A Unified Urban Model With Non-Homothetic Housing Demand*

Frédéric Robert-Nicoud†, Pierre-Philippe Combes‡, Gilles Duranton§, Laurent Gobillon¶

June 26, 2026

Abstract

Canonical urban models fail to jointly account for flexible housing demand, non-degenerate city sizes, and observed urban systems. We introduce a unifying urban framework based on price-independent generalized linear (PIGL) preferences in which housing is a necessity. Non-homothetic housing demand generates income effects that cause urban costs to scale more strongly with population than wages, restoring a unique interior efficient city size under standard assumptions. The framework nests existing canonical models, remains tractable, and is consistent with key empirical regularities, including Zipf's law and observed housing price and income elasticities. We also show that a standard monocentric city model maps exactly into our framework.

JEL: R14, R21 and R31. **Keywords**: Urban modeling, PIGL preferences, Monocentric model

*We are grateful to Michael Peters for highly valuable comments and suggestions, and to Clement Bohr, Ziyang Chen, Nicolas Coeurdacier, Axelle Ferrière, Martí Mestieri, and Jacques-François Thisse for discussions and feedback. We also thank audiences at the 2026 Zurich-Bern-Wisconsin Conference on Real Estate and Urban Economics. All the authors benefited from the financial support of the French National Research Agency (ANR) through the MAPHIS (ANR-20-ORAR-0007) and LANDURB (ANR-23-CE26-0016) projects. Duranton was also supported by the Zell-Lurie Center for Real Estate at the Wharton School and acknowledges the hospitality of CEMFI during the 2024-2025 academic year. Gobillon acknowledges the support of the EUR grant ANR-17-EURE-0001.

†Université de Genève, Institute of Economics and Econometrics, Boulevard du Pont d'Arve 40, 1211 Genève 4, Switzerland; CEPR. Frederic.Robert-Nicoud@unige.ch. https://www.unige.ch/gsem/en/research/faculty/all/frederic-robert-nicoud/

‡Sciences Po–CNRS, Department of Economics, 28, Rue des Saints-Pères, 75007 Paris, France; CEPR. PPCombes@gmail.com. https://www.sciencespo.fr/department-economics/en/researcher/pierre-philippe-combes.html

§Wharton School, University of Pennsylvania, 3620 Locust Walk, Philadelphia, PA 19104, USA; CEPR; NBER. Duranton@wharton.upenn.edu. https://real-estate.wharton.upenn.edu/profile/21470/

¶Paris School of Economics – CNRS, 48 Boulevard Jourdan, 75014 Paris, France; CEPR; IZA. Laurent.Gobillon@psemail.eu. http://laurent.gobillon.free.fr/

1 Introduction

Informed by a large body of empirical work, a useful urban model must satisfy a set of core empirical and theoretical requirements. Housing should be a necessity, and, within cities, housing prices, land prices, and densities should decline with distance to jobs or amenities. Across cities, both housing costs and the overall cost of living should rise with population, while wages and productivity increase with workforce size due to agglomeration economies. Internal consistency also requires that cities exist: welfare cannot be monotonically increasing or decreasing in city population, so that a finite interior city size emerges in equilibrium. At the same time, urban systems feature substantial heterogeneity in city sizes, with distributions that are well approximated by Zipf's law. To date, no existing model reconciles these facts within a single, tractable equilibrium structure.

This paper introduces such a framework. We develop PIGL-UM (Price-Independent Generalized Linear Urban Model), which combines non-homothetic PIGL preferences with a competitive housing sector. We show that a realistic departure from homotheticity (treating housing as a necessity) reconciles these empirical regularities within a tractable model. In contrast to Cobb–Douglas specifications, the model generates income effects in housing demand; in contrast to fixed-housing approaches, it preserves price effects. This combination implies that urban costs scale more strongly with population than wages, restoring a unique interior efficient city size. The framework nests canonical models with either fixed housing consumption or fixed expenditure shares as polar cases, preserves analytical tractability, and naturally accounts for the empirical regularities documented in the urban and housing literature (section 2). It also aggregates naturally to an urban system with heterogeneous cities and workers, and admits microfoundations from a fully specified monocentric city model.

Urban economics relies on stylized preferences to analyze city size, housing markets, and the spatial allocation of workers. Following Alonso (1964), Fujita (1989), Lucas and Rossi-Hansberg (2002), and Solow (1973), the dominant approach assumes homothetic Cobb–Douglas preferences, which ensures analytical convenience but imposes a unit income elasticity in housing demand. This counterfactual implication yields a degenerate efficient city size in extant formulations (Allen and Arkolakis 2025; Redding 2025).[1] Fol-

[1]Models combining Cobb–Douglas preferences with explicit urban spatial structure are also known to aggregate poorly once heterogeneity is introduced, except under knife-edge assumptions on preferences and technologies. See Duranton and Handbury (2023) for a discussion in the context of work-from-home shocks.

lowing Fujita and Ogawa (1982), Henderson (1974), and Mills (1967), an alternative strand assumes fixed housing consumption, which restores income effects and a non-zero optimal city scale, but shuts down price effects (Albouy, Behrens, Robert-Nicoud, and Seegert 2019; Duranton and Puga 2023).

PIGL-UM bridges these extremes. The model features non-homothetic PIGL preferences over a numeraire and a local housing service produced competitively from land and a numeraire input. Housing is a necessity, consistent with the empirical evidence (Combes, Duranton, and Gobillon 2019; Couture, Gaubert, Handbury, and Hurst 2023; Finlay and Williams 2025; Fretz, Parchet, and Robert-Nicoud 2022). The resulting structure preserves isoelastic functional forms throughout, allowing income and price effects to operate while remaining analytically tractable.

We abstract from city geography in the baseline environment with identical households and a fixed supply of city land (section 3). The model delivers closed-form expressions for housing demand, housing supply, equilibrium prices, land rents, and the cost of living. When wages depend on population through agglomeration economies, the model yields a unique interior efficient city size provided housing-market crowding eventually dominates agglomeration forces. In the homothetic Cobb–Douglas limit, the efficient city size collapses to zero, highlighting the role of non-homotheticity in sustaining finite cities.

We then embed the city model into three distinct urban systems (section 4). First, with heterogeneous city fundamentals and identical households, efficient city sizes follow either a Pareto or a log-normal distribution under standard assumptions (Duranton and Puga 2014). Second, introducing heterogeneous location preferences yields an extensive margin with an endogenous mass of cities that is novel to the quantitative spatial model literature (Ahlfeldt, Redding, Sturm, and Wolf 2015; Allen and Arkolakis 2025; Diamond 2016; Fajgelbaum, Morales, Suárez Serrato, and Zidar 2019; Monte, Redding, and Rossi-Hansberg 2018; Redding 2016; Redding 2025). PIGL-UM also generalizes results from Albouy, Behrens, Robert-Nicoud, and Seegert (2019). Third, allowing for heterogeneous worker talent generates a sorting equilibrium in which higher skilled workers reside in larger, higher-wage, and higher-cost cities. This variant of the model generates Zipf's law as a limiting but empirically relevant case. It also generalizes the sorting framework of Behrens, Duranton, and Robert-Nicoud (2014) by restoring price effects without sacrificing tractability.

A final contribution of the paper is to show that PIGL-UM, which has no internal urban structure, is not only a reduced-form shortcut, but the exact equilibrium representation of

a fully specified monocentric city model with commuting costs, endogenous land use, and spatial housing price gradients (section 5). We label this monocentric city model PIGL-AMM in reference to the seminal contributions of Alonso (1964), Mills (1967), and Muth (1969). Under standard assumptions, the within-city spatial equilibrium collapses to a single sufficient statistic, the urban cost of living, which enters utility and housing demand exactly as in the space-free model of sections 3 and 4.

Throughout we build on the class of price-independent generalized linear preferences introduced by Muellbauer (1975; 1976), and further developed in Deaton and Muellbauer (1980), who provide a flexible yet tractable framework for non-homothetic demand that aggregates exactly across heterogeneous households. PIGL preferences have since become a workhorse in the modern structural change literature, where they are employed to reconcile income-driven expenditure shifts with balanced-growth or representative-agent environments, most notably in Alder, Boppart, and Müller (2022), Boppart (2014), and Fan, Peters, and Zilibotti (2023).[2] Beyond traditional structural change, the PIGL class has also been adopted in spatial contexts: Chatterjee, Giannone, Kleineberg, Kuno, and Looser (2025) and Eckert and Peters (2025) explicitly embed PIGL preferences to generate tractable income-dependent expenditure shares across sectors in their spatial structural change models.[3] Empirically, the cost of living varies across cities and income groups because housing is a necessity (Albouy, Ehrlich, and Liu 2016; Chatterjee, Giannone, Kleineberg, Kuno, and Looser 2025; Finlay and Williams 2025; Gaubert and Robert-Nicoud 2025; Ioannides and Ngai 2025), and this mechanism links rising inequality to spatial sorting patterns.

In benchmark macro–housing models, housing typically enters preferences in homothetic form (Piazzesi and Schneider 2016). Deviations from unit income elasticity typically arise either from financial frictions and heterogeneity (Berger, Guerrieri, Lorenzoni, and Vavra 2018; Favilukis, Ludvigson, and Van Nieuwerburgh 2017; Iacoviello 2005; Kaplan,

[2] A growing macro literature introduces non-homotheticity in multi-sector frameworks through non-homothetic CES preferences (Comin, Lashkari, and Mestieri 2021). This alternative approach does not yield closed-form characterizations of equilibrium objects such as efficient city size.

[3] Eeckhout, Pinheiro, and Schmidheiny (2014) and Gaigné, Koster, Moizeau, and Thisse (2022) use Stone-Geary preferences instead. The former develop a quantitative spatial equilibrium model with non-homothetic preferences to study the distribution of city sizes and spatial sorting, emphasizing how subsistence housing demand shapes urban structure and inequality. The latter study the sorting of heterogeneous individuals inside a monocentric city. Unlike PIGL, Stone–Geary preferences yield affine Engel curves (which are convenient but counterfactual) and generally preclude closed-form solutions for the efficient size of cities, a central focus of our analysis.

Mitman, and Violante 2020) or through the multi-sector calibration (Iacoviello and Neri 2010), rather than from the utility specification itself. By contrast, our approach introduces non-homotheticity through a PIGL structure, allowing for Engel effects while preserving tractability. The PIGL-UM framework is therefore well suited to bridge the gap between these approaches by providing a tractable microfoundation for income- and price-inelastic housing demand, consistent with the empirical evidence summarized in Ioannides and Ngai (2025).

2 Stylized facts

This section confronts the key mechanisms and predictions of the model with the data. We combine qualitative evidence on within- and across-city patterns with parameter values drawn from the existing literature, which we use to discipline the model through direct and indirect inference. Table 1 collects the empirical benchmarks.

2.1 Urban wage premium parameter

Wages increase with urban population size and density, and observed urban wage differentials reflect both agglomeration economies and endogenous worker composition (i.e., sorting). In the model wages are increasing in city size with a constant elasticity, denoted α. Surveying the empirical literature Combes and Gobillon (2015) report a central estimate for the agglomeration elasticity of

$$\alpha \in [0.02, 0.05] \tag{1}$$

across studies using individual worker panel data from multiple countries. Identification typically relies on instrumenting density with long-lagged population variables and exploiting worker mobility across cities to control for time-invariant unobserved heterogeneity.

2.2 Housing supply and Cobb-Douglas technology parameter

Housing supply is elastic, but imperfectly so. Multiple sources of variation in the data can be exploited to estimate this parameter.

Consider a constant-returns-to-scale housing production function combining land and non-land inputs. Combes, Duranton, and Gobillon (2019; 2021), using French data, and

Ahlfeldt and McMillen (2014), using data for Berlin, Chicago, and Pittsburgh, show that a Cobb–Douglas specification provides a reasonable approximation, with an elasticity of housing output with respect to non-land inputs of about $\frac{2}{3}$. In equilibrium, this elasticity corresponds to $\frac{\sigma}{1+\sigma}$, where σ is the local housing supply elasticity, implying

$$\sigma = 2. \tag{2}$$

Second, direct estimates of housing supply elasticities range from near zero to about 10, depending on the method, context, and time horizon (Baum-Snow and Han 2024; DiPasquale and Wheaton 1994; Finlay and Williams 2025; Glaeser, Gyourko, and Saks 2006; Saiz 2010; Wu, Gyourko, and Deng 2016). Estimates based on construction data identify σ conditional on housing development, whereas most of the literature recovers an unconditional elasticity (namely, the conditional elasticity scaled by the probability of development), which is typically much lower, especially over short horizons. See Baum-Snow and Duranton (2025) for a comprehensive discussion.

Finally, in the model σ also governs the ratio of the land price gradient to the housing price gradient (see section 2.7). There we report $\sigma = 2.43$, remarkably close to the value in equation (2). Taken together, these estimates imply that $\alpha\sigma$ is well below one, ensuring a finite efficient city size.

2.3 Local labor supply parameter

Households hold heterogeneous preferences for location characteristics in an extension of the model in section 4; if these preferences are multiplicatively separable in the common component of utility, and idiosyncratic preferences shocks are assumed to be Frechet distributed, then the combination of these assumptions yields a finite local labor supply elasticity (sometimes referred to as the migration elasticity), denoted κ. Existing estimates of the local labor supply elasticity typically lie in the interval

$$\kappa \in [2.1, 5.1],$$

with Beaudry, Green, and Sand (2014) reporting estimates between 2.3 and 3.2, Diamond (2016) finding values around 2.1 to 4 depending on skill group, and Finlay and Williams (2025) obtaining around 5.1.[4] Taken together, these estimates discipline the admissible

[4]Fretz, Parchet, and Robert-Nicoud (2022) obtain values from 2.7 to 5.4.

range of the local labor supply elasticity in the model.

2.4 Model prediction: Housing demand and PIGL preferences parameter

Housing expenditure shares decline with income and increase with housing prices. Under PIGL preferences over housing and a non-housing composite (taken as the numeraire), the model delivers log-linear, closed-form relationships between the housing expenditure share, total expenditure.

Using the 2005–2017 biennial surveys of the Panel Study of Income Dynamics in the United States, Finlay and Williams (2025) estimate:

$$\ln e = -0.25 \ln w + 0.39 \ln p + \text{controls} + \text{residual}, \tag{3}$$

where e denotes the housing expenditure share, w total expenditure, and p housing prices.[5] These estimates imply that housing demand is both income- and price-inelastic.

