The Role of Market Liquidity in Public Debt Sustainability

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I solve a model with a government subject to exceptional spending shocks. Positive debt is sustained solely by the threat of losing access to borrowing. The key assumption is that rapid asset sales used to finance exceptionally large government expenditures entail liquidity costs, giving a comparative advantage to more liquid sovereign bond issuance. Surprisingly, less patient governments can sustain more debt.

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1 Introduction

Why do governments repay their debts and why do financial markets believe they will do so? Are there natural limits to this ability to borrow? These questions have generated renewed interest after recent substantial spending increases in many jurisdictions have pushed up public debt, often starting from already elevated levels.

Since direct enforcement by creditors is limited, a common explanation is the cost associated to default.[1] In the seminal model of Eaton and Gersovitz (1981) the threat of total financial autarky is sufficient to incentivize repayments. However, this punishment is less realistic than the impossibility to borrow, which Bulow and Rogoff (1989b) found to be insufficient to sustain positive debt. This result and its proof have been well studied, and relies on the existence of opportunities to default and to guarantee larger consumption almost surely with cash-in-advance financial contracts. This replication strategy relies on the fact that (i) restriction to cash-in-advance contracts does not dampen market completeness, and (ii) cash-in-advance contracts are as liquid as debt contracts.[2] In this paper, (i) holds because only non-contingent financial contracts are considered: only non-contingent debt is available, and cash-in-advance contracts are simple risk-free assets.[3] However, this paper departs from the assumption (ii) by assuming that large asset sales are more frictional than large debt emissions.

I model a benevolent government in continuous time that receives a constant income, which can be used for normal day-to-day spending, debt repayments, or asset accumulation. Spending opportunities – such as wars or financial crises – arise according to a Poisson process. Expenditures on these opportunities generate high social value and can be financed either through debt issuance or through asset fire sales, which incur a proportional cost. The government is subject to a debt limit and continuously chooses how to allocate its income, how much to spent on opportunities when they arise, and whether to default. This dynamic optimization problem can be written with a Hamilton-Jacobi-Bellman (HJB) equation.

First, I solve the HJB equation and find the optimal policy for any exogenous debt

[1] Schumacher, Trebesch, and Enderlein (2021) argue sovereign debt is becoming more enforceable in courts.

[2] See Bloise, Polemarchakis, and Vailakis (2017) for a in-depth treatment of (i).

[3] Auclert and Rognlie (2016) showed that Bulow and Rogoff (1989b) replication proof and result still hold.

limit, conditional on the government never defaulting.

Second, I allow the government to default. Following the approach of Bulow and Rogoff (1989b), if a default ever occurs, the government will not able to borrow until the end of time – while still being able to accumulate assets. Losing access to borrowing is costly since it allows to circumvent asset fire-sale costs.[4] The endogenous debt limit is the maximum amount of debt that can be sustained by this threat. In other words, the government never defaults when it has a lower debt than this endogenous debt limit, and immediately defaults when its debt is higher than this endogenous debt limit. I completely characterize the debt sustainability region, i.e. the set of parameters that allow to sustain a positive debt (Proposition 1). I also study the impact of each parameter on the size of the endogenous debt limit (Proposition 2). When there is no cost associated with asset fire sales, no debt can be sustained. The higher this cost, the larger the debt limit is. The more frequent spending opportunities are, and the higher their social value is compared to normal spending, the larger the debt limit is. Lower interest rates make debt cheaper and asset accumulation more expensive, and so yield a larger debt limit. More surprisingly, for a large set of parameter values, a less patient government can sustain a larger debt limit.

Finally, I show that this mechanism is able to sustain large debt to GDP ratios.

The central assumption of this paper is that large asset sales are more frictional than large debt emissions when financing public expenditures that are large and unanticipated, such as wars, financial crises, recessions, or pandemics. When is this assumption justified? It may not be justified for small countries which may use, without concern for liquidity drying up, large countries' debt as liquid assets.[5] [6] [7] Accordingly, this paper focuses on large countries. Historically, spending shocks amounting to several percentage points of annual GDP over a period of months or a few years were indeed financed primarily through public debt, and government bond prices have resisted large debt emissions. Volumes as large have not been observed for other asset classes,

[4] To avoid these fire-sale costs, governments can also spend less or raise taxes. Both imply welfare losses that are indirect consequences of fire-sale costs.

[5] This is especially true for countries whose debt is already denominated in the currencies of large economies.

[6] Other asset classes can be used, e.g., Norway tapped around 3% of its sovereign wealth fund to cope with Covid public expenditures. See Annual address by Governor Øystein Olsen to the Supervisory Council of Norges Bank on Thursday, 18 February 2021.

[7] Liquidity could still dry up if many small countries sell at the same time.

considered as less liquid (and more risky) than government bonds. Large public asset sales have occurred in developed countries, but they have generally been smaller in scale and, crucially, anticipated and planned (e.g., the UK privatizations of the 1980s).

Related Literature

The paper contributes to the broader effort to understand why governments are able to borrow and to identify the limits of public indebtedness. This paper studies a departure of Bulow and Rogoff (1989b) by introducing a fire-sale friction. Other departures have been studied. Blanchard (2019); Domeij and Ellingsen (2018); Hellwig and Lorenzoni (2009) study interest rates that are lower than growth rate and allow carrying large public debt at small cost. Bloise et al. (2017) allows for incomplete financial markets such that some insurance opportunities are not available after default. In Bloise, Polemarchakis, and Vailakis (2021) interest rates are low and financial markets are incomplete.

Reputation mechanisms, although elegant and simple, have been found to be insufficient to sustain significant debt (Uribe & Schmitt-Grohé, 2017). Since Arellano (2008), a standard assumption in quantitative models that aim to match observed debt levels is that default entails large costs, modeled as output losses. Hébert and Schreger (2017) found causal evidence of these costs. Many mechanisms – potentially operating simultaneously – linking default to costs have been studied. In Cole and Kehoe (1998), a loss of reputation following a default erodes the trust in government that is necessary for carrying out other governmental functions. Rose (2005) shows that default causes a decline in trade.

My model features no default in equilibrium and bonds are priced with the risk-free rate. Of course, in reality there are many defaults, and sovereign spreads movements. An important topic is to document spread and default patterns and find models able to replicate them. Defaults can be triggered by shocks to revenues (Arellano, 2008), by shocks to the after-default continuation value (Aguiar, Amador, Hopenhayn, & Werning, 2019; Aguiar, Chatterjee, Cole, & Stangebye, 2022), or by commitment governments separating themselves from opportunistic governments (Amador & Phelan, 2021). Defaults can be excusable and part of a risk sharing arrangement with lenders (Dovis, 2019; Grossman & Van Huyck, 1985), Defaults can also be self-fulfilling (Aguiar

et al., 2022; Calvo, 1988; Cole & Kehoe, 2000; Lorenzoni & Werning, 2019).

