Linkage-Specific Fiscal Policies in Production Networks

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Abstract

This paper studies linkage-specific fiscal policies in economies with input–output production networks and link-level distortions. Using a tractable sufficient-statistics approach, it derives closed-form expressions that map link-level interventions into equilibrium outcomes across a broad set of fiscal instruments, including production, consumption, and factor subsidies, as well as targeted government purchases. The analysis highlights a simple but powerful empirical statistic for policy design—scale-to-efficiency—defined as the ratio of a linkage's efficient size relative to GDP to its realized size in the distorted allocation. The paper shows analytically why this statistic is informative for subsidy targeting. For government-spending shocks, the closed-form solution implies that spending multipliers are linear in a weighted Leontief inverse, with cross-sector variation governed by factor-specific Leontief-inverse exposures. Applying this approach to U.S. input–output data surrounding the COVID-19 shock, the paper shows that scale-to-efficiency performs strongly as an empirical guide for subsidy targeting and that spending multipliers vary systematically with the structure of factor markets.

Keywords: Production networks, Input–output linkages, Sufficient statistics, Fiscal multipliers

JEL Classification: D24, D57, D85, E62, H21

1 Introduction

Fiscal policy interventions deployed during economic crises are rarely uniform. Instead of relying on broad, economy-wide tools, governments typically implement highly targeted measures that operate at the level of specific sectors, activities, or buyer–seller links within production networks. During the COVID-19 pandemic, for example, policymakers introduced industry-specific consumption subsidies, sector-targeted employment support programs, and large-scale government procurement initiatives aimed at stabilizing fragile supply-chain relationships and sustaining essential production. Similar recent interventions include China's consumer subsidy programs for vehicles and household durables, as well as the European Union's energy subsidy schemes directed at households and energy-intensive industries.

These policies are linkage-specific fiscal interventions: measures that condition on a particular buyer–seller relationship or on a producer's use of a specific input, rather than applying uniformly at the level of a sector or final-demand node. A link-level policy targets the flow from one node in the network to another; such nodes may be producers, households, government entities, or factor suppliers. Although many policies are described in sectoral terms, their implementation is often linkage-specific in practice. Examples include a consumption subsidy tied to purchases

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from designated producers by households, a factor subsidy applied to the flow of labor or capital to selected industries, a production subsidy triggered when one identified producer buys from another, and government purchases directed to specific firms or contracts. In contrast, node-level instruments—such as a sales tax, a uniform marginal-cost subsidy within a sector, an economy-wide consumption subsidy, or a proportional increase in all components of government demand—do not depend on the identity of the trading partner at the link level.

The core motivation for targeting fiscal policy at the link level is that distortions are highly heterogeneous across individual buyer–seller relationships. Large shocks, such as supply-chain disruptions, mandated shutdowns, or abrupt reallocations of private and public demand, can displace the economy from its pre-crisis allocation in ways that uniform or proportional (node-level) wedges cannot adequately capture. Even when the pre-crisis allocation is efficient, nonlinear interactions in production networks can generate substantial misallocation following large shocks. In these environments, targeted fiscal policy can raise real economic activity only if it acts on the specific margins that shape shock propagation. This heterogeneity thus creates the central policy design challenge: How should the government design linkage-specific fiscal interventions in production networks with arbitrary distortions?

To address this question, I adopt a tractable analytical approach for studying linkage-specific fiscal policy in production networks with arbitrary link-level pre-existing wedges. The approach accommodates a broad class of fiscal instruments—including production, consumption, and factor subsidies as well as targeted government purchases—and yields closed-form sufficient statistics for evaluating these policies around a distorted steady state.

The analysis builds on the nonparametric production-network model of D. R. Baqaee & Farhi (2020), which characterizes how Hicks-neutral productivity shocks and markup disturbances propagate through distorted economies. This paper, however, applies the production-network perspective to a different set of questions. Rather than analyzing the propagation of node-level exogenous shocks, it adapts and extends the framework to fiscal-policy analysis at the level of individual buyer–seller relationships. Addressing this problem requires two modeling ingredients that are not central in existing shock-propagation approaches.

First, the framework treats government demand as a separate economic agent rather than embedding public purchases within private consumption. This modification removes the representative-agent structure and introduces additional propagation channels. These channels allow fiscal interventions to interact with public demand and are essential for studying government-spending multipliers. Second, the framework incorporates distortions and policy instruments directly at the level of specific input–output links. Using the cost duality of the production structure, together with a tractable transformation of the underlying cost problems, I derive explicit nonparametric expressions for the first-order responses of prices and Hicksian input demands in the presence of linkage-specific distortions, productivity shocks, and policy interventions. These responses form the essential building blocks for characterizing how consumption shares, factor shares, Domar weights, and real output adjust throughout the network. The same derivations can also be applied to demand shocks once a specific preference structure is specified.

By working directly with economically fundamental buyer–seller relationships, rather than introducing auxiliary constructs to represent distortions, the framework yields transparent mappings from fiscal interventions to equilibrium outcomes. This direct approach produces closed-form sufficient statistics for link-level policies, identifying which specific production relationships should be subsidized, taxed, or supported through government purchases in order to restore efficiency or stimulate economic activity.

In addition to the model-implied sufficient statistics, the link-level analysis motivates a second, simple, yet highly informative measure: scale-to-efficiency. This statistic measures the ratio

between the optimal (efficient) size of a linkage relative to GDP and its realized size in the distorted steady state. I show analytically why this measure is informative for subsidy design. Conceptually, scale-to-efficiency compares the economic importance of a linkage in the efficient allocation with its distorted counterpart, thereby pinpointing which links are "too small" or "too large" relative to efficiency.

I contrast this statistic with distortion centrality, a measure inspired by Liu (2019) but defined here as the ratio of efficient to distorted Domar weights. While distortion centrality is a useful proxy in the environments studied by Liu (2019), where distortions resemble financial-friction–type wedges, the results in this paper show that it remains informative for proportional (node-level) production subsidies but is substantially less informative for linkage-specific targeting. By contrast, scale-to-efficiency explains a much larger share of the variation in optimal targeting for production subsidies, consumption subsidies, and subsidies on factors. Thus, scale-to-efficiency provides a broader and more powerful guide for fiscal-policy design.

Moreover, when the government intends to subsidize a particular sector, the scale-to-efficiency statistic also identifies the relative importance of each production linkage from that sector to the rest of the economy in shaping aggregate real-GDP responses. This provides a direct criterion for determining when a link-level subsidy can outperform a sector-level subsidy. In this sense, scale-to-efficiency is highly informative for designing input subsidies: for example, if the government considers an energy subsidy but has not yet decided which sector should receive it, the scale-to-efficiency measure directly indicates where the subsidy should be targeted.

The paper also characterizes the structure of government-spending multipliers. While a subsidy has a direct effect on real output, the effect of a government-spending shock on real GDP operates entirely through its impact on factor shares and consumption shares. The analysis shows that the key determinants of the cross-sectoral variation in these multipliers are factor-specific Leontief-inverse exposures, which measure the total (direct and indirect) requirement of each primary factor to the production of a given producer. Under integrated factor markets with labor as the sole primary input, the spending multiplier is negatively linear in the labor Leontief-inverse exposure of the targeted sector. The frictionless economy is a special case of this environment: with only one factor, every producer has a labor Leontief-inverse exposure of one, implying that government-spending multipliers are identical across all industries.[1]

When multiple factors are present or factor markets are segmented, the multiplier depends linearly on a weighted sum of factor-specific Leontief-inverse exposures. The weights are negative and invariant to the shock, but they depend on substitution elasticities, factor-supply elasticities, pre-existing wedges, and the full structure of production networks.

Another instructive special case arises in a frictionless economy with two factors, elastic labor and inelastic capital. Under standard calibrations, the decision about where to purachse becomes positively correltated to the labor Leontief-inverse exposure of the targeted sector. This occurs because the labor and capital Leontief-inverse exposures of any sector sum to one, while the fixed weights associated with both factors are negative, with the weight on the inelastic factor (capital) being substantially more negative. As a result, sectors with higher labor exposure end up having larger multipliers.[2] These cases underscore the importance of factor-market structure for multiplier heterogeneity: changes in factor-market assumptions can reverse the sign of the relationship between government-spending multipliers and labor Leontief-inverse exposure.

[1] This property also holds as long as $\sum_{j \in N \cup F} \Omega_{ij} = 1$ for all producers $i \in N$. If labor is the only primary factor in the one-factor case, the condition $\sum_{j \in N \cup F} \Omega_{ij} = 1$ implies that $\Psi_{iL} = 1$ for all i, and therefore the spending multipliers coincide across all producers.

[2] In our calibration, the input-output matrix rows satisfy $\sum_{j \in N \cup F} \Omega_{ij} = 1$. When factor markets are assumed to be integrated and labor and capital are the only primary factors, this condition implies that $\Psi_{iL} + \Psi_{iK} = 1$ for all $i \in N$, so the special-case logic applies directly.

Building on these theoretical insights, I apply the framework to U.S. input–output data from 2019 (pre-COVID) and 2022 (post-COVID) and document substantial dispersion in fiscal multipliers across both sectors and buyer–seller linkages. Allowing fiscal instruments to operate at the granularity of individual links delivers markedly larger multipliers than uniform sector-wide policies, underscoring the importance of granular, network-aware fiscal design. Quantitatively, the scale-to-efficiency statistic performs strongly as an empirical guide for subsidy targeting, whereas sector-level distortion measures are far less informative for link-specific interventions such as consumption and factor subsidies. Finally, I show that the ranking of government-spending multipliers depends systematically on the structure of factor markets. Under integrated factor markets with elastic labor and inelastic capital, the largest multipliers arise in labor-intensive service industries such as nursing and residential care facilities, warehousing and storage, and educational services. In contrast, when factor markets are segmented, government-dominated sectors—federal and state general government, as well as federal government enterprises—move to the top of the ranking. These differences highlight that both the ranking and the effectiveness of government spending critically hinge on whether factor inputs can reallocate freely across sectors.

Related Literature. A growing body of research studies how production networks shape the transmission and amplification of shocks from the micro level to the macroeconomy.

The closest strand to this paper examines wedge-like distortions in production networks, including markups, taxes, financial frictions, and other sector- or link-specific wedges. A pioneering contribution in this area is Liu (2019), who analyzes how governments should target interventions in economies with financial frictions. He shows that, starting from a decentralized equilibrium without policy intervention, the marginal subsidy should be allocated to the sector with the highest distortion centrality, which in this paper corresponds to the ratio of efficient to distorted Domar weights. Because of his modeling assumptions—most importantly, frictions generate deadweight losses—the link-level subsidy multiplier coincides with the sector-level multiplier, implying that additional link-level detail does not change the structure of optimal targeting in his environment. Bigio & La'o (2020) study how distortions propagate through production networks and characterize how such wedges affect, among other things, labor wedges. D. R. Baqaee & Farhi (2020) broaden this line of work by characterizing how productivity and markup shocks propagate through production networks when the economy is inefficient. Building on this contribution, D. R. Baqaee et al. (2024) extend the analysis from a closed-economy environment to an international setting with heterogeneous agents at the country level, thereby allowing for cross-border propagation and policy interactions.

Although D. R. Baqaee et al. (2024) consider tariff shocks that are link-specific in implementation, their theoretical characterization is formulated at the sector (node) level. The present paper develops a link-level theory of fiscal intervention that treats distortions and policy instruments directly as objects defined on individual buyer–seller relationships. Using a tractable transformation of the original cost-duality approach, I derive nonparametric expressions for the first-order responses of prices and Hicksian input demands to changes in underlying wedges and policy instruments. These results provide the foundation for analytical sufficient statistics that map link-level fiscal interventions to equilibrium outcomes and optimal targeting rules. In addition, based on the closed-form characterization and under a set of simplifying assumptions, I derive a second simple yet powerful link-level statistic—the scale-to-efficiency measure—and show that it provides a substantially more informative guide for determining where subsidies should be directed. Intuitively, the statistic resembles its node-level analogue, and one might think it could be obtained by rewriting a link as a node through the introduction of a fictitious producer. However, I show that its informativeness becomes transparent only through the link-level analysis.

A second strand examines government spending in production networks, documenting that network structure can materially raise spending multipliers relative to one-sector benchmarks. In multi-sector environments with intermediate inputs, multipliers vary systematically with upstreamness, input intensity, and sectoral supply elasticities, and tend to be larger when markets are segmented (Bouakez et al., 2023; Proebsting, 2022). Related work emphasizes that the location of spending in the production network, rather than its aggregate size alone, critically shapes aggregate outcomes (Cox et al., 2024; Bouakez et al., 2025).

The closest paper in this strand to my analysis is D. R. Baqaee (2015), who shows that network-adjusted labor intensity, corresponding to the labor Leontief exposure in my notation, is the key statistic governing employment multipliers from targeted government purchases. Their analysis focuses on labor market responses and the employment effects of sector-specific spending shocks. By contrast, this paper is concerned with real output (GDP) and the government-spending GDP multiplier. I show that government spending has no direct effect on real GDP in equilibrium and that the GDP multiplier is linear in factor-specific Leontief inverse exposures. This result extends the logic of network-adjusted factor intensities to a broader environment with arbitrary distortions and provides an output-based sufficient statistic for fiscal effectiveness. Rubbo (2025) shows that the industry composition of government spending matters only when factor markets are segmented and sectors employ different combinations of primary factors. Within the environment studied in this paper, I clarify a mechanism consistent with that result: when factor markets are segmented and sectors differ in factor intensities, factor-specific Leontief inverse exposures vary across industries, and this heterogeneity is the channel through which differences in spending multipliers arise. Moreover, pre-existing distortions can also generate variation in these exposures across sectors, and therefore variation in spending multipliers. In this model, despite rich production-network structure, general substitution patterns, endogenous factor-supply responses, distortions, and potential market segmentation, differences in factor-specific Leontief inverse exposures are the key statistics governing heterogeneity in government-spending multipliers.

This paper also contributes to the broader literature on the transmission of productivity and preference shocks in production networks (Hulten, 1978; King et al., 1988; Acemoglu et al., 2012; Foerster et al., 2011; Atalay, 2017; Carvalho, 2014; Carvalho & Tahbaz-Salehi, 2019; D. R. Baqaee & Farhi, 2019; D. Baqaee & Farhi, 2020, 2022). In particular, it provides a theoretical treatment of these shocks within the economy with arbitrary link-level distortions. Most importantly, beyond standard productivity and demand disturbances, it also analyzes shocks to input-augmenting technology, which helps capture reduced-form spillovers of policy interventions through the network.

The remainder of the paper is organized as follows. Section 2 presents the model and defines the notation. Sections 3 and 4 derive sufficient statistics for policy interventions when the distorted steady state is observable. Section 5 shows how these distortion statistics can be expressed as functions of the model primitives when only the initial steady state is observed and extends the analysis by incorporating indirect policy effects. Section 6 applies the results from Sections 3 and 4 to U.S. data from the COVID-19 crisis and discusses the multipliers associated with different policies. Section 7 concludes.

2 Model

2.1 A Semiparametric Model with Input–Output Linkages

The model builds on the nonparametric production network framework of D. R. Baqaee & Farhi (2020), while addressing a distinct set of questions and adopting a different modeling strategy. In contrast to their use of fictitious producers to represent linkage-specific distortions, productivity

shocks, and tax instruments, I incorporate these wedges and shocks directly at the level of production links. This design keeps the underlying network transparent and allows the policy objects of interest to map one-for-one into observables at the linkage level. The framework explicitly treats the government as a separate economic agent, which enables public-demand subsidies to interact with a broader set of fiscal distortions and is essential for analyzing government-spending shocks. Finally, I allow for preference shocks and input-augmenting technology shocks around an arbitrarily distorted steady state. This generalization enlarges the set of admissible disturbances and enables the evaluation of both ex-ante and indirect policy effects in environments with pervasive wedges, as discussed in Section 5.

The economy consists of a single representative household, denoted by C, a government, denoted by G, a set of N producers, a set of F primary factors, and a set of wedge collectors. In what follows, I use C, G, N, and F to refer both to the corresponding sets of agents and to their cardinalities. Variables indexed by t denote endogenous variables or exogenous shocks, whereas variables without a time subscript denote parameters or steady-state values.

2.2 Household

The representative household derives utility from aggregate consumption and experiences disutility from supplying primary factors. The household's utility function is

$$U_t \equiv \log y_{Ct} - \sum_{f \in F} \frac{L_{ft}^{\frac{\eta_f+1}{\eta_f}}}{\frac{\eta_f+1}{\eta_f}}$$

where y_{Ct} is the household's aggregate consumption, L_{ft} is the supply of primary factor $f \in F$, and η_f denotes the Frisch elasticity of factor for factor f. When a factor is supplied inelastically, $\eta_f = 0$; otherwise, $\eta_f > 0$. This framework accommodates an arbitrary number of primary factors and nests both segmented-factor and integrated-factor markets, depending on how the primary factors are defined. For example, if labor markets are fully integrated, labor can be represented as a single aggregate factor. If labor markets are segmented, in contrast, labor is represented by producer-specific factor supplies.

The household maximizes utility subject to the budget constraint

$$P_{Ct} y_{Ct} \leq \sum_{f \in F} p_{ft} L_{ft} + \sum_{i \in C \cup N} \sum_{j \in N \cup F} \Pi_{ijt} - T_t$$

where p_{Ct} is the price index of the aggregate consumption bundle, p_{ft} is the price of factor f, Π_{ijt} is the wedge income that arises when producer i purchases inputs from producer or factor j, and T_t denotes lump-sum taxes paid to the government.

Aggregate consumption, y_{Ct}, is defined through a CES aggregator over products

$$y_{Ct} = \left(\sum_{j \in N} w_{Cjt}^{\frac{1}{\theta_C}} x_{Cjt}^{\frac{\theta_C - 1}{\theta_C}} \right)^{\frac{\theta_C}{\theta_C - 1}} \tag{1}$$

where $\sum_{j \in N} w_{Cjt} = 1$. Here, x_{Cjt} denotes consumption of goods produced by producer j, and w_{Cjt} determines the relative importance of each producer in the household's aggregate consumption bundle. The parameter θ_C is the elasticity of substitution between goods produced by different producers. I adopt a parametric specification for the aggregate consumption aggregator because it allows for a transparent and tractable characterization of preference shocks, i.e. shocks to the consumption weights w_{Cjt}.

The representative household minimizes total expenditure on consumption goods, net of subsidies

$$p_{Ct}y_{Ct} \equiv \min_{\{x_{Cjt}\}} \sum_{j \in N} (1 - \tau_{Cjt}) M_{Cj} p_{jt} x_{Cjt} \quad \text{s.t. (1)}$$

where p_{jt} denotes the price of the good produced by sector j, τ_{Cjt} is the consumption subsidy received when purchasing that good, and M_{Cj} is a pre-existing wedge that distorts the effective price the household faces when buying from producer j.

2.3 Producers

Apart from the aggregate consumption aggregator, the economy consists of N individual producers that each produce a distinct gross output good.

Each producer $i \in N$ operates a constant-returns-to-scale production technology and takes pre-existing wedges and policy instruments as given. Variable returns to scale can be represented as constant returns to scale through the inclusion of additional producer-specific factors that are supplied inelastically (McKenzie, 1959; D. R. Baqaee & Farhi, 2020).

Each producer $i \in N$ chooses its input vector $\{x_{ijt}\}_{j \in N \cup F}$, where x_{iN+ft} denotes the quantity of factor $f \in F$ used by producer i, in order to minimize total cost net of subsidies. The producer's cost-minimization problem is

$$c_{it}y_{it} \equiv \min_{\{x_{ijt}\}} \sum_{j \in N \cup F} (1 - \tau_{ijt}) M_{ij} p_{jt} x_{ijt}$$

$$\text{s.t.} \quad A_{it} F_i(Z_{i1t}x_{i1t}, Z_{i2t}x_{i2t}, ..., Z_{iNt}x_{iNt}, Z_{iN+1t}x_{iN+1t} ..., Z_{iN+Ft}x_{iN+Ft}) \geq y_{it}$$

where A_{it} is a Hicks-neutral productivity shock, Z_{ijt} is an input-augmenting productivity shock associated with input $j \in N \cup F$, τ_{ijt} is the linkage-specific subsidy that producer i receives when purchasing input j, and M_{ij} is a pre-existing wedge that distorts the effective cost of sourcing from j.

I denote by c_{it} the unit cost of producing y_{it} and assume that producer i charges a markup μ_{it}, so that its price satisfies $p_{it} = \mu_{it} c_{it}$. Without loss of generality, I normalize markups and production subsidies to one in the steady state. When they differ from one, I absorb them into the pre-existing wedge M_{ij}. I assume that producers do not retain profits. Instead, the economy includes a set of wedge collectors who receive wedge income and rebate it to the representative household.

The wedges in this economy may arise from a variety of sources, including markup dispersion, financial frictions, or other distortions that cause producers to face different effective prices for the same input. Wedges may also capture shocks that disrupt the efficiency of resource allocation across production links. The combination of pre-existing wedges and tax instruments allows us to study the effects of arbitrary wedge-like shocks in an economy with arbitrary distortions. Importantly, the wedge collectors in this paper differ from the fictitious producers in D. R. Baqaee & Farhi (2020) and D. R. Baqaee et al. (2024): unlike those fictitious producers, wedge collectors do not alter the structure of the production network.

2.4 Government

The government chooses which links in the production network to subsidize or tax. Specifically, τ_{ijt} for $i \in C \cup N$ and $j \in N \cup F$ denotes the set of policy instruments applied to each demander-supplier pair; this set also encompasses consumption subsidy policies and factor subsidy policies. In addition to subsidy policies, the government undertakes government purchases g_{it} for $i \in N$, which represent quantities produced by firm i and purchased by the government.

