A Theory of Land Finance and Investment-Led Growth

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Abstract

Managing land has been a challenge for governments throughout history. I develop a growth model in which the government finances public investment with land sale and rent. A benevolent government optimally front-loads investment but maintains a *constant stock* of land supplied to the private sector, balancing fundraising through controlled supply against the welfare loss from reduced land utilization. This optimal allocation can be implemented on a balanced budget by adjusting the mix of land sale and rent, or in a time-consistent manner by issuing a full set of bonds across the maturity structure. A discretionary government on a balanced budget, however, would gradually increase land supply, consistent with historical patterns. Finally, I show that a contingent land contract linking land supply to the provision of public capital can restore the optimal allocation.

Keywords: neoclassical growth, public investment, optimal fiscal policy, financial constraints, time inconsistency, land contracts, Coase conjecture.

JEL: E62, G10, H27, H54, H63, L12, O23, O40, Q24

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1 Introduction

Throughout history, land has been one of the most important assets for governments. Its management has been a pressing concern, particularly for developing nations, with seminal scholarly contributions marking the discourse (George, 1879; Tideman et al., 1990). Various governments—including Singapore, Hong Kong, mainland China, and the 19th-century US—have engaged in large-scale land transfers to the private sector to fund public infrastructure investment during periods of rapid economic growth, a practice known as "land finance". This practice is gaining further traction across the developing world (Peterson, 2008; UK FCDO, 2015). This paper develops a dynamic general equilibrium model to study the mechanism of land finance. Under realistic frictions, it predicts that the government front-loads public investment while continuously expanding land supply as observed in reality. It further proposes a welfare-improving design of land contract which links land supply to the provision of public capital.

I consider an economy where production uses public capital as an external input, provided by a benevolent government, as in Barro (1990). Differing from it, the government finances its investment through land sale or rent rather than tax, and land is valued by households as housing consumption. The model formalizes the positive feedback loop between economic growth and land finance intuited by Tideman et al. (1990): current land value embeds future economic growth, and a higher land value enables the government to invest more which aids economic growth. Assuming complete markets and that the government can commit to its future actions, the model predicts that while the government optimally front-loads public investment, it supplies a *constant stock* of land while leaving the rest idle. The constant supply prediction follows from a fundraising motive and a consumption-smoothing motive of the government. Under inelastic demand for land, the benevolent government restricts land supply to raise funds for public investment while balancing this against the welfare loss of reduced land consumption. Given the need for restricting land supply, it does so in a time-invariant way to minimize the welfare loss, i.e., consumption smoothing.

Admittedly, the assumptions of complete markets and commitment are both strong. Nonetheless, I show that the optimal allocation can be implemented on a balanced budget or in a time-consistent manner. The balanced-budget implementation is possible thanks to the dual property of land as both a consumption good and an asset. In particular, a land sale front-loads all future rents into current revenue. For the government, the optimal

investment is front-loaded, but it can be matched by adjusting the mixture of land sale and rent while maintaining the total supply. Tideman et al. (1990) previously suggested that the government should retain land ownership and raise revenue only from rent, concerned that large-volume land sales would depress prices and exclude credit-constrained buyers. My model illustrates that land rent brings a constant stream of income in a normalized sense. However, if the desired government investment is time-varying, as is generally the case in reality, there will be a mismatch in timing between income and investment. The model elucidates that this timing mismatch can be resolved by selling land at a gradual pace, without necessitating government or private sector borrowing.

The optimal allocation can also be implemented in a time-consistent manner. This is against the backdrop of the renowned Coase (1972) conjecture that a durable good monopoly would flood the market with supply rather than restricting it.[1] It is known that if the monopoly only rents the good out instead of selling it, they are not subject to the time consistency issue (Bulow, 1982). However, that does not implement the optimal allocation, since the rent income mismatches investment need in timing as I have argued. I build on the insights of Lucas and Stokey (1983) to offer a general equilibrium resolution of the Coase conjecture. In my model, the interest rate—rather than taken as given in the Coase conjecture literature—is endogenous to consumption growth, which in turn depends on the government's investment. Consequently, if there are long-term bonds outstanding, the government's investment decision would alter the interest rate and revaluate the bonds. When markets are complete, the government can issue a set of bonds of different maturities up to infinity such that they would revaluate adversarially if future governments were to deviate from the optimal course of action. That is, it uses bond revaluation as a commitment device.

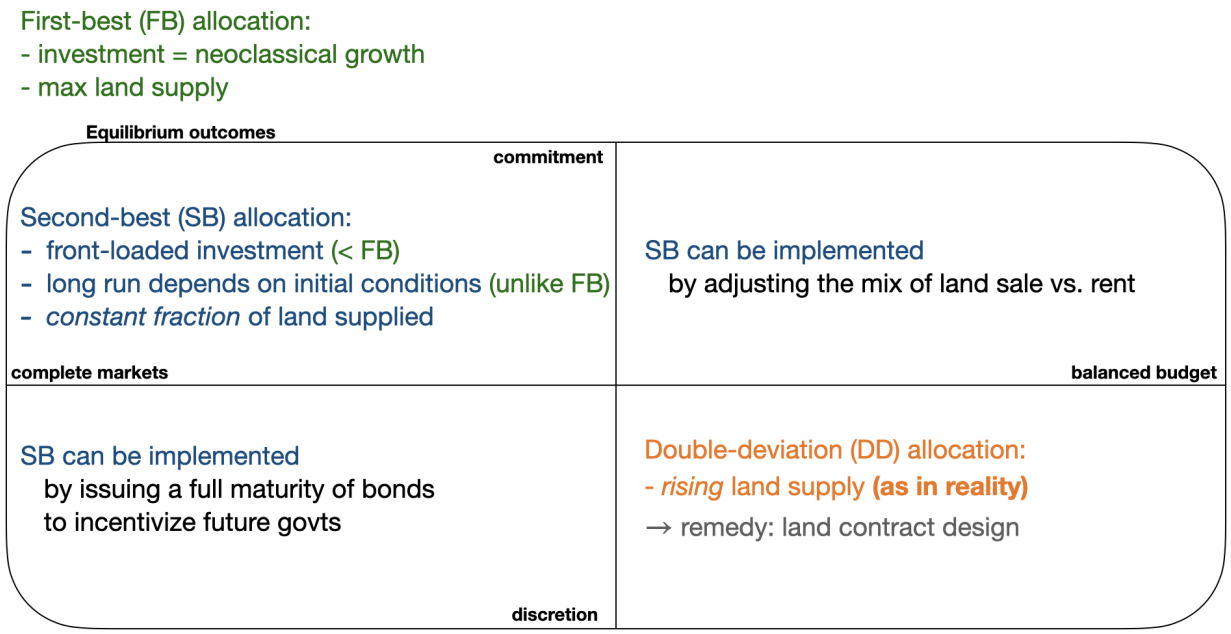
However, I show that when the government is simultaneously discretionary and on a balanced budget, it will still front-load investment but continuously increase land supply to the private sector. This phenomenon conforms to the Coase conjecture but in a smooth manner, as the monopoly gradually rather than immediately flood the market. The intuition is that a discretionary government always takes past sales as given and always wants to sell more to fund investment.

Finally, given that continual expansion of land supply indicates a deviation from the

[1]Interestingly, Coase makes his argument with an example of a monopoly owning all the land in the US, attesting to the importance of land. Despite his example, the literature that follows typically considers settings that are partial equilibrium in nature, whereas this paper studies land in general equilibrium.

optimal allocation, I ask if an improvement is possible. I propose a novel land contract that links land supply to the provision of public capital to discipline future governments. Specifically, the optimal land contract penalizes future governments' deviation by obliging them to give more land to the contract-holders if the public capital stock is lower. This contract, whose institutional requirement is the same as administering land sale, helps illustrate the direction of policy design that can improve welfare. These theoretical predictions are summarized in Figure 1.

Figure 1: Summary of results

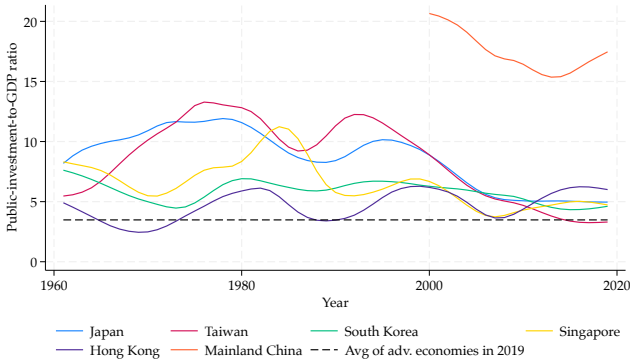


While this paper's contribution is theoretical, its predictions are broadly in line with reality. First, the public-investment-to-GDP ratio declines during convergence. Gurara et al. (2017) and Adarov, Clements and Jalles (2024), among others, document that this ratio is significantly higher for developing countries than advanced economies. Over time, individual economies show similar patterns. Figure 2a suggests that, across Asian growth miracles, the public-investment-to-GDP ratios tend to converge downward over time. Second, in economies where public investment is financed by proceeds from land supply, the government supplies land to the private sector in a gradual and continuous manner. Figure 2b shows the case for mainland China, and Figure A1 shows Singapore and Hong Kong. Importantly, the gradual expansion is not constrained by the availability

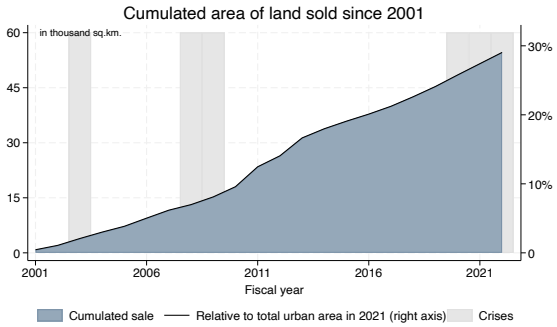
of land—rather, much land sits idle, suggested by Figure A2. Similarly, land was only gradually transferred by the US governments to private use over the course of 19th century (Anderson and Martin, 1987; Libecap, 2007).

Figure 2: Declining public investment and continual land supply

(a) Public-investment-to-GDP ratio



(b) Land supply in mainland China



Notes: (a) This plots the time series of public-investment-to-GDP ratio using the IMF Investment and Capital Stock Dataset. The time series are smoothened using the Hodrick and Prescott (1997) filter with a smoothing parameter of 6.25 for annual data following Ravn and Uhlig (2002). But the qualitative patterns are the same if one uses the raw data of public-investment-to-GDP ratio or that averaged over several years.
(b) Data are from China's National Bureau of Statistics and Ministry of Natural Resources.

Literature review. First and foremost, this paper contributes to our understanding of the role of public capital and land in economic growth. The idea that government should use land revenue to finance public goods is well articulated in the open letter to Gorbachev by a group of renowned economists (Tideman et al., 1990). This paper provides a model to examine its mechanism and qualify the intuitive proposal of never selling land. It also studies the potential time inconsistency issue overlooked in that letter.[2] Barro (1990) develops the seminal model to study income-tax-financed public capital for economic growth.[3] Building on that, I study the finance of public capital through land supply. Land revenue is different in nature from income tax, as land is a stock variable in fixed supply and income is a flow variable.

[2] I also briefly discuss how land tax as proposed by George (1879) may interact with a motive that restricts supply to increase the total market value of land to raise public funds in Appendix B.

[3] Ratner (1983), Aschauer (1989) and Munnell (1990*a*,*b*) estimate the output elasticity of public capital. See Bom and Ligthart (2014) for a review and Leduc and Wilson (2012) and Ramey (2021) for recent advances.

It is widely recognized that land finance and public investment have played a key role in China's growth miracle and have important implications for its real estate market. See Liu (2018) and Gyourko et al. (2022) for reviews and Fang et al. (2016), Glaeser et al. (2017), Rogoff and Yang (2024), He et al. (2025), and Chang, Wang and Xiong (2026) for empirical studies. Song and Xiong (2026) build a model of how career incentives of government officials affect public investment in China's growth. They treat land revenue as proportional to output for simplicity—an approach closer in spirit to Barro (1990) than my study of land as a durable asset in fixed supply. Jiang, Miao and Zhang (2022) build a quantitative model of land finance and public investment by assuming an exogenous land supply. This paper first examines the mechanism and optimal implementation of land finance in dynamic general equilibrium.

My study of land relates to but differs significantly from the study of exhaustible resources (such as oil and minerals), which was once a prominent topic of economic research (Hotelling, 1931; Solow, 1974), because land does not depreciate. Consumption smoothing of land calls for a time-invariant supply, i.e., a constant stock, as opposed to a constant flow. Moreover, the demand of land is local and endogenous to the country's growth, while the demand for oil is largely from the international market. Finally, in practice, only a few countries are endowed (or burdened) with significant exhaustible resources,[4] whereas managing land is a challenge for every nation.

This paper further connects to the literature on optimal fiscal policy since Ramsey (1927), which is primarily concerned with taxation and typically assumes an exogenous stream of government expenditure. While I highlight various conceptual links to this literature, this paper differs prominently in its subject of study—the role of land supply and land contracts as opposed to taxation—and endogenously determines the stream of public spending rather than taking it as given.[5] I draw connection of my result of optimal time-invariant land supply to the tax smoothing result of Barro (1979) and Aiyagari et al. (2002) in optimal taxation. Kydland and Prescott (1977) first note the potential time inconsistency of optimal policy. Lucas and Stokey (1983) show that in a model without capital, the optimal fiscal policy to finance exogenous spending can be made time-consistent under complete markets, and Debortoli, Nunes and Yared (2021) further qualify this statement. Angeletos (2002) and Buera and Nicolini (2004) investigate optimal fiscal

[4]Economies with abundant exhaustible resources may grow less rapidly than resource-scarce economies, known as the "resource curse" (Sachs and Warner, 1995; van der Ploeg, 2011; Venables, 2016).

[5]This paper also broadly relates to the study of optimal development policies, such as Acemoglu, Aghion and Zilibotti (2006), Liu (2019), and Itskhoki and Moll (2019).

policy under commitment with non-contingent debt only. Debortoli, Nunes and Yared (2017, 2022) instead study optimal fiscal policy under complete markets in the absence of commitment. These papers make use of quantitative methods to arrive at some of their results. I offer analytical insights into the optimal policy in my model when removing either complete markets, commitment, or both.

Last, this paper treats the Coase (1972) conjecture in general equilibrium. Given the importance of land, it is no wonder that Coase opens his essay by considering a monopoly owning all the land in the United States. The following literature that formalizes his argument (Stokey, 1981; Bulow, 1982; Gul, Sonnenschein and Wilson, 1986), however, typically considers examples that are more microeconomic in nature, such as cars and diamonds. They posit a constant interest rate and a durable good demand curve that is independent of other consumption goods. In contrast, this paper endogenizes both the interest rate and the land price from household optimization in general equilibrium, which trades off consumption across time and balances land consumption against non-durable good consumption.

2 The Economy and First-Best Allocation

I first set up the economy and discuss the first-best allocation that maximizes welfare only subject to resource constraints. Under the first-best allocation, the government does not restrict land supply.

2.1 The Economy

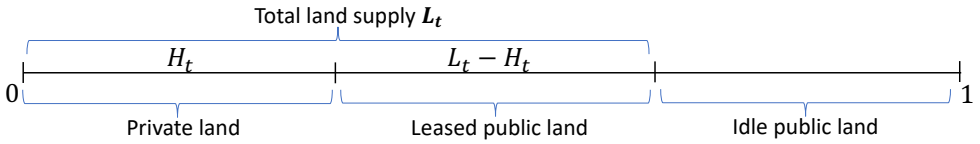
The economy is fully deterministic. An infinitely-lived representative household derives utility from (non-durable) consumption C_t and land L_t

$$\mathcal{U}_0 \equiv \int_0^\infty \mathrm{e}^{-\rho t} U\left(C_t, L_t\right) \mathrm{d}t = \int_0^\infty \mathrm{e}^{-\rho t} \left(\ln C_t + \nu \frac{L_t^{1-\sigma} - 1}{1-\sigma}\right) \mathrm{d}t, \tag{1}$$

with $\rho > 0, \sigma > 1$. For simplicity, I assume that the household derives utility from land without modeling housing structure. The assumption of $\sigma > 1$ ensures that the demand for housing, holding fixed consumption C, is inelastic, as will become clear later. I normalize the total amount of land in this economy to one without loss of generality. Time-t housing consumption consists of H_t amount of owned land that is valued at price P_t, and $L_t - H_t$

amount leased from the government at rent D_t, as illustrated in Figure 3. Allowing the government to both sell and rent captures various forms of land contracts in reality such as sale, long-term and short-term leases. At time t, the household could also buy or sell housing at rate $\dot{H}_t$ at the prevailing price P_t which will determine the owned amount $H_{t+\mathrm{d}t}$ in the next instance. Land does not depreciate.

Figure 3: Land utilization in the model



The household supplies one fixed unit of labor. The production uses public capital stock Z_t as an external input with total factor productivity (TFP) A.[6] The stock Z_t depreciates at rate $\delta \geq 0$. For concreteness, I may refer to the public capital as infrastructure, but one may also include other forms of capital or public service with a higher δ. For the production, the household takes Z_t as a public good, which is the only externality in this model. The household earns labor income Y_t. The infrastructure investment is thus carried out by the government. The time-t resource constraint is

$$C_t + \dot{Z}_t + \delta Z_t \leq Y_t(Z_t) = AZ_t^{\alpha}, \tag{2}$$

with $\alpha \in (0,1)$. The production function is of a neoclassical nature, with an incarnation of public investment à la Barro (1990). I denote the consumption rate (consumption-to-GDP ratio) as

$$\chi_t \equiv \frac{C_t}{Y_t}$$

and thus the investment rate (investment-to-GDP ratio) is $\frac{\dot{Z}_t + \delta Z_t}{Y_t} = 1 - \chi_t$. The reality of declining government-investment-to-GDP ratio means that χ_t increases over time.

The economy starts at time 0 with public capital stock Z_0 and land in private hands of the amount $H_0 < 1$. The government has no debt when the economy starts, and is endowed with the rest of the land $(1 - H_0)$. At each time t, the amount of land the

[6] Growth in TFP and population will be accommodated in an extension considered in Appendix E.1, but they do not change the prediction that the second-best allocation features *constant* amount of land supplied to the private sector.

government supplies L_t and the household owns H_t satisfy

$$L_t \in [0, 1], \tag{3}$$

$$H_t \in [0, 1]. \tag{4}$$

The government derives income from land sale and rent as their only revenue source to fund government investment. The government is allowed to rebate extra money to the household in a lump-sum manner (but they may not want to), but cannot levy lump-sum tax.

Calibration. $\delta = 5\%, \rho = 2\%$, $\alpha = 10\%$ (Bom and Ligthart, 2014), $\sigma = 2$ (which corresponds to a demand elasticity of land/housing of one half), $A = 1$. To calibrate ν, using the FOC $U_C = U_L D^{-1}$ at the steady state and $P = \rho^{-1} D$ to get

$$\nu = \rho \frac{PL}{C} L^{\sigma - 1}$$

Take $\rho = 0.02$, housing-value-to-GDP ratio of 2 and $L = 1$, we get $\nu = 0.04$.

2.2 Remarks on Model Assumptions

Before presenting results, I provide some remarks on the model. This model is designed to minimize deviation from a neoclassical growth model while offering insights into land finance. Appendix E offers a few formal extensions.

1. I assume away lump-sum tax and land tax as proposed by George (1879). Pure land tax is practically non-existent. Property tax exists, but it distorts investment in housing structure, and is hard to implement (especially for developing countries) as it requires pricing off-market houses. An insight behind the celebrated Henry George theorem is that land tax is *non-distortionary* as land is in fixed supply. In Appendix B, I suggest a limit of this logic: while land tax is non-distortionary, it may be *insufficient* to fund "best outcome" if the tax base (total land market value) is low. The incentive analyzed in this paper that the government may lower land supply in order to raise its market value can still play a role, in which case there will be welfare loss from reduced land utilization, even if the government can collect land tax.

2. I assume the land is exclusively designated for housing purposes. In reality, land

is used for production too and is typically zoned for different uses. The insight that a less-than-maximum supply of land ($L_t < 1$) may be chosen to increase public revenue under inelastic demand even if it incurs welfare cost does not hinge on the exact use of land. Appendix E.4 considers land used as factor of production instead of housing. I further assume for simplicity that building infrastructure does not use land, which is harmless if the land used for infrastructure does not exceed the amount of land optimally left idle.

3. I abstract away from various distortionary taxes for simplicity. This model could be augmented to feature endogenous labor supply and income tax. In that case, it is desirable to use both instruments (labor tax and land supply) to raise funds and balance between multiple sources of welfare loss (labor supply distortion and lowered land utilization). A further reason not to include private capital and capital tax is that the time consistency issue with capital tax is well-known. Shutting that down helps isolate the time consistency issue with land.

4. This model is interesting only when there is a non-trivial amount of land owned by the government at time 0. This is the case for the US and Hong Kong due to their colonial histories. For other economies this model can be viewed as analyzing the policy problem *after* the government acquires land from the private sector. In Singapore, the government already owned 44% of land in 1960 (before independence), and it passed the Land Acquisition Act in 1966 (after independence) to acquire more land for public purpose at a fixed price. In mainland China, the government acquires land from rural collectives (groups of local farmers) and "rezones" it as urban land. Through acquisitions at a fixed price, a government can obtain the land, engage in infrastructure investment which raises the land value, and thus profit from the appreciation of land. We directly place ourselves in the second stage where the government already has land at its disposal.

2.3 First-Best Allocation

The first-best (FB) allocation maximizes household utility (1) subject to the resource constraint (2) and land constraints (3, 4) only.

Proposition 1. (First best allocation) *The FB allocation* $\left(C_t^{FB}, Z_t^{FB}, L_t^{FB}, H_t^{FB}\right)$ *is completely characterized as follows, given* Z_0*:*

1. *the consumption growth follows*

$$\frac{\dot{C}_t^{FB}}{C_t^{FB}} = Y_t'\left(Z_t^{FB}\right) - (\delta + \rho), \tag{5}$$

subject to the transversality condition and the resource constraint

$$\dot{Z}_t^{FB} = Y_t\left(Z_t^{FB}\right) - C_t^{FB} - \delta Z_t^{FB}; \tag{6}$$

2. *the land supply is maximum*

$$L_t^{FB} = 1; \tag{7}$$

3. *the amount of land held in private hands* H_t^{FB} *for* $t > 0$ *can take any value in* $[0, 1]$.

The path of consumption and public capital at the first-best allocation in this economy is identical to the path of consumption and (private) capital in a neoclassical growth model. However, the market outcome in a neoclassical model coincides with the first best, whereas this economy embeds a market failure that the household takes the public capital as given, which may inhibit the FB allocation. I will show that, in order to generate more profits, the government may lower land supply. This generates a tradeoff between fundraising and welfare loss from reduced land consumption.

Under constant TFP A, the economy converges to a steady-state that features

$$Y'\left(Z_\infty^{FB}\right) = \delta + \rho \quad \Rightarrow \quad Z_\infty^{FB} = \left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}. \tag{8}$$

At the steady state, the consumption rate is

$$\chi_\infty^{FB} = 1 - \frac{\delta Z_\infty^{FB}}{Y_\infty^{FB}} = \frac{\delta(1-\alpha) + \rho}{\delta + \rho} \tag{9}$$

and the investment rate is thus its complement $\frac{\delta\alpha}{\delta+\rho}$.

Throughout this paper, I assume that the initial capital stock Z_0 falls short of the first-best steady-state level, i.e.,

$$Z_0 < Z_\infty^{FB} = \left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}} \tag{10}$$

so that the economy grows over time.

3 Second-Best Allocation

Now I restrict attention to allocations that can be sustained as equilibrium outcomes. That is, the government needs to fund its investment with land sale and rent. I start with household optimization which determines the equilibrium prices. Then I study the Ramsey planning problem in primal form for the government, which will give rise to the second-best (SB) allocation, under the assumptions of complete financial markets and commitment by the government.

3.1 Household Behavior and Premises for Second Best

Under complete markets, the household borrowing is unconstrained other than by the transversality condition. Thus the household is subject to a present-value budget constraint

$$\begin{aligned} 0 &= \int_0^\infty Q_t \left[C_t - Y_t(Z_t) + P_t \dot{H}_t + D_t(L_t - H_t) - T_t \right] \mathrm{d}t \\ &= \int_0^\infty \left\{ Q_t \left[C_t - Y_t(Z_t) + D_t(L_t - H_t) - T_t \right] - H_t \frac{\mathrm{d}(P_t Q_t)}{\mathrm{d}t} \right\} \mathrm{d}t - P_0 H_0 \end{aligned} \tag{11}$$

in which Q_t denotes the bond price under normalization $Q_0 = 1$, P_t is the land price, D_t is the land rent, and $T_t \geq 0$ is the lump-sum transfer from the government.

Equilibrium prices. The household optimality conditions pin down bond price and land rent as

$$Q_t = \mathrm{e}^{-\rho t} U_{C,0}^{-1} U_{C,t} \tag{12}$$

$$D_t = U_{C,t}^{-1} U_{L,t} \tag{13}$$

and thus the land price satisfies

$$P_t = Q_t^{-1} \int_t^\infty Q_s D_s \mathrm{d}s = U_{C,t}^{-1} \mathcal{U}_{L,t} \tag{14}$$

in which $\mathcal{U}_{L,t} \equiv \int_t^\infty \mathrm{e}^{-\rho(s-t)} U_{L,s} \mathrm{d}s$ incorporates the discounted sum of marginal utility from housing. Land rent D_t declines in current supply L_t, due to diminishing marginal housing consumption utility. Land price P_t decreases in future supply $L_{\tau \geq t}$ too, since land does not depreciate. Thus the government is a durable good monopolist à la Coase (1972) and faces a time consistency issue: in order to raise current price P_t, the government wants to hold future supply low; but when future comes, there is an incentive to deviate since what is sold is sold. In this section, I assume that the government can commit, which I relax later.

Feedback between growth and land finance. Holding fixed the path of land supply $\{L_t\}$, land price increases in consumption C_t: as the economy grows but land is ultimately in fixed supply, land price appreciates. Having fiscal income derived from land allows the government to invest in public capital Z_t and helps the economy grow. Hence, there is a positive feedback between land finance and economic growth the government can activate, capturing the intuition from Tideman et al. (1990).

Fiscal requirement of FB. Now I clarify assumptions such that the first best cannot be achieved as market outcome, and that the government may profit from restricting land supply.

I assess the fiscal position evaluated at the market price determined by household optimization (12-14) under the FB allocation. The present-value of time-t government investment in infrastructure, multiplied by $U_{C,0}$ to transform into unit of utility, is

$$\mathcal{E}_t^{FB} = Q_t \left(\dot{Z}_t^{FB} + \delta Z_t^{FB} \right) U_{C,0} = \mathrm{e}^{-\rho t} U_{C,t} \left(\dot{Z}_t^{FB} + \delta Z_t^{FB} \right) = \mathrm{e}^{-\rho t} \left(\frac{1}{\chi_t^{FB}} - 1 \right),$$

where the last step uses the resource constraint (2) and the consumption rate $\chi_t^{FB} = \frac{C_t^{FB}}{Y_t^{FB}}$. The present-value util-unit of total government spending is thus

$$\mathcal{E}^{FB} = \int_0^\infty \mathcal{E}_t^{FB} \mathrm{d}t = \mathcal{E}^{FB}(Z_0),$$

which implicitly depends on the initial infrastructure stock Z_0.

Holding $H_t = H_0$ fixed,[7] the government derives income from land lease at each time

[7] In fact, the claim that follows holds even if H_t varies over time, because the housing price is "fair" in incorporating all rental revenue when household borrowing is unconstrained.

t whose present value, again multiplied by $U_{C,0}$ to transform into util, is

$$\mathcal{I}_t^{FB} = Q_t D_t (1 - H_0) U_{C,0} = \mathrm{e}^{-\rho t} \nu (1 - H_0),$$

and the present-value util-unit of total government income is

$$\mathcal{I}^{FB} = \int_0^\infty \mathcal{I}_t^{FB} \mathrm{d}t = \frac{\nu (1 - H_0)}{\rho}$$

Proposition 2. Given a low initial capital stock Z_0 that satisfies (10), the first-best cannot be achieved if the initial household land holding H_0 *is high, i.e.,*

$$\nu (1 - H_0) < \rho \int_0^\infty \mathrm{e}^{-\rho t} \left(\frac{1}{\chi_t^{FB}} - 1 \right) \mathrm{d}t \tag{15}$$

In particular, this threshold is lower if the initial capital stock Z_0 is lower.

