \geq 0$, (IA.B.41) can be simplified as

$$V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) - p = \frac{1}{r}\left[\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,\xi k^a)}\Big)^\beta\mathcal{B}(k^a,\xi k^a) - \mathcal{B}(k^a,k^b)\right] = \frac{G(k^a,k^b)}{r}, \tag{IA.B.43}$$

where $G(k^a,k^b)$ is given by

$$G(k^a,k^b) = \Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,\xi k^a)}\Big)^\beta\mathcal{B}(k^a,\xi k^a) - \mathcal{B}(k^a,k^b). \tag{IA.B.44}$$

We observe that (IA.B.37) is equivalent to

$$G(k^a,k^b) < 0, \quad \xi k^a > k^b \geq 0. \tag{IA.B.45}$$

Denote $\psi(k^a) = [(2-\frac{\theta_1^b}{\theta_2^b})k^a+pr+c-(\theta_1^b k^a+pr+c)\frac{1}{\beta-1}(\beta-\frac{1}{\theta_2^b})]^{\frac{\beta-1}{\beta}}\frac{1}{\theta_2^b-1}$. We have $\mathcal{B}(k^a,\psi(k^a)) = 0$. Given that $\theta_2^b > 1$, we find $\mathcal{B}_{k^b} = \frac{\beta}{\beta-1}(1-\theta_2^b) < 0$, and $\mathcal{B}(k^a, -\frac{\theta_1^b k^a+pr+c}{\theta_2^b}) = 2(1-\frac{\theta_1^b}{\theta_2^b})k^a + (pr+$

$c)(1-\frac{1}{\theta_2^b})>0$. From $\mathcal{B}_{k^b}<0$, $\mathcal{B}(k^a,\psi(k^a))=0$, and $\mathcal{B}(k^a,-\frac{\theta_1^b k^a+pr+c}{\theta_2^b})>0$, we conclude that $-\frac{\theta_1^b k^a+pr+c}{\theta_2^b}<\psi(k^a)$. Since $\mathcal{X}(k^a,-\frac{\theta_1^b k^a+pr+c}{\theta_2^b})=0$, it follows that $\mathcal{X}(k^a,k^b)>0$ for $k^b\geq\psi(k^a)$ and $k^a\geq 0$. Additionally, from $rp+c>0$, we deduce that $\mathcal{X}(k^a,k^b)>0$ for $k^b\geq 0$ and $k^a\geq 0$. Therefore, $G(k^a,k^b)$, as given by (IA.B.44), is well-defined for any $k^b\in[0\wedge\psi(k^a),\xi k^a)$ with $k^a>0$.

For any $k^b\in[0\wedge\psi(k^a),\xi k^a)$, $k^a>0$, we have

$$\begin{aligned}G_{k^b}(k^a,\xi k^a)=&\beta\frac{\vartheta_2}{\mathcal{X}(k^a,\xi k^a)}\mathcal{B}(k^a,\xi k^a)-\frac{\beta}{\beta-1}(1-\theta_2^b)\\=&\frac{\vartheta_2}{\mathcal{X}(k^a,\xi k^a)}\beta k^a\Big[2-\frac{\theta_1^b}{\theta_2^b}-\theta_1^b+\xi(1-\theta_2^b)\Big]\geq 0,\end{aligned}\tag{IA.B.46}$$

where the inequality follows from the right inequality in (D.9). Additionally, we have

$$G_{k^bk^b}(k^a,k^b)=\beta(\beta-1)\Big(\frac{\vartheta_2}{\mathcal{X}(k^a,\xi k^a)}\Big)^2\Big(\frac{\mathcal{X}(k^a,k^b)}{\mathcal{X}(k^a,\xi k^a)}\Big)^{\beta-2}\mathcal{B}(k^a,\xi k^a).\tag{IA.B.47}$$

Fix $k^a\geq 0$. If $\mathcal{B}(k^a,\xi k^a)=0$, then from $\mathcal{B}_{k^b}<0$, we derive that $\mathcal{B}(k^a,k^b)>0$ for $\xi k^a>k^b$, and thus $G(k^a,k^b)=-\mathcal{B}(k^a,k^b)<0$. If $\mathcal{B}(k^a,\xi k^a)<0$, then we derive from (IA.B.47) that $G_{k^bk^b}(k^a,k^b)<0$, which, together with (IA.B.46), implies that $G_{k^b}(k^a,k^b)>G_{k^b}(k^a,\xi k^a)\geq 0$ for $k^b\in[0\wedge\psi(k^a),\xi k^a)$. Hence, we have $G(k^a,k^b)<G(k^a,\xi k^a)=0$ for any $k^b\in[0\wedge\psi(k^a),\xi k^a)$.

To prove (IA.B.45), it suffices to consider any given $k^a\geq 0$ such that $\mathcal{B}(k^a,\xi k^a)>0$. From (IA.B.47), we deduce that $G(k^a,k^b)$ is convex in k^b, which implies that

$$G(k^a,k^b)<G(k^a,\xi k^a)\vee G(k^a,0\wedge\psi(k^a))=0\vee G(k^a,0\wedge\psi(k^a)),\tag{IA.B.48}$$

for any $k^b\in(0\wedge\psi(k^a),\xi k^a)$. Setting $k^b=0\wedge\psi(k^a)$ in (IA.B.44), we obtain

$$\begin{aligned}&G(k^a,0\wedge\psi(k^a))=\Big(\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,\xi k^a)}\Big)^\beta\mathcal{B}(k^a,\xi k^a)-\mathcal{B}(k^a,0\wedge\psi(k^a))\\\leq&\Big(\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,\xi k^a)}\Big)^\beta[\mathcal{B}(k^a,\xi k^a)-\mathcal{B}(k^a,0\wedge\psi(k^a))]<0,\quad k^a>0,\end{aligned}\tag{IA.B.49}$$

where the first inequality uses $\frac{\mathcal{X}(k^a,0\wedge\psi(k^a))}{\mathcal{X}(k^a,\xi k^a)}\in(0,1]$ and $\mathcal{B}(k^a,0\wedge\psi(k^a))\geq\mathcal{B}(k^a,\psi(k^a))=0$, and the second inequality follows from $\mathcal{B}_{k^b}<0$ and $k^a>0$. Combining (IA.B.48) with (IA.B.49), we conclude that $G(k^a,k^b)<0$ for any $k^b\in[0\wedge\psi(k^a),\xi k^a)$ and $k^a>0$.

Therefore, (IA.B.45) holds. Referring to (IA.B.43), we see that (IA.B.37) holds.