Total subsidy is given by the sum of the subsidies to consumers and producers

$$S_t = \sum_{j \in N} \tau_{Cjt} M_{Cj} p_{jt} x_{Cjt} + \sum_{i \in N} \sum_{j \in F} \tau_{ijt} M_{ij} p_{jt} x_{ijt}$$

Total government expenditure is given by

$$G_t = \sum_{i \in N} p_{it} g_{it}$$

I assume that the government maintains a balanced budget in every period, so that total subsidy expenditures and government purchases are financed by lump-sum taxes, $T_t = G_t + S_t$, collected from the representative household.

2.5 Market Clearing

The following market-clearing conditions hold. In the output markets, the output of each producer $i \in N$ is allocated to final household consumption, government demand, or intermediate input use by other producers, including itself. Formally, the output of producer i satisfies

$$y_{it} = x_{Cit} + g_{it+} \sum_{j \in N} x_{jit},$$

where x_{jit} denotes the quantity of goods produced by producer i and used by producer j.

In the factor markets, the total demand for each primary factor $f \in F$ equals its supply. That is, for each factor f, the sum of the factor used across all producers satisfies

$$\sum_{i \in N} x_{ift} = L_{ft}.$$

2.6 Equilibrium

The equilibrium of this economy is defined as follows. Given the Hicks-neutral productivities $\{A_{it}\}_{i \in N}$, the input-augmenting productivities $\{Z_{ijt}\}_{i \in N \& j \in N \cup F}$, the preference weights $\{w_{Cjt}\}_{j \in N}$, the linkage specific subsidy policies $\{\tau_{ijt}\}_{i \in C \cup N \& j \in N \cup F}$, the markups $\{\mu_{it}\}_{i \in N}$, and the government purchases $\{g_{it}\}_{i \in N}$, a general equilibrium consists of a set of prices $\{p_{jt}\}_{j \in N \cup F}$, input choices $\{x_{ijt}\}_{i \in C \cup N, j \in N \cup F}$, factor supplies $\{L_{ft}\}_{f \in F}$, and outputs $\{y_{it}\}_{i \in C \cup N}$ such that, given these prices, the household chooses consumption and factor supplies to maximize expected utility subject to its budget constraint, each producer chooses inputs to maximize profits, the government's budget constraint is satisfied, and markets for all goods and factors clear.

2.7 Normalization of the Model

I take nominal GDP as the numeraire and normalize all steady-state prices to one. Under this normalization, steady-state real GDP is also equal to one, so percentage changes in real GDP can be interpreted directly as changes in its monetary value. The model's results are invariant to the choice of numeraire, since all objects of interest are expressed in real terms and in percentage changes. As will become clear when the Domar weights are introduced, taking nominal GDP as the numeraire is especially convenient for analyzing production networks.

2.8 Definitions

I define the variables of interests and notations used in the analysis.

Nominal GDP. The definition of nominal GDP is defined through the expenditure approach and income approach as

$$\begin{aligned} GDP_t &= \sum_{i\in N} (1-\tau_{Cit})\, M_{Ci} p_{it} x_{Cit} + \sum_{i\in N} p_{it} g_{it} \\ &= \sum_{f\in F} p_{ft} L_{ft} + \sum_{i\in N}\sum_{j\in N\cup F} \Pi_{ijt} - \sum_{j\in N} \tau_{Cjt} M_{Ci} p_{jt} x_{Cjt} - \sum_{i\in N}\sum_{j\in N\cup F} \tau_{ijt} M_{ij} p_{jt} x_{ijt} \end{aligned}$$

The real GDP, denoted by Y_t, is defined when the prices and the tax rates are kept constant at the steady state.

Input-Output Matrices. The input-output matrix is denoted by Ω_t. It is a $(C+G+N+F)\times(C+G+N+F)$ matrix whose (i,j)-th element is given by

$$\Omega_{ijt} = \frac{p_{jt} x_{ijt}}{p_{it} y_{it}}, \qquad i \in C\cup N\cup F,\ j\in C\cup G\cup N\cup F,$$

and

$$\Omega_{Gjt} = \frac{p_{jt} g_{jt}}{G_t}, \qquad j\in N.$$

As an illustrative example, Table 1 presents an input–output matrix of dimension $(C+G+N+F)\times(C+G+N+F)$, where Ω_{iN+ft} denotes the entries corresponding to the usage of factor $f\in F$ by the producer i.

Table 1: Input–Output Matrix

$$\Omega_t = \left[\begin{array}{cccccccccc} 0 & 0 & \Omega_{C1t} & \Omega_{C2t} & \cdots & \Omega_{CNt} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Omega_{G1t} & \Omega_{G2t} & \cdots & \Omega_{GNt} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \Omega_{11t} & \Omega_{12t} & \cdots & \Omega_{1Nt} & \Omega_{1N+1t} & \Omega_{1N+2t} & \cdots & \Omega_{1N+Ft} \\ 0 & 0 & \Omega_{21t} & \Omega_{22t} & \cdots & \Omega_{2Nt} & \Omega_{2N+1t} & \Omega_{2N+2t} & \cdots & \Omega_{2N+Ft} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \Omega_{N1t} & \Omega_{N2t} & \cdots & \Omega_{NNt} & \Omega_{NN+1t} & \Omega_{NN+2t} & \cdots & \Omega_{NN+Ft} \\ \hline\hline 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right]$$

The presence of wedges such as pre-existing distortions M_{ij}, markups μ_{it}, and input taxes or subsidies τ_{ijt} implies that the decentralized equilibrium generally departs from the first best. I use variables without a check mark (e.g., x_{kjt}, p_{it}, y_{it}) to denote allocations in the distorted economy, and variables denoted with a check mark (e.g., $\check{x}_{kjt}, \check{p}_{it}, \check{y}_{it}$) to denote the corresponding objects in the efficient allocation.

To facilitate the statement of the main results, I introduce an auxiliary input-output matrix $\tilde{\Omega}_t$, whose (i,j)-th element is given by

$$\tilde{\Omega}_{ijt} = \frac{\check{p}_{jt}\check{x}_{ijt}}{\check{p}_{it}\check{y}_{it}} = \frac{\mu_{it}\left(1-\tau_{ijt}\right)M_{ij}p_{jt}x_{ijt}}{p_{it}y_{it}}, \quad i \in C \cup N, j \in C \cup G \cup N \cup F$$

and

$$\tilde{\Omega}_{Gjt} = \frac{p_{jt}g_{jt}}{G_t}, \quad j \in N$$

For $i \in C \cup N$, this definition coincides with the cost share that would prevail in the efficient economy. The second equality in $\tilde{\Omega}_{ijt}$ shows that this efficient cost share can also be written as a function of distorted allocations. Appendix B.1 establishes this relationship formally.

For the government-spending row, the model features no government production function; aggregate public expenditure is simply the sum of purchases from producers. Hence the government row of the auxiliary matrix is defined as above. Under this definition, $\tilde{\Omega}_{Gjt}$ coincides with the government-spending distribution observed in the distorted economy.

Leontief Inverse Matrices. I define the auxiliary and distorted Leontief-inverse matrices as

$$\tilde{\Psi}_t \equiv (I - \tilde{\Omega}_t)^{-1} \quad \text{and} \quad \Psi_t \equiv (I - \Omega_t)^{-1}$$

Domar weights. The Domar weight of a producer is defined as the ratio of its nominal sales to nominal GDP. I extend this standard definition to incorporate the aggregate consumption share, the aggregate government spending share, and factor income shares. Specifically, the Domar weight of entity i is given by

$$\lambda_{it} = \frac{p_{it}y_{it}}{GDP_t} \quad \text{for} \quad i \in \{C \cup N\}, \quad \lambda_{ft} = \frac{p_{ft}L_{ft}}{GDP_t} \quad \text{for} \quad f \in F \quad \text{and} \quad \lambda_{Gt} = \frac{G_t}{GDP_t}$$

For $k \in N \cup F$, the Domar weight satisfies

$$\lambda_{kt} = \sum_{i \in N} \left(\lambda_{Ct}\Omega_{Cit} + \lambda_{Gt}\Omega_{Git}\right)\Psi_{ikt}.$$

I define the auxiliary aggregate consumption share and the auxiliary aggregate government spending share to match their counterparts in the distorted allocation,

$$\tilde{\lambda}_{Ct} = \frac{p_{Ct}y_{Ct}}{GDP_t}, \quad \tilde{\lambda}_{Gt} = \frac{G_t}{GDP_t}.$$

Thus, the auxiliary Domar weight $\tilde{\lambda}_{kt}$ for $k \in N \cup F$ is

$$\tilde{\lambda}_{kt} = \sum_{i \in N} \left(\tilde{\lambda}_{Ct}\tilde{\Omega}_{Cit} + \tilde{\lambda}_{Gt}\tilde{\Omega}_{Git}\right)\tilde{\Psi}_{ikt} = \sum_{i \in N} \left(\lambda_{Ct}\Omega_{Cit} + \lambda_{Gt}\Omega_{Git}\right)\tilde{\Psi}_{ikt}.$$

As with the auxiliary government-spending share allocated to each producer, $\tilde{\Omega}_{Gjt}$, the auxiliary aggregate private-consumption share and the auxiliary aggregate public-consumption share do not appear directly in the model's optimality or accounting conditions. However, the steady-state auxiliary Domar weights of producers and factors depend on these auxiliary aggregates.

Steady state. To simplify the exposition, and consistent with calibration practice, I assume that the government spending shares are identical in the distorted and undistorted economies at the steady state. Under this assumption,

$$\tilde{\Omega}_{Gj} = \frac{\check{p}_j \check{g}_j}{\check{G}} = \frac{p_j g_j}{G}, \quad j \in N$$

Thus, at the steady state the government spending shares that enter $\tilde{\Omega}$ can also be interpreted as those that would prevail in the efficient allocation.

In addition, I assume that the aggregate government spending share coincides with its distorted counterpart at the steady state, that is,

$$\check{\lambda}_G = \frac{\check{G}}{G\check{D}P} = \frac{G}{GDP}$$

With this additional assumption, the entire collection of auxiliary variables at the steady state is identical to the efficient steady-state allocation that would arise in an economy without wedges. These two steady-state assumptions do not affect any of the theoretical results in the paper; they matter only for interpretation. Strictly speaking, the resulting allocation in the auxiliary economy is only partially efficient, because the government-spending row of $\tilde{\Omega}$ is constructed from the (assumed identical) distorted spending shares, and the aggregate government spending share may differ from the fully efficient one.[3]

I adopt these two assumptions for the remainder of the paper. Consequently, any variable with a tilde at the steady state can also be interpreted as the corresponding efficient steady-state object. I refer to a steady state as efficient if the associated allocation is efficient; otherwise, the steady state is distorted.

In an efficient steady state of the static production-network model, the envelope theorem implies that tax policy has no effect on real output to a first-order approximation.[4] By contrast, when the steady state is distorted, appropriately designed tax instruments can improve the allocation.

The relations of the efficient steady state input-output matrix and distorted steady state input-output matrix satisfy that

$$\tilde{\Omega} = M \otimes \Omega$$

where $\otimes$ represented the elementwise product. M is the wedge matrix where M_{ij} is defined for $i \in C \cup N \& j \in N \cup F$ in the subsection 2.3, 1 otherwise.

The definition of the efficient production network used here is closely related to the cost-based input-output matrix in D. R. Baqaee & Farhi (2020). Because this paper focuses on fiscal policy, adopting this definition provides a transparent benchmark for assessing when and why policy intervention is justified. It clarifies the sources of distortions and the channels through which welfare-improving interventions operate.

[3] Accordingly, the steady-state auxiliary Domar weights and the steady-state auxiliary Leontief inverse computed from $\tilde{\Omega}$ should also be interpreted as partially efficient objects in the absence of these two assumptions.

[4] This result also applies to the partially efficient allocation in which the government-spending distribution and the aggregate government-spending share may differ from their fully efficient values. Because the model contains no government production function, and because government output is not used as an input in any producer's technology, the distribution of public expenditure does not enter the envelope-theorem argument. When evaluating the effects of a policy instrument under arbitrary distortions, information about the fully efficient allocation is therefore unnecessary; the partially efficient and distorted input-output matrices defined in the paper are sufficient.

I also distinguish between two steady-state Domar weights. The Domar weights associated with the efficient steady state are denoted by $\tilde{\lambda}_i$ and referred to as efficient Domar weights. In contrast, the Domar weights obtained under a distorted steady state are referred to as distorted Domar weights.

Because the government-purchase distribution is exogenously given in the model, both the efficient steady state and the corresponding efficient Domar weights depend on the steady-state government-spending distribution and the aggregate government-spending share.

Allen–Uzawa Elasticity of Substitution (AUES). Denote the Allen-Uzawa elasticity of substitution (AUES) between inputs j and i for producer k by $\sigma^{AU}_{k,ji,t}$. It satisfies

$$\tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t} = \frac{\partial \log a_{kjt}}{\partial \log p_{it}}$$

where a_{kjt} denotes the unit requirement of input j needed to produce one unit of output k in equilibrium. Thus, $\frac{\partial \log a_{kjt}}{\partial \log p_{it}}$ is the compensated (Hicksian) price elasticity of the input requirement. I use $\tilde{\Omega}_{ki}\sigma^{AU}_{k,ji}$ to represent this Hicksian price elasticity at the steady state. For the formal definition and derivation of these elasticities, see Appendix B.2.

3 Sufficient Statistics for Subsidy Shocks

3.1 Change in Domar Weight and Input–Output Linkages

Before presenting the effects of the policy shocks, I first outline some fundamental properties of the production network and the model. These properties provide intuition and facilitate the understanding of the subsequent propositions.

Lemma 1 (Network Decomposition of Changes in Domar Weights)**.** *Let $i \in C \cup G \cup N \cup F$. The first-order change in the Domar weight of i satisfies*

$$\mathrm{d}\lambda_{it} = \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \lambda_b \Omega_{bl} \, \mathrm{d}\log \Omega_{blt} \right) \Psi_{li} + \mathrm{d}\lambda_{Ct}\Psi_{Ci} + \mathrm{d}\lambda_{Gt}\Psi_{Gi}$$

Proof. See Appendix B.4. ■

Lemma 1 shows that the change in the Domar weight of entity i, $\mathrm{d}\lambda_{it}$, can be decomposed into two components: (i) changes in input–output linkages, $\mathrm{d}\log\Omega_{blt}$, and (ii) changes in aggregate consumption and government spending shares, $\mathrm{d}\lambda_{Ct}$ and $\mathrm{d}\lambda_{Gt}$. Importantly, these linkage and share adjustments are endogenous and, in general, depend on $\mathrm{d}\lambda_{it}$ itself through the production network.

Lemma 2 (First-Order Change in Input–Output Linkages)**.** *For any pair (b, l) with $b \in C \cup G \cup N$ and $l \in N \cup F$, the first-order change in the input–output linkage Ω_{blt} satisfies*

$$\begin{aligned} dlog\Omega_{blt} = &- d\log\mu_{bt} + \sum_{s \in N \cup F} \left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right) \right] dlogp_{st} - \mathbb{1}\{b \notin G\} \sum_{s \in N \cup F} \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right) d\tau_{bst} \\ &- \sum_{s \in N \cup F} \left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right) \right] dlogZ_{bst} + \mathbb{1}\{b \in C\} dlogw_{blt} - \mathbb{1}\{b \in C\} \sum_{s \in N \cup F} \tilde{\Omega}_{bs} d\log w_{bst} \\ &+ \mathbb{1}\{b \in G\}(dlogp_{lt} + dlogg_{lt} - dlog\lambda_{Gt}) \end{aligned}$$

where δ_{ls} denotes the Kronecker delta.

Proof. See Appendix B.2. ■

Lemma 2 follows from the dual approach, using a tractable transformation of the original unit-cost function. It shows that changes in input–output linkages are functions of exogenous shocks as well as endogenous variables—namely, prices and aggregate government spending—so that $\mathrm{d}\log\Omega_{blt}$ depends on both policy shocks and equilibrium adjustments.

3.2 First-Order Effects of Linkage-Specific Subsidies

In this subsection, I derive sufficient statistics characterizing the effects of linkage-specific subsidy shocks on price changes, changes in Domar weights, and changes in real GDP. The analysis is conducted using a first-order approximation that allows for arbitrary pre-existing wedges.

Proposition 1 (First-Order Effects of a Linkage-Specific Subsidy)**.** *Let $i \in C \cup G \cup N \cup F$, $k \in C \cup N$, and $j \in N \cup F$. Consider a marginal subsidy τ_{kjt} to the $j \to k$ linkage, evaluated around an arbitrary distorted state with pre-existing wedges. Then the first-order change in Domar weights is*

$$\frac{dlog\lambda_{it}}{d\tau_{kjt}} = -\sum_{l\in C\cup G\cup N\cup F}\Big(\frac{\lambda_k}{\lambda_i}\,\Omega_{kl}\,\tilde{\Omega}_{kj}\,(\sigma^{AU}_{k,lj}-1)\Big)\Psi_{li}$$

$$-\,\mathbb{1}\{k\notin C\}\,\tilde{\Omega}_{kj}\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\,\Omega_{bl}\Big[\sum_{s\in N\cup F}\big(\delta_{ls}+\tilde{\Omega}_{bs}(\sigma^{AU}_{a,ls}-1)\big)\tilde{\Psi}_{sk}\Big]\right)\Psi_{li}$$

$$-\,\mathbb{1}\{k\notin C\}\,\tilde{\Omega}_{kj}\sum_{s\in N\cup F}\left(\Omega_{Gs}\,\frac{\lambda_G}{\lambda_i}\sum_{n\in N}\tilde{\Psi}_{sn}\left(\Psi_{si}+\mathbb{1}\{i\in G\}-\Psi_{Ci}\right)\right)$$

$$+\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\,\Omega_{bl}\Big[\sum_{s\in N\cup F}\big(\delta_{ls}+\tilde{\Omega}_{bs}(\sigma^{AU}_{b,ls}-1)\big)\sum_{f\in F}\tilde{\Psi}_{sf}\Big(\frac{1}{1+\eta_f}\frac{d\log\lambda_{ft}}{d\tau_{kjt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\Big)\Big]\right)\Psi_{li}$$

$$+\sum_{s\in N\cup F}\left(\Omega_{Gs}\,\frac{\lambda_G}{\lambda_i}\sum_{f=1}^{F}\tilde{\Psi}_{sf}\Big(\frac{1}{1+\eta_f}\frac{d\log\lambda_{ft}}{d\tau_{kjt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\Big)\right)\left(\Psi_{si}+\mathbb{1}\{i\in G\}-\Psi_{Ci}\right)$$

The induced first-order change in prices satisfies

$$\frac{d\log p_{it}}{d\tau_{kjt}} = -\mathbb{1}\{k\notin C\}\tilde{\Psi}_{ik}\tilde{\Omega}_{kj}+\sum_{f\in F}\tilde{\Psi}_{if}\Big(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{d\tau_{kjt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\Big) \tag{2}$$

and the corresponding effect on real output (GDP) is

$$\frac{d\log Y_t}{d\tau_{kjt}} = \tilde{\lambda}_k\tilde{\Omega}_{kj}-\sum_{f\in F}\tilde{\lambda}_f\Big(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{d\tau_{kjt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\Big) \tag{3}$$

Proof. See Appendix B.6. ■

Proposition 1 characterizes the first-order change in Domar weights for consumers, the government, producers, and factors in the economy following a linkage-specific subsidy shock. To solve the system implied by the proposition, the procedure begins by deriving the response of factor-income shares, $\mathrm{d}\log\lambda_{it}$ for $i \in C \cup F$. Once the factor share adjustments and the change in the private consumption share are obtained, I substitute them back into the expressions above. This delivers the resulting changes in government expenditure shares, producer-level Domar weights, prices, and ultimately real GDP.

3.2.1 Matrix Representation of the Solution

To illustrate how the system can be solved in practice and in closed form, I now present the model in a compact matrix representation. For expositional clarity, it focus on the case with two primary factors, labor L and capital K.

Real GDP responds to a linkage-specific subsidy shock τ_{kjt} according to

$$\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}} = \tilde{\lambda}_k\tilde{\Omega}_{kj} - \tilde{\lambda}^\top W\frac{\mathrm{d}\log \Lambda_t}{\mathrm{d}\tau_{kjt}},$$

where $\tilde{\lambda}^\top = [\tilde{\lambda}_L \quad \tilde{\lambda}_K]$ denotes the corresponding efficient Domar-weight (factor share) vector,

$$\frac{\mathrm{d}\log \Lambda_t}{\mathrm{d}\tau_{kjt}} = \begin{bmatrix} \frac{\mathrm{d}\log \lambda_L}{\mathrm{d}\tau_{kjt}} \\ \frac{\mathrm{d}\log \lambda_K}{\mathrm{d}\tau_{kjt}} \\ \frac{\mathrm{d}\log \lambda_C}{\mathrm{d}\tau_{kjt}} \end{bmatrix}$$

and

$$W = \begin{bmatrix} \frac{1}{1+\eta_L} & 0 & \frac{\eta_L}{1+\eta_L} \\ 0 & \frac{1}{1+\eta_K} & \frac{\eta_K}{1+\eta_K} \end{bmatrix}$$

.

