A Unified Urban Model With Non-Homothetic Housing Demand*

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Abstract

Canonical urban models fail to jointly account for flexible housing demand, non-degenerate city sizes, and observed urban systems. We introduce a unifying urban framework based on price-independent generalized linear (PIGL) preferences in which housing is a necessity. Non-homothetic housing demand generates income effects that cause urban costs to scale more strongly with population than wages, restoring a unique interior efficient city size under standard assumptions. The framework nests existing canonical models, remains tractable, and is consistent with key empirical regularities, including Zipf's law and observed housing price elasticities. We also encapsulate a monocentric city model into our framework.

JEL code : R14, R21 and R31

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1 Introduction

Informed by a large body of empirical work, a unifying urban model should satisfy a number of requirements. Housing should be a necessity. Within cities, housing prices, land prices, built-up density, and population density should all decline with distance to jobs or amenities. Across cities, housing costs and overall urban cost of living should rise with population; and wages and productivity should increase with workforce size due to agglomeration economies. Internal consistency further requires the existence of cities to be justified: the welfare of urban dwellers cannot be monotonically increasing or decreasing in population. Furthermore, cities do not exist in isolation and instead interact within systems: so the model should be consistent with empirical regularities such as Zipf's law. Finally, models are useful only if they are tractable. To date, no framework jointly satisfies all these properties.

This paper introduces such a unified and parsimonious framework, which we label PIGL-UM (Price-Independent Generalized Linear Urban Model). It combines non-homothetic PIGL preferences with a competitive housing sector. When housing is a necessity, income effects in housing demand cause urban costs to scale more strongly with population than wages, thereby restoring a unique interior efficient city size under otherwise standard assumptions. The framework nests canonical urban models as polar cases, preserves analytical tractability, and naturally accounts for empirical regularities (documented in section 2). We discipline key parameters of the model using empirical estimates from the urban and housing literature, which guides the analysis toward economically relevant equilibria. It also extends seamlessly to urban systems with heterogeneous cities and workers, as well as to a fully specified monocentric city model.

Urban economics relies on stylized preference structures to analyze city size, housing markets, and the spatial allocation of workers. Following Alonso (1964), Fujita (1989), Lucas and Rossi-Hansberg (2002), and Solow (1973), the dominant approach assumes homothetic Cobb–Douglas preferences, which ensures analytical convenience but imposes a unit income elasticity in housing demand. This counterfactual implication yields a degenerate efficient city size in canonical formulations (Allen and Arkolakis 2025; Redding 2025).[1] Following Fujita and Ogawa (1982), Henderson (1974), and Mills (1967), an alternative

[1] Models combining Cobb–Douglas preferences with explicit urban spatial structure are also known to aggregate poorly once heterogeneity is introduced, except under knife-edge assumptions on preferences and technologies. See Duranton and Handbury (2023) for a discussion in the context of work-from-home shocks.

strand assumes fixed housing consumption, which restores income effects and a non-trivial optimal city scale, but shuts down price effects (Albouy, Behrens, Robert-Nicoud, and Seegert 2019; Duranton and Puga 2023).

PIGL-UM bridges these extremes. The model features non-homothetic PIGL preferences over a traded numeraire and a local housing service produced competitively from land and a numeraire input. Housing is a necessity, consistent with the empirical evidence (Combes, Duranton, and Gobillon 2019; Couture, Gaubert, Handbury, and Hurst 2023; Finlay and Williams 2025; Fretz, Parchet, and Robert-Nicoud 2022). The resulting structure preserves isoelastic functional forms throughout, allowing income and price effects to operate while remaining analytically tractable.

We abstract from city geography in the baseline environment with identical households and a fixed supply of city land (section 3). The model delivers closed-form expressions for housing demand, housing supply, equilibrium prices, land rents, and the cost of living. When wages depend on population through agglomeration economies, the model yields a unique interior efficient city size provided housing-market crowding eventually dominates agglomeration forces. In the homothetic Cobb–Douglas limit, the efficient city size collapses to zero, highlighting the role of non-homotheticity in sustaining finite cities.