Step 4: We prove that

$$V^a_{k^a}(x,k^a,k^b)<p,\quad (k^a,k^b)\in \mathbf{N}_a(x),\ x>0. \tag{IA.B.50}$$

For $\xi k^a\geq k^b\geq 0$, we have

$$\begin{aligned}&\mathcal{X}(k^a,k^b)-\rho[2k^a+k^b+pr+c]=\rho[(\theta_2^b-1)k^b+(\theta_1^b-2)k^a]\\ \leq&\rho[(\theta_2^b-1)\xi k^a+(\theta_1^b-2)k^a]=\rho k^a(\theta_1^b+\theta_2^b\xi-2-\xi)\leq 0,\end{aligned} \tag{IA.B.51}$$

where the first inequality holds due to $\theta_2^b>1$, and the second inequality follows from $\theta_1^b+\theta_2^b\xi\leq \xi+2-\frac{\theta_1^b}{\theta_2^b}<\xi+2$ as given in the right inequality in (D.9) with $i=b$. For $k^b\geq \xi k^a\geq 0$, we have

$$\begin{aligned}&\mathcal{X}(k^a,k^b)-\rho[2k^a+k^b+pr+c]=\rho[(\theta_2^a-2)k^a+(\theta_1^a-1)k^b]\\ \leq&\rho[(\theta_2^a-2)k^a+(\theta_1^a-1)\xi k^a]=\rho k^a(\theta_1^a\xi+\theta_2^a-2-\xi)\leq 0,\end{aligned} \tag{IA.B.52}$$

where the first inequality uses $\theta_1^a\leq 1$, and the second inequality uses $\theta_1^a\xi+\theta_2^a=\theta_1^b+\theta_2^b\xi\leq \xi+2-\frac{\theta_1^b}{\theta_2^b}$ as given in the right inequality in (D.9) with $i=b$. Combining (IA.B.51), and (IA.B.52), we have

$$\mathcal{X}(k^a,k^b)\leq \rho[2k^a+k^b+pr+c],\quad (k^a,k^b)\in\mathbb{R}^2_+. \tag{IA.B.53}$$

Next, for each fixed $(k^a,k^b)\in\mathbb{R}^2_+$, we prove that $V^a_{k^a}(x,k^a,k^b)<p$ for all $x\in[0,\mathcal{X}(k^a,k^b))$. Using (34), we obtain

$$V^a_{k^a}(x,k^a,k^b)=\frac{x}{r-\mu}-\frac{2k^a+k^b+c}{r}+H^a_{k^a}(k^a,k^b)x^\beta,\quad x\in[0,\mathcal{X}(k^a,k^b)). \tag{IA.B.54}$$

If $H^a_{k^a}(k^a,k^b)\geq 0$, then using $r>\mu$, we see that $V^a_{k^a}(x,k^a,k^b)$ is increasing in $x\in(0,\mathcal{X}(k^a,k^b))$ and

$$V^a_{k^a}(x,k^a,k^b)<V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b)\leq p, \tag{IA.B.55}$$

for any $x\in(0,\mathcal{X}(k^a,k^b))$, where the second inequality uses (IA.B.16) and (IA.B.37).

If $H^a_{k^a}(k^a,k^b)<0$, then $\frac{x}{r-\mu}+H^a_{k^a}(k^a,k^b)x^\beta$ is first increasing and then decreasing in the region $x>0$. Denote $\widehat{x}=\arg\max_{x\in[0,\mathcal{X}(k^a,k^b)]}\frac{x}{r-\mu}+H^a_{k^a}(k^a,k^b)x^\beta$. We have $\widehat{x}>0$. If $\widehat{x}=\mathcal{X}(k^a,k^b)$, then $\frac{x}{r-\mu}+H^a_{k^a}(k^a,k^b)x^\beta$ is increasing in $x\in[0,\mathcal{X}(k^a,k^b)]$. Hence, we conclude from (IA.B.54) that $V^a_{k^a}(x,k^a,k^b)<V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b)\leq p$, $\forall\ x\in(0,\mathcal{X}(k^a,k^b))$. If $\widehat{x}<\mathcal{X}(k^a,k^b)$, then we have $\widehat{x}\in(0,\mathcal{X}(k^a,k^b))$ and $\frac{\partial V^a_{k^a}(x,k^a,k^b)}{\partial x}\mid_{x=\widehat{x}}=0$, which implies that

$\frac{1}{r-\mu} + \beta H^a_{k^a}(k^a, k^b)\widehat{x}^{\beta-1} = 0$. It follows that

$$\frac{\widehat{x}}{r-\mu} + H^a_{k^a}(k^a, k^b)\widehat{x}^{\beta} = \frac{\beta-1}{\beta}\frac{\widehat{x}}{r-\mu} < \frac{\beta-1}{\beta}\frac{\mathcal{X}(k^a, k^b)}{r-\mu} \leq \frac{2k^a + k^b + pr + c}{r}, \quad \text{(IA.B.56)}$$

where the first inequality follows from $r - \mu > 0$ and $\beta > 2$, and the second inequality uses (IA.B.53). Combining (IA.B.56) with (IA.B.54), we have $V^a_{k^a}(x, k^a, k^b) \leq V^a_{k^a}(\widehat{x}, k^a, k^b) < p$ for any $x \in (0, \mathcal{X}(k^a, k^b))$.

Therefore, we have shown $V^a_{k^a}(x, k^a, k^b) < p$, for all $x \in (0, \mathcal{X}(k^a, k^b))$ and $(k^a, k^b) \in \mathbb{R}^2_+$, i.e., for any $(k^a, k^b) \in \mathbf{N}_{ab}(x)$ with $x \geq 0$. Using (IA.B.37), we have $V^a_{k^a}(x, k^a, k^b) = V^a_{k^a}(x, k^a, \widehat{k}^b) < p$ for any $(k^a, k^b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b$ is uniquely determined by the following linear equation:

$$\mathcal{X}(k^a, \widehat{k}^b) = x. \quad \text{(IA.B.57)}$$

Note that $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)) = \mathbf{N}_a(x)$. Then (IA.B.50) holds.

In summary, by combining our analyses in Steps 1-4 with (IA.B.18)-(IA.B.19), we have verified Condition A.1.

□

Proof of Lemma 1 According to Lemma 2 in Internet Appendix IA.A, we have $\theta_2 > 1 > \theta_1$ when $\theta_2 > \theta_1$, and $\theta_1 = \theta_2 = 1$ when $\theta_2 = \theta_1$. Thus, the result in this lemma is a special case of Lemma 4 in Internet Appendix IA.A.