The vector of Domar-weight responses satisfies the linear system

$$\frac{\mathrm{d}\log \Lambda_t}{\mathrm{d}\tau_{kjt}} = \left(\Gamma + \Gamma^G\right) W\frac{\mathrm{d}\log \Lambda_t}{\mathrm{d}\tau_{kjt}} + \boldsymbol{\delta}^\tau_{kj},$$

where the matrices Γ and Γ^G collect the endogenous propagation resulting from intermediate-input linkages and government purchases, respectively. Both matrices are independent of the particular shock (k, j). Given the expression of Γ, Γ^G and $\boldsymbol{\delta}^\tau_{kj}$, the linear can be solved as

$$\frac{\mathrm{d}\log \Lambda_t}{\mathrm{d}\tau_{kjt}} = \left[I_{F+C} - (\Gamma + \Gamma^G)W\right]^{-1}\boldsymbol{\delta}^\tau_{kj},$$

where I_{F+C} is the identity matrix with dimension $(F + C) \times (F + C)$.

The **closed-form** expression characterizing the impact of the subsidy shock on real GDP is given by

$$\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}} = \tilde{\lambda}_k\tilde{\Omega}_{kj} - \tilde{\lambda}^\top W\left[I_{F+C} - (\Gamma + \Gamma^G)W\right]^{-1}\boldsymbol{\delta}^\tau_{kj}$$

I explain each terms in turn below.

Propagation matrices. The baseline propagation matrix Γ, of dimension $(F + C) \times F$, is defined elementwise as

$$\Gamma_{if} = \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\Omega_{bl}\sum_{s\in N\cup F}\left(\delta_{ls} + \tilde{\Omega}_{bs}(\sigma^{AU}_{a,ls} - 1)\right)\tilde{\Psi}_{sf}\right)\Psi_{li}, \quad i \in F\cup C, f \in F$$

This propagation matrix arise because a change in a subsidy τ_{kjt} can affect factor shares and share of private consumption endogenously, which in turn alter producer prices (see equation 2). This leads to changes in the input shares used by each producer (see Lemma 2), which ultimately feed back into factor prices throught the production networks (see Lemma 1).

The government-related propagation matrix Γ^G, of dimension $(F+C)\times F$, is captured by

$$\Gamma^G_{if} = \frac{\lambda_G}{\lambda_i} \sum_{s\in N\cup F} \Omega_{Gs}\tilde{\Psi}_{sf}\left(\Psi_{si} + \mathbb{1}\{i\in G\} - \Psi_{Ci}\right), \quad i\in F\cup C, f\in F.$$

This term in the model arises from the introduction of a second agent, the government. In a one-agent model, the consumption share is equal to one, implying $\mathrm{d}\log\lambda_{Ct} = 0$. As a result, the terms $\mathrm{d}\lambda_{Ct}\Psi_{Cl} + \mathrm{d}\lambda_{Gt}\Psi_{Gl}$ do not appear in the corresponding decomposition in Lemma 1. The changes in aggregate spending shares are not only driven solely by the quantities purchased by the government, g_{it}, but also by endogenous price movements. Because subsidy shocks affect producer prices heterogeneously, they alter the elements of the government-demand row of the input-output matrix. This generates a new propagation channel that operates through the joint effect of changes in Ω_{Glt} for $l\in N$ and the terms $\mathrm{d}\lambda_{Ct}\Psi_{Cl} + \mathrm{d}\lambda_{Gt}\Psi_{Gl}$.

Shock-specific term. For each subsidy shock τ_{kjt}, the term $\boldsymbol{\delta}^{\tau}_{kj}$ denotes an $(F+C)\times 1$ vector whose $\boldsymbol{i}$-th entry is $\delta^{\tau}_{i,kj}$. Equivalently, $\boldsymbol{\delta}^{\tau}_{kj} = \begin{bmatrix}\boldsymbol{\delta}^{\tau}_{L,kj}\\ \boldsymbol{\delta}^{\tau}_{K,kj}\\ \boldsymbol{\delta}^{\tau}_{C,kj}\end{bmatrix}$.

The effect of the subsidy enters through the vector

$$\delta^{\tau}_{i,kj} = \delta^{m}_{i,kj} + \mathbb{1}\{k\notin C\cup G\}\,\tilde{\Omega}_{kj}\,(\delta^{a}_{ik} - \delta^{G}_{ik}),$$

where

$$\delta^{m}_{i,kj} = -\sum_{l\in C\cup G\cup N\cup F}\left(\frac{\lambda_k}{\lambda_i}\Omega_{kl}\tilde{\Omega}_{kj}\left(\sigma^{AU}_{k,lj}-1\right)\right)\Psi_{li}$$

This term can be written as the two separate term, i.e.

$$\delta^{m}_{i,kj} = -\sum_{l\in C\cup G\cup N\cup F}\left(\frac{\lambda_k}{\lambda_i}\Omega_{kl}\tilde{\Omega}_{kj}\sigma^{AU}_{k,lj}\right)\Psi_{li} + \sum_{l\in C\cup G\cup N\cup F}\left(\frac{\lambda_k}{\lambda_i}\Omega_{kl}\tilde{\Omega}_{kj}\right)\Psi_{li}$$

The first component of $\delta^{m}_{i,kj}$ captures the fact that the linkage-specific subsidy $\mathrm{d}\tau_{kjt}$ changes the wedge between $j\to k$, which reduces the effective price that producer k pays for input j. This effective price change alters the demand for all inputs used by producer k. In particular, the induced change in the demand for input l is governed by the Hicksian price elasticity $-\tilde{\Omega}_{kj}\,\sigma^{AU}_{k,lj}$ (see Appendix equation 14), which describes the direct effect of the subsidy shock $\mathrm{d}\tau_{kjt}$ on $\mathrm{d}\log\Omega_{klt}$.

The second component of $\delta^{m}_{i,kj}$ captures the output price effect: a change in $\mathrm{d}\log\Omega_{klt}$ affects the output price of producer k by an amount equal to $\tilde{\Omega}_{kj}$ (see equation 2). This price change, in turn, feeds back into the input share $\mathrm{d}\log\Omega_{klt}$.

Finally, changes in $\mathrm{d}\log\Omega_{klt}$ propagate to entities' Domar weights via Lemma 1.

The next term, δ^{a}_{ik}, captures the price effects that propagate through the production network by altering both the input prices and the input demands of downstream producers.

$$\delta^{a}_{ik} = \sum_{l\subset C\cup G\cup N\cup F}\left(\sum_{b\subset C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\Omega_{bl}\left[\sum_{s\subset N\cup F}\left(\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\right)\tilde{\Psi}_{sk}\right]\right)\Psi_{li}$$

First, the subsidy $\mathrm{d}\tau_{kjt}$ affects the price of producer l through the production network, with the induced change transmitted according to Ψ_{sl}. This change in $\mathrm{d}\log p_{lt}$ alters the input shares $\mathrm{d}\log \Omega_{blt}$ of all producers b that use input l.

In addition, the subsidy shock modifies the price of producer s, $\mathrm{d}\log p_{st}$, which in turn affects Ω_{blt} through the Hicksian substitution channel. The magnitude of this effect is given by $\tilde{\Omega}_{bs}\,\sigma^{AU}_{b,ls}$, the Hicksian price elasticity of substitution between inputs l and s for producer b.

Finally, the resulting change in $\mathrm{d}\log \Omega_{blt}$ influences the Domar weight of producer i via Lemma 1.

The final term, δ^G_{ik}, arises from introducing the government as a second agent. It captures how the subsidy shock affects the government's spending shares, $\mathrm{d}\log \Omega_{Glt}$, through price changes that propagate across all gross-output producers, as well as its effect on the aggregate government spending share, $dlog\lambda_{Gt}$.

$$\delta^G_{ik} = \frac{\lambda_G}{\lambda_i} \sum_{s \in N \cup F} \Omega_{Gs}\, \tilde{\Psi}_{sk}\, (\Psi_{si} - \Psi_{Ci})$$

3.2.2 Subsidy Multiplier and Scale-to-Efficiency

The multiplier of a linkage-specific subsidy is defined as the change in real GDP relative to the fiscal cost of the intervention. Formally, for a marginal subsidy on the $j \to k$ linkage, the multiplier is

$$\mathcal{M}^{\tau}_{kj} \;\equiv\; \frac{\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}}}{\lambda_k \Omega_{kj}},$$

where the denominator $\lambda_k \Omega_{kj}$ is the additional expenditure required to lower the effective price of good $j \to k$ by one percent through a subsidy on link (k, j). Proposition 1 provides the fully fledged sufficient-statistics characterization of $\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}}$, and therefore delivers an exact sufficient-statistics expression for the subsidy multiplier $\mathcal{M}^{\tau}_{kj}$.

While the general sufficient-statistics formula is transparent, it is also rich: it depends on the production-network structure, substitution elasticities, the definition of factor markets (through factor-supply responses), and the additional propagation channel generated by the government as a distinct demand agent. To build intuition for the sources of multiplier heterogeneity, and in particular for what determines which links the government should subsidize, I derive a simplified expression under three assumptions: (i) a Cobb–Douglas economy ; (ii) no propagation through government demand ($\mathrm{d}\log \Omega_{Gkt} = 0$ and $\mathrm{d}\log \lambda_{Gt} = 0$); and (iii) for expositional clarity, only two primary factors, aggregate labor L and aggregate capital K.

Under these assumptions, the first-order response of Domar weights simplifies to

$$\frac{\mathrm{d}\log \lambda_{it}}{\mathrm{d}\tau_{kjt}} = \lambda_k \Omega_{kj}\, \frac{\Psi_{ji}}{\lambda_i} \tag{4}$$

Substituting into the first-order expression for real GDP yields

$$\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}} = \tilde{\lambda}_k \tilde{\Omega}_{kj} - \tilde{\lambda}_K \frac{1}{1+\eta_K}\, \lambda_k \Omega_{kj} \frac{\Psi_{jK}}{\lambda_K} - \tilde{\lambda}_L \frac{1}{1+\eta_L}\, \lambda_k \Omega_{kj} \frac{\Psi_{jL}}{\lambda_L}$$

Dividing by the fiscal cost $\lambda_k \Omega_{kj}$ yields a compact decomposition of the linkage-specific subsidy multiplier

$$\mathcal{M}^{\tau}_{kj} = \frac{\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kjt}}}{\lambda_k \Omega_{kj}} = \underbrace{\frac{\tilde{\lambda}_k \tilde{\Omega}_{kj}}{\lambda_k \Omega_{kj}}}_{\text{scale-to-efficiency}} + \underbrace{\left(-\frac{1}{1+\eta_K}\, \tilde{\lambda}_K \frac{\Psi_{jK}}{\lambda_K} - \frac{1}{1+\eta_L}\, \tilde{\lambda}_L \frac{\Psi_{jL}}{\lambda_L} \right)}_{\text{factor adjustment}}$$

Even in the Cobb–Douglas benchmark (with the government-propagation channel switched off), optimal subsidy design depends on two conceptually distinct forces.

Scale-to-efficiency. I refer to the first term as the *scale-to-efficiency* component. The numerator $\tilde{\lambda}_k\tilde{\Omega}_{kj}$ measures the size of the $j \to k$ linkage to GDP in the efficient steady state, whereas $\lambda_k\Omega_{kj}$ measures its relative size to GDP in the distorted economy. Hence,

$$\frac{\tilde{\lambda}_k\tilde{\Omega}_{kj}}{\lambda_k\Omega_{kj}}$$

captures the relative scale of that linkage under efficiency versus distortion.

Intuitively, a value above one indicates that the linkage is inefficiently small in the distorted economy, whereas a value below one indicates that it is inefficiently large.

This statistic has a particularly appealing property under the simplified setting (No government propagation and Cobb-Douglas production). In this benchmark, the scale-to-efficiency ratio becomes a sufficient statistic for determining where factor subsidies should be targeted.

To see this, suppose the government is considering a labor-subsidy program but has not yet decided which producer should receive the subsidy. Under the simplified assumptions, the multiplier of a subsidy τ_{kLt} applied to producer k's labor use is

$$\mathcal{M}^{\tau}_{kL} = \frac{\frac{\mathrm{d}\log Y_t}{\mathrm{d}\tau_{kLt}}}{\lambda_k\Omega_{kL}} = \underbrace{\frac{\tilde{\lambda}_k\tilde{\Omega}_{kL}}{\lambda_k\Omega_{kL}}}_{\text{scale-to-efficiency}} + \underbrace{\left(-\frac{1}{1+\eta_K}\tilde{\lambda}_K\frac{\Psi_{LK}}{\lambda_K} - \frac{1}{1+\eta_L}\tilde{\lambda}_L\frac{\Psi_{LL}}{\lambda_L}\right)}_{\text{factor adjustment}}$$

Because the factor-adjustment term does not depend on the identity of the subsidized producer k, the variation between producers in the subsidy multiplier is driven entirely by the scale-to-efficiency component.

$$\mathcal{M}^{\tau}_{kL} = \underbrace{\frac{\tilde{\lambda}_k\tilde{\Omega}_{kL}}{\lambda_k\Omega_{kL}}}_{\text{scale-to-efficiency}} + t.i.w.p.$$

where *t.i.w.p.* is terms independent of where the policy targets. Thus, in the simplified economy, the scale-to-efficiency ratio is a sufficient statistic for determining which sector should receive the labor subsidy.

The same logic applies to proportional cost or sales subsidies. Suppose the government implements a proportional (node-level) subsidy for sector J. Under the same assumptions, it has

$$\mathcal{M}^{\tau}_{kJ} = \underbrace{\frac{\tilde{\lambda}_k\tilde{\Omega}_{kJ}}{\lambda_k\Omega_{kJ}}}_{\text{scale-to-efficiency}} + t.i.w.p.$$

Hence, among all outbound linkages from sector J, those with the highest scale-to-efficiency ratios deliver the largest gains in real GDP when subsidized. The scale-to-efficiency statistic therefore identifies the linkages through which proportional subsidies operate most effectively.

Factor adjustment. The second component is the *factor-adjustment* term, which reflects how the subsidy shifts primary-factor income shares through the production network. From equation 4,

$$\frac{\frac{\mathrm{d}\log\lambda_{ft}}{\mathrm{d}\tau_{kjt}}}{\lambda_k\Omega_{kj}} = \frac{\Psi_{jf}}{\lambda_f}, \quad f \in \{K, L\}$$

the ratios $\frac{\Psi_{jf}}{\lambda_f}$ measure the multiplier of the subsidy τ_{kjt} on factor share f . Multiplying these terms by the efficient Domar weights $\tilde{\lambda}_f$ and the factor-supply elasticities η_f yields the general-equilibrium contribution of factor-market adjustments to the subsidy GDP multiplier.

Importance of the two terms. In practice, variation in the scale-to-efficiency component is typically much larger than variation in the factor-adjustment component, especially when pre-existing distortions are substantial and linkages are disproportionately distorted, and when factor supplies are inelastic. For this reason, I refer to the scale-to-efficiency ratio as the informative statistic for determining which linkages should be subsidized for a given type of fiscal intervention. The quantitative results in Section 6 verify the performance of this simple statistic.

Comparison to the node-level analysis. One might think that the scale-to-efficiency statistic could also be recovered from a node-level analysis, since in production networks any input–output link can, in principle, be represented as a separate node by introducing a fictitious producer. To clarify why this statistic can be obtained only from the link-level analysis, I rewrite the link-level subsidy problem in its node-level form using the example of an energy-subsidy program.

Suppose the government intends to subsidize energy but has not yet decided which sector should receive the subsidy. In the link-level formulation, the multiplier of a subsidy to link $(E \to i)$ decomposes, under the same simplifying assumptions, into a *scale-to-efficiency* component and a *factor-adjustment* term:

$$\frac{\frac{d\log Y_t}{d\tau_{iEt}}}{\lambda_i \Omega_{iE}} = \underbrace{\frac{\tilde{\lambda}_i \tilde{\Omega}_{iE}}{\lambda_i \Omega_{iE}}}_{\text{scale to efficiency}} + \underbrace{\left(-\frac{1}{1+\eta_K} \tilde{\lambda}_K \frac{\Psi_{EK}}{\lambda_K} - \frac{1}{1+\eta_L} \tilde{\lambda}_L \frac{\Psi_{EL}}{\lambda_L} \right)}_{\text{factor adjustment}}$$

A key feature revealed only in the link-level decomposition is that the factor-adjustment term does not depend on the identity of the targeted link i. Thus, all cross-link variation in the subsidy multiplier arises entirely from the scale-to-efficiency term. When the same question is reformulated at the node level by rewriting the link $(E \to i)$ as a node j, the multiplier becomes

$$\frac{\frac{d\log Y_t}{d\mu_{jt}}}{\lambda_j} = \underbrace{\frac{\tilde{\lambda}_j}{\lambda_j}}_{\text{scale to efficiency}} + \underbrace{\left(-\frac{1}{1+\eta_K} \tilde{\lambda}_K \frac{\Psi_{jK}}{\lambda_K} - \frac{1}{1+\eta_L} \tilde{\lambda}_L \frac{\Psi_{jL}}{\lambda_L} \right)}_{\text{factor adjustment}}$$

In this sector-level representation, it is not apparent that the factor-adjustment term should be independent of the target fictitious sector j, because the terms Ψ_{jL} and Ψ_{jK} are indexed by j. Additionally, there is no economic explanation for the factor-specific Leontief inverses Ψ_{jL} and Ψ_{jK}. Even if one defines them as the direct and indirect labor or capital required to produce the output of fictitious producer j, it is not obvious why these terms are independent of j. Thus, even though scale-to-efficiency may appear numerically similar to a node-level ratio, its conceptual role becomes transparent only in the link-level analysis, where the decomposition isolates the policy-relevant object and makes clear that it is the sole source of variation in the link-specific multiplier.

4 Sufficient Statistics for Spending Shocks

4.1 Spending Shocks

In this subsection, I characterize the model's response to a shock to government purchases, g_{kt}.

Proposition 2 (First-Order Change in Domar Weights from Government Spending Shocks)**.** *Let $i \in C \cup G \cup N \cup F$, and consider a marginal change in government spending on product $k \in N$, represented by $\mathrm{d}\log g_{kt}$. The first-order change in the Domar weight of i is given by*

$$\frac{d\log\lambda_{it}}{d\log g_{kt}} = \Omega_{Gk}\frac{\lambda_G}{\lambda_i}\Big[\Psi_{ki} + \mathbb{1}\{i\in G\} - \Psi_{Ci}\Big]$$
$$+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\Omega_{bl}\Big[\sum_{s\in N\cup F}\left(\delta_{ls}+\tilde{\Omega}_{bs}(\sigma^{AU}_{a,ls}-1)\right)\sum_{f=1}^{F}\tilde{\Psi}_{sf}\Big(\frac{1}{1+\eta_f}\frac{d\log\lambda_{ft}}{d\tau_{kjt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\Big)\Big]\right)\Psi_{li}$$
$$+ \sum_{s\in N\cup F}\left(\Omega_{Gs}\frac{\lambda_G}{\lambda_i}\left[\sum_{f=1}^{F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{dlogg_{kt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{dlogg_{kt}}\right)\right]\right)\Big[\Psi_{si}+\mathbb{1}\{i\in G\}-\Psi_{Ci}\Big]$$

The induced first-order change in prices satisfies

$$\frac{d\log p_{it}}{dlogg_{kt}} = \sum_{f\in F}\tilde{\Psi}_{if}\Big(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{dlogg_{kt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{dlogg_{kt}}\Big)$$

and the corresponding effect on real output (GDP) is

$$\frac{d\log Y_t}{dlogg_{kt}} = -\sum_{f\in F}\tilde{\lambda}_f\Big(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{dlogg_{kt}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{dlogg_{kt}}\Big) \tag{5}$$

Proof. See Appendix B.7. ■

As in the analysis of subsidy shocks, this system of equations can be solved by first computing the changes in factor and private consumption shares and then using these changes to determine the resulting adjustments in Domar weights, prices, and real GDP.

The effect of a government spending shock on factor shares can be decomposed into three components. The second and third components correspond to the network propagation channels already discussed in the subsidy-shock analysis: government spending changes factor demands, which in turn alter prices, modify input–output shares, and adjust aggregate consumption and government spending shares, feeding back into factor shares through the network. The first component captures the direct quantity effect of government purchases on input–output shares and the aggregate government spending share, which then affects factor shares via Lemma 1.

Proposition 2 (Equation 5) shows that the effect of a government spending shock on real GDP depends entirely on its impact on factor and consumption shares.

4.1.1 Matrix Representation of the Solution

To illustrate how the analytical expressions derived above map into the closed-form solution and to show what determines the appropriate targeting of government purchases, I rewrite the equilibrium system in matrix form.

Up to a first-order approximation, the effect of a marginal government purchases shock $d\log g_{kt}$ on real GDP is

$$\frac{\mathrm{d}\log Y_t}{\mathrm{d}\log g_{kt}} = -\tilde{\lambda}^{\top} W \frac{\mathrm{d}\log\Lambda_t}{\mathrm{d}\log g_{kt}},$$

where $\tilde{\lambda}^{\top}$ denotes the vector of efficient Domar weights for factors, and W is the block matrix collecting factor-supply elasticities. The vector $\mathrm{d}\log\Lambda_t$ collects the responses of all factor and consumer Domar weights.

The change in these Domar weights satisfies the linear system

$$\frac{\mathrm{d}\log\Lambda_t}{\mathrm{d}\log g_{kt}} = \left(\Gamma + \Gamma^{G}\right) W \frac{\mathrm{d}\log\Lambda_t}{\mathrm{d}\log g_{kt}} + \boldsymbol{\delta}^{G}_{k}$$

where Γ and Γ^G are the baseline and government-related propagation matrices introduced previously. Under the assumption of two primary factors, aggregate labor and aggregate capital, their interpretation carries over directly to the spending-shock environment. The vector $\boldsymbol{\delta}_k^G$, defined below, collects the shock-specific contributions of a government purchase of good k.