We then embed the city model into three distinct urban systems (section 4). First, with heterogeneous city fundamentals and identical households, efficient city sizes follow either a Pareto or a log-normal distribution under standard assumptions (Duranton and Puga 2014). Second, introducing heterogeneous location preferences yields an endogenous extensive margin that is unique in the quantitative spatial model literature (Ahlfeldt, Redding, Sturm, and Wolf 2015; Allen and Arkolakis 2025; Diamond 2016; Fajgelbaum, Morales, Suárez Serrato, and Zidar 2019; Monte, Redding, and Rossi-Hansberg 2018; Redding 2016; Redding 2025). It also generalizes results from Albouy, Behrens, Robert-Nicoud, and Seegert (2019). Third, allowing for heterogeneous worker talent generates a sorting equilibrium in which higher skilled workers reside in larger, higher-wage, and higher-cost cities. This variant of the model generates Zipf's law as a limiting but empirically relevant case. It also generalizes the sorting framework of Behrens, Duranton, and Robert-Nicoud (2014) by restoring income and price effects without sacrificing tractability.

A final contribution of the paper is to show that PIGL-UM, which has no urban structure, is not a reduced-form shortcut, but the exact equilibrium representation of a fully specified monocentric city model with commuting costs, endogenous land use, and spatial housing price gradients (section 5). We label this monocentric city model PIGL-AMM in

reference to the seminal contributions of Alonso (1964), Mills (1967), and Muth (1969). Under standard assumptions, the within-city spatial equilibrium collapses to a single sufficient statistic, the urban cost of living, which enters utility and housing demand exactly as in the space-free model of section 3.

The preference structure used in this paper builds on the class of price-independent generalized linear preferences introduced by Muellbauer (1975; 1976), and further developed in Deaton and Muellbauer (1980), who provide a flexible yet tractable framework for non-homothetic demand that aggregates exactly across heterogeneous households.[2] PIGL preferences have since become a workhorse in the modern structural change literature, where they are employed to reconcile income-driven expenditure shifts with balanced-growth or representative-agent environments, most notably in Alder, Boppart, and Müller (2022), Boppart (2014), and Fan, Peters, and Zilibotti (2023).

Beyond traditional structural change, the PIGL class has also been adopted in spatial contexts: Eckert and Peters (2025) explicitly embed PIGL preferences to generate tractable income-dependent expenditure shares across sectors (farming and manufacturing) in their spatial structural change model. Chatterjee, Giannone, Kleineberg, Kuno, and Looser (2025), Finlay and Williams (2025), and Gaubert and Robert-Nicoud (2025) use non-homothetic housing demand to link rising inequality to spatial sorting patterns. Most works use a fixed housing consumption combined with a housing Leontief production function.

2 Stylized facts

In this section, we document a set of stylized facts that emerge as equilibrium outcomes of the model. We draw primarily on estimates from Combes, Duranton, and Gobillon (2008; 2019; 2021) for France and from Finlay and Williams (2025) for the United States. These estimates have direct structural counterparts in our framework.

[2] Eeckhout, Pinheiro, and Schmidheiny (2014) and Gaigné, Koster, Moizeau, and Thisse (2022) use Stone-Geary preferences instead. he former develop a quantitative spatial equilibrium model with non-homothetic preferences to study the distribution of city sizes and spatial sorting, emphasizing how subsistence housing demand shapes urban structure and inequality. The latter study the sorting of heterogeneous individuals inside a monocentric city. Unlike PIGL, Stone–Geary preferences yield affine Engel curves (which are convenient but counterfactual) and prevent closed-form solutions for the efficient size of cities, a central focus of our analysis.

2.1 Cross-household and cross-city regularities on housing expenditure shares and housing pries

Housing is a necessity. Housing expenditure shares decline with income and increase with housing prices.