Note that (IA.B.18)-(IA.B.19), and (IA.B.21)-(IA.B.25) also hold for the case $\theta^a_1 = \theta^a_2 = \theta^b_1 = \theta^b_2 = 1$. In particular, for $\mathcal{G}(k^a, k^b)$ as defined in (IA.B.24), we have $\mathcal{G}(k^a, k^b) = 0$ for all k^a and k^b. Substituting it into (IA.B.25), we have $V^a_{k^b}(\mathcal{X}(k^a, k^b), k^a, k^b) = 0$. Thus, for any $(k^a, k^b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$, we derive from (B.6) that $V^a_{k^b}(x, k^a, k^b) = V^a_{k^b}(x, \widehat{k}^a, k^b) = 0$, where $\widehat{k}^a = \frac{x}{\rho} - (rp + c) - k^b$. Combining this with (IA.B.19), we have

$$V^a_{k^b}(x, k^a, k^b) = 0, \quad x \geq \mathcal{X}(k^a, k^b) \quad \text{(IA.B.58)}$$

for any $(k^a, k^b) \in \mathbb{R}^2_+$.

Observe that (IA.B.38)-(IA.B.44) also apply when $\theta^a_1 = \theta^a_2 = \theta^b_1 = \theta^b_2 = 1$. In particular, for $\mathcal{B}(k^a, k^b)$ as defined in (IA.B.42), we have $\mathcal{B}(k^a, k^b) \equiv 0$. Substituting this into (IA.B.43), we find that $V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p$ for any $k^a \geq k^b \geq 0$. Thus $V^a_{k^a}(x, k^a, k^b) = V^a_{k^a}(x, k^a, \widehat{k}^b) = p$ for any $(k^a, k^b) \in \mathbf{I}_b(x) \setminus \mathbf{I}_{ab}(x)$, where $\widehat{k}^b = \frac{x}{\rho} - (rp + c) - k^a$. Combining this with (IA.B.18), we have

$$V^a_{k^a}(x, k^a, k^b) = p, \quad x \geq \mathcal{X}(k^a, k^b) \quad \text{(IA.B.59)}$$

for any $(k^a, k^b) \in \mathbb{R}^2_+$.

Substituting $\theta_1 = \theta_2 = 1$ into (39), we have

$$H^a(k,k) = -\frac{\mathcal{X}(k,k)^{1-\beta}}{\beta(r-\mu)}k. \tag{IA.B.60}$$

Substituting (IA.B.60) and $\mathcal{X}(k^a, k^b) = \rho(k^a + k^b + rp + c)$ into (34) with $k^a = k^b \geq 0$, $x = \mathcal{X}(k^a, k^a)$, we have $V^a(\mathcal{X}(k^a, k^a), k^a, k^a) = pk^a$. Combining this with (IA.B.58) and (IA.B.59), we obtain $V^a(x, k^a, k^b) = pk^a$ for any $x \geq \mathcal{X}(k^a, k^b)$ and $(k^a, k^b) \in \mathbb{R}^2_+$. Hence, (E.1) holds for $i = a$. Due to symmetry, we see that (E.1) also holds for $i = b$.

Finally, by following similar procedures in Step 2 and Step 4 in the proof of Lemma 4, we can demonstrate that (41)-(42) holds for the case $\theta_1 = \theta_2 = 1$.

□

Proof of Proposition 3 To highlight the dependence on (θ_1, θ_2), we let $\varphi^{\theta_1,\theta_2}$ and $V^i(z; \varphi^{\theta_1,\theta_2})$ be respectively the strategy and firm i's value function associated with the linear trigger function given in (43). Due to symmetry, we only prove $V^a(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2})$ is increasing in $\theta_1 + \theta_2 \in [2, 4]$.

For any $\theta_2 \geq \theta_1 > 0$ and strategy $\boldsymbol{\varphi}^{\theta_1,\theta_2}$, the corresponding firm value is characterized by Theorem 4. In particular, the option value $H^i(k^a, k^b)$ is given by (39)-(40).

We can see that $H^a(k,k)$ given by (39) depends on (θ_1, θ_2) only via $\theta_1 + \theta_2$. We denote it as $H^a(k, k; \theta_1 + \theta_2)$. First, we show that $H^a(k, k; \theta_1 + \theta_2)$ is increasing in $\theta_1 + \theta_2 \in (0, 4]$. To this end, we define $f(\theta) := -\frac{(\theta k + rp + c)^{1-\beta}}{\theta}\left[(4(\beta-1) - \beta\theta)k + (rp + c)(\frac{4}{\theta} - 2)\right]$. We can see that $H^a(k, k; \theta) = (\rho)^{1-\beta}\frac{f(\theta)}{\beta(\beta-2)(r-\mu)}$. A direct calculation shows that

$$f'(\theta) = \frac{(\theta k + rp + c)^{-\beta}}{\theta^3}(4 - \theta)\left[(\theta k)^2(\beta - 1)\beta + 2\theta k\beta(rp + c) + 2\Big(rp + c\Big)^2\right].$$

Thanks to $\beta > 2$ and $r > 0$, we see that $f'(\theta) > 0$ for any $\theta \in [0, 4)$, and $f'(\theta) < 0$ for any $\theta > 4$. Hence, $H^a(k, k; \theta)$ is increasing in $\theta \in (0, 4]$.

Second, we can see $\mathcal{K}(x)$ given by (50) is decreasing in $\theta_1 + \theta_2$ for any $x \geq \rho(rp + c)$. To highlight the dependence of $\mathcal{K}(x)$ on $\theta_1 + \theta_2$, we denote it as $\mathcal{K}(x; \theta_1 + \theta_2)$.

Next, we fix $\theta_2 \geq \theta_1 > 0$ and $\widetilde{\theta}_2 \geq \widetilde{\theta}_1 > 0$ with $4 \geq \theta := \theta_1 + \theta_2 > \widetilde{\theta} := \widetilde{\theta}_1 + \widetilde{\theta}_2$. For any $k \geq \mathcal{K}(x; \widetilde{\theta})$, we conclude from (34) that

$$V^a(x, k, k; \boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x, k, k; \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = \Big[H^a(k, k; \theta) - H^a(k, k; \widetilde{\theta})\Big]x^\beta > 0. \tag{IA.B.61}$$