Solving the system yields the **closed-form** expression for the effect of a government-spending shock on real GDP:

$$\frac{\mathrm{d}\log Y_t}{\mathrm{d}\log g_{kt}} = -\tilde{\lambda}^\top W \left[I_{C+F} - \left(\Gamma + \Gamma^G\right) W\right]^{-1} \boldsymbol{\delta}_k^G$$

Shock-specific term. For each government-purchases shock g_{kt}, the vector $\boldsymbol{\delta}_k^G$ is an $(F+C)\times 1$ vector whose i-th entry is

$$\delta_{i,k}^G = \Omega_{Gk} \frac{\lambda_G}{\lambda_i} \left[\Psi_{ki} + \mathbb{1}\{i \in G\} - \Psi_{Ci}\right]$$

This term captures the direct effect of a marginal increase in government purchases of good k on the Domar weight of i through its direct effect on $dlog\Omega_{Gkt}$ and $dlog\lambda_{Gt}$ (see Lemma 1).

4.2 Spending Multiplier

The government-spending multiplier on sector k is defined as the change in real GDP relative to the fiscal cost of increasing government purchases of good k. Formally,

$$\mathcal{M}_k^G = \frac{\frac{\mathrm{d}\log Y_t}{\mathrm{d}\log g_{kt}}}{\lambda_G \Omega_{Gk}},$$

where the denominator $\lambda_G \Omega_{Gk}$ is the additional government expenditure required to raise purchases of good k by one percent.

To connect this definition to the closed-form solution, note that in the matrix representation of spending shocks the only object that varies with the *location* of spending is the shock-specific vector $\boldsymbol{\delta}_k^G$. The i-th entry of this vector is

$$\delta_{i,k}^G = \Omega_{Gk} \frac{\lambda_G}{\lambda_i} \Psi_{ki} + t.i.w.p., \qquad i \in F \cup C,\ k \in N,$$

where *t.i.w.p.* again denotes terms independent of where the policy targets, so that the dependence on k enters only through the Leontief-inverse exposure Ψ_{ki} and the steady state government spending share Ω_{Gk}.

Substituting the closed-form solution for $\mathrm{d}\log Y_t/\mathrm{d}\log g_{kt}$ and dividing by $\lambda_G \Omega_{Gk}$ yields

$$\mathcal{M}_k^G = -\tilde{\lambda}^\top W \Big[I_{C+F} - (\Gamma + \Gamma^G) W\Big]^{-1} \boldsymbol{m}_k^G + t.i.w.p.$$

where $\boldsymbol{m}_k^G \in \mathbb{R}^{F+C}$ is the shock-location vector with entries

$$m_{i,k}^G \equiv \frac{\Psi_{ki}}{\lambda_i}, \qquad i \in F \cup C$$

Finally, since $\Psi_{kC} = 0$ for $k \in N$, only factor components matter. In particular, the spending multiplier can be written as a linear function of factor Leontief-inverse exposures

$$\mathcal{M}_k^G \;=\; \sum_{f \in F} O_f\, \Psi_{kf} + t.i.w.p.$$

where O_f is defined by

$$O_f \equiv \frac{1}{\lambda_f}\left(-\tilde{\lambda}^\top W\left[I_{C+F} - (\Gamma + \Gamma^G)W\right]^{-1}\right)_f,$$

and $(\cdot)_f$ denotes the component associated with factor $f \in F$. Hence, sectoral differences in spending multipliers are driven by the factors' Leontief-inverse exposures, Ψ_{kf}, whereas the weights O_f are negative, fixed, and do not depend on the location of the spending shock.

Sufficient Statistics Cases. Based on the closed-form solution above, there are several special cases in which *labor Leontief-inverse exposure* serves as a sufficient statistic for identifying where government purchases have the largest multiplier.

Single-factor economy (labor only). If labor is the only primary factor, then the labor Leontief-inverse exposure of sector k, Ψ_{kL}, is sufficient for determining which government purchases have a larger effect on aggregate output. In this case, the ranking of government-spending multipliers is *negatively related* to Ψ_{kL}. Thus, sectors with larger labor Leontief-inverse exposure have smaller spending multipliers in this environment. Moreover, in a frictionless economy, each producer's labor Leontief-inverse exposure equals one, $\Psi_{kL} = 1$, implying that spending multipliers are identical across producers in this special case.

Two-factor frictionless economy (elastic labor, inelastic capital). Another instructive case arises in a frictionless economy with two primary factors: elastic labor and inelastic capital. Under frictionless markets, the labor and capital Leontief-inverse exposures for any sector sum to one. Since the fixed weights on both factors are negative and the weight on the inelastic factor (capital) is substantially more negative under standard calibrations, the spending multiplier becomes *positively related* to the labor Leontief-inverse exposure of the targeted sector. In this case, sectors with higher labor exposure tend to exhibit larger spending multipliers. Straightforwardly, if the ratio of the output elasticity of labor to the output elasticity of capital is the same across producers, then all sectors share the same factor Leontief-inverse exposure, and government spending multipliers are identical across producers.

Taken together, these cases show *why* factor-market structure is central to multiplier heterogeneity. In particular, changes in factor-market assumptions can even *reverse the sign* of the relationship between the government-spending multiplier and labor Leontief-inverse exposure.

5 Extensions and Discussion

In most applications, the *distorted* steady state can be read directly from the data, since the real economy typically contains wedges arising from markups, taxes, policy interventions, and frictions. By contrast, recovering the *efficient* steady state generally requires information about these wedges or additional identifying assumptions. This section develops general results that allow the researcher to infer a new efficient steady state using only the initial steady state and the model's primitives, and to incorporate indirect effects of policy interventions that operate through induced changes in productivity, demand, markups, or other wedges.

Proposition 3 in Appendix A characterizes the general first-order responses of a semiparametric production network to small perturbations in fundamentals. In addition to the subsidy and government spending shocks discussed above, it encompasses markup shocks, Hicksian-neutral productivity shocks, input-augmenting productivity shocks, and preference (demand) shocks. Among these shocks, productivity and demand shocks alter the efficient steady state; markup and wedge shocks instead move the distorted steady state away from efficiency. Proposition 3 provides the mapping needed to trace how prices, Hicksian input demands, Domar weights, and input–output linkages adjust under any of these shocks.

Implementation with pre-crisis data. The results furnish a practical guide when only pre-crisis information is available. Taking the COVID-19 crisis as an example, if the pre-COVID allocation is taken as efficient, then the new efficient steady state associated with COVID-19 can be represented by a combination of input-augmenting productivity shocks Z_{kLt} for $k \in N$ and demand shocks $\mathrm{d}\log w_{Cjt}$ for $j \in N$. Given such shocks, either observed or posited, Proposition 3 provides the mapping from the initial efficient steady state to the new efficient benchmark via the implied adjustments in Domar weights and input–output linkages. The new distorted steady state can then be taken from the post-shock real economy data. With these two objects in hand (the counterfactual efficient network and the observed distorted network), the fiscal policy analysis from the previous section applies directly.

Large shocks and iterative updating. Although our analysis is conducted at first order, it can accommodate larger shocks if the shock path is known or can be approximated linearly. In that case, the model can be solved iteratively using *Differential Exact-Hat Algebra*, as described in D. R. Baqaee et al. (2024).

Indirect policy effects. Proposition 3 also makes it straightforward to incorporate *indirect* policy effects, including situations where the policy itself changes productivity, demand, markups, or input-augmenting productivity shocks. These indirect effects can be obtained through the endogenous responses in the model or from reduced-form analysis, which lies outside the scope of the current paper. Since the analysis is linear, the direct and indirect effects are additive. For example, if a fiscal policy changes the average markup in an industry, the aggregate effect of this policy shock equals the sum of the direct policy shock and the induced markup effect. Thus, Proposition 3 makes the sufficient-statistics approach operational for analyzing both direct and indirect consequences of fiscal interventions in production network analysis.

6 Fiscal Multipliers in a COVID-Induced Distorted Economy

6.1 Calibration

In this section, I employ the theoretical results derived above to quantify fiscal multipliers in the U.S. economy around the COVID-19 period. The model is calibrated using data from the Bureau of Economic Analysis (BEA) Input-Output Accounts. I use the most recent BEA release that spans the years before, during, and after the COVID-19 shock. In particular, the efficient steady-state input-output matrix Ω_{ji} for $j \in C \cup N$ and $i \in N \cup F$ is calibrated using data from 2019, the last year before the pandemic materially took effect. The distorted input-output linkages, along with consumption and government expenditure shares, are calibrated using 2022 data. Years 2020–2021 are excluded from the distorted calibration because they coincide with the acute phase of the shock, when sectoral productivities and final-demand compositions changed abruptly and transitorily, thereby shifting the model's efficient steady state. The assumption of the current calibration is that, absent persistent wedges, productivity and demand revert toward their pre-COVID levels, so that 2022 provides a cleaner snapshot of a post-shock allocation that embeds residual distortions rather than contemporaneous transitory shocks in productivity and demands. In this interpretation, lockdowns, information frictions, price rigidities, and heterogeneous policy interventions during the pandemic generated wedge-like departures from efficiency, creating scope for fiscal policy to improve aggregate outcomes. Fiscal multipliers are evaluated around this distorted steady state.

This calibration strategy differs from the standard approach used in models featuring markup distortions. For example, in a model where markups are embedded directly in the firm's cost structure, each row of the distorted input-output matrix sums to less than one. In contrast, my calibration removes the effects of markups from the cost structure: even in the distorted

equilibrium, each row of the input-output matrix continues to sum to one. This feature is enabled by the general framework developed above, which accommodates arbitrary distortions at the link-level directly. In models with purely node-level distortions, such a calibration would be considerably more restrictive and difficult to implement.

The model's quantitative responses depend on the Allen–Uzawa elasticities of substitution. To obtain these elasticities, I adopt a nested CES production structure. Following standard practice in the literature, I use a two-nest specification. In the first nest, a value-added producer combines capital and labor to generate value added, and an intermediate-mix producer aggregates material inputs sourced from all sectors. In the second nest, a gross-output producer combines value added with the aggregate of intermediate inputs to produce gross output.

I calibrate the production elasticities following D. Baqaee & Farhi (2022). The elasticity of substitution between labor and capital is set to 0.6; the elasticity between value added and intermediates is 0.6; and the elasticity across intermediate inputs is 0.2. On the household side, I assume an elasticity of substitution across consumption goods of $\theta_C = 0.9$ and a Frisch elasticity of labor supply of 2, consistent with Rubbo (2023).

Regarding factor markets, I focus on the two primary factors, capital and labor, and consider two benchmark cases: (i) fully integrated factor markets and (ii) fully segmented factor markets.

6.2 Subsidy Policy

Before presenting the quantitative results, I introduce an additional statistic, *distortion centrality*, inspired by Liu (2019). Distortion centrality is defined as the ratio of an industry's efficient Domar weight to its distorted Domar weight. Empirically, I proxy this ratio by the industry's share of nominal GDP after the COVID period relative to its pre-COVID benchmark.

Thus, the analysis employs three related statistics. First, the paper's *sufficient statistics* are derived from the full model and characterize the exact first-order responses and multipliers. Second, the *informative statistic*, the scale-to-efficiency ratio, is shown to be informative under a set of simplifying assumptions. Third, *distortion centrality* is motivated by the literature and provides a parsimonious, empirically implementable sector-level index.

In the quantitative exercise, the scale-to-efficiency statistic also has a direct mapping to the data: it corresponds to the ratio of a linkage's size relative to GDP after COVID to its pre-COVID benchmark. I report the resulting multipliers and examine how well distortion centrality and scale-to-efficiency correlate with the model-implied sufficient statistics.

Regrading the subsidy-policy multipliers, I consider several distinct objects corresponding to different fiscal instruments. The BEA Input–Output Accounts contain $N = 71$ sectors.

(i) Proportional production subsidy (node-level). A proportional production subsidy to sector k subsidizes all of its input purchases uniformly. The associated multiplier is defined as

$$\frac{\sum_{j \in N \cup F} \frac{\mathrm{d} \log Y_t}{\mathrm{d} \tau_{kjt}}}{\lambda_k},$$

and yields an $N \times 1$ vector of multipliers, one for each producing sector.[5]

[5] In the model we have

$$\frac{\sum_{j \in N \cup F} \frac{\mathrm{d} \log Y_t}{\mathrm{d} \tau_{kjt}}}{\lambda_k} = \frac{\sum_{b \in C \cup N} \frac{\mathrm{d} \log Y_t}{\mathrm{d} \tau_{bkt}}}{\lambda_k} = -\frac{\frac{d \ log Y_t}{d log \mu_{kt}}}{\lambda_k}$$

(ii) Linkage-specific production subsidy. These multipliers correspond to the main theoretical results of the paper. A subsidy can be applied to any specific buyer–seller linkage $j \to k$, generating an $N \times N$ matrix of multipliers.

(iii) Labor and capital subsidies (linkage-specific). These policies subsidize the use of labor or capital by each producer. Since every sector can receive a subsidy on each factor, labor and capital subsidies each produce an $N \times 1$ vector of multipliers.

(iv) Consumption subsidy (linkage-specific). A consumption subsidy applies to the purchase of final goods by the household. Because each sector can receive such a subsidy, this yields another $N \times 1$ vector of multipliers.

Results. Appendices C.1 and C.2 report the fiscal multipliers derived from the full model using the calibration described in the previous subsection, under both integrated and segmented factor market environments.

Table 2 reports the correlations between the distortion centrality statistic and the various multipliers. It shows that the distortion centrality measure is relatively informative in explaining variation in the proportional-subsidy multiplier. However, as expected, its explanatory power declines substantially for more targeted policy instruments, such as factor-specific subsidies (labor or capital).

Table 3 reports the correlation between the scale-to-efficiency statistic and the multipliers. In contrast to distortion centrality, the scale-to-efficiency measure is consistently more informative for predicting link-specific multipliers, particularly consumption subsidies and subsidies applied to factors that are integrated and inelastically supplied.

Table 2: Correlation of Distortion Centrality with Multiplier

		Proportional Subsidy	Consumption Multiplier	Labor Multiplier	Capital Multiplier
Distortion Centrality	(I)	0.9206	0.7865	0.4019	0.3531
Distortion Centrality	(II)	0.7555	0.7699	0.3968	0.3647
Distortion Centrality	(III)	0.6824	0.7435	0.2509	0.3763
Distortion Centrality	(IV)	0.6561	0.6590	0.2818	NaN

Note: (I) Cobb–Douglas economy with inelastic factors and integrated markets; (II) Cobb–Douglas economy with elastic labor and inelastic capital under integrated markets; (III) Nested-CES economy with elastic labor and inelastic capital under integrated markets; (IV) Nested-CES economy with elastic labor and inelastic capital under segmented markets.

Table 3: Correlation of Scale-to-Efficiency with Multiplier

		Production Subsidy	Consumption Multiplier	Labor Multiplier	Capital Multiplier
Scale-to-Efficiency	(I)	0.9998	0.9992	1	1
Scale-to-Efficiency	(II)	0.9734	0.9310	0.8318	0.9591
Scale-to-Efficiency	(III)	0.9373	0.8861	0.7748	0.9051
Scale-to-Efficiency	(IV)	0.9207	0.7985	0.7071	NaN

Note: (I) Cobb–Douglas economy with inelastic factors and integrated markets; (II) Cobb–Douglas economy with elastic labor and inelastic capital under integrated markets; (III) Nested-CES economy with elastic labor and inelastic capital under integrated markets; (IV) Nested-CES economy with elastic labor and inelastic capital under segmented markets.

It is worth highlighting the difference between cases (I) and (II) in Table 3. The only change from (I) to (II) is that labor supply becomes elastic rather than inelastic. The theoretical results

in Section 3.2.2 imply that, in the absence of propagation through government demand, scale-to-efficiency remains a sufficient statistic for factor subsidy multipliers even when factors are elastically supplied. Therefore, the fact that the correlation between scale-to-efficiency and the factor multipliers is below one in case (II) is driven precisely by the introduction of the government as an additional demand agent. This highlights an interaction between subsidy policy and the presence of government demand: government-induced propagation breaks the exact sufficiency result that would otherwise hold.

6.3 Spending Multiplier

Table 4 reports the government spending multipliers under integrated factor markets, while Table 5 presents the corresponding results for segmented factor markets. Comparing the two sets of results shows that the ranking of sectors with respect to their spending multipliers is highly sensitive to whether labor markets are integrated or segmented.

Our calibration also implies that the labor and capital Leontief-inverse exposures sum to one. Thus, when factor markets are integrated, the size of the government-spending multiplier is positively aligned with the structure of the labor Leontief-inverse exposure across sectors: sectors that require more labor, directly or indirectly, tend to exhibit larger multipliers. By contrast, when factor markets are segmented, the ranking of spending multipliers is determined by a richer combination of factor-specific Leontief inverses across sectors, weighted by a vector that depends on substitution elasticities, factor-supply elasticities, pre-existing wedges, and the entire input–output network, yet is invariant to the specific spending shock. Consequently, the mapping from production network structure to fiscal transmission becomes substantially richer and less linear when markets are segmented.

Table 4: Government Spending Multipliers with Integrated Factor Markets

Code	**Industry**	**Spending Multiplier**
493	Warehousing and storage	0.58853
5415	Computer systems design and related services	0.56768
623	Nursing and residential care facilities	0.56299
55	Management of companies and enterprises	0.54973
523	Securities, commodity contracts, and investments	0.54528
624	Social assistance	0.54373
61	Educational services	0.52359
GSLG	State and local general government	0.50776
621	Ambulatory health care services	0.50636
622	Hospitals	0.50503
452	General merchandise stores	0.49822
GFE	Federal government enterprises	0.49374
561	Administrative and support services	0.48137
81	Other services, except government	0.46710
722	Food services and drinking places	0.45665
113FF	Forestry, fishing, and related activities	0.45412
54120P	Miscellaneous professional, scientific, and technical services	0.44994

Table 5: Government Spending Multipliers with Segmented Factor Markets

Code	Industry	Spending Multiplier
GFGD	Federal general government (defense)	0.68986
GFGN	Federal general government (nondefense)	0.65220
493	Warehousing and storage	0.63683
GSLG	State and local general government	0.62179
5415	Computer systems design and related services	0.55351
55	Management of companies and enterprises	0.55164
523	Securities, commodity contracts, and investments	0.53268
525	Funds, trusts, and other financial vehicles	0.52456
4870S	Other transportation and support activities	0.52261
321	Wood products	0.51187
GFE	Federal government enterprises	0.51068
561	Administrative and support services	0.50470
624	Social assistance	0.50378
623	Nursing and residential care facilities	0.49875
315AL	Apparel and leather and allied products	0.48542
54120P	Miscellaneous professional, scientific, and technical services	0.47803
113FF	Forestry, fishing, and related activities	0.46662
452	General merchandise stores	0.45989
335	Electrical equipment, appliances, and components	0.44873
622	Hospitals	0.43867

7 Conclusion

This paper develops a tractable framework for analyzing linkage-specific fiscal policies in production networks with link-level distortions. By modeling fiscal instruments directly at the level of buyer–seller relationships and by treating the government as a distinct demand agent, the framework moves beyond representative-agent environments and node-level analyses, permitting a direct characterization of how fiscal interventions propagate through individual input–output linkages.

The central theoretical contribution is a set of closed-form sufficient statistics that characterize how link-level interventions (production, consumption, and factor subsidies, as well as targeted government purchases) affect equilibrium prices, Domar weights, and real output. These statistics make clear why the impact of a subsidy applied to a specific input-output link generally cannot be reduced to the impact of a node-level wedge: optimal targeting depends on the interaction between distortions and network propagation at the link level.

Building on these sufficient statistics, the paper develops a simple yet powerful empirical guide for subsidy design: the scale-to-efficiency measure, defined as the ratio of a linkage's efficient size relative to GDP to its realized size in the distorted allocation. I show analytically why this statistic is informative for subsidy policy targets. In particular, it identifies which sectors and linkages are too small or too large relative to their efficient scales and therefore where marginal fiscal support delivers the greater gains in real GDP.

Quantitatively, applying the framework to U.S. input–output data from 2019 and 2022, which bracket the COVID-19 shock, I document substantial dispersion in fiscal multipliers across sectors and instruments. A sector-level measure of distortion centrality, motivated by Liu (2019), is informative for proportional production subsidies, but it is less sufficient for linkage-specific subsidies. Instead, scale-to-efficiency provides a markedly more informative guide for subsidy

targeting across a broader range of instruments, consistent with the paper's theoretical prediction.

Under the calibration, a linkage's scale-to-efficiency is proportional to the ratio of its realized scale during the shock to its pre-shock scale. This mapping provides a rationale for the widespread crisis-era practice of supporting the most adversely affected sectors and firms. Sectors or linkages that contract the most during a crisis tend to be those that are most distorted relative to their efficient scales, assuming the pre-crisis allocation reflects the efficient benchmark. Targeting fiscal support toward these "most-hit" activities can therefore generate favorable general-equilibrium effects by alleviating distortions and improving the allocation of resources.

The paper also characterizes the structure of government-spending multipliers. When factor markets are integrated, this characterization implies that sectoral spending multipliers are closely aligned with labor Leontief-inverse exposure: the relationship is negative in a single-factor (labor-only) economy, but becomes positive in a frictionless two-factor economy with elastic labor and inelastic capital. When factor markets are segmented, spending multipliers depend on a weighted sum of factor-specific Leontief-inverse exposures, where the weights are invariant to the spending location. Thus, to understand heterogeneity in the sector-level spending multipliers, differences in factor-specific Leontief inverse exposures are the key statistics governing cross-sector variation. Quantitatively, both the ranking and the magnitude of spending multipliers shift systematically with the degree of factor-market integration.