Using the 2005–2017 biennial surveys of the Panel Study of Income Dynamics in the United States, Finlay and Williams (2025) estimate:

$$\ln e = \underset{(0.035)}{-0.25} \ln w + \underset{(0.057)}{0.39} \ln p + \text{demographic controls}, \tag{1}$$

where e denotes the housing expenditure share, w total expenditure, and p housing prices (Table 1, col. 4; 2SLS, standard errors clustered at the MSA level in parenthesis). Housing demand is both income- and price-inelastic: the negative coefficient on total expenditure is negative, implying that housing is a necessity, while the positive coefficient on prices implies that housing and non-housing goods are complements.

In the model, these estimates map into income and price elasticities of housing demand equal to

$$1 - 0.25 = 0.75 = \varepsilon \quad \text{and} \quad 1 - 0.39 = 0.61 = \nu. \tag{2}$$

Table 1 collects the empirical values of the parameters of the model.

Consistent evidence is found in other settings. Using French Family Expenditure Surveys for 2006 and 2011, Combes, Duranton, and Gobillon (2019) document that housing expenditure shares decrease with income across the distribution.[3]

Housing prices are increasing in city size. Housing prices rise with urban population and decline with land availability. Housing costs are also a central determinant of real income differences across space, as emphasized by Card, Rothstein, and Yi (2025) and Moretti (2013).

Combes, Duranton, and Gobillon (2019) use French census, household expenditure surveys, and housing and land transaction records for 2000–2012. In their baseline specification

[3]See their table 1. Fretz, Parchet, and Robert-Nicoud (2022) document similar patterns for Switzerland. Finlay and Williams (2025) also report consistent estimates of the income elasticity of housing demand for the United Kingdom, Canada, Norway, and the United States based on earlier contributions (see their Table B.3).

(Table 4, col. 1; standard errors clustered at the urban area level), they estimate:

$$\ln p = \underset{(0.021)}{0.217} \ln N - \underset{(0.022)}{0.151} \ln S + \text{year effects}, \tag{3}$$

where here p denotes the unit housing price at the city center, N the urban area population, and S the urban land area. Larger cities exhibit higher housing prices, reflecting stronger demand for housing, while a larger land area reduces prices through increased supply.

Combined with the non-homotheticity of housing demand, these estimates imply that housing expenditure shares increase with city size and decrease with income. In the model, the coefficients above map into the reduced-form relationship between population, income, and the cost of living, and jointly discipline the parameters governing housing demand and urban costs.

Housing expenditure shares are increasing in city size. Using the same data sources, Combes, Duranton, and Gobillon (2019) regress the expenditure share e on urban area population size N, household income w, geographic controls, and household demographic characteristics. In their baseline specification (Table 6, column 1), they obtain (standard errors clustered at the urban-area level in parentheses):

$$\ln e = \underset{(0.013)}{-0.282} \ln w + \underset{(0.001)}{0.028} \ln N + \text{controls}. \tag{4}$$

Thus, the housing expenditure shares decline with income (confirming that housing is a necessity in France) and increase with urban population. In our framework, both coefficients have structural interpretations as functions of the underlying parameters. Combining the regression coefficients in equation (4) with an external estimate of the housing supply elasticity $\sigma = 2$ from the review of Baum-Snow and Duranton (2025), we recover the implied values $\varepsilon = 0.7$ and $\nu = 0.9$ using equation (15) in section 3. These are close to the estimates obtained by Finlay and Williams (2025) for the United States, suggesting that the parameter values used in the model are empirically plausible across contexts.

These findings consistently support $\varepsilon < 1$. We thus conclude that housing is a necessity, which governs the mechanism that yields a finite and positive efficient city scale in our framework.

2.2 Estimates of agglomeration economies and housing supply elasticity

Urban wage premium: agglomeration economies and sorting. Wages increase with urban population size and density, and observed urban wage differentials reflect both agglomeration economies and endogenous worker composition.