It ensures that if the government supplies all the land to the private sector ($L_t = 1$), it does not get enough funds to fulfill the required FB investment. On the left-hand side, $1 - H_0$ is the amount of land government sells or rents to the household, and ν is the total market value of land in unit of utility. The right-hand side implicitly depends on model parameters that are unrelated to land (TFP A, time preference ρ, output elasticity α, initial capital stock Z_0 and depreciation rate δ). Under Assumption 2, the government is too poor relative to FB, and thus the optimal lump-sum transfer T_t will be zero.

Corollary 1. *FB cannot be achieved if*

$$\nu < \frac{\delta \alpha}{\delta (1 - \alpha) + \rho}$$

In this case, the required replenishing of capital cannot be afforded by the government.

3.2 Inelastic Demand and Quantity Lerner Rule

Assumption 2 ensures that the FB allocation cannot be achieved with maximum land supply $L_t = 1$. I further focus on a case that is empirically relevant where the government can raise more money by restricting land supply.

I first take a small detour to derive a useful variant of the celebrated Lerner rule, which will also provide useful intuition for our later results.

Lemma 1. (A quantity Lerner rule) *Consider a monopoly that faces a demand curve $q(p)$. Assume that it produces with zero marginal cost, but the buyer has already owned some quantity q_0, so the monopoly profit is*

$$\max \pi = (q(p) - q_0)\, p \tag{16}$$

The supply to maximize profit π satisfies

$$\frac{q - q_0}{q} = \underbrace{-\frac{\mathrm{d}\log q(p)}{\mathrm{d}\log p}}_{\textit{demand elasticity}} \tag{17}$$

Formula (17) suggests that, in problem (16), a solution with finite quantity exists when the demand is *inelastic*. This is meaningful as decades of empirical research almost all point to inelastic demand for housing.[8] This requirement is in contrast to the canonical Lerner rule that gives rise to a solution only when demand is elastic and characterizes price relative to marginal cost. The high-level difference is that it is a pre-owned quantity q_0 that erodes the monopoly's profit in problem (16), rather than a positive marginal cost as in the textbook case. Appendix C discusses this in greater detail. While our quantity Lerner rule works under inelastic demand and the price Lerner rule is only applicable to elastic demand, they share qualitatively similar comparative statics: as the demand elasticity rises, the supply q increases and the price p decreases.

Coming back to our analysis of land finance, taking household demand of housing (13) and holding fixed non-durable consumption C_t, we can derive the demand elasticity of housing

$$-\frac{\partial \log L_t}{\partial \log D_t} = \sigma^{-1}$$

which is less than one under our assumption $\sigma > 1$. That is, an ϵ percentage demand shock to the housing market changes the price of housing by $\sigma\epsilon$ percentage. If the government were to maximize its time-t profit from land lease, i.e., $\max_{L_t} (L_t - H_t)\, D_t(L_t)$, the optimal land supply would follow from Lemma 1, if interior,

$$\frac{L_{mono} - H_t}{L_{mono}} = -\frac{\partial \log L_t}{\partial \log D_t} = \sigma^{-1} \quad \Rightarrow \quad L_{mono} = \frac{\sigma}{\sigma - 1} H_t \tag{18}$$

Thus a government that maximizes profits would set $L_{mono} = \frac{\sigma}{\sigma-1} H_t$ if $\frac{\sigma}{\sigma-1} H_t < 1$. This

[8] Albouy, Ehrlich and Liu (2016) estimate an elasticity of about two thirds and review earlier estimates.

suggests that the total supply will be proportional to the outstanding amount H_t under constant elasticity, which is parallel to the price Lerner rule that prescribes a price proportional to the marginal cost.

Later, when I introduce a benevolent government that maximizes the household's utility, it would balance the need to raise money for investment against the household utility gain from living in larger houses. Relative to this profit-maximizing benchmark, it will supply more land to the household.

Assumption 1. The following inequality implies that the government can profit from restricting land supply

$$\frac{\sigma}{\sigma - 1} H_0 < 1. \tag{19}$$

I make Assumptions 2 and 1 henceforth. They can hold simultaneously, since 2 hinges on ν while 1 relates to σ, in addition to the dependence of both on H_0.

3.3 Second-Best Allocation

Using the household optimality conditions (12-14) to substitute prices in the budget constraint (11), as is a standard technique in the optimal fiscal policy literature, I get the following implementability condition

$$0 = \int_0^\infty \mathrm{e}^{-\rho t} \left[\left(C_t - Y_t \left(Z_t \right) - T_t \right) U_{C,t} + \left(L_t - H_0 \right) U_{L,t} \right] \mathrm{d}t. \tag{20}$$

The absence of P_t and H_t in this condition results from the "fair price" of housing as the household faces no borrowing constraints. The discounted sum of payment to housing consumption L_t is the same, regardless of how much the household buys or rents. When both the implementability constraint (20) and the resource constraint (2) hold, the government's budget constraint holds as a result of the Walras' law.

Definition 1. *(Ramsey problem under commitment and complete markets)* A Ramsey planner maximizes the household's life-time utility (1), subject to the resource constraint (2), land constraint (3), as well as the implementability constraint (20).

I attach multiplier $\mu_t^* \mathrm{e}^{-\rho t}$ to the time-t resource constraint (2), $\zeta_t^* \mathrm{e}^{-\rho t}$ to the time-t land constraint (3), and multiplier λ^* to the implementability constraint (20). λ^* measures the *social value of public fund*. I use superscript * to indicate the optimal allocation and policy,

decided at time 0 under commitment. Formally, the Lagrangian is

$$\mathcal{L}^* = \max_{\{C_{t\geq 0}, L_{t\geq 0}, T_{t\geq 0}, H_{t>0}, Z_{t>0}\}} \int_0^\infty \mathrm{e}^{-\rho t} U(C_t, L_t)\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\rho t}\left\{\mu_t^*\left[Y_t(Z_t) - C_t - (\delta+\rho)Z_t\right] + \dot{\mu}_t^* Z_t\right\}\mathrm{d}t + \mu_0^* Z_0$$
$$+ \int_0^\infty \mathrm{e}^{-\rho t}\zeta_t^*(1-L_t)\,\mathrm{d}t + \lambda^* \int_0^\infty \mathrm{e}^{-\rho t}\left[(C_t - Y_t(Z_t) - T_t)U_{C,t} + (L_t - H_0)U_{L,t}\right]\mathrm{d}t \tag{21}$$

where I have used the present-value sum version of capital stock in place of its flow version for a more intuitive representation.[9]

The SB policy satisfies

$$0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial C_t} = (1+\lambda^*)U_{C,t} + \lambda^*(C_t - Y_t)U_{CC,t} - \mu_t^* = U_{C,t} - \lambda^* Y_t U_{CC,t} - \mu_t^* \tag{22}$$

$$0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial L_t} = (1+\lambda^*)U_{L,t} + \lambda^*(L_t - H_0)U_{LL,t} - \zeta_t^* \tag{23}$$

$$0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial Z_t} = \mu_t^*\left[Y_t' - (\delta+\rho)\right] + \dot{\mu}_t^* - \lambda^* Y_t' U_{C,t} \tag{24}$$

First observe that H_t appears in neither the objective (1) nor the constraints (2, 3, 20). Thus, given the total supply $\left\{L_t^*\right\}$, the division between sale and rent is indeterminate under SB, formalized in Lemma 2. The reason behind is that as households optimize freely intertemporally, the housing price P_t depends on the total supply L_t but not on the owned quantity H_t and the household is willing to hold any quantity $H_t \in [0,1]$ at the market price. However, the level of H_t will matter for the amount of borrowing at each point in time between the household and the government, as sale can substitute for borrowing in raising fund for the government.

Lemma 2. (Indeterminate sale/rent ratio under SB) *The optimal policy may feature any* $H_t \in [0,1]$.

Using the FOC w.r.t. total land supply L_t (23), we can characterize the optimal land supply which equals $L^* = \min\left\{1, L_{interior}^*\right\}$ with

$$\frac{L_{interior}^* - H_0}{L_{interior}^*} = \left[1 + (\lambda^*)^{-1}\right]\sigma^{-1}. \tag{25}$$

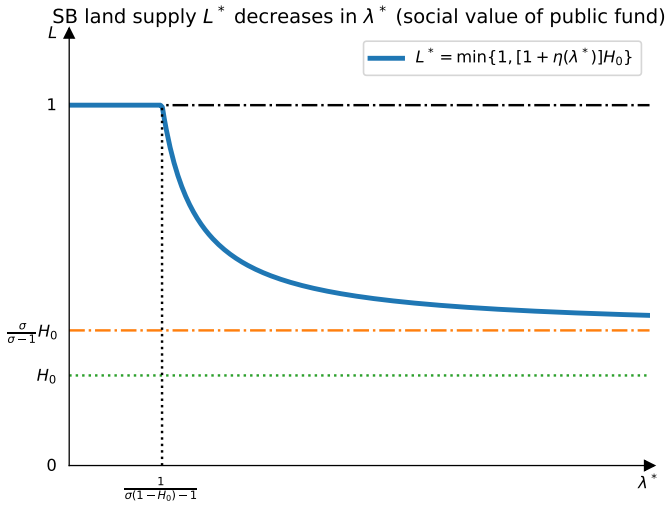
$L_{interior}^*$ decreases in both the social value of public fund λ^* and σ, depicted in Figure 4. Comparing the SB land supply (25) to the profit-maximizing land supply (18), the former

[9] See derivation of (D1) in Appendix D.1.

converges to the latter when $\lambda^* \to \infty$. In that case, an additional dollar to the government leads to an infinite increase of social welfare and thus the benevolent government acts as if it maximizes profits. In the other extreme, if $\lambda^* \to 0$, it is guaranteed that $L^* = 1$: if a transfer to the government does not raise social welfare, the government should not make money by restricting land supply that hurts housing consumption.

Indeed, there exists a threshold $\bar{\lambda} \equiv \frac{1}{\sigma(1-H_0)-1}$ which is positive under Assumption 1 such that the optimal land supply is interior if and only if $\lambda^* > \bar{\lambda}$. When the marginal welfare of an extra dollar to the government is small ($\lambda^* < \bar{\lambda}$), the government prefers not to distort the land consumption. We could see this by evaluating (23) at $L_t = 1$ to get $e^{\rho t}\frac{\partial \mathcal{L}_0^*}{\partial L_t}\Big|_{L_t=1} = (1+\lambda^*)\,U_{L,t} + \lambda^*(1-H_0)\,U_{LL,t}$ which is positive if λ^* is small. When the marginal welfare improvement λ^* of government making money is small, the government avoids welfare loss from reduction in housing consumption $U_{L,t}$.

Figure 4: Optimal land supply in the model



Lemma 3. (Land supply under SB) *The optimal land supply is time-invariant and satisfies*

$$L^* = \begin{cases} 1, & \lambda^* \le \bar{\lambda} = \frac{1}{\sigma(1-H_0)-1}, \\ L^*_{interior} = [1+\eta(\lambda^*)]\,H_0 < 1, & \lambda^* > \bar{\lambda}, \end{cases} \tag{26}$$

with $\eta(\lambda^*) = \left(\frac{\lambda^*\sigma}{1+\lambda^*} - 1\right)^{-1}$ *which is decreasing in* λ^* *denoting the supply in excess of initial household-owned land* H_0.

At the interior solution, the optimal land supply is time-invariant. That is because the utility $U(C, L)$ is separable and hence the convergence of consumption C_t does not alter the marginal utility from housing $U_{L,t}$. It is thus optimal to smooth out the housing consumption, much like the tax-smoothing result from Barro (1979).

To focus on an interior solution, henceforth I make Assumption 2 as follows, which implies that the SB land supply is interior based on Lemma 3. Assumption 2 supersedes Assumptions 2 and 1.

Assumption 2. *(Interior SB)* Assume (Z_0, H_0) are such that $\lambda^* > \frac{1}{\sigma(1-H_0)-1}$.

So far we have analyzed the land supply. Making use of the FOCs w.r.t. consumption C_t and public capital Z_t (22, 24), we can derive the path of consumption. Further, λ^* is determined by the implementability condition (20). Proposition 3 completely characterizes the SB allocation, whose proof is in Appendix D.

Proposition 3. (Interior SB allocation) *The SB allocation* $\left(C_t^*, Z_t^*, L_t^*, H_t^*\right)$ *is completely characterized as follows, given* (Z_0, H_0)*:*

1. *the consumption growth follows*

$$\frac{\dot{C}_t^*}{C_t^*} = \left[Y_t'\left(Z_t^*\right) - (\delta + \rho)\right] - \frac{\lambda^*}{\lambda^* + \chi_t^*}\left[\chi_t^* Y_t'\left(Z_t^*\right) + \frac{\dot{\chi}_t^*}{\chi_t^*}\right], \tag{27}$$

   with consumption rate $\chi_t^* \equiv \frac{C_t^*}{Y_t^*}$ *that strictly increases over time, subject to the transversality condition and the resource constraint*

$$\dot{Z}_t^* = Y_t\left(Z_t^*\right) - C_t^* - \delta Z_t^*; \tag{28}$$

2. *the land supply is time-invariant and satisfies*

$$L^* = \left[1 + \eta\left(\lambda^*\right)\right] H_0, \tag{29}$$

   with excess supply function $\eta(\lambda) \equiv \left(\frac{\lambda\sigma}{1+\lambda} - 1\right)^{-1}$ *which decreases in* λ*;*

3. *the amount of land held in private hands* H_t^* *for* $t > 0$ *can take any value in* $[0, 1]$*;*

4. *the multiplier* λ^* *is positive and satisfies*

$$\nu H_0^{1-\sigma} \frac{\eta\left(\lambda^*\right)}{\left[1 + \eta\left(\lambda^*\right)\right]^\sigma} = \rho \int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t^*} - 1\right) \mathrm{d}t \tag{30}$$

whose LHS is increasing in λ^*.

The consumption growth process (27) coupled with the resource constraint (2) is a family of ordinary differential equations (ODEs) parameterized by $\lambda^* > 0$ that meets the boundary condition Z_0 and the transversality condition. The limit of $\lambda^* \to 0$ describes the FB allocation (Proposition 1). When λ^* is positive, there is a wedge between the net marginal product of public capital ($Y_t'\left(Z_t^*\right) - \delta - \rho$) and consumption growth rate $\dot{C}_t^*/C_t^*$. That is a result of the government optimally lowering the private rate of return to fund its investment.

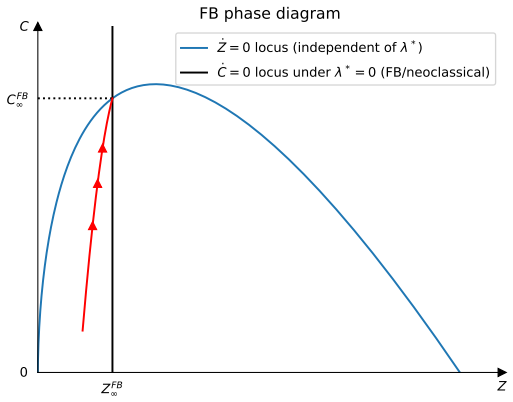
Further, extending a result from Barro and Sala-i-Martin (2004, Chapter 2.6), I show that χ_t^* strictly increases during convergence. Barro and Sala-i-Martin give conditions for monotonicity of χ_t^{FB} during transition under efficient allocation, i.e. the ODE system (27, 28) with $\lambda^* = 0$. I generalize their result to $\lambda^* \geq 0$. Even when the government cannot achieve the first best, it still chooses to front-load investment (investment rate $1 - \chi_t^*$ strictly decreasing over time as the economy grows).[10]

Based on (27, 28), Figure 5 plots the phase diagram in the (Z, C) space. The shape of $\dot{Z} = 0$ is independent of λ^*, as it arises from the resource constraint. When $\lambda^* = 0$, the $\dot{C} = 0$ locus is a vertical line, as in the neoclassical growth model. Denote the intersection of these two loci as $(Z_\infty^{FB}, C_\infty^{FB})$. When $\lambda^* > 0$, the $\dot{C} = 0$ locus rotates counterclockwise around exactly $(Z_\infty^{FB}, C_\infty^{FB}/2)$ and its interaction with the $\dot{Z} = 0$ locus shifts to the left. Indeed, I establish the following comparative statics for the steady state value.

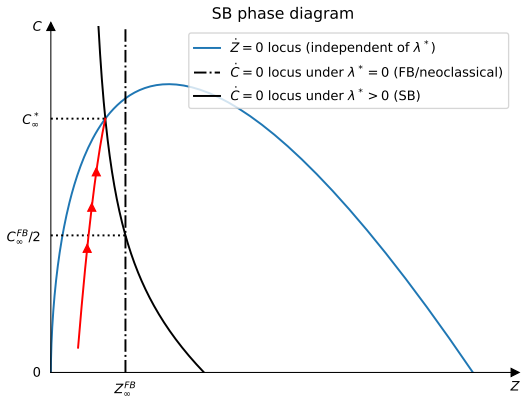
[10] I assumed log utility in my model for simplicity. This functional form can be relaxed without disturbing the monotonicity of χ_t^*, as long as the elasticity of intertemporal substitution (EIS) is lower than $1/\alpha$. This inequality holds with a large margin: Havranek et al. (2015) reviews a vast literature that estimates EIS which mostly produce estimates less than or around one; Bom and Ligthart (2014) survey estimates of output elasticity of public capital α and suggest an average estimate of 0.106.

Figure 5: Phase diagram

(a) First best



(b) Second best



Corollary 2. *At the steady state, if the multiplier λ^* is higher, both the consumption C_∞^* and the infrastructure stock Z_∞^* are lower, but the consumption rate $\chi_\infty^* \equiv C_\infty^*/Y_\infty^*$ is higher.*

Specifically, steady-state consumption rate χ_∞^* follows

$$\chi_\infty^*(\lambda^*) = \frac{(1-\lambda^*) + \sqrt{(1+\lambda^*)^2 + \dfrac{4\delta\alpha}{\delta(1-\alpha)+\rho}\lambda^*}}{2} \frac{\delta(1-\alpha)+\rho}{\delta+\rho} \in [\chi_\infty^{FB}, 1) \tag{31}$$

which increases in λ^* unless $\delta = 0$ (in which case $\chi_\infty^* = 1$ obviously holds as infrastructure does not depreciate). The steady-state infrastructure and consumption satisfy

$$Z_\infty^*(\lambda^*) = \begin{cases} \left[\frac{\alpha}{(1+\lambda^*)\rho}\right]^{\frac{1}{1-\alpha}} & \delta = 0 \\ \left[\frac{A(1-\chi_\infty^*)}{\delta}\right]^{\frac{1}{1-\alpha}} & \delta > 0 \end{cases} \tag{32}$$

$$C_\infty^*(\lambda^*) = \begin{cases} A\left[\frac{\alpha}{(1+\lambda^*)\rho}\right]^{\frac{\alpha}{1-\alpha}} & \delta = 0 \\ A\chi_\infty^*\left[\frac{A(1-\chi_\infty^*)}{\delta}\right]^{\frac{\alpha}{1-\alpha}} & \delta > 0 \end{cases} \tag{33}$$

both of which decrease in λ^* under any δ.

When the marginal value of a dollar to the government is higher (higher λ^*), the economy converges to a steady state with lower consumption and lower infrastructure stock but higher consumption-to-output rate. Obviously, the steady state consumption

and infrastructure are both lower under SB than under FB, and the consumption rate is higher, since the FB allocation can be viewed as the limit of $\lambda^* \to 0$.

The steady-state housing price $P_\infty = U_{C,\infty}^{-1} \mathcal{U}_{L,\infty}$ is subject to two offsetting forces. On the supply side, when λ^* is higher, land supply L^* is lower, which pushes up the housing price. However, as a higher λ^* also leads to lower consumption C_∞^*, the demand force depresses the housing price. One cannot simply take P_∞ as an indicator of welfare.

4 Constrained Implementations of Second Best

Previously I allow the government to flexibly choose between land sale or lease, to borrow, and to commit to its future actions. Here I analyze what role each of these assumptions play. The assumption of commitment directly circumvents the time inconsistency issue from Coase (1972). As for forms of land contracts and financial markets, recall that the investment-to-GDP ratio $1-\chi_t^*$ strictly declines over time. Using (13), the land-rent-to-GDP ratio $\frac{D_t^*}{Y_t^*} = U_L^* \frac{C_t^*}{Y_t^*} = U_L^* \chi_t^*$ strictly declines. Thus the fiscal expenditure is front-loaded but the fiscal income from lease is back-loaded. Though the total fiscal income and expenditure are equal in terms of present value, the timing mismatch suggests that the government's lease income is less than expenditure early on. Hence the government has to transfer funds intertemporally. This can be done by borrowing (in which case having one bond is enough), or by selling rather than leasing land (which amounts to collecting all future rents from a piece of land today).

One may wonder if the optimal policy is changed once one of these assumptions is removed. I establish that land contract forms, financial markets, and the ability to commit, individually, are not essential for implementing the SB allocation.

4.1 Implementation with a Single Land Contract

The SB allocation can be implemented with a single land contract, i.e. either sale or lease. This is an immediate implication of Lemma 2 and Proposition 3, since H_t is indeterminate. In fact, it can be implemented by any form of land contract, as it will be priced in the financial markets. If the government sells L^* amount of land directly to the household at time 0, this maximizes the debt owed by the household and the government can save the proceeds. If the government keeps leasing L^* amount of land, it has to borrow against its future lease income.

An important piece of policy advice from Tideman et al. (1990) is that, if there are potential borrowing constraints faced by the household, the government could rent the land out on an annual basis. While it is intuitive that outright sale can be problematic under borrowing constraints, it takes a model to see if it is a good idea for the government to hold on to the land forever. On this proposal, the previous paragraph suggests that the government would run a fiscal deficit early on since the lease income is insufficient to fund expenditure. A slightest friction or reason against government deficit or borrowing could render this undesirable. Next, I show that there is a way to implement the SB allocation without necessitating government or private sector borrowing.

4.2 Balanced-Budget Implementation

I show that SB can be implemented by a government on balanced budget by adjusting the amount of land sale and rent while holding the total supply fixed.

First, we evaluate the fiscal position under SB. The time-t income from land lease relative to GDP is, assuming $H_t = H_0$,

$$\frac{D_t^* (L^* - H_0)}{Y_t^*} = \frac{U_L^*}{U_{C,t}^*} \frac{L^* - H_0}{Y_t^*} = U_L^* \eta H_0 \chi_t^*,$$

which is increasing over time. That is because the household values future housing L_t more as their future consumption C_t is higher relative to future GDP Y_t. If the government derives rental income at each time t without selling any land, the fiscal income is back-loaded. In contrast, the time-t expenditure relative to GDP $\frac{\dot{Z}_t^* + \delta Z_t^*}{Y_t^*} = 1 - \chi_t^*$ is front-loaded. This maturity mismatch is an immediate consequence of monotonicity of χ_t^* established in Proposition 3, stated as Corollary 3 below.

Corollary 3. (Fiscal mismatch under SB) *Under the SB allocation, the fiscal expenditure relative to GDP strictly decreases over time. If the government does not sell any land ($H_t = H_0$), the rental income relative to GDP strictly increases over time.*

Nonetheless, acknowledge that the housing price will incorporate all the future rental income and thus the government could access future rental income via land sale, a point made in Lemma 2—if the government sells $H_t = L^*$, it can extract all the land value at time 0 and roll over the savings for future expenditure. Further, as the time profiles of sale income and lease income "span" the time profile of SB expenditure, the government

can tune the sale/lease ratio to exactly fund its expenditure as each time t. Proposition 4 formalizes this intuition to illustrate that borrowing is not necessary to implement SB.

Proposition 4. *SB allocation could be achieved without borrowing by either the government or the household, if the government can commit to its future actions, by maintaining for all* $s \geq 0$

$$H_s = H_s^{NB} \equiv H_0 + \left[1 - \frac{\int_s^\infty \mathrm{e}^{-\rho(t-s)}\left(\frac{1}{\chi_t^*} - 1\right)\mathrm{d}t}{\int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t^*} - 1\right)\mathrm{d}t}\right]\eta\left(\lambda^*\right)H_0, \tag{34}$$

The threshold H_s^{NB} *increases in* s *from* H_0 *to* $H_\infty^{NB} = H_0 + \left[1 - \frac{\frac{1}{\chi_\infty^*} - 1}{\rho\int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t^*} - 1\right)\mathrm{d}t}\right]\eta\left(\lambda^*\right)H_0$ *as* s *goes from* 0 *to* ∞*. That is, the government sells land from* H_0 *until* H_∞^{NB}*.*

When (34) holds, the fiscal income exactly offsets the expenditure at any point in time, and no financial transaction is needed. As $s \to \infty$, the amount of land the government owns $L^* - H_\infty^{NB}$ generates exactly enough revenue to replenish infrastructure depreciation. If $\delta = 0$ holds in addition, and thus $\chi_\infty^* = 1$, then the government owns no land in the long run ($H_\infty^{NB} = L^*$). When (34) holds as inequality with H_s larger than H_s^{NB}, the government saves the fiscal surplus for future investment. In this case, it suffices to have only one debt instrument for the government to use.

4.3 Time-Consistent Implementation

Last, I show that SB can be implemented in a time-consistent manner by using a full maturity of bonds as a commitment device. This builds on the insights of Lucas and Stokey (1983) and Debortoli, Nunes and Yared (2021).

Let B_s^τ be a zero-coupon bond with maturity $\tau - s$ issued at time s, which pays one dollar at time τ to the household. If the government reoptimizes at time s, taking as given $\left(Z_s, H_s, \{B_s^\tau\}_{\tau \geq s}\right)$, its implementability constraint (20) takes the following form

$$0 = \int_s^\infty \mathrm{e}^{-\rho(t-s)}\left[\left(C_t - Y_t - B_s^t\right)U_{C,t} + \left(L_t - H_s\right)U_{L,t}\right]\mathrm{d}t, \tag{35}$$

which is a discounted sum from time s to ∞. The time-s Lagrangian is

$$\mathcal{L}^{s*} = \max \int_s^\infty \mathrm{e}^{-\rho(t-s)} U\left(C_t, L_t\right) \mathrm{d}t + \int_s^\infty \mathrm{e}^{-\rho(t-s)} \left\{\mu_t^{s*} \left[Y_t\left(Z_t\right) - C_t - \left(\delta + \rho\right) Z_t\right] + \dot{\mu}_t^{s*} Z_t\right\} \mathrm{d}t + \mu_s^{s*} Z_s$$
$$+ \int_s^\infty \mathrm{e}^{-\rho(t-s)} \zeta_t^{s*} \left(1 - L_t\right) \mathrm{d}t + \lambda^{s*} \int_s^\infty \mathrm{e}^{-\rho(t-s)} \left[\left(C_t - Y_t\left(Z_t\right) - T_t\right) U_{C,t} + \left(L_t - H_s\right) U_{L,t}\right] \mathrm{d}t \tag{36}$$

The optimality conditions (22-24) become, for any $t \geq s$,

$$0 = U_{C,t} - \lambda^{s*} \left(Y_t + B_s^t\right) U_{CC,t} - \mu_t^{s*} \tag{37}$$
$$0 = \left(1 + \lambda^{s*}\right) U_{L,t} + \lambda^{s*} \left(L_t - H_s\right) U_{LL,t} - \zeta_t^{s*} \tag{38}$$
$$0 = \mu_t^{s*} \left[Y_t' - \left(\delta + \rho\right)\right] + \dot{\mu}_t^{s*} - \lambda^{s*} Y_t' U_{C,t} \tag{39}$$

in which λ^{s*} is the multiplier attached to the implementability constraint (35), μ_t^{s*} is the multiplier attached to the time-t resource constraint, and ζ_t^{s*} regards the upper bound on time-t land supply. The subscript $s*$ indicates that the optimization is done at time s.