In the model, these estimates map into income and price elasticities of housing demand equal to

$$\varepsilon = 0.75 = 1 - 0.25 \quad \text{and} \quad \nu = 0.61 = 1 - 0.39. \tag{4}$$

Consistent evidence is found in other settings.[6] Taken together, these estimates imply $\alpha\kappa\,(1-\varepsilon) < 1$, so, with unbounded city size in the model, large cities are below their efficient scale in spatial equilibrium with heterogeneous location preferences.

2.5 Model prediction: Housing expenditure shares are increasing in city size.

The model makes several parametric equilibrium predictions: we start with the relationship between individual housing expenditure as a function of earnings and city size. Combes, Duranton, and Gobillon (2019) use French census, household expenditure surveys, and housing and land transaction records for 2000–2012. They regress the household housing

[5] See their Table 1, col. 4; 2SLS. All variables are measured at the household level; the specification also includes demographic controls.

[6] Combes, Duranton, and Gobillon (2019) and Fretz, Parchet, and Robert-Nicoud (2022) document similar patterns for France and Switzerland, respectively. Finlay and Williams (2025) also report consistent estimates of the income elasticity of housing demand for the United Kingdom, Canada, Norway, and the United States based on earlier contributions (see their Table B.3).

expenditure share e on urban area population size N, household income w, geographic controls, and household demographic characteristics. In their baseline specification (Table 6, column 4), they obtain:

$$\ln e = -0.286 \ln w + 0.039 \ln N + \text{controls} + \text{residual}. \tag{5}$$

Thus, the housing expenditure shares decline with income (confirming that housing is a necessity in France) and increase with urban population. In our framework, both coefficients have structural interpretations as functions of the underlying parameters. Combining the regression coefficients in equation (5) with an external estimate of the housing supply elasticity $\sigma = 2$ from the review of Baum-Snow and Duranton (2025), we recover the implied values $\varepsilon = 0.7$ and $\nu = 0.9$ using equation (15) in section 3. Chen, Combes, Démurger, and Liu (2025) estimate equation (5) by skill group for China and the implied values are $\varepsilon = 0.9$ and $\nu = 0.9$. Both pairs of values are quite close to the estimates for the United States reported in equation (4), suggesting that the parameter values used in the model are empirically plausible across contexts.

These findings consistently support that housing is a necessity, which governs the mechanism that yields a finite and positive efficient city scale in section 3.

2.6 Model prediction: Zipf's law

As a first-order approximation, the distribution of city populations follows Zipf's law in many countries: city sizes are Pareto distributed with a shape parameter ζ close to one (Rozenfeld, Rybski, Gabaix, and Makse 2011). Some authors instead argue that the distribution is better described by a log-normal (Eeckhout 2004). Following Duranton and Puga (2014), we do not take a stance in this debate.

As an illustration, consider ranking cities in descending order of population and estimating the standard rank–size relationship as in Gabaix and Ibragimov (2011):

$$\ln\left(R - \frac{1}{2}\right) = -\zeta \ln N + \text{constant} + \text{residual},$$

where the correction $R - \frac{1}{2}$ removes the small sample bias of the OLS estimator. In our model, ζ is a combination of structural parameters $(\alpha, \varepsilon, \nu, \sigma)$ and of the shape parameter of the distribution of amenity and productivity fundamentals across cities.

Using digitized historical French maps and censuses, Combes, Duranton, Gobillon,

Gorin, and Robert-Nicoud (2026) estimate the Pareto shape parameter at different points in time (subscripts denote years):

$$\zeta_{1760} = 1.43, \quad \zeta_{1860} = 1.09, \quad \zeta_{1960} = 0.88, \quad \zeta_{2020} = 0.84. \tag{6}$$

Three lessons emerge. First, the Pareto distribution provides a reasonable approximation of the upper tail of the city size distribution (the R^2 is close to one). Second, Zipf's law ($\zeta = 1$) is rejected at conventional confidence levels. Third, the French urban system has become increasingly concentrated over time, with a growing share of the urban population residing in the largest cities. This pattern is consistent with a framework in which larger cities offer sufficiently strong productivity advantages to offset rising urban costs, thereby sustaining a thick upper tail in the city size distribution.

These findings discipline the distribution of city fundamentals in section 4, which map into empirically plausible distributions for city sizes in our model.

2.7 Model prediction: Estimates of density gradients

Within cities, housing prices, land prices, built-up density, and population density all decline with distance to the center (see Duranton and Puga 2015 for a review of this literature).

In the model, housing prices $p(x)$ decline with distance x to the city center according to

$$\ln p(x) = \ln P - \theta \ln x,$$

where P is a city-specific intercept capturing the local cost of living. Land prices r satisfy a similar isoelastic relationship with a steeper gradient ρ,

$$\ln r(x) = \ln R - \rho \ln x,$$

some $R > 0$. The model links the two gradients through $\rho = (1 + \sigma)\theta$, where σ is the local housing supply elasticity.

Using French data on house prices and land transactions, Combes, Duranton, and Gobillon (2019) estimate these within-city gradients by regressing prices per square meter on distance to the urban center, allowing both intercepts and slopes to vary across urban areas. In their preferred specification (Table 3, column 9), which controls for house or parcel characteristics as well as a rich set of municipal geography, geology, income, education,

land-use, and amenity variables, the median log-distance gradient is -0.0088 for housing prices and -0.087 for land prices. These imply (rounding to two decimals) $\theta = 0.01$ and $\sigma = 8.89$. The corresponding first city quartile gradients are -0.0417 for house prices and -0.143 for land prices, implying

$$\theta = 0.04, \quad \sigma = 2.43. \tag{7}$$

Both values for the housing supply elasticity σ lie within the range $[0, 10]$ typically found in the literature (table 1). The first-quartile estimates are more informative for our purposes: cities with steeper price gradients are closer to the monocentric benchmark underlying the model, whereas flatter gradients at the median may reflect more polycentric or spatially dispersed urban structures.

These estimates support the monocentric structure underlying PIGL–AMM in section 5.

3 A tractable urban model with PIGL preferences

3.1 Preferences and housing demand

A representative household in a given city consumes a traded good c, produced with labor only and which we use as the numeraire, and a local non-traded housing service h, produced with land and the numeraire. Let $w > 0$ denote income (wage) and $p > 0$ the unit price of housing.

Following Boppart (2014), preferences take a parsimonious PIGL form, represented by the indirect utility:

$$V(w, p) = \frac{w^{1-\varepsilon} - 1}{1 - \varepsilon} - \mu \frac{p^{1-\nu} - 1}{1 - \nu}, \qquad 0 < \mu \leq 1, \quad 0 \leq \varepsilon, \nu. \tag{8}$$

From Roy's identity, Marshallian demand and expenditure share for housing are:

$$h^D(p, w) = \mu w^{\varepsilon} p^{-\nu}, \qquad e(p, w) = \mu w^{-1+\varepsilon} p^{1-\nu}. \tag{9}$$

Utility is defined for $e \leq 1$. This requires assuming wages are are sufficiently high to cover some subsistence level of housing in the sense $w^{1-\varepsilon} p^{-1+\nu} > \mu$. Empirical estimates reported in section 2 imply values for ε and ν strictly below one, consistent with housing being a necessity and price-inelastic. Parameter μ governs the strength of housing demand.

These PIGL preferences also nest two polar cases used in most of the literature in urban economics:

Lemma 1. *The PIGL indirect utility in equation* (8) *nests Cobb-Douglas preferences with* $e(p,w) = \mu$ *as* $\varepsilon, \nu \to 1$ *and fixed housing consumption* ($h = \mu$) *when* $\varepsilon, \nu = 0$.

Proof. In appendix A. □

In particular, households spend a constant fraction μ of their income on housing if $\varepsilon, \nu \to 1$, or they consume exactly μ units of housing independently of prices and income and utility is linear in the consumption of c if $\varepsilon, \nu = 0$. In the former case, the price- and income-elasticity of housing demand are both equal to one: both relative price and income effects are absent. In the latter, the price- and income-elasticity of housing demand are both zero, an extreme form of a necessary good.

The preferences in equation (8) nests two further special cases. First, if $\nu = \varepsilon$, then the direct utility admits a closed form; specifically, the dual of V in equation (8) is:

$$\nu = \varepsilon \;\; \Rightarrow \;\; U(c,h) = \frac{1}{1-\varepsilon}\left\{c^{1-\varepsilon}\left[1-\left(\frac{h_{\min}}{h}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{\varepsilon} - \left[1-(h_{\min})^{1-\varepsilon}\right]\right\}, \tag{10}$$

where $h_{\min} \equiv \mu^{\frac{1}{1-\varepsilon}}$ can be interpreted as the subsistence level of housing. Second, V defines quasi-linear preferences with price effects if $\varepsilon = 0$ and $\nu \in (0,1)$.

3.2 Housing supply

Housing is produced competitively from land and a numeraire input z. The quantity of land available for housing development L is supplied inelastically. Assume a constant returns to scale Cobb-Douglas production function, written in intensive form (per unit of land) as follows:

$$h = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{\sigma}{1+\sigma}} B^{\frac{1}{1+\sigma}} z^{\frac{\sigma}{1+\sigma}}, \qquad \sigma > 0, \tag{11}$$

where $B > 0$ governs construction total factor productivity, z is numeraire input, and $\frac{\sigma}{1+\sigma}$ is the share of non-land inputs in the production of housing.

Lemma 2. *Under competitive production with technology given in equation* (11) *and exogenous land endowment* L*, aggregate urban housing supply is*

$$H^S(p) = BLp^{\sigma}. \tag{12}$$

Proof. In appendix A. □

Hence, σ is the (constant) housing supply elasticity in equilibrium. This formulation encompasses models that assume the quantity of housing to be proportional to the quantity of land as the limiting case $\sigma \to 0$. Empirical estimates reported in section 2 suggest single-digit values for σ.

Land developers competitively bid for land, hence their profits are zero, and aggregate land rent is a share $\frac{1}{1+\sigma}$ of aggregate revenue pH in the housing sector. Using equation (12), we obtain:

$$R \equiv rL = \frac{B}{1+\sigma} L p^{1+\sigma}. \tag{13}$$

The aggregate land rent is increasing and convex in housing prices and proportional to land supply.

3.3 Equilibrium and the cost of living

Let N denote the number of identical households residing in the city. Market clearing in the housing market requires $Nh^D(p,w) = H^S(p)$. Using equations (9) and (12), we obtain:

Lemma 3. *Assume the housing sector is competitive and produces with the constant-returns-to-scale Cobb-Douglas production function in equation* (11)*. In equilibrium, the price and quantity of housing satisfy*

$$p = \left(\mu \frac{N}{BL}\right)^{\frac{1}{\nu+\sigma}} w^{\frac{\varepsilon}{\nu+\sigma}}, \qquad H = (BL)^{\frac{\nu}{\nu+\sigma}} (\mu N)^{\frac{\sigma}{\nu+\sigma}} w^{\frac{\varepsilon\sigma}{\nu+\sigma}}. \tag{14}$$

Proof. In appendix A. □

The cost of living in the city coincides with the housing price. It increases with population and income, and decreases with land availability and housing productivity. Because housing is a necessity, prices respond more strongly to population than to income.

Combining equations (9) and (14), we obtain:

$$e(N,w) = \mu^{\frac{1+\sigma}{\nu+\sigma}} w^{-1+\varepsilon\frac{1+\sigma}{\nu+\sigma}} \left(\frac{N}{BL}\right)^{\frac{1-\nu}{\nu+\sigma}}. \tag{15}$$

Parameter values are disciplined by the empirical estimates of section 2. In particular equation (15) directly yields the specification in equation (5). We show in online appendix OA.1 that the isoelastic properties of the model (equations (9) and (14) in particular) are preserved if the city is composed of households with heterogeneous incomes.

Henceforth we set $\nu = \varepsilon \in (0, 1)$ and we normalize $\mu = 1$ to streamline equilibrium expressions. This simplifies notation without affecting the core economic mechanisms. Online appendix OA.2 generalizes the results of Proposition 1 below for the case $\nu \neq \varepsilon$.

3.4 Agglomeration and efficient city size

We now introduce agglomeration economies. Suppose that income depends on city population as

$$w = AN^{\alpha}, \qquad A \geq A_{min} > 0, \quad \alpha \geq 0, \tag{16}$$

where A is fundamental productivity and α measures the strength of agglomeration effects. Substituting equations (14) and (16) into the indirect utility (8) yields:

$$V(N) = \frac{1}{1-\varepsilon}\left\{[AN^{\alpha}]^{1-\varepsilon} - \left[A^{\frac{\varepsilon}{\varepsilon+\sigma}}(BL)^{-\frac{1}{\varepsilon+\sigma}}N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right]^{1-\varepsilon}\right\}. \tag{17}$$

This expression reflects the so-called fundamental trade-off in urban economics, whereby larger cities increase both productivity and the cost of living (Henderson 1974; Fujita and Thisse 2013). Utility in the city is maximized at a unique interior city size whenever the elasticity of the cost of living with respect to population exceeds the agglomeration elasticity:

$$\alpha < \frac{1+\alpha\varepsilon}{\varepsilon+\sigma} \;\Leftrightarrow\; \alpha\sigma < 1.$$

This condition ensures that dispersion forces, which arise from housing market crowding, dominate agglomeration forces for city populations large enough. Given $\alpha = 0.03$ and $\sigma \leq 10$ (see table 1), it follows that $\alpha\sigma$ is at about an order of magnitude smaller than 1. We write:

Proposition 1. ***Efficient city size.*** *Suppose wages satisfy equation* (16) *and* $\alpha\sigma < 1$ *holds, then: (i) There exists a unique interior efficient city size, given by*

$$N = \left[\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}}A^{\frac{\sigma}{\varepsilon+\sigma}}(BL)^{\frac{1}{\varepsilon+\sigma}}\right]^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}}. \tag{18}$$

(ii) In the homothetic Cobb-Douglas case, the efficient city size is zero:

$$\lim_{\varepsilon \to 1} N = 0.$$

Proof. In appendix B. □

Cities are at their efficient size in equilibrium if local governments control their city population either directly or indirectly (Albouy, Behrens, Robert-Nicoud, and Seegert 2019; Duranton and Puga 2023). This efficient size is finite whenever the elasticity of the cost of living with respect to population, $\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}$, exceeds the agglomeration elasticity, α, that is, whenever $\alpha\sigma < 1$. The elasticity of efficient city size with respect to technology is increasing in agglomeration elasticity α and housing supply elasticity σ, as a higher α strengthens agglomeration forces and a higher σ relaxes dispersion forces.