For $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$, we conclude from Theorem 4-(ii) that

$$\begin{aligned} &V^a(x, \mathcal{K}(x;\widetilde{\theta}), \mathcal{K}(x;\widetilde{\theta}); \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) - V^a(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p[\mathcal{K}(x;\widetilde{\theta}) - k] \\ \geq &V^a(x, \mathcal{K}(x;\widetilde{\theta}), \mathcal{K}(x;\widetilde{\theta}); \boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}), \end{aligned} \tag{IA.B.62}$$

where the equality follows from $V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p$ and $V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = 0$ for $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$, and the inequality follows from $V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) \leq p$, $V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) \leq 0$. Then we derive from (IA.B.62) that for $k \in [\mathcal{K}(x;\theta), \mathcal{K}(x;\widetilde{\theta})]$,

$$\begin{aligned} &V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) \\ \geq &V^a(x, \mathcal{K}(x;\widetilde{\theta}), \mathcal{K}(x;\widetilde{\theta}); \boldsymbol{\varphi}^{\theta_1,\theta_2}) - V^a(x, \mathcal{K}(x;\widetilde{\theta}), \mathcal{K}(x;\widetilde{\theta}); \boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) > 0, \end{aligned} \tag{IA.B.63}$$

where the second inequality uses (IA.B.61). Since $V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) = V^a_{k^a}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = p$, and $V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) = V^a_{k^b}(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2}) = 0$ for any $k \in [0, \mathcal{K}(x;\theta)]$, we derive from (IA.B.63) that $V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2}) > V^a(x,k,k;\boldsymbol{\varphi}^{\widetilde{\theta}_1,\widetilde{\theta}_2})$ for any $k \in [0, \mathcal{K}(x;\theta)]$.

As a result, we have shown that $V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2})$ is increasing in $\theta_1 + \theta_2 \in [2,4]$. For any (θ_1,θ_2) satisfying Condition 1, $(\theta_1,\theta_2) \neq (1,1)$, and $(\theta_1,\theta_2) \neq (\theta_1^*,\theta_2^*)$, we have

$$3 > \theta_1^* + \theta_2^* > \theta_1 + \theta_2 > 2,$$

where the first inequality uses $\theta_1^* + \theta_2^* = 3 - \frac{1}{w^*} < 3$, the second inequality uses Lemma 2-(iii), and the last inequality uses Lemma 2-(iv).

Using $V^{\text{mon}}(x,2k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{2,2})$, $V^{\text{duo}}_{\text{open}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{1,2})$, $V^{\text{duo}}_{\text{best}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1^*,\theta_2^*})$, $V^{\text{duo}}_{\text{closed}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{\theta_1,\theta_2})$, $V^{\text{duo}}_{\text{BP}}(x,k,k) = 2V^a(x,k,k;\boldsymbol{\varphi}^{1,1})$, we conclude that (56) holds.

Finally, we prove that (58) holds.

□

Proof of Proposition 4 (57) can be derived from (9), (E.7) and $dK_t^{\text{duo}} = dK_t^a + dK_t^b$.

Using Corollary 2, we can obtain (58).

□

Proof of Corollary 1 Note that (IA.B.38)-(IA.B.43) hold for $\xi = 1$, and any $\theta_1^a = \theta_1^b = \theta_1$, $\theta_2^a = \theta_2^b = \theta_2$, where $\theta_2 \geq \theta_1 > 0$. Recall the function $G(k^a,k^b)$ defined in (IA.B.44) and the equalities in (IA.B.46). If the right inequality in (44) is reversed, then we derive from (IA.B.46) that $G_{k^b}(k^a,k^a) < 0$. Since (IA.B.44) implies $G(k^a,k^a) = 0$, there exists $\epsilon \in (0,k^a)$

such that $G(k^a, k^b) > 0$ for any $k^b \in (k^a - \epsilon, k^a)$. Using (IA.B.43), we have

$$V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) > p \tag{IA.B.64}$$

for any $k^b \in (k^a - \epsilon, k^a)$. Therefore, the right inequality in (44) is a necessary condition for (41).

Note that (IA.B.38)-(IA.B.44) hold for $\xi = 1$, and any $\theta^a_1 = \theta^b_1 = \theta_1$, $\theta^a_2 = \theta^b_2 = \theta_2$, where $\theta_2 \geq \theta_1 > 0$. Recall the function $\mathcal{H}(k^b)$ defined in (IA.B.27). If the left inequality in (44) is reversed, then Lemma 3 implies that the inequality (IA.A.2) is reversed. Then, we derive from the second equality in (IA.B.33) that $\lim_{k\to+\infty} \frac{\mathcal{H}(k)}{k} > 0$. It follows that $\mathcal{H}(k^b) > 0$ for some $k^b > 0$. Combining (IA.B.25) and (IA.B.27), we obtain $V^a_{k^b}(\mathcal{X}(0, k^b), 0, k^b) > 0$. Therefore, the left inequality in (44) is a necessary condition for (42).

□

Proof of Corollary 2 According to Theorem 4, for $x < \mathcal{X}(0, 0) = \rho(rp + c)$, we have $V^i(x, 0, 0) = \Psi^i(x, 0, 0) + H^i(0, 0)x^\beta = H^i(0, 0)x^\beta$. According to the proof of Proposition 3, we have

$$H^a(0, 0) = (\rho(rp + c))^{1-\beta} \frac{-\frac{1}{\theta}\left[(rp + c)(\frac{4}{\theta} - 2)\right]}{\beta(\beta - 2)(r - \mu)},$$

where $\theta = \theta_1 + \theta_2$. Then we derive (E.56).

According to Theorem 4, for $x \geq \mathcal{X}(0, 0) = \rho(rp + c)$, we have

$$V^i(x, 0, 0) = V^i(x, \mathcal{K}(x), \mathcal{K}(x)) - p\mathcal{K}(x).$$

Using $\mathcal{K}(x) = \frac{1}{\theta_1+\theta_2}\left[\frac{x}{\rho} - (rp + c)\right]$, we can derive (E.57).