Given the initial condition $\left(Z_s, H_s, \{B_s^\tau\}_{\tau \geq s}\right)$, the time-$s$ planner solves (35-39) subject to the resource constraint and land constraint to arrive at the optimal allocation. The resulting multipliers $\left(\lambda^{s*}, \left\{\mu_t^{s*}\right\}_{t \geq s}, \left\{\zeta_t^{s*}\right\}_{t \geq s}\right)$ must all be non-negative. For the time-0 optimal allocation analyzed in Section 3.3 to be carried out by time-s planner, the time-0 planner solves the inverse problem and sets $\left(Z_s, H_s, \{B_s^\tau\}_{\tau \geq s}\right)_{s \geq 0}$ to incentivize future planners. If there exists a solution whose implied multipliers $\left(\lambda^{s*}, \left\{\mu_t^{s*}\right\}_{t \geq s}, \left\{\zeta_t^{s*}\right\}_{t \geq s}\right)_{s \geq 0}$ are all non-negative, then the time-0 optimal allocation can be carried out.

I characterize the set of solutions in Proposition 5, and in doing so, uncover an important property of any time-consistent implementation regarding the amount of land in private hands H_t.

Proposition 5. *SB allocation could be achieved without commitment, if the government can issue bonds in complete markets with a specific maturity structure.*

Specifically, there exists a set of implementation parameterized by $\left\{\lambda^{s*} | \lambda^{s*} \in \left(\frac{1}{\sigma - 1}, \infty\right]\right\}_{s>0}$ *such that the land in private hands and the amount of* $(\tau - s)$*-maturity zero-coupon bond the government owes at time s satisfy*

$$H_s = \frac{L^*}{1 + \eta\left(\lambda^{s*}\right)} = \frac{\sigma - 1 - \left(\lambda^{s*}\right)^{-1}}{\sigma} L^* \leq \frac{\sigma - 1}{\sigma} L^* \tag{40}$$
$$B_s^\tau = \left(\bar{b}_s^\tau + \tilde{b}_s^\tau\right) C_\tau^*, \ \forall \tau \geq s \tag{41}$$

with

$$\bar{b}_s^\tau = \frac{C_\tau^* m_\tau^{-1}}{\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} \left(\mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right) \tag{42}$$

$$\tilde{b}_s^\tau = \left(\frac{C_\tau^* m_\tau^{-1}}{\rho \int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} - 1\right)\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) \tag{43}$$

In the expressions above, m_t *is an integrating factor defined as* $m_t \equiv \mathrm{e}^{\int_0^t [Y_\tau' - (\delta+\rho)]\mathrm{d}\tau}$ *that increases in* t*,* $\mathcal{I}\left(\lambda^{s*}; L^*\right) \equiv \frac{\nu (L^*)^{1-\sigma}}{\rho} \frac{1+\lambda^{s*}}{\lambda^{s*}\sigma}$ *is the discounted sum of future land income under* $H_s = \frac{L^*}{1+\eta(\lambda^{s*})}$*, and* $\mathcal{E}^{s*} \equiv \int_s^\infty \mathrm{e}^{-\rho(t-s)} \left(\frac{1}{\lambda_t^*} - 1\right) \mathrm{d}t$ *is the discounted sum of future fiscal expenditure. Last, for any* $s > 0$ *and* λ^{s*}*,* B_s^τ / C_τ^* *is either a positive constant across* τ *or strictly monotonic (increasing or decreasing) in* τ*.*

In (41), the $\bar{b}_s^\tau$ term indicates a baseline position whose sum across the maturity structure equals fiscal surplus (land income minus fiscal expenditure), and the $\tilde{b}_s^\tau$ term is a residual position that averages to zero to fine-tune the maturity structure. The baseline position $\bar{b}_s^\tau$ is positive (/negative) across τ if λ^{s*} is larger (/smaller) than a threshold $\bar{\lambda}^s$, which increases over s from value $\bar{\lambda}^0 = \lambda^*$ at $s = 0$. Further, the $\bar{b}_s^\tau$ term decreases in absolute value over τ as $C_\tau^* m_\tau^{-1}$ decreases in τ — the further into the future, the smaller bond position relative to consumption. The residual position $\tilde{b}_s^\tau$ is monotonically decreasing (/increasing) if λ^{s*} is less (/greater) than λ^*. If setting $\lambda^{s*} = \lambda^*$ (which implies $H_s = H_0$), we would exactly have $B_s^\tau = C_\tau^* \bar{b}_s^\tau = C_\tau^* \frac{C_\tau^* m_\tau^{-1}}{\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} \left(\mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right)$ amount of government borrowing for all $\tau \geq s$. In this case, as time $s \to 0$, the bond position approaches zero since $\mathcal{I}\left(\lambda^*; L^*\right) = \mathcal{E}^*$ from Proposition 3.

Moreover, (40) implies that the land in private hands H_s never exceeds $\frac{\sigma-1}{\sigma} L^*$. To see the intuition, applying Lemma 3 to the time-s decision, the optimal land supply satisfies

$$L^{s*} = \begin{cases} 1, & \lambda^{s*} \leq \frac{1}{\sigma(1-H_s)-1}, \\ \left[1 + \eta\left(\lambda^{s*}\right)\right] H_s < 1, & \lambda^{s*} > \frac{1}{\sigma(1-H_s)-1}, \end{cases}$$

with a decreasing function $\eta\left(\lambda^{s*}\right) = \left(\frac{\lambda^{s*}\sigma}{1+\lambda^{s*}} - 1\right)^{-1}$. As λ^{s*} increases from 0 to ∞, L^{s*} decreases from 1 to $\frac{\sigma}{\sigma-1} H_s$. The logic behind is that, depending on the need for money, the government may supply any amount of land from $\frac{\sigma}{\sigma-1} H_s$ (maximizing profits) to 1 (maximizing housing consumption), but it will never set L^{s*} below the level $\frac{\sigma}{\sigma-1} H_s$. In order for L^{s*} to equal L^*,

we must have $H_s \leq \frac{\sigma-1}{\sigma}L^*$; otherwise there is no non-negative solution λ^{s*}. That is, the time-0 planner has to leave $H_s \leq \frac{\sigma-1}{\sigma}L^*$ in private hands to incentivize future planners.

Recalling that any no-borrowing implementation of the SB allocation characterized in Proposition 4 has the property that the government gradually sells land, we establish the following impossibility result.

Corollary 4. *There exists an implementation of the SB allocation that is both time-consistent and involving non-positive* total value *of government debt (i.e. government as net creditor), if and only if*

$$\frac{1}{\chi^*_\infty} - 1 \geq \frac{\nu\,(L^*)^{1-\sigma}}{\sigma}. \tag{44}$$

If δ is small (i.e., $\frac{\alpha\delta}{(1-\alpha)\delta+\rho} \leq \frac{\nu}{\sigma}$), this inequality never holds. Otherwise it holds only when λ^* is small.

Intuitively, if infrastructure never depreciates ($\delta = 0$), in any time-consistent implementation, the government needs no fiscal income at the steady state. Thus it should put all land supply in private hands ($H^{NB}_\infty = L^*$), since it has no debt to repay either. In this case, the SB allocation can never be implemented in a time-consistent way, since $H^{NB}_\infty > \frac{\sigma-1}{\sigma}L^*$.

If δ is large, the government needs lease income at the steady state to replenish infrastructure. When λ^* is large, the steady-state consumption ratio χ^*_∞ is high and investment ratio is low, suggesting a low need of fiscal income. Further, as λ^* is large, the land supply is depressed and the land income is high, the government needs to hold less land to derive a certain amount of income. Taking together, the government chooses H_∞ that exceeds $\frac{\sigma-1}{\sigma}L^*$ when λ^* is large enough, denying (44).

I conclude this section with a remark that, in cases where (44) does hold, even though the total value of government debt is non-positive, the government needs the whole maturity structure to implement SB allocation in a time-consistent way, based on Proposition 5.

5 Optimal Policy under Discretion and Balanced Budget

I have established that land contracts, complete markets, and commitment, individually, are not essential prerequisites for the implementation of SB allocation. While in reality, governments do use various land contracts (sale and lease, or long-term and short-term leases), complete markets and commitment are both unrealistically strong assumptions.

Here I show that the combination of discretion and financial constraints denies SB allocation. I call this case *double deviation (DD)* for simplicity.

I assume that the government has to run a balanced budget, using income from land sale and lease to exactly pay for investment at each point in time.[11] Thus we have two state variables (Z_t, H_t). To tractably model discretion, I take the limit of $\Delta t \to 0$ of a discrete-time Markov Perfect Equilibrium (MPE) in which each governor is in charge for for one period with period length Δt. Each governor takes into account how her choice of (Z_t, H_t) affects the future governors' choices.

Now the budget constraint holds period by period as

$$0 = (C_t - Y_t(Z_t) - T_t)\Delta t + (L_t - H_t) D_t \Delta t + \Delta H_{t+\Delta t} P_t$$

Plugging in the discretized household optimality conditions (13, 14), I rewrite this as a period-t implementability constraint

$$0 = (C_t - Y_t(Z_t) - T_t) U_{C,t} \Delta t + (L_t - H_t) U_{L,t} \Delta t + e^{-\rho \Delta t} \Delta H_{t+\Delta t} \mathcal{P}^{DD}_{L,t+\Delta t} \tag{45}$$

with $\mathcal{P}^{DD}_{L,t} \equiv \sum_{s \geq 0} e^{-\rho s \Delta t} U_{L,t+s\Delta t} \Delta t$.

Definition 2. *(Markov perfect equilibrium in discrete time)* Each governor maximizes the household's life-time utility

$$\mathcal{V}^{DD}(H_t, Z_t) = \max_{C_t, L_t, T_t, H_{t+\Delta t}, Z_{t+\Delta t}} U(C_t, L_t)\Delta t + e^{-\rho \Delta t} \mathcal{V}^{DD}(H_{t+\Delta t}, Z_{t+\Delta t}), \tag{46}$$

subject to the resource constraint (2), land constraints (3, 4), as well as period-t implementability constraint (45).

The discrete-time Lagrangian with period length Δt is

$$\begin{aligned} \mathcal{V}^{DD}(H_t, Z_t) = \max_{C_t, L_t, H_{t+\Delta t}, Z_{t+\Delta t}} & U(C_t, L_t)\Delta t + e^{-\rho \Delta t} \mathcal{V}^{DD}(H_{t+\Delta t}, Z_{t+\Delta t}) \\ & + \mu^{DD}_t \left[Y(Z_t)\Delta t + (1 - \delta \Delta t) Z_t - C_t \Delta t - Z_{t+\Delta t} \right] \\ & + \lambda^{DD}_t \left[(C_t - Y(Z_t)) U_{C,t} \Delta t + (L_t - H_t) U_{L,t} \Delta t + e^{-\rho \Delta t} \Delta H_{t+\Delta t} \mathcal{P}^{DD}_{t+\Delta t} \right] \\ & + \zeta^{DD}_t \Delta t (1 - L_t) + \xi^{DD}_t (1 - H_{t+\Delta t}) \end{aligned} \tag{47}$$

[11] If the government is allowed to save but not borrow, I conjecture that it is optimal not to save. The logic is that expenditure is optimally front-loaded, and if the government sells more than it needs for expenditure, this exacerbates the time inconsistency problem.

where I attach multiplier μ_t^{DD} to the resource constraint (2), λ_t^{DD} to the time-t implementability condition (45), and ζ_t^{DD}, ξ_t^{DD} to the land constraints. ζ_t^{DD} ($/\xi_t^{DD}$) is equal to zero when L_t ($/H_{t+\Delta t}$) is less than one. Differing from the SB policy analyzed in Section 3.3, here there is a multiplier λ_t^{DD}, which measures the social value of public fund for each time t, instead of a single λ^*.

Further note that $\mathcal{P}_t^{DD} = \mathcal{U}_{L,t}^{DD}(H_t, Z_t)$ is itself recursive and depends on the optimal policy rule $L^{DD}(H, Z)$, as

$$\mathcal{P}^{DD}(H_t, Z_t) = U_L\left(L^{DD}(H_t, Z_t)\right)\Delta t + \mathrm{e}^{-\rho\Delta t}\mathcal{P}^{DD}(H_{t+\Delta t}, Z_{t+\Delta t}). \tag{48}$$

This is because the time-t planner decides on $H_{t+\Delta t}, Z_{t+\Delta t}$, acknowledging that this choice affects future planners' optimal choices of $L_{t+s\Delta t}$ and thus $U_{L,t+s\Delta t}$.

I characterize the optimality conditions w.r.t. $C_t, L_t, H_{t+\Delta t}, Z_{t+\Delta t}$ and take the limit of $\Delta t \to 0$ to arrive at the optimal equilibrium allocation in continuous time. See Appendix D for detailed derivations.

Proposition 6. (Allocation under double deviation) *The DD allocation* $\left(C_t^{DD}, Z_t^{DD}, L_t^{DD}, H_t^{DD}\right)$ *is completely characterized as follows, given* (Z_0, H_0)*:*

1. *the consumption growth follows*

$$\frac{\dot{C}_t^{DD}}{C_t^{DD}} = \left[Y_t'\left(Z_t^{DD}\right) - (\delta+\rho)\right] - \frac{\lambda_t^{DD}}{\lambda_t^{DD} + \chi_t^{DD}}\left[\chi_t^{DD} Y_t'\left(Z_t^{DD}\right) + \frac{\dot{\chi}_t^{DD}}{\chi_t^{DD}} - \frac{\dot{\lambda}_t^{DD}}{\lambda_t^{DD}} - \chi_t^{DD} C_t^{DD} \dot{H}_t^{DD}\left(\frac{\partial \mathcal{U}_L^{DD}}{\partial Z}\right)_t\right] \tag{49}$$

with consumption rate $\chi_t^{DD} \equiv \frac{C_t^{DD}}{Y_t^{DD}}$ *and* $\mathcal{U}_L^{DD}$ *capturing present value of future marginal utility as (48), subject to the transversality condition and the resource constraint*

$$\dot{Z}_t^{DD} = Y_t\left(Z_t^{DD}\right) - C_t^{DD} - \delta Z_t^{DD}; \tag{50}$$

2. *the land supply satisfies*

$$L_t^{DD} = \min\left\{1, \left[1 + \eta\left(\lambda_t^{DD}\right)\right] H_t^{DD}\right\}, \tag{51}$$

with excess supply function $\eta(\lambda) \equiv \left(\frac{\lambda\sigma}{1+\lambda} - 1\right)^{-1}$ *which decreases in* λ*;*

3. *the amount of land held in private hands evolves according to, when* $H_t^{DD} < 1$,

$$\dot{H}_t^{DD} = \frac{\mathcal{U}_{L,t}^{DD}}{\left(\partial \mathcal{U}_L^{DD}/\partial H\right)_t} \frac{\dot{\lambda}_t^{DD}}{\lambda_t^{DD}} \tag{52}$$

4. *when the multiplier* λ_t^{DD} *is positive, it satisfies*

$$\nu \left(L_t^{DD}\right)^{-\sigma} \left(L_t^{DD} - H_t^{DD}\right) + \dot{H}_t^{DD} \mathcal{U}_{L,t}^{DD} = \frac{1}{\chi_t^{DD}} - 1 \tag{53}$$

whose LHS is increasing in λ_t^{DD} *holding* H_t^{DD} *fixed, when* $L_t^{DD} < 1$.

We notice a few interesting differences between the DD allocation in Proposition 6 to the SB allocation in Proposition 3. In terms of consumption growth (49), in addition to the wedge between $Y'\left(Z_t^{DD}\right) - (\delta + \rho)$ and $\dot{C}_t^{DD}/C_t^{DD}$ caused by non-zero λ_t^{DD}, which exists in the SB allocation too, another wedge emerges due to time-varying λ_t^{DD} and H_t^{DD}. In terms of total land supply (51), time-t government always supplies an extra amount relative to H_t^{DD}, which is what is already sold by time t, rather than H_0. And the added supply $\eta\left(\lambda_t^{DD}\right)$ changes over time too, as the social value of public fund λ_t^{DD} varies.

Further, I show in Corollary 7 that when the depreciation rate δ is low, the land market will eventually be saturated ($L_\infty^{DD} = 1$). If the economy grows but the land market is never saturated, assuming that the consumption rate χ_t^{DD} is eventually monotonic and a few other regularity conditions, I establish in Corollary 8 that both H_t^{DD} and L_t^{DD} will strictly increase during convergence.

Proposition 7. *If* $\frac{\alpha\delta}{(1-\alpha)\delta+\rho} \leq 4\frac{\nu}{\sigma}\left(1 + \frac{\nu}{\sigma}\right)$, *land supply is saturated in the long run, i.e.,* $L_\infty^{DD} = 1$.

Proposition 8. *If the economy grows but land market is never saturated, under certain regularity conditions (i.e., there existing* τ *such that* $\dot{Z}_t^{DD} > 0$, $\left(\frac{\partial \mathcal{U}_L^{DD}}{\partial Z}\right)_t < 0$, $\left(\frac{\partial \mathcal{U}_L^{DD}}{\partial H}\right)_t < 0$ *for all* $t \geq \tau$, $L_t^{DD} \in \left(\frac{\sigma}{\sigma-1} H_s^{DD}, 1\right)$ *for all* $t \geq s \geq \tau$, *and that* χ_t^{DD} *is strictly monotonic for all* $t \geq \tau$*), then it must be that, for* $t \geq \tau$, χ_t^{DD} *strictly rises,* λ_t^{DD} *strictly declines, and* H_t^{DD}, L_t^{DD} *both strictly increase.*

6 Contingent Design of Land Contract

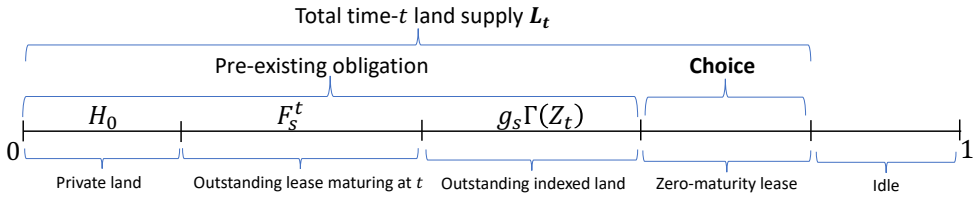
Last, I show that there exists a better design of land contracts that can help restore the SB allocation, under discretion and incomplete markets. Before deriving the result, I first

discuss the time inconsistency issue of the no-borrowing implementation of SB. In order to maintain a balanced budget, the government needs to gradually sell land, i.e., following a specific increasing path H_t^{NB}. Suppose the time-s planner wants to reoptimize, if she has the same λ^* as the time-0 planner, she would supply a total amount of land equal to $[1+\eta(\lambda^*)]H_s^{NB}$, which will be larger than $L^* = [1+\eta(\lambda^*)]H_0$. The reason she does not do this under the no-borrowing implementation is that she keeps a promise made by past self; otherwise the past land price would have been lower. In order for her to still choose L^* when reoptimizing, she needs a specific $\lambda^s > \lambda^*$ to ensure $[1+\eta(\lambda^s)]H_s^{NB} = L^*$. However, if she has such a λ^s that is larger than λ^*, that means she is even poorer relative to FB. In that case, she prefers a lower investment rate $\dot{Z}$ and private return than SB.

This reasoning above clearly illustrates the dilemma between balanced budget and time consistency, and indicates that the specific direction of deviation future governments want is to supply less public capital. When markets are complete, the time-0 governments can leave a specific debt portfolio to future governments which will revaluate adversarially if future governments deviate. Can we achieve the same goal via a land contract, rather than requiring the full maturity structure of debt? The answer is yes, and the optimal land contract penalizes deviation by obliging future governments to give more land to household if the stock of public capital Z_t is low, per unit of land contract outstanding.

We could view land contracts as financial securities that pay land (instead of dollar) to the household by the government. A lease (that is valid for an infinitesimal amount of time at present) is a zero-maturity bond. A sale amounts to a consol that pays one unit of land indefinitely. By no means these are the only possibilities. The government can also sell a future lease F^t, which is a right to use one unit of land for an infinitesimal amount of time at future time t. A future lease F^t is like a zero-coupon bond maturing at time t. A full set of future leases spans all bond-like contracts, including current lease and sale. I denote the outstanding amount at time s of a future lease paying off at time t as F_s^t. Further, the government can sell an indexed land contract that pays a variable amount $\Gamma(Z_t)$ at all t depending on the prevailing level of public capital Z_t. I denote its outstanding amount at time s as g_s. Figure 6 illustrates the land allocation, faced by a time-s planner who chooses time-t allocation for all $t \geq s$. The time-s planner inherits pre-existing obligations including private land H_0, outstanding lease maturing at time t of amount F_s^t, and she is obliged to pay a certain amount of land per unit of outstanding indexed land contract, depending on the time-t stock of public capital she chooses.

Figure 6: Land utilization under multiple land contracts



Proposition 9 suggests that an indexed land contract with a specific dependence $\Gamma(\cdot)$ can be combined with future leases to provide incentives for future governments to continue carrying out the time-0 optimal plan.

Proposition 9. (Optimal design of land contract)

1. *When* $\frac{1}{\chi^*_\infty} - 1 \geq \frac{\nu(L^*)^{1-\sigma}}{\sigma}$ *(or equivalently,* $\frac{\sigma}{\sigma-1} H^{NB}_\infty \leq L^*$*), the SB allocation can be implemented under double deviation using future leases and Z-indexed land if and only if*

$$\Gamma'(Z_t) = -k \frac{\left(C^*_t\right)^{-1}}{\nu (L^*)^{-\sigma}} \Delta r^*_t$$

with k being an arbitrary positive constant and $\Delta r^*_t \equiv \left[Y'\left(Z^*_t\right) - (\delta + \rho)\right] - \frac{\dot{C}^*_t}{C^*_t} > 0$ *capturing the wedge between private return and net marginal product of capital at any given time t along the SB convergence path. The amount outstanding at time t of such Z-indexed land* g_s *is increasing over time*

$$g_s = k^{-1}\left(\frac{1}{\lambda^*} - \frac{1}{\lambda^{s*}}\right)$$
$$F^t_s = \frac{L^*}{1 + \eta(\lambda^{s*})} - H_0 - g_s\Gamma(Z_t)$$

2. *When* $\frac{1}{\chi^*_\infty} - 1 < \frac{\nu(L^*)^{1-\sigma}}{\sigma}$*, the same land contract can achieve SB if the government can borrow using at least one bond.*

The two-part obligation consists of an indexed part $g_s\Gamma(Z_t)$ that penalizes future governments supplying less capital (by requiring them to pay out more land since $\Gamma' < 0$), and future leases F^t_s that make up to H^{NB}_s which is the amount of land sale needed to implement SB with no borrowing. This design essentially splits the rights associated with land sale into two parts, so that it can provide incentives without changing the land income. When

$\frac{\sigma}{\sigma-1}H_{\infty}^{NB} \leq L^*$ is violated, it suffices to have one bond for the government to borrow. This is in contrast to the time-consistent implementation of SB allocation discussed in Section 4, which requires a full maturity structure of debt to provide incentives. Here, the incentive is provided by the indexed land contract.

Last, I provide some remarks on the feasibility of such an indexed land contract. The required legal protection of future leases and indexed land contract is the same as outright sale, in that they are all subject to expropriation. In order for this contract design to work, the indexed land contract has to be honored by future governments. This is a fair requirement, as we note that in order to implement the SB allocation under complete markets à la Lucas and Stokey (1983) and Debortoli, Nunes and Yared (2021), debt contracts issued by the current government have to be acknowledged by future governments too. Further, we note that, while a land contract indexed on capital stock Z_t can help resume the SB allocation, there exist other indexed contracts based on other variables that could achieve the same goal. The advantage of an indexed land contract based on capital stock is that Z_t is easily verifiable, since public investment is an important item in national accounts, and is directly controlled by the government. The timeless dependence $\Gamma(\cdot)$ is easy to enforce too: since it is differentiable, there is no need to argue about a small amount of measurement error in court.

7 Conclusion

Better understanding land finance can help improve policies concerning many developing economies. This paper makes three contributions regarding the mechanism and design of land finance. First, I provide a model to understand land finance and show that the optimal policy should front-load public investment while supplying a constant amount of land over time. Second, I show that the combination of discretion and financial constraints implies that the government will continuously expand land supply, as observed in reality. Lastly, I propose a realistic design of land contracts that links land supply to the provision of public goods to restore the optimal allocation.

This paper focuses on the efficiency aspect of land finance in a deterministic environment. It presents a neoclassical growth model augmented with the role of land finance, which can serve as a starting point for other models. Future research may fruitfully explore the pricing and supply of land in the presence of risk and their role in cyclical fluctuations. If the government is subject to collateral constraints, the use of land finance

may create a financial accelerator (Bernanke, Gertler and Gilchrist, 1999) that goes through the public sector rather than the private sector. While this paper studies efficiency in a representative agent model, housing and land play crucial roles in driving wealth inequality across countries (Bonnet et al., 2021). In light of that, how should the policymaker design land finance policies to take into account its distributional impact? Last but not least, this paper addresses a monopoly problem. In practice, land is supplied by many local governments instead of a consolidated government. It is of interest to explore how imperfect competition and migration matter for land finance.

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Online Appendix for

"A Theory of Land Finance and Investment-Led Growth"

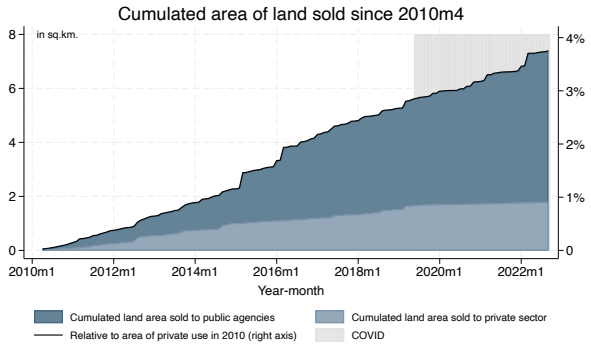
Lingxuan Wu

March 26, 2026

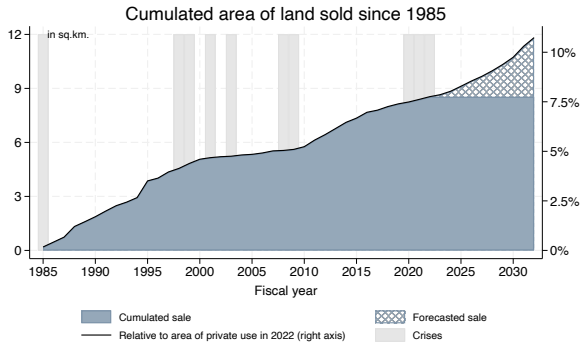
A Additional Figures

Figure A1: Added land supply since beginning of sample

(a) Singapore, 2010-2022



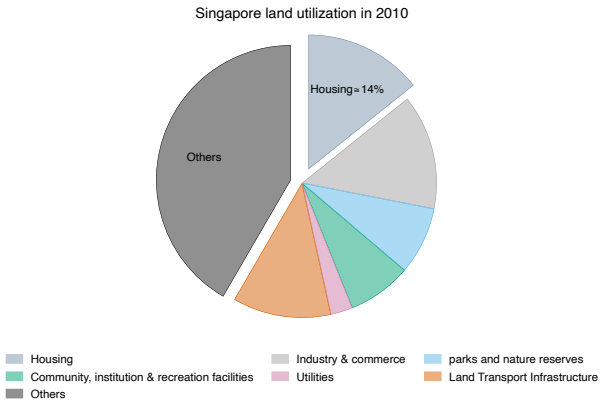
(b) Hong Kong, 1985-2032 (with 10-year forecast)



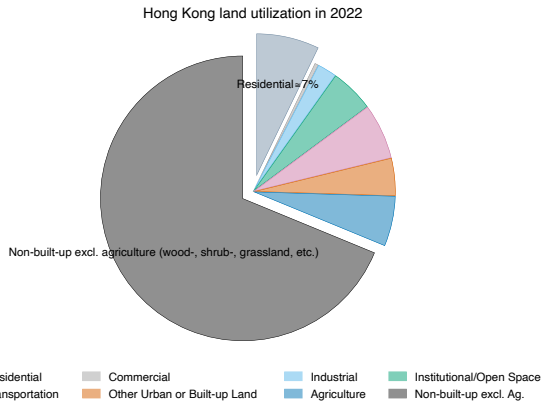
Notes: (a) Singapore data since 2010 are from the aggregate statistics published on Singapore Land Authority's website. Detailed transfer data by sites going back to last century can be found in websites of various government agencies (Urban Redevelopment Authority, Jurong Town Corporation, and Housing Development Board).
(b) Hong Kong land sale records and forecasts are from the Hong Kong Lands Department.