Under the isoelastic functional forms for agglomeration and dispersion forces that are ubiquitous in the literature, the absence of a finite efficient city size under homothetic Cobb–Douglas preferences is the result of the combination of homothetic preferences, $\varepsilon = 1$, and unit price elasticity of housing demand, $\nu = 1$. Breaking *either* assumption is sufficient to restore a finite efficient size. Focusing on empirically relevant parameter values yields the following result:

Corollary 1. *Consider the preferences in equation* (8)*. Suppose wages satisfy equation* (16) *and* $\alpha\sigma < 1$ *holds. Then the efficient city size is strictly positive, finite, and increasing in* A*,* B*, and* L *if and only if* $\varepsilon, \nu \in (0, 1)$*.*

Proof. In appendix B. □

Breaking either homotheticity in income or unit price elasticity is sufficient to ensure that congestion forces eventually dominate agglomeration forces, thereby ensuring a finite efficient city size. However, both are jointly needed to generate realistic outcomes where city sizes increase with productivity. If housing demand is homothetic in income and price-inelastic ($\varepsilon = 1$, $\nu < 1$), higher productivity scales housing demand proportionally, so that rising housing costs offset agglomeration gains. If instead housing demand is unit-elastic with respect to prices and a necessity ($\nu = 1$, $\varepsilon < 1$), then households strongly adjust their housing demand to price changes, again allowing productivity gains to be absorbed through housing costs rather than city population. Only when housing demand is inelastic

with respect to both income and prices do increases in productivity translate into larger cities as well as in higher housing costs; appendix B provides algebraic details.

This result clarifies the difference between PIGL preferences and Stone–Geary or fixed-housing specifications commonly used in urban models. Stone–Geary preferences introduce income effects in housing demand and are therefore sufficient to restore a finite efficient city size. However, because Engel curves are linear in income and prices, productivity improvements are capitalized into higher housing prices rather than into population increases. By contrast, PIGL preferences generate empirically relevant income and price effects simultaneously, ensuring not only the existence of finite cities but also the empirically consistent prediction that more productive cities are larger.

4 Urban systems

We now turn to the aggregation of cities as modeled above into an urban system. The literature has considered several equilibrium concepts for aggregation (Albouy, Behrens, Robert-Nicoud, and Seegert 2019; Behrens, Duranton, and Robert-Nicoud 2014; Duranton and Puga 2023; Lee and Li 2013), and in each case we follow the relevant concept. We start by allowing for heterogeneity across locations.

4.1 Heterogeneous cities and identical households

Assume that cities have heterogeneous fundamental amenities, productivity, and land. In online appendix OA.3, we show:

Result. ***Zipf's law.*** *Assume cities are at their efficient scale. If city fundamentals are jointly log-normally distributed then the size distribution of cities is log-normal. Alternatively, if the appropriate weighted geometric average of these fundamentals is Pareto distributed, then the size distribution of cities is also Pareto.*

We formalize this result as proposition 4 in online appendix OA.3. There we also show that the result holds approximately when the size of cities is not their efficient scale, but the outcome of the decentralized equilibrium of the next subsection.

4.2 Heterogeneous cities and heterogeneous preferences

We now allow for idiosyncratic location preferences (Ahlfeldt, Redding, Sturm, and Wolf 2015; Allen and Arkolakis 2025; Diamond 2016; Fajgelbaum, Morales, Suárez Serrato, and

Zidar 2019; Monte, Redding, and Rossi-Hansberg 2018; Redding 2016; Redding 2025), and let the size of cities be determined by location decisions of individual households without constraining them to be at their efficient size. Here, we reduce equation clutter by normalizing the exogenous total urban population to one, and by setting $B = L = 1$ for all cities; hence cities are indexed by the scalar $A \in [A_{min}, \infty)$. Denote the mass of potential sites to host cities by $M > 0$, the equilibrium mass of inhabited cities by $M_+ < M$, the equilibrium set of inhabited cities by $\mathbb{A}_+ \subset [A_{min}, \infty)$, and the associated distribution function by $F_{\mathbb{A}_+}$. Assume households draw idiosyncratic preference shocks that multiply the common component of utility $V(A)$ and that these shocks are Frechet distributed with shape parameter $\kappa > 1$. Then the fraction of households choosing to live in city A is

$$N(A) = \left(\frac{V(A)}{\mathbb{V}}\right)^{\kappa}, \qquad \mathbb{V} \equiv \left[M_+ \int_{x \in \mathbb{A}_+} (V(x))^{\kappa} \, \mathrm{d}F_{\mathbb{A}_+}(x)\right]^{\frac{1}{\kappa}}. \tag{19}$$

The expected utility of households is equal to $\mathbb{E}U = \Gamma\left(1 - \frac{1}{\kappa}\right)\mathbb{V}$, where Γ is the gamma function, and it increases in the mass of cities M_+. This property can be interpreted as reflecting a taste for variety of local amenities.

A useful property of Frechet-distributed multiplicative preference shocks is that the expected utility in inhabited city A is proportional to $(N(A))^{-\frac{1}{\kappa}} V(A)$, which is equal to $\mathbb{V}$ by inversion of the first expression in equation (19). Thus, in spatial equilibrium, the expected utility is equalized across inhabited cities and $N(A)$ solves:

$$\frac{1}{1-\varepsilon}\left[(AN^{\alpha})^{1-\varepsilon} - \left(A^{\frac{\varepsilon}{\varepsilon+\sigma}} N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right)^{1-\varepsilon}\right] = \mathbb{V} N^{\frac{1}{\kappa}}. \tag{20}$$

The left-hand side of this expression trades off agglomeration benefits against urban costs, and is akin to a local labor demand. The right-hand side is the local labor supply, and a lower κ (higher preference heterogeneity) yields a lower labor supply elasticity. Preference heterogeneity across locations thus acts as an additional dispersion force.[7]

Define the wedge between the equilibrium city size $N(A)$ in equation (20) and the efficient scale, in equation (18), denoted $N^*(A)$, as

$$\omega(A) \equiv \frac{N(A)}{N^*(A)}. \tag{21}$$

[7] As $\kappa \to 1$, heterogeneity in the preference for amenities of households rises, increasing the ex-ante value of variety under idiosyncratic taste uncertainty and reinforcing this dispersion force.

We obtain:

Proposition 2. ***Spatial equilibrium with heterogeneous location preferences.*** *Assume*

$$(1-\varepsilon)\,\kappa < \sigma, \quad \textit{and} \quad \frac{1+\alpha\varepsilon}{\varepsilon+\sigma}\sigma < 1. \tag{22}$$

Then (i) given total population, there is a unique equilibrium in which all cities are populated, (namely, $\mathbb{A}_+ = [A_{min}, \infty)$ and $M_+ = M$), and expected utility in any city is equal to $\Gamma\left(1-\frac{1}{\kappa}\right)\mathbb{V}$; (ii) for every $A \geq A_{min}$ there exists a unique unique equilibrium wedge $\omega(A) \in (0, \overline{\omega})$; (iii) $N(A)$ is increasing in A and $\omega(A)$ is decreasing in A; (iv) if $\omega(A_{min}) > 1$ then there exists a unique thresholds $\overline{A} > A_{min}$ such that

$$\omega\left(\overline{A}\right) = 1, \qquad \forall A \neq \overline{A}:\ \left(\omega(A) - 1\right)\left(\overline{A} - A\right) > 0.$$

Proof. In appendix B. □

With heterogeneous location preferences, households trade off agglomeration benefits against housing-market crowding and idiosyncratic tastes for locations. For any given city productivity A, expected utility is decreasing in population because the dispersion induced by taste heterogeneity dominates agglomeration effects net of housing costs when the first condition in equation (22) holds. Specifically, $N^{-1/\kappa}V(N)$ decreases monotonically from $+\infty$ as $N \to 0$, so that equation (20) admits a unique solution for N for any $\mathbb{V} > 0$, as in quantitative spatial models with Cobb-Douglas preferences.

In spatial equilibrium with individual household mobility, households sort across cities until expected utility is equalized across all inhabited locations. Empirical estimates of the local labor supply $\frac{1}{\kappa}$ are small enough that highly productive cites attract residents below their efficient size, while less productive cities are larger than their efficient scale. In this sense, large cities are too small and small cities are too large.

If the first condition in equation (22) is violated, by contrast, agglomeration effects dominate both housing costs and the dispersion induced by taste heterogeneity at very small city sizes. At larger sizes, housing-market crowding and preference dispersion dominate. In spatial equilibrium with individual mobility, highly productive cities attract residents beyond their efficient size, while less productive cities remain uninhabited, as we show in online appendix OA.4. The marginal inhabited city is exactly at its efficient size, as it is just attractive enough to draw residents. This result highlights a key distinction with existing approaches. Models with fixed housing supply generate an extensive margin, with only a

subset of locations populated in equilibrium, but at the cost of ruling out adjustment in individual housing consumption. By contrast, models with homothetic preferences imply that all locations are inhabited, as housing consumption adjusts without bound. Our framework combines both features: housing demand is price-sensitive, while preference heterogeneity may select a finite set of inhabited cities. As a result, the model can avoid both the zero efficient scale of homothetic specifications and the counterfactual rigidity of fixed-housing environments.

4.3 Sorting across cities

In online appendix OA.5 we embed our urban model into an economy with many cities with identical city fundamentals but workers endowed with heterogeneous efficiency units of labor. We characterize sorting across cities in the spirit of Behrens, Duranton, and Robert-Nicoud (2014), and we show:

Result. ***Sorting equilibrium.*** *In the sorting equilibrium, cities hosting more talented workers are larger, pay higher wages, and feature higher housing costs, reflecting stronger agglomeration forces and tighter housing markets.*

We formalize this result as proposition 7 in online appendix OA.5. These equilibrium properties are in line with empirical evidence (Combes, Duranton, and Gobillon 2008; Diamond 2016; Moretti 2013).

4.4 Redistribution

Most OECD countries combine progressive income taxation with housing policies that reduce the effective cost of housing for at least some households (e.g., housing vouchers, rent subsidies, tax deductions, or social housing). We show in online appendix OA.6 that the combination of progressive income taxation with a flat housing subsidy compresses equilibrium city-size differences and increase land use relative to the benchmark economy without fiscal distortions.

5 PIGL-AMM: Monocentric city model

The PIGL-UM model abstracts from within-city spatial structure. In this section, we introduce urban space, commuting costs, and endogenous land use, and show that the

space-free equilibrium characterized in section 3 arises as the exact reduced form of a monocentric city model when commuting costs enter multiplicatively with housing costs, rather than from the more standard additive formulation. We label this spatial model PIGL-AMM, in reference to Alonso (1964), Mills (1967), and Muth (1969). See Duranton and Puga (2015) for a recent treatment. Here we only provide a brief sketch of the model; online appendix section OA.7 provides the comprehensive analysis.

5.1 Preferences

Households work and consume in a central business district (CBD) and reside at distance $x \geq 0$ from the CBD. Commuting is costly. Space extends in all directions on the Euclidean space. Let $p(x)$ denote the unit price of housing at location x.

Preferences extend the PIGL structure in equation (8) by allowing commuting costs and housing consumption to enter utility jointly. Setting $\nu = \varepsilon$ and $\mu = 1$, indirect utility is given by:

$$V(w,x) = \frac{1}{1-\varepsilon}\left[w^{1-\varepsilon} - \left(p(x)\,\tau x^{\theta}\right)^{1-\varepsilon}\right], \tag{23}$$

where $\tau > 0$ is a commuting cost shifter and $\theta > 0$ governs the elasticity of commuting cost with respect to distance. When $\theta = 0$ and $\tau = 1$, this formulation collapses to the space-free PIGL-UM preferences of section 3.

At the spatial equilibrium within a city, residents must be indifferent across all city locations, so $V(w,x) = V(w,x')$ for all $x, x' \leq \overline{x}$, where $\overline{x}$ denotes the endogenous urban fringe. Then, the expression in equation (23) implies

$$\forall x \leq \overline{x}: \qquad p(x)\,\tau x^{\theta} = P, \tag{24}$$

where, like p in the model of section 3, P is a sufficient statistic for the urban *cost of living* faced by all residents. Specifically, $P \equiv p(1)\,\tau$ combines the housing price and the commuting cost at the reference location $x = 1$. Substituting the spatial equilibrium condition of equation (24) into equation (23) yields

$$V(P,w,x) = \frac{1}{1-\varepsilon}\left(w^{1-\varepsilon} - P^{1-\varepsilon}\right). \tag{25}$$

5.2 Equilibrium cost of living

Housing supply and the rest of the model are similar to that of section 3. We show in online appendix section OA.7 that the equilibrium cost of living obeys:

$$P = \left(\frac{N}{\Lambda}\right)^{\frac{1}{\varepsilon+\eta}} w^{\frac{\varepsilon}{\varepsilon+\eta}}, \tag{26}$$

where $\eta \equiv \frac{2-\theta}{\theta}$ and Λ collects parameters.

Crucially, equations (26) and (14) are formally isomorphic, with P (the cost of living combining housing prices and commuting costs) replacing p (the equilibrium housing price). The housing supply elasticity σ and land endowment L of the space-free PIGL-UM model are replaced here by η and Λ, which summarize commuting technology, housing production technology, and land availability. We have shown:

Proposition 3. ***PIGL-UM and PIGL-AMM.*** *The space-free PIGL-UM model of section 3 arises as the exact reduced form of the PIGL-AMM model of this section.*

Proof. In appendix B. □

From this point onward, the monocentric city PIGL-AMM model collapses to PIGL-UM, and hence all the results of sections 3 and 4 apply to PIGL-AMM, too.