Overall, the results underscore a clear policy lesson: where fiscal support is placed in the network can matter as much as, and often more than, how much is spent. Subsidy policies implemented at the granularity of individual buyer–seller links can generate substantially larger multipliers than uniform and sector-wide interventions, precisely because they can be aligned with the distorted margins that govern propagation. When considering government-spending (purchase) policies, the structure of factor markets must be taken into account, as it fundamentally shapes how shocks diffuse through the network. The sufficient-statistics toolkit and the scale-to-efficiency measure developed in this paper provide a practical foundation for designing granular, network-aware fiscal interventions during crises.

Beyond these findings, it is important to emphasize the scope of the analysis. The framework is designed to study the cross-sectional allocation of fiscal support conditional on a given policy instrument. The contribution is to determine, for any specified instrument, which sectors or network linkages the government should optimally target. The question of which instrument to use is distinct and has intertemporal, distributional, and institutional dimensions that lie outside the scope of this paper. Likewise, the analysis focuses on the short-run, static effects of linkage-specific policies. Although the sufficient-statistics results provide comparative-static insights, a full dynamic treatment would require modeling intertemporal choices, capital adjustment, and expectations. Finally, several features abstracted from here, such as heterogeneity in price stickiness or differential investment rates across producers, may also influence optimal targeting both through the model mechanism and in practice. Incorporating these elements represents a natural direction for future work.

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A Additional Proposition

Proposition 3 (First-Order Responses in a Semiparametic Economy)**.** *Consider a semiparametic economy with arbitrary pre-existing wedges and let $i \in C \cup G \cup N \cup F$. To first order, the effects of linkage-specific subsidy shocks, government purchase shocks, markup shocks, Hicks-neutral productivity shocks, input-augmenting productivity shocksa and preference shocks are given by:*

Prices. *For any $i \in C \cup G \cup N \cup F$,*

$$\begin{aligned}\mathrm{d}\log p_i = &\sum_{k\in N} \tilde{\Psi}_{ik}\left(-\mathrm{d}\log A_k + \ \mathrm{d}\log \mu_k - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kj} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kf}\right)\\ &+ \sum_{f\in F} \tilde{\Psi}_{if}\left(\frac{1}{1+\eta_f}\ \mathrm{d}\log \lambda_f + \frac{\eta_f}{1+\eta_f} dlog\lambda_C\right) - \mathbb{1}\{i\in C\}\sum_{g}^{N\cup F} \tilde{\Omega}_{ig} dlog\tau_{ig} + \mathbb{1}\{i\in C\}\frac{1}{1-\theta_i}\sum_{g}^{N\cup F} \tilde{\Omega}_{ig} dlogw_{ig}\end{aligned}$$

Domar weights. *For any $i \in C \cup G \cup N \cup F$,*

$$\begin{aligned}\mathrm{d}log\lambda_{it} = &-\left(\sum_{b\in C\cup G\cup N\cup F} \frac{\lambda_b}{\lambda_i} d\log \mu_{bt}\right)(\Psi_{bi} + \delta_{bi})\\ &+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F} \frac{\lambda_b}{\lambda_i}\Omega_{bl}\left[\sum_{s\in N\cup F}\left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right)\right] dlogp_{st}\right]\right)\Psi_{li}\\ &+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F} \frac{\lambda_b}{\lambda_i}\Omega_{bl}\left[-\sum_{s\in N\cup F} \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right) d\tau_{bst}\right]\right)\Psi_{li}\\ &- \frac{\lambda_G}{\lambda_i}\sum_{l\in N\cup F} \Omega_{Gl}[\mathbb{1}\{i\in G\} - \Psi_{Ci}] dlog\tau_{Glt}\\ &+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F} \frac{\lambda_b}{\lambda_i}\Omega_{bl}\left[-\sum_{s\in N\cup F}\left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right)\right] dlogZ_{bst}\right]\right)\Psi_{li}\\ &+ \frac{\lambda_C}{\lambda_i}\sum_{l\in C\cup G\cup N\cup F}\left(\Omega_{Cl} dlogw_{Clt}\right)\Psi_{li} - \frac{\lambda_C}{\lambda_i}\left(\sum_{s\in N\cup F} \tilde{\Omega}_{Cs}\ d\log w_{Cst}\right)\left(\sum_{l\in C\cup G\cup N\cup F} \Omega_{Cl}\Psi_{li}\right)\\ &+ \sum_{l\in N\cup F}\left(\Omega_{Gl}\frac{\lambda_G}{\lambda_i}(dlogp_l + dlogg_l)\right)\left[\Psi_{li} + [\mathbb{1}\{i\in G\} - \Psi_{Ci}]\right]\end{aligned}$$

Real GDP.

$$d\log Y = \sum_{k\in N} \tilde{\lambda}_k \left(d\log A_k - d\log \mu_k + \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} + \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kj} + \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kf} \right)$$
$$- \sum_{f\in F} \tilde{\lambda}_f \left(\frac{1}{1+\eta_f} \, \mathrm{d}\log \lambda_f + \frac{\eta_f}{1+\eta_f} dlog\lambda_C \right) + \sum_{i=1}^{N} \lambda_C \Omega_{Ci} d\log \tau_{Ci}$$

Input–output linkages. *The first-order change in linkages* $\mathrm{d}\log \Omega_{jit}$ *is given by Lemma 2.*

Proof. See Appendix B.5. ■

B Proofs

B.1 Price Changes

B.1.1 Price Change for $i \in N$ (Nonparametric)

In the paper, x_{ijt} denotes the total amount of input j used by producer i to produce y_{it} units of output.

I use the a_{ijt} denotes the amount of input j used by sector i to produce 1 unit of output y_{it} below.

In the following expression, the variables with a $\hat{}$ represent the variables in the original cost duality problem, the variables with a $\check{}$ represent the variables in the efficient allocation problem, and the variables without a hat or check are the variables under arbitrary distortion.

The original cost duality problem. For each producer i, let $c_{it}(\hat{p}_{i\cdot t})$ be the unit cost function, dual to the non-parametric production function (without productivity shock). It gives the minimal cost of producing one unit of output using prices $\hat{p}_{i\cdot t} \in \mathbb{R}_+^{N+F}$. $c_{it}(\cdot)$ is differentiable and homogeneous of degree one in $\hat{p}_{i\cdot t}$.

Then the unit cost-minimization problem non-parametrically is

$$c_{it}\left(1, \hat{p}_{i\cdot t}\right) \equiv \min_{\{\hat{a}_{ijt}\}} \left\{ \sum_{j\in N\cup F} \hat{p}_{ijt}\hat{a}_{ijt} \quad s.t. \quad F\left(\hat{a}_{i1t}, \hat{a}_{i2t}, ..., \hat{a}_{iNt}, \hat{a}_{iN+1t}, \hat{a}_{iN+2t}, ..., \hat{a}_{iN+Ft}\right) = 1 \right\}$$

By Shephard's Lemma: For $i \in N$ and $j \in N \cup F$, Hicksian input demands are given by

$$\hat{a}_{ijt} = \frac{\partial c_{it}}{\partial \hat{p}_{jt}}$$

Therefore, cost shares of intermediate and primary usage for producer i are

$$\tilde{\Omega}_{ijt} \equiv \frac{\hat{p}_{ijt}\hat{a}_{ijt}}{c_i} = \frac{\partial \log c_{it}}{\partial \log \hat{p}_{ijt}}$$

The unit cost function $c_{it}\left(1, \hat{p}_{i\cdot t}\right)$ is homogeneous of degree 1 in prices $\hat{p}_{i\cdot t}$ (from the cost minimization problem).

$$c_{it}(\lambda \hat{p}_{i\cdot t}) = \lambda c_i(\hat{p}_{i\cdot t})$$

Apply Euler's theorem, it has

$$\sum_{j \in N \cup F} \hat{p}_{ijt} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}} = c_{it}(\hat{p}_{i\cdot t})$$

Total differential of $\log c_{it}$

$$d \log c_{it} = \frac{1}{c_{it}} dc_{it} = \frac{1}{c_{it}} \sum_{j \in N \cup F} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}} d\hat{p}_{ijt}$$

where

$$\frac{1}{c_{it}} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}} d\hat{p}_{ijt} = \frac{\hat{p}_{ijt}}{c_{it}} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}} \frac{d\hat{p}_{ijt}}{\hat{p}_{ijt}} = \underbrace{\left(\frac{\hat{p}_{ijt}}{c_{it}} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}} \right)}_{\tilde{\Omega}_{ijt}} d \log \hat{p}_{ijt}$$

Define cost shares (from Shephard's lemma + homogeneity):

$$\tilde{\Omega}_{ijt} \equiv \frac{\hat{p}_{ijt}}{c_{it}} \frac{\partial c_{it}}{\partial \hat{p}_{ijt}}$$

Substitute this to the above expression, it has

$$d\log c_{it} = \sum_{j\in N+F} \tilde{\Omega}_{ijt} d\log \hat{p}_{ijt}$$

The above solution is exact. I now derive the first-order approximation, assuming that time 0 corresponds to the original steady state

Take a first-order Taylor expansion of

$$\log c_{it}\left(\hat{p}_{i\cdot t}\right) \text{ around } \hat{p}_{i\cdot 0}$$

The first-order expansion is:

$$\log c_{it}\left(\hat{p}_{i\cdot t}\right) \approx \log c_{it}\left(\hat{p}_{i\cdot 0}\right) + \sum_{j\in N\cup F} \left.\frac{\partial \log c_{it}}{\partial \log \hat{p}_{ijt}}\right|_{\hat{p}_{i\cdot 0}} \left(\log \hat{p}_{ijt} - \log \hat{p}_{ij0}\right)$$

Differentiate it, it has

$$d\log c_{it} \approx \sum_{j\in N\cup F} \underbrace{\left.\frac{\partial \log c_{it}}{\partial \log \hat{p}_{ijt}}\right|_{\hat{p}_{i\cdot 0}}}_{\tilde{\Omega}_{ij}} d\log \hat{p}_{ijt}$$

where $\frac{\partial \log c_{it}}{\partial \log \hat{p}_{ijt}}|_{\hat{p}_{i\cdot 0}} = \frac{\hat{p}_{ij0}}{c_{i0}} \frac{\partial c_{i0}}{\partial \hat{p}_{ij0}} = \frac{\hat{p}_{ij0} x_{ij0}}{c_{i0}} = \tilde{\Omega}_{ij}$

Thus,

$$d\log c_{it} \approx \sum_{j\in N\cup F} \tilde{\Omega}_{ij} d\log \hat{p}_{ijt}$$

Problem with no distortion When there is no distortion and wedge, the problem in our model is

$$c_{it}\left(1, \check{p}_{i\cdot t}, \right) \equiv \min_{\{\check{a}_{ijt}\}} \sum_{j\in N\cup F} \check{p}_{jt}\check{a}_{ijt}$$

$$s.t. \quad \check{A}_{it} F\left(\check{Z}_{i1t}\check{a}_{i1t}, \check{Z}_{i2t}\check{a}_{i2t}, ..., \check{Z}_{iNt}\check{a}_{iNt}, \check{Z}_{iN+1t}\check{a}_{iN+1t}, \check{Z}_{iN+2t}\check{a}_{iN+2t}, ..., \check{Z}_{iN+Ft}\check{a}_{iN+Ft}\right) = 1.$$

Since the production function is homogeneous of degree 1 (constant return to scale),

First step, replace $\check{A}_{it}\check{Z}_{ijt}\check{a}_{ijt} \equiv \hat{a}_{ijt}$, then $\check{a}_{ijt} \equiv \frac{\hat{a}_{ijt}}{\check{A}_t\check{Z}_{ijt}}$

The problem can be written as

$$c_{it}\left(1, p_{i\cdot t}\right) \equiv \min_{\{\check{a}_{ijt}\}} \sum_{j \in N \cup F} \check{p}_{jt} \frac{\hat{a}_{ijt}}{\check{A}_{it}\check{Z}_{ijt}}$$

$$s.t. \quad F\left(\hat{a}_{i1t}, \hat{a}_{i2t}, ..., \hat{a}_{iNt}, \hat{a}_{iN+1t}, \hat{a}_{iN+2t}, ..., \hat{a}_{iN+Ft}\right) = 1$$

Now define $\hat{p}_{ijt} = \frac{\check{p}_{jt}}{\check{A}_{it}\check{Z}_{ijt}}$

Since there is no distortion, it has $\check{p}_{it} = c_{it}$

Cost share

$$\tilde{\Omega}_{ijt} \equiv \frac{\hat{p}_{ijt}}{c_{it}}\hat{a}_{ijt} = \frac{\frac{\check{p}_{jt}}{\check{A}_{it}\check{Z}_{ijt}}\check{A}_{it}\check{Z}_{ijt}\check{a}_{ijt}}{c_{it}} = \frac{\check{p}_{jt}\check{a}_{ijt}}{c_{it}} = \frac{\check{p}_{jt}\check{a}_{ijt}}{\check{p}_{it}} = \frac{\check{p}_{jt}\check{x}_{ijt}}{\check{p}_{it}\check{y}_{it}}$$

where $\check{x}_{ijt} = \check{a}_{ijt}\check{y}_{it}$.

$$dlog\check{p}_{it} = d\log c_{it} = \sum_{j \in N+F} \tilde{\Omega}_{ij} d\log \hat{p}_{ijt}$$

$$dlog\check{p}_{it} = \sum_{j \in N+F} \tilde{\Omega}_{ij} d\log\left(\frac{\check{p}_{jt}}{\check{A}_{it}\check{Z}_{ijt}}\right)$$

Since Euler's theorem implies $\sum_{j \in N \cup F} \tilde{\Omega}_{ij} = 1$.

Up to the first order approximation, it has

$$d\log \check{p}_{it} = -d\log \check{A}_{it} + \sum_{j\in N\cup F} \tilde{\Omega}_{ij} \left(d\log \check{p}_{jt} - dlog\check{Z}_{ijt} \right)$$

Problem with distortion The problem in our model is

$$c_{it}\left(1, p_{i\cdot t}, \tau_{i\cdot t}, M_{i\cdot}\right) \equiv \min_{\{a_{ijt}\}} \sum_{j\in N\cup F} \left(1-\tau_{ijt}\right) M_{ij} p_{jt} a_{ijt}$$

$$s.t. \quad A_{it} F\left(Z_{i1t}a_{i1t}, Z_{i2t}a_{i2t}, ..., Z_{iNt}a_{iNt}, Z_{iN+1t}a_{iN+1t}, Z_{iN+2t}a_{iN+2t}, ..., Z_{iN+Ft}a_{iN+Ft}\right) = 1.$$

Since the production function is homogeneous of degree 1 (constant return to scale),

First step, replace $A_{it}Z_{ijt}a_{ijt} \equiv \hat{a}_{ijt}$, then $a_{ijt} \equiv \frac{\hat{a}_{ijt}}{A_{it}Z_{ijt}}$

The problem can be written as

$$c_{it}\left(1, p_{i\cdot t}, \tau_{i\cdot t}, M_{i\cdot}\right) \equiv \min_{\{a_{ijt}\}} \sum_{j\in N\cup F} \left(1-\tau_{ijt}\right) M_{ij} p_{jt} \frac{\hat{a}_{ijt}}{A_{it}Z_{ijt}}$$

$$s.t. \quad F\left(\hat{a}_{i1t}, \hat{a}_{i2t}, ..., \hat{a}_{iNt}, \hat{a}_{iN+1t}, \hat{a}_{iN+2t}, ..., \hat{a}_{iN+Ft}\right) = 1$$

Now define $\hat{p}_{ijt} = \frac{(1-\tau_{ijt})M_{ij}p_{jt}}{A_{it}Z_{ijt}}$

Using the above equation and $p_{it} = \mu_{it}c_{it}$

Cost share

$$\tilde{\Omega}_{ijt} \equiv \frac{\hat{p}_{ijt}}{c_{it}}\hat{a}_{ijt} = \frac{\frac{(1-\tau_{ijt})M_{ij}p_{jt}}{A_{it}Z_{ijt}} A_{it}Z_{ijt}a_{ijt}}{c_{it}} = \frac{\left(1-\tau_{ijt}\right) M_{ij}p_{jt}a_{ijt}}{c_{it}} = \frac{\mu_{it}\left(1-\tau_{ijt}\right) M_{ij}p_{jt}a_{ijt}}{p_{it}} = \frac{\mu_{it}\left(1-\tau_{ijt}\right) M_{ij}p_{jt}x_{ijt}}{p_{it}y_{it}}$$

$$\begin{aligned} dlogp_{it} &= dlog\mu_{it} + d\log c_{it} = dlog\mu_{it} + \sum_{j\in N+F} \tilde{\Omega}_{ij} d\log \hat{p}_{ijt} \\ &= dlog\mu_{it} + \sum_{j\in N+F} \tilde{\Omega}_{ij} d\log \hat{p}_{ijt} \end{aligned}$$

$$dlogp_{it} = dlog\mu_{it} + \sum_{j \in N+F} \tilde{\Omega}_{ij} d\log\left(\frac{(1-\tau_{ijt})\, M_{ij} p_{jt}}{A_{it} Z_{ijt}}\right)$$

Since Euler's theorem implies $\sum_{j \in N \cup F} \tilde{\Omega}_{ij} = 1$.

Up to the first order approximation, it has

$$d\log p_{it} = -d\log A_{it} + d\log \mu_{it} + \sum_{j \in N \cup F} \tilde{\Omega}_{ij} \left(d\log p_{jt} - d\tau_{ijt} - dlog Z_{ijt}\right)$$

For $i \in N$

$$d\log p_{it} = -d\log A_{it} + d\log \mu_{it} - \sum_{j \in N \cup F} \tilde{\Omega}_{ij} d\log Z_{ijt} + \sum_{j \in N} \tilde{\Omega}_{ij} \left(d\log p_{jt} - d\tau_{ijt}\right) + \sum_{f \in F} \tilde{\Omega}_{if} \left(d\log p_{ft} - d\tau_{ift}\right)$$

Interpretation of $\tilde{\Omega}_{ijt}$. In the problem without distortions, we have that

$$\tilde{\Omega}_{ijt} = \frac{\check{p}_{jt} \check{x}_{ijt}}{\check{p}_{it} \check{y}_{it}}$$

Thus, $\tilde{\Omega}_{ijt}$ can be explained as the element of the input-output matrix when the economy is efficient.

By combining the two expressions for $\tilde{\Omega}_{ijt}$ in the problem without distortion and the problem with distortion. It has

$$\tilde{\Omega}_{ijt} = \frac{\check{p}_{jt} \check{x}_{ijt}}{\check{p}_{it} \check{y}_{it}} = \frac{\mu_{it}\,(1-\tau_{ijt})\, M_{ij} p_{jt} x_{ijt}}{p_{it} y_{it}}, \qquad i \in C \cup N,\ j \in C \cup G \cup N \cup F,$$

Regarding the government sector G, there is no production function in the model. At any arbitrary steady state, we therefore have

$$d\log p_{Gt} = \sum_{j \in N} \frac{p_{jt} g_{jt}}{G_t}\bigg|_{p_{\cdot 0}, g_{\cdot 0}} d\log p_{jt} = \sum_{j \in N} \frac{p_j g_j}{G} d\log p_{jt}$$

Thus, I define the entry of $\tilde{\Omega}_t$ corresponding to the government row as

$$\tilde{\Omega}_{Gjt} = \frac{p_{jt} g_{jt}}{G_t}, \quad j \in N$$

This row represents the governnment spending shares across producers in aggregate government expenditure, evaluated in the distorted allocation.

First order effect on prices Define the $\tilde{\Omega}^p$ as the $N \times N$ submatrix of $\tilde{\Omega}$ that contains only the producer-to-producer. It has

$$\begin{aligned} d\log p_{it} = &\sum_{k\in N} \left[\left(I - \tilde{\Omega}^p\right)^{-1}\right]_{ik} \left(-d\log A_{kt} + d\log \mu_{kt} - \sum_{j\in N\cup F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft}\right) \\ &+ \sum_{f\in F}\sum_{k\in N} \left[\left(I - \tilde{\Omega}^p\right)^{-1}\right]_{ik} \tilde{\Omega}_{kf} d\log p_{ft} \end{aligned}$$

Using that $\left[\left(I - \tilde{\Omega}^p\right)^{-1}\right]_{ik} = \tilde{\Psi}_{ik}$

$$\begin{aligned} d\log p_{it} = &\sum_{k\in N} \tilde{\Psi}_{ik} \left(-d\log A_{kt} + d\log \mu_{kt} - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft}\right) \\ &+ \sum_{f\in F}\sum_{k\in N} \tilde{\Psi}_{ik}\tilde{\Omega}_{kf} d\log p_{ft} \end{aligned}$$

Using the identity that $\sum_{k\in N} \tilde{\Psi}_{ik}\tilde{\Omega}_{kf} = \tilde{\Psi}_{if}$, it has

$$d\log p_{it} = \sum_{k\in N} \tilde{\Psi}_{ik} \left(-d\log A_{kt} + d\log \mu_{kt} - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft}\right) + \sum_{f\in F} \tilde{\Psi}_{if} d\log p_{ft}$$

B.1.2 Price Change for $i \in C \cup G \cup N \cup F$

Recalling the identify,

$$\tilde{\Psi}_{jk} = \mathbf{1}(j = k) + \sum_{i \in N \cup F} \tilde{\Omega}_{ji} \tilde{\Psi}_{ik}$$

Thus, $\tilde{\Psi}_{ff} = 1$ for $f \in F$ and $\tilde{\Psi}_{if} = 1$ for $f \in F$, $i \in F$ but $i \neq f$.