In our model, the government's only decision – aside from the choice of spending levels – is whether to default. Many other decisions have been modeled. Chari, Dovis, and Kehoe (2020) studies financial repression. Aguiar et al. (2019); Chatterjee and Eyigungor (2012) study maturity management. Bulow and Rogoff (1991) studies bond buy-backs. (Bulow & Rogoff, 1989a; Yue, 2010) study debt renegotiation. (Arellano, Mateos-Planas, & Ríos-Rull, 2023) models partial default.

The rest of the article is organized as follows. Section 2 lays down the setup. Section 3 derives the optimal government policy for any exogenous debt limit, conditional on the government never defaulting. Section 4 allows for defaults and studies the endogenous debt limit. Section 5 concludes.

2 The Setup

We study a government in continuous time, which earns a constant income y and discounts the future at rate ρ. Most of the time this income y can be spent with a marginal social value of 1. With a Poisson arrival rate λ, a spending opportunity appears and offers a marginal social value of public spending of $g > 1$. Spending opportunities are short lived and must be financed quickly with either debt emissions or asset sales. The key assumption of this paper is that these asset sales are instantaneous fire sales and incurs a proportional loss c. This assumption favors the use of debt emissions.

At each time t, the net asset position of the government is denoted a_t. The return on assets (when $a_t \geq 0$) and the interest rate on debt (when $a_t \leq 0$) are both equal to r. We assume a lower bound $\overline{a} \leq 0$ on the net asset position. In other words, $-\overline{a} \geq 0$ is a debt limit. Ruling out Ponzi schemes yields, without loss of generality,

$$-\frac{y}{r} \leq \overline{a} \leq a_t \tag{1}$$

At most times t there is no spending opportunity and the government spends $s_t \geq 0$ on normal spending. Therefore, the low of motion of the asset position a_t is

$$\dot{a}_t = y + ra_t - s_t \tag{2}$$

When an spending opportunity appears at a time t, then the government chooses to jump to a new asset level $a' \leq a_t$ and so spends

$$(1-c)\left[(a_t)^+ - (a')^+\right] + \left[(a')^- - (a_t)^-\right] \tag{3}$$

on this spending opportunity.[8] The first term represents assets fire sales and the second term represents new debt emissions.

3 Forbidding default

We denote by $V_{\overline{a}}(a)$ the value of a government with an asset position a and a debt limit $\overline{a}$ which is not authorized to default.

The problem of the government is stationary and can be written with an Hamilton-Jacobi-Bellman (HJB) equation. For any net asset position $a > \overline{a}$,

$$\rho V_{\overline{a}}(a) = \max_{s,a'} \quad s + V'_{\overline{a}}(a)\left[y + ra - s\right] + \lambda\left[g(1-c)\left[(a)^+ - (a')^+\right] + g\left[(a')^- - (a)^-\right] + V_{\overline{a}}(a') - V_{\overline{a}}(a)\right] \tag{4}$$

$$s.t. \qquad 0 \leq s \quad and \quad \overline{a} \leq a' \leq a$$

When $a = \overline{a}$, the government cannot spend more than $y + r\overline{a}$ on normal spending and cannot spend anything on spending opportunities. The HJB equation is then

$$\rho V_{\overline{a}}(\overline{a}) = \max_{s} \quad s + V'_{\overline{a}}(\overline{a})\left[y + r\overline{a} - s\right] \tag{5}$$

$$s.t. \qquad 0 \leq s \leq y + r\overline{a}$$

In this model with linear utility, if $r > \rho$, the optimal government policy is to never spend anything and to back-load all resources indefinitely, so we assume $r \leq \rho$. To derive a closed-form solution $V_{\overline{a}}(a)$ to the HJB equations (4) and (5), we adopt a guess and verify approach. The solution varies with the social value of spending opportunity g. I treat three cases in turn.

[8] We use the standard notations $x^+ \equiv \max(0, x)$ and $x^- \equiv -\min(x, 0)$.

High g. Consider a social value of spending opportunity g high enough such that

$$\rho - r \leq [g(1-c) - 1]\lambda \tag{6}$$

The right-hand-side term in (6) represents the instantaneous expected gain of saving and waiting to spend on a spending opportunity rather than on normal spending. The left-hand-side is the cost of waiting. Therefore, the optimal policy is to spend as much as possible on spending opportunities, i.e. it is optimal to never spend anything on normal spending (i.e. $s_t = 0$), to pay back debt and save. When an spending opportunity arises, the optimal policy is to spend as much as possible (because $r \leq \rho$), so the net asset position a jumps to the lower bound $\overline{a}$. Say this happens at $t = 0$, then, for all $t \geq 0$, $a_t = [\overline{a} + \frac{y}{r}]e^{rt} - \frac{y}{r}$ until a spending opportunity arises. So $V_{\overline{a}}(\overline{a})$ solves

$$V_{\overline{a}}(\overline{a}) = \int_{t=0}^{T} \Big[\, g(a_t - \overline{a}) + V_{\overline{a}}(\overline{a}) \,\Big] \lambda e^{-(\lambda+\rho)t}\, dt + \int_{t=T}^{+\infty} \Big[\, g(1-c)a_t + g(-\overline{a}) + V_{\overline{a}}(\overline{a}) \,\Big] \lambda e^{-(\lambda+\rho)t}\, dt \tag{7}$$

where $T \equiv -\frac{\ln(1+\frac{r\overline{a}}{y})}{r}$ is such that $a_T = 0$. Calculus in the Appendix yields

$$V_{\overline{a}}(\overline{a}) = \frac{gy}{\rho} \frac{\lambda}{\lambda + \rho - r} \Big[\, 1 + \frac{r\overline{a}}{y} - c(1 + \frac{r\overline{a}}{y})^{\frac{\rho+\lambda}{r}} \,\Big] \tag{8}$$

With this policy (i.e. $s = 0$ and $a' = \overline{a}$), the HJB equation (4) becomes an ordinary differential equation with the value function $V_{\overline{a}}(\cdot)$ as the unknown and $V_{\overline{a}}(\overline{a})$ as the initial condition.[9] For $a \in [\overline{a}, 0]$, the solution is

$$V_{\overline{a}}(a) = \frac{\lambda g}{\rho(\lambda + \rho - r)} \Big[y + r\overline{a} + \rho(a - \overline{a}) \Big] - \frac{\lambda g c y}{(\rho + \lambda)(\lambda + \rho - r)} \left[\left(1 + \frac{ra}{y}\right)^{\frac{\rho+\lambda}{r}} + \frac{\lambda}{\rho} \left(1 + \frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}} \right] \tag{9}$$