Using the French worker–firm administrative panel DADS for selected years between 1976 and 1996, Combes, Duranton, and Gobillon (2008) estimate wage equations of the form (standard errors in parentheses):

$$\ln w = \underset{(0.0063)}{0.03} \ln D + \underset{(0.0154)}{0.004} \ln S + \text{controls} + \text{individual effects},$$

where D denotes population density. Using the identity $D \equiv \frac{N}{S}$, these estimates imply an elasticity of wages with respect to urban population equal to

$$\alpha = 0.03, \tag{5}$$

and an elasticity of wages with respect to land area equal to -0.026.[4] Population density and land area are instrumented using long-lagged population variables (from 1831) and identification relies on workers moving across urban areas, thereby controlling for time-invariant unobserved worker heterogeneity.

Taken together, these results imply that the urban wage premium combines causal agglomeration effects with the sorting of workers across cities along both observed and unobserved characteristics. In the model, this evidence disciplines the agglomeration elasticity α.

Housing supply elasticity. Housing supply is elastic, but imperfectly so.

Consider a constant-returns-to-scale housing production function combining land and non-land inputs. Combes, Duranton, and Gobillon (2019; 2021), using French data, and Ahlfeldt and McMillen (2014), using data for Berlin, Chicago, and Pittsburgh, show that a Cobb–Douglas specification provides a reasonable approximation, with an elasticity of housing output with respect to non-land inputs of about $\frac{2}{3}$. In competitive equilibrium, this

[4] Using worker-level panel data and instruments based on history and geology, Combes, Duranton, Gobillon, and Roux (2010) find that agglomeration economies are modest but robust, with wage elasticities with respect to density (or city size) typically in the range of 2–5 percent once sorting and unobserved worker heterogeneity are controlled for.

elasticity corresponds to $\frac{\sigma}{1+\sigma}$, where σ denotes the local housing supply elasticity, implying

$$\sigma = 2. \tag{6}$$

Direct estimates of housing supply elasticities range from near zero to above 10, depending on the method, context, and time horizon (Baum-Snow and Han 2024; DiPasquale and Wheaton 1994; Finlay and Williams 2025; Glaeser, Gyourko, and Saks 2006; Saiz 2010; Wu, Gyourko, and Deng 2016). Estimates based on construction data identify σ conditional on housing development, whereas most of the literature recovers an unconditional elasticity (namely, the conditional elasticity scaled by the probability of development), which is typically much lower, especially over short horizons. See Baum-Snow and Duranton (2025) for a comprehensive discussion.

These estimates of σ discipline the admissible values of the housing supply elasticity in the model. We find that the product $\alpha\sigma$ is much smaller than 1, which is a necessary and sufficient condition to obtain a finite efficient city scale in the model.

2.3 Zipf's law

As a first-order approximation, the distribution of city populations follows Zipf's law in many countries: city sizes are Pareto distributed with a shape parameter ζ close to one (Rozenfeld, Rybski, Gabaix, and Makse 2011). Some authors instead argue that the distribution is better described by a log-normal (Eeckhout 2004). Following Duranton and Puga (2014), we do not take a stance in this debate.

As an illustration, consider ranking cities in descending order of population and estimating the standard rank–size relationship as in Gabaix and Ibragimov (2011):

$$\ln\left(R - \frac{1}{2}\right) = -\zeta \ln N + \text{constant},$$

where the correction $R - \frac{1}{2}$ removes the small sample bias of the OLS estimator.