5.3 Gradients

Equations (24) and (45) from the online appendix give us the housing price gradient and building height gradient::

$$\mathcal{E}\left(p, x\right) \equiv \frac{\partial \ln p\left(x\right)}{\partial \ln x} = -\theta, \qquad \mathcal{E}\left(h^S, x\right) \equiv \frac{\partial \ln h^S\left(x\right)}{\partial \ln x} = -\sigma\theta,$$

where σ is the housing supply elasticity as in section 3. Summing the two, we obtain the gradient of the land price $r\left(x\right)$:

$$\mathcal{E}\left(r, x\right) \equiv \frac{\partial \ln r\left(x\right)}{\partial \ln x} = -\left(1+\sigma\right)\theta.$$

It follows that housing prices, land prices, built-up density, and population density all decay with distance to the CBD, while housing consumption increases. Furthermore, a higher θ makes proximity to the center more desirable, resulting in steeper price and density

gradients. These properties are consistent with the large body of literature reviewed in Duranton and Puga (2015).

6 Summary and conclusion

This paper develops PIGL-UM, a unifying and analytically tractable framework for urban analysis that incorporates empirically relevant income and price effects in housing demand. By adopting non-homothetic PIGL preferences, the model nests the two polar cases dominating the literature and restores a non-degenerate interior efficient city size under weak and realistic conditions. The framework delivers closed-form solutions for housing markets, land rents, and city scale, and extends naturally to urban systems with heterogeneous fundamentals, preferences, and worker talent.

Another central contribution of the paper is to show that PIGL-UM is not a reduced-form shortcut, but the exact equilibrium representation of a fully specified monocentric city model with commuting costs, endogenous land use, and spatial price gradients. This spatial extension, labeled PIGL-AMM, collapses to a single sufficient statistic — the urban cost of living — so that all equilibrium conditions governing city size and welfare carry over unchanged. Together, PIGL-UM and PIGL-AMM provide a unified foundation for analyzing city size, housing markets, sorting, and internal urban structure within a single framework.

References

Ahlfeldt, Gabriel M. and Daniel P. McMillen (2014). "New Estimates of the Elasticity of Substitution of Land for Capital". In: *Regional Science and Urban Economics* 48, pp. 1–15. DOI: 10.1016/j.regsciurbeco.2014.03.002.

Ahlfeldt, Gabriel M., Stephen Redding, Daniel M. Sturm, and Nikolaus Wolf (2015). "The Economics of Density: Evidence from the Berlin Wall". In: *Econometrica* 83.6, pp. 2127–2189. DOI: 10.3982/ECTA10876. URL: https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA10876.

Albouy, David (2009). "The Unequal Geographic Burden of Federal Taxation". In: *Journal of Political Economy* 117.4, pp. 635–667. DOI: 10.1086/605309.

Albouy, David, Kristian Behrens, Frédéric Robert-Nicoud, and Nathan Seegert (2019). "The optimal distribution of population across cities". In: *Journal of Urban Economics* 110, pp. 102–113. DOI: 10.1016/j.jue.2018.08.004.

Albouy, David, Gabriel Ehrlich, and Yingyi Liu (Nov. 2016). *Housing Demand, Cost-of-Living Inequality, and the Affordability Crisis*. NBER Working Paper 22816. National Bureau of Economic Research. DOI: 10.3386/w22816. URL: https://www.nber.org/papers/w22816.

Alder, Simon, Timo Boppart, and Andreas Müller (Apr. 2022). "A Theory of Structural Change That Can Fit the Data". In: *American Economic Journal: Macroeconomics* 14.2, pp. 160–206. DOI: 10.1257/mac.20190303.

Allen, Treb and Costas Arkolakis (2025). "Quantitative Regional Economics". In: *Handbook of Regional and Urban Economics*. Elsevier, pp. 1–72. DOI: 10.1016/bs.hesreg.2025.05.002.

Alonso, William (1964). *Location and Land Use: Toward a General Theory of Land Rent*. Cambridge, MA: Harvard University Press.

Baum-Snow, Nathaniel and Gilles Duranton (2025). "Housing Supply and Housing Affordability". In: *Handbook of Regional and Urban Economics*. Ed. by Dave Donaldson and Stephen J. Redding. Vol. 6. 1. Elsevier, pp. 353–461. DOI: 10.1016/bs.hesreg.2025.05.003.

Baum-Snow, Nathaniel and Lu Han (2024). "The Microgeography of Housing Supply". In: *Journal of Political Economy* 132.6, pp. 1897–1946. DOI: 10.1086/728110.

Beaudry, Paul, David A. Green, and Benjamin M. Sand (2014). "Spatial equilibrium with unemployment and wage bargaining: Theory and estimation". In: *Journal of Urban Economics* 79, pp. 2–19. DOI: 10.1016/j.jue.2013.08.005.

Behrens, Kristian, Gilles Duranton, and Frédéric Robert-Nicoud (2014). "Productive Cities: Sorting, Selection, and Agglomeration". In: *Journal of Political Economy* 122.3, pp. 507–553. DOI: 10.1086/675534.

Berger, David, Veronica Guerrieri, Guido Lorenzoni, and Joseph Vavra (2018). "House Prices and Consumer Spending". In: *Review of Economic Studies* 85.3, pp. 1502–1542. DOI: 10.1093/restud/rdx060.

Boppart, Timo (2014). "Structural Change and the Kaldor Facts in a Growth Model With Relative Price Effects and Non-Gorman Preferences: Structural Change and the Kaldor Facts". In: *Econometrica* 82.6, pp. 2167–2196. DOI: 10.3982/ecta11354.

Chatterjee, Shoumitro, Elisa Giannone, Tatjana Kleineberg, Kan Kuno, and Luca Looser (2025). "Unequal Global Convergence: Structural Change, Economic Geography, and Regional Inequality". Working paper. URL: https://sites.google.com/view/elisagiannone/research/papers.

Chen, Ziyang, Pierre-Philippe Combes, Sylvie Démurger, and Xiuyan Liu (2025). "Housing Costs and Real Income Differences across Chinese Cities". In: *Journal of Development Economics* 176, p. 103527. ISSN: 0304-3878. DOI: 10.1016/j.jdeveco.2025.103527.

Combes, Pierre-Philippe, Gilles Duranton, and Laurent Gobillon (2008). "Spatial wage disparities: Sorting matters!" In: *Journal of Urban Economics* 63.2, pp. 723–742. DOI: 10.1016/j.jue.2007.04.004.

— (2019). "The Costs of Agglomeration: House and Land Prices in French Cities". In: *The Review of Economic Studies* 86.4, pp. 1556–1589. DOI: 10.1093/restud/rdy063.

— (2021). "The Production Function for Housing: Evidence from France". In: *Journal of Political Economy* 129.10, pp. 2766–2816. DOI: 10.1086/715076.

Combes, Pierre-Philippe, Gilles Duranton, Laurent Gobillon, Clément Gorin, and Frédéric Robert-Nicoud (2026). "Urbanisation and urban divergence: France 1760–2020." In: *in progress*.

Combes, Pierre-Philippe, Gilles Duranton, Laurent Gobillon, and Sébastien Roux (2010). "Estimating Agglomeration Economies with History, Geology, and Worker Effects". In: *Agglomeration Economics*. Ed. by Edward L. Glaeser. Chicago, IL: University of Chicago Press, pp. 15–66. DOI: 10.7208/chicago/9780226297927.003.0002.

Combes, Pierre-Philippe and Laurent Gobillon (2015). "The Empirics of Agglomeration Economies". In: *Handbook of Regional and Urban Economics*. Ed. by Gilles Duranton, J. Vernon Henderson, and William C. Strange. Vol. 5. Elsevier, pp. 247–348. DOI: 10.1016/B978-0-444-59517-1.00005-2.

Comin, Diego, Danial Lashkari, and Martí Mestieri (2021). "Structural Change with Long-Run Income and Price Effects". In: *Econometrica* 89.1, pp. 311–374. DOI: 10.3982/ECTA16317.

Couture, Victor, Cécile Gaubert, Jessie Handbury, and Erik Hurst (2023). "Income Growth and the Distributional Effects of Urban Spatial Sorting". In: *Review of Economic Studies* 91.2, pp. 858–898. DOI: 10.1093/restud/rdad048.

Deaton, Angus S. and John Muellbauer (1980). "An Almost Ideal Demand System". In: *American Economic Review* 70.3, pp. 312–326.

Diamond, Rebecca (2016). "The Determinants and Welfare Implications of US Workers' Diverging Location Choices by Skill: 1980–2000". In: *American Economic Review* 106.3, pp. 479–524. DOI: 10.1257/aer.20131706.

DiPasquale, Denise and William C. Wheaton (1994). "Housing Market Dynamics and the Future of Housing Prices". In: *Journal of Urban Economics* 35.1, pp. 1–27.

Duranton, Gilles and Jessie Handbury (Apr. 2023). *Covid and Cities, Thus Far*. Working Paper 31158. National Bureau of Economic Research.

Duranton, Gilles and Diego Puga (2014). "The Growth of Cities". In: *Handbook of Economic Growth*. Elsevier, pp. 781–853. DOI: 10.1016/b978-0-444-53540-5.00005-7.

— (2015). "Urban Land Use". In: *Handbook of Regional and Urban Economics*. Ed. by Gilles Duranton, J. Vernon Henderson, and William S. Strange. Vol. 5. Amsterdam: Elsevier. Chap. 8, pp. 467–560. DOI: 10.1016/B978-0-444-59517-1.00008-1.

— (2023). "Urban Growth and Its Aggregate Implications". In: *Econometrica* 91.6, pp. 2219–2259. DOI: 10.3982/ecta17936.

Eckert, Fabian and Michael Peters (2025). *Spatial Structural Change*. Working Paper.

Eeckhout, Jan (2004). "Gibrat's Law for (All) Cities". In: *American Economic Review* 94.5, pp. 1429–1451. DOI: 10.1257/0002828043052303.

Eeckhout, Jan, Roberto Pinheiro, and Kurt Schmidheiny (2014). "Spatial Sorting". In: *Journal of Political Economy* 122.3, pp. 554–620. DOI: 10.1086/675534.

Eriksen, Michael D. and Amanda Ross (2015). "Housing Vouchers and the Price of Rental Housing". In: *American Economic Journal: Economic Policy* 7.3, pp. 154–176. DOI: 10.1257/pol.20130064.

Fajgelbaum, Pablo D., Eduardo Morales, Juan Carlos Suárez Serrato, and Owen M. Zidar (2019). "State Taxes and Spatial Misallocation". In: *The Review of Economic Studies* 86.1. Published version of the spatial misallocation paper., pp. 333–376. DOI: 10.1093/restud/rdy050.

Fan, Xiaoxuan, Michael Peters, and Fabrizio Zilibotti (2023). "Growing Like India: Structural Change with Non-Homothetic Preferences". In: *Econometrica* 91.4, pp. 1457–1494. DOI: 10.3982/ECTA20964.

Favilukis, Jack, Sydney C. Ludvigson, and Stijn Van Nieuwerburgh (Feb. 2017). "The Macroeconomic Effects of Housing Wealth, Housing Finance, and Limited Risk Sharing in General Equilibrium". In: *Journal of Political Economy* 125.1, pp. 140–223. DOI: 10.1086/689606.

Finlay, John and Trevor C. Williams (2025). "Housing Demand, Inequality, and Spatial Sorting". In: *Journal of International Economics* 158, p. 104171. DOI: 10.1016/j.jinteco.2025.104171.

Fretz, Stephan, Raphaël Parchet, and Frédéric Robert-Nicoud (2022). "Highways, Market Access and Spatial Sorting". In: *The Economic Journal* 132.643, pp. 1011–1036.

Fujita, Masahisa (1989). *Urban Economic Theory: Land Use and City Size*. Cambridge: Cambridge University Press.

Fujita, Masahisa and Hideaki Ogawa (1982). "Multiple Equilibria and Structural Transition of Non-Monocentric Urban Configurations". In: *Regional Science and Urban Economics* 12.2, pp. 161–196. DOI: 10.1016/0166-0462(82)90031-X.

Fujita, Masahisa and Jacques-François Thisse (2013). *Economics of Agglomeration: Cities, Industrial Location, and Globalization*. 2nd ed. Cambridge, United Kingdom: Cambridge University Press.

Gabaix, Xavier and Rustam Ibragimov (2011). "Rank-1/2: A Simple Way to Improve the OLS Estimation of Tail Exponents". In: *Journal of Business & Economic Statistics* 29.1, pp. 24–39. DOI: 10.1198/jbes.2009.06157.

Gaigné, Carl, Hans R. A. Koster, Fabien Moizeau, and Jacques-François Thisse (2022). "Who Lives Where in the City? Amenities, Commuting and Income Sorting". In: *Journal of Urban Economics* 128, p. 103394. DOI: 10.1016/j.jue.2021.103394.

Gaubert, Cécile and Frédéric Robert-Nicoud (2025). *Sorting to Expensive Cities*. NBER Working Paper. DOI: 10.3386/w33652.

Glaeser, Edward L., Joseph Gyourko, and Raven E. Saks (2006). "Why Have Housing Prices Gone Up?" In: *Journal of Urban Economics* 60.1, pp. 1–26.

Harari, Mariaflavia (2020). "Cities in Bad Shape: Urban Geometry in India". In: *American Economic Review* 110.8, pp. 2377–2421. DOI: 10.1257/aer.20171673.

Heathcote, Jonathan, Kjetil Storesletten, and Giovanni L. Violante (2017). "Optimal Tax Progressivity: An Analytical Framework". In: *Quarterly Journal of Economics* 132.4, pp. 1693–1754. DOI: 10.1093/qje/qjx018.

Henderson, J. Vernon (1974). "The Sizes and Types of Cities". In: *American Economic Review* 64.4, pp. 640–656. DOI: 10.2307/1813316.

Hilber, Christian A. L. and Frédéric Robert-Nicoud (2013). "On the origins of land use regulations: Theory and evidence from US metro areas". In: *Journal of Urban Economics* 75.75, pp. 29–43. DOI: 10.1016/j.jue.2012.10.002.

Iacoviello, Matteo (2005). "House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle". In: *American Economic Review* 95.3, pp. 739–764. DOI: 10.1257/0002828054201477.

Iacoviello, Matteo and Stefano Neri (2010). "Housing Market Spillovers: Evidence from an Estimated DSGE Model". In: *American Economic Journal: Macroeconomics* 2.2, pp. 125–164. DOI: 10.1257/mac.2.2.125.