When $a \geq 0$, each marginal increase in asset will be saved and spent on a spending opportunity when it arises, so it has a social value of $\frac{\lambda g(1-c)}{\lambda+\rho-r}$, which yields, for $a > 0$,

$$V_{\overline{a}}(a) = V_{\overline{a}}(0) + \frac{\lambda g(1-c)}{\lambda + \rho - r} a \tag{10}$$

[9] The assumptions of the standard Cauchy–Lipschitz theorem are satisfied, and so existence and uniqueness of the solution of the differential equation are guaranteed.

using (9) to replace $V_{\bar{a}}(0)$ yields, for $a > 0$,

$$V_{\bar{a}}(a) = \frac{\lambda g}{\rho(\lambda+\rho-r)}\left[y-(\rho-r)\bar{a}\right] - \frac{\lambda g c y}{(\rho+\lambda)(\lambda+\rho-r)}\left[1+\frac{\lambda}{\rho}\left(1+\frac{r\bar{a}}{y}\right)^{\frac{\rho+\lambda}{r}}\right] + \frac{\lambda g(1-c)}{\lambda+\rho-r}a \tag{11}$$

We verify in the Appendix that equations (9) and (11) provide a solution to the HJB equations (4) and (5) for all asset level $a \geq \bar{a}$.[10]

Medium g. Consider a social value of spending opportunity g such that

$$[g(1-c)-1]\lambda \leq \rho - r \leq [g-1]\lambda \tag{12}$$

Then it is not optimal to accumulate assets for spending opportunities. However, it is still optimal to borrow as much as possible to finance spending opportunities. The optimal policy is to pay back debt and to spend all positive wealth instantly on normal spending. $V_{\bar{a}}(\bar{a})$ solves

$$V_{\bar{a}}(\bar{a}) = \int_{t=0}^{T}\Big[\, g\left[a_t - \bar{a}\right] + V_{\bar{a}}(\bar{a})\,\Big]\lambda e^{-(\lambda+\rho)t}\,dt \tag{13}$$

$$+\int_{t=T}^{+\infty}\Big[\, g\left[-\bar{a}\right]e^{-\rho t} + V_{\bar{a}}(\bar{a})e^{-\rho t} + \int_{s=T}^{t} y e^{-\rho s}\,ds\,\Big]\lambda e^{-\lambda t}\,dt$$

Calculus in the Appendix yields

$$V_{\bar{a}}(\bar{a}) = \frac{gy\lambda}{\rho(\lambda+\rho-r)}\left(1+\frac{r\bar{a}}{y}\right) - \frac{y\left[\lambda(g-1)+r-\rho\right]}{\rho(\lambda+\rho-r)}\left(1+\frac{r\bar{a}}{y}\right)^{\frac{\rho+\lambda}{r}} \tag{14}$$

With this policy, the solution to the HJB equation is for $a \in [\bar{a}, 0]$,

$$V_{\bar{a}}(a) = \frac{\lambda g}{\rho(\lambda+\rho-r)}\left[y+r\bar{a}+\rho(a-\bar{a})\right] - \frac{y\left[\lambda(g-1)+r-\rho\right]}{(\rho+\lambda)(\lambda+\rho-r)}\left[\left(1+\frac{ra}{y}\right)^{\frac{\rho+\lambda}{r}} + \frac{\lambda}{\rho}\left(1+\frac{r\bar{a}}{y}\right)^{\frac{\rho+\lambda}{r}}\right]. \tag{15}$$

When $a \geq 0$, each marginal increase in asset is spent immediately with a social value of 1, which yields, for $a > 0$,

$$V_{\bar{a}}(a) = V_{\bar{a}}(0) + a \tag{16}$$

Again, one could check that the HJB equations (4) and (5) are verified.

[10] Note that $V_{\bar{a}}(\cdot)$ is differentiable at $a = 0$.

Low g. Consider a social value of spending opportunity g low enough such that

$$[g-1]\lambda \leq \rho - r \tag{17}$$

The optimal policy is then to spend all resources instantly on normal spending. The solution to the HJB equation is

$$V_{\overline{a}}(a) = \frac{y + r\overline{a}}{\rho} + (a - \overline{a}) \tag{18}$$

Again, one could check that the HJB equations (4) and (5) are verified.

4 Debt sustainability

We now allow the government to default. Following the approach of Bulow and Rogoff (1989b), if a default ever occurs, the government will not able to borrow until the end of time – while still being able to accumulate assets. In our setup, default translates into $\overline{a} = a = 0$. Since $V_{\overline{a}}(\cdot)$ is increasing, a government never chooses to default if and only if

$$V_{\overline{a}}(\overline{a}) \geq V_0(0) \tag{19}$$

4.1 Debt sustainability and the social value of spending opportunity g

When the social value of spending opportunity g is high (i.e. (6) holds), equation (19) is written

$$V_{\overline{a}}(\overline{a}) = \frac{g}{\rho}\frac{\lambda}{\lambda+\rho-r}\left[\, y + r\overline{a} - cy\left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}} \,\right] \quad \geq \quad \frac{g}{\rho}\frac{\lambda}{\lambda+\rho-r}\left[\, y(1-c) \,\right] = V_0(0) \tag{20}$$

Since $\overline{a} \mapsto V_{\overline{a}}(\overline{a})$ is strictly concave, a necessary and sufficient condition for the existence of a strictly negative endogenous asset limit $\overline{a} < 0$ such that this inequality holds is that $\frac{dV_{\overline{a}}(\overline{a})}{d\overline{a}}\Big|_{\overline{a}=0} < 0$, or equivalently,

$$r < c(\rho + \lambda) \tag{21}$$

Similarly, when the social value of spending opportunity g is medium (i.e. (12) holds), $\frac{dV_{\overline{a}}(\overline{a})}{d\overline{a}}\Big|_{\overline{a}=0} < 0$ is equivalent to

$$\rho < \lambda(g-1) \tag{22}$$

When the social value of spending opportunity g is low (i.e. (17) holds), for all a, $\frac{dV_{\overline{a}}(\overline{a})}{d\overline{a}}\Big|_{\overline{a}=a} = \frac{r}{\rho} > 0$, so debt is never sustainable.

4.2 Existence of debt and the fire-sale costs c

It will be convenient to distinguish two cases for the fire-sale costs c. Fire-sale costs are *low* when

$$c \leq \overline{c} \equiv \frac{r}{\rho}\left(1 - \frac{1}{g}\right) \in (0,1) \tag{23}$$

and fire-sale costs are *high* when $c > \overline{c}$.

Low c. When the fire-sale costs c are low, we have

$$\frac{\rho - r}{g(1-c)-1} \leq \frac{\rho}{g-1} \leq \frac{r}{c} - \rho \tag{24}$$

So, if g is high (i.e. (6) holds), then the sustainability condition (21) is verified if and only if

$$\frac{r}{c} - \rho < \lambda \tag{25}$$

So, if g is medium (i.e. (12) holds), then the sustainability condition (22) is never verified.