Using digitized historical French maps and censuses, Combes, Duranton, Gobillon, Gorin, and Robert-Nicoud (2026) estimate the Pareto shape parameter at different points in time (subscripts denote years):

$$\zeta_{1760} = 1.43, \quad \zeta_{1860} = 1.09, \quad \zeta_{1960} = 0.88, \quad \zeta_{2020} = 0.84. \tag{7}$$

Three lessons emerge. First, the Pareto distribution provides a reasonable approximation of the upper tail of the city size distribution. Second, Zipf's law ($\zeta = 1$) is rejected at conventional confidence levels. Third, the French urban system has become increasingly concentrated over time, with a growing share of the urban population residing in the largest cities. This pattern is consistent with a framework in which larger cities offer sufficiently strong productivity advantages to offset rising urban costs, thereby sustaining a thick upper tail in the city size distribution.

These findings discipline the distribution of city fundamentals in section 4, which map into empirically plausible distributions for city sizes in our model.

2.4 Estimates of density gradients

Within cities, housing prices, land prices, built-up density, and population density all decline with distance to the center (see Duranton and Puga 2015 for a review of this literature).

In the model, housing prices $p(x)$ decline with distance x to the city center according to

$$\ln p(x) = -\theta \ln x + \ln P,$$

where P is a city-specific intercept capturing the local cost of living. Land prices satisfy a similar isoelastic relationship with a steeper gradient ρ. The model links the two through $\rho = (1 + \sigma)\theta$, where σ is the local housing supply elasticity.

Using French data on house prices and land transactions, Combes, Duranton, and Gobillon (2019) estimate these within-city gradients by regressing prices per square meter on distance to the urban center, allowing both intercepts and slopes to vary across urban areas. In their preferred specification (Table 3, column 9), which controls for house or parcel characteristics as well as a rich set of municipal geography, geology, income, education, land-use, and amenity variables, the median log-distance gradient is -0.0088 for housing prices and -0.087 for land prices These imply (rounding to two decimals) $\theta = 0.01$ and $\sigma = 8.89$. The corresponding first quartile gradients are -0.0417 for house prices and -0.143 for land prices, implying

$$\theta = 0.04, \quad \sigma = 2.43. \tag{8}$$

Both values for the housing supply elasticity σ lie within the range $(0, 10)$ typically found

in the literature (table 1). Furthermore, the values of σ reported in equations (6) and (8) are remarkably close to one another. The first-quartile estimates are more informative for our purposes: cities with steeper price gradients are closer to the monocentric benchmark underlying the model, whereas flatter gradients at the median reflect more polycentric or spatially dispersed urban structures.

These estimates discipline θ and σ and support the monocentric structure underlying PIGL–AMM in section 5.

Taken together, these empirical relationships discipline the key parameters of the model, namely the income and price elasticities of housing demand (ε, ν), the agglomeration elasticity α, the housing supply elasticity σ, and the within-city gradient θ. In the following sections, we show that a parsimonious framework consistent with these estimates yields a unified and tractable model of urban structure.

3 A tractable urban model with PIGL preferences

3.1 Preferences and housing demand

A representative household in a given city consumes a traded good c, produced with labor only and which we use as the numeraire, and a local non-traded housing service h, produced with land and the numeraire. Let $w > 0$ denote income (wage) and $p > 0$ the price of one unit of housing.

Following Boppart (2014), preferences take a parsimonious PIGL form, represented by the indirect utility:

$$V(w,p) = \frac{w^{1-\varepsilon} - 1}{1-\varepsilon} - \mu \frac{p^{1-\nu} - 1}{1-\nu}, \qquad 0 < \mu \leq 1, \quad 0 \leq \varepsilon, \nu. \tag{9}$$

From Roy's identity, Marshallian demand and expenditure share for housing are:

$$h^D(p,w) = \mu w^{\varepsilon} p^{-\nu}, \qquad e(p,w) = \mu w^{-1+\varepsilon} p^{1-\nu}. \tag{10}$$

Utility is defined for wages and prices that satisfy $e \leq 1$. This requires assuming wages are sufficiently high to cover the subsistence level of housing: we henceforth assume $w^{1-\varepsilon} p^{-1+\nu} > \mu$. Empirical estimates reported in section 2 imply values for ε and ν strictly below one, consistent with housing being a necessity and price-inelastic. Parameter μ governs the strength of housing demand.