Ioannides, Yannis M. and L. Rachel Ngai (2025). "Housing and Inequality". In: *Journal of Economic Literature* 63.3, pp. 916–963. DOI: 10.1257/jel.20251659.

Kaplan, Greg, Kurt Mitman, and Giovanni L. Violante (2020). "The Housing Boom and Bust: Model Meets Evidence". In: *Journal of Political Economy* 128.9, pp. 3285–3345. DOI: 10.1086/708816.

Lee, Sanghoon and Qiang Li (2013). "Uneven landscapes and city size distributions". In: *Journal of Urban Economics* 78, pp. 19–29. DOI: 10.1016/j.jue.2013.05.001.

Lucas, Robert E. Jr. and Esteban Rossi-Hansberg (2002). "On the Internal Structure of Cities". In: *Econometrica* 70.4, pp. 1445–1476. DOI: 10.1111/1468-0262.00336.

Mills, Edwin S. (1967). "An Aggregative Model of Resource Allocation in a Metropolitan Area". In: *American Economic Review* 57.2, pp. 197–210.

Monte, Ferdinando, Stephen Redding, and Esteban Rossi-Hansberg (2018). "Commuting, Migration, and Local Employment Elasticities". In: *American Economic Review* 108.12, pp. 3855–3890. DOI: 10.1257/aer.20151507. URL: https://www.aeaweb.org/articles?id=10.1257/aer.20151507.

Moretti, Enrico (2013). "Real Wage Inequality". In: *American Economic Journal: Applied Economics* 5.1, pp. 65–103. DOI: 10.1257/app.5.1.65.

Muellbauer, John (1975). "Aggregation, Income Distribution and Consumer Demand". In: *The Review of Economic Studies* 42.4, p. 525. DOI: 10.2307/2296792.

— (1976). "Community Preferences and the Representative Consumer". In: *Econometrica* 44.5, p. 979. DOI: 10.2307/1911540.

Muth, Richard F. (1969). *Cities and Housing*. Chicago, IL: University of Chicago Press.

Piazzesi, Monika and Martin Schneider (2016). "Housing and Macroeconomics". In: *Handbook of Macroeconomics*. Ed. by John B. Taylor and Harald Uhlig. Vol. 2. Elsevier, pp. 1547–1640. DOI: 10.1016/bs.hesmac.2016.03.002.

Pines, David and Efraim Sadka (1986). "Comparative Statics Analysis of a Fully Closed City". In: *Journal of Urban Economics* 20.1, pp. 1–20. DOI: 10.1016/0094-1190(86)90021-3.

Redding, Stephen (2016). "Goods Trade, Factor Mobility and Welfare". In: *Journal of International Economics* 101, pp. 148–167. DOI: 10.1016/j.jinteco.2016.04.006. URL: https://www.sciencedirect.com/science/article/pii/S0022199616300274.

— (2025). "Quantitative Urban Economics". In: *Handbook of Regional and Urban Economics*. Elsevier, pp. 73–141. DOI: 10.1016/bs.hesreg.2025.06.007.

Rozenfeld, Hernán D., Diego Rybski, Xavier Gabaix, and Hernán A. Makse (2011). "The Area and Population of Cities: New Insights from a Different Perspective on Cities". In: *American Economic Review* 101.5, pp. 2205–2225. DOI: 10.1257/aer.101.5.2205.

Saiz, Albert (2010). "The Geographic Determinants of Housing Supply". In: *Quarterly Journal of Economics* 125.3, pp. 1253–1296.

Solow, Robert M. (1973). "Congestion Cost and the Use of Land for Streets". In: *The Bell Journal of Economics and Management Science* 4.2, pp. 602–618. URL: https://www.jstor.org/stable/3003055.

Wu, Jing, Joseph Gyourko, and Yongheng Deng (2016). "Evaluating the Risk of Chinese Housing Markets: What We Know and What We Need to Know". In: *Journal of Economic Perspectives* 30.1, pp. 163–190.

Table

Table 1: Empirical discipline and implied parameter values.

Object	Parameter(s)	Value	Sources
Income elasticity of housing demand	ε	0.7–0.9	CCDL 2025, CDG 2019, 2021 (implied); FW 2025.
Price elasticity of housing demand	ν	0.6–0.9	CDG 2019, 2021 (implied); FW 2025.
Housing supply elasticity	σ	$0 - 10$	BSD 2025; DW 1994; FW 2025; GGS 2006; S 2010; WGD 2016.
Land share in housing supply	$\frac{1}{1+\sigma}$	$\frac{1}{3} \to \sigma = 2$	AM 2014; BSD 2025; CDG 2021.
Ratio of land-price to housing-price gradients	$1+\sigma$	$3.43 \to \sigma = 2.43$	CDG 2019.
Labor productivity elasticity wrt population	α	0.02–0.05	CDG 2008, CDGR 2010.
Urban cost elasticity wrt population	$\frac{1-\nu}{\varepsilon+\sigma}$	0.03–0.08	CDG 2019, 2021.
Local labor supply elasticity	κ	$2.1 - 5.1$	BGS 2014; D 2016; FW 2025.
Pareto shape of city size distribution	ζ	$0.84 - 1.43$	CDGGRN 2026.
Commuting cost elasticity wrt distance to CBD	θ	0.04–0.08	CDG 2019; DP 2023.

Notes: Reported elasticities from existing work (Ahlfeldt and McMillen 2014; Baum-Snow and Duranton 2025; Beaudry, Green, and Sand 2014; Chen, Combes, Démurger, and Liu 2025; Combes, Duranton, and Gobillon 2008; Combes, Duranton, and Gobillon 2019; Combes, Duranton, and Gobillon 2021; Combes, Duranton, Gobillon, Gorin, and Robert-Nicoud 2026; Combes, Duranton, Gobillon, and Roux 2010; Diamond 2016; DiPasquale and Wheaton 1994; Duranton and Puga 2023; Finlay and Williams 2025; Glaeser, Gyourko, and Saks 2006; Saiz 2010; Wu, Gyourko, and Deng 2016). These map into the equilibrium relationships of the model and jointly discipline the key preference and technology parameters. Ranges reflect cross-study variation.

Appendix

A Proofs of lemmas

Lemma 1.

Proof. Taking limits of (8), and using L'Hôpital's rule when $\varepsilon, \nu \to 1$, we get:

$$\lim_{\varepsilon,\nu\to 0} V(p,w) = w - \mu p - (1-\mu), \qquad \lim_{\varepsilon,\nu\to 1} V(p,w) = \ln w - \mu \ln p.$$

□

Using the direct form of utility in (10), we get $\lim_{\varepsilon\to 1} U(c,h) = (1-\mu)\ln c + \mu \ln h$. Turning to $\varepsilon \to 0$, we find

$$h > \mu^{\frac{1}{1-\varepsilon}} \;\Rightarrow\; \lim_{\varepsilon\to 0} U(c,h) = c \lim_{\varepsilon\to 0} \left[1 - \left(\mu^{-\frac{1}{1-\varepsilon}} h\right)^{-\frac{1}{\varepsilon}+1}\right]^{\varepsilon} = c,$$

where the second equality holds for any $h > h_{\min} \equiv \mu^{\frac{1}{1-\varepsilon}}$. The results then follow by standard limits of (9).

Lemma 2.

Proof. A developer chooses z to maximize profit per unit of land:

$$\pi(z) = p\left(\frac{1+\sigma}{\sigma}\right)^{\frac{\sigma}{1+\sigma}} B^{\frac{1}{1+\sigma}} z^{\frac{\sigma}{1+\sigma}} - z - r,$$

where r is the unit bid rent. Solving the first-order condition for z gives

$$z = \frac{\sigma}{1+\sigma} B p^{1+\sigma}.$$

Substituting into (11)yields the result. □

Lemma 3.

Proof. Equations (9) and (12) imply (14). □

B Proofs of propositions

Proposition 1. Efficient city size.

Proof. (i) V is concave and increasing in N at zero whenever

$$\alpha(1-\varepsilon) < \frac{1+\alpha\varepsilon}{\varepsilon+\sigma}(1-\varepsilon) \quad \Leftrightarrow \quad \alpha\sigma < 1.$$

(ii) Note that $\frac{1-\alpha\sigma}{1+\alpha\epsilon}$ is lower than 1 if $\alpha\sigma < 1$. Hence, taking the limit $\varepsilon \to 1$ from below, we have:

$$\lim_{\varepsilon\to 1} N = \left[A^{\frac{\sigma}{1+\sigma}}(BL)^{\frac{1}{1+\sigma}} \lim_{\varepsilon\to 1}\left\{\left(1-\frac{1-\alpha\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}}\right\}\right]^{\frac{1+\alpha}{1-\alpha\sigma}} = 0.$$

□

Corollary 1.

Proof. The efficient size N is strictly positive and finite if and only if

$$(1+\sigma)\varepsilon > \nu + \sigma - \frac{1-\nu}{\alpha}.$$

Under this condition, the efficient size N is increasing in A if and only if

$$\nu + \sigma > (1+\sigma)\varepsilon.$$

Thus both conditions hold simultaneously only if $\nu \in (0,1)$. Then the inequality above holds only if $\varepsilon < 1$. □

Proposition 2. Spatial equilibrium with heterogeneous location preferences.

Proof. (i) Rewrite equation (20) as follows:

$$\begin{aligned}\mathbb{V} = v(A,N) &\equiv \frac{1}{1-\varepsilon}N^{-\frac{1}{\kappa}}\left[(AN^{\alpha})^{1-\varepsilon} - \left(A^{\frac{\varepsilon}{\varepsilon+\sigma}}N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right)^{1-\varepsilon}\right] \\ &= \frac{1}{1-\varepsilon}A^{1-\varepsilon}N^{\frac{\alpha\kappa(1-\varepsilon)-1}{\kappa}}\left[1-\left(A^{-\frac{\sigma}{\varepsilon+\sigma}}N^{\frac{1-\alpha\sigma}{\varepsilon+\sigma}}\right)^{1-\varepsilon}\right].\end{aligned}$$

The conditions in equation (22) imply that v is unbounded as $N \to 0$, namely, $\lim_{N\to 0} v(A,N) = +\infty$. More generally:

$$\lim_{N\to 0} v(A,N) = \begin{cases} +\infty, & \alpha\kappa(1-\varepsilon) < 1 \\ \frac{1}{1-\varepsilon}A^{1-\varepsilon}, & \alpha\kappa(1-\varepsilon) = 1 \\ 0, & \alpha\kappa(1-\varepsilon) > 1. \end{cases}$$

Under these conditions v is also decreasing in N for all N such that $v \geq 0$. Thus all cities are populated and $N(A)$ is the unique, finite solution to $v(A, N) = \mathbb{V}$ given $\mathbb{V} > 0$. (ii) Since $N(A)$ and $N^*(A)$ are both strictly positive and finite, it follows that the wedge $\omega(A)$ defined in equation (21) is strictly positive and finite, hence $\omega(A) \in (0, \bar{\omega})$ for all $A \in (A_{min}, \infty)$, some $\bar{\omega} > 0$. (iii) That $N(A)$ is increasing in A follows from $\partial v/\partial N < 0$ and $\partial v/\partial A > 0$. Using equation (18), we plug $N(A) = \omega N^*(A)$ into equation (21) and obtain:

$$\mathbb{V} = v(A, \omega) = \frac{1}{1-\varepsilon}\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right)^{-\frac{\varepsilon+\sigma}{1-\alpha\sigma}\frac{1-\alpha\kappa(1-\varepsilon)}{\kappa(1-\varepsilon)}} A^{-\frac{\sigma-\kappa(1-\varepsilon)}{\kappa(1-\alpha\sigma}}\omega^{-\frac{1}{\kappa}}\left[\omega^{\alpha(1-\varepsilon)} - \alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\omega^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}(1-\varepsilon)}\right],$$

which is decreasing in both A and ω for all values of A and ω such that $v \geq 0$. Hence, the unique solution $\omega(A)$ to $v(A, \omega) = \mathbb{V}$ is decreasing in A. (iv) By continuity, then, $\omega(A_{min}) = \bar{\omega}$ and $\lim_{A\to\infty}\omega(A) = 0$. If $\bar{\omega} > 1$, then $\exists \bar{A} > A_{min}$ such that $\omega\left(\overline{A}\right) = \bar{\omega}$. □

Proposition 3. PIGL-UM and PIGL-AMM.

Proof. The level of indirect utility in the PIGL-UM model of section 3, given by equation (8), and the equilibrium level of indirect utility in the PIGL-AMM model of section 5, given by equation (43), are isomorphic. In turn, the equilibrium expressions for the cost of living – equations (14) for PIGL-UM and (47) for PIGL-AMM – are isomorphic, too. □

Online Appendix (not for publication)

OA.1 Heterogeneous incomes in the city

Consider the city of section 3 inhabited by households with heterogeneous incomes, with cdf $G_w(w)$ over support Ω. Summing individual housing demand in equation (3) yields aggregate housing demand in the city equal to

$$H^D(p, N, G(w)) = \mu N (\bar{w}_\varepsilon)^\varepsilon p^{-\nu}, \ \bar{w}_\varepsilon \equiv \left(\int_\Omega w^\varepsilon \mathrm{d}G_w(w) \right)^{\frac{1}{\varepsilon}}.$$

Thus, aggregation preserves the isoelastic properties of individual housing demand and the result in lemma 1 if average income is defined appropriately. Note that the generalized average wage $\bar{w}$ is approximately equal to the arithmetic average wage $\bar{w}_1$ corrected by the Theil index measure of wage inequality $T \in (0, 1)$:

$$\bar{w}_\varepsilon \simeq \bar{w}_1 [1 - (1 - \varepsilon) T], \qquad T \equiv \int_\Omega \frac{w}{\bar{w}_1} \ln \left(\frac{w}{\bar{w}_1} \right) \mathrm{d}G_w(w).$$

In turn, the equilibrium housing price and quantity in equation (14) of lemma 3 becomes

$$p = \left(\mu \frac{N}{BL} \right)^{\frac{1}{\nu+\sigma}} (\bar{w}_\varepsilon)^{\frac{\varepsilon}{\varepsilon+\sigma}}, \qquad H = (BL)^{\frac{\nu}{\nu+\sigma}} (\mu N)^{\frac{\sigma}{\nu+\sigma}} (\bar{w}_\varepsilon)^{\frac{\varepsilon\sigma}{\nu+\sigma}}.$$

Again, the isoelastic properties of the model of the main text are preserved.