Figure A2: Land utilization in Singapore, Hong Kong and mainland China

(a) Singapore in 2010



(b) Hong Kong in 2022



(c) Mainland China in 2022

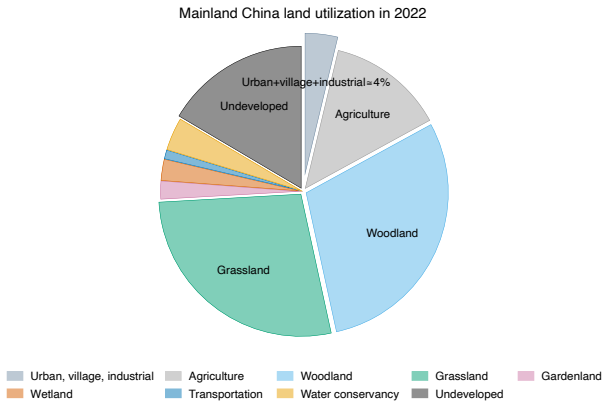
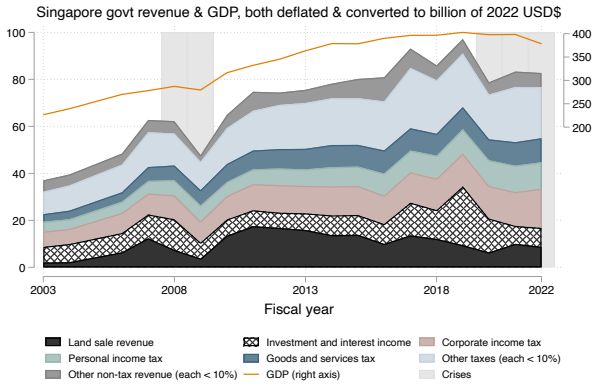
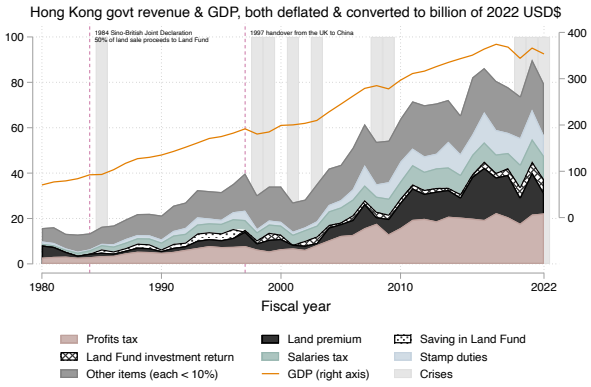


Figure A3: Land sale in government revenue of Asian growth miracles

(a) Singapore, 2003-2022



(b) Hong Kong, 1980-2022



(c) Mainland China, 2001-2022

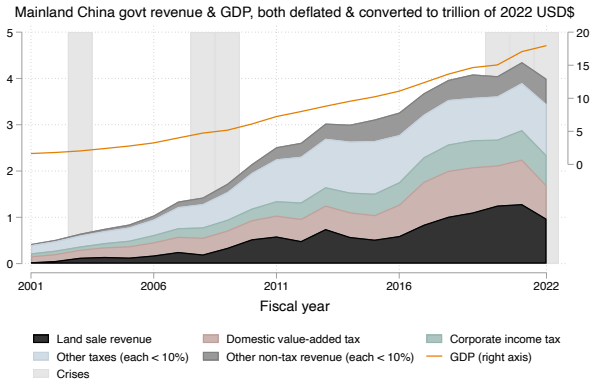
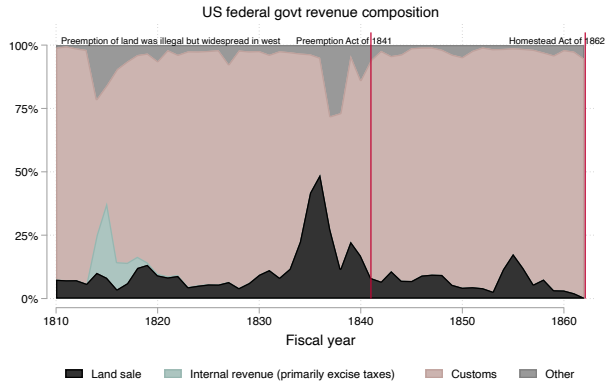
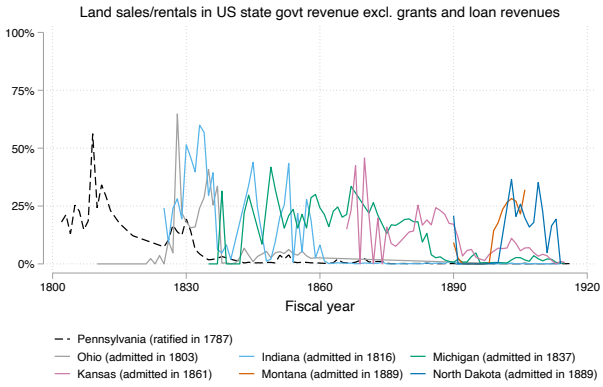


Figure A4: Land sale in US and Hong Kong government revenue in 19th century

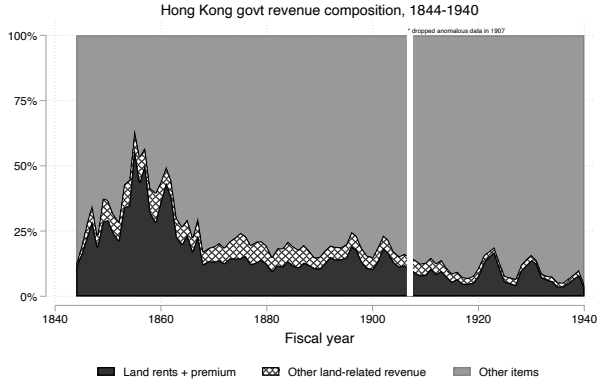
(a) US federal government



(b) US state governments



(c) Hong Kong, 1844-1940



Notes: Data of US federal and state government revenues are from Wallis (2006) and Sylla, Legler and Wallis (1993) respectively.

B Land Tax

It is known since George (1879) that a tax on land is non-distortionary when land is in fixed supply. I show that this argument may have its limitations as the amount of revenue it raises depends on the total market value of land. Under certain parameters, the government may not raise enough money to support the first-best allocation. In that case, land tax interacts with a motive that restricts supply to increase land value, in order to raise public funds.

Proposition B1. *Assume that the government can impose a land tax of τ_p. The first best allocation cannot be achieved if*

$$\underbrace{v}_{\text{land market value}} \left[\underbrace{(1-H_0)}_{\text{sale/rent revenue}} + \underbrace{\frac{\tau^p}{\rho\left(\tau^p+\rho\right)}}_{\text{tax revenue}} \right] < \int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t^{FB}}-1\right)\mathrm{d}t \tag{B1}$$

Comparing this to (15), with a land tax of τ_p, the government can get additional fiscal income of present value of $v\frac{\tau_p}{\rho\left(\tau_p+\rho\right)}$. This income increases in the land tax rate τ_p the government sets, which in reality is quite low, and scales in proportion to the land market value v (in unit of utility). When (B1) holds, the first best still cannot be implemented. In that case, the government has an incentive to shrink land supply to raise the land market value under inelastic demand. It could be interesting for future work to study the optimal joint use of land tax and land supply to finance public investment.

C Price vs. Quantity Lerner Rule

One may wonder why the requirement of demand elasticity less than one behind Lemma 1 seems to go against the usual intuition that the demand elasticity has to be larger than one to admit a monopoly pricing solution. Let us indeed consider such a textbook example (Tirole, 1988, Chapter 1.1): a monopoly quotes a price p under constant marginal cost mc and demand function $q\left(p\right)$ to maximize profits,

$$\max \pi = \left(p-mc\right) \underbrace{q\left(p\right)}_{\text{demand}} \tag{C1}$$

The optimal price satisfies the familiar Lerner condition

$$\frac{p-mc}{p} = \left(-\frac{\partial \log q\left(p\right)}{\partial \log p}\right)^{-1} \tag{C2}$$

The demand elasticity $-\frac{\partial \log q}{\partial \log p}$ has to be larger than one to admit an interior solution. We observe that (C2) and (17) are isomorphic once we swap the price and quantity. In the problem (16), the price declines in supply, but the monopoly does not profit from the pre-owned amount q_0, which acts to erode the profit in a way much like the marginal

cost in problem (C1). Thus the requirement for (16) to admit a solution is exactly the opposite: the demand for housing has to be inelastic rather than elastic, which is also empirically relevant. Note that the key difference is not in whether the monopoly sets price or quantity, which maps into each other via the demand function $q(p)$, but rather where a quantity-like term (q_0) or a price-like term (mc) erodes profits.

D Derivations and Proofs

D.1 Optimality and Implementability Conditions

Derivation of household optimality conditions (12-14).

$$\begin{aligned}
&\max \int_0^\infty \mathrm{e}^{-\rho t} U(C_t)\,\mathrm{d}t \\
&\text{s.t.} \int_0^\infty Q_t \left[C_t - Y_t + P_t \dot{H}_t + D_t (L_t - H_t)\right] \mathrm{d}t = 0
\end{aligned}$$

where $Q_0 = 1$. Integrate the budget constraint by parts to get

$$\begin{aligned}
0 &= \int_0^\infty Q_t \left[C_t - Y_t + P_t \dot{H}_t + D_t (L_t - H_t)\right] \mathrm{d}t \\
&= \int_0^\infty Q_t \left[C_t - Y_t + D_t (L_t - H_t)\right] \mathrm{d}t - P_0 H_0 - \int_0^\infty H_t \mathrm{d}\left(P_t Q_t\right)
\end{aligned}$$

Thus the FOCs w.r.t. C_t, L_t, H_t are

$$\begin{aligned}
\frac{\mathrm{e}^{-\rho t} U_{C,t}}{U_{C,0}} &= Q_t \\
\frac{U_{L,t}}{U_{C,t}} &= D_t \\
\mathrm{d}\left(P_t Q_t\right) &= -Q_t D_t \mathrm{d}t
\end{aligned}$$

We iterate the last expression to get

$$\begin{aligned}
P_t &= Q_t^{-1} \int_t^\infty Q_s D_s \mathrm{d}s \\
&= \frac{U_{C,0}}{\mathrm{e}^{-\rho t} U_{C,t}} \int_t^\infty \frac{\mathrm{e}^{-\rho s} U_{C,s}}{U_{C,0}} \frac{U_{L,s}}{U_{C,s}} \mathrm{d}s
\end{aligned}$$

$$= U_{C,t}^{-1} \mathcal{U}_{L,t}$$

with $\mathcal{U}_{L,t} \equiv \int_t^\infty \mathrm{e}^{-\rho(s-t)} U_{L,s} \mathrm{d}s$. □

Proof of Lemma 1. The first-order condition is

$$0 = \frac{\mathrm{d}\pi}{\mathrm{d}q} = p + (q - q_0) \frac{\mathrm{d}p}{\mathrm{d}q}$$

and thus

$$\frac{q}{q - q_0} = -\frac{\mathrm{d}\log p}{\mathrm{d}\log q}$$

□

Derivation of implementability constraint (20). Plugging (12, 13, 14) into (11) and setting $T_t = 0$ yields

$$\begin{aligned}
0 &= \int_0^\infty \mathrm{e}^{-\rho t} U_{C,0}^{-1} \left[(C_t - Y_t) U_{C,t} + \dot{H}_t \mathcal{U}_{L,t} + (L_t - H_t) U_{L,t} \right] \mathrm{d}t \\
0 &= \int_0^\infty \mathrm{e}^{-\rho t} \left[(C_t - Y_t) U_{C,t} + (L_t - H_t) U_{L,t} \right] \mathrm{d}t - H_0 \mathcal{U}_{L,0} - \int_0^\infty H_t \mathrm{d}\left(\mathrm{e}^{-\rho t} \mathcal{U}_{L,t} \right) \\
&= \int_0^\infty \mathrm{e}^{-\rho t} \left[(C_t - Y_t) U_{C,t} + L_t U_{L,t} \right] \mathrm{d}t - H_0 \mathcal{U}_{L,0} \\
&= \int_0^\infty \mathrm{e}^{-\rho t} \left[(C_t - Y_t) U_{C,t} + (L_t - H_0) U_{L,t} \right] \mathrm{d}t
\end{aligned}$$

□

Present-value sum of public capital, useful for (21). I establish a relation that will be helpful in forming a Lagrangian instead of a Hamiltonian, to make the continuous-time model more intuitive and relatable to a discrete-time model. Take the continuous-time resource constraint (2)

$$0 = Y_t(Z_t) - C_t - \delta Z_t - \dot{Z}_t,$$

attach an arbitrary multiplier $\{\mu_t \mathrm{e}^{-\rho t}\}_{t \geq 0}$, and integrate by parts to get

$$0 = \int_0^\infty \mu_t \mathrm{e}^{-\rho t} \left[Y_t(Z_t) - C_t - \delta Z_t - \dot{Z}_t \right] \mathrm{d}t$$

$$= \int_0^\infty \mu_t e^{-\rho t} \left[Y_t(Z_t) - C_t - \delta Z_t \right] dt - \left[\mu_t e^{-\rho t} Z_t |_0^\infty - \int Z_t d\left(\mu_t e^{-\rho t} \right) \right]$$
$$= \int_0^\infty e^{-\rho t} \left\{ \mu_t \left[Y_t(Z_t) - C_t - (\delta + \rho) Z_t \right] + \dot{\mu}_t Z_t \right\} dt + \mu_0 Z_0. \tag{D1}$$

This is a present-value constraint that depends on Z_t but not $\dot{Z}_t$.□

D.2 Proofs

Proof of Proposition 1.

$$\begin{aligned} \max \mathcal{L}^{FB} &= \int_0^\infty e^{-\rho t} U(C_t, L_t) dt + \int_0^\infty \mu_t e^{-\rho t} \left[Y_t(Z_t) - C_t - \delta Z_t - \dot{Z}_t \right] dt \\ &= \int_0^\infty e^{-\rho t} U(C_t, L_t) dt + \int_0^\infty e^{-\rho t} \left\{ \mu_t \left[Y_t(Z_t) - C_t - (\delta + \rho) Z_t \right] + \dot{\mu}_t Z_t \right\} dt + \mu_0 Z_0 \end{aligned}$$

The optimality conditions are

$$0 = e^{\rho t} \frac{\partial \mathcal{L}^{FB}}{\partial C_t} = U_{C,t} - \mu_t$$
$$0 = e^{\rho t} \frac{\partial \mathcal{L}^{FB}}{\partial Z_t} = \dot{\mu}_t + \left[Y_t' - (\delta + \rho) \right] \mu_t$$

Thus, the consumption growth rate is

$$\frac{\dot{C}_t}{C_t} = Y_t' - (\delta + \rho)$$

□

Proof of Proposition 3.

Consumption growth and fiscal balance. Use (22) to get

$$\mu_t^* = (1 + \lambda^* Y_t / C_t) C_t^{-1} = \left(1 + \frac{\lambda^*}{\chi_t} \right) C_t^{-1}$$

Plug this into (24) and arrive at

$$0 = \left(1 + \frac{\lambda^*}{\chi_t}\right) C_t^{-1} \left[Y_t' - (\delta + \rho)\right] - \left(1 + \frac{\lambda^*}{\chi_t}\right) C_t^{-2} \dot{C}_t - \frac{\lambda^*}{(\chi_t)^2} C_t^{-1} \dot{\chi}_t - \lambda^* Y_t' C_t^{-1}$$

which simplifies to

$$\frac{\dot{C}_t^*}{C_t^*} = \left[Y_t'\left(Z_t^*\right) - (\delta + \rho)\right] - \frac{\lambda^*}{\lambda^* + \chi_t^*}\left[\chi_t^* Y_t'\left(Z_t^*\right) + \frac{\dot{\chi}_t^*}{\chi_t^*}\right]$$

For any (Z_0, H_0), the multiplier λ^* satisfies the implementability condition (20). I write (20) as equalizing fiscal income and expenditure in unit of utility

$$\underbrace{\int_0^\infty \mathrm{e}^{-\rho t}\left(L_t - H_0\right) U_{L,t} \mathrm{d}t}_{\equiv \mathcal{I}^*} = \underbrace{\int_0^\infty \mathrm{e}^{-\rho t}\left(Y_t - C_t\right) U_{C,t} \mathrm{d}t}_{\equiv \mathcal{E}^*} \tag{D2}$$

The present-value util-unit sum of fiscal income $\mathcal{I}^*$ equals

$$\mathcal{I}^* = \int_0^\infty \mathrm{e}^{-\rho t} \eta H_0 U_L\left((1+\eta) H_0\right) \mathrm{d}t = \frac{\nu H_0^{1-\sigma}}{\rho} \frac{\eta}{(1+\eta)^\sigma}$$

which only depends on (H_0, λ^*), i.e., $\mathcal{I}^* = \mathcal{I}^*\left(H_0, \lambda^*\right)$. It satisfies $\frac{\partial \mathcal{I}^*}{\partial H_0} < 0$ and

$$\frac{\partial \mathcal{I}^*}{\partial \lambda^*} = \frac{\nu H_0^{1-\sigma}}{\rho} \frac{1 - (\sigma - 1)\eta}{(1+\eta)^{\sigma+1}} \frac{\mathrm{d}\eta}{\mathrm{d}\lambda^*} = -\frac{\nu H_0^{1-\sigma}}{\rho} (1+\eta)^{-(\sigma+1)} \frac{\eta\sigma}{1+\lambda^*} \frac{\mathrm{d}\eta}{\mathrm{d}\lambda^*} > 0$$

Intuitively, the larger the need for fiscal income (higher λ^*), the larger the fiscal income ($\mathcal{I}^*$). $\mathcal{I}^*$ is independent of Z_0 except through its dependence on λ^*.

The present-value util-unit sum of fiscal expenditure $\mathcal{E}^*$ equals

$$\mathcal{E}^* = \int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t} - 1\right) \mathrm{d}t$$

which implicitly depends on Z_0 and λ^* through the ODE system (2, 27), i.e., $\mathcal{E}^* = \mathcal{E}^*\left(Z_0, \lambda^*\right)$.

Phase diagram in (Z, C) space. The $\dot{Z} = 0$ locus is from (28) as

$$C_t^* = Y\left(Z_t^*\right) - \delta Z_t^*$$

which is concave and positive between 0 and $\left(\frac{A}{\delta}\right)^{\frac{1}{1-\alpha}}$, the same as in the neoclassical growth model. Its apex value is $C^a = \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\left(\frac{A}{\delta^\alpha}\right)^{\frac{1}{1-\alpha}}$ at $Z^a = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}$.

To arrive at the $\dot{C} = 0$ locus, note that

$$\frac{\dot{\chi}_t^*}{\chi_t^*} = \frac{\dot{C}_t^*}{C_t^*} - \frac{\dot{Y}_t^*}{Y_t^*} = \frac{\dot{C}_t^*}{C_t^*} - \alpha\frac{\dot{Z}_t^*}{Z_t^*} = \frac{\dot{C}_t^*}{C_t^*} - \alpha\frac{Y_t^* - C_t^* - \delta Z_t^*}{Z_t^*}$$

where I have plugged in the production function and (28). Then, using (27) and setting $\dot{C}_t^* = 0$, I get

$$\begin{aligned} 0 &= Y'\left(Z_t^*\right) - (\delta + \rho) - \frac{\lambda^*}{\lambda^* + C_t^*/Y\left(Z_t^*\right)}\left[\frac{C_t^*}{Y\left(Z_t^*\right)}Y'\left(Z_t^*\right) - \alpha\frac{Y\left(Z_t^*\right) - C_t^* - \delta Z_t^*}{Z_t^*}\right] \\ &= Y'\left(Z_t^*\right) - (\delta + \rho) + \frac{\alpha\lambda^*}{\lambda^* + C_t^*/Y\left(Z_t^*\right)}\frac{Y\left(Z_t^*\right) - 2C_t^* - \delta Z_t^*}{Z_t^*} \end{aligned} \tag{D3}$$

In the case with $\lambda^* = 0$, this locus is $0 = Y'\left(Z_t^*\right) - (\delta + \rho)$ which specifies a constant $Z_t^* \equiv Z_\infty^{FB} = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$ (a vertical line in (Z, C) space) as in the neoclassical model. Its intersection with the $\dot{Z} = 0$ locus is $C_\infty^{FB} = Y(Z_\infty^{FB}) - \delta Z_\infty^{FB}$. For any $\lambda^* > 0$, the $\dot{C} = 0$ locus is no longer a vertical line at Z_∞^{FB}, but it will cross that vertical line once at exactly $(Z_\infty^{FB}, C_\infty^{FB}/2)$.[12] I will prove in Corollary 2 that the steady state C_∞^*, Z_∞^* both decrease in λ^*, suggesting that the intersection of $\dot{Z} = 0$ locus and $\dot{C} = 0$ locus under $\lambda^* > 0$ is to the left of Z_∞^{FB}. That is, the $\dot{C} = 0$ locus is rotated counterclockwise around $(Z_\infty^{FB}, C_\infty^{FB}/2)$ for $\lambda^* > 0$.

Monotonicity of consumption rate χ_t. Omit the time subscript and $*$ superscript here for simplicity. We have

$$\begin{aligned} g^\chi \equiv \frac{\dot{\chi}}{\chi} &= \frac{\dot{C}}{C} - \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} - \alpha\frac{\dot{Z}}{Z} = \frac{\dot{C}}{C} - \alpha\frac{Y - C - \delta Z}{Z} \\ &= \left[Y' - (\delta + \rho)\right] - \frac{\lambda}{1 + \frac{\lambda}{\chi}}\left[\frac{\dot{\chi}}{\chi^2} + Y'\right] - Y'(1 - \chi) + \alpha\delta \\ &= Y'\chi - \left[(1 - \alpha)\delta + \rho\right] - \frac{\lambda}{1 + \frac{\lambda}{\chi}}\left(\frac{\dot{\chi}}{\chi^2} + Y'\right) \end{aligned}$$

[12] To be precise, there is only one crossing in the $\dot{Z} > 0$ subspace ($C \leq Y(Z) - \delta Z$). Outside that, there is another uninteresting crossing at $C \to \infty$ which corresponds to infinitely fast depreciation of Z.

$$= Y' \frac{\chi^2}{\chi + \lambda} - [(1-\alpha)\delta + \rho] - \frac{\lambda}{\chi + \lambda} g^\chi$$

$$\frac{\chi + 2\lambda}{\chi + \lambda} g^\chi = Y' \frac{\chi^2}{\chi + \lambda} - [(1-\alpha)\delta + \rho] \tag{D4}$$

Let $\gamma \equiv \frac{\chi+2\lambda}{\chi+\lambda} g^\chi$ which has the same sign as g^χ. We have

$$\begin{aligned}
\gamma &= Y' \frac{\chi^2}{\chi + \lambda} - [(1-\alpha)\delta + \rho] \\
\dot{\gamma} &= Y'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + Y' \frac{(\chi + 2\lambda)\chi^2}{(\chi + \lambda)^2} g^\chi \\
&= Y'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + Y' \frac{\chi^2}{\chi + \lambda} \gamma
\end{aligned}$$

If $\gamma \leq 0$ for some t, then $\dot{\gamma} < 0$ since $Y'' < 0, \dot{Z} > 0, Y' > 0$ at that time, as Z monotonically increases when starting from the left of Z^*_∞ in the third quadrant on the phase diagram. It suggests that $\gamma < 0$ holds forever and γ keeps decreasing, a result inconsistent with convergence. Thus we must have $\gamma > 0$ for all t, which means $g^\chi > 0$ and thus $\dot{\chi} > 0$.□

Proof of Corollary 2. At the long-run steady state, (27) gives

$$\left(1 - \frac{\lambda^*}{1 + \frac{\lambda^*}{\chi^*_\infty}}\right) Y'_\infty = \delta + \rho \tag{D5}$$

Case with $\delta = 0$. If $\delta = 0$, then $\chi^*_\infty = 1$ from the resource constraint and thus (D5) reduces to $Y'_\infty = (1 + \lambda^*)\rho$. Hence

$$Z^*_\infty = (Y')^{-1}((1 + \lambda^*)\rho) = \left[\frac{\alpha}{(1 + \lambda^*)\rho}\right]^{\frac{1}{1-\alpha}}$$

and $C^*_\infty = Y^*_\infty = A(Z^*_\infty)^\alpha$. Obviously, both decrease in λ^*.

Case with $\delta > 0$. If $\delta > 0$, the resource constraint implies that $Z^*_\infty = \frac{Y^*_\infty - C^*_\infty}{\delta}$ and thus the marginal product is

$$Y'_\infty = \frac{\alpha Y^*_\infty}{Z^*_\infty} = \frac{\alpha\delta}{1 - \chi^*_\infty} \tag{D6}$$

Plugging this in (D5) gives

$$\left(1-\frac{\lambda^*}{1+\frac{\lambda^*}{\chi^*_\infty}}\right)\frac{\alpha\delta}{1-\chi^*_\infty}=\delta+\rho$$
$$\left(1+\frac{\chi^*_\infty}{1-\chi^*_\infty-\lambda^*+\frac{\lambda^*}{\chi^*_\infty}}\right)\alpha\delta=\delta+\rho$$
$$\frac{\chi^*_\infty}{1-\chi^*_\infty-\lambda^*+\frac{\lambda^*}{\chi^*_\infty}}=\frac{\delta(1-\alpha)+\rho}{\alpha\delta}$$

It is a quadratic equation in χ^*

$$(\delta+\rho)(\chi^*_\infty)^2-(1-\lambda^*)[\delta(1-\alpha)+\rho]\chi^*_\infty-\lambda^*[\delta(1-\alpha)+\rho]=0$$

which gives two solutions

$$\chi^*_\infty=\frac{(1-\lambda^*)[\delta(1-\alpha)+\rho]\pm\sqrt{(1-\lambda^*)^2[\delta(1-\alpha)+\rho]^2+4(\delta+\rho)\lambda^*[\delta(1-\alpha)+\rho]}}{2(\delta+\rho)}$$
$$=\frac{(1-\lambda^*)\pm\sqrt{(1-\lambda^*)^2+4\frac{\delta+\rho}{\delta(1-\alpha)+\rho}\lambda^*}}{2}\frac{\delta(1-\alpha)+\rho}{\delta+\rho}$$
$$=\frac{(1-\lambda^*)\pm\sqrt{(1+\lambda^*)^2+4\frac{\delta\alpha}{\delta(1-\alpha)+\rho}\lambda^*}}{2}\frac{\delta(1-\alpha)+\rho}{\delta+\rho}$$

We take the solution with plus sign since consumption can only be positive.

χ^*_∞ increases in λ^* since

$$\frac{\mathrm{d}\chi^*_\infty}{\mathrm{d}\lambda^*}\propto-1+\frac{(1+\lambda^*)+2\frac{\delta\alpha}{\delta(1-\alpha)+\rho}}{\sqrt{(1+\lambda^*)^2+4\frac{\delta\alpha}{\delta(1-\alpha)+\rho}\lambda^*}}>0$$

(D6) gives infrastructure

$$Z^*_\infty=(Y')^{-1}\left(\frac{\alpha\delta}{1-\chi^*_\infty}\right)=\left[\frac{A(1-\chi^*_\infty)}{\delta}\right]^{\frac{1}{1-\alpha}}$$

which decreases in χ^*_∞ and thus decreases in λ^*.