These PIGL preferences nest two polar cases used in most of the literature in urban economics:

Lemma 1. *The PIGL indirect utility in equation* (9) *nests Cobb-Douglas preferences with* $e(p,w) = \mu$ *as* $\varepsilon, \nu \to 1$ *and fixed housing consumption* ($h = \mu$) *when* $\varepsilon, \nu = 0$.

Proof. In Appendix A. □

In particular, households consume exactly μ units of housing independently of prices and income and utility is linear in the consumption of c if $\varepsilon, \nu = 0$, or spend a constant fraction μ of their income on housing if $\varepsilon, \nu \to 1$. In the former case, the price- and income-elasticity of housing demand are both zero, an extreme form of a necessary good. In the latter, the price- and income-elasticity of housing demand are both equal to one: both relative price and income effects are absent.

The preferences in equation (9) nests two further special cases. First, if $\nu = \varepsilon$, then the direct utility admits a closed form; specifically, the dual of V in equation (9) is:

$$\nu = \varepsilon \;\;\Rightarrow\;\; U(c,h) = \frac{1}{1-\varepsilon}\left\{c^{1-\varepsilon}\left[1 - \left(\frac{h_{\min}}{h}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{\varepsilon} - (1-\mu)\right\}, \tag{11}$$

where $h_{\min} \equiv \mu^{\frac{1}{1-\varepsilon}}$ can be interpreted as the subsistence level of housing. Second, V defines quasi-linear preferences with price effects if $\varepsilon = 0$ and $\nu \in (0,1)$.

3.2 Housing supply

Housing is produced competitively from land and a numeraire input z. The quantity of land available for housing development L is supplied inelastically. Assume a constant returns to scale Cobb-Douglas production function, written in intensive form (per unit of land) as follows:

$$h = \left(\frac{1+\sigma}{\sigma}\right)^{\frac{\sigma}{1+\sigma}} B^{\frac{1}{1+\sigma}} z^{\frac{\sigma}{1+\sigma}}, \qquad \sigma > 0, \tag{12}$$

where $B > 0$ governs construction total factor productivity, z is numeraire input, and $\frac{\sigma}{1+\sigma}$ is the share of non-land inputs in the production of housing.

Lemma 2. *Under competitive production with technology given in equation* (12) *and exogenous land endowment* L*, aggregate urban housing supply is*

$$H^S(p) = BLp^{\sigma}. \tag{13}$$

Proof. In Appendix A. □

Hence, σ is the (constant) housing supply elasticity in equilibrium. This formulation encompasses models that assume the quantity of housing to be proportional to the quantity of land as the limiting case when $\sigma \to 0$. Empirical estimates reported in section 2 suggest single-digit values for σ.

Land developers competitively bid for land, hence their profits are zero in equilibrium, and aggregate land rent is a share $\frac{1}{1+\sigma}$ of aggregate revenue pH in the housing sector. Using equation (13), we obtain:

$$R \equiv rL = \frac{B}{1+\sigma} L p^{1+\sigma}.$$

The aggregate land rent is increasing and convex in housing prices and proportional to land supply.

3.3 Equilibrium and the cost of living

Let N denote the number of identical households residing in the city. Market clearing in the housing market requires $Nh^D(p, w) = H^S(p)$. Using equations (10) and (13), we obtain:

Lemma 3. *Assume the housing sector is competitive and produces with the constant-returns-to-scale Cobb-Douglas production function in equation* (12)*. In equilibrium, the price and quantity of housing satisfy*

$$p = \left(\mu \frac{N}{BL}\right)^{\frac{1}{\nu+\sigma}} w^{\frac{\varepsilon}{\nu+\sigma}}, \qquad H = (BL)^{\frac{\nu}{\nu+\sigma}} (\mu N)^{\frac{\sigma}{\nu+\sigma}} w^{\frac{\varepsilon\sigma}{\nu+\sigma}}. \tag{14}$$

Proof. In Appendix A. □

The cost of living in the city coincides with the housing price. It increases with population and income, and decreases with land availability and housing productivity. Because housing is a necessity, prices respond more strongly to population than to income.