Finally, assume that households derive their earnings from heterogeneous endowments of effective units of labor, denoted by t. Consider a city with fundamental productivity A and inhabited by N households of heterogeneous type t with cdf $G_t(t)$ over support $\mathcal{T}$. Define:

$$\bar{t}_\varepsilon \equiv \left(\int_{\mathcal{T}} t^\varepsilon \mathrm{d}G_t(t) \right)^{\frac{1}{\varepsilon}}.$$

Then the earnings and indirect utility of a household of type t are respectively equal to

$$w = AtN^\alpha$$

and

$$V = \frac{1}{1-\varepsilon} \left\{ [AtN^\alpha]^{1-\varepsilon} - \left[(A\bar{t}_\varepsilon)^{\frac{\varepsilon}{\varepsilon+\sigma}} (BL)^{-\frac{\varepsilon}{\varepsilon+\sigma}} N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}} \right]^{1-\varepsilon} \right\},$$

which replace equations (16) and (17). Thus, the hump shape of the fundamental trade-off in urban economics is also preserved.

OA.2 More general PIGL preferences

Assume preferences obey equation (8). Plugging the equilibrium price of equation (14) into the indirect utility function V yields, for $\mu = 1$:

$$V(N) = (1-\varepsilon)^{-1}\left[A^{1-\varepsilon}N^{\alpha(1-\varepsilon)} - 1\right] - (1-\nu)^{-1}\left[A^{\varepsilon\frac{1-\nu}{\nu+\sigma}}(BL)^{-\frac{1-\nu}{\nu+\sigma}}N^{\frac{(1+\alpha\varepsilon)(1-\nu)}{\nu+\sigma}} - 1\right],$$

which is concave in N if and only if $0 < \alpha(1-\varepsilon) < \frac{(1+\alpha\varepsilon)(1-\nu)}{\nu+\sigma}$, or, equivalently,

$$\alpha(1+\sigma)(\nu-\varepsilon) < (1-\alpha\sigma)(1-\nu) \tag{27}$$

for $\varepsilon, \nu > 0$. Note that this condition is implied by the combined assumptions $\nu = \varepsilon \in (0,1)$ and $\alpha\sigma < 1$ in the main text. In this case, the efficient population scale of the city is finite and positive, and equal to

$$N = \left[\left(\alpha\frac{\nu+\sigma}{1+\alpha\varepsilon}\right)^{\nu+\sigma}(BL)^{1-\nu}A^{\sigma(1-\varepsilon)+\nu-\epsilon}\right]^{\frac{1}{(1-\alpha\sigma)(1-\nu)-\alpha(1+\sigma)(\nu-\varepsilon)}}.$$

The efficient size is degenerate in the limit case, $\nu = \varepsilon = 1$, that is, if preferences are Cobb-Douglas. By contrast, it is positive and finite if housing demand is income-inelastic ($0 < \varepsilon < 1$) and $\nu = 1$, or if housing demand is price-inelastic ($0 < \nu < 1$) and homothetic ($\varepsilon = 1$). In both cases, however, it is decreasing in A. See corollary 1 for details.

OA.3 Urban systems I: City size distribution

We maintain the assumption of identical households, and consider a large set of potential cities, endowed with heterogeneous fundamentals $\mathbf{X} \equiv (A, B, L)$. Let also

$$X \equiv \left(\alpha \frac{\varepsilon + \sigma}{1 + \alpha\varepsilon}\right)^{\frac{\varepsilon+\sigma}{\sigma(1-\varepsilon)}} A\,(BL)^{\frac{1}{\sigma}} \tag{28}$$

collect the productivity and housing affordability advantages of the city, and rewrite the efficient size in equation (18) as

$$N = X^{\psi}, \qquad \psi \equiv \frac{\sigma}{1 - \alpha\sigma}.$$

Assume that the equilibrium population size of individual cities is their efficient size, as would be the case if the local government maximizes the well-being of its inhabitants, as in Albouy, Behrens, Robert-Nicoud, and Seegert (2019) and Duranton and Puga (2023).

Let m_j denote the mean of some random variable j and η_{ij} denote the covariance between variables i, j, and define $\mathbf{M} \equiv \{m_j\}$ and $\mathbf{\Sigma} \equiv \{\eta_{ij}\}$ for $i, j = A, B, L$. We obtain:

Proposition 4. ***City size distribution.*** *(i) If the city scalar X is Pareto distributed with shape parameter $k > \psi$, then efficient city sizes are Pareto distributed with shape parameter $\zeta \equiv \frac{k}{\psi}$. (ii) If the components of the vector of city fundamentals $\mathbf{X}$ jointly follow a log normal distribution as $(\ln A, \ln B, \ln L)^{\top} \sim \mathcal{N}(\mathbf{M}, \mathbf{\Sigma})$, then efficient city sizes follow a log-normal distribution as $\ln N \sim \mathcal{N}(m_N, \eta_N)$, where*

$$m_N \equiv \ln\left\{\left(\alpha \frac{\varepsilon + \sigma}{1 + \alpha\varepsilon}\right)^{\frac{\varepsilon+\sigma}{\sigma(1-\varepsilon)}}\right\} + \psi m_A + \frac{\psi}{\sigma}(m_B + m_L), \quad \eta_N \equiv \psi^2 \left(1, \frac{1}{\sigma}, \frac{1}{\sigma}\right) \mathbf{\Sigma} \left(1, \frac{1}{\sigma}, \frac{1}{\sigma}\right)^{\top}.$$

Proof. (i) If $X \sim \mathcal{P}(X_0, k)$, namely, if X follows a Pareto distribution with scale parameter X_0 and scale parameter k, then $G(X) = 1 - \left(\frac{X}{X_0}\right)^{-k}$ and $g(X) = k\,(X_0)^k\, X^{-k-1}$, where G and g denote the cumulative distribution and probability density functions of X, respectively. Let F denote the cumulative distribution function of N. Using equations (18) and (28) and a change of variable, we have:

$$\begin{aligned} F(N) &= 1 - \int_N^{\infty} k\,(X(N_0))^k\,(X(n))^{-k-1}\,\mathrm{d}X(n) \\ &= 1 - \zeta\,(N_0)^{\zeta} \int_N^{\infty} n^{-\zeta-1}\mathrm{d}n \\ &= 1 - \left(\frac{N_0}{N}\right)^{\zeta}. \end{aligned}$$

The integral above converges if $\zeta \equiv \frac{k}{\psi} > 1$. (ii) From equation (27) and taking logs, we have:

$$\begin{aligned}\ln N &= \ln\left\{\left[\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}} A^{\frac{\sigma}{\varepsilon+\sigma}} (BL)^{\frac{1}{\varepsilon+\sigma}}\right]^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}}\right\} \\ &= \frac{\varepsilon+\sigma}{(1-\alpha\sigma)(1-\varepsilon)}\ln\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right) + \psi\ln A + \frac{\psi}{\sigma}\ln(BL).\end{aligned}$$

If $\ln A$, $\ln B$ and $\ln L$ are jointly normally distributed, then $\ln N$ is also normally distributed. □

Consistent with empirical evidence discussed in section 2, city size distributions are well approximated by Pareto or log-normal forms. This proposition notes that the model can both account for Zipf's law if the scalar X in equation (28) is Pareto distributed with shape parameter k close to ψ, or a log-normal distribution if the vector $\mathbf{X}$ is log-normally distributed.[8]

Consider now the urban system with household mobility of section 4.2, so that equilibrium city sizes are now given by equation (20). Here we ask: given that the empirical size distribution of cities is approximately Pareto or lognormal, what does that imply for the unobserved distribution of urban productivity? For simplicity, consider the case $B = L = 1$ for all cities. We obtain the following result:

Proposition 5. ***City size distribution in spatial equilibrium.*** *(i) If N is distributed Pareto or Frechet with shape parameter ζ_N, then the upper tail of city productivity A is asymptotically Pareto:*

$$\Pr(A > x) \sim x^{-\zeta_A} \text{ as } x \to \infty, \qquad \zeta_A \equiv \zeta_N\left[\frac{1}{\kappa} - \alpha\kappa(1-\varepsilon)\right]^{-1}.$$

(ii) If instead $\ln N \sim \mathcal{N}(m_N, \eta_N)$, then the upper tail of A is asymptotically lognormal:

$$\Pr(A > x) \sim 1 - \Phi\left(\frac{\ln x - m_A}{\eta_A}\right) \text{ as } x \to \infty,$$

where Φ is the standard normal cdf, $m_A \equiv \frac{1}{1-\varepsilon}\ln((1-\varepsilon)\mathbb{V}) + \left[\frac{1}{\kappa} - \alpha\kappa(1-\varepsilon)\right] m_N$, and $\eta_A \equiv \left[\frac{1}{\kappa} - \alpha\kappa(1-\varepsilon)\right]\eta_N$.

[8] We can further extend the model by allowing each component of the vector of city fundamentals to be the product of multiple random underlying factors — such as amenities, topographic features, and productivity characteristics, following Lee and Li (2013). When city fundamentals aggregate many such independent or weakly correlated components, the central limit theorem implies that their logarithm is approximately normally distributed. As a result, the efficient size of cities follows a log-normal distribution.

Proof. Define

$$z \equiv AN^{-r}, \qquad r \equiv \frac{1-\alpha\kappa\left(1-\varepsilon\right)}{\kappa\left(1-\varepsilon\right)},$$

and rewrite equation (20) as

$$\frac{1}{1-\varepsilon}\left[z^{1-\varepsilon}-z^{\frac{\varepsilon}{\varepsilon+\sigma}(1-\varepsilon)}N^{-\Delta}\right]=\mathbb{V}, \qquad \Delta \equiv \frac{\sigma-\kappa\left(1-\varepsilon\right)}{\kappa\left(\varepsilon+\sigma\right)}>0.$$

The second term in the brackets converges to zero as $N \to \infty$, so the equilibrium condition implies

$$z \to C \equiv \left[\left(1-\varepsilon\right)\mathbb{V}\right]^{\frac{1}{1-\varepsilon}}.$$

Therefore,

$$A \sim CN^{r}.$$

(i) If N is Pareto or Frechet with shape parameter ζ_N, then

$$\Pr\left(N>n\right) \sim C_N n^{-\zeta_N},$$

some $C_N > 0$. Hence the upper tail of A is asymptotically Pareto,

$$\Pr\left(A>x\right) \sim C_N\left(\frac{x}{C}\right)^{-\zeta_A}, \qquad \zeta_A \equiv \frac{\zeta_N}{r}=\zeta_N\frac{\kappa\left(1-\varepsilon\right)}{1-\alpha\kappa\left(1-\varepsilon\right)}.$$

(ii) If instead $\ln N \sim \mathcal{N}\left(m_N,\eta_N\right)$, then

$$\ln A=\ln C+r\ln N+o\left(1\right).$$

Therefore, the upper tail of A is asymptotically lognormal with

$$m_A \equiv \ln C+rm_N, \qquad \eta_A \equiv r\eta_N.$$

□

OA.4 Urban systems II: Endogenous extensive margin

Consider the model with individual household location choices of section 4.2, but assume that the local labor supply $\frac{1}{\kappa}$ is low (in a sense made precise below). This case encompasses the limit case with perfect labor mobility $\kappa \to \infty$, in which all households are indifferent in equilibrium. To avoid a typology of cases, suppose A_{min} is arbitrarily small. We obtain:

Proposition 6. ***Spatial equilibrium with heterogeneous location preferences and an extensive margin.*** *Assume*

$$1 < \alpha (1 - \varepsilon) \kappa, \quad \textit{and} \quad \alpha\sigma < 1. \tag{29}$$

Then (i) the size $\tilde{N}(A)$ that maximizes expected utility in the city with urban productivity A is positive and finite; (ii) there exists a threshold urban productivity $\underline{A} > 0$ and a scalar $\mathbb{V} > 0$ such that the populations of cities with a productivity larger than $\underline{A}$ are strictly larger than $\tilde{N}(A)$, (iii) the (potential) population of the city with urban productivity $\underline{A}$ is of size $\tilde{N}(\underline{A})$, (iv) the (potential) populations of cities with a productivity below $\underline{A}$ are zero, and (v) expected utility in any inhabited city is equal to $\Gamma\left(1 - \frac{1}{\kappa}\right)\mathbb{V}$.

Proof. First, the conditions for a spatial equilibrium are:

$$(N(A))^{-\frac{1}{\kappa}} V(A) \leq \mathbb{V}, \qquad 0 \leq N(A), \qquad 0 = N(A)\left[(N(A))^{-\frac{1}{\kappa}} V(A) - \mathbb{V}\right]. \tag{30}$$

Second, using equation (17), we write the expected utility in a city as a function of its population as

$$\mathbb{V}(A, N) \equiv N^{-\frac{1}{\kappa}} V(A, N) = \frac{1}{1 - \varepsilon} N^{-\frac{1}{\kappa}} \left[(AN^{\alpha})^{1-\varepsilon} - \left(A^{\frac{\varepsilon}{\varepsilon+\sigma}} N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right)^{1-\varepsilon}\right], \quad \varepsilon \in (0,1), \ 1 < \kappa. \tag{31}$$

$\mathbb{V}(A, N)$ is increasing in N at the origin and concave in N if and only if

$$\frac{1}{\kappa} < \alpha (1 - \varepsilon) < 1 + \frac{1}{\kappa}.$$

The second inequality in the expression above is redundant by $\alpha, \varepsilon \in (0, 1)$. If the condition in equation (22) holds, then the city size that maximizes expected utility $\mathbb{V}(A, N)$ is equal to

$$\tilde{N}(A) = \left\{ \left[1 - \frac{(1 - \alpha\sigma)(1 - \varepsilon)\kappa}{(1 + \alpha\varepsilon)(1 - \varepsilon)\kappa - (\varepsilon + \sigma)}\right]^{\frac{1}{1-\varepsilon}} A^{\frac{\sigma}{\varepsilon+\sigma}} \right\}^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}},$$

which yields property (i). In turn, expected utility evaluated at the efficient size, is increasing in A. Together with equation (30), this result implies properties (ii) and (v).