Further, the consumption is

$$C^*_\infty = \chi^*_\infty Y^*_\infty = A\chi^*_\infty \left[\frac{A\left(1-\chi^*_\infty\right)}{\delta}\right]^{\frac{\alpha}{1-\alpha}}$$

To show that it is increasing in λ^*, it suffices to show that $\chi^*_\infty \left(1-\chi^*_\infty\right)^{\frac{\alpha}{1-\alpha}}$ decreases in χ^*_∞. We check

$$\frac{\mathrm{d}\log\left[\chi^*_\infty\left(1-\chi^*_\infty\right)^{\frac{\alpha}{1-\alpha}}\right]}{\mathrm{d}\chi^*_\infty} = \frac{1}{\chi^*_\infty} - \frac{\alpha}{1-\alpha}\frac{1}{1-\chi^*_\infty} = \frac{1-\alpha-\chi^*_\infty}{(1-\alpha)\left(1-\chi^*_\infty\right)\chi^*_\infty}$$

which is negative since $\chi^*_\infty > \frac{\delta(1-\alpha)+\rho}{\delta+\rho} > 1-\alpha$.□

Proof of Proposition 4. The cumulative expenditure from time 0 to time t is

$$\int_0^t \mathrm{e}^{-\rho s}\left(Y_s - C_s\right)U_{C,s}\mathrm{d}s = \int_0^t \mathrm{e}^{-\rho s}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s$$

The cumulated income from time 0 to time t is

$$\begin{aligned}
&\int_0^t \mathrm{e}^{-\rho s}\left(L_s - H_t\right)U_{L,s}\mathrm{d}s + \int_0^\infty \mathrm{e}^{-\rho s}\left(H_t - H_0\right)U_{L,s}\mathrm{d}s \\
=&U^*_L\left[\frac{1-\mathrm{e}^{-\rho t}}{\rho}\left(L^* - H_t\right) + \frac{1}{\rho}\left(H_t - H_0\right)\right] \\
=&\frac{\nu H_0^{1-\sigma}}{\rho\left(1+\eta\right)^\sigma}\left[\left(1-\mathrm{e}^{-\rho t}\right)\eta + \mathrm{e}^{-\rho t}\frac{H_t - H_0}{H_0}\right] \\
=&\left[\int_0^\infty \mathrm{e}^{-\rho s}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s\right]\left[\left(1-\mathrm{e}^{-\rho t}\right) + \mathrm{e}^{-\rho t}\frac{H_t - H_0}{\eta H_0}\right]
\end{aligned}$$

in which the last step uses (30). In order for the cumulated income to be larger than the cumulative expenditure (so that the government is net creditor), we need

$$\begin{aligned}
\left(1-\mathrm{e}^{-\rho t}\right) + \mathrm{e}^{-\rho t}\frac{H_t - H_0}{\eta H_0} &\geq \frac{\int_0^t \mathrm{e}^{-\rho s}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s}{\int_0^\infty \mathrm{e}^{-\rho s}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s} \\
\frac{H_t - H_0}{\eta H_0} &\geq 1 - \frac{\int_t^\infty \mathrm{e}^{-\rho(s-t)}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s}{\int_0^\infty \mathrm{e}^{-\rho s}\left(\frac{1}{\chi_s}-1\right)\mathrm{d}s}
\end{aligned}$$

the RHS of which increases in t since χ_t increases in t. When this holds as equality, SB can be achieved without borrowing by either the government or the household.□

Proof of Proposition 5. Define an integrating factor $m_t \equiv \mathrm{e}^{\int_0^t [Y'_\tau - (\delta+\rho)]\mathrm{d}\tau}$ and we have, using (39)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_t \frac{\mu_t^{s*}}{\lambda^{s*}}\right) &= \left[Y'_t - (\delta+\rho)\right] m_t \frac{\mu_t^{s*}}{\lambda^{s*}} + m_t \frac{\dot{\mu}_t^{s*}}{\lambda^{s*}} \\ &= m_t Y'_t U_{C,t}\end{aligned}$$

As its RHS is independent of s, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_t \frac{\mu_t^{s*}}{\lambda^{s*}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(m_t \frac{\mu_t^{*}}{\lambda^{*}}\right) \\ m_t \frac{\mu_t^{s*}}{\lambda^{s*}} - m_s \frac{\mu_s^{s*}}{\lambda^{s*}} &= m_t \frac{\mu_t^{*}}{\lambda^{*}} - m_s \frac{\mu_s^{*}}{\lambda^{*}}\end{aligned}$$

and thus

$$\frac{\mu_t^{s*}}{\lambda^{s*}} = \frac{\mu_t^{*}}{\lambda^{*}} + m_t^{-1} m_s \left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}}\right) \tag{D7}$$

To determine the bond position, rewrite (37) as

$$B_s^t = \frac{U_{C,t} - \mu_t^{s*}}{\lambda^{s*} U_{CC,t}} - Y_t$$

whose $s = 0$ case is $0 = \frac{U_{C,t} - \mu_t^{*}}{\lambda^{*} U_{CC,t}} - Y_t$. Taking their difference to cancel Y_t and plugging in (D7) gives

$$\begin{aligned}B_s^t &= \frac{U_{C,t} - \mu_t^{s*}}{\lambda^{s*} U_{CC,t}} - \frac{U_{C,t} - \mu_t^{*}}{\lambda^{*} U_{CC,t}} = -C_t^*\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + (C_t^*)^2 \left(\frac{\mu_t^{s*}}{\lambda^{s*}} - \frac{\mu_t^{*}}{\lambda^{*}}\right) \\ &= -C_t^*\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + (C_t^*)^2 m_t^{-1} m_s \left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}}\right)\end{aligned} \tag{D8}$$

As for H_s, (38) implies $L^* = \left[1 + \eta\left(\lambda^{s*}\right)\right] H_s$. The logic is the same as Lemma 3.

The chosen portfolio has to satisfy the time-s implementability constraint

$$\underbrace{\int_s^\infty \mathrm{e}^{-\rho(t-s)}\left(L_t - H_s\right) U_{L,t}\mathrm{d}t}_{\equiv \mathcal{I}^{s*}} = \underbrace{\int_s^\infty \mathrm{e}^{-\rho(t-s)}\left(Y_t - C_t\right) U_{C,t}\mathrm{d}t}_{\equiv \mathcal{E}^{s*}} + \underbrace{\int_s^\infty \mathrm{e}^{-\rho(t-s)} B_s^t U_{C,t}\mathrm{d}t}_{\equiv \mathcal{B}^{s*}}$$

The land income term is, using $L_t = L^*, H_s = \left[1+\eta\left(\lambda^{s*}\right)\right]^{-1} L^*$,

$$\mathcal{I}^{s*} = \frac{\nu U_L^* L^*}{\rho}\frac{\eta\left(\lambda^{s*}\right)}{1+\eta\left(\lambda^{s*}\right)} = \frac{\nu H_0^{1-\sigma}}{\rho}\frac{\eta\left(\lambda^{s*}\right)}{\left[1+\eta\left(\lambda^{s*}\right)\right]\left[1+\eta\left(\lambda^*\right)\right]^{\sigma-1}} = \frac{\nu\left(L^*\right)^{1-\sigma}}{\rho}\frac{1+\lambda^{s*}}{\lambda^{s*}\sigma} \equiv \mathcal{I}\left(\lambda^{s*}; L^*\right)$$

which decreases in λ^{s*} and otherwise does not depend on calendar time s.[13]

The present-sum util-unit of fiscal expenditure equals

$$\mathcal{E}^{s*} = \int_s^\infty \mathrm{e}^{-\rho(t-s)}\left(\frac{1}{\chi_t^*} - 1\right)\mathrm{d}t$$

which decreases in s.

The debt repayment term is, plugging in (D8),

$$\begin{aligned}\mathcal{B}^{s*} &= \int_s^\infty \mathrm{e}^{-\rho(t-s)}\left[-\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + C_t^* m_t^{-1} m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^*}{\lambda^*}\right)\right]\mathrm{d}t \\ &= -\frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^*}{\lambda^*}\right)\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1}\mathrm{d}t\end{aligned}$$

Taking stock, the time-s implementability constraint is

$$\mathcal{I}\left(\lambda^{s*}; L^*\right) = \mathcal{E}^{s*} - \frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^*}{\lambda^*}\right)\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1}\mathrm{d}t$$

which determines

$$\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^*}{\lambda^*} = \left(m_s\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1}\mathrm{d}t\right)^{-1}\left[\frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + \mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right]$$

[13] Note that in this analysis, we hold fixed L^* and find $\mathcal{I}^{s*}$ to be decreasing in λ^{s*}, which does not contradict the finding in Section 3.3 that $\mathcal{I}$ increases in λ^* when holding H_0 fixed.

Plugging this back into (D8), we get

$$B_s^\tau = -C_\tau^* \left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + \left(C_\tau^*\right)^2 m_\tau^{-1} \left(\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t\right)^{-1} \left[\frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) + \mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right]$$
$$= C_\tau^* \left[\frac{C_\tau^* m_\tau^{-1}}{\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} \left(\mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right) + \left(\frac{C_\tau^* m_\tau^{-1}}{\rho \int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} - 1\right)\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right)\right]$$

The second term in the bracket averages to zero, weighted by $\mathrm{e}^{-\rho(t-s)}$, since $\rho \int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t$ is exactly the weighted average of $C_\tau^* m_\tau^{-1}$ across all $\tau \geq s$. As for the first term, since $\mathcal{E}^{s*}$ decreases in s and $\mathcal{I}\left(\lambda^{s*}; L^*\right)$ decreases in λ^{s*}, there exists a threshold $\bar{\lambda}^s$ such that the first term is positive at time s if and only if $\lambda^{s*} > \bar{\lambda}^s$. The threshold $\bar{\lambda}^s$ increases over s, and in particular, its time-0 value is $\bar{\lambda}^0 = \lambda^*$.

We conclude with an observation that, for any $s > 0$ and λ^{s*}, B_s^τ / C_τ^* is either a positive constant across τ or strictly monotonic in τ. This is because

$$B_s^\tau / C_\tau^* = \frac{C_\tau^* m_\tau^{-1}}{\int_s^\infty \mathrm{e}^{-\rho(t-s)} C_t^* m_t^{-1} \mathrm{d}t} \left[\left(\mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right) + \frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right)\right] - \left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right)$$

as a sum of a term that varies in τ and a constant. The coefficient of the first term strictly decreases in τ, since

$$\frac{\mathrm{d}\left(C_t^*/m_t\right)/\mathrm{d}t}{C_t^*/m_t} = \frac{\dot{C}_t^*}{C_t^*} - \frac{\dot{m}_t}{m_t} = -\frac{\lambda^*}{\lambda^* + \chi_t^*}\left[\chi_t^* Y_t'\left(Z_t^*\right) + \frac{\dot{\chi}_t^*}{\chi_t^*}\right] < 0$$

using (27) and $\dot{\chi}_t^* > 0$ from Proposition 3. Further, $\left(\mathcal{I}\left(\lambda^{s*}; L^*\right) - \mathcal{E}^{s*}\right) + \frac{1}{\rho}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right)$ is decreasing in λ^{s*} and positive when $\lambda^{s*} = \lambda^*$, since $\mathcal{I}\left(\lambda^{s*}; L^*\right) = \mathcal{I}^* = \mathcal{E}^* > \mathcal{E}^{s*}$ for $s > 0$. Thus it admits only one zero that is larger than λ^*. At that zero, the constant $-\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^*}\right) > 0$, suggesting that the government borrows with a "flat" (relative to consumption C_τ^*) maturity structure. For any λ^{s*} below (/above) that zero, B_s^τ / C_τ^* strictly decreases (/increases) in τ.□

Proof of Corollary 4. Proposition 5 indicates that SB can be implemented in a time-consistent way, if and only if

$$H_s \leq \frac{\sigma - 1}{\sigma} L^*$$

Proposition 4 suggests that the threshold H_s^{NB} for no-borrowing implementation increases over time. Thus in order to find an implementation that is both time-consistent and free

from borrowing, we need

$$H_\infty^{NB} \leq \frac{\sigma-1}{\sigma} L^*$$

$$H_0 + \left[1 - \frac{\frac{1}{\chi_\infty^*} - 1}{\rho \int_0^\infty \mathrm{e}^{-\rho t} \left(\frac{1}{\chi_t^*} - 1\right) \mathrm{d}t}\right] \eta(\lambda^*) H_0 \leq \frac{\sigma-1}{\sigma} L^*$$

Using (29) and (30), we simplify it to

$$\frac{1}{1+\eta(\lambda^*)} + \left[1 - \frac{\frac{1}{\chi_\infty^*} - 1}{\nu (L^*)^{1-\sigma} \frac{\eta(\lambda^*)}{1+\eta(\lambda^*)}}\right] \frac{\eta(\lambda^*)}{1+\eta(\lambda^*)} \leq \frac{\sigma-1}{\sigma}$$

$$-\frac{\frac{1}{\chi_\infty^*} - 1}{\nu (L^*)^{1-\sigma} \frac{\eta(\lambda^*)}{1+\eta(\lambda^*)}} \frac{\eta(\lambda^*)}{1+\eta(\lambda^*)} \leq -\frac{1}{\sigma}$$

$$\frac{1}{\chi_\infty^*} - 1 \geq \frac{\nu (L^*)^{1-\sigma}}{\sigma} \tag{D9}$$

Note that the LHS decreases in λ^* from Corollary 2, and the RHS increases in λ^* from Lemma 3. Thus this inequality (D9) can only hold if λ^* is small.

Last, we establish a sufficient condition under which the inequality (D9) cannot hold for any $\lambda^* > 0$. We simply need (D9) to hold with weak inequality to the opposite direction under $\lambda^* = 0$, i.e.

$$\frac{1}{\chi_\infty^{FB}} - 1 \leq \frac{\nu}{\sigma}$$

$$\frac{\alpha\delta}{(1-\alpha)\delta + \rho} \leq \frac{\nu}{\sigma}$$

□

Proof of Proposition 6. For simplicity I suppress the superscript DD. The FOCs w.r.t. C_t, L_t of (47) are

$$0 = U_{C,t}\Delta t + \lambda_t U_{C,t}\Delta t + \lambda_t C_t U_{CC,t}\Delta t - \lambda_t Y_t U_{CC,t}\Delta t - \mu_t \Delta t = U_{C,t}\Delta t - \lambda_t \Delta t Y_t U_{CC,t} - \mu_t \Delta t$$

$$0 = (1+\lambda_t) U_{L,t}\Delta t + \lambda_t (L_t - H_t) U_{LL,t}\Delta t - \zeta_t \Delta t$$

and the limits of $\Delta t \to 0$ are, assuming an interior solution $L_t < 1$,

$$\begin{aligned} 0 &= U_{C,t} - \lambda_t Y_t U_{CC,t} - \mu_t \\ 0 &= (1+\lambda_t) U_{L,t} + \lambda_t (L_t - H_t) U_{LL,t} - \zeta_t \end{aligned}$$

Thus

$$\mu_t = U_{C,t} - \lambda_t Y_t U_{CC,t} \tag{D10}$$

and

$$L_t = \min\{1, [1 + \eta(\lambda_t)] H_t\}$$

with excess supply function $\eta(\lambda) \equiv \left(\frac{\lambda\sigma}{1+\lambda} - 1\right)^{-1}$ which decreases in λ;

The envelope conditions from (47) are

$$\begin{aligned} \mathcal{L}_H^{DD}(H_t, Z_t) &= -\lambda_t \left(U_{L,t}\Delta t + \mathrm{e}^{-\rho\Delta t}\mathcal{U}_{L,t+\Delta t}\right) = -\lambda_t \mathcal{U}_{L,t} \\ \mathcal{L}_Z^{DD}(H_t, Z_t) &= \mu_t \left[Y_t' \Delta t + (1 - \delta\Delta t)\right] - \lambda_t Y_t' U_{C,t} \Delta t \end{aligned}$$

The optimality conditions w.r.t. $H_{t+\Delta t}, Z_{t+\Delta t}$ of (47) are

$$\begin{aligned} 0 &= \mathrm{e}^{-\rho\Delta t}\mathcal{L}_{H,t+\Delta t}^{DD} + \lambda_t \mathrm{e}^{-\rho\Delta t}\mathcal{U}_{L,t+\Delta t} + \mathrm{e}^{-\rho\Delta t}\lambda_t \Delta H_{t+\Delta t}\frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial H_{t+\Delta t}} - \xi_t \\ &= \mathrm{e}^{-\rho\Delta t}\left[(-\lambda_{t+\Delta t} + \lambda_t)\mathcal{U}_{L,t+\Delta t} + \lambda_t \Delta H_{t+\Delta t}\frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial H_{t+\Delta t}}\right] - \xi_t \end{aligned} \tag{D11}$$

$$\begin{aligned} 0 &= \mathrm{e}^{-\rho\Delta t}\mathcal{L}_{Z,t+\Delta t}^{DD} - \mu_t + \mathrm{e}^{-\rho\Delta t}\lambda_t \Delta H_{t+\Delta t}\frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial Z_{t+\Delta t}} \\ &= \mathrm{e}^{-\rho\Delta t}\left\{\mu_{t+\Delta t}\left[Y_{t+\Delta t}'\Delta t + (1 - \delta\Delta t)\right] - \lambda_{t+\Delta t} Y_{t+\Delta t}' U_{C,t+\Delta t}\Delta t + \lambda_t \Delta H_{t+\Delta t}\frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial Z_{t+\Delta t}}\right\} - \mu_t \end{aligned} \tag{D12}$$

In simplifying last two expressions, we have plugged in the envelope conditions.

When $H_{t+\Delta t} < 1$ and thus $\xi_t = 0$, the limit of $\Delta t \to 0$ of $\mathrm{e}^{\rho\Delta t}\times$(D11) is

$$0 = -\frac{\dot{\lambda}_t}{\lambda_t}\mathcal{U}_{L,t} + \dot{H}_t\left(\frac{\partial \mathcal{U}_L}{\partial H}\right)_t$$

which implies

$$\dot{H}_t = \frac{\frac{\dot{\lambda}_t}{\lambda_t}\mathcal{U}_{L,t}}{\left(\frac{\partial \mathcal{U}_L}{\partial H}\right)_t} \tag{D13}$$

The limit of $\Delta t \to 0$ of $e^{\rho \Delta t} \times$(D12) is

$$\begin{aligned}
0 &= \mu_{t+\Delta t}\left[Y'_{t+\Delta t}\Delta t + (1-\delta\Delta t)\right] - \lambda_{t+\Delta t} Y'_{t+\Delta t} U_{C,t+\Delta t}\Delta t + \lambda_t \Delta H_{t+\Delta t}\frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial Z_{t+\Delta t}} - \mu_t e^{\rho\Delta t} \\
&= \mu_{t+\Delta t}\Delta t\left(Y'_{t+\Delta t} - \delta\right) + \dot{\mu}_t \Delta t - \rho\mu_t\Delta t - \lambda_{t+\Delta t} Y'_{t+\Delta t} U_{C,t+\Delta t}\Delta t + \lambda_t \dot{H}_t \Delta t \frac{\partial \mathcal{U}_{L,t+\Delta t}}{\partial Z_{t+\Delta t}} \\
0 &= \mu_t\left(Y'_t - \rho - \delta\right) + \dot{\mu}_t - \lambda_t Y'_t U_{C,t} + \lambda_t \dot{H}_t \left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t
\end{aligned}$$

Plugging in (D10) and then (D13), we get

$$\begin{aligned}
0 &= \left(U_{C,t} - \lambda_t Y_t U_{CC,t}\right)\left(Y'_t - \rho - \delta\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(U_{C,t} - \lambda_t Y_t U_{CC,t}\right) - \lambda_t Y'_t U_{C,t} + \lambda_t \dot{H}_t \left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t \\
&= C_t^{-1}\left(1 + \lambda_t \chi_t^{-1}\right)\left(Y'_t - \rho - \delta\right) - C_t^{-2}\dot{C}_t\left(1 + \lambda_t\chi_t^{-1}\right) + C_t^{-1}\dot{\lambda}_t\chi_t^{-1} - C_t^{-1}\lambda_t\chi_t^{-2}\dot{\chi}_t - \lambda_t Y'_t U_{C,t} + \lambda_t \dot{H}_t\left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t \\
\frac{\dot{C}_t}{C_t} &= \left[Y'_t(Z_t) - (\delta+\rho)\right] - \frac{\lambda_t}{\lambda_t + \chi_t}\left[\chi_t Y'_t(Z_t) + \frac{\dot{\chi}_t}{\chi_t} - \frac{\dot{\lambda}_t}{\lambda_t} - \chi_t C_t \dot{H}_t \left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t\right] \\
&= \left[Y'_t(Z_t) - (\delta+\rho)\right] - \frac{\lambda_t}{\lambda_t + \chi_t}\left[\chi_t Y'_t(Z_t) + \frac{\dot{\chi}_t}{\chi_t} - \frac{\dot{\lambda}_t}{\lambda_t} - \chi_t C_t \frac{\mathcal{U}_{L,t}}{(\partial \mathcal{U}_L/\partial H)_t}\frac{\dot{\lambda}_t}{\lambda_t}\left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t\right] \\
&= \left[Y'_t(Z_t) - (\delta+\rho)\right] - \frac{\lambda_t}{\lambda_t + \chi_t}\left[\chi_t Y'_t(Z_t) + \frac{\dot{\chi}_t}{\chi_t} - \left(1 + \chi_t C_t \frac{\mathcal{U}_{L,t}\left(\partial \mathcal{U}_L/\partial Z\right)_t}{(\partial \mathcal{U}_L/\partial H)_t}\right)\frac{\dot{\lambda}_t}{\lambda_t}\right]
\end{aligned}$$

Last, the limit under $\Delta t \to 0$ of the time-t implementability condition is

$$\begin{aligned}
0 &= \left(C_t - Y_t(Z_t)\right) U_{C,t}\Delta t + \left(L_t - H_t\right) U_{L,t}\Delta t + e^{-\rho\Delta t}\Delta H_{t+\Delta t}\mathcal{U}_{L,t+\Delta t} \\
0 &= \left(C_t - Y_t(Z_t)\right) U_{C,t} + \left(L_t - H_t\right) U_{L,t} + \dot{H}_t \mathcal{U}_{L,t}
\end{aligned}$$

which simplifies to

$$\nu L_t^{-\sigma}\left(L_t - H_t\right) + \dot{H}_t \mathcal{U}_{L,t} = \frac{1}{\chi_t} - 1$$

□

Proof of Proposition 7. For simplicity I suppress the superscript DD when there is no ambiguity. If the steady state $L_\infty < 1$, then we must have $\lambda_\infty > 0$ and $L_\infty = \left[1 + \eta\left(\lambda_\infty\right)\right] H_\infty$ from (51). The fiscal budget (53) implies that

$$\nu\left(L_\infty\right)^{1-\sigma}\frac{\eta\left(\lambda_\infty\right)}{1+\eta\left(\lambda_\infty\right)} = \frac{1}{\chi_\infty} - 1 \tag{D14}$$

Now we seek a sufficient condition under which (D14) can never hold. First, note that the DD steady state χ_∞ follows the same expression as in Corollary 2 if we replace λ^* by λ_∞^{DD}. Thus the property in Corollary 2 implies that

$$\chi_\infty^{DD} > \chi_\infty^{FB} = \frac{\delta(1-\alpha)+\rho}{\delta+\rho}$$

and hence

$$\frac{1}{\chi_\infty^{DD}} - 1 < \frac{\alpha\delta}{(1-\alpha)\delta+\rho}$$

Second, the LHS (D14) decreases in both L_∞ and λ_∞, and thus

$$\nu(L_\infty)^{1-\sigma}\frac{\eta(\lambda_\infty)}{1+\eta(\lambda_\infty)} > \nu\frac{\eta(\infty)}{1+\eta(\infty)} = \frac{\nu}{\sigma}$$

There, (D14) never holds if

$$\frac{\alpha\delta}{(1-\alpha)\delta+\rho} \leq \frac{\nu}{\sigma}$$

That means L_∞ must be equal to one.□

Proof of Proposition 8. Though the Corollary is stated for a generic τ, note that we are studying a Markov Perfect Equilibrium and thus without loss of generality we can start our analysis at time τ, i.e. relabeling $t-\tau$ as t and focusing on later periods.

For simplicity I suppress the superscript DD. Assume that $\dot{Z}_t > 0$, $\left(\frac{\partial \mathcal{U}_L}{\partial Z}\right)_t < 0$ and $\left(\frac{\partial \mathcal{U}_L}{\partial H}\right)_t < 0$ for all t. Assume that $L_t \in \left(\frac{\sigma}{\sigma-1}H_s, 1\right)$ for all $t \geq s \geq 0$, and thus (51) takes the interior solution $L_t = [1+\eta(\lambda_t)]H_t$. Further, assume that χ_t is strictly monotonic and λ_t is twice differentiable.

Step 1: If $\dot{\lambda}$ has the same sign as $\dot{\chi}$ at some time s, they must have the same sign for all $t \geq s$. For any τ such that $\dot{\lambda}_\tau = 0$, plug (51, 52) into (53) and evaluate its time derivative at τ,

$$\frac{(\mathcal{U}_{L,\tau})^2}{(\partial \mathcal{U}_L/\partial H)_\tau}\frac{\ddot{\lambda}_\tau}{\lambda_\tau} = -\frac{\dot{\chi}_\tau}{(\chi_\tau)^2}$$

As $\left(\frac{\partial \mathcal{U}_L}{\partial H}\right)_\tau < 0$, $\ddot{\lambda}_\tau$ must have the same sign as $\dot{\chi}_\tau$. Thus $\dot{\lambda}$ crosses zero at most once, and after that point, $\dot{\lambda}$ must have the same sign as $\dot{\chi}$.

Step 2: $\dot{\lambda}_t$ must have the opposite sign from $\dot{\chi}_t$ for all t. From step 1, if $\dot{\lambda}_t$ ever has the same sign as $\dot{\chi}_t$ at some point, there must exists $s \geq 0$ such that $\dot{\lambda}_t$ has the same sign as $\dot{\chi}_t$ for all $t \geq s$. I show that this cannot happen. Integrate $e^{-\rho(s-t)} \times$(53) from such a time s to infinity and get the LHS as

$$\begin{aligned}
\mathcal{I}^s &\equiv \int_s^\infty e^{-\rho(t-s)} \left[U_{L,t} (L_t - H_t) + \mathcal{U}_{L,t} \dot{H}_t \right] dt \\
&= \int_s^\infty e^{-\rho(t-s)} U_{L,t} (L_t - H_t) \, dt + \int_s^\infty e^{-\rho(t-s)} \mathcal{U}_{L,t} dH_t \\
&= \int_s^\infty e^{-\rho(t-s)} U_{L,t} (L_t - H_t) \, dt - \mathcal{U}_{L,s} H_s - \int_s^\infty H_t d\left(e^{-\rho(t-s)} \mathcal{U}_{L,t} \right) \\
&= \int_s^\infty e^{-\rho(t-s)} U_{L,t} (L_t - H_s) \, dt
\end{aligned}$$

The intuition for this transformation is that future land sale is fairly priced too, the same as why H_t does not appear in the time-0 implementability condition (20) under SB allocation. Now we plug in the optimal land supply (51) and write

$$\mathcal{I}^s = \int_s^\infty e^{-\rho(t-s)} \iota_{t,s} dt$$

with $\iota_{t,s} = \nu \left[(1+\eta_t)^{1-\sigma} H_t^{1-\sigma} - (1+\eta_t)^{-\sigma} H_t^{-\sigma} H_s \right], \eta_t = \eta(\lambda_t)$. That implies that $\rho \mathcal{I}^s$ is a weighted average of $\iota_{t,s}$ over $t \geq s$. If $\iota_{t,s}$ is strictly decreasing (/increasing) for all $t \geq s$, we would have $\iota_{s,s}$ to be strictly larger (/smaller) than $\rho \mathcal{I}^s$. We take the derivative of $\iota_{t,s}$ w.r.t. t to get

$$\begin{aligned}
\frac{d\iota_{t,s}}{dt} &= \nu (1+\eta_t)^{-\sigma} H_t^{-\sigma} \left[(1-\sigma) \eta_t' \dot{\lambda}_t H_t + (1-\sigma)(1+\eta_t) \dot{H}_t + \sigma \frac{\eta_t' \dot{\lambda}_t}{1+\eta_t} H_s + \sigma \frac{\dot{H}_t}{H_t} H_s \right] \\
&= -\nu (\sigma - 1)(1+\eta_t)^{-\sigma} H_t^{-\sigma-1} \left[(1+\eta_t) H_t - \frac{\sigma}{\sigma-1} H_s \right] \left(\dot{H}_t + \frac{\eta_t' \dot{\lambda}_t}{1+\eta_t} H_t \right) \\
&= -\nu (\sigma - 1)(1+\eta_t)^{-\sigma} H_t^{-\sigma-1} \left(L_t - \frac{\sigma}{\sigma-1} H_s \right) \left(\frac{\mathcal{U}_{L,t}}{(\partial \mathcal{U}_L / \partial H)_t} \frac{1}{\lambda_t} + \frac{\eta_t'}{1+\eta_t} H_t \right) \dot{\lambda}_t
\end{aligned}$$

of which the last step uses (51) and (52). Since $L_t > \frac{\sigma}{\sigma-1} H_s$, $(\partial \mathcal{U}_L / \partial H)_t < 0$, $\eta' < 0$, and $\sigma > 1$, we observe that $\frac{d\iota_{t,s}}{dt}$ has the same sign as $\dot{\lambda}_t$. Thus if λ_t is strictly decreasing (/increasing) for all $t \geq s$, we would have $\iota_{s,s}$ to be strictly larger (/smaller) than $\rho \mathcal{I}^s$.