□

Proof of Theorem 4 The proof is straightforward thanks to Theorems 2-3. □

Proof of Lemma 5 Denote

$$\mathcal{G}(k^a, k^b) = \mathcal{X}(k^a, k^b)(k^a + k^b)^{-1/\gamma} - \rho. \tag{IA.B.65}$$

One can show that (F.13) is equivalent to

$$\begin{aligned} &\mathcal{G}_{k^b}(k^a, k^b)k^b\left[\mathcal{G}(k^a, k^b) + \rho\right] + \mathcal{G}_{k^a}(k^a, k^b)\left[\mathcal{G}(k^a, k^b)\left(\gamma(k^a + k^b) - k^b\right) - \rho k^b\right] \\ &+\mathcal{G}(k^a, k^b)(\mathcal{G}(k^a, k^b) + \rho) = 0. \end{aligned} \tag{IA.B.66}$$

Let $g(k^b,k)=\mathcal{G}(k^a,k^b)k^b$, where $k=k^a+k^b$. Then (IA.B.66) is equivalent to

$$g_{k^b}(k^b,k)(g(k^b,k)k^b+\rho(k^b)^2)+g_k(k^b,k)g(k^b,k)\gamma k=0. \tag{IA.B.67}$$

Note that (IA.B.67) is a quasilinear first-order PDE, and the coefficients of $g_{k^b}(k^b,k)$ and $g_k(k^b,k)$ cannot be zero simultaneously given $k\geq k^b>0$ and $\gamma>1$. The general (local) solution of (IA.B.67) can be obtained by solving the following characteristic equations (see e.g., Chapter 1 in McOwen (2004)):

$$\begin{cases} \frac{dk^b}{gk^b+\rho(k^b)^2}=\frac{dk}{g\gamma k}, \\ dg=0. \end{cases} \tag{IA.B.68}$$

Solving (IA.B.68) yields the following two independent solutions:

$$\frac{1}{g}\ln\frac{k^b}{(g+\rho k^b)k^{\frac{1}{\gamma}}}=\lambda_1, \quad g=\lambda_2, \tag{IA.B.69}$$

where λ_1, λ_2 are constants. Then, the general solution to the PDE (IA.B.66) is characterized by (F.9). Hence, the PDE (F.13) admits a general solution as given by (F.14).

In addition, if $\mathcal{G}(k^a,k^b)$ satisfies (F.9) for any $k^b\geq k^a\geq \underline{k}>0$, then for $k=k^a+k^b$, $g=g(k^b,k)=\mathcal{G}(k^a,k^b)k^b$ satisfies

$$\phi\Big(g,\frac{1}{g}\ln\frac{k^b}{(g+\rho k^b)k^{\frac{1}{\gamma}}}\Big)=0. \tag{IA.B.70}$$

Denote $U(k^b,k,z)=\frac{1}{z}\ln\frac{k^b}{(z+\rho k^b)k^{\frac{1}{\gamma}}}$. Taking partial derivatives with respect to k^b and k in (IA.B.70), we have

$$\phi_{z_1}\Big(g,U(k^b,k,g)\Big)g_{k^b}+\phi_{z_2}\Big(g,U(k^b,k,g)\Big)\big(U_{k^b}+U_zg_{k^b}\big)=0, \tag{IA.B.71}$$

$$\phi_{z_1}\Big(g,U(k^b,k,g)\Big)g_{k}+\phi_{z_2}\Big(g,U(k^b,k,g)\Big)\big(U_{k}+U_zg_{k}\big)=0, \tag{IA.B.72}$$

where $g=g(k^b,k)$. Using $\phi_{z_1}\neq 0$ or $\phi_{z_2}\neq 0$, we derive from (IA.B.71) and (IA.B.72) that $\big(U_k+U_zg_k\big)g_{k^b}=\big(U_{k^b}+U_zg_{k^b}\big)g_k$, which is equivalent to

$$g_{k^b}U_k=g_kU_{k^b}. \tag{IA.B.73}$$

Substituting $U_{k^b}(k^b,k,g)=\frac{1}{g}(\frac{1}{k^b}-\frac{\rho}{g+\rho k^b})$ and $U_k(k^b,k,g)=\frac{-1}{g\gamma k}$ into (IA.B.73), we obtain (IA.B.67) and thus (IA.B.66) holds for any $k^b\geq k^a\geq\underline{k}$.

□

Proof of Lemma 6 For convenience, we denote

$$\mathbf{I}_i(x) = \{(k^a, k^b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : x \geq \mathcal{X}(k^a, k^b), k^i \leq \mathcal{K}(x)\}, \tag{IA.B.74}$$

$$\mathbf{N}_i(x) = \{(k^a, k^b) \in [\underline{k}, \infty) \times [\underline{k}, \infty) : (k^a, k^b) \notin \mathbf{I}_i(x)\}, \tag{IA.B.75}$$

$\mathbf{I}_{ab}(x) = \mathbf{I}_a(x) \cap \mathbf{I}_b(x)$, and $\mathbf{N}_{ab}(x) = \mathbf{N}_a(x) \cap \mathbf{N}_b(x)$.

We only prove the case for $V^a(z)$ as the case for $V^b(z)$ can be treated similarly. It is straightforward to verify that $V^a \in \mathcal{C}^{0,1,1}(\mathbb{Z})$, $V^a_{k^a}(x, k^a, k^b) = p$ for $(k^a, k^b) \in \mathbf{I}_a(x)$, $V^a_{k^b}(x, k^a, k^b) = 0$ for $(k^a, k^b) \in \mathbf{I}_b(x)$, and that Condition A.2 holds. The proof of (E.32) is similar to that in the proof of Theorem 2. Hence, we only need to prove the two inequalities in Condition A.1.

Next, we complete our proof in three steps.

Step 1: We prove that (E.37) *holds.*

Using (34) and (F.2), we obtain

$$V^a_{k^a}(x, k^a, k^b) = \psi^a_{k^a}(k^a, k^b)x + H^a_{k^a}(k^a, k^b)x^\beta, \quad x \in (0, \mathcal{X}(k^a, k^b)], \tag{IA.B.76}$$

where $\psi^a_{k^a}(k^a, k^b) = (k^a + k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a + k^b}{r-\mu}$. Using $r > \mu$ and $\gamma > 1$, we have $\psi^a_{k^a}(k^a, k^b) > 0$. If $H^a_{k^a}(k^a, k^b) \geq 0$, then $V^a_{k^a}(x, k^a, k^b)$ is increasing in $x \in [0, \mathcal{X}(k^a, k^b)]$. Hence, for any $x \in (0, \mathcal{X}(k^a, k^b))$,

$$V^a_{k^a}(x, k^a, k^b) < V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p, \tag{IA.B.77}$$

where the equality uses (E.32).