The proof of properties (iii) and (iv) is by contradiction, as shown by Albouy, Behrens, Robert-Nicoud, and Seegert (2019) in the limit case $\varepsilon \to 0$, $\kappa \to \infty$. $\square$

OA.5 Urban systems III: Sorting

We embed our urban model into an economy with many cities with identical city fundamentals but heterogeneous workers. Here also we let the size of cities be determined by the free mobility of households. We characterize sorting across cities in the spirit of Behrens, Duranton, and Robert-Nicoud (2014). They work with the model featuring unitary housing consumption and a production function linear in land use, which is the special case $\varepsilon = \sigma = 0$ of our PIGL-UM model. For simplicity we abstract from selection into occupations, which is an important focus of their paper, but we add human capital spillovers, which are absent in theirs.

Without further loss of generality, we normalize the value of all components of city fundamental to one, $A = B = L = 1$. Workers are endowed with heterogeneous amounts of effective units of labor, or "talent" for short, $s, t \in \mathcal{T} \subset \mathbb{R}_{++}$; we assume that the cumulative distribution function of talent $F_{\mathcal{T}}$ is twice continuously differentiable. Consider for now that each city is inhabited by identical households, so with some abuse of notation we may refer to t as the type of the city as well. Denote the size, wage per unit of effective labor, and housing price of city t by $N(t)$, $w(t)$, and $p(t)$, respectively. Wages per unit effective labor reflect both agglomeration economies and human capital spillovers :

$$w(t) = (N(t))^{\alpha} t^{\lambda}.$$

If $\lambda \geq 0$ then the productivity of a worker is (weakly) increasing in that of the workforce. Using equations (14) and (16), the nominal earnings of the representative worker in that city is $w(t)\, t = (N(t))^{\alpha} t^{1+\lambda}$, and the housing price is:

$$p(t) = (N(t))^{\gamma} t^{(1+\lambda)\frac{\varepsilon}{\varepsilon+\sigma}}, \qquad \gamma \equiv \alpha + \frac{1-\alpha\sigma}{\varepsilon+\sigma}, \tag{32}$$

where $\gamma > \alpha$ under $\alpha\sigma < 1$. The nominal earnings a worker of type s living in a city of type t is $w(t)\, s = (N(t))^{\alpha} t^{\lambda} s$. Thus, the utility of a worker of type s living in a city of type t, which we denote by $V(s,t)$, is

$$V(s,t) = \frac{1}{1-\varepsilon}\left\{\left[(N(t))^{\alpha} t^{\lambda} s\right]^{1-\varepsilon} - \left[(N(t))^{\gamma} t^{(1+\lambda)\frac{\varepsilon}{\varepsilon+\sigma}}\right]^{1-\varepsilon}\right\}.$$

An equilibrium features talent-homogeneous cities if type-s workers choose to live in type-s cities, that is, if $s = \text{argmax}_t V(s,t)$. We also contrast the equilibrium size of each talent-homogeneous city with that chosen by a utilitarian planner. We obtain:

Proposition 7. ***Sorting across cities.*** *In an equilibrium with talent-homogeneous cities,*

(i) city population is equal to:

$$N^*(t) = \left\{ \left(\frac{\alpha\sigma + \lambda}{1+\lambda} \right)^{\frac{\varepsilon+\sigma}{1-\varepsilon}} t^{\sigma(1+\lambda)} \right\}^{\frac{1}{1-\alpha\sigma}}. \tag{33}$$

(ii) Cities hosting more talented workers are larger, pay higher wages, and feature higher housing costs. (iii) If $\lambda(1-\alpha\sigma) \leq \alpha\varepsilon$*, then the equilibrium size of talent-homogeneous cities* $N^*(t)$ *is smaller than the optimal size of talent-homogeneous cities* $N^0(t)$*:*

$$\lambda < \alpha\varepsilon \;\;\Rightarrow\;\; N^*(t) < N^0(t) \equiv \left[\left(\frac{\alpha\sigma + \alpha\varepsilon}{1+\alpha\varepsilon} \right)^{\frac{\varepsilon+\sigma}{1-\varepsilon}} t^{\sigma(1+\lambda)} \right]^{\frac{1}{1-\alpha\sigma}}. \tag{34}$$

(iv) The size distribution of cities converges to Zipf's law as $\alpha\sigma \to 1$*.*

Proof. (i) Assuming for now that $N(t)$ is continuously differentiable, type-s workers solve the program $\max_t V(s,t)$. The first order condition to this program is:

$$\begin{aligned} 0 = {} & (N(t))^{\alpha(1-\varepsilon)} s^{1-\varepsilon} t^{\lambda(1-\varepsilon)-1} \left[\alpha \frac{N'(t)}{N(t)} t + \lambda \right] \\ & - (N(t))^{\gamma(1-\varepsilon)} t^{\frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda)(1-\varepsilon)-1} \left[\gamma \frac{N'(t)}{N(t)} t + \frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda) \right]. \end{aligned}$$

The equilibrium features talent-homogeneous cities if type-t workers choose to live in type-t cities, that is, if $s = t$. Plugging $s = t$ into the first order condition above yields an ordinary differential equation:

$$\begin{aligned} 0 = \frac{1}{t} \Bigg\{ & \left[\alpha \frac{N'(t)}{N(t)} t + \lambda \right] \left[(N(t))^{\alpha} t^{1+\lambda} \right]^{1-\varepsilon} \\ & - \left[(N(t))^{\gamma} t^{\frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda)} \right]^{1-\varepsilon} \left[\gamma \frac{N'(t)}{N(t)} t + \frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda) \right] \Bigg\}. \end{aligned}$$

This equation is homogeneous in logs, so that a natural solution to this differential equation is of the form $N(t) = ct^f$, some $c, f \in \mathbb{R}$. Plugging this conjecture into the expression above yields:

$$(\alpha f + \lambda) c^{\alpha(1-\varepsilon)} t^{(1+\lambda+\alpha f)(1-\varepsilon)} = c^{\gamma(1-\varepsilon)} t^{\left[\frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda)+\gamma f\right](1-\varepsilon)} \left[\gamma f + \frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda) \right].$$

Thus c, f solve this two-equation system:

$$(\alpha f + \lambda)\, c^{\alpha(1-\varepsilon)} = c^{\gamma(1-\varepsilon)} \left[\left(\gamma f + \frac{\varepsilon}{\varepsilon + \sigma} (1 + \lambda) \right) \right]$$
$$1 + \lambda + \alpha f = \frac{\varepsilon}{\varepsilon + \sigma} (1 + \lambda) + \gamma f.$$

Solving and using equation (32) yields the expression in equation 33. The second order condition is satisfied (the algebraic expression is too unwieldy to be revealing). (ii) N^* is increasing in t by inspection of equation (33). In turn, $p^*(t) = (N^*(t))^{\gamma} t^{\frac{\varepsilon}{\varepsilon+\sigma}}$ and $w^*(t) = (N^*(t))^{\alpha}$ are increasing in t by inspection. (iii) A planner aiming to create talent-homogeneous cities would solve the following problem:

$$\max_{N(t)} \int_{t \in \mathcal{T}} V(t,t)\, \mathrm{d}F_{\mathcal{T}}(t) = \frac{1}{1-\varepsilon} \int_{t \in \mathcal{T}} \Big[t^{(1+\lambda)(1-\varepsilon)} (N(t))^{\alpha(1-\varepsilon)} - t^{\frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda)(1-\varepsilon)} (N(t))^{\gamma(1-\varepsilon)} \Big] \mathrm{d}F_{\mathcal{T}}(t).$$

The solution to this problem gives the optimal talent-homogeneous city sizes in equation 34. Finally,

$$\left(\frac{N^*(t)}{N^0(t)} \right)^{\frac{(1-\alpha\sigma)(1-\varepsilon)}{\varepsilon+\sigma}} = \frac{\alpha\sigma + \lambda}{\alpha\sigma + \alpha\varepsilon} \frac{1 + \alpha\varepsilon}{1 + \lambda},$$

which is smaller than one if and only if $\lambda < \alpha\varepsilon$. (iv) Rewrite expression (33) as $N^*(t) \propto t^{\frac{\sigma(1+\lambda)}{1-\alpha\sigma}}$, which is isomorphic to the corresponding expression in Behrens, Duranton, and Robert-Nicoud (2014). They show that the size distribution of cities converges to Zipf's law as $\alpha\sigma \to 1$ regardless of the distribution $F_{\mathcal{T}}$ under mild regularity conditions.. □

These equilibrium properties are in line with empirical evidence (Combes, Duranton, and Gobillon 2008; Diamond 2016; Moretti 2013). Some properties of this equilibrium deserve special emphasis. First, the elasticity of city size with respect to worker talent is increasing in agglomeration elasticity α and housing supply elasticity σ, as a higher α strengthens agglomeration forces and a higher σ relaxes dispersion forces.

Second, this equilibrium is degenerate in the limiting cases of homothetic Cobb-Douglas preferences ($\lim_{\varepsilon \to 1} N^*(t) = 0$), or if dispersion forces dominate agglomeration forces at any city size, which arises if housing supply is inelastic or in the absence of agglomeration economies ($\lim_{\alpha\sigma \to 0} N^*(t) = 0$).

Third, as in the model of Behrens, Duranton, and Robert-Nicoud (2014), the equilibrium location decision of workers is inefficient. Consider first the case $\lambda = 0$. Housing market crowding dominates agglomeration forces at the margin by $\alpha\sigma < 1$, and the worker does not internalize the full productivity gain of their location decisions. Conversely, if knowledge spillovers are sufficiently large in the sense $\lambda > \alpha\varepsilon$, then the pecuniary externality a worker imposes on fellow city dwellers is negative at the margin. Hence, upon making

location decisions, the marginal urban dweller imposes a net cost on her fellow citizens and talent-homogeneous cities are over-sized in equilibrium.

Fourth, productivity is embedded in people rather than in places: if $\lambda = 0$, then the expression for $N^0(t)$ in equation (34) above is isomorphic to the efficient city size $N(A)$ in equation (18).

Finally, Behrens, Duranton, and Robert-Nicoud (2014) show that the size distribution of cities follows Zipf's law regardless of the distribution $F_{\mathcal{T}}$ if returns to city size are close to constant, under mild additional regularity conditions (their method of proof applies to our model, too). This case is not a mere intellectual curiosity: it is also a good representation of extant empirical evidence, which shows that the magnitude of the elasticity of urban costs with respect to city size γ, when estimated directly, is close to that of the elasticity of wages with respect to city size α (Combes, Duranton, and Gobillon 2019; Duranton and Puga 2023).

OA.6 Redistribution

Most OECD countries combine progressive income taxation with housing policies that reduce the effective cost of housing for at least some households (e.g., housing vouchers, rent subsidies, tax deductions, or social housing). To capture these policies parsimoniously, assume after-tax income obeys

$$\tilde{w} = \phi w^{\omega}, \qquad \phi > 0,\ \omega \in (0,1)\,, \tag{35}$$

and after-subsidy housing prices obey

$$\tilde{p} = \rho p^{\tau}, \qquad \rho > 0,\ \tau \in (0,1)\,. \tag{36}$$

Our reduced-form policy wedges are motivated by existing empirical evidence on tax progressivity and housing subsidies. Following the public-finance literature, we model disposable income as a concave function of pre-tax income, where $\omega < 1$ in equation (35) captures progressivity. This specification is closely related to the nonlinear tax schedules estimated by Heathcote, Storesletten, and Violante (2017), whose estimates imply values around $\omega = 0.8$ for the U.S. tax system.

Housing subsidies are captured parsimoniously by a reduced-form housing-price wedge in equation (36), motivated by evidence on housing vouchers and subsidy incidence (e.g., Eriksen and Ross 2015). Since many housing-support programs are approximately proportional to rent over relevant ranges, we view values of τ close to one as a natural benchmark. It follows that the condition $\omega < \tau$ is likely to hold in practice.

Finally, we assume that ϕ and ρ are jointly chosen sot that this policy is revenue neutral in aggregate, given (τ, ω). Below we consider the implications of redistributive taxation on the allocation of population across cities, and on land use in an arbitrary city.

OA.6.1 Redistribution and the distribution of city population size

We show that progressive income taxation reduces the relative size of the largest, most productive cities.

Proposition 8. ***Progressive income taxation and the distribution of city size.*** *Assume after-tax income and after-subsidy housing prices satisfy equations* (35) *and* (36)*, respectively, and suppose*

$$\omega < \tau.$$

Then the efficient city scale exists, is unique, and satisfies

$$\tilde{N} = \left[1 - \frac{1}{1+\alpha\varepsilon\omega}\left(1 - \alpha\sigma\frac{\omega}{\tau}\right)\right]^{\frac{\varepsilon\tau+\sigma}{(1-\varepsilon)(\tau-\alpha\sigma\omega)}} \left(\frac{\phi}{\rho}\right)^{\frac{\sigma}{\tau-\alpha\sigma\omega}} A^{\frac{\omega\sigma}{\tau-\alpha\sigma\omega}}.$$

Relative to the benchmark economy without fiscal distortions ($\omega = \tau = 1$), fiscal policy compresses equilibrium city-size differences.

Proof. Combining housing market clearing with the subsidy schedule yields the equilibrium housing price faced by households,

$$\tilde{p}(A, N) = \left[\rho^{\sigma}\phi^{\varepsilon\tau}A^{\varepsilon\tau\omega}N^{\tau(1+\alpha\varepsilon\omega)}\right]^{\frac{1}{\varepsilon\tau+\sigma}}.$$

Indirect utility is therefore equal to

$$\tilde{V} = \frac{1}{1-\varepsilon}\left\{\left[\phi\left(AN^{\alpha}\right)^{\omega}\right]^{1-\varepsilon} - \left[\rho^{\sigma}\phi^{\varepsilon\tau}A^{\varepsilon\tau\omega}N^{\tau(1+\alpha\varepsilon\omega)}\right]^{\frac{1-\varepsilon}{\varepsilon\tau+\sigma}}\right\}.$$

In turn, the efficient city scale is equal to

$$\tilde{N} = \left[1 - \frac{1}{1+\alpha\varepsilon\omega}\left(1 - \alpha\sigma\frac{\omega}{\tau}\right)\right]^{\frac{\varepsilon\tau+\sigma}{(1-\varepsilon)(\tau-\alpha\sigma\omega)}}\left(\frac{\phi}{\rho}\right)^{\frac{\sigma}{\tau-\alpha\sigma\omega}}A^{\frac{\omega\sigma}{\tau-\alpha\sigma\omega}}.$$

Therefore, assuming $\alpha\sigma\frac{\omega}{\tau} < 1$ (which guarantees existence and uniqueness and is satisfied under the benchmark calibration in table 1), the elasticity of efficient city size with respect to productivity is

$$\frac{\partial \ln \tilde{N}}{\partial \ln A} = \frac{\sigma}{\frac{\tau}{\omega} - \alpha\sigma} < \frac{\sigma}{1-\alpha\sigma},$$

where the inequality follows from $\omega < \tau$. □

Intuitively, progressive income taxation dampens nominal wage differences across cities, reducing the attractiveness of high-productivity locations, whereas stronger housing subsidies partly offset congestion costs in expensive cities.