Note that $\mathcal{E}^s \equiv \int_s^\infty e^{-\rho(t-s)} \left(\frac{1}{\chi_t} - 1 \right) dt$ also implies that $\rho \mathcal{E}^s$ is a weighted average of $\frac{1}{\chi_t} - 1$

for all $t \geq s$. Thus if χ_t is strictly increasing (/decreasing), then $\frac{1}{\chi_s} - 1$ is strictly larger (/smaller) than $\rho \mathcal{E}^s$.

As $\mathcal{I}^s = \mathcal{E}^s$ holds for all s, we take the derivative w.r.t. s and get

$$\rho \mathcal{I}^s - \iota_{s,s} - \mathcal{U}_{L,s} \dot{H}_s = \rho \mathcal{E}^s - \left(\frac{1}{\chi_s} - 1 \right)$$

Now supposing that $\dot{\chi}_t > 0$ for all t, we would have $\rho \mathcal{E}^s - \left(\frac{1}{\chi_s} - 1 \right) < 0$ and thus $\rho \mathcal{I}^s - \iota_{s,s} - \mathcal{U}_{L,s} \dot{H}_s < 0$. If $\dot{\lambda}_t \geq 0$ for all t, we would have $\dot{H}_s \leq 0$ from (52) and $\iota_{s,s} \leq \rho \mathcal{I}^s$, which together imply $\rho \mathcal{I}^s - \iota_{s,s} - \mathcal{U}_{L,s} \dot{H}_s \leq 0$ — a contradiction! Thus if $\dot{\chi}_t > 0$ for all t, we must have $\dot{\lambda}_t < 0$ for all t. Similarly, we can prove that if $\dot{\chi}_t < 0$ for all t, we must have $\dot{\lambda}_t > 0$ for all t.

Step 3: $\dot{\chi}_t < 0$ for all t. Here I further suppress the time subscript when there is no ambiguity. Similar to the proof of Proposition 3, from (49), we have

$$g^\chi = \Upsilon' \frac{\chi^2}{\chi + \lambda} - [(1 - \alpha)\delta + \rho] - \frac{\lambda}{\chi + \lambda} g^\chi + \frac{\lambda}{\chi + \lambda} \left(1 + \chi C \frac{\mathcal{U}_L \left(\partial \mathcal{U}_L / \partial Z \right)}{\left(\partial \mathcal{U}_L / \partial H \right)} \right) g^\lambda$$

with $g^\chi \equiv \frac{\dot{\chi}}{\chi}, g^\lambda \equiv \frac{\dot{\lambda}}{\lambda}$. Let $K_t \equiv \chi_t C_t \frac{\mathcal{U}_{L,t} (\partial \mathcal{U}_L / \partial Z)_t}{(\partial \mathcal{U}_L / \partial H)_t} > 0$ and $\gamma \equiv \frac{\chi + 2\lambda}{\chi + \lambda} g^\chi - \frac{\lambda}{\chi + \lambda} (1 + K) g^\lambda$ which has the same sign as g^χ. We have

$$\begin{aligned}
\gamma &= \Upsilon' \frac{\chi^2}{\chi + \lambda} - [(1 - \alpha)\delta + \rho] \\
\dot{\gamma} &= \Upsilon'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + \Upsilon' \frac{(\chi + 2\lambda)\chi^2}{(\chi + \lambda)^2} g^\chi - \Upsilon' \frac{\lambda \chi^2}{(\chi + \lambda)^2} g^\lambda \\
&= \Upsilon'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + \Upsilon' \frac{(\chi + 2\lambda)\chi^2}{(\chi + \lambda)^2} g^\chi + \Upsilon' \frac{\chi^2}{\chi + \lambda} \frac{\gamma - \frac{\chi + 2\lambda}{\chi + \lambda} g^\chi}{1 + K} \\
&= \Upsilon'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + \Upsilon' \frac{\chi^2}{\chi + \lambda} \frac{\gamma}{1 + K} + \Upsilon' \frac{(\chi + 2\lambda)\chi^2}{(\chi + \lambda)^2} g^\chi \frac{K}{1 + K}
\end{aligned}$$

Next I show that $g^\chi < 0$ cannot hold. Suppose for contradiction that it holds, then $\gamma < 0$ and

$$\dot{\gamma} < \Upsilon'' \dot{Z} \frac{\chi^2}{\chi + \lambda} + \Upsilon' \frac{\chi^2}{\chi + \lambda} \frac{\gamma}{1 + K}$$

whose RHS is less than zero too. Thus γ will keep decreasing, which contradicts conver-

gence. Therefore, we conclude that $\dot{\chi} > 0$. Consequently, from step 2, $\dot{\lambda} < 0$. (52) suggests that $\dot{H} > 0$. As H increases over time and $\eta(\lambda)$ decreases in λ and thus increases in time, (51) implies that $\dot{L} > 0$.□

Proof of Proposition 9. The implementability constraint is

$$0 = \int_s^\infty e^{-\rho(t-s)} \left[(C_t - Y_t) U_{C,t} + \left(L_t - H_0 - F_s^t - g_s \Gamma(Y_t) \right) U_{L,t} \right] dt, \tag{D15}$$

I construct the Lagrangian as

$$\begin{aligned} \mathcal{L}^{s*} = \max & \int_s^\infty e^{-\rho(t-s)} U(C_t, L_t)\, dt + \int_s^\infty e^{-\rho(t-s)} \left\{ \mu_t^{s*} \left[Y(Z_t) - C_t - (\delta + \rho) Z_t \right] + \dot{\mu}_t^{s*} Z_t \right\} dt + \mu_s^{s*} Z_s \\ & + \int_s^\infty e^{-\rho(t-s)} \zeta_t^{s*} (1 - L_t)\, dt + \lambda^{s*} \int_s^\infty e^{-\rho(t-s)} \left[(C_t - Y(Z_t) - T_t) U_{C,t} + \left(L_t - F_s^t - g_s \Gamma(Z_t) \right) U_{L,t} \right] dt \end{aligned} \tag{D16}$$

The optimality conditions (22-24) become, for any $t \geq s$,

$$0 = U_{C,t} - \lambda^{s*} Y_t U_{CC,t} - \mu_t^{s*} \tag{D17}$$

$$0 = (1 + \lambda^{s*}) U_{L,t} + \lambda^{s*} \left(L_t - H_0 - F_s^t - g_s \Gamma(Y_t) \right) U_{LL,t} - \zeta_t^{s*} \tag{D18}$$

$$0 = \mu_t^{s*} \left[Y_t' - (\delta + \rho) \right] + \dot{\mu}_t^{s*} - \lambda^{s*} \left(Y_t' U_{C,t} + g_s \Gamma_t' U_{L,t} \right) \tag{D19}$$

We need (D19) and (D15) both to hold, which means

$$H_0 + F_s^t + g_s \Gamma(Z_t) = \frac{L^*}{1 + \eta(\lambda^{s*})} \tag{D20}$$

and

$$\mathcal{I}(\lambda^{s*}; L^*) = \mathcal{E}^{s*}$$

with $\mathcal{I}(\lambda^{s*}; L^*) \equiv \frac{\nu (L^*)^{1-\sigma}}{\rho} \frac{1 + \lambda^{s*}}{\lambda^{s*} \sigma}$. This latter equation pins down $\lambda^{s*} > \lambda^*$ and the former one determines F_s^t for any $g_s \Gamma(Z_t)$. Then we need to find $g_s \Gamma(Z_t)$ that satisfies (D17) and (D19).

Define an integrating factor $m_t \equiv e^{\int_0^t [Y_\tau' - (\delta + \rho)] d\tau}$ and we have, using (D19)

$$\begin{aligned} \frac{d}{dt} \left(m_t \frac{\mu_t^{s*}}{\lambda^{s*}} \right) &= \left[Y_t' - (\delta + \rho) \right] m_t \frac{\mu_t^{s*}}{\lambda^{s*}} + m_t \frac{\dot{\mu}_t^{s*}}{\lambda^{s*}} \\ &= m_t \left(Y_t' U_{C,t} + g_s \Gamma_t' U_{L,t} \right) \end{aligned}$$

for all s. Take the difference between a generic s and $s = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_t \frac{\mu_t^{s*}}{\lambda^{s*}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(m_t \frac{\mu_t^{*}}{\lambda^{*}}\right) = m_t \Upsilon_t' H_s^F F_t' U_{L,t}$$

$$m_t \frac{\mu_t^{s*}}{\lambda^{s*}} - m_s \frac{\mu_s^{s*}}{\lambda^{s*}} = m_t \frac{\mu_t^{*}}{\lambda^{*}} - m_s \frac{\mu_s^{*}}{\lambda^{*}} + g_s \int_s^t m_\tau \Gamma_\tau' U_{L,\tau} \mathrm{d}\tau$$

Rewrite (D17) as

$$0 = \frac{U_{C,t} - \mu_t^{s*}}{\lambda^{s*} U_{CC,t}} - \Upsilon_t$$

and take the difference between a generic s and $s = 0$ to get

$$\begin{aligned} 0 &= \frac{U_{C,t} - \mu_t^{s*}}{\lambda^{s*} U_{CC,t}} - \frac{U_{C,t} - \mu_t^{*}}{\lambda^{*} U_{CC,t}} = -C_t^*\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + (C_t^*)^2 \left(\frac{\mu_t^{s*}}{\lambda^{s*}} - \frac{\mu_t^{*}}{\lambda^{*}}\right) \\ &= -C_t^*\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + (C_t^*)^2 m_t^{-1}\left[m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}}\right) + H_s^F \int_s^t m_\tau \Gamma_\tau' U_{L,\tau} \mathrm{d}\tau\right] \\ 0 &= -\frac{m_t}{C_t^*}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}}\right) + g_s \int_s^t m_\tau \Gamma_\tau' U_{L,\tau} \mathrm{d}\tau \end{aligned} \tag{D21}$$

Evaluate (D21) at $t = s$

$$0 = -\frac{m_s}{C_s^*}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right) + m_s\left(\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}}\right)$$

which determines μ_s^{s*} with $\frac{\mu_s^{s*}}{\lambda^{s*}} - \frac{\mu_s^{*}}{\lambda^{*}} < 0$ since $\lambda^{s*} > \lambda^{*}$.

Take the derivative of (D21) w.r.t. t

$$0 = \frac{m_t}{C_t^*}\left(\frac{1}{\lambda^{s*}} - \frac{1}{\lambda^{*}}\right)\left(\frac{\dot{C}_t^*}{C_t^*} - \frac{\dot{m}_t}{m_t}\right) + g_s m_t \Gamma_t' U_{L,t}$$

In order for this to hold, we need

$$m_t \Gamma_t' U_{L,t} = k \frac{m_t}{C_t^*}\left(\frac{\dot{C}_t^*}{C_t^*} - \frac{\dot{m}_t}{m_t}\right)$$

holds with a positive constant k. Then

$$g_s = k^{-1}\left(\frac{1}{\lambda^{*}} - \frac{1}{\lambda^{s*}}\right)$$

We need

$$\begin{aligned}
\Gamma_t' U_{L,t} &= k\frac{1}{C_t^*}\left(\frac{\dot{C}_t^*}{C_t^*} - \left[Y'\left(Z_t^*\right) - \left(\delta+\rho\right)\right]\right) \\
&= -k\frac{\lambda^*}{\lambda^*+\chi_t^*}\left(\chi_t^* Y'\left(Z_t^*\right) + \frac{\dot{\chi}_t^*}{\chi_t^*}\right) \qquad \text{(D22)} \\
&= -k\frac{\lambda^*}{\lambda^*+\chi_t^*}\left(\chi_t^* Y'\left(Z_t^*\right) + Y'\left(Z_t^*\right)\frac{\left(\chi_t^*\right)^2}{\chi_t^*+2\lambda^*} - \frac{\chi_t^*+\lambda^*}{\chi_t^*+2\lambda^*}\left[\left(1-\alpha\right)\delta+\rho\right]\right) \\
&= -k\frac{\lambda^*}{\lambda^*+2\chi_t^*}\left[2\chi_t^* Y'\left(Z_t^*\right) - \left[\left(1-\alpha\right)\delta+\rho\right]\right] \\
\Gamma_t' &= -k\nu^{-1}\left(L^*\right)^{\sigma}\frac{\lambda^*}{\lambda^*+2\chi_t^*}\left[2\chi_t^* Y'\left(Z_t^*\right) - \left[\left(1-\alpha\right)\delta+\rho\right]\right] \qquad \text{(D23)}
\end{aligned}$$

in which we first use (27) and then plug in (D4). (D22) implies that $\Gamma_t' < 0$ — the more output, the less land payment. (D20) then pins down F_s^t.□

Proof of Proposition B1. The household lifetime budget constraint is

$$\begin{aligned}
0 &= \int_0^{\infty} Q_t\left[C_t - Y_t\left(Z_t\right) + P_t\dot{H}_t + \tau^p P_t H_t + D_t\left(L_t - H_t\right) - T_t\right]\mathrm{d}t \\
&= \int_0^{\infty}\left\{Q_t\left[C_t - Y_t\left(Z_t\right) + D_t\left(L_t - H_t\right) + \tau^p P_t H_t - T_t\right] - H_t\frac{\mathrm{d}\left(P_t Q_t\right)}{\mathrm{d}t}\right\}\mathrm{d}t - P_0 H_0
\end{aligned}$$

Their consumption optimization implies

$$\begin{aligned}
Q_t &= \mathrm{e}^{-\rho t} U_{C,0}^{-1} U_{C,t} \\
D_t &= U_{C,t}^{-1} U_{L,t}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}\left(P_t Q_t\right)}{\mathrm{d}t} &= Q_t\left(\tau^p P_t - D_t\right) \\
P_t &= Q_t^{-1}\int_t^{\infty}\mathrm{e}^{-\tau^p(s-t)} Q_s D_s \mathrm{d}s
\end{aligned}$$

The present value of land tax income multiplied by $U_{C,0}$ is, assuming $H_t = L_t = 1$,

$$\begin{aligned} U_{C,0}\int_0^\infty \tau^p Q_t P_t \mathrm{d}t &= U_{C,0}\int_0^\infty \tau^p \int_t^\infty \mathrm{e}^{-\tau^p(s-t)} Q_s D_s \mathrm{d}s\mathrm{d}t \\ &= \nu \int_0^\infty \tau^p \int_t^\infty \mathrm{e}^{-\tau^p(s-t)}\mathrm{e}^{-\rho s}\mathrm{d}s\mathrm{d}t \\ &= \nu \frac{\tau^p}{\rho\left(\tau^p+\rho\right)} \end{aligned}$$

which is weakly less than $\frac{\nu}{\rho}$.

The present value of land sale/lease income multiplied by $U_{C,0}$ is, as in main text,

$$\nu\left(1-H_0\right)\frac{1}{\rho}$$

If $\nu\left[\left(1-H_0\right)+\frac{\tau^p}{\rho\left(\tau^p+\rho\right)}\right] < \int_0^\infty \mathrm{e}^{-\rho t}\left(\frac{1}{\chi_t^{FB}}-1\right)\mathrm{d}t$, FB cannot be achieved with land tax plus land sale/lease.□

E Theory Extensions

The paper offers a theoretical framework to study land finance and growth, which generates a declining public-investment-to-GDP ratio $(1-\chi_t)$ and a continuously rising land supply L_t. It implies that L_t and χ_t comove positively. Here I entertain a few extensions. In each of these cases, I characterize the optimal land supply and discuss if it can be rising over time or comoving positively with χ_t, without determining the path of χ_t. While this paper is mainly concerned with establishing the mechanism of land finance and its interaction with complete markets and commitment, it turns out that most of these extensions also make counterfactual predictions under complete markets and commitment.

Table E1: Summary of extensions

Section	Extension	Consistent w/ facts?	Details: land supply of second-best policy
E.1	Baseline under separable utility, w/ growth in population, productivity, and building height	No	Time-invariant
E.1	Baseline under non-separable utility, w/ growth in population, productivity, and building height	No	Negative comovement between L_t and χ_t, or $L_t = 1$, under popular functional form
E.2	Small open economy (SOE)	No	Time-invariant
E.3.1	Perpetual youth, with lump-sum redistribution across generations	No	Time-invariant
E.3.2	Perpetual youth, w/o lump-sum redistribution, in SOE	Potentially	$\dot{L}_0 > 0$ but the whole process is intractable
E.4	Land as factor of production	No	Negative comovement between L_t and χ_t under Cobb-Douglas production function
E.5	Land development cost	Not really	The amount of land consumed is rising, but the amount of land supplied is indeterminate
E.6	Urban externality	No	Negative comovement between L_t and χ_t, or $L_t = 1$, under popular functional form

E.1 Non-Separable Utility and Growth in Population, Productivity, and Building Height

In the baseline model, there is a representative household consuming non-durable c_t and floor area f_t. Population grows at rate n with newborns entering into the only dynasty in the economy. Buildings get higher on each parcel of land at exogenous rate g. More broadly, g can capture the improvement in land quality and thus f_t means the quality-adjusted land consumption. Household maximizes

$$\max \mathcal{U}_0 \equiv \int_0^\infty \mathrm{e}^{-(\rho-n)t} u\left(c_t, f_t\right) \mathrm{d}t$$

with $f_t\left(L_t\right) = \mathrm{e}^{-(n-g)t} L_t, c_t\left(C_t\right) = \mathrm{e}^{-nt} C_t$, in which f_t is the per capita consumption of floor area. I use capital letters for aggregate variables, and lower-case ones for per capita quantities. When total land supply is L_t, each person consumes $l_t = \mathrm{e}^{-nt} L_t$ amount. The land constraints is still

$$L_t \leq 1 \tag{E1}$$

Production follows

$$Y_t = A_t Z_t^\alpha N_t$$

with growth in TFP and labor supply as $A_t = \mathrm{e}^{at}, N_t = \mathrm{e}^{nt}$. Output per capita is $y_t \equiv Y_t/N_t = A_t Z_t^{\alpha}$. The resource constraint is

$$Y_t = C_t + \dot{Z}_t + \delta Z_t \tag{E2}$$

Household optimality and budget constraint. The household life-time budget constraint is

$$\begin{aligned} 0 &= \int_0^{\infty} Q_t \left[C_t - Y_t\left(Z_t, N_t\right) + D_t\left(L_t - H_t\right) + P_t \dot{H}_t \right] \mathrm{d}t \\ &= \int_0^{\infty} Q_t \mathrm{e}^{nt} \left[c_t - y_t\left(Z_t\right) + D_t\left(\mathrm{e}^{-gt} f_t - \mathrm{e}^{-nt} H_t\right) \right] \mathrm{d}t - P_0 H_0 - \int_0^{\infty} H_t \mathrm{d}\left(P_t Q_t\right) \end{aligned}$$

whose second step integrates $Q_t P_t \dot{H}_t$ by parts. Household optimization determines state price Q_t, rent per unit of land D_t and price per unit of land P_t

$$\begin{aligned} Q_t &= \mathrm{e}^{-\rho t} u_{c,0}^{-1} u_{c,t} \\ D_t &= \mathrm{e}^{gt} u_{c,t}^{-1} u_{f,t} \\ P_t &= Q_t^{-1} \int_t^{\infty} Q_s D_s \mathrm{d}s = \mathrm{e}^{gt} u_{c,t}^{-1} \int_t^{\infty} \mathrm{e}^{-(\rho - g)(s-t)} u_{f,s} \mathrm{d}s \end{aligned}$$

Time preference ρ affects the state price Q_t, and building height growth rate g affects rent and price directly as each unit of land hosts more housing and indirectly through u_f. Plugging the prices into the present-value budget constraint, we arrive at the implementability constraint

$$\begin{aligned} 0 &= \int_0^{\infty} Q_t \left[C_t - Y_t\left(Z_t, N_t\right) + D_t\left(L_t - H_0\right) \right] \mathrm{d}t \\ 0 &= \int_0^{\infty} \mathrm{e}^{-(\rho - n)t} \left[\left(c_t - y_t\right) u_{c,t} + \left(L_t - H_0\right) \frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{f,t} \right] \mathrm{d}t \end{aligned} \tag{E3}$$

In the first equation, H_t drops out since housing is priced fairly. We multiply that by $u_{c,0}$ to get the second expression in unit of utility. $\frac{\mathrm{d}f_t}{\mathrm{d}L_t} = \mathrm{e}^{-(n-g)t}$ captures the rate at which time-t land supply transforms into time-t housing.

Ramsey planning. The Ramsey planning problem is to pick $\{C_t, Z_t, L_t\}$ to maximize household utility, subject to (E1, E2, E3)

Proposition E1. (Baseline theory) *The optimal land supply satisfies, if it is interior ($L_t < 1$),*

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_{f,t} + \left(\frac{1}{\chi_t} - 1\right)\sigma_{cf,t} \tag{E4}$$

with $\sigma_{f,t} \equiv -f_t \frac{u_{ff,t}}{u_{f,t}} > 0, \sigma_{cf,t} \equiv c_t \frac{u_{cf,t}}{u_{f,t}}$. The amount of land in private hand H_t is indeterminate for all $t > 0$.

Under separable isoelastic utility, $\sigma_{f,t} \equiv \sigma, \sigma_{cf,t} \equiv 0$, the optimal land supply is time-invariant as

$$L_t = \left[1 + \eta\left(\lambda^*\right)\right] H_0 \tag{E5}$$

with $\eta\left(\lambda^*\right) = \left(\frac{\lambda^*\sigma}{1+\lambda^*} - 1\right)^{-1}$ decreasing in λ^*.

The proof is short and worth going through, which will also set the stage for all the extensions, before we discuss the results.

Proof of Proposition E1. We form the Lagrangian of Ramsey planning, by attaching multiplier λ^* to the implementability constraint (E3), ζ_t^* to the land constraint, μ_t^* to the resource constraint as follows

$$\begin{aligned}\mathcal{L}^* = & \max_{\{C_{t\geq 0}, L_{t\geq 0}, H_{t>0}, Z_{t>0}\}} \int_0^\infty \mathrm{e}^{-(\rho-n)t} u\left(c_t, f_t\right) \mathrm{d}t \\ & + \lambda^* \int_0^\infty \mathrm{e}^{-(\rho-n)t} \left[\left(c_t - y_t\right) u_{c,t} + \left(L_t - H_0\right) \frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{f,t}\right] \mathrm{d}t + \int_0^\infty \mathrm{e}^{-(\rho-n)t} \zeta_t^* \left(1 - L_t\right) \mathrm{d}t \\ & + \int_0^\infty \mathrm{e}^{-(\rho-n)t} \mu_t^* \left(Y_t - C_t - \dot{Z}_t - \delta Z_t\right) \mathrm{d}t\end{aligned} \tag{E6}$$

H_t is indeterminate as housing is fairly priced. Now we only characterize the optimal land supply

$$0 = \mathrm{e}^{(\rho-n)t} \frac{\partial \mathcal{L}^*}{\partial L_t} = \left(1 + \lambda^*\right) \frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{f,t} + \lambda^* \left(c_t - y_t\right) \frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{cf,t} + \lambda^* \left(L_t - H_0\right) \left(\frac{\mathrm{d}f_t}{\mathrm{d}L_t}\right)^2 u_{ff,t} - \zeta_t^* \tag{E7}$$

The μ_t^* multiplier is irrelevant as supply of land is independent of the production of non-durable good.

Let $\sigma_{f,t} \equiv -f_t \frac{u_{ff,t}}{u_{f,t}} > 0, \sigma_{cf,t} \equiv c_t \frac{u_{cf,t}}{u_{f,t}}$, we simplify (E7) to, if L_t is interior ($\zeta_t^* = 0$),

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_{f,t} + \left(\frac{1}{\chi_t} - 1\right)\sigma_{cf,t}$$

with $\chi_t \equiv c_t/y_t = C_t/Y_t$. This further simplifies to (E5) when $\sigma_{f,t} \equiv \sigma, \sigma_{cf,t} \equiv 0$.□

To build intuition, first acknowledge that the time-0 market value of private land $P_0 H_0$ enters negatively into the Ramsey planning Lagrangian with multiplier λ^*, which measures the social welfare of public fund. It is as if the government buys back private land at time 0. Now think about the tradeoff of the government to lower time-t land supply by ϵ amount which would raise the time-t rent, embedded in (E7). This perturbation has a few implications. It directly lowers utility and thus social welfare by $\frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{f,t}\epsilon$, which is part of the first term. When utility is non-separable, it also indirectly changes the state price $u_{c,t}$ (i.e., the price of investment) by $\frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{cf,t}\epsilon$, which matters for the government budget constraint. This perturbation further changes government income in present value in the following ways: lowering land supply by ϵ lowers government profits by $D_t\epsilon$ holding fixed rent (the $\lambda^* \frac{\mathrm{d}f_t}{\mathrm{d}L_t} u_{f,t}$ part of the first term), but it marginally raises rent (the $u_{ff,t}$ term). As rent goes up, the government gets more money on L_t amount supplied, but it also pushes up price the government "as if" pays to buy back private land H_0. The optimal land supply balances these force, characterized in (E4).

When utility is separable and isoelastic, we have

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_f$$

leading to a time-invariant L_t that declines in λ^*, simplified to (E5). There is an interior solution L_t when σ is larger than one (i.e. inelastic housing demand) and λ^* is not too small. This is the neutral benchmark the paper adopts. Starting from there, I show that adding financial constraints and discretion tilts the path of L_t to be increasing.

Now we consider some other possibilities. From (E5), for L_t to increase over time, specific time variation of elasticities $\sigma_{f,t}, \sigma_{cf,t}$ is needed. While this is a possibility, so far we do not have great evidence on the shape of utility function other than some local estimates. One commonly used functional form, for example by Berger et al. (2018), specifies

$$u(c, f) = \frac{\left(c^\phi f^{1-\phi}\right)^{1-\sigma} - 1}{1-\sigma}$$

supported by empirical literature that finds an elasticity between housing and non-durable consumption close to one. In this case, we have constant elasticities $\sigma_f = \left[1-\left(1-\phi\right)(1-\sigma)\right] > 0, \sigma_{cf} = \phi(1-\sigma)$. $\sigma = 1$ means utility is indeed separable. If $\sigma > 1$, which is the common calibration, then $\sigma_{cf} < 0$ and (E5) implies that L_t moves in the opposite direction as χ_t, at odds with the facts. The intuition is that, the government supplies more land L_t to lower the state price $u_{c,t}$ at times of high investment to GDP ratio (low χ_t) to make investment cheaper. If $\sigma < 1$, then σ_f and σ_{cf} are both positive but less than one. Though in this case L_t does comove positively with χ_t, this combination appears incompatible with (E5) as an equality: its LHS $\frac{1+\lambda^*}{\lambda^*}$ is larger than one, $1-\frac{H_0}{L_t}$ is less than one, and $\frac{1}{\chi_t}-1$ is a number that is much smaller than one in reality.[14] If the LHS of (E5) is larger than the RHS, that means the land constraint $L_t \leq 1$ is binding. That is, the government supplies all the land to economy.[15]

The role of growth in technology, population, and building height. The supply of land characterized in (E5) does not explicitly depend on the TFP growth rate a, population growth rate n, or the rate g at which buildings get taller, except through λ^*. The irrelevance of TFP growth rate is the most obvious, as it affects the production side only. The reason population growth rate is absent is that (E5) is strictly about elasticities, which govern the relative strengths of different implications from perturbing land supply as analyzed before. It is not about the level of utility. The housing consumption f_t can be rising or declining under constant L_t depending on whether g is larger or smaller than n. As population grows, each person will have a smaller piece of land $l_t = \mathrm{e}^{-nt}L_t$, but they also divide the initial private land H_0 among themselves into thinner pieces $h_0 = \mathrm{e}^{-nt}H_0$. As a result, the proportion of per capita land consumption l_t to endowment h_0 is the same as the aggregate proportion L_t/H_0. The logic with growth of building height g is similar. If buildings are getting taller on all land including land H_0 that is initially owned by the household, the proportion of per capita floor consumption $\mathrm{e}^{gt}l_t$ to endowment $\mathrm{e}^{gt}h_0$ is the same as L_t/H_0. If buildings are getting taller only on land owned by the government at

[14] The typical infrastructure investment to GDP ratio is about 5% for developed countries and slightly over 10% for Asian growth miracles during their high growth periods. That means χ_t is close than one, and we have $\frac{1}{\chi_t}-1 \approx 1-\chi_t$, which is the infrastructure investment to GDP ratio.