If $H^a_{k^a}(k^a, k^b) < 0$, then $\psi^a_{k^a}(k^a, k^b)x + H^a_{k^a}(k^a, k^b)x^\beta$ is first increasing and then decreasing in the region $x \geq 0$. Denote $\hat{x} := \arg\max_{x \in [0, \mathcal{X}(k^a, k^b)]}[\psi^a_{k^a}(k^a, k^b)x + H^a_{k^a}(k^a, k^b)x^\beta]$. We have $\hat{x} > 0$. If $\hat{x} = \mathcal{X}(k^a, k^b)$, then $\psi^a_{k^a}(k^a, k^b)x + H^a_{k^a}(k^a, k^b)x^\beta$ is increasing in $x \in [0, \mathcal{X}(k^a, k^b)]$. Hence, we conclude from (IA.B.76) that $V^a_{k^a}(x, k^a, k^b) < V^a_{k^a}(\mathcal{X}(k^a, k^b), k^a, k^b) = p$ for all $x \in (0, \mathcal{X}(k^a, k^b))$. If $\hat{x} < \mathcal{X}(k^a, k^b)$, then we have $\hat{x} \in (0, \mathcal{X}(k^a, k^b))$ and $\frac{\partial V^a_{k^a}(x, k^a, k^b)}{\partial x} \mid_{x=\hat{x}} = 0$, which implies that $\psi^a_{k^a}(k^a, k^b) + \beta H^a_{k^a}(k^a, k^b)\hat{x}^{\beta-1} = 0$. It follows that

$$\begin{aligned} \psi^a_{k^a}(k^a, k^b)\hat{x} + H^a_{k^a}(k^a, k^b)\hat{x}^\beta &= \frac{\beta - 1}{\beta}\psi^a_{k^a}(k^a, k^b)\hat{x} \\ < \frac{\beta - 1}{\beta}(k^a + k^b)^{-\frac{1}{\gamma}-1}&\frac{(1-\frac{1}{\gamma})k^a + k^b}{r-\mu}\mathcal{X}(k^a, k^b) \leq p, \end{aligned} \tag{IA.B.78}$$

where the first inequality uses $\psi^a_{k^a}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}>0$ and $\beta>\gamma>1$, and the second inequality uses (F.10) and (F.4). Combining (IA.B.78) with (IA.B.76), we have $V^a_{k^a}(x,k^a,k^b)\le V^a_{k^a}(\hat{x},k^a,k^b)<p$ for any $x\in(0,\mathcal{X}(k^a,k^b))$. Therefore, (E.37) holds.

Step 2: We prove that (E.41) *holds.*

Using (34) and (F.2), we have

$$V^a_{k^b}(x,k^a,k^b)=\psi^a_{k^b}(k^a,k^b)x+H^a_{k^b}(k^a,k^b)x^\beta,\quad x\in(0,\mathcal{X}(k^a,k^b)], \tag{IA.B.79}$$

where $\psi^a_{k^b}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$. Thus, (E.41) is equivalent to $H^a_{k^b}(k^a,k^b)<\mathcal{H}(k^a,k^b)$ for $k^b>k^a\ge\underline{k}$, where $\mathcal{H}(k^a,k^b):=-\psi^a_{k^b}(k^a,k^b)\mathcal{X}(k^a,k^b)^{1-\beta}$. Additionally, we can derive from (F.7) that $H^a_{k^b}(k^a,k^b)=\mathcal{H}(k^a,k^b)$ for $k^a=k^b$. As a result, (E.41) is equivalent to (E.42).

Then, we only need to show that $H^a_{k^ak^b}(k^a,k^b)>\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b>k^a\ge\underline{k}$. Using (F.6), we have

$$\begin{aligned}
&H^a_{k^ak^b}(k^a,k^b)-\frac{\partial\mathcal{H}(k^a,k^b)}{\partial k^a}\\
=&\frac{(\beta-1)\Big[\mathcal{X}(k^a,k^b)\big[\mathcal{X}_{k^b}(k^a,k^b)\psi^a_{k^a}(k^a,k^b)-\mathcal{X}_{k^a}(k^a,k^b)\psi^a_{k^b}(k^a,k^b)\big]-p\frac{\beta}{\beta-1}\mathcal{X}_{k^b}(k^a,k^b)\Big]}{\mathcal{X}(k^a,k^b)^{\beta+1}}\\
=&\frac{(\beta-1)\Big[\frac{\mathcal{X}(k^a,k^b)}{(k^a+k^b)^{\frac{1}{\gamma}}}\Big[\mathcal{X}_{k^b}(k^a,k^b)+(\mathcal{X}_{k^a}(k^a,k^b)-\mathcal{X}_{k^b}(k^a,k^b))\frac{k^a}{\gamma(k^a+k^b)}\Big]-\rho\mathcal{X}_{k^b}(k^a,k^b)\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}}\\
=&\frac{(\beta-1)\mathcal{X}_{k^b}(k^a,k^b)\Big[\frac{\mathcal{X}(k^a,k^b)}{(k^a+k^b)^{\frac{1}{\gamma}}}\Big[1+\big(\frac{\vartheta_2(k^a,k^b)}{\vartheta_1(k^a,k^b)}-1\big)\frac{k^a}{\gamma(k^a+k^b)}\Big]-\rho\Big]}{(r-\mu)\mathcal{X}(k^a,k^b)^{\beta+1}},
\end{aligned}\tag{IA.B.80}$$

where the second equality uses $\psi^a_{k^a}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(1-\frac{1}{\gamma})k^a+k^b}{r-\mu}$ and $\psi^a_{k^b}(k^a,k^b)=(k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu}$, and the last equality uses $\vartheta_2(k^a,k^b)=\mathcal{X}_{k^a}(k^a,k^b)$, $\vartheta_1(k^a,k^b)=\mathcal{X}_{k^b}(k^a,k^b)$ for $k^b>k^a$.

According to part (ii) of Condition A.6, we have $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}\ge0$ for any $k^b\ge k^a\ge\underline{k}$. By combining (F.13) with (F.8), we obtain $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}=1-\frac{\gamma(k^a+k^b)\mathcal{G}(k^a,k^b)}{k^b(\rho+\mathcal{G}(k^a,k^b))}$ for $k^b\ge k^a\ge\underline{k}$. Since $\mathcal{G}(k^a,k^b)>0$, we have $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}<1$ and $\mathcal{X}(k^a,k^b)>\rho(k^a+k^b)^{\frac{1}{\gamma}}$ for any $k^b\ge k^a\ge\underline{k}$. Further, since $\mathcal{G}(k^a,k^b)<\frac{\rho k^a}{\gamma(k^a+k^b)-k^a}\le\frac{\rho k^b}{\gamma(k^a+k^b)-k^b}$ as implied by part (i) of Condition A.6, we have $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}>0$ for any $k^b\ge k^a\ge\underline{k}$. Hence,

$$\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)}\in(0,1),\quad\forall k^a\ge\underline{k},\quad k^b\ge\underline{k}. \tag{IA.B.81}$$

It follows that $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a}>0$, $\frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^b}>0$ for any $k^b\ge k^a\ge\underline{k}$. By the symmetry of $\mathcal{X}(k^a,k^b)$,

we conclude $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} > 0$, $\frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^b} > 0$ and

$$\vartheta_2(k^a,k^b) > \vartheta_1(k^a,k^b) > 0, \quad \forall k^a \geq \underline{k}, \quad k^b \geq \underline{k}. \tag{IA.B.82}$$

Plugging $\mathcal{X}(k^a,k^b) > \rho(k^a+k^b)^{\frac{1}{\gamma}}$ and $\frac{\vartheta_1(k^a,k^b)}{\vartheta_2(k^a,k^b)} \in (0,1)$ into (IA.B.80), we have $H^a_{k^ak^b}(k^a,k^b) > \frac{\partial \mathcal{H}(k^a,k^b)}{\partial k^a}$ for any $k^b > k^a \geq \underline{k}$. Hence, (E.41) holds.