Thus, the following equation holds for For $i \in N \cup F$,

$$\mathrm{d}\log p_{it} = \sum_{k \in N} \tilde{\Psi}_{ik} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j \in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j \in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f \in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) + \sum_{f \in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft}$$

From the cost-minimization problem of the aggregate consumer, the consumer price is

For $i \in C$,

$$\begin{aligned} dlogp_{Ct} = & \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} (\mathrm{d}\log p_{gt} - dlog\tau_{Cgt} + \frac{1}{1-\theta_C} dlogw_{Cgt}) = \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} \Bigg[\sum_{k \in N} \tilde{\Psi}_{gk} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j \in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j \in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f \in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) \\ & + \sum_{f \in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft} \Bigg] - \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} dlog\tau_{Cgt} + \frac{1}{1-\theta_C} \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} dlogw_{Cgt} \end{aligned}$$

$$\begin{aligned} dlogp_{Ct} = & \Bigg[\sum_{k \in N} \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} \tilde{\Psi}_{gk} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j \in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j \in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f \in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) \\ & + \sum_{f \in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft} \Bigg] - \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} dlog\tau_{Cgt} + \frac{1}{1-\theta_C} \sum_{g}^{N \cup F} \tilde{\Omega}_{Cg} dlogw_{Cgt} \end{aligned}$$

$$dlogp_{Ct} = = \Big[\sum_{k\in N} \tilde{\Psi}_{Ck} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) + \sum_{f\in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft} \Big] - \sum_{g}^{N\cup F} \tilde{\Omega}_{Cg} dlog\tau_{Cgt} + \frac{1}{1-\theta_C} \sum_{g}^{N\cup F} \tilde{\Omega}_{Cg} dlogw_{Cgt}$$

For $i \in G$, by the definition that $\Omega_{Gdt} = \tilde{\Omega}_{Gdt}$

$$dlogp_{Gt} = \sum_{d}^{N\cup F} \tilde{\Omega}_{Gd} (\mathrm{d}\log p_{dt} - dlog\tau_{Gdt}) = \sum_{d}^{N\cup F} \tilde{\Omega}_{Gd} \Big[\sum_{k\in N} \tilde{\Psi}_{dk} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) + \sum_{f\in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft} \Big] - \sum_{d}^{N\cup F} \tilde{\Omega}_{Gd} dlog\tau_{Gdt}$$

Thus, for $i \in C \cup G \cup N \cup F$

$$\mathrm{d}\log p_{it} = \sum_{k\in N} \tilde{\Psi}_{ik} \left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log \mu_{kt} - \sum_{j\in N+F} \tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N} \tilde{\Omega}_{kj} d\tau_{kjt} - \sum_{f\in F} \tilde{\Omega}_{kf} d\tau_{kft} \right) + \sum_{f\in F} \tilde{\Psi}_{if} \ \mathrm{d}\log p_{ft} - \mathbb{1}\{i \in C \cup G\} \sum_{g}^{N\cup F} \tilde{\Omega}_{ig} dlog\tau_{igt} + \mathbb{1}\{i \in C\} \frac{1}{1-\theta_i} \sum_{g}^{N\cup F} \tilde{\Omega}_{ig} dlogw_{igt}$$

From the utility maximization problem of the household, it has

$$\frac{1}{\eta_f} dlogL_{ft} = dlogp_{ft} - dlog\lambda_{C_t}$$

Since the nominal GDP is the numeraire, we have $\frac{P_f L_f}{GDP} = \lambda_f$. Using that $dlog\lambda_{ft} = dlogp_{ft} + dlogL_{ft}$,

$$\frac{1}{\eta_f} dlogL_{ft} = dlog\lambda_{ft} - dlogL_{ft} - dlog\lambda_{C_t}$$

Thus,

$$d\log L_{ft} = \frac{\eta_f}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} - \frac{\eta_f}{1+\eta_f} d\log\lambda_{Ct}$$

It has

$$d\log p_{ft} = d\log\lambda_{ft} - d\log L_{ft} = d\log\lambda_{ft} - \frac{\eta_f}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} d\log\lambda_{Ct} = \frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} d\log\lambda_{Ct}$$

Replace $d\log p_{ft} = \frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} dlog\lambda_{Ct}$ to the expression, it has

$$\begin{aligned}\mathrm{d}\log p_{it} = &\sum_{k\in N}\tilde{\Psi}_{ik}\left(-\mathrm{d}\log A_{kt} + \ \mathrm{d}\log\mu_{kt} - \sum_{j\in N+F}\tilde{\Omega}_{kj} dlogZ_{kjt} - \sum_{j\in N}\tilde{\Omega}_{kjt} d\tau_{kjt} - \sum_{f\in F}\tilde{\Omega}_{kf} d\tau_{kft}\right)\\ &+\sum_{f\in F}\tilde{\Psi}_{if}\ \mathrm{d}\left(\frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} dlog\lambda_{Ct}\right) - \mathbb{1}\{i\in C\cup G\}\sum_{g}^{N\cup F}\tilde{\Omega}_{ig} dlog\tau_{igt} + \mathbb{1}\{i\in C\}\frac{1}{1-\theta_i}\sum_{g}^{N\cup F}\tilde{\Omega}_{ig} dlogw_{igt}\end{aligned}$$

where $d\log A_{it}$, $d\log Z_{ijt}$, and $d\log\mu_{it}$ are equal to zero for all $i \in C \cup G$.

B.2 Nonparametric first-order approximation of Hicksian input demands

AUES. Let c_{kt} be twice differentiable in $\hat{p}_t$. Define the Allen–Uzawa elasticity of substitution (AUES) between input j and i for the production of k by

$$\sigma^{AU}_{k,ji,t} \equiv \frac{c_{kt}\, c_{k,ij,t}}{c_{k,i,t}\, c_{k,j,t}}, \qquad c_{k,i,t} \equiv \frac{\partial c_{kt}}{\partial \hat{p}_{kit}}, \quad c_{k,ij,t} \equiv \frac{\partial^2 c_{kt}}{\partial \hat{p}_{kit}\partial \hat{p}_{kjt}}$$

Then the compensated (Hicksian) price elasticity satisfies

$$\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = \tilde{\Omega}_{kit}\, \sigma^{AU}_{k,ji,t}$$

Proof. By Shephard's lemma, it has $c_{k,i,t} = \hat{a}_{kit}$ and $c_{k,j,t} = \hat{a}_{kjt}$.

By the definition, it has

$$\frac{\partial \hat{a}_{kjt}}{\partial \hat{p}_{kit}} = \frac{\partial^2 c_{kt}}{\partial \hat{p}_{kit} \partial \hat{p}_{kjt}} = c_{k,ij,t}$$

Thus, the Hicksian elasticity is

$$\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = \frac{\hat{p}_{kit}}{\hat{a}_{kjt}} \frac{\partial \hat{a}_{kjt}}{\partial \hat{p}_{kit}} = \frac{\hat{p}_{kit}}{\hat{a}_{kjt}} c_{k,ij,t}$$

From the AUES definition and $c_{k,i,t} = \hat{a}_{kit}$, $c_{k,j,t} = \hat{a}_{kjt}$, It obtains

$$c_{k,ij,t} = \sigma^{AU}_{k,ji,t} \frac{c_{k,i,t} c_{k,j,t}}{c_{kt}} = \sigma^{AU}_{k,ji,t} \frac{\hat{a}_{kit} \hat{a}_{kjt}}{c_{kt}}$$

Substituting,

$$\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = \frac{\hat{p}_{kit}}{\hat{a}_{kjt}} \left(\sigma^{AU}_{k,ji,t} \frac{\hat{a}_{kit} \hat{a}_{kjt}}{c_{kt}} \right) = \sigma^{AU}_{k,ji,t} \frac{\hat{p}_{kit} \hat{a}_{kit}}{c_{kt}} = \tilde{\Omega}_{kit} \, \sigma^{AU}_{k,ji,t}$$

■

Since c_{kt} is a function depends on all prices $p_{k \cdot t}$, thus

$$\hat{a}_{kjt} = \frac{\partial c_{kt}}{\partial \hat{p}_{kjt}} = \hat{a}_{kj} \left(\hat{p}_{k1t}, \ldots, \hat{p}_{kN+Ft} \right)$$

Thus, up to the first order approximation, it has

$$d \log \hat{a}_{kjt} = \sum_{i \in N \cup F} \underbrace{\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} \mid_{\hat{p}_{k \cdot 0}}}_{\tilde{\Omega}_{ki} \, \sigma^{AU}_{k,ji}} d \log \hat{p}_{kit}. \tag{6}$$

where $\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} \mid_{\hat{p}_{k \cdot 0}} = \tilde{\Omega}_{ki} \, \sigma^{AU}_{k,ji}$

$$d \log \hat{a}_{kjt} = \sum_{i \in N \cup F} \tilde{\Omega}_{ki} \, \sigma^{AU}_{k,ji} d \log \hat{p}_{kit}. \tag{7}$$

Now replace $\hat{p}_{kit} = \frac{(1 - \tau_{kit}) M_{iki} p_{it}}{A_{kt} Z_{kit}}$ and $\hat{a}_{kjt} = A_{kt} Z_{kjt} a_{kjt}$

$$d\log(A_{kt}Z_{kjt}a_{kjt}) = \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\,\sigma^{AU}_{k,ji} d\log\left(\frac{(1-\tau_{kit})\,M_{iki}p_{it}}{A_{kt}Z_{kit}}\right) \tag{8}$$

$$d\log a_{kjt} = -dlogA_{kt} - dlogZ_{kjt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\,\sigma^{AU}_{k,ji}\Big(dlogp_{it} - d\tau_{kit} - dlogA_{kt} - dlogZ_{kit}\Big) \tag{9}$$

Using that $\sum_i \tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t} = 0$

From this equation, it has that

$$d\log a_{kjt} = -dlogA_{kt} - dlogZ_{kjt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\,\sigma^{AU}_{k,ji}\Big(dlogp_{it} - d\tau_{kit} - dlogZ_{kit}\Big) \tag{10}$$

$$\tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t} = \frac{\partial \log a_{kjt}}{\partial \log p_{kit}}$$

To avoid introducing the original dual problem (the one with hat notation) in the main text, I use this equation to Hicksian price elasticity of the input requirement in this euqation to express the AUES.

Mapping to a CES environment with preference and weight shocks.. The aggregate consumption block is parameterized with a CES aggregator and may be subject to preference shocks. This subsection shows how to map the nonparametric theory to a parametric CES environment with shocks either to production weights or to preference weights.

Each producer or consumption $k \in C \cup N$ chooses its inputs $\{x_{klt}\}_{l\in N\cup F}$ to minimize total cost, net of subsidies

$$\min_{\{x_{klt}\}} \sum_{l\in N\cup F} (1-\tau_{klt})\,M_{kl}p_{lt}x_{klt} \quad \text{s.t.} \quad A_{kt}\left(\sum_{l\in N\cup F} w_{klt}^{\frac{1}{\theta_k}}\left(Z_{klt}x_{klt}\right)^{\frac{\theta_k-1}{\theta_i}}\right)^{\frac{\theta_k}{\theta_k-1}} \geq y_{kt}$$

where A_{kt} is a Hicks-neutral productivity shock, Z_{klt} is a linkage-specific productivity shock associated with input $l \in N \cup F$, M_{kl} is a pre-existing wedge that affects the cost of purchasing from l and w_{klt} determines the relative importance of product or factor l in the production of producer or consumer k.

Given the Nested CES structure, now replace $\hat{p}_{klt} = \frac{(1-\tau_{klt})M_{kl}p_{lt}}{w_{klt}^{\frac{1}{\theta_k-1}}A_{kt}Z_{klt}}$ and $\hat{a}_{klt} = w_{klt}^{\frac{1}{\theta_k-1}}A_{kt}Z_{klt}a_{klt}$

It gives

$$d\log(w_{klt}^{\frac{1}{\theta_k-1}}A_{kt}Z_{klt}a_{klt}) = \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\,\sigma_{k,li}^{AU}d\log\left(\frac{(1-\tau_{kit})\,M_{ki}p_{it}}{w_{kit}^{\frac{1}{\theta_k-1}}A_{kt}Z_{kit}}\right) \tag{11}$$

Thus, for the CES economy with preferrene/weight shocks, it has

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \sum_{j\in N\cup F}\tilde{\Omega}_{kj}\,\sigma_{k,lj}^{AU}d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogA_{kt} - dlogZ_{kjt} - \frac{1}{\theta_k-1}d\log w_{kjt}\right) - \frac{1}{\theta_k-1}d\log w_{klt} \tag{12}$$

Change in $dlog\Omega_{jit}$ for $j \in C\cup G\cup N\cup F$ In the main text of the paper, there is a parametric production function for aggregate consumption, which is subject to preference shocks. For the producers, there is a nonparametric production function.

Thus, for $k \in C+N$,

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \sum_{j\in N\cup F}\tilde{\Omega}_{kj}\,\sigma_{k,lj}^{AU}d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{kjt}\right) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{klt} \tag{13}$$

Using that $\sum_i \tilde{\Omega}_{kit}\sigma_{k,ji,t}^{AU} = 0$

$$d\log a_{kjt} = -dlogA_{kt} - dlogZ_{kjt} + \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\,\sigma_{k,ji}^{AU}\left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{kjt}\right) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{klt} \tag{14}$$

Using that $dlogx_{klt} = d\log a_{klt} + d\log y_{kt}$

$$d\Omega_{klt} = dlogp_{lt} + dlogx_{klt} - dlogp_{kt} - dlogy_{kt} = dlogp_{lt} + d\log a_{klt} + d\log y_{kt} - dlogp_{kt} - dlogy_{kt}$$

$$\begin{aligned}\Omega_{klt} &= dlogp_{lt} + d\log a_{klt} - dlogp_{kt} \\ &= dlogp_{lt} - dlogA_{kt} - dlogZ_{klt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\, \sigma^{AU}_{k,li} \left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{klt} - dlogp_{kt}\end{aligned}$$

Using

$$d\log p_{kt} = -d\log A_{kt} + d\log \mu_{kt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki} \left(d\log p_{it} - d\tau_{kit} - dlogZ_{kit} + \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right)$$

$$\begin{aligned}dlog\Omega_{klt} &= dlogp_{lt} - dlogA_{kt} - dlogZ_{klt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\, \sigma^{AU}_{k,li} \left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{klt} - dlogp_{kt} \\ &= dlogp_{lt} - dlogA_{kt} - dlogZ_{klt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\, \sigma^{AU}_{k,li} \left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) \\ &\quad - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{klt} - \left(- d\log A_{kt} + d\log \mu_{kt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki} \left(d\log p_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) \right) \\ &= dlogp_{lt} - dlogA_{kt} - dlogZ_{klt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\, \sigma^{AU}_{k,li} \left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) \\ &\quad - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{klt} + d\log A_{kt} - d\log \mu_{kt} - \sum_{i\in N\cup F} \tilde{\Omega}_{ki} \left(d\log p_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) \Bigg) \\ &= dlogp_{lt} - dlogZ_{klt} - d\log \mu_{kt} + \sum_{i\in N\cup F} \tilde{\Omega}_{ki}\, (\sigma^{AU}_{k,li} - 1) \left(dlogp_{it} - d\tau_{kit} - dlogZ_{kit} - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{kit} \right) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k - 1} d\log w_{klt}\end{aligned}$$

$$
\begin{aligned}
dlog\Omega_{klt} &= dlogp_{lt} - dlogZ_{klt} - d\log\mu_{kt} + \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)dlogp_{it} - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big(d\tau_{kit} + dlogZ_{kit}\Big) \\
&\quad - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big(\mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{kit}\Big) - \mathbb{1}\{k\in C\}\frac{1}{\theta_k-1}d\log w_{klt} \\
&= dlogp_{lt} - dlogZ_{klt} - d\log\mu_{kt} + \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)dlogp_{it} - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big(d\tau_{kit} + dlogZ_{kit}\Big) \\
&\quad - \sum_{i\in N\cup F}\tilde{\Omega}_{Ci}\left(\sigma^{AU}_{C,li}-1\right)\Big(\frac{1}{\theta_C-1}d\log w_{Cit}\Big) - \frac{1}{\theta_C-1}d\log w_{Clt} \\
&= dlogp_{lt} - dlogZ_{klt} - d\log\mu_{kt} + \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)dlogp_{it} - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big(d\tau_{kit} + dlogZ_{kit}\Big) \\
&\quad + \mathbb{1}\{k\in C\}d\log w_{klt} - \mathbb{1}\{k\in C\}\sum_{i\in N\cup F}\tilde{\Omega}_{ki}\; d\log w_{kit}
\end{aligned}
$$

See Appendix B.10 for the simplification of the last equality.

For $k \in C \cup N$,

$$
\begin{aligned}
dlog\Omega_{klt} &= -d\log\mu_{kt} + \sum_{i\in N\cup F}\Big[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big]dlogp_{it} - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)d\tau_{kit} - \sum_{i\in N\cup F}\Big[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li}-1\right)\Big]dlogZ_{kit} \\
&\quad + \mathbb{1}\{k\in C\}d\log w_{klt} - \mathbb{1}\{k\in C\}\sum_{i\in N\cup F}\tilde{\Omega}_{ki}\; d\log w_{kit}
\end{aligned}
$$

$$dlog\Omega_{klt} = -d\log\mu_{kt} + \sum_{i\in N\cup F}\left[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right)\right] dlogp_{it} - \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right) d\tau_{kit} - \sum_{i\in N\cup F}\left[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right)\right] dlogZ_{kit}$$
$$+ \mathbb{1}\{k \in C\} d\log w_{klt} - \mathbb{1}\{k \in C\} \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\; d\log w_{kit}$$

Note that this equation also holds for $k \in F$ since $\Omega_{flt} = 0$ for $f \in F$.

Note that (for $j \in G$ and $l \in C \cup G \cup N \cup F$)

$$\Omega_{Glt} = \frac{p_{lt} g_{lt}}{p_{Gt} G_t}$$

$$dlog\Omega_{Glt} = dlogp_{lt} + dlogg_{lt} - dlog\lambda_{Gt}$$

I assume that the cross Allen-Uzawa elasticities for the government are identical with $\sigma^{AU}_{G,li} = 1$ if $l \neq i$, and the own Allen-Uzawa elasticities are given by $\sigma^{AU}_{G,ll} = -\frac{\left(1-\tilde{\Omega}_{Gl}\right)}{\tilde{\Omega}_{Gl}}$. This ensures that $\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right) = 0$ for $k \in G$. Thus,

For $k \in C \cup G \cup N \cup F$

$$dlog\Omega_{klt} = -d\log\mu_{kt} + \sum_{i\in N\cup F}\left[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right)\right] dlogp_{it} - (k \notin G) \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right) d\tau_{kit} - \sum_{i\in N\cup F}\left[\delta_{li} + \tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right)\right] dlogZ_{kit}$$
$$+ \mathbb{1}\{k \in C\} d\log w_{klt} - \mathbb{1}\{k \in C\} \sum_{i\in N\cup F}\tilde{\Omega}_{ki}\; d\log w_{kit} + \mathbb{1}\{k \in G\}(dlogp_{lt} + dlogg_{lt} - dlog\lambda_{Gt})$$

where $d\log A_{kt}, d\log Z_{kjt}$, and $d\log\mu_{kt}$ are equal to zero for all $k \in C \cup G$.

Note: We are not interested in subsidy shocks to government purchases. Nevertheless, the expression developed above can also be applied to this case. The term $-(k \notin G)\sum_{i\in N\cup F}\tilde{\Omega}_{ki}\left(\sigma^{AU}_{k,li} - 1\right) d\tau_{kit}$ makes sure that $d\tau_{Glt}$ does not enter the expression for $d\log\Omega_{Glt}$ directly. In fact, it only

changes the $d\log\Omega_{Glt}$ through the $dlog\lambda_{Gt}$. Solving the model shows that the effect of a subsidy to government purchases is zero. Intuitively, since the subsidy to the government does not enter producers' pricing conditions or the first-order conditions of producers and consumers, it has no effect on producer prices or any quantity variables. It only affects the price index of government spending and, consequently, measured nominal GDP.

B.3 Proof for the change in real GDP

Normalization at the steady state I normalize the model such that, at the steady state, all prices are equal to one. The linkage-specific pre-existing wedges at the steady state are set equal to the product of the Hicks-neutral productivity and the factor-augmenting productivity evaluated at the steady state.

Normalization for the full model I assume that nominal GDP serves as the numeraire.