Our mechanism is conceptually related to the spatial distortions induced by nominal income taxation emphasized by Albouy (2009), who shows that progressive federal taxation disproportionately burdens high-wage, high-cost locations and thereby distorts migration incentives. Our framework complements his by characterizing in closed form how the interaction between tax progressivity and housing subsidies shapes the elasticity of city size with respect to local productivity.

OA.6.2 Redistribution and land use

Consider now a city in isolation, inhabited with a population with heterogeneous incomes. Budget-neutral progressive income taxation increases housing demand and urban land use. To see this result, set $\rho = \tau = 1$ and define $H(\omega)$ and $L(\omega)$ as the equilibrium housing and land consumption with tax progressivity, and $H(1)$ and $L(1)$ as the equilibrium housing and land consumption without it. We obtain:

Proposition 9. ***Progressive income taxation and land use****. Assume urban population N is fixed, land supply satisfies*

$$L = L_0 r^{\lambda}, \qquad L_0 > 0,\ \lambda > 0, \tag{37}$$

after-tax income satisfies equation (35)*, incomes are heterogeneous with CDF F_w defined over the compact set Ω, and the budget is balanced so that aggregate after-tax earnings are equal to aggregate pretax earnings:*

$$N \int_{\Omega} \left(\phi w^{\omega} - w\right) \mathrm{d}F_w\left(w\right) = 0. \tag{38}$$

Then progressive income taxation increases housing demand and land use if housing is a necessity:

$$\varepsilon, \omega \in (0,1) \ \Rightarrow\ H\left(\omega\right) > H\left(1\right),\ L\left(\omega\right) > L\left(1\right).$$

Proof. Define

$$D\left(\omega\right) \equiv N \int_{\Omega} \left(\phi w^{\omega}\right)^{\varepsilon} \mathrm{d}F_w\left(w\right) \tag{39}$$

as the aggregate housing-demand shifter so that aggregate housing demand is equal to $p^{-\nu} D\left(\omega\right)$. Using equations (13), (14), and (37) yields

$$H\left(\omega\right) = B\left[L_0\left(\frac{B}{1+\sigma}\right)^{\lambda}\right]^{1-\Lambda_H}\left(\frac{\mu D\left(\omega\right)}{B}\right)^{\Lambda_H}, \qquad \Lambda_H \equiv \frac{\sigma + \lambda\left(1+\sigma\right)}{\nu + \sigma + \lambda\left(1+\sigma\right)},$$

and

$$L\left(\omega\right) = \left[L_0\left(\frac{B}{1+\sigma}\right)^{\lambda}\right]^{1-\Lambda_L}\left(\frac{\mu D\left(\omega\right)}{B}\right)^{\Lambda_L}, \qquad \Lambda_L \equiv \frac{\lambda\left(1+\sigma\right)}{\nu + \sigma + \lambda\left(1+\sigma\right)}.$$

Given ω and (F_w, Ω), it follows from equation (38) that ϕ satisfies:

$$\phi = \frac{\int_{\Omega} w \mathrm{d}F_w\left(w\right)}{\int_{\Omega} w^{\omega} \mathrm{d}F_w\left(w\right)}.$$

Plugging this expression into equation (39) yields

$$D\left(\omega\right) = N\left(\frac{\int_{\Omega} w \mathrm{d}F_w\left(w\right)}{\int_{\Omega} w^{\omega} \mathrm{d}F_w\left(w\right)}\right)^{\varepsilon} \int_{\Omega} \left(w^{\omega}\right)^{\varepsilon} \mathrm{d}F_w\left(w\right),$$

with $D\left(1\right) = N \int_{\Omega} w^{\varepsilon} \mathrm{d}F_w\left(w\right)$ and

$$\frac{\partial D\left(\omega\right)}{\partial \omega} < 0$$

by $\varepsilon \in (0,1)$. Thus, the combination of tax progressivity and housing being a necessity

implies that housing and land consumption increase with redistribution progressivity. □

The distribution of earnings has an implication for aggregate housing and land consumption. The mechanism is different from that in Pines and Sadka (1986), who study the implications of redistributing land rents to local, homogeneous residents. In our model, if land rents are redistributed to local residents proportionately to their after-tax labor income, then it can be shown that proposition 9 still holds if parameters obey

$$0 < \nu - \varepsilon + (1 - \varepsilon)(\lambda + \lambda\sigma + \sigma).$$

Using the empirically plausible values of table 1, this condition holds for any combination of housing- and land-supply elasticities that are "not too low" in the sense

$$\lambda + \lambda\sigma + \sigma \geq 0.6.$$

In particular, this sufficient condition is satisfied for any $\lambda \geq 0$ in the benchmark case $\sigma = 2$.

OA.7 PIGL-AMM: Monocentric city model

OA.7.1 Preferences and housing demand

Households work and consume in a central business district (CBD) and reside at distance $x \geq 0$ from the CBD. Commuting is costly. Space extends in all directions on the Euclidean space. Let $p(x)$ denote the unit price of housing at location x.

Preferences extend the PIGL structure in equation (8) by allowing commuting costs and housing consumption to enter utility jointly. Indirect utility is given by:

$$V(w,x) = \frac{1}{1-\varepsilon}\left[w^{1-\varepsilon} - \mu\left(p(x)\,\tau x^{\theta}\right)^{1-\varepsilon} - (1-\mu)\right], \tag{40}$$

where $\tau > 0$ is a commuting cost shifter and $\theta > 0$ is the elasticity of commuting cost with respect to distance. When $\theta = 0$ and $\tau = 1$, this formulation collapses to the space-free PIGL-UM preferences of section 3.

This formulation assumes that commuting costs and housing costs enter indirect utility in a complementary way. The direct utility, that is, the dual of V in equation (40), is:

$$U(c,h) = \frac{1}{1-\varepsilon}\left\{c^{1-\varepsilon}\left[1 - \left(\frac{h_{\min}(x)}{h}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{\varepsilon} - (1-\mu)\right\},$$

where $h_{\min}(x) \equiv \mu^{\frac{1}{1-\varepsilon}}\tau x^{\theta}$: the "subsistence" amount of housing increases with distance to the CBD. Thus, the utility cost of distance is incurred in units of housing (rather than in units of the numeraire good, as is often assumed).

From Roy's identity, Marshallian demand and housing expenditure share at location x are:

$$h^{D}(w,x) = \mu\left(\tau x^{\theta}\right)^{1-\varepsilon}\left(\frac{w}{p(x)}\right)^{\varepsilon}, \qquad e(w,x) = \mu\left(\frac{\tau x^{\theta} p(x)}{w}\right)^{1-\varepsilon}. \tag{41}$$

We normalize the reference location within each city to $x = 1$ to simplify notation; all results are invariant to this choice. In a within city spatial equilibrium, residents must be indifferent across all city locations, so $V(w,x) = V(w,1)$ for all $x \leq \overline{x}$, where $\overline{x}$ denotes the endogenous urban fringe. Then, the expression in equation (40) implies

$$\forall x \leq \overline{x}: \qquad p(x)\,\tau x^{\theta} = P, \tag{42}$$

where, like p in the model of section 3, P is a sufficient statistic for the urban *cost of living* faced by all residents. Specifically, $P \equiv p(1)\,\tau$ combines the housing price and the commuting cost at the reference location.

Substituting the spatial equilibrium condition of equation (42) into equations (40) and

(41) yield

$$V(P,w,x) = \frac{1}{1-\varepsilon}\left[w^{1-\varepsilon} - \mu P^{1-\varepsilon} - (1-\mu)\right] \tag{43}$$

and

$$h^D(P,w,x) = \mu\tau x^\theta \left(\frac{w}{P}\right)^\varepsilon, \qquad e(P,w,x) = \mu\left(\frac{P}{w}\right)^{1-\varepsilon}, \tag{44}$$

respectively. Hence, the housing expenditure share is constant across space and is isomorphic to that in the space-free PIGL-UM model. Also, housing price $p(x)$ declines and housing consumption $h^D(x)$ increases with distance to the CBD. Empirical estimates reported in section 2 imply small but positive values of θ. As in section 4, we henceforth set $\mu = 1$.

OA.7.2 Housing supply

Housing is produced competitively using land and a numeraire input with the same Cobb–Douglas technology as in section 3. Because housing prices vary with distance, housing supply per unit of land at location x is:

$$h^S(x) = B\left(p(x)\right)^\sigma = B\left(\frac{P}{\tau x^\theta}\right)^\sigma. \tag{45}$$

Thus, floor space density declines monotonically with distance from the CBD.

With free-entry, builders earn zero profits. The equilibrium land rent at location x is therefore

$$r(x) = \frac{B}{1+\sigma}\left(p(x)\right)^{1+\sigma}.$$

Just like the price of housing, the price of land declines with distance to the center. Let $\frac{\overline{r}}{1+\sigma} > 0$ denote the opportunity cost of land, with $\overline{r} > 0$. The urban fringe $\overline{x}$ is defined by the no-arbitrage condition $r(\overline{x}) = \frac{\overline{r}}{1+\sigma}$. Let

$$\overline{p} \equiv p(\overline{x}) = \left(\frac{\overline{r}}{B}\right)^{\frac{1}{1+\sigma}}$$

denote the housing price at the city boundary. Using equation (42), the city radius and total urbanized area are

$$\overline{x} = \left(\frac{P}{\overline{p}\tau}\right)^{\frac{1}{\theta}}, \qquad S = \pi\left(\frac{P}{\overline{p}\tau}\right)^{\frac{2}{\theta}},$$

where π denotes the mathematical constant, $\pi \approx 3.1416$.

OA.7.3 Equilibrium and the cost of living

Following Combes, Duranton, Gobillon, Gorin, and Robert-Nicoud (2026), assume that the share of land convertible to urban use at distance x is equal to $\psi x^{-\xi}$ to account for topographical and institutional features of the city (Harari 2020; Hilber and Robert-Nicoud 2013; Saiz 2010). Here, χ governs the fraction of land within the city boundaries that can be used for urban development (e.g., $\psi = \frac{1}{2}$ for a city that grows along a linear coast and $\psi = 1$ for a city that can expand freely in all directions), and ξ captures the extent to which this fraction evolves with the distance to the CBD: for instance, $\xi > 0$ if terrain becomes steeper and more difficult to build and/or if land use regulations become more stringent as distance to the CBD increases. We show below that $\xi < 0$ if the physical density gradient is steeper than the population gradient in equilibrium. In the main text we work with the special case $\xi = 0$.

Population density at location x is defined as the number of residents per square meter of land, which is equal to the fraction of land converted to urban use times housing supply, divided by housing demand. Using equations (44) and (45) , we obtain:

$$D(x) \equiv \psi x^{-\xi} \frac{h^S(x)}{h^D(x)} = \frac{\chi B}{\mu \tau^{1+\sigma}} w^{-\varepsilon} P^{\varepsilon+\sigma} x^{-\xi-\theta(1+\sigma)}. \tag{46}$$

Total urban population satisfies

$$N = 2\pi \int_0^{\overline{x}} x D(x) \, \mathrm{d}x.$$

The integral above converges if and only if $\xi < 2 - \theta(1+\sigma)$, which we henceforth impose, and diverges to $-\infty$ otherwise. Using equations (42) and (46), this condition yields a closed-form expression for the equilibrium cost of living:

$$P = \left(\frac{N}{\Lambda}\right)^{\frac{1}{\varepsilon+\eta}} w^{\frac{\varepsilon}{\varepsilon+\eta}}, \tag{47}$$

where

$$\eta \equiv \frac{2-\xi}{\theta} - 1, \qquad \Lambda \equiv \frac{2\pi\psi}{\eta - \sigma} \theta^{-1} \tau^{-1-\eta} \overline{r}^{-\frac{\eta-\sigma}{1+\sigma}} B^{\frac{1+\eta}{1+\sigma}}.$$

Crucially, equations (47) and (14) are formally isomorphic, with P (the cost of living combining housing prices and commuting costs) replacing p (the equilibrium housing price). The housing supply elasticity σ and land endowment L of the space-free PIGL-UM model are replaced here by η and Λ, which summarize commuting technology and land availability. From this point onward, the monocentric city PIGL-AMM model collapses to PIGL-UM, and hence all the results of propositions 1 (efficient city size), 4 (city size distribution), 2 (heterogeneous location preferences), and 7 (sorting across cities) apply to PIGL-AMM,

too. Summarizing, we have established proposition 3.

OA.7.4 Gradients

Equations (42) and (45) give us the housing price gradient and building height gradient:

$$\mathcal{E}(p,x) \equiv \frac{\partial \ln p(x)}{\partial \ln x} = -\theta, \qquad \mathcal{E}\left(h^S,x\right) \equiv \frac{\partial \ln h^S(x)}{\partial \ln x} = -\sigma\theta.$$

Summing the two, we obtain the gradient of the land price $r(x) \equiv \frac{\sigma}{1+\sigma} p(x) h^S(x)$:

$$\mathcal{E}(r,x) \equiv \frac{\partial \ln r(x)}{\partial \ln x} = -(1+\sigma)\theta.$$

Using equations (41), (42), and (45), the gradient of population density is equal to

$$\mathcal{E}(D,x) \equiv \frac{\partial \ln D(x)}{\partial \ln x} = -\xi - \theta(1+\sigma).$$

Empirical estimates of price and land gradients reported in section 2 imply values of θ and σ consistent with the model.

From these we conclude that the housing price, land prices, built-up density, and population density all decay with distance to the CBD. Furthermore, a higher θ makes proximity to the center more desirable, resulting in steeper price and density gradients. These properties are consistent with the large body of literature reviewed in Duranton and Puga (2015).