Step 3: We prove that

$$V^a_{k^b}(x,k^a,k^b) < 0, \quad (k^a,k^b) \in \mathbf{N}_b(x),\ k^a \geq \underline{k},\ k^b \geq \underline{k},\ x > 0. \tag{IA.B.83}$$

We can see from (IA.B.79) that for $x \in (0, \mathcal{X}(k^a,k^b)]$, $\frac{U^a_{k^b}(x,k^a,k^b)}{x} = \psi^a_{k^b}(k^a,k^b) + H^a_{k^b}(k^a,k^b)x^{\beta-1}$ is linear in $x^{\beta-1} > 0$ and satisfies $\lim_{x\to 0+} \frac{U^a_{k^b}(x,k^a,k^b)}{x} = \psi^a_{k^b}(k^a,k^b) = (k^a+k^b)^{-\frac{1}{\gamma}-1}\frac{(-\frac{1}{\gamma})k^a}{r-\mu} < 0$. Using (36) and (E.41), we have $\lim_{x\to\mathcal{X}(k^a,k^b)-} U^a_{k^b}(x,k^a,k^b)/x \leq 0$. Hence, we have $V^a_{k^b}(x,k^a,k^b) = U^a_{k^b}(x,k^a,k^b) < 0$ for any $x \in (0,\mathcal{X}(k^a,k^b))$.

In addition, for any $(k^a,k^b) \in \mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)$ with $k^a \geq \underline{k}$, $k^b \geq \underline{k}$, it follows from equation (B.6) that

$$V^a_{k^b}(x,k^a,k^b) = V^a_{k^b}(x,\widehat{k}^a,k^b) < 0, \tag{IA.B.84}$$

where $\widehat{k}^a$ is determined by $\mathcal{X}(\widehat{k}^a,k^b) = x$, the equality is derived from $V^a_{k^a}(\mathcal{X}(k^a,k^b),k^a,k^b) = p$, and the inequality follows from (E.41). Since $\mathbf{N}_{ab}(x) \cup (\mathbf{I}_a(x) \setminus \mathbf{I}_{ab}(x)) \supseteq \mathbf{N}_b(x)$, we infer (IA.B.83). $\square$

Proof of Lemma 7 We only prove that $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$. The proof of Conditions A.3-A.4 and Footnote 23 is similar to that of Theorem 1.

We derive from (63) and (IA.B.82) that there exists a unique pair $(u^a,u^b) \in \mathbb{R}^2_+$ satisfying (21). In addition, we can see that holds. Hence, Definition 2-(i) holds.

By (51), we have $K^i_t \leq K^i_0 \vee \mathcal{K}(M_t)$, $t \geq 0$. Recall that Condition A.6 implies $\mathcal{X}(k^a,k^b) \geq \rho(k^a+k^b)^{\frac{1}{\gamma}}$, and $\mathcal{K}(x)$ is defined in (65). It follows that there exists a constant $C_1 > 0$ such that $K^i_t \in [K^i_0, C_1(1+|M_t|^\gamma)]$, $\forall\ t \geq 0$, $i = a,b$. Using $F^i(x,k^a,k^b) = x(k^a+k^b)^{-1/\gamma}k^i$, $K^i_t \in [K^i_0, C_1(1+|M_t|^\gamma)]$ and $X_t \in [0,M_t]$, $\forall\ t \geq 0$, we can see there exists a constant $C_2 > 0$ such that $|F^i(X_t,K^a_t,K^b_t)| \leq C_2(1+|M_t|^{\gamma+1})$, $\forall\ t \geq 0$. Recalling (E.8), (E.9), and (E.10), we can see the conditions in Definition 2-(ii) are satisfied, and $(\boldsymbol{\varphi}^a_*, \boldsymbol{\varphi}^b_*) \in \mathcal{S}$. $\square$

Proof of Theorem 7 We can see that Theorem 3 holds for the region $\mathbb{Z} = (0,\infty) \times [\underline{k},\infty) \times [\underline{k},\infty)$. Hence, using Theorem 3, Lemmas 6 and 7, we can see that Theorem 7 holds. $\square$

Proof of Proposition 7 We only need to verify that Condition A.6 holds.

We first consider the case $\phi(z_1, z_2) = z_1 - \lambda$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \frac{\lambda}{k^b}$ solves the equation (F.9). Since $\frac{\rho k^a k^b}{\gamma(k^a+k^b)-k^a} = \frac{\rho}{\gamma/k^a+(\gamma-1)/k^b}$ is increasing in $k^a \geq \underline{k}$ and $k^b \geq \underline{k}$, we have

$$\frac{\rho k^a k^b}{\gamma(k^a + k^b) - k^a} \geq \frac{\rho \underline{k}^2}{2\gamma \underline{k} - \underline{k}} = \frac{\rho \underline{k}}{2\gamma - 1} > \lambda, \tag{IA.B.85}$$

where the second inequality uses $\underline{k} > \frac{\lambda(2\gamma-1)}{\rho}$. It follows that $\frac{\rho k^a}{\gamma(k^a+k^b)-k^a} > \frac{\lambda}{k^b}$. Hence, Condition A.6-(i) holds. Using $\gamma > 1$ and $\rho > 0$, we can see that $\big(\rho + \mathcal{G}(k^a, k^b)\big)(k^a + k^b)^{\frac{1}{\gamma}} = \big(\rho + \frac{\lambda}{k^b}\big)(k^a + k^b)^{\frac{1}{\gamma}}$ is increasing in k^a and thus $\vartheta_2(k^a, k^b) = \frac{\partial \mathcal{X}(k^a,k^b)}{\partial k^a} > 0$ for $k^b \geq k^a \geq \underline{k}$. Using (IA.B.81), we conclude that $\vartheta_1(k^a, k^b) > 0$ from Condition A.6-(i) and $\vartheta_2(k^a, k^b) > 0$. Thus, Condition A.6-(ii) holds.