Combining equations (10) and (14), we obtain:

$$e(N, w) = \mu^{\frac{1+\sigma}{\nu+\sigma}} w^{-1+\varepsilon\frac{1+\sigma}{\nu+\sigma}} \left(\frac{N}{BL}\right)^{\frac{1-\nu}{\nu+\sigma}}. \tag{15}$$

Parameter values are disciplined by the empirical estimates of section 2; recall equation (4) in particular.

Henceforth we set $\nu = \varepsilon \in (0,1)$ and we normalize $\mu = 1$ to streamline equilibrium expressions. Online Appendix OA.1 generalizes the results of Proposition 1 below for the case $\nu \neq \varepsilon$.

3.4 Agglomeration and efficient city size

We now introduce agglomeration economies. Suppose that income depends on city population as

$$w = AN^{\alpha}, \qquad A > 0, \quad \alpha \geq 0, \tag{16}$$

where A is fundamental productivity and α measures the strength of agglomeration effects. Substituting equations (14) and (16) into the indirect utility (9) yields:

$$V(N) = \frac{1}{1-\varepsilon}\left\{[AN^{\alpha}]^{1-\varepsilon} - \left[A^{\frac{\varepsilon}{\varepsilon+\sigma}}(BL)^{-\frac{1}{\varepsilon+\sigma}}N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right]^{1-\varepsilon}\right\}. \tag{17}$$

This expression reflects the so-called fundamental trade-off in urban economics, whereby a larger city size increases productivity as well as the cost of living (Henderson 1974; Fujita and Thisse 2013). Utility in the city is maximized at a unique interior city size whenever the elasticity of the cost of living with respect to population exceeds the agglomeration elasticity:

$$\alpha < \frac{1+\alpha\varepsilon}{\varepsilon+\sigma} \quad \Leftrightarrow \quad \alpha\sigma < 1.$$

This condition ensures that dispersion forces, which arise from housing market crowding, dominate agglomeration forces for city populations large enough. Given $\alpha = 0.03$ and $\sigma \leq 10$ (see table 1), it follows that $\alpha\sigma$ is at about an order of magnitude smaller than 1. We write:

Proposition 1. ***Efficient city size.*** *Suppose wages satisfy equation* (16) *and* $\alpha\sigma < 1$ *holds, then: (i) There exists a unique interior efficient city size, given by*

$$N = \left[\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}}A^{\frac{\sigma}{\varepsilon+\sigma}}(BL)^{\frac{1}{\varepsilon+\sigma}}\right]^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}}. \tag{18}$$

(ii) In the homothetic Cobb-Douglas case, the efficient city size is zero:

$$\lim_{\varepsilon \to 1} N = 0.$$

Proof. In Appendix B. □

The efficient city size is finite whenever the elasticity of the cost of living with respect to population, $\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}$, exceeds the agglomeration elasticity, α, that is, whenever $\alpha\sigma < 1$. The elasticity of efficient city size with respect to technology is increasing in agglomeration elasticity α and housing supply elasticity σ, as a higher α strengthens agglomeration forces and a higher σ relaxes dispersion forces.

Under the maintained isoelastic functional forms for agglomeration and dispersion forces, the absence of a finite efficient city size under homothetic Cobb–Douglas preferences is the result of the combination of homothetic preferences, $\varepsilon = 1$, and unit price elasticity of housing demand, $\nu = 1$. Breaking *either* assumption is sufficient to restore a finite efficient size. However, with either $\varepsilon = 1$ or $\nu = 1$, the efficient size is decreasing in A. Focusing on empirically relevant parameter values yields the following result:

Corollary 1. *Consider the preferences in equation* (9)