Since nominal GDP is the numeraire, we have

$$d\log GDP_t = 0 = d\log Y_t + d\log P_t^Y$$

which implies

$$d\log Y_t = -d\log P_t^Y$$

where P_t^Y denotes the GDP deflator.

$$\begin{aligned} d\log Y_t = -dlogP_t^Y =& -\lambda_C\Omega^{(C)}(d\log p_t - dlog\tau_{Ct}) - \lambda_G\Omega^{(G)}(d\log p_t - dlog\tau_{Gt}) \\ =& -\sum_{i=1}^{N}\lambda_C\Omega_{Ci}d\log p_{it} + \sum_{i=1}^{N}\lambda_C\Omega_{Ci}d\log\tau_{Cit} - \sum_{i=1}^{N}\lambda_G\Omega_{Gi}d\log p_{it} + \sum_{i=1}^{N}\lambda_G\Omega_{Gi}d\log\tau_{Git} \end{aligned}$$

It has

$$d\log Y_t = -\sum_{i=1}^{N}(\lambda_C\Omega_{Ci} + \lambda_G\Omega_{Gi})d\log p_{it} + \sum_{i=1}^{N}\lambda_C\Omega_{Ci}d\log\tau_{Cit} + \sum_{i=1}^{N}\lambda_G\Omega_{Gi}d\log\tau_{Git}$$

Thus,

$$d\log Y_t = -\sum_{i=1}^{N}(\lambda_C\Omega_{Ci}+\lambda_G\Omega_{Gi})\Bigg[\sum_{k\in N}\tilde{\Psi}_{ik}\left(-d\log A_{kt}+d\log\mu_{kt}-\sum_{j\in N+F}\tilde{\Omega}_{kj}dlogZ_{kjt}-\sum_{j\in N}\tilde{\Omega}_{kj}d\tau_{kjt}-\sum_{f\in F}\tilde{\Omega}_{kf}d\tau_{kft}\right)$$
$$+\sum_{f\in F}\tilde{\Psi}_{if}d\log p_{ft}\Bigg]+\sum_{i=1}^{N}\lambda_C\Omega_{Ci}d\log\tau_{Cit}+\sum_{i=1}^{N}\lambda_G\Omega_{Gi}d\log\tau_{Git}$$

Since

$$(\lambda_C\Omega^{(C)}+\lambda_G\Omega^{(G)})\tilde{\Psi}=\tilde{\lambda}'-e_2'$$

where $e_2'=[\lambda_C,\lambda_G,0,...,0]_{1\times(C\cup G\cup N\cup F)}$

It finally has

$$d\log Y_t=\sum_{k\in N}\tilde{\lambda}_k d\log A_{kt}-\sum_{k\in N}\tilde{\lambda}_k\left(d\log\mu_{kt}-\sum_{j\in N+F}\tilde{\Omega}_{kj}dlogZ_{kjt}-\sum_{j\in N}\tilde{\Omega}_{kj}d\tau_{kjt}-\sum_{f\in F}\tilde{\Omega}_{kf}d\tau_{kft}\right)$$
$$-\sum_{f\in F}\tilde{\lambda}_f d\log p_{ft}+\sum_{i=1}^{N}\lambda_C\Omega_{Ci}d\log\tau_{Cit}+\sum_{i=1}^{N}\lambda_G\Omega_{Gi}d\log\tau_{Git}$$

Note The expression for $d\log Y_t$ derived above is invariant to the choice of numeraire. Under the alternative normalization $d\log P_t^Y=0$, the same algebraic steps apply, with the only modification being the replacement of the numeraire condition by $d\log P_t^Y=0$ and the corresponding adjustment

$$d\log p_{ft}=d\log p_{ft}+d\log Y_t$$

This normalization yields an identical final result.

From the utility maximization problem of the household and the above proof, it has

$$d\log p_{ft} = \frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} dlog\lambda_{Ct}$$

Replace $dlogp_{ft}$ into the GDP expression, it gives

$$\begin{aligned} d\log Y_t = &\sum_{k\in N}\tilde{\lambda}_k\left(d\log A_{kt} - d\log\mu_{kt} + \sum_{j\in N+F}\tilde{\Omega}_{kj} dlog Z_{kjt} + \sum_{j\in N}\tilde{\Omega}_{kj} d\tau_{kjt} + \sum_{f\in F}\tilde{\Omega}_{kf} d\tau_{kft}\right) \\ &- \sum_{f\in F}\tilde{\lambda}_f\left(\frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_{ft} + \frac{\eta_f}{1+\eta_f} dlog\lambda_{Ct}\right) + \sum_{i=1}^{N}\lambda_C\Omega_{Ci} d\log\tau_{Cit} + \sum_{i=1}^{N}\lambda_G\Omega_{Gi} d\log\tau_{Git} \end{aligned}$$

B.4 Proof for the expression of $d\lambda_{it}$

$$\lambda'\Omega = \lambda' - e_2'$$

where $e_2 = [\lambda_C, \lambda_G, 0, 0]'$

$$\lambda'\Omega = \lambda' - e_2'$$

Differentiate it,

$$d\lambda'\Omega + \lambda' d\Omega = d\lambda' - de_2'$$

$$d\lambda' - d\lambda'\Omega = -\lambda' d\Omega + de_2'$$

$$d\lambda'(I-\Omega) = -\lambda' d\Omega + de_2'$$

$$d\lambda' = -\lambda' d\Omega (I-\Omega)^{-1} + de_2'(I-\Omega)^{-1}$$

$$d\lambda' = -\lambda' d\Omega \Psi + de_2' \Psi$$

For $i \in C \cup G \cup N \cup F$, we have

$$\mathrm{d}\lambda_{it} = \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \lambda_b \ \mathrm{d}\Omega_{blt} \right) \Psi_{li} + \mathrm{d}\lambda_{Ct} \Psi_{Ci} + \mathrm{d}\lambda_{Gt} \Psi_{Gi}$$

$$\mathrm{d}\lambda_{it} = \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \lambda_b \Omega_{bl} \ \mathrm{d}\log \Omega_{blt} \right) \Psi_{li} + \mathrm{d}\lambda_{Ct} \Psi_{Ci} + \mathrm{d}\lambda_{Gt} \Psi_{Gi}$$

B.5 Proof of general expression

For $b \in C \cup G \cup N \cup F$

$$\begin{aligned} dlog\Omega_{blt} = & -d\log \mu_{bt} + \sum_{s \in N \cup F} \Big[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) \Big] dlogp_{st} - (b \notin G) \sum_{s \in N \cup F} \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) d\tau_{bst} - \sum_{s \in N \cup F} \Big[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) \Big] dlogZ_{bst} \\ & + \mathbb{1}\{b \in C\} d\log w_{blt} - \mathbb{1}\{b \in C\} \sum_{s \in N \cup F} \tilde{\Omega}_{bs} \ d\log w_{bst} + \mathbb{1}\{b \in G\} (dlogp_{lt} + dlogg_{lt} - dlog\lambda_{Gt}) \end{aligned}$$

For $i \in C \cup G \cup N \cup F$, we have

$$\mathrm{d}\lambda_{it} = \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \lambda_b \Omega_{bl} \ \mathrm{d}\log \Omega_{blt} \right) \Psi_{li} + \mathrm{d}\lambda_{Ct} \Psi_{Ci} + \mathrm{d}\lambda_{Gt} \Psi_{Gi}$$

$$\begin{aligned} \mathrm{d}\lambda_{it} = \sum_{l \in C \cup G \cup N \cup F} \Bigg(\sum_{b \in C \cup G \cup N \cup F} \lambda_b \Omega_{bl} \Big[& - d\log \mu_{bt} + \sum_{s \in N \cup F} \big[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) \big] dlogp_{st} \\ & - (b \notin G) \sum_{s \in N \cup F} \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) d\tau_{bst} - \sum_{s \in N \cup F} \big[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma_{b,ls}^{AU} - 1 \right) \big] dlogZ_{bst} + \mathbb{1}\{b \in C\} d\log w_{blt} \\ & - \mathbb{1}\{b \in C\} \sum_{s \in N \cup F} \tilde{\Omega}_{bs} \ d\log w_{bst} + \mathbb{1}\{b \in G\} (dlogp_{lt} + dlogg_{lt} - dlog\lambda_{Gt}) \Big] \Bigg) \Psi_{li} + \mathrm{d}\lambda_{Ct} \Psi_{Ci} + \mathrm{d}\lambda_{Gt} \Psi_{Gi} \end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\lambda_{it} = & \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-d\log\mu_{bt}\Big]\right)\Psi_{li} \\
& + \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]dlogp_{st}\Big]\right)\Psi_{ls} \\
& + \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-(b\notin G)\sum_{s\in N\cup F}\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)d\tau_{bst}\Big]\right)\Psi_{ls} \\
& + \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]dlogZ_{bst}\Big]\right)\Psi_{ls} \\
& + \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\mathbb{1}\{b\in C\}dlogw_{blt}-\mathbb{1}\{b\in C\}\sum_{s\in N\cup F}\tilde{\Omega}_{bs}\ d\log w_{bst}\Big]\right)\Psi_{li} \\
& + \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\mathbb{1}\{b\in G\}(dlogp_{lt}+dlogg_{lt}-dlog\lambda_{Gt})\Big]\right)\Psi_{li} \\
& + \mathrm{d}\lambda_{Ct}\Psi_{Ci}+\mathrm{d}\lambda_{Gt}\Psi_{Gi}
\end{aligned}
$$

the first term

$$
\begin{aligned}
& \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-d\log\mu_{bt}\Big]\right)\Psi_{li} \\
= & -\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b d\log\mu_{bt}\right)\sum_{l\in C\cup G\cup N\cup F}\Omega_{bl}\Psi_{li} \\
= & -\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b d\log\mu_{bt}\right)(\Psi_{bi}+\delta_{bi})
\end{aligned}
$$

the fifth term

$$\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[\mathbb{1}\{b\in C\}dlogw_{blt}-\mathbb{1}\{b\in C\}\sum_{s\in N\cup F}\tilde{\Omega}_{bs}\ d\log w_{bst}\right]\right)\Psi_{li}$$
$$=\sum_{l\in C\cup G\cup N\cup F}\left(\lambda_C\Omega_{Cl}\left[dlogw_{Clt}-\sum_{s\in N\cup F}\tilde{\Omega}_{Cs}\ d\log w_{Cst}\right]\right)\Psi_{li}$$
$$=\lambda_C\sum_{l\in C\cup G\cup N\cup F}\left(\Omega_{Cl}dlogw_{Clt}\right)\Psi_{li}-\lambda_C\left(\sum_{s\in N\cup F}\tilde{\Omega}_{Cs}\ d\log w_{Cst}\right)\left(\sum_{l\in C\cup G\cup N\cup F}\Omega_{Cl}\Psi_{li}\right)$$

$$\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[\mathbb{1}\{b\in C\}dlogw_{bl}\right]\right)\Psi_{li}$$
$$=\sum_{l\in C\cup G\cup N\cup F}\left(\lambda_C\Omega_{Cl}dlogw_{Cl}\right)\Psi_{li}$$

government spending terms - the sixth with the seventh term

$dlogG_l = dlogp_l + dlogg_l$

$$\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[\mathbb{1}\{b\in G\}(dlogp_{lt}+dlogg_{lt}-dlog\lambda_G)\right]\right)\Psi_{li}+\mathrm{d}\lambda_{Ct}\Psi_{Ci}+\mathrm{d}\lambda_{Gt}\Psi_{Gi}$$
$$=\sum_{l\in C\cup G\cup N\cup F}\left(\lambda_G\Omega_{Gl}(dlogp_{lt}+dlogg_{lt}-dlog\lambda_G)\right)\Psi_{li}+\mathrm{d}\lambda_{Ct}\Psi_{Ci}+\mathrm{d}\lambda_{Gt}\Psi_{Gi}$$

$$+\sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_{lt}+dlogg_{lt}-dlog\lambda_G)\right)\Psi_{li}$$
$$+d\lambda_C\Psi_{Ci}+d\lambda_G\Psi_{Gi}$$

$$
\begin{aligned}
&+ \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G (dlogp_{lt} + dlogg_{lt}) \right) \Psi_{li} \\
&- \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G dlog\lambda_G \right) \Psi_{li} \\
&\quad + d\lambda_C \Psi_{Ci} + d\lambda_G \Psi_{Gi}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G (dlogp_{lt} + dlogg_{lt}) \right) \Psi_{li} \\
&- \left(\lambda_G dlog\lambda_G \right) \left(\Psi_{Gi} - \mathbb{1}\{i \in G\} \right) \\
&\quad + d\lambda_C \Psi_{Ci} + d\lambda_G \Psi_{Gi}
\end{aligned}
$$

$$
+ \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G (dlogp_{lt} + dlogg_{lt}) \right) \Psi_{li} + \lambda_C dlog\lambda_C \Psi_{Ci} + (\mathbb{1}\{i \in G\}) \lambda_G dlog\lambda_G
$$

Using that $\lambda_C + \lambda_G = 1$

we have $d\lambda_C = -\lambda_G$

$$
+ \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G (dlogp_{lt} + dlogg_{lt}) \right) \Psi_{li} - \lambda_G dlog\lambda_G \Psi_{Ci} + (\mathbb{1}\{i \in G\}) \lambda_G dlog\lambda_G
$$

$$
+ \sum_{l \in N \cup F} \left(\Omega_{Gl} \lambda_G (dlogp_{lt} + dlogg_{lt}) \right) \Psi_{li} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}] \lambda_G dlog\lambda_G
$$

using that $dlog\lambda_G = \sum_{l\in N\cup F}\Omega_{Gl}(dlogp_{lt} - dlog\tau_{Glt} + dlogg_{lt})$

$$+\sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_{lt}+dlogg_{lt})\right)\Psi_{li} + [\mathbb{1}\{i\in G\}-\Psi_{Ci}]\lambda_G\sum_{l\in N\cup F}\left(\Omega_{Gl}(dlogp_{lt}+dlogg_{lt})\right) - \lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}$$

$$+\sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_{lt}+dlogg_{lt})\right)\Big[\Psi_{li}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\Big] - \lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}$$

$$+\sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_{l}+dlogg_{l})\right)\Big[\Psi_{li}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\Big] - \lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}$$

General Expression Thus, it has that for $i \in C\cup G\cup N\cup F$

$$\begin{aligned}
\mathrm{d}\lambda_{it} = &-\Big(\sum_{b\in C\cup G\cup N\cup F}\lambda_b d\log\mu_{bt}\Big)(\Psi_{bi}+\delta_{bi})\\
&+\sum_{l\in C\cup G\cup N\cup F}\Big(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]dlogp_{st}\Big]\Big)\Psi_{li}\\
&+\sum_{l\in C\cup G\cup N\cup F}\Big(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-(b\notin G)\sum_{s\in N\cup F}\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)d\tau_{bst}\Big]\Big)\Psi_{li}\\
&+\sum_{l\in C\cup G\cup N\cup F}\Big(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[-\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]dlogZ_{bst}\Big]\Big)\Psi_{li}\\
&+\lambda_C\sum_{l\in C\cup G\cup N\cup F}\Big(\Omega_{Cl}dlogw_{Clt}\Big)\Psi_{li} - \lambda_C\Big(\sum_{s\in N\cup F}\tilde{\Omega}_{Cs}\ d\log w_{Cst}\Big)\Big(\sum_{l\in C\cup G\cup N\cup F}\Omega_{Cl}\Psi_{li}\Big)\\
&+\sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_{l}+dlogg_{l})\right)\Big[\Psi_{li}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\Big]\\
&-\lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}
\end{aligned}$$

B.6 Subsidy Shocks

$$
\begin{aligned}
\mathrm{d}\lambda_{it} = &+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[\sum_{s\in N\cup F}\left[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\right]dlogp_{st}\right]\right)\Psi_{li}\\
&+ \sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[-\sum_{s\in N\cup F}\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)d\tau_{bst}\right]\right)\Psi_{li}\\
&+ \sum_{l\in N\cup F}\left(\Omega_{Gl}\lambda_G(dlogp_l+dlogg_l)\right)\left[\Psi_{li}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\right]\\
&-\lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}
\end{aligned}
$$

Differentiate the third term to τ_{kjt}

In this term, only $dlogp_l$ depends on τ_{kjt}

it gives

$$
\begin{aligned}
&-\mathbb{1}\{k\notin C\cup G\}\tilde{\Omega}_{kj}\sum_{s\in N\cup F}\left(\Omega_{Gs}\lambda_G(\sum_{k\in N}\tilde{\Psi}_{sk}\left[\Psi_{si}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\right]\right.\\
&\left.+\sum_{s\in N\cup F}\left(\Omega_{Gs}\lambda_G(\sum_{g=1}^{F}\tilde{\Psi}_{sg}\left(\frac{1}{1+\eta_g}\frac{\mathrm{d}\log\lambda_{gt}}{d\tau_{kjt}}+\frac{\eta_g}{1+\eta_g}\frac{d\log\lambda_{Ct}}{d\tau_{kjt}}\right))\right)\left[\Psi_{si}+[\mathbb{1}\{i\in G\}-\Psi_{Ci}]\right]\right.
\end{aligned}
$$

the second term

$$
-\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\left[(b\notin G)\sum_{s\in N\cup F}[\tilde{\Omega}_{bs}(\sigma^{AU}_{b,ls}-1)]\tau_{bst}\right]\right)\Psi_{li}
$$

when differentiate it with τ_{kj}

$$- (k \notin G) \sum_{l \in C \cup G \cup N \cup F} \left(\lambda_k \Omega_{kl} \left[[\tilde{\Omega}_{kj} (\sigma^{AU}_{k,lj} - 1)] \tau_{kjt} \right] \right) \Psi_{li}$$

remove τ_{kjt}

$$- (k \notin G) \sum_{l \in C \cup G \cup N \cup F} \left(\lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} (\sigma^{AU}_{k,lj} - 1) \right] \right) \Psi_{li}$$

$$= -(k \notin G) \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \sigma^{AU}_{k,lj} \right] \Psi_{li} + (k \notin G) \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \right] \Psi_{li}$$

now assume it is the case of CES

$$\begin{aligned}
&= - \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \sigma^{AU}_{k,lj} \right] \Psi_{li} + \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \right] \Psi_{li} \\
&= - \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \theta_k \right] \Psi_{li} - \lambda_k \Omega_{kj} \left[\tilde{\Omega}_{kj} (-\theta_k - \frac{\theta_k \left(1 - \tilde{\Omega}_{kl} \right)}{\tilde{\Omega}_{kl}}) \right] \Psi_{ji} + \sum_{l \in C \cup G \cup N \cup F} \lambda_k \Omega_{kl} \left[\tilde{\Omega}_{kj} \right] \Psi_{li} \\
&= -\lambda_k \left[\tilde{\Omega}_{kj} \theta_k \right] [\Psi_{ki} - 1(k = i)] - \lambda_k \Omega_{kj} \left[-\theta_k \right] \Psi_{ji} + \lambda_k \left[\tilde{\Omega}_{kj} \right] [\Psi_{ki} - 1(k = i)] \\
&= -(\theta_k - 1) \lambda_k \tilde{\Omega}_{kj} [\Psi_{ki} - 1(k = i)] + \theta_k \lambda_k \Omega_{kj} \Psi_{ji}
\end{aligned}$$

Thus, the second term is consistent with the expression

the first term

The change in prices such that

$$\begin{aligned}\mathrm{d}\log p_i = &\sum_{k\in N}\tilde{\Psi}_{ik}\left(-\mathrm{d}\log A_k + \ \mathrm{d}\log\mu_k - \sum_{j\in N+F}\tilde{\Omega}_{kj}dlogZ_{kjt} - \sum_{j\in N}\tilde{\Omega}_{kj}d\tau_{kj} - \sum_{f\in F}\tilde{\Omega}_{kf}d\tau_{kf}\right)\\ &+\sum_{f\in F}\tilde{\Psi}_{if}\ \mathrm{d}\left(\frac{1}{1+\eta_f}\ \mathrm{d}\log\lambda_f + \frac{\eta_f}{1+\eta_f}dlog\lambda_C\right) - \mathbb{1}\{i\in C\cup G\}\sum_{g}^{N\cup F}\tilde{\Omega}_{ig}dlog\tau_{ig} + \mathbb{1}\{i\in C\}\frac{1}{1-\theta_i}\sum_{g}^{N\cup F}\tilde{\Omega}_{ig}dlogw_{ig}\end{aligned}$$

$$\frac{d\log p_i}{d\tau_{kjt}} = -1(k\notin C)\tilde{\Psi}_{ik}\tilde{\Omega}_{kj} + \sum_{f=1}^{F}\tilde{\Psi}_{if}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\Lambda_f}{d\tau_{kj}} + \frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{d\tau_{kj}}\right)$$

$$\frac{d\log p_s}{d\tau_{kjt}} = -\mathbb{1}\{k\notin C\}\tilde{\Psi}_{sk}\tilde{\Omega}_{kj} + \sum_{f=1}^{F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\Lambda_f}{d\tau_{kj}} + \frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{d\tau_{kj}}\right)$$

for a shock to τ_{kj}, differentite it to τ_{kj},

$$\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]d\log p_s\Big]\right)\Psi_{li}$$

$$=\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]\Big[-\mathbb{1}\{k\notin C\}\tilde{\Psi}_{sk}\tilde{\Omega}_{kj}+\sum_{f=1}^{F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\Lambda_f}{d\tau_{kj}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{d\tau_{kj}}\right)\Big]\Big]\right)\Psi_{li}$$

$$=-\mathbb{1}\{k\notin C\}\tilde{\Omega}_{kj}\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]\tilde{\Psi}_{sk}\Big]\right)\Psi_{li}$$

$$+\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\lambda_b\Omega_{bl}\Big[\sum_{s\in N\cup F}\Big[\delta_{ls}+\tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls}-1\right)\Big]\Big[\sum_{f=1}^{F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\Lambda_f}{d\tau_{kj}}+\frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{d\tau_{kj}}\right)\Big]\Big]\right)\Psi_{li}$$