Next, we consider the case $\phi(z_1, z_2) = \frac{\ln(\lambda/z_1)}{z_1} + z_2$ for a constant $\lambda > 0$. Then, $\mathcal{G} = \sqrt{(\frac{\rho}{2})^2 + \frac{\lambda}{k^b(k^a+k^b)^{\frac{1}{\gamma}}}} - \frac{\rho}{2}$ solves the equation (F.9). Since $\lambda > 0$, we can see $\mathcal{G} > 0$ for any $k^a \geq \underline{k}$, $k^b \geq \underline{k}$. Next, we prove the second inequality in Condition A.6-(i), which is equivalent to

$$\sqrt{\Big(\frac{\rho}{2}\Big)^2 + \frac{\lambda}{k^b(k^a + k^b)^{\frac{1}{\gamma}}}} < \frac{\rho k^a}{\gamma(k^a + k^b) - k^a} + \frac{\rho}{2}. \tag{IA.B.86}$$

Using $\frac{\rho k^a}{\gamma(k^a+k^b)-k^a} + \frac{\rho}{2} = \frac{\rho}{2}\frac{\gamma(k^a+k^b)+k^a}{\gamma(k^a+k^b)-k^a}$, one can see that (IA.B.86) is equivalent to

$$\Big(\frac{\rho}{2}\Big)^2 + \frac{\lambda}{k^b\big(k^a + k^b\big)^{\frac{1}{\gamma}}} < \Big(\frac{\rho}{2}\Big)^2 \Big(\frac{\gamma(k^a + k^b) + k^a}{\gamma(k^a + k^b) - k^a}\Big)^2. \tag{IA.B.87}$$

Using $\left(\frac{\gamma(k^a+k^b)+k^a}{\gamma(k^a+k^b)-k^a}\right)^2 - 1 = \frac{4\gamma(k^a+k^b)k^a}{(\gamma(k^a+k^b)-k^a)^2}$, one can see that (IA.B.87) is equivalent to

$$\lambda < \gamma\rho^2 \frac{k^a k^b (k^a + k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a + k^b) - k^a)^2}. \tag{IA.B.88}$$

We prove that $\frac{k^a k^b (k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}$ is increasing in k^a and k^b in the region $k^b \geq k^a \geq \underline{k}$. Denote $f(k^a, k^b) := \ln\left(\frac{k^a k^b (k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}\right) = \ln k^a + \ln k^b + (1 + \frac{1}{\gamma}) \ln(k^a + k^b) - 2\ln(\gamma(k^a + k^b) - k^a)$. We have

$$\frac{\partial f(k^a, k^b)}{\partial k^a} = \frac{1}{k^a} + \Big(1 + \frac{1}{\gamma}\Big)\frac{1}{k^a + k^b} - \frac{2(\gamma - 1)}{\gamma(k^a + k^b) - k^a}, \tag{IA.B.89}$$

$$\frac{\partial f(k^a, k^b)}{\partial k^b} = \frac{1}{k^b} + \Big(1 + \frac{1}{\gamma}\Big)\frac{1}{k^a + k^b} - \frac{2\gamma}{\gamma(k^a + k^b) - k^a}. \tag{IA.B.90}$$

We can rewrite $\frac{\partial f(k^a,k^b)}{\partial k^b}$ as follows:

$$\frac{\partial f(k^a,k^b)}{\partial k^b} = \frac{(k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b}{k^b(k^a+k^b)(\gamma(k^a+k^b)-k^a)}. \tag{IA.B.91}$$

For any $k^b \geq k^a \geq \underline{k}$, we have $\frac{\partial\left((k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b\right)}{\partial k^b} = 2k^b+(\gamma-1-\frac{1}{\gamma})k^a \geq 2k^a+(\gamma-1-\frac{1}{\gamma})k^a > 0$, and $((k^b)^2+(\gamma-1)(k^a)^2+(\gamma-1-\frac{1}{\gamma})k^ak^b)\mid_{k^a=k^b} = (2\gamma-1-\frac{1}{\gamma})(k^a)^2 > 0$. Then, we derive from (IA.B.91) that $\frac{\partial f(k^a,k^b)}{\partial k^b} > 0$. Comparing (IA.B.89) with (IA.B.90), we have $\frac{\partial f(k^a,k^b)}{\partial k^a} \geq \frac{\partial f(k^a,k^b)}{\partial k^b} > 0$ for any $k^b \geq k^a \geq \underline{k}$.

Hence, $f(k^a,k^b)$, and consequently $\frac{k^ak^b(k^a+k^b)^{1+\frac{1}{\gamma}}}{(\gamma(k^a+k^b)-k^a)^2}$, are increasing in k^a and k^b in the region $k^b \geq k^a \geq \underline{k}$. It follows that (IA.B.88) holds for any $k^b \geq k^a \geq \underline{k}$ if and only if (IA.B.88) holds for $k^a = k^b = \underline{k}$, which is equivalent to $\underline{k} > \frac{1}{2}\left(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\right)^{\frac{\gamma}{\gamma+1}}$. Hence, Condition A.6-(i) holds for any $\underline{k} > \frac{1}{2}\left(\frac{(2\gamma-1)^2\lambda}{\gamma\rho^2}\right)^{\frac{\gamma}{\gamma+1}}$.

Since $\mathcal{G}(k^a,k^b) = \frac{\lambda(k^a+k^b)^{\frac{-1}{\gamma}}}{\sqrt{\left(\frac{\rho k^b}{2}\right)^2+\lambda k^b(k^a+k^b)^{\frac{-1}{\gamma}}}+\frac{\rho k^b}{2}}$, we have

$$\mathcal{G}(k^a,k^b)(k^a+k^b)^{\frac{1}{\gamma}} = \frac{\lambda}{\sqrt{\left(\frac{\rho k^b}{2}\right)^2+\lambda k^b(k^a+k^b)^{\frac{-1}{\gamma}}}+\frac{\rho k^b}{2}}, \tag{IA.B.92}$$

which is increasing in k^a. It follows that $\mathcal{X}(k^a,k^b) = \Big(\rho+\mathcal{G}(k^a,k^b)\Big)(k^a+k^b)^{\frac{1}{\gamma}}$ is increasing in k^a and $\vartheta_2(k^a,k^b) = \frac{\partial\mathcal{X}(k^a,k^b)}{\partial k^a} > 0$ for $k^b \geq k^a \geq \underline{k}$. Using (IA.B.81), we conclude that $\vartheta_1(k^a,k^b) > 0$ from Condition A.6-(i) and $\vartheta_2(k^a,k^b) > 0$. Thus, Condition A.6-(ii) holds. □