High c. When the fire-sale costs c are high, we have

$$\frac{r}{c} - \rho < \frac{\rho}{g-1} < \frac{\rho - r}{g(1-c)-1} \tag{26}$$

If g is high (i.e. (6) holds), then

$$\frac{r}{c} - \rho < \frac{\rho - r}{g(1-c)-1} \leq \lambda \tag{27}$$

and so the sustainability condition (21) is always met.

If g is medium (i.e. (12) holds), then the sustainability condition (22) is verified when

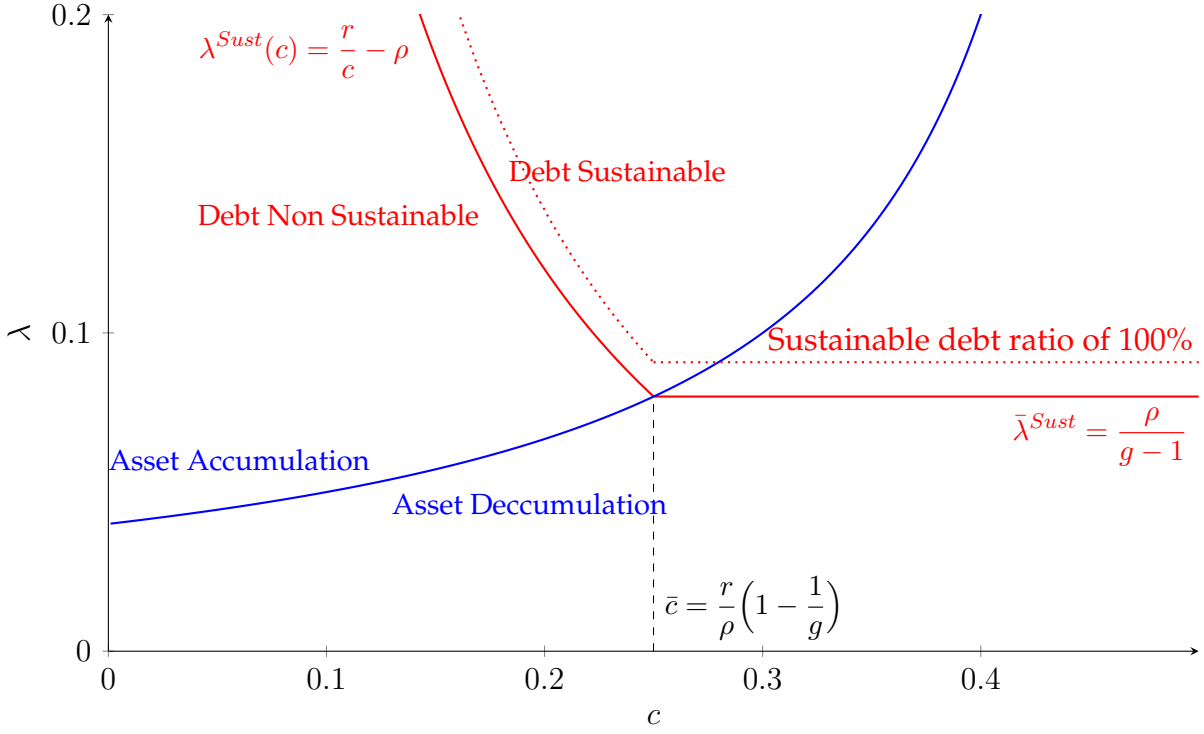
$$\frac{\rho}{g-1} < \lambda \tag{28}$$

All previous results are summarized in the following Proposition and illustrated in Figure 1.

Proposition 1. ***(Debt Sustainability)***

- *When c is low (i.e. (23) holds, there exists a positive sustainable debt iff $\lambda > \frac{r}{c} - \rho$.*
- *When c is high, there exists a positive sustainable debt iff $\lambda > \frac{\rho}{g-1}$.*

Figure 1: Debt sustainability region when in (c, λ) plan with $r = 4\%,\ \rho = 8\%,\ g = 2$.



4.3 Endogenous debt limit

Until now, we have focused on the sustainability of an infinitesimally small level of debt. We now turn to the *endogenous debt limit*, defined as the maximum level of debt that can be sustained. In this model, it corresponds to the smallest solution to

$$V_{\overline{a}}(\overline{a}) = V_0(0) \tag{29}$$

denoted $\tilde{a}(\lambda, c, g, r, \rho)$. The comparative statics of this endogeneous debt limit are summarized in the following Proposition.

Proposition 2. ***(Endogenous Debt Limit)***

Consider parameters (λ, c, g, r, ρ) such that a positive debt is sustainable, i.e. $\tilde{a}(\lambda, c, g, r, \rho) < 0$. If g is high (i.e. (6) holds), then

$$\tilde{a}(\ \bar{\lambda}\ ,\ \bar{c}\ ,\ \bar{\bar{g}}\ ,\ \overset{+}{r}\ ,\ \bar{\rho}\ )$$

If g is medium (i.e. (12) holds), then

$$\tilde{a}(\ \bar{\lambda}\ ,\ \bar{\bar{c}}\ ,\ \bar{g}\ ,\ \overset{+}{r}\ ,\ \overset{+}{\rho}\ )$$

The proof is in the Appendix.

We now discuss Propositions 1 and 2, and Figure 1.

Arrival rate of spending opportunities λ. When the arrival rate of spending opportunities λ increases, the debt sustainability region (in the (c, g, r, ρ) space) increases, and more debt can be sustained. When spending needs are more often shocked, access to borrowing has more value and losing it becomes a larger threat that helps sustain debt. When $\lambda \leq \bar{\lambda}^{Sust} \equiv \frac{\rho}{g-1}$, no debt is sustainable.[11]

Fire-sale cost c . First, when the fire-sale cost c is zero, no debt is sustainable. The standard replication proof of Bulow and Rogoff (1989b) stands.[12] Second, when $c \in [0, \bar{c}]$ increases, the debt sustainability region (in the (λ, g, r, ρ) space) increases, and more debt can be sustained. This shows that it is indeed the financial market friction that allows debt sustainability. Third, increasing $c > \bar{c}$ has no impact on the debt sustainability region because the government with parameters (λ, c, g, r, ρ) close to sustainability must have a medium g, never accumulates assets and never pays the fire-sale cost c. More debt can be sustained only if g is high.

[11] Of course, as opportunities become more frequent, the government spends less on each and the fire-sale cost assumption becomes less valid.

[12] Optimal times t to default and start a replication strategy are when $a_t = \bar{a}$.

Social value of spending opportunities g. First, when $c \in (0, \overline{c})$, the social value of spending opportunities g has no impact on the debt sustainability region (in the (λ, c, r, ρ) space with $c \in (0, \overline{c})$), nor on endogenous debt limits. When $c \in (0, \overline{c})$, a government with parameters (λ, c, g, r, ρ) in or close to the sustainability region must have a high g. Thus, it spends all its income y on spending opportunities, whether or not it has access to borrowing. Therefore, changes in g have a proportional impact on all utility streams, whether or not the government has access to borrowing. Thus, default decisions are unchanged and so is debt sustainability. Second, when $c > \overline{c}$, a government with parameters (λ, c, g, r, ρ) close to sustainability must have a medium g, which implies it never accumulates positive assets and spends all its income y on normal spending of social value 1 instead. Increasing g makes access to borrowing – the only way spending opportunities are worth financing – relatively more valuable, and the debt sustainability region (in the (λ, c, r, ρ) space with $c > \overline{c}$) increases. Similarly, more debt can be sustained only if g is medium.

Interest rate r. With a higher interest rate r, less debt can be sustained. The debt sustainability region (in the (λ, c, g, ρ) space) decreases only when $c \in (0, \overline{c})$. In contrast, when $c > \overline{c}$, r has no effect on the sustainability region. The interest rate r affects debt sustainability through two distinct mechanisms. First, a higher interest rate r makes accumulating assets a better way to cope with spending shocks, therefore losing access to borrowing is a less effective threat. Second, it makes debt more costly. Both mechanisms negatively affect debt sustainability. However, a government with parameters (λ, c, g, r, ρ) close to sustainability and such that $c > \overline{c}$ must have a medium g. Therefore, it does not accumulate assets, and the first mechanism is not at play. When it comes to the existence of sustainable debt (however small), the second mechanism is not at play either. Indeed, if some debt is sustainable for a given r, then, for any $r' > r$, a sufficiently smaller debt would also be sustainable because interest payments can be made arbitrarily small.

Discount rate ρ. First, when $c \in (0, \overline{c})$, a higher discount rate ρ increases the debt sustainability region (in the (λ, c, g, r) space with $c \in (0, \overline{c})$), and more debt can be sus-

tained. This surprising result requires some explanation.[13] Consider a government contemplating a default. Defaulting implies saving $-r\overline{a}$ on interest payments until the end of time. Not defaulting implies saving on fire-sale costs c until the end of time. The interest savings are constant, whereas savings on fire-sale costs c are front-loaded since fire-sale costs arise later as the government must pay back its debt $\overline{a}$ first. Ceteris paribus, the less patient the government, the more it values short-term savings on fire-sale costs c relative to the permanent savings on interest payments $-r\overline{a}$, and the less appealing default becomes. In contrast, when $c > \overline{c}$, a higher discount rate ρ decreases the debt sustainability region (in the (λ, c, g, r) space with $c > \overline{c}$), and less debt can be sustained. When $c > \overline{c}$, a government with parameters (λ, c, g, r, ρ) close to sustainability must have a medium s, and so never accumulates positive assets and spends all its income y on normal spending instead. Such a government at its debt limit must pay back its debt to be able to spend on opportunities, and so spending is back-loaded, whereas a defaulting government spends its constant income y on normal spending. Ceteris paribus, the less patient the government, the less it values back-loaded spending and the more appealing default becomes.

4.4 Sustaining large debt ratios

As the Figure 1 displays, the fire-sale cost allows to sustain large debt to GDP ratios, for parameters that are not very far from the sustainability frontier. Using equation (32), one gets that, when $c < \overline{c}$, sustaining at least a debt to GDP ratio $d \equiv \frac{-\overline{a}}{y}$ requires

$$\lambda \quad \geq \quad r\frac{\ln(1-\frac{rd}{c})}{\ln(1-rd)} - \rho \quad = \quad \lambda^{Sust}(c) + \frac{1}{2}\frac{r^2}{c^2}(1-c)d + \circ(d^2) \tag{30}$$

When parameters are such that some debt is sustainable, the term $\frac{1}{2}\frac{r^2}{c^2}(1-c)$ is typically small compared to 1. This implies that, to sustain large debt to GDP ratios (e.g. $d = 1$), λ does not have to be much higher than $\lambda^{Sust}(c)$. The same logic applies when $c > \overline{c}$.

[13] According to Folk Theorems, patience (low ρ) helps sustain cooperative behaviors. This difference can be explained by the fact that leaving the cooperative equilibrium here means defaulting on debt, and it does not yield an immediate utility gain for the government but rather long term savings on interest rates payments.

5 Conclusion

The main contribution of this paper is to introduce a realistic friction, fire-sale costs, that allows sustaining large sovereign debt with a reputation punishment mechanism à la Bulow and Rogoff (1989b) that have been found insufficient to sustain large debt (Arellano, 2008).

Estimating government debt limits for the many countries that have not yet reached theirs is difficult, as observations are unavailable and extrapolations from borrowing-constrained countries raise concerns about external validity. Theory papers such as this one may be helpful.

The mechanism in this paper requires that a government faces similar fire-sale costs, whether or not it has access to borrowing. One concern is that these frictions could be much lower when the government has no access access to borrowing. Indeed, a Lucas-critique argument may apply: assets are not liquid by nature, they are liquid because many agents trade them. In the absence of government bonds, economic agents would substitute into another asset class – such as corporate bonds – to conduct the transactions they currently carry out with government bonds, making that asset class very liquid.

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Appendix

Proofs for expressions (8) and (14)

Proof of (8).

$$V_{\overline{a}}(\overline{a}) = \int_{t=0}^{+\infty} \Big[\, g(a_t - \overline{a}) + V_{\overline{a}}(\overline{a}) \,\Big] \lambda e^{-(\lambda+\rho)t}\, dt - \int_{t=T}^{+\infty} gca_t \lambda e^{-(\lambda+\rho)t}\, dt$$

$$V_{\overline{a}}(\overline{a}) = g\lambda[\overline{a}+\frac{y}{r}] \int_{t=0}^{+\infty} \Big[\, e^{rt} - 1 \,\Big] e^{-(\lambda+\rho)t}\, dt + V_{\overline{a}}(\overline{a}) \frac{\lambda}{\lambda+\rho} - gc\lambda \int_{t=T}^{+\infty} \Big[\, [\overline{a}+\frac{y}{r}] e^{rt} - \frac{y}{r} \,\Big] e^{-(\lambda+\rho)t}\, dt$$

$$V_{\overline{a}}(\overline{a}) \frac{\rho}{\lambda+\rho} = g\lambda[\overline{a}+\frac{y}{r}] \Big[\, \frac{1}{\lambda+\rho-r} - \frac{1}{\lambda+\rho} \,\Big] - gc \Big[\, [\overline{a}+\frac{y}{r}] \frac{e^{-(\lambda+\rho-r)T}}{\lambda+\rho-r} - \frac{y}{r} \frac{e^{-(\lambda+\rho)T}}{\lambda+\rho} \,\Big]$$

$$V_{\overline{a}}(\overline{a}) \frac{\rho}{\lambda+\rho} = g\lambda[\overline{a}+\frac{y}{r}] \frac{r}{(\lambda+\rho-r)(\lambda+\rho)}$$

$$-gc\frac{y}{r} \Big[\, [1+\frac{\overline{a}r}{y}][1+\frac{\overline{a}r}{y}]^{\frac{\lambda+\rho}{r}-1} \frac{1}{\lambda+\rho-r} - [1+\frac{\overline{a}r}{y}]^{\frac{\lambda+\rho}{r}} \frac{1}{\lambda+\rho} \,\Big]$$

$$V_{\overline{a}}(\overline{a}) \frac{\rho}{\lambda+\rho} = g\lambda[\overline{a}+\frac{y}{r}] \frac{r}{(\lambda+\rho-r)(\lambda+\rho)} - gc\frac{y}{r}[1+\frac{\overline{a}r}{y}]^{\frac{\lambda+\rho}{r}} \frac{r}{(\lambda+\rho-r)(\lambda+\rho)}$$

$$V_{\overline{a}}(\overline{a}) = \frac{y}{\rho} \frac{g\lambda}{\lambda+\rho-r} \Big[\, [1+\frac{\overline{a}r}{y}] \;-\; c[1+\frac{\overline{a}r}{y}]^{\frac{\lambda+\rho}{r}} \,\Big]$$

Proof of (14).

$$V_{\overline{a}}(\overline{a}) = \int_{t=0}^{+\infty} \Big[\, g(a_t - \overline{a}) + V_{\overline{a}}(\overline{a}) \,\Big] \lambda e^{-(\lambda+\rho)t}\, dt - \int_{t=T}^{+\infty} ga_t \lambda e^{-(\lambda+\rho)t}\, dt \tag{31}$$

$$+ \int_{t=T}^{+\infty} \int_{s=T}^{t} ye^{-\rho s}\, ds\, \lambda e^{-\lambda t}\, dt$$

With the same algebra as for (8) with $c = 1$, one gets

$$V_{\overline{a}}(\overline{a}) = \frac{gy}{\rho} \frac{\lambda}{\lambda+\rho-r} \Big[\, 1 + \frac{r\overline{a}}{y} - (1+\frac{r\overline{a}}{y})^{\frac{\rho+\lambda}{r}} \,\Big] + \frac{\lambda+\rho}{\rho} \int_{t=T}^{+\infty} \int_{s=T}^{t} ye^{-\rho s}\, ds\, \lambda e^{-\lambda t}\, dt$$

Define

$$I \equiv \frac{\lambda+\rho}{\rho} \int_{t=T}^{+\infty} \int_{s=T}^{t} ye^{-\rho s}\, ds\, \lambda e^{-\lambda t}\, dt = y\lambda \frac{\lambda+\rho}{\rho} \int_{t=T}^{+\infty} \Big[\, \frac{e^{-\rho T} - e^{-\rho t}}{\rho} \,\Big] e^{-\lambda t}$$

$$I = y\lambda \frac{\lambda+\rho}{\rho^2} \Big[\, \frac{e^{-(\rho+\lambda)T}}{\lambda} - \frac{e^{-(\rho+\lambda)T}}{\lambda+\rho} \,\Big]$$

$$I = y\lambda \frac{\lambda+\rho}{\rho^2} \frac{\rho}{\lambda(\lambda+\rho)} \left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}}$$

$$I = \frac{y}{\rho} \left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}}$$

Finally,

$$V_{\overline{a}}(\overline{a}) = \frac{gy}{\rho} \frac{\lambda}{\lambda+\rho-r} \left[\, 1+\frac{r\overline{a}}{y} - (1+\frac{r\overline{a}}{y})^{\frac{\rho+\lambda}{r}} \,\right] + \frac{y}{\rho} \left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}}$$

$$V_{\overline{a}}(\overline{a}) = \frac{gy}{\rho} \frac{\lambda}{\lambda+\rho-r} \left(1+\frac{r\overline{a}}{y}\right) - [\frac{gy}{\rho} \frac{\lambda}{\lambda+\rho-r} - \frac{y}{\rho}] \left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}}$$

Proof that equations (9) and (11) provide a solution to the HJB equations (4) and (5) with high g

Let us check it is indeed optimal to save i.e. to choose $s = 0$ when $a \in [\overline{a}, 0]$. A sufficient condition is $\frac{dV_{\overline{a}}(a)}{da} \geq 1$

$$\frac{dV_{\overline{a}}(a)}{da} = \frac{\lambda g}{\lambda+\rho-r} \left[1 - c\left(1+\frac{ra}{y}\right)^{\frac{\lambda+\rho-r}{r}}\right] \geq \frac{\lambda g}{\lambda+\rho-r} \left[1-c\right] > 1$$

Where the last inequality stands because g is high, i.e. (6) is verified. The case $a \geq 0$ is checked similarly using $\frac{dV_{\overline{a}}(a)}{da} = \frac{\lambda g}{\lambda+\rho-r} \geq \frac{\lambda g}{\lambda+\rho-r} \left[1-c\right] > 1$.

Let us now check that when $a \in [\overline{a}, 0]$, it is optimal to spend as much as possible on any spending opportunity by selecting $a' = \overline{a}$. A sufficient condition is $g \geq \frac{dV_{\overline{a}}(a)}{da}$ for all a. For $a < 0$,

$$\frac{dV_{\overline{a}}(a)}{da} = \frac{\lambda g}{\lambda+\rho-r} \left[1 - c\left(1+\frac{ra}{y}\right)^{\frac{\lambda+\rho-r}{r}}\right] \leq \frac{\lambda g}{\lambda+\rho-r} \leq g$$

where we used the natural debt limit $0 \leq y + ra$ and the impatience of the government relative to market $\rho - r \geq 0$. The case $a \geq 0$ is checked similarly using $\frac{dV_{\overline{a}}(a)}{da} = \frac{\lambda g}{\lambda+\rho-r} \left[1-c\right] \leq g \left[1-c\right] \leq g$.