[15] In this case, the reason $L_t = 1$ binds is that demand is elastic under $\sigma < 1$ and lowering land supply does not increase government profits.

time 0,[16] then (E5) changes to

$$L_t = [1 + \eta(\lambda^*)] H_0 e^{-gt}$$

which prescribes a declining path of L_t. This is because the government aims at a constant proportion of total floor area supplied to endowed. If the added supply is of better quality (higher buildings per unit of land), then the government does not need to supply as much.

E.2 Small Open Economy

Now I consider a small open economy with a representative household instead of a closed economy. Let there be a frictionless international capital market with interest rate r. Assume time preference $\rho = r$; otherwise there is no steady state. Instead of having a resource constraint related to production $Y_t = C_t + \dot{Z}_t + \delta Z_t$ at each point in time in a closed economy, there is just one constraint which equates the present value of consumption and investment to that of production

$$0 = \int_0^\infty e^{-rt} \left(Y_t - C_t - \dot{Z}_t - \delta Z_t \right) dt \tag{E8}$$

The imbalance $Y_t - C_t - \dot{Z}_t - \delta Z_t$ at any point of time is made up for by imports and exports.

In this economy, since there is a constant interest rate r, housing is priced by it too. As before, the housing ownership H_t for $t > 0$ is indeterminate.

First best. The government maximizes social welfare subject to (E8) and land constraints only

$$\begin{aligned}
\mathcal{L}^{FB} &= \max \int_0^\infty e^{-\rho t} U(C_t, L_t)\, dt + \mu^{FB} \int_0^\infty e^{-rt} \left(Y_t - C_t - \dot{Z}_t - \delta Z_t \right) dt + \int_0^\infty e^{-\rho t} \zeta_t^{FB} (1 - L_t)\, dt \\
&= \max \int_0^\infty e^{-\rho t} U(C_t, L_t)\, dt + \mu^{FB} \int_0^\infty e^{-rt} \left(Y_t - C_t - (r + \delta) Z_t \right) dt + \mu^{FB} Z_0 + \int_0^\infty e^{-\rho t} \zeta_t^{FB} (1 - L_t)\, dt
\end{aligned}$$

[16] This can be a very crude proxy for the idea that time-0 private land is already built up, whereas the land owned by the government is empty. We will see a more serious extension on land development cost later.

leading to

$$Y_t' = r + \delta$$
$$L_t = 1$$

The economy converges instantly to an infrastructure level $Z^{FB} = (Y')^{-1}(r+\delta)$. If the economy starts with $Z_0 < Z^{FB}$, then there is a jump in Z_t at $t = 0$. This jump is possible since the economy borrows and buys from the international market and later repays the debt. The first-best consumption C_t^{FB} is a constant over time (since $\rho = r$) and satisfies (E8). This instant convergence is known in growth textbooks (Barro and Sala-i-Martin, 2004) and can be slowed down by adding frictions, such as constraints on international credit or capital adjustment cost.

Second best. For the second best, the government maximizes welfare subject to (E8) and land constraints, as well as the household optimality and budget constraint. The government budget constraint is implied by Walras' law.

Proposition E2. (Small open economy) *The optimal land supply satisfies, if it is interior*

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_{L,t} + \left(\frac{1}{\chi_t} - 1\right)\sigma_{CL,t} \tag{E9}$$

with $\sigma_{L,t} \equiv -L_t \frac{u_{LL}(C_t,L_t)}{u_L(C_t,L_t)} > 0, \sigma_{CL,t} \equiv C_t \frac{u_{CL}(C_t,L_t)}{u_L(C_t,L_t)}$ *as in the closed economy. And the infrastructure stock instantly jumps to* $Z^* = (Y')^{-1}\left(\frac{r+\delta}{1-\lambda^* U_C/\mu^*}\right)$ *at time* 0.

In particular, if utility is separable and isoelastic, then land supply is constant as

$$L_t = [1 + \eta(\lambda^*)] H_0 \tag{E10}$$

with $\eta(\lambda^*) = \left(\frac{\lambda^*\sigma}{1+\lambda^*} - 1\right)^{-1}$.

Under separable isoelastic utility, the land supply is constant because given the amount of money the government needs to raise, it does it in the least distortionary way. The logic is the same as the closed economy case, even though the government may have a different λ^* in the open economy case. If the economy starts with infrastructure stock $Z_0 < Z^*$, then there is a discrete jump in infrastructure stock at time 0 (a Dirac delta investment flow). This is still directly consistent with our fact that countries invest in infrastructure early on, albeit in an extreme form.

E.3 Perpetual Youth

Here I entertain a perpetual youth model in the style of Yaari (1965) and Blanchard (1985), where there is no altruistic link between generations and Ricardian equivalence does not hold. I shut down technological growth and growth of land quality (growth of floor area per unit of land) for simplicity with $A_t = A$, $f_t = l_t$. Suppose the death rate is x and the birth rate is the same, to maintain a constant population of size one. Based on the discussion we have had, I do not expect any of these normalizations matters qualitatively.

The key departure from the baseline model is that newborns are separate households rather than entering into old dynasties. Use superscript s to denote cohort born at time s. The measure of the people born at time $s < t$ who are alive at time t is

$$N_t^s = x\mathrm{e}^{-x(t-s)}$$

which is also the measure of households with age $t - s$. At time 0, suppose people born at time $s < 0$ who are still alive each own h_0^s amount of land, with $H_0 = \int_{-\infty}^{0} h_0^s N_0^s \mathrm{d}s$ denoting total endowment, and have no debt. Assume that no future cohorts are born with any land ($h_s^s = 0$ for all $s \geq 0$) or debt. At each time t, everyone alive earns the same labor income $y_t = AZ_t^\alpha$. We have $Y_t = y_t$ since there is unit measure of households alive.

Now the Ramsey planning requires the government to pick allocation for each generation, subject to their optimality conditions and budget constraints. I start by characterizing and defining the Ramsey problem. The constraints on aggregate resources and allocation are

$$C_t = \int_0^\infty x\mathrm{e}^{-x\tau} c_t^{t-\tau} \mathrm{d}\tau \tag{E11}$$

$$L_t = \int_0^\infty x\mathrm{e}^{-x\tau} l_t^{t-\tau} \mathrm{d}\tau \tag{E12}$$

$$H_t = \int_0^\infty x\mathrm{e}^{-x\tau} h_t^{t-\tau} \mathrm{d}\tau \tag{E13}$$

$$Y_t = C_t + \delta Z_t + \dot{Z}_t \tag{E14}$$

$$H_t, L_t \in [0, 1] \tag{E15}$$

Following Calvo and Obstfeld (1988), the social welfare is, taking into account cohorts

born before time 0,

$$\mathcal{W}_0 = \int_{-\infty}^{\infty} \mathrm{e}^{-\rho s} x U_0^s \mathrm{d}s \tag{E16}$$

with $U_0^s = \int_s^{\infty} \mathrm{e}^{-(\rho+x)(t-s)} u\left(c_t^s, l_t^s\right) \mathrm{d}t$ for $s \geq 0$ and $U_0^s = \int_0^{\infty} \mathrm{e}^{-(\rho+x)(t-s)} u\left(c_t^s, l_t^s\right) \mathrm{d}t$ for $s < 0$. To simplify notation, introduce $s^+ \equiv \max(s, 0)$ as the starting point of cohort s's optimization when the government optimizes at time 0. We can write $U_0^s = \int_{s^+}^{\infty} \mathrm{e}^{-(\rho+x)(t-s)} u\left(c_t^s, l_t^s\right) \mathrm{d}t$ for all s.

Household optimality and budget constraints. Each agent born at time s maximizes

$$U_0^s = \int_{s^+}^{\infty} \mathrm{e}^{-(\rho+x)(t-s)} u\left(c_t^s, l_t^s\right) \mathrm{d}t \tag{E17}$$

with the probability of death x raising the effective discount rate.

Each agent's present-value budget constraint is[17]

$$0 = \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t - x P_t h_t^s + P_t \dot{h}_t^s + D_t \left(l_t^s - h_t^s\right)\right] \mathrm{d}t - \mathcal{T}^s \tag{E18}$$

with $\hat{Q}_t = \mathrm{e}^{-\int_0^t (r_\tau + x) \mathrm{d}\tau}$ denoting the state price using cumulative return a surviving household faces from time 0 to t, and $\mathcal{T}_t^s$ denoting the time-0 present value of government transfer to cohort s. The return surviving households face and thus the discount factor $\hat{Q}_t$ features an insurance premium x, which is paid by a perfectly competitive insurance firm in exchange for the right to receive dying households' wealth. In addition, the household receives a flow payment $x P_t h_t^s$ from the insurance firm for the claim to the houses of dying households, like a reverse mortgage. Differing from $\hat{Q}_t$, we define the risk-free state price $Q_t = \mathrm{e}^{-\int_0^t r_\tau \mathrm{d}t} = \mathrm{e}^{xt} \hat{Q}_t$.

For households to be indifferent between holding housing and bond, housing must be fairly priced by the same return r_t as bond,

$$P_t = Q_t^{-1} \int_t^{\infty} Q_\tau D_\tau \mathrm{d}\tau \tag{E19}$$

[17] To arrive at this present-value budget constraint, we use the flow budget constraint $\dot{b}_t^s = (r_t + x) b_t^s + x P_t h_t^s + y_t - c_t^s - D_t \left(l_t^s - h_t^s\right) - P_t \dot{h}_t^s$ and integrate $\hat{Q}_t b_t^s$ from $t = s^+$ to ∞.

otherwise they would buy or sell an infinite amount of housing given an unfair price.[18] Since the household is indifferent between holding housing and bond, we can set $h_t^s = h_{s^+}^s$ without loss of generality to get from (E18)

$$\begin{aligned}
0 &= \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t + D_t \left(l_t^s - h_{s^+}^s \right) - x P_t h_{s^+}^s \right] \mathrm{d}t - \mathcal{T}_{s^+}^s \\
&= \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t + D_t \left(l_t^s - h_{s^+}^s \right) \right] \mathrm{d}t - h_{s^+}^s \int_{s^+}^{\infty} x \int_{s^+}^{\tau} \mathrm{e}^{x(\tau - t)} \mathrm{d}t \hat{Q}_\tau D_\tau \mathrm{d}\tau - \mathcal{T}_{s^+}^s \qquad \text{(E20)} \\
&= \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t + D_t \left(l_t^s - \mathrm{e}^{x(t - s^+)} h_{s^+}^s \right) \right] \mathrm{d}t - \mathcal{T}_{s^+}^s \qquad \text{(E21)}
\end{aligned}$$

in which I plug in $\hat{Q}_t P_t = \mathrm{e}^{-xt} \int_t^{\infty} Q_\tau D_\tau \mathrm{d}\tau = \int_t^{\infty} \mathrm{e}^{x(\tau - t)} \hat{Q}_\tau D_\tau \mathrm{d}\tau$ and change the order of integral. The result is intuitive too: for households endowed with $h_{s^+}^s$ amount of land at time s^+ who are alive at t, it is as if they inherit the land from dying households in the same cohort and thus have $\mathrm{e}^{x(t - s^+)} h_{s^+}^s$ unit of land per person.

Given this constraint and the utility (E17), the optimal non-durable and housing consumption satisfy

$$\frac{\mathrm{e}^{-\rho t} u_{c,t}^s}{Q_t} = \frac{\mathrm{e}^{-\rho s^+} u_{c,s^+}^s}{Q_{s^+}} \qquad \text{(E22)}$$

$$u_{c,t}^s = \frac{u_{l,t}^s}{D_t} \qquad \text{(E23)}$$

(E22) does not feature x since its presence in the discount rate cancels off with the insurance premium in saving.

[18] To see this, we integrate (E18) by parts

$$\begin{aligned}
0 &= \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t - x P_t h_t^s + P_t \dot{h}_t^s + D_t \left(l_t^s - h_t^s \right) \right] \mathrm{d}t - \hat{Q}_{s^+} \mathcal{T}_{s^+}^s \\
&= \int_{s^+}^{\infty} \hat{Q}_t \left[c_t^s - y_t - x P_t h_t^s + D_t \left(l_t^s - h_t^s \right) \right] \mathrm{d}t - \hat{Q}_{s^+} P_{s^+} h_{s^+}^s - \int_{s^+}^{\infty} h_t^s \mathrm{d} \left(\hat{Q}_t P_t \right) - \hat{Q}_{s^+} \mathcal{T}_{s^+}^s
\end{aligned}$$

Since the constraint is linear in h_t^s, we must have

$$\begin{aligned}
\mathrm{d} \left(\hat{Q}_t P_t \right) &= -\hat{Q}_t \left(x P_t + D_t \right) \mathrm{d}t \\
\mathrm{d} \left(Q_t P_t \right) &= -Q_t D_t \mathrm{d}t
\end{aligned}$$

Ramsey planning. The government chooses $\left\{c^s_{t\geq s^+}, l^s_{t\geq s^+}, h^s_{t>s^+}, \mathcal{T}^s\right\}_{s\in\mathbb{R}}$ and $\{Q_{t>0}, D_{t\geq 0}, P_{t\geq 0}\}$ to maximize social welfare (E16), subject to constraints on aggregate resources and allocation (E11-E15), household budget constraints (E18), and household optimality conditions (E22, E23). The government budget constraint is implied by these conditions due to Walras' law. I will study two cases separately:

1. With lump-sum redistribution: following Calvo and Obstfeld (1988) and Itskhoki and Moll (2019) among others on optimal fiscal policy in canonical macro models (without land), I allow the government to use lump-sum taxes and transfers to redistribute among generations. However, the government cannot use lump-sum taxes to fund public investment. This leads to a clean separation between aggregate efficiency and distributional consideration of optimal policy.

2. Without lump-sum redistribution

E.3.1 With lump-sum redistribution

Changing variable from cohort s to age $\tau = t - s$, we can rewrite (E16) as

$$\mathcal{W}_0 = \int_0^\infty \mathrm{e}^{-\rho t} V_t \mathrm{d}t, \quad V_t = \int_0^\infty x\mathrm{e}^{-x\tau} u\left(c_t^{t-\tau}, l_t^{t-\tau}\right) \mathrm{d}\tau \tag{E24}$$

in which V_t represents the utility flow of all households alive at time t, aggregated across the age distribution with density $x\mathrm{e}^{-x\tau}$.

In studying the optimal fiscal policy, Calvo and Obstfeld (1988) and Itskhoki and Moll (2019) allow the government to redistribute among generations with lump-sum taxes and transfers so that the optimization can be separated into two stages: first, at each time t, allocate resources among households to statically maximize time-t instantaneous utility flow V_t; and second, choose the path of aggregate variables. To apply this insight in my setting, I assume that the government can use lump-sum taxes and transfers solely for redistribution purpose, but not for funding public investment, i.e.

$$0 = \int_{-\infty}^0 x\mathrm{e}^{xs}\mathcal{T}^s \mathrm{d}s + \int_0^\infty x\mathcal{T}^s \mathrm{d}s$$

Therefore, there is a separation between aggregate efficiency and distributional consideration of optimal policy.

Proposition E3. (Perpetual youth with redistribution) *The optimal land supply satisfies, if it is interior ($L_t < 1$)*

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_{L,t} + \left(\frac{1}{\chi_t} - 1\right)\sigma_{CL,t} \tag{E25}$$

with $\sigma_{L,t} \equiv -L_t \frac{u_{LL}(C_t,L_t)}{u_L(C_t,L_t)} > 0, \sigma_{CL,t} \equiv C_t \frac{u_{CL}(C_t,L_t)}{u_L(C_t,L_t)}$.

Under separable utility with constant elasticity ($\sigma_{L,t} \equiv \sigma, \sigma_{CL,t} \equiv 0$), when the optimal land supply $L_t < 1$, it is time-invariant as

$$L_t = [1 + \eta(\lambda^*)] H_0 \tag{E26}$$

with $\eta(\lambda^*) = \left(\frac{\lambda^*\sigma}{1+\lambda^*} - 1\right)^{-1}$.

The proof is in Appendix F. The intuition is pretty straightforward. As the government can redistribute among generations, the optimal allocation has $c_t^s = C_t, l_t^s = L_t$ for all $s \leq t$ at any time t. That is, the government equalizes consumption across agents. Thus there is as if a representative agent consuming aggregate C_t, L_t. One multiplier λ^* still measures the marginal social value of public fund, since the government cannot use lump-sum tax to fund infrastructure investment. The results here are the same as in the baseline theory with a representative household.

E.3.2 Without lump-sum redistribution in small open economy

Next I assume that the government cannot redistribute across generations, i.e. $\mathcal{T}^s \equiv 0$ in (E18). For tractability, I assume $u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} + \nu\frac{l^{1-\sigma}}{1-\sigma}$ (separable homothetic utility) and $h_0^s = H_0$ for all $s < 0$. Thus all households still alive at time 0 have the same land endowment and the same future labor income, whereas newborns have no land and start earning labor income and consumption starting from different birthtimes. Further, we study a small open economy with interest rate $r = \rho$ instead of a closed economy.

Proposition E4. (Perpetual youth without redistribution in small open economy) *The optimal land supply is rising at time* 0, *i.e.*, $\dot{L}_0 > 0$.

At a high level, this is because only the cohorts born before time 0 have land. Thus they are richer than households that are born immediately after time 0 without land, since they have the same labor income processes. As time goes from 0 to 0^+, the government is less eager to get money from households as the newborns are poor and have high marginal utilities. Thus the government supplies slightly more land to the economy. Note

that this logic only applies to the neighborhood near $t = 0$. To say anything about future land supply requires to pin down the aggregate consumption and investment paths and compare among generations born after time 0, which is a fairly intractable problem even under the various simplifying assumptions.

E.4 Land as Factor of Production

Now instead of having land as consumption, suppose the only use of land is as factor of production in this economy ($U_t = U(C_t)$ for household). Specifically, let the production function be

$$Y_t = AZ_t^{\alpha} y(N_t, L_t)$$

Still assume that labor is in fixed supply $N_t = 1$. The household buys or rents land from the government to use in production

$$\begin{aligned} &\max \int_0^{\infty} \mathrm{e}^{-\rho t} U(C_t)\,\mathrm{d}t \\ &\text{s. t.} \int_0^{\infty} Q_t \left[C_t - Y_t + P_t \dot{H}_t + D_t (L_t - H_t)\right] \mathrm{d}t = 0 \end{aligned}$$

leading to state price and rent of land

$$\begin{aligned} Q_t &= \mathrm{e}^{-\rho t} U_{C,0}^{-1} U_{C,t} \\ D_t &= Y_{L,t} \end{aligned}$$

Thus the implementability condition is, in which we set $H_t = H_0$ without loss of generality since housing is fairly priced,

$$0 = \int_0^{\infty} \mathrm{e}^{-\rho t} U_{C,t} \left[C_t - Y_t + (L_t - H_0) Y_{L,t}\right] \mathrm{d}t$$

Proposition E5. (Land as factor of production) *The optimal land supply satisfies, if it is interior ($L_t < 1$),*

$$0 = 1 + \lambda^* + \lambda^* \left(\frac{1}{\chi_t} - 1\right) \sigma_{C,t} - \lambda^* \frac{1}{\chi_t} \frac{L_t - H_0}{L_t} \sigma_{C,t} \epsilon_{L,t} - \lambda^* \frac{L_t - H_0}{L_t} \sigma_{L,t} \tag{E27}$$

with $\sigma_{C,t} = -\frac{C_t U_{CC,t}}{U_{C,t}} > 0, \sigma_{L,t} \equiv -\frac{L_t Y_{LL,t}}{Y_{L,t}} > 0, \epsilon_{L,t} = \frac{L_t Y_{L,t}}{Y_t} > 0$. The amount of land in private hand H_t is indeterminate for all $t > 0$.

Under isoelastic utility $U(C) = \frac{C^{1-\sigma_C}}{1-\sigma_C}$ *and Cobb-Douglas production function* $y_t = N_t^{\phi} L_t^{1-\phi}$, *the optimal land supply moves in the* opposite *direction as* χ_t, *following*

$$0 = 1 + \lambda^* - \lambda^* \sigma_C + \lambda^* \frac{\sigma_C}{\chi_t} \left[1 - \left(1 - \phi \right) \frac{L_t - H_0}{L_t} \right] - \lambda^* \phi \frac{L_t - H_0}{L_t} \tag{E28}$$

Similar to our analysis in E.1, time variation in land supply L_t can be driven by variations of elasticities $\sigma_{C,t}, \sigma_{L,t}, \epsilon_{L,t}$ over time. However, under the most commonly used functional forms (isoelastic utility and Cobb-Douglas production function), L_t moves in the opposite direction as χ_t, at odds with our facts.

E.5 Land Development Cost

Suppose that land needs to be developed under adjustment cost. In particular, assume that g amount of land can be freely converted from undeveloped land to developed land without cost, while any more development will be infinitely costly. This is a specific convex adjustment cost (zero below g and infinite above g) that maximally simplifies our analysis, but the same intuition would go through to less extreme cost functional form. Assume that the initial private land H_0 is already developed. Let L_t be the amount of developed land at each point in time and $\tilde{L}_t$ be the total amount of land made available through sale or lease to the household. The constraint on L_t is that

$$L_t \leq \min \left(\tilde{L}_t, H_0 + gt \right)$$

Since land is fairly priced as in all previous extensions, without loss of generality we set $H_t = H_0$ and think about land supply through lease only. Further, we assume separable isoelastic utility which leads to a constant supply of land in the baseline theory.

Proposition E6. (Land development cost) *The amount of developed land* L_t *is equal to* $H_0 + gt$ *before* $T = g^{-1} \eta(\lambda^*) H_0$ *and equal to* $[1 + \eta(\lambda^*)] H_0$ *after* T, *with* $\eta(\lambda^*) = \left(\frac{\lambda^* \sigma}{1+\lambda^*} - 1 \right)^{-1}$. *The amount of land available to the private sector* $\tilde{L}_t$ *can take any value in* $[L_t, 1]$ *before* T *and is equal to* $[1 + \eta(\lambda^*)] H_0$ *after* T.

This result suggests that a constraint on land development matters for the amount of developed land L_t, but not the amount of land available to the private sector $\tilde{L}_t$, before time T. When we introduce the distinction between developed vs undeveloped land, we note that both are increasing in reality. In particular, the data I have on government land

transfers is about $\tilde{L}_t$ rather than L_t, as in reality real estate developers get the land first from the government and then build on it. So this extension does not really explain the pattern of rising $\tilde{L}_t$, as there is a large indifference zone for $\tilde{L}_t$ before T.

Further, if there is an epsilon amount of benefit of having undeveloped land in private hands (for example, that could be used as collateral, or just that the private sector is better at figuring out which piece of land to develop first), that breaks the tie in favor of $\tilde{L}_t = 1$ before T.

E.6 Urban Externality

Now we consider a case with urban externality. Assume household takes density as given and maximizes consumption

$$\max_{c_t, l_t} \mathcal{U}_0 \equiv \int_0^\infty \mathrm{e}^{-(\rho-n)t} u\left(c_t, l_t; d_t\right) \mathrm{d}t$$

with $l_t = \mathrm{e}^{-nt} L_t$ denoting the per-capita land area and $d_t = \mathrm{e}^{nt}/L_t$ denoting the density. While we do not have much evidence on how the externality exactly enters into utility, a convenient functional form used by Owens, Rossi-Hansberg and Sarte (2020) is

$$u\left(c, l; d\right) = d^{\epsilon} c^{\phi} l^{1-\phi} \tag{E29}$$

in which ϵ parametrizes the degree of externality. In this extension, the benevolent government aims to fix two externalities: one related to infrastructure as public good funded by the government, and another related to the urban externality.

Proposition E7. (Urban externality) *The optimal land supply satisfies, if interior,*

$$0 = 1+\lambda^*-\lambda^*\frac{L_t - H_0}{L_t}\cdot\left(-\frac{l_t u_{ll,t}}{u_{l,t}}\right)+\lambda^*\left(1-\frac{1}{\chi_t}\right)\frac{c_t u_{cl,t}}{u_{l,t}}-d_t^2\left\{\frac{u_{d,t}}{u_{l,t}}+\lambda^*\left[\left(1-\frac{1}{\chi_t}\right)\frac{c_t u_{cd,t}}{u_{l,t}}+\frac{L_t - H_0}{L_t}\frac{l_t u_{ld,t}}{u_{l,t}}\right]\right\} \tag{E30}$$

Under functional form $u\left(c, l; d\right) = d^{\epsilon} c^{\phi} l^{1-\phi}$ (Owens, Rossi-Hansberg and Sarte, 2020), it simplifies to

$$0 = 1 + \lambda^* - \frac{\epsilon}{1-\phi} - \lambda^*\frac{L_t - H_0}{L_t}\left(\phi+\epsilon\right) - \lambda^*\left(1-\frac{1}{\chi_t}\right)\phi\left(\frac{\epsilon}{1-\phi}-1\right) \tag{E31}$$

Depending on whether ϵ is larger or smaller than $1-\phi$, L_t comoves negatively with χ_t or is always

equal to one.

If $\epsilon > 1 - \phi$,[19] then this expression decreases in both L_t and χ_t and thus they comove negatively, contradicting our facts.

If $\epsilon \leq 1 - \phi$,[20] then

$$1 + \lambda^* - \frac{\epsilon}{1-\phi} - \lambda^* \frac{L_t - H_0}{L_t}\left(\phi + \epsilon\right) - \lambda^* \left(1 - \frac{1}{\chi_t}\right)\phi\left(\frac{\epsilon}{1-\phi} - 1\right) \geq \lambda^* - \lambda^* \frac{L_t - H_0}{L_t} > 0$$

implying binding land constraint, i.e. $L_t = 1$, once again contradicting our facts.

F Proofs for Theory Extensions

Proof of Proposition E2. Household intertemporal optimization leads to $e^{-rt} = e^{-\rho t} U_{C,0}^{-1} U_{C,t}$. Under $\rho = r$, it has to be the case that U_C is a constant over time. If utility is separable, this means $C_t = C$; otherwise C_t and L_t comove in a way that keeps $U_{C,t}$ constant. In forming the Lagrangian, I attach multiplier μ^* to the aggregate resource constraint and λ^* to the household budget constraint expressed in quantities.

$$\begin{aligned}
\mathcal{L}^* &= \max \int_0^\infty e^{-\rho t} U\left(C_t, L_t\right) dt + \mu^* \int_0^\infty e^{-rt}\left(Y_t - C_t - \dot{Z}_t - \delta Z_t\right) dt \\
&+ \lambda^* \int_0^\infty e^{-\rho t}\left[\left(C_t - Y_t\right) U_C + \left(L_t - H_0\right) U_{L,t}\right] \\
&= \max \int_0^\infty e^{-\rho t} U\left(C_t, L_t\right) dt + \mu^* \int_0^\infty e^{-rt}\left(Y_t - C_t - \left(r + \delta\right) Z_t\right) dt + \mu^* Z_0 \\
&+ \lambda^* \int_0^\infty e^{-\rho t}\left[\left(C_t - Y_t\right) U_C + \left(L_t - H_0\right) U_{L,t}\right]
\end{aligned}$$

The FOCs are

$$\begin{aligned}
0 &= e^{\rho t} \frac{\partial \mathcal{L}^*}{\partial Z_t} = \mu^*\left[Y_t' - \left(r + \delta\right)\right] - \lambda^* Y_t' U_C \\
0 &= e^{\rho t} \frac{\partial \mathcal{L}^*}{\partial L_t} = \left(1 + \lambda^*\right) U_{L,t} + \lambda^* U_{LL,t}\left(L_t - H_0\right) + \lambda^*\left(C_t - Y_t\right) U_{CL,t}
\end{aligned}$$

[19] Owens, Rossi-Hansberg and Sarte (2020)'s calibration satisfies this, but their ϵ is about externality in a neighborhood, instead of economy-wide average density.