The last term

$-\lambda_G\sum_{l\in N\cup F}\Omega_{Gl}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]dlog\tau_{Glt}$

when differentiate it with τ_{kj}

$-(k\in G)\lambda_k\Omega_{kj}[\mathbb{1}\{i\in G\}-\Psi_{Ci}]$

Thus, for the subsidy shocks

$$
\begin{aligned}
\frac{dlog\lambda_{it}}{d\tau_{kjt}} =& -(k \notin G) \sum_{l \in C \cup G \cup N \cup F} \left(\frac{\lambda_k}{\lambda_i} \Omega_{kl} \left[[\tilde{\Omega}_{kj} (\sigma^{AU}_{k,lj} - 1)] \right] \right) \Psi_{li} - (k \in G) \lambda_k \Omega_{kj} [\mathbb{1}\{i \in G\} - \Psi_{Ci}] \\
& - \mathbb{1}\{k \notin C \cup G\} \tilde{\Omega}_{kj} \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \frac{\lambda_b}{\lambda_i} \Omega_{bl} \left[\sum_{s \in N \cup F} \left[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma^{AU}_{b,ls} - 1 \right) \right] \tilde{\Psi}_{sk} \right] \right) \Psi_{li} \\
& - \mathbb{1}\{k \notin C \cup G\} \tilde{\Omega}_{kj} \sum_{s \in N \cup F} (\Omega_{Gs} \frac{\lambda_G}{\lambda_i} (\sum_{k \in N} \tilde{\Psi}_{sk} \left[\Psi_{si} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}] \right] \\
& + \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \frac{\lambda_b}{\lambda_i} \Omega_{bl} \left[\sum_{s \in N \cup F} \left[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma^{AU}_{b,ls} - 1 \right) \right] \left[\sum_{f=1}^{F} \tilde{\Psi}_{sf} \left(\frac{1}{1+\eta_f} \frac{\mathrm{d} \log \Lambda_f}{d\tau_{kj}} + \frac{\eta_f}{1+\eta_f} \frac{d \log \lambda_C}{d\tau_{kj}} \right) \right] \right] \right) \Psi_{li} \\
& + \sum_{s \in N \cup F} \left(\Omega_{Gs} \frac{\lambda_G}{\lambda_i} (\sum_{g=1}^{F} \tilde{\Psi}_{sg} \left(\frac{1}{1+\eta_g} \frac{\mathrm{d} \log \lambda_{gt}}{d\tau_{kjt}} + \frac{\eta_g}{1+\eta_g} \frac{d \log \lambda_{Ct}}{d\tau_{kjt}} \right)) \right) \left[\Psi_{si} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}] \right]
\end{aligned}
$$

B.7 Government Spending Shocks

for the government spending shock

for $i \in C \cup G \cup N \cup F$

$$
\begin{aligned}
\mathrm{d}log\lambda_{it} =& \sum_{l \in C \cup G \cup N \cup F} \left(\sum_{b \in C \cup G \cup N \cup F} \frac{\lambda_b}{\lambda_i} \Omega_{bl} \left[\sum_{s \in N \cup F} \left[\delta_{ls} + \tilde{\Omega}_{bs} \left(\sigma^{AU}_{b,ls} - 1 \right) \right] dlogp_{st} \right] \right) \Psi_{li} \\
& + \sum_{l \in N \cup F} \left(\Omega_{Gl} \frac{\lambda_G}{\lambda_i} (dlogp_l + dlogg_l) \right) \left[\Psi_{li} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}] \right]
\end{aligned}
$$

government spending terms - the second term

$$+\mathbb{1}\{k \notin C \cup G\}\Omega_{Gk}\frac{\lambda_G}{\lambda_i}\left[\Psi_{ki} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}]\right]$$
$$+\sum_{s\in N\cup F}\left(\Omega_{Gs}\frac{\lambda_G}{\lambda_i}(\sum_{g=1}^{F}\tilde{\Psi}_{sg}(\frac{1}{1+\eta_g}\frac{\mathrm{d}\log\lambda_{gt}}{dlogg_{kt}} + \frac{\eta_g}{1+\eta_g}\frac{d\log\lambda_{Ct}}{dlogg_{kt}}))\right)\left[\Psi_{si} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}]\right]$$

the first term is

$$+\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\Omega_{bl}\Big[\sum_{s\in N\cup F}\left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right)\right]\Big[\sum_{f\in F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_f}{dlogg_{kt}} + \frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{dlogg_{kt}}\right)\Big]\Big]\right)\Psi_{li}$$

Put all together

$$\begin{aligned}\frac{d\log\lambda_{it}}{d\log g_{kt}} =& \mathbb{1}\{k \notin C \cup G\}\Omega_{Gk}\frac{\lambda_G}{\lambda_i}\left[\Psi_{ki} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}]\right] \\ &+\sum_{l\in C\cup G\cup N\cup F}\left(\sum_{b\in C\cup G\cup N\cup F}\frac{\lambda_b}{\lambda_i}\Omega_{bl}\Big[\sum_{s\in N\cup F}\left[\delta_{ls} + \tilde{\Omega}_{bs}\left(\sigma^{AU}_{b,ls} - 1\right)\right]\Big[\sum_{f\in F}\tilde{\Psi}_{sf}\left(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_f}{dlogg_{kt}} + \frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_C}{dlogg_{kt}}\right)\Big]\Big]\right)\Psi_{li} \\ &+\sum_{s\in N\cup F}\left(\Omega_{Gs}\frac{\lambda_G}{\lambda_i}(\sum_{f=1}^{F}\tilde{\Psi}_{sf}(\frac{1}{1+\eta_f}\frac{\mathrm{d}\log\lambda_{ft}}{dlogg_{kt}} + \frac{\eta_f}{1+\eta_f}\frac{d\log\lambda_{Ct}}{dlogg_{kt}}))\right)\left[\Psi_{si} + [\mathbb{1}\{i \in G\} - \Psi_{Ci}]\right]\end{aligned}$$

B.8 Proof of $\sum_i \tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t} = 0$

Hicksian (compensated) demands $a_{kjt}(p_{k\cdot t})$ satisfy zero-degree homogeneity in prices

The cost-minimizing input bundle is homogeneous of degree zero in prices, since the cost-minimizing problem does not depend on the scale of prices

$$a_{kjt}(\lambda p_{k\cdot t}) = a_{kjt}(p_{k\cdot t}).$$

If $f(p)$ is homogeneous of degree 0

$$\sum_i \frac{\partial f}{\partial p_i} p_i = 0$$

Apply this to $a_{kjt}(p_{k\cdot t})$

$$\sum_{i \in N \cup F} \frac{\partial \hat{a}^H_{kjt}}{\partial \hat{p}_{kit}} \hat{p}_{kit} = 0$$

Divide by $\hat{a}_{kjt}$ and rewrite

$$\sum_{i \in N \cup F} \frac{\hat{p}_{kit}}{\hat{a}_{kjt}} \frac{\partial \hat{a}_{kjt}}{\partial \hat{p}_{kit}} = 0.$$

Since

$$\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = \frac{\hat{p}_{kit}}{\hat{a}_{kjt}} \frac{\partial \hat{a}_{kjt}}{\partial \hat{p}_{kit}}$$

Therefore:

$$\sum_i \frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = 0$$

Since

$$\frac{\partial \log \hat{a}_{kjt}}{\partial \log \hat{p}_{kit}} = \tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t}.$$

Thus it has

$$\sum \tilde{\Omega}_{kit}\sigma^{AU}_{k,ji,t} = 0$$

B.9 Comparing the parametric and non-parametric results in the CES economy.

Since the production function is parametric, the Hicksian demand can be solved through the cost-minimization problem. Combining this with $p_{kt} = \mu_{kt} + c_{kt}$, it follows that, for any pair (k, l) with $k \in C \cup N$ and $l \in N \cup F$, the first-order change in the input-output linkage Ω_{klt} satisfies

$$\mathrm{d}\log \Omega_{klt} = (1-\theta_k)\left(-\mathrm{d}\log A_{kt} - \mathrm{d}\log Z_{klt} + \mathrm{d}\log p_{lt} - \mathrm{d}\log p_{kt}\right) - \theta_k dlog\mu_{kt} + \theta_k\ \mathrm{d}\tau_{klt} + \mathrm{d}\log w_{klt}$$

Now i show that the expression this paper got based on the non-parametric derivation is consistent with this result.

In the Nested CES economy with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\sigma^{AU}_{k,li} = \theta_k$ if $l \neq i$, and the own Allen-Uzawa elasticities are given by $\sigma^{AU}_{k,ll} = -\frac{\theta_k\left(1-\tilde{\Omega}_{kl}\right)}{\tilde{\Omega}_{kl}}$.

Now i show that For $k \in C \cup N$, we could write the unit input requirement in the nested CES economy as the expression below is associated with the correspondin

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \sum_{j\in N\cup F} \tilde{\Omega}_{kj}\ \sigma^{AU}_{k,lj} d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \frac{1}{\theta_k - 1} d\log w_{kjt}\right) - \frac{1}{\theta_k - 1} dlogw_{klt} \tag{15}$$

In the Nested CES economy with elasticity θ_k, all the cross Allen-Uzawa elasticities are identical with $\sigma^{AU}_{k,li} = \theta_k$ if $l \neq i$, and the own Allen-Uzawa elasticities are given by $\sigma^{AU}_{k,ll} = -\frac{\theta_k\left(1-\tilde{\Omega}_{kl}\right)}{\tilde{\Omega}_{kl}}$.

Now i show that For $k \in C \cup N$, we could write the unit input requirement in the nested CES economy as the expression below is associated with the correspondin

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \sum_{j\in N\cup F}\tilde{\Omega}_{kj}\theta_k d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \frac{1}{\theta_k-1}d\log w_{kjt}\right) - \tilde{\Omega}_{kl}\theta_k d\log\left(dlogp_{lt} - d\tau_{klt} - dlogZ_{klt} - \frac{1}{\theta_k-1}d\log w_{klt}\right)$$
$$- \tilde{\Omega}_{kl}\frac{\theta_k\left(1-\tilde{\Omega}_{kl}\right)}{\tilde{\Omega}_{kl}}\left(dlogp_{lt} - d\tau_{klt} - dlogZ_{klt} - \frac{1}{\theta_k-1}d\log w_{klt}\right) - \frac{1}{\theta_k-1}dlogw_{klt}$$

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \sum_{j\in N\cup F}\tilde{\Omega}_{kj}\theta_k d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \frac{1}{\theta_k-1}d\log w_{kjt}\right)$$
$$- \tilde{\Omega}_{kl}\left(\theta_k + \frac{\theta_k\left(1-\tilde{\Omega}_{kl}\right)}{\tilde{\Omega}_{kl}}\right)\left(dlogp_{lt} - d\tau_{klt} - dlogZ_{klt} - \frac{1}{\theta_k-1}d\log w_{klt}\right) - \frac{1}{\theta_k-1}dlogw_{klt}$$

$$d\log a_{klt} = -dlogA_{kt} - dlogZ_{klt} + \theta_k\sum_{j\in N\cup F}\tilde{\Omega}_{kj} d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \frac{1}{\theta_k-1}d\log w_{kjt}\right)$$
$$- \theta_k\left(dlogp_{lt} - d\tau_{klt} - dlogZ_{klt} - \frac{1}{\theta_k-1}d\log w_{klt}\right) - \frac{1}{\theta_k-1}dlogw_{klt}$$

$$\sum_{j\in N\cup F}\tilde{\Omega}_{kj} d\log\left(dlogp_{jt} - d\tau_{kjt} - dlogZ_{kjt} - \frac{1}{\theta_k-1}d\log w_{kjt}\right) = dlogp_k + dlogA_k - dlog\mu_k$$

$$\begin{aligned} d\log a_{klt} =& - dlogA_{kt} - dlogZ_{klt} + \theta_k(dlogp_k + dlogA_k - dlog\mu_k) - \theta_k\left(dlogp_{lt} - d\tau_{klt} - dlogZ_{klt} - \frac{1}{\theta_k-1}d\log w_{klt}\right) - \frac{1}{\theta_k-1}dlogw_{klt} \\ =&(\theta_k-1)dlogA_{kt} + (\theta_k-1)dlogZ_{klt} - \theta_k dlog\mu_k - \theta_k(dlogp_{lt} - d\tau_{klt}) + \theta_k dlogp_k + \frac{\theta_k}{\theta_k-1}dlogw_{klt} - \frac{1}{\theta_k-1}dlogw_{klt} \\ =&(\theta_k-1)dlogA_{kt} + (\theta_k-1)dlogZ_{klt} - \theta_k dlog\mu_k - \theta_k(dlogp_{lt} - d\tau_{klt}) + \theta_k dlogp_k + dlogw_{klt} \end{aligned}$$

$$dlogx_{klt} = d\log a_{klt} + d\log y_{kt} = (\theta_k - 1)dlogA_{kt} + (\theta_k - 1)dlogZ_{klt} - \theta_k dlog\mu_k - \theta_k(dlogp_{lt} - d\tau_{klt}) + \theta_k dlogp_k + d\log y_{kt} + dlogw_{klt}$$

$$dlogx_{klt} = d\log a_{klt} + d\log y_{kt} = (\theta_k - 1)dlogA_{kt} + (\theta_k - 1)dlogZ_{klt} - \theta_k dlog\mu_k - \theta_k(dlogp_{lt} - d\tau_{klt}) + \theta_k dlogp_k + d\log y_{kt} + dlogw_{klt}$$

$$\begin{aligned}
dlog\Omega_{klt} &= dlogp_{lt} + d\log a_{klt} - dlogp_{kt} \\
&= dlogp_{lt} + (\theta_k - 1)dlogA_{kt} + (\theta_k - 1)dlogZ_{klt} - \theta_k dlog\mu_k - \theta_k(dlogp_{lt} - d\tau_{klt}) + \theta_k dlogp_k + dlogw_{klt} - dlogp_{kt} \\
&= -(\theta_k - 1)dlogp_{lt} + (\theta_k - 1)dlogA_{kt} + (\theta_k - 1)dlogZ_{klt} - \theta_k dlog\mu_k + \theta_k d\tau_{klt} + dlogw_{klt} + (\theta_k - 1)dlogp_k \\
&= (1 - \theta_k)(-dlogA_{kt} - dlogZ_{klt} + dlogp_{lt} - dlogp_{kt}) - \theta_k dlog\mu_k + \theta_k d\tau_{klt} + dlogw_{klt}
\end{aligned}$$

B.10 Simplifying $-\sum_{i\in N\cup F}\tilde{\Omega}_{Ci}\left(\sigma^{AU}_{C,li}-1\right)\left(\frac{1}{\theta_C-1}d\log w_{Cit}\right)-\frac{1}{\theta_C-1}d\log w_{Clt}$

$$\begin{aligned}
dlog\Omega_{klt} &= ... - \sum_{i\in N\cup F}\tilde{\Omega}_{Ci}\left(\sigma^{AU}_{C,li}-1\right)\left(\frac{1}{\theta_C-1}d\log w_{Cit}\right) - \frac{1}{\theta_C-1}d\log w_{Clt} \\
&= ... - \sum_{i\in N\cup F}\tilde{\Omega}_{Ci}\left(\theta_C-1\right)\left(\frac{1}{\theta_C-1}d\log w_{Cit}\right) + \Omega_{Cl}\theta_C\frac{1}{\theta_C-1}d\log w_{Clt} + \tilde{\Omega}_{Cl}\frac{\theta_C\left(1-\tilde{\Omega}_{Cl}\right)}{\tilde{\Omega}_{Cl}}\frac{1}{\theta_C-1}d\log w_{Clt} - \frac{1}{\theta_C-1}d\log w_{Clt} \\
&= ... - \sum_{i\in N\cup F}\tilde{\Omega}_{Ci}d\log w_{Cit} + d\log w_{Clt}
\end{aligned}$$

C Additional Tables

C.1 Subsidy Policy - Integrated Market

Table 6: Industry-Level Proportional Production Subsidy Multipliers

Code	Industry	Proportional Subsidy
213	Support activities for mining	0.62109
525	Funds, trusts, and other financial vehicles	0.42598
483	Water transportation	0.42548
3361MV	Motor vehicles, bodies and trailers, and parts	0.32051
322	Paper products	0.24198
623	Nursing and residential care facilities	0.20987
313TT	Textile mills and textile product mills	0.20206
ORE	Other real estate	0.20069
GSLE	State and local government enterprises	0.19791
311FT	Food and beverage and tobacco products	0.19723
333	Machinery	0.19514
523	Securities, commodity contracts, and investments	0.19466
622	Hospitals	0.19071
485	Transit and ground passenger transportation	0.19026
721	Accommodation	0.18622
512	Motion picture and sound recording industries	0.18604
325	Chemical products	0.18560

Table 7: Linkage-Specific Consumption Multipliers

Code	Industry	C Multiplier
213	Support activities for mining	1.22930
113FF	Forestry, fishing, and related activities	0.73881
486	Pipeline transportation	0.71178
483	Water transportation	0.66778
623	Nursing and residential care facilities	0.65862
493	Warehousing and storage	0.65499
322	Paper products	0.61595
323	Printing and related support activities	0.59231
61	Educational services	0.59197
GFE	Federal government enterprises	0.56873
GSLG	State and local general government	0.55319
621	Ambulatory health care services	0.54671
3361MV	Motor vehicles, bodies and trailers, and parts	0.54665
561	Administrative and support services	0.54088
523	Securities, commodity contracts, and investments	0.54020
485	Transit and ground passenger transportation	0.53879
711AS	Performing arts, spectator sports, museums, and related activities	0.53878
622	Hospitals	0.53823
624	Social assistance	0.52190

Table 8: Linkage-Specific Labor Multipliers

Code	Industry	L Multiplier
211	Oil and gas extraction	1.15570
213	Support activities for mining	0.95342
324	Petroleum and coal products	0.79518
483	Water transportation	0.74978
482	Rail transportation	0.73896
485	Transit and ground passenger transportation	0.73528
486	Pipeline transportation	0.72530
721	Accommodation	0.71585
623	Nursing and residential care facilities	0.71381
532RL	Rental and leasing services and lessors of intangible assets	0.70315
325	Chemical products	0.69304
GSLE	State and local government enterprises	0.69220
622	Hospitals	0.68501
22	Utilities	0.68471
61	Educational services	0.68114
315AL	Apparel and leather and allied products	0.67447
3361MV	Motor vehicles, bodies and trailers, and parts	0.66017
621	Ambulatory health care services	0.65881

Table 9: Linkage-Specific Capital Multipliers

Code	Industry	K Multiplier
493	Warehousing and storage	0.62253
61	Educational services	0.43331
3361MV	Motor vehicles, bodies and trailers, and parts	0.37592
623	Nursing and residential care facilities	0.29989
213	Support activities for mining	0.27949
622	Hospitals	0.25856
5415	Computer systems design and related services	0.24580
713	Amusements, gambling, and recreation industries	0.23357
524	Insurance carriers and related activities	0.23257
711AS	Performing arts, spectator sports, museums, and related activities	0.22699
513	Broadcasting and telecommunications	0.22096
621	Ambulatory health care services	0.22072
311FT	Food and beverage and tobacco products	0.21514
337	Furniture and related products	0.20023
722	Food services and drinking places	0.17451
481	Air transportation	0.14824
624	Social assistance	0.13797
81	Other services, except government	0.13560

C.2 Subsidy Policy - Segmented Market

Table 10: Industry-Level Proportional Production Subsidy Multipliers

Code	Industry	Proportional Subsidy
213	Support activities for mining	0.59388
525	Funds, trusts, and other financial vehicles	0.43300
483	Water transportation	0.41923
3361MV	Motor vehicles, bodies and trailers, and parts	0.30276
322	Paper products	0.24125
311FT	Food and beverage and tobacco products	0.22272
313TT	Textile mills and textile product mills	0.20401
GSLE	State and local government enterprises	0.20104
623	Nursing and residential care facilities	0.19675
ORE	Other real estate	0.19568
333	Machinery	0.19310
325	Chemical products	0.18642
523	Securities, commodity contracts, and investments	0.18547
337	Furniture and related products	0.18320
111CA	Farms	0.18084
486	Pipeline transportation	0.17865

Table 11: Linkage-Specific Consumption Multipliers

Code	Industry	C Multiplier
213	Support activities for mining	1.03520
113FF	Forestry, fishing, and related activities	0.74855
483	Water transportation	0.71682
493	Warehousing and storage	0.69720
GSLG	State and local general government	0.65459
322	Paper products	0.61826
623	Nursing and residential care facilities	0.59951
GFE	Federal government enterprises	0.58272
486	Pipeline transportation	0.58111
323	Printing and related support activities	0.56748
561	Administrative and support services	0.56064
523	Securities, commodity contracts, and investments	0.52769
61	Educational services	0.50003
485	Transit and ground passenger transportation	0.49950
624	Social assistance	0.48480
55	Management of companies and enterprises	0.48240
621	Ambulatory health care services	0.47765
622	Hospitals	0.47729
315AL	Apparel and leather and allied products	0.47406

Table 12: Linkage-Specific Labor Multipliers

Code	Industry	L Multiplier
211	Oil and gas extraction	0.92020
213	Support activities for mining	0.82405
486	Pipeline transportation	0.71521
324	Petroleum and coal products	0.69623
483	Water transportation	0.68624
482	Rail transportation	0.67900
485	Transit and ground passenger transportation	0.67692
532RL	Rental and leasing services and lessors of intangible assets	0.67348
721	Accommodation	0.66088
623	Nursing and residential care facilities	0.65406
315AL	Apparel and leather and allied products	0.65099
GSLE	State and local government enterprises	0.64524
325	Chemical products	0.64420
22	Utilities	0.63178
622	Hospitals	0.63132
61	Educational services	0.62628
3361MV	Motor vehicles, bodies and trailers, and parts	0.62249
711AS	Performing arts, spectator sports, museums, and related activities	0.62047
621	Ambulatory health care services	0.61011
713	Amusements, gambling, and recreation industries	0.59922
452	General merchandise stores	0.59859