By construction, $V_{\overline{a}}(\cdot)$ verifies the HJB equation when $a < 0$. Let us check that it is also the case when $a > 0$. We need to verify

$$\rho V_{\overline{a}}(a) = V'_{\overline{a}}(a) \left[y + ra\right] + \lambda \big[g(-\overline{a}) + g(1-c)a + V_{\overline{a}}(\overline{a}) - V_{\overline{a}}(a)\big]$$

Using equation (10), we get

$$\rho[V_{\overline{a}}(0)+\frac{\lambda g(1-c)}{\lambda+\rho-r}a]=\frac{\lambda g(1-c)}{\lambda+\rho-r}\left[y+ra\right]+\lambda\big[g(-\overline{a})+g(1-c)a+V_{\overline{a}}(\overline{a})-[V_{\overline{a}}(0)+\frac{\lambda g(1-c)}{\lambda+\rho-r}a]\big]$$

All the terms in a are linear and cancel out, we are left to prove

$$\rho V_{\overline{a}}(0)=\frac{\lambda g(1-c)}{\lambda+\rho-r}y+\lambda\big[g(-\overline{a})+V_{\overline{a}}(\overline{a})-V_{\overline{a}}(0)\big]$$

Using equations (8) and (9), i.e.

$$V_{\overline{a}}(\overline{a})=\frac{gy}{\rho}\frac{\lambda}{\lambda+\rho-r}\big[\,1+\frac{r\overline{a}}{y}-c(1+\frac{r\overline{a}}{y})^{\frac{\rho+\lambda}{r}}\,\big]$$

$$V_{\overline{a}}(0)=\frac{\lambda g}{\rho(\lambda+\rho-r)}\Big[y-(\rho-r)\overline{a}\Big]-\frac{\lambda gcy}{(\rho+\lambda)(\lambda+\rho-r)}\left[1+\frac{\lambda}{\rho}\left(1+\frac{r\overline{a}}{y}\right)^{\frac{\rho+\lambda}{r}}\right]$$

we find that indeed this equality holds.

Proof of Proposition 2

This proof relies on the Implicit Function Theorem. When a debt is sustainable, g can be high or medium.

High g

When g is high (i.e. (6) holds), we have

$$V_{\tilde{a}}(\tilde{a})=\frac{g}{\rho}\frac{\lambda}{\lambda+\rho-r}\big[\,y+r\tilde{a}-cy(1+\frac{r\tilde{a}}{y})^{\frac{\rho+\lambda}{r}}\,\big]\quad=\quad\frac{g}{\rho}\frac{\lambda}{\lambda+\rho-r}\big[\,y(1-c)\,\big]=V_0(0)$$

So $\tilde{a}(\lambda,c,g,r,\rho)<0$ is implicitly defined by

$$f(\tilde{a},\lambda,c,g,r,\rho)\equiv\frac{r\tilde{a}}{y}-c(1+\frac{r\tilde{a}}{y})^{\frac{\rho+\lambda}{r}}+c\quad=\quad0\tag{32}$$

Since $\frac{dV_{\overline{a}}(\overline{a})}{d\overline{a}}\Big|_{\overline{a}=\tilde{a}}>0$, $\frac{r}{y}\Big[\,1-c\frac{\rho+\lambda}{r}(1+\frac{r\tilde{a}}{y})^{\frac{\rho+\lambda-r}{r}}\,\Big]>0$, and so $\frac{df(\overline{a})}{d\overline{a}}\Big|_{\overline{a}=\tilde{a}}>0$.

$$\frac{df(\overline{a})}{dr}\Big|_{\overline{a}=\tilde{a}}=\frac{\overline{a}}{y}\Big[\,1-c\frac{\rho+\lambda}{r}(1+\frac{r\tilde{a}}{y})^{\frac{\rho+\lambda-r}{r}}\,\Big]+\frac{d\ -c(1+\frac{r\overline{a}}{y})^{\frac{\rho+\lambda}{r}}}{dr}\Big|_{\overline{a}=\tilde{a}}$$

Both terms are negative so $\frac{df(\overline{a})}{dr}\Big|_{\overline{a}=\tilde{a}} < 0$. Finally, g has no impact on $\tilde{a}(\lambda, c, g, r, \rho)$. Since $(1 + \frac{r\tilde{a}}{y}) \in [0, 1)$, $\frac{df(\overline{a})}{d\lambda}\Big|_{\overline{a}=\tilde{a}} = \frac{df(\overline{a})}{d\rho}\Big|_{\overline{a}=\tilde{a}} > 0$ and $\frac{df(\overline{a})}{dc}\Big|_{\overline{a}=\tilde{a}} > 0$.

Medium g

When g is medium (i.e. (12) holds), we have

$$V_{\tilde{a}}(\tilde{a}) = \frac{gy\lambda}{\rho(\lambda+\rho-r)}\left(1+\frac{r\tilde{a}}{y}\right) - \frac{y\left[\lambda(g-1)+r-\rho\right]}{\rho(\lambda+\rho-r)}\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} \quad = \quad \frac{y}{\rho} = V_0(0)$$

So $\tilde{a}(\lambda, c, g, r, \rho) < 0$ is implicitly defined by

$$f(\tilde{a}, \lambda, c, g, r, \rho) \equiv g\lambda\left(1+\frac{r\tilde{a}}{y}\right) - \left[\lambda(g-1)+r-\rho\right]\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} - \lambda - \rho + r$$

Proof for λ, g and c. First, g is medium, so $[\lambda(g-1)+r-\rho] \geq 0$. Since $(1+\frac{r\tilde{a}}{y}) \in [0,1)$, we have $\frac{df(\overline{a})}{d\lambda}\Big|_{\overline{a}=\tilde{a}} = (g-1)\Big[\ \left(1+\frac{r\tilde{a}}{y}\right) - \left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}}\ \Big] - [\lambda(g-1)+r-\rho]\ \frac{d\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}}}{d\lambda}\Big|_{\overline{a}=\tilde{a}} > 0$

Similarly, $\frac{df(\overline{a})}{dg}\Big|_{\overline{a}=\tilde{a}} = \lambda\left(1+\frac{r\tilde{a}}{y}\right) - \lambda\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} > 0.$

$$\frac{df(\overline{a})}{d\rho}\Big|_{\overline{a}=\tilde{a}} = \left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} - \left[\lambda(g-1)+r-\rho\right]\frac{\ln(1+\frac{r\tilde{a}}{y})}{r}\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} - 1$$

c has no impact on $\tilde{a}(\lambda, c, g, r, \rho)$.

Proof for ρ. We are going to show that the permanent utility stream equivalent $\rho V_{\overline{a}}(\overline{a})$ is decreasing in ρ for any $\overline{a} < 0$ when the government does not accumulate assets. This is because the utility stream is back-loaded.

$$\frac{1}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} = \frac{-g\lambda\left(1+\frac{r\tilde{a}}{y}\right)}{(\lambda+\rho-r)^2} - \frac{-\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}} + \left[\lambda(g-1)+r-\rho\right]\frac{\ln(1+\frac{r\tilde{a}}{y})}{r}\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}}}{\lambda+\rho-r}$$

$$+\frac{\left[\lambda(g-1)+r-\rho\right]\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{\rho+\lambda}{r}}}{(\lambda+\rho-r)^2}$$

$$\frac{(\lambda+\rho-r)^2}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} = -g\lambda\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{r-\rho-\lambda}{r}} +(\lambda+\rho-r)-(\lambda+\rho-r)[\lambda(g-1)+r-\rho]\frac{\ln(1+\frac{r\tilde{a}}{y})}{r}$$

$$+[\lambda(g-1)+r-\rho]$$

$$\frac{(\lambda+\rho-r)^2}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} = -g\lambda\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{r-\rho-\lambda}{r}} +g\lambda-(\lambda+\rho-r)[\lambda(g-1)+r-\rho]\frac{\ln(1+\frac{r\tilde{a}}{y})}{r}$$

Using the fact that $\left(1+\frac{r\tilde{a}}{y}\right)^{\frac{r-\rho-\lambda}{r}} = e^{\frac{r-\rho-\lambda}{r}\ln(1+\frac{r\tilde{a}}{y})} \geq 1+\frac{r-\rho-\lambda}{r}\ln(1+\frac{r\tilde{a}}{y})$, we get

$$\frac{(\lambda+\rho-r)^2}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} \leq -g\lambda\frac{r-\rho-\lambda}{r}\ln(1+\frac{r\tilde{a}}{y})-(\lambda+\rho-r)[\lambda(g-1)+r-\rho]\frac{\ln(1+\frac{r\tilde{a}}{y})}{r}$$

$$\frac{(\lambda+\rho-r)^2}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} \leq \frac{(\lambda+\rho-r)\ln(1+\frac{r\tilde{a}}{y})}{r}\Big[\ g\lambda-[\lambda(g-1)+r-\rho]\ \Big]$$

$$\frac{(\lambda+\rho-r)^2}{y}\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} \leq \frac{(\lambda+\rho-r)^2\ln(1+\frac{r\tilde{a}}{y})}{r}$$

So $\frac{d[\ \rho V_{\overline{a}}(\overline{a})\ ]}{d\rho} < 0$. Since $\rho V_{\tilde{a}}(\tilde{a}) = \rho V_0(0) = y$, and $V_{\overline{a}}(\overline{a}) = V_0(0)$, we get $\left.\frac{d\left[\ \rho(V_{\overline{a}}(\overline{a})-V_0(0))\ \right]}{d\rho}\right|_{\overline{a}=\tilde{a}} < 0$.

Proof for r. We want to show that $\frac{dV_{\overline{a}}(\overline{a})}{dr} < 0$ for all $\overline{a} < 0$. Equation (31) can be written as

$$V_{\overline{a}}(\overline{a})\frac{\lambda}{\lambda+\rho} = \int_{t=0}^{T} g a_t \lambda e^{-(\lambda+\rho)t}\,dt + \int_{t=0}^{+\infty} -\overline{a}\lambda e^{-(\lambda+\rho)t}\,dt \tag{33}$$
$$+\int_{t=T}^{+\infty}\int_{s=T}^{t} y e^{-\rho s}\,ds\ \lambda e^{-\lambda t}\,dt$$

The second term is invariant with r. The last term is decreasing in r because, using the fact that $[\ -x+\ln(1+x)(1+x)\ ] > 0$ for $x \in (-1,0)$ (because $\frac{d[\ -x+\ln(1+x)(1+x)\ ]}{dx} = \ln(1+x) < 0$),

$$\frac{dT}{dr} = \frac{d-\frac{\ln(1+\frac{r\overline{a}}{y})}{r}}{dr} = -\frac{\frac{\overline{a}}{y}}{r(1+\frac{r\overline{a}}{y})} + \frac{\ln(1+\frac{r\overline{a}}{y})}{r^2}$$

$$\frac{dT}{dr} = \frac{-\frac{r\overline{a}}{y}+\ln(1+\frac{r\overline{a}}{y})(1+\frac{r\overline{a}}{y})}{r^2(1+\frac{r\overline{a}}{y})} > 0$$

i.e. the time to pay back $\overline{a}$ increases with r. The first term is also decreasing because

$$\frac{d[\int_{t=0}^{T} g a_t \lambda e^{-(\lambda+\rho)t}\, dt\,]}{dr} = -\frac{dT}{dr} g a_T \lambda e^{-(\lambda+\rho)T} + \int_{t=0}^{T} g \frac{da_t}{dr} \lambda e^{-(\lambda+\rho)t}\, dt$$

By definition of T, $a_T = 0$ and, for $t \in [0, T]$

$$\frac{da_t}{dr} = \frac{d\Big[\ (\overline{a} + \frac{y}{r})e^{rt} - \frac{y}{r}\ \Big]}{dr} = -\frac{y}{r^2}e^{rt} + (\overline{a} + \frac{y}{r})te^{rt} + \frac{y}{r^2} = (\overline{a} + \frac{y}{r})te^{rt} - \frac{y}{r^2}(e^{rt} - 1)$$

By denoting $\alpha \equiv \frac{t}{T} \in [0, 1]$, and using the fact that $e^{rT} = -\ln(1 + \frac{r\overline{a}}{y})$, we get

$$\frac{da_t}{dr} = -(\overline{a} + \frac{y}{r})\alpha \ln(1 + \frac{r\overline{a}}{y})(1 + \frac{r\overline{a}}{y})^{-\alpha} - \frac{y}{r^2}((1 + \frac{r\overline{a}}{y})^{-\alpha} - 1)$$

$$\frac{r^2}{y}(1 + \frac{r\overline{a}}{y})^{\alpha}\frac{da_t}{dr} = -(1 + \frac{r\overline{a}}{y}) \ln\Big(\ (1 + \frac{r\overline{a}}{y})^{\alpha}\ \Big) - (1 - (1 + \frac{r\overline{a}}{y})^{\alpha})$$

Adding $\big[\ (1 + \frac{r\overline{a}}{y}) - (1 + \frac{r\overline{a}}{y})^{\alpha}\ \big] \ln\Big(\ (1 + \frac{r\overline{a}}{y})^{\alpha}\ \Big) \geq 0$ to the right-hand-side term yields

$$\frac{r^2}{y}(1 + \frac{r\overline{a}}{y})^{\alpha}\frac{da_t}{dr} \leq -(1 + \frac{r\overline{a}}{y})^{\alpha} \ln\Big(\ (1 + \frac{r\overline{a}}{y})^{\alpha}\ \Big) - (1 - (1 + \frac{r\overline{a}}{y})^{\alpha}) \leq 0$$

where we used the fact that $-y\ln(y) - 1 + y \leq 0$ for all $y \in [0, 1]$. So $\frac{da_t}{dr} \leq 0$, i.e. a higher interest rate r slows down the payback of $\overline{a}$. Thus, $\frac{dV_{\overline{a}}(\overline{a})}{dr} < 0$ for all $\overline{a} < 0$. Since $V_{\tilde{a}}(\tilde{a}) = V_0(0) = \frac{y}{\rho}$, we get $\frac{d\Big[\ V_{\overline{a}}(\overline{a}) - V_0(0)\ \Big]}{dr}\Big|_{\overline{a}=\tilde{a}} < 0$, which implies $\frac{d\tilde{a}}{dr} > 0$.