[20] While normally people suggest a net positive urban externality, this case accommodates negative externality ($\epsilon < 0$) as well.

which implies that the economy converges instantly to $Z^* = (Y')^{-1}\left(\frac{r+\delta}{1-\lambda^* U_C/\mu^*}\right)$ which is less than Z^{FB} when $\lambda^* > 0$. In terms of land supply, we still have

$$\frac{1+\lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right)\sigma_{L,t} + \left(\frac{1}{\chi_t} - 1\right)\sigma_{CL,t}$$

as in the closed economy. In particular, if utility is separable, then land supply is constant as

$$L_t = \left[1 + \eta\left(\lambda^*\right)\right] H_0$$

□

Proof of Proposition E3. For the first stage, the aggregate resource constraints at any t are $\int_0^\infty x\mathrm{e}^{-x\tau} c_t^{t-\tau}\mathrm{d}\tau = C_t, \int_0^\infty x\mathrm{e}^{-x\tau} l_t^{t-\tau}\mathrm{d}\tau = L_t$, in which $x\mathrm{e}^{-x\tau}$ is the measure of τ-year-old households. Thus, to maximize $V_t = \int_0^\infty x\mathrm{e}^{-x\tau} u\left(c_t^{t-\tau}, l_t^{t-\tau}\right)\mathrm{d}\tau$, the planner equalizes non-durable and housing consumption across people alive of all ages, $c_t^{t-\tau} = C_t, l_t^{t-\tau} = L_t$. Therefore, $V_t = u\left(C_t, L_t\right)$ and $\mathcal{W}_0 = \int_0^\infty \mathrm{e}^{-\rho t} u\left(C_t, L_t\right)\mathrm{d}t$. That is, the social welfare is equal to the utility of an infinitely-lived household with time discount ρ, even though various generations of households all have finite lives (in expectation). Next, we consider the government budget constraint. At this allocation, the household optimality conditions (E22, E23) hold in a way independent of cohort s as

$$\begin{aligned}
\hat{Q}_t &= \mathrm{e}^{-(\rho+x)t} u_{c,0}^{-1} u_{c,t} \\
D_t &= u_{c,t}^{-1} u_{l,t} \\
P_t &= Q_t^{-1}\int_t^\infty Q_\tau D_\tau \mathrm{d}\tau = u_{c,t}^{-1}\int_t^\infty \mathrm{e}^{-\rho(\tau-t)} u_{l,\tau}\mathrm{d}\tau
\end{aligned}$$

Note that the bond return the government faces is r_t without the insurance premium x. We have the discount factor $Q_t = \mathrm{e}^{-\int_0^t r_t \mathrm{d}\tau}$ satisfying $Q_t = \mathrm{e}^{xt}\hat{Q}_t = \mathrm{e}^{-\rho t} u_{c,0}^{-1} u_{c,t}$. Thus the government budget constraint in present value is

$$\begin{aligned}
0 &= \int_0^\infty Q_t\left[\left(C_t - Y_t\right) + \left(L_t - H_t\right) D_t + \dot{H}_t P_t\right]\mathrm{d}t \\
&= \int_0^\infty Q_t\left[\left(C_t - Y_t\right) + \left(L_t - H_t\right) D_t\right]\mathrm{d}t - P_0 H_0 - \int_0^\infty H_t \mathrm{d}\left(P_t Q_t\right) \\
&= \int_0^\infty Q_t\left[\left(C_t - Y_t\right) + \left(L_t - H_t\right) D_t\right]\mathrm{d}t - P_0 H_0 + \int_0^\infty H_t Q_t D_t \mathrm{d}t
\end{aligned}$$

$$= \int_0^\infty Q_t \left[(C_t - Y_t) + L_t D_t \right] \mathrm{d}t - P_0 H_0$$

$$0 = \int_0^\infty \mathrm{e}^{-\rho t} \left[(C_t - Y_t) u_{c,t} + L_t u_{l,t} \right] - \left(\int_0^\infty \mathrm{e}^{-\rho \tau} u_{l,\tau} \mathrm{d}\tau \right) H_0$$

The Lagrangian is

$$\mathcal{L}^* = \max_{\{C_{t\geq 0}, L_{t\geq 0}, H_{t>0}, Z_{t>0}\}} \int_0^\infty \mathrm{e}^{-\rho t} u\left(C_t, L_t\right) \mathrm{d}t + \int_0^\infty \mathrm{e}^{-\rho t} \mu_t^* \left(Y_t - C_t - \dot{Z}_t - \delta Z_t \right) \mathrm{d}t$$
$$+ \int_0^\infty \mathrm{e}^{-\rho t} \zeta_t^* \left(1 - L_t\right) \mathrm{d}t + \lambda^* \left\{ \int_0^\infty \mathrm{e}^{-\rho t} \left[(C_t - Y_t) u_{c,t} + L_t u_{l,t} \right] \mathrm{d}t - \left(\int_0^\infty \mathrm{e}^{-\rho \tau} u_{l,\tau} \mathrm{d}\tau \right) H_0 \right\}$$

The optimal land supply satisfies

$$0 = \mathrm{e}^{\rho t} \frac{\partial \mathcal{L}^*}{\partial L_t} = \left(1 + \lambda^*\right) u_{l,t} + \lambda^* L_t u_{ll,t} - \lambda^* H_0 u_{ll,t} + \lambda^* \left(C_t - Y_t\right) u_{cl,t} - \zeta_t^*$$

It simplifies to, if interior ($\zeta_t^* = 0$), under $\sigma_{L,t} \equiv -L_t \frac{u_{LL}(C_t, L_t)}{u_L(C_t, L_t)} > 0, \sigma_{CL,t} \equiv C_t \frac{u_{CL}(C_t, L_t)}{u_L(C_t, L_t)}$

$$\frac{1 + \lambda^*}{\lambda^*} = \left(1 - \frac{H_0}{L_t}\right) \sigma_{L,t} + \left(\frac{1}{\chi_t} - 1\right) \sigma_{CL,t}$$

identical to the baseline theory.□

Proof of Proposition E4.

Step 1: reducing the dimensionality of Ramsay planning. Since $\rho = r$ and utility is separable, for each generation we have $c_t^s = c^s$. (E23) suggests

$$u_c\left(c^s\right) = \frac{u_l\left(l_t^s\right)}{D_t}$$

$$\left(c^s\right)^{-\sigma} = \nu \frac{\left(l_t^s\right)^{-\sigma}}{D_t}$$

$$c^s = l_t^s \left(\frac{\nu}{D_t}\right)^{-\frac{1}{\sigma}} \tag{F1}$$

Aggregating this across all households alive at time t, we get

$$C_t = L_t \left(\frac{\nu}{D_t} \right)^{-\frac{1}{\sigma}}$$

which pins down

$$D_t = \nu \frac{L_t^{-\sigma}}{C_t^{-\sigma}} = \left[u_c \left(C_t \right) \right]^{-1} u_l \left(L_t \right) \tag{F2}$$

as if an representative agent with $u\left(C_t, L_t\right)$ is pricing the land rent. I denote $u\left(C_t, L_t\right)$ without a superscript as the consumption utility using aggregate consumption (C_t, L_t). Plugging rent into (F1) implies

$$l_t^s = \frac{L_t}{C_t} c^s \tag{F3}$$

Generation s's life-time budget constraint is, plugging in l_t^s,

$$0 = \left(C_0 \right)^{-\sigma} \int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t} \left[c^s - Y_t + \nu \frac{L_t^{-\sigma}}{C_t^{-\sigma}} \left(\frac{L_t}{C_t} c^s - \mathrm{e}^{x(t-s^+)} h_{s^+}^s \right) \right] \mathrm{d}t - \left(C_0 \right)^{-\sigma} \mathcal{T}^s \tag{F4}$$

which is multiplied by marginal utility $\left(C_0 \right)^{-\sigma}$ to change its unit to utility, or equivalently

$$c^s = \frac{\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t} \left(Y_t + \nu \frac{L_t^{-\sigma}}{C_t^{-\sigma}} \mathrm{e}^{x(t-s^+)} h_{s^+}^s \right) \mathrm{d}t + \mathcal{T}^s}{\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t} \left(1 + \nu \frac{L_t^{1-\sigma}}{C_t^{1-\sigma}} \right) \mathrm{d}t} \tag{F5}$$

If $h_0^s = H_0$ for all $s < 0$, then we have c^s being the same for all $s < 0$, which we denote as $\bar{c}$.

In the small open economy, the aggregate resource constraint is

$$0 = \int_0^{\infty} \mathrm{e}^{-rt} \left[Y\left(Z_t\right) - C_t - \delta Z_t - \dot{Z}_t \right] \mathrm{d}t \tag{F6}$$

The constraint on aggregate consumption is

$$C_t = \int_{-\infty}^{t} x \mathrm{e}^{x(s-t)} c^s \mathrm{d}s \tag{F7}$$

When $c^s = \bar{c}$ for all $s < 0$, we have $C_0 = \bar{c}$.

Taking stock, we have reduced the dimensionality of the Ramsey planning problem to the following: the government chooses $\left\{ c^s, \mathcal{T}^s \right\}_{s \in \mathbb{R}}, \left\{ C_t, L_t \right\}$ subject to (F4, F7) and (F6, E15).

If the government can use lump-sum taxes and transfers to redistribution across generations (the case analyzed in Section E.3.1), then the constraint on $\{\mathcal{T}^s\}$ is $0 = \int_{-\infty}^{0} x\mathrm{e}^{xs}\mathcal{T}^s\mathrm{d}s + \int_0^{\infty} x\mathcal{T}^s\mathrm{d}s$. The difference in treatment of generations with $s < 0$ and $s \geq 0$ stems from defining $\mathcal{T}^s$ as the time-0 present value of lump-sum transfer to an individual alive or yet to be born. Generation s born before time 0 has a $x\mathrm{e}^{xs}$ measure of people alive at time 0.

If the government cannot use lump-sum redistribution, we simply set $\mathcal{T}^s = 0$ for all s.

Step 2: optimal land supply from Ramsey planning.

$$\begin{aligned}\mathcal{L}^* = \max_{\{C_{t\geq 0}, L_{t\geq 0}, Z_{t>0}, \mathcal{T}^{s\in\mathbb{R}}, \omega^{s\in\mathbb{R}}\}} & \int_{-\infty}^{\infty} \mathrm{e}^{-\rho s} x \left[\int_{s^+}^{\infty} \mathrm{e}^{-(\rho+x)(t-s)} u\left(c^s, \frac{L_t}{C_t}c^s\right)\mathrm{d}t\right]\mathrm{d}s \\ & + \mu^* \int_0^{\infty} \mathrm{e}^{-rt}\left[Y\left(Z_t\right) - C_t - (\delta + r)\, Z_t\right]\mathrm{d}t + \mu^* Z_0 \\ & + \int_0^{\infty} \mathrm{e}^{-(\rho+x)t}\zeta_t^*\left(1 - L_t\right)\mathrm{d}t \\ & + \int_{-\infty}^{\infty} \mathrm{e}^{xs} x\hat{\lambda}^s \left(C_0\right)^{-\sigma}\left[\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t}\left[c^s - Y_t + \nu\frac{L_t^{-\sigma}}{C_t^{-\sigma}}\left(\frac{L_t}{C_t}c^s - \mathrm{e}^{x(t-s^+)}h_{s^+}^s\right)\right]\mathrm{d}t - \mathcal{T}^s\right]\mathrm{d}s \\ & + \int_0^{\infty} \mathrm{e}^{xt}\kappa_t\left(C_t - \int_{-\infty}^{t} x\mathrm{e}^{x(s-t)}c^s\mathrm{d}s\right)\mathrm{d}t\end{aligned} \tag{F8}$$

The optimal c^s for any s satisfies

$$\begin{aligned}0 &= \frac{\partial \mathcal{L}^*}{\partial c^s} \\ &= \mathrm{e}^{-\rho s} x\int_{s^+}^{\infty} \mathrm{e}^{-(\rho+x)(t-s)}\left(c^s\right)^{-\sigma}\left(1 + \nu\frac{L_t^{1-\sigma}}{C_t^{1-\sigma}}\right)\mathrm{d}t + \mathrm{e}^{xs} x\hat{\lambda}^s\left(C_0\right)^{-\sigma}\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t}\left(1 + \nu\frac{L_t^{1-\sigma}}{C_t^{1-\sigma}}\right)\mathrm{d}t - \int_{s^+}^{\infty} \mathrm{e}^{xt}\kappa_t x\mathrm{e}^{x(s-t)}\mathrm{d}t \\ 0 &= \left(c^s\right)^{-\sigma}\int_{s^+}^{\infty} \mathrm{e}^{-(\rho+x)t}\left(1 + \nu\frac{L_t^{1-\sigma}}{C_t^{1-\sigma}}\right)\mathrm{d}t + \hat{\lambda}^s\left(C_0\right)^{-\sigma}\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t}\left(1 + \nu\frac{L_t^{1-\sigma}}{C_t^{1-\sigma}}\right)\mathrm{d}t - \int_{s^+}^{\infty}\kappa_t\mathrm{d}t\end{aligned}$$

which determines

$$\hat{\lambda}^s = \left[\frac{\int_{s^+}^{\infty}\kappa_t\mathrm{d}t}{\int_{s^+}^{\infty} \mathrm{e}^{-(r+x)t}\left(1 + \nu\frac{L_t^{1-\sigma}}{C_t^{1-\sigma}}\right)\mathrm{d}t} - \left(c^s\right)^{-\sigma}\right]\left(C_0\right)^{\sigma} \tag{F9}$$

We must have $\hat{\lambda}^s \geq 0$ for all s so that the government sets $\mathcal{T}^s = 0$. From (F5), since $h_0^0 = 0$ while $h_0^s > 0$ for all $s < 0$, we know $c^0 < c^s$ for all $s < 0$, and thus $\hat{\lambda}^0 < \hat{\lambda}^s$. If $h_0^s = H_0$ for all

$s < 0$, then we have $\hat{\lambda}^s$ being the same for all $s < 0$, which we denote as $\bar{\lambda}$.

The optimal Z_t satisfies

$$0 = \mathrm{e}^{(\rho+x)t}\frac{\partial \mathcal{L}^*}{\partial Z_t} = \mu^* \mathrm{e}^{xt}\left[Y'\left(Z_t\right) - (\delta + r)\right] - \int_{-\infty}^{t} \mathrm{e}^{xs} x \hat{\lambda}^s \left(C_0\right)^{-\sigma} Y'\left(Z_t\right) \mathrm{d}s$$

which leads to

$$Z_t = \left(Y'\right)^{-1}\left(\frac{\delta + r}{1 - \left(\mu^*\right)^{-1}\left(C_0\right)^{-\sigma}\Lambda_t}\right) \tag{F10}$$

where $\Lambda_t \equiv \int_{-\infty}^{t} \mathrm{e}^{x(s-t)} x \hat{\lambda}^s \mathrm{d}s$ is the population average of $\hat{\lambda}^s$. We have Z_t decreases in Λ_t. Further $\dot{\Lambda}_0 = x\left(\hat{\lambda}^0 - \Lambda_0\right) < 0$ since $\hat{\lambda}^0 < \hat{\lambda}^s$ for all $s < 0$. Thus $\dot{Z}_0 > 0$.

The optimal L_t satisfies, if interior

$$\begin{aligned}
0 &= \mathrm{e}^{(\rho+x)t}\frac{\partial \mathcal{L}^*}{\partial L_t} \\
&= \int_{-\infty}^{t} \mathrm{e}^{-\rho s} x \mathrm{e}^{(\rho+x)s} \nu \frac{L_t^{-\sigma}\left(c^s\right)^{1-\sigma}}{\left(C_t\right)^{1-\sigma}} \mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{xs} x \hat{\lambda}^s \left(C_0\right)^{-\sigma} \nu \frac{L_t^{-\sigma}}{C_t^{-\sigma}}\left(\frac{1-\sigma}{C_t} c^s + \frac{\sigma}{L_t}\mathrm{e}^{x(t-s^+)} h^s_{s^+}\right)\mathrm{d}s \\
0 &= \int_{-\infty}^{t} \mathrm{e}^{xs} x \left(\frac{c^s}{C_0}\right)^{1-\sigma} C_0 \mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{xs} x \hat{\lambda}^s \left((1-\sigma)\, c^s + \sigma \frac{C_t}{L_t}\mathrm{e}^{x(t-s^+)} h^s_{s^+}\right)\mathrm{d}s
\end{aligned}$$

which simplifies to, under $h^s_{s^+} = 0$ for all $s \geq 0$

$$\Lambda^h \sigma \frac{H_0}{L_t} = C_t^{-1} \int_{-\infty}^{t} \mathrm{e}^{x(s-t)} x c^s \left[(\sigma - 1)\,\hat{\lambda}^s - \left(\frac{c^s}{C_0}\right)^{-\sigma}\right]\mathrm{d}s \tag{F11}$$

with $\Lambda^h \equiv H_0^{-1}\int_{-\infty}^{0} \mathrm{e}^{xs} x h_0^s \hat{\lambda}^s \mathrm{d}s$ denoting the land-endowment-weighted $\hat{\lambda}^s$ among all generations born before time 0. If $h_0^s = H_0$ for all $s < 0$, then $\Lambda^h = \bar{\lambda}$.

Evaluating (F11) at time 0 yields

$$\bar{\lambda}\sigma\frac{H_0}{L_0} = (\sigma - 1)\,\bar{\lambda} - 1$$

Moreover, taking time derivative of (F11) at time 0 gives, using (F7)

$$-\frac{\dot{L}_0}{L_0} = -\frac{\dot{C}_0}{C_0} - x + x\frac{c^0}{C_0}\frac{(\sigma-1)\,\hat{\lambda}^0 - \left(\frac{c^0}{C_0}\right)^{-\sigma}}{(\sigma-1)\,\bar{\lambda} - 1}$$

$$= x\frac{c^0}{C_0}\left[-1+\frac{(\sigma-1)\,\hat{\lambda}^0-\left(\frac{c^0}{C_0}\right)^{-\sigma}}{(\sigma-1)\,\bar{\lambda}-1}\right]$$

Since $c^0 < \bar{c} = C_0, \hat{\lambda}^0 < \bar{\lambda}$, the RHS is negative and thus $\dot{L}_0 > 0$.□

Proof of Proposition E5. The Lagrangian is

$$\begin{aligned}\mathcal{L}^* = \max_{\{C_{t\geq 0}, L_{t\geq 0}, H_{t>0}, Z_{t>0}\}} &\int_0^\infty \mathrm{e}^{-\rho t} U(C_t)\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\rho t}\left\{\mu_t^*\left[Y(Z_t,L_t) - C_t - (\delta+\rho)Z_t\right] + \dot{\mu}_t^* Z_t\right\}\mathrm{d}t + \mu_0^* Z_0 \\ &+ \int_0^\infty \mathrm{e}^{-\rho t}\zeta_t^*(1-L_t)\,\mathrm{d}t + \lambda^*\int_0^\infty \mathrm{e}^{-\rho t}\left[C_t - Y(Z_t,L_t) + (L_t-H_0)\,Y_L(Z_t,L_t)\right]U_{C,t}\mathrm{d}t\end{aligned}$$

The FOCs are

$$\begin{aligned}0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial C_t} &= (1+\lambda^*)\,U_{C,t} + \lambda^*\left[C_t - Y_t + (L_t-H_0)\,Y_{L,t}\right]U_{CC,t} - \mu_t^* \\ 0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial L_t} &= (\mu_t^* - \lambda^* U_{C,t})\,Y_{L,t} + \lambda^* Y_{L,t}U_{C,t} + \lambda^*(L_t-H_0)\,Y_{LL,t}U_{C,t} - \zeta_t^* \\ 0 = \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial Z_t} &= \mu_t^*\left[Y_{Z,t} - (\delta+\rho)\right] + \dot{\mu}_t^* + \lambda^*\left[-Y_{Z,t} + (L_t-H_0)\,Y_{LZ,t}\right]U_{C,t}\end{aligned}$$

Take the L_t FOC, assuming interior

$$\begin{aligned}0 &= (\mu_t^* - \lambda^* U_{C,t})\,Y_{L,t} + \lambda^* Y_{L,t}U_{C,t} + \lambda^*(L_t-H_0)\,Y_{LL,t}U_{C,t} \\ &= \left[U_{C,t} + \lambda^*(C_t-Y_t)\,U_{CC,t} + \lambda^*(L_t-H_0)\,Y_{L,t}U_{CC,t}\right]Y_{L,t} + \lambda^* Y_{L,t}U_{C,t} + \lambda^*(L_t-H_0)\,Y_{LL,t}U_{C,t}\end{aligned}$$

Define $\sigma_{C,t} = -\frac{C_t U_{CC,t}}{U_{C,t}} > 0, \sigma_{L,t} \equiv -\frac{L_t Y_{LL,t}}{Y_{L,t}} > 0, \epsilon_{L,t} = \frac{L_t Y_{L,t}}{Y_t}$, and $\chi_t = \frac{C_t}{Y_t}$, and we get

$$0 = 1 + \lambda^* + \lambda^*\left(\frac{1}{\chi_t} - 1\right)\sigma_{C,t} - \lambda^*\frac{1}{\chi_t}\frac{L_t - H_0}{L_t}\sigma_{C,t}\epsilon_{L,t} - \lambda^*\frac{L_t-H_0}{L_t}\sigma_{L,t}$$

If $U(C)$ is isoelastic with constant σ_C and $y(N_t,L_t) = N_t^{\phi}L_t^{1-\phi}$, then $\epsilon_L = 1-\phi, \sigma_L = \phi$, and

$$0 = 1 + \lambda^* - \lambda^*\sigma_C + \lambda^*\frac{\sigma_C}{\chi_t}\left[1 - \left(1-\phi\right)\frac{L_t-H_0}{L_t}\right] - \lambda^*\phi\frac{L_t-H_0}{L_t}$$

which decreases in both χ_t and L_t. Therefore, χ_t and L_t must move in opposite directions to meet this optimality condition.□

Proof of Proposition E6. The Lagrangian is

$$\begin{aligned}\mathcal{L}^* = \max_{\{C_{t\geq 0}, L_{t\geq 0}, H_{t>0}, Z_{t>0}\}} & \int_0^\infty \mathrm{e}^{-\rho t} U(C_t)\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\rho t}\mu_t^*\left(Y_t - C_t - \dot{Z}_t - \delta Z_t\right)\mathrm{d}t \\ & + \int_0^\infty \mathrm{e}^{-\rho t}\zeta_t^*\left(1-\tilde{L}_t\right)\mathrm{d}t + \lambda^*\int_0^\infty \mathrm{e}^{-\rho t}\left[\left(C_t - Y_t\right)U_{C,t} + \left(L_t - H_0\right)U_{L,t}\right]\mathrm{d}t \\ & + \int_0^\infty \mathrm{e}^{-\rho t}\tilde{\zeta}_t\left(\tilde{L}_t - L_t\right)\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\rho t}\zeta_t^g\left(H_0 + gt - L_t\right)\mathrm{d}t\end{aligned}$$

The FOCs w.r.t. $L_t, \tilde{L}_t$ are, under separable utility

$$\begin{aligned}0 &= \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial L_t} = \left(1+\lambda^*\right)U_{L,t} + \lambda^*\left(L_t - H_0\right)U_{LL,t} - \tilde{\zeta}_t - \zeta_t^g \\ 0 &= \mathrm{e}^{\rho t}\frac{\partial \mathcal{L}^*}{\partial \tilde{L}_t} = \tilde{\zeta}_t - \zeta_t^*\end{aligned}$$

Whenever $L_t \leq \tilde{L}_t$ binds and $\tilde{\zeta}_t > 0$, increasing $\tilde{L}_t$ is desirable. Thus $L_t \leq \tilde{L}_t$ should never bind. When $L_t \leq H_0 + gt$ is slack, the optimal land supply is time-invariant as

$$L_t = \left[1 + \eta\left(\lambda^*\right)\right]H_0$$

with $\eta\left(\lambda^*\right) = \left(\frac{\lambda^*\sigma}{1+\lambda^*} - 1\right)^{-1}$ decreasing in λ^*. But this level apparently violates $L_t \leq H_0 + gt$ early on. We conclude that $L_t \leq H_0 + gt$ must bind until time $T = \frac{\eta(\lambda^*)H_0}{g}$, after which it is slack and L_t is constant.□

Proof of Proposition E7. Household optimization leads to state price, land rent and land price, under given d_t

$$\begin{aligned}Q_t &= \mathrm{e}^{-\rho t}u_{c,0}^{-1}u_{c,t} \\ D_t &= u_{c,t}^{-1}u_{l,t} \\ P_t &= Q_t^{-1}\int_t^\infty Q_s D_s \mathrm{d}t = u_{c,t}^{-1}\int_t^\infty \mathrm{e}^{-\rho(s-t)}u_{l,s}\mathrm{d}s\end{aligned}$$

$$\begin{aligned}\mathcal{L}^* = &\max_{\{C_{t\geq 0}, L_{t\geq 0}, T_{t\geq 0}, H_{t>0}, Z_{t>0}\}} \int_0^\infty \mathrm{e}^{-(\rho-n)t} u\left(c_t, l_t; d_t\right) \mathrm{d}t \\ &+ \int_0^\infty \mathrm{e}^{-(\rho-n)t} \left\{\mu_t^* \left[Y_t\left(Z_t\right) - C_t - \left(\delta + \rho - n\right) Z_t\right] + \dot{\mu}_t^* Z_t\right\} \mathrm{d}t + \mu_0^* Z_0 \\ &+ \int_0^\infty \mathrm{e}^{-(\rho-n)t} \zeta_t^* \left(1 - L_t\right) \mathrm{d}t + \lambda^* \int_0^\infty \mathrm{e}^{-(\rho-n)t} \left[\left(c_t - y_t\right) u_{c,t} + \left(L_t - H_0\right) \mathrm{e}^{-nt} u_{l,t}\right] \mathrm{d}t\end{aligned}$$

$$\begin{aligned}0 &= \mathrm{e}^{(\rho-n)t} \frac{\partial \mathcal{L}^*}{\partial L_t} \\ &= \left(1 + \lambda^*\right) \mathrm{e}^{-nt} u_{l,t} + \lambda^* \left(L_t - H_0\right) \mathrm{e}^{-2nt} u_{ll,t} + \lambda^* \left(c_t - y_t\right) \mathrm{e}^{-nt} u_{cl,t} + \frac{\mathrm{d}d_t}{\mathrm{d}L_t} \left\{u_{d,t} + \lambda^* \left[\left(c_t - y_t\right) u_{cd,t} + \left(L_t - H_0\right) \mathrm{e}^{-nt} u\right.\right.\end{aligned}$$

$$0 = 1 + \lambda^* - \lambda^* \frac{L_t - H_0}{L_t} \cdot \left(-\frac{l_t u_{ll,t}}{u_{l,t}}\right) + \lambda^* \left(1 - \frac{1}{\chi_t}\right) \frac{c_t u_{cl,t}}{u_{l,t}} - d_t^2 \left\{\frac{u_{d,t}}{u_{l,t}} + \lambda^* \left[\left(1 - \frac{1}{\chi_t}\right) \frac{c_t u_{cd,t}}{u_{l,t}} + \frac{L_t - H_0}{L_t} \frac{l_t u_{ld,t}}{u_{l,t}}\right]\right\}$$

If $u\left(c, l; d\right) = d^\epsilon c^\phi l^{1-\phi}$, then

$$\begin{aligned}0 &= 1 + \lambda^* - \lambda^* \frac{L_t - H_0}{L_t} \cdot \phi + \lambda^* \left(1 - \frac{1}{\chi_t}\right) \phi - \left\{\frac{\epsilon}{1-\phi} + \lambda^* \left[\left(1 - \frac{1}{\chi_t}\right) \frac{\epsilon\phi}{1-\phi} + \frac{L_t - H_0}{L_t} \epsilon\right]\right\} \\ &= 1 + \lambda^* - \frac{\epsilon}{1-\phi} - \lambda^* \frac{L_t - H_0}{L_t} \left(\phi + \epsilon\right) - \lambda^* \left(1 - \frac{1}{\chi_t}\right) \phi \left(\frac{\epsilon}{1-\phi} - 1\right)\end{aligned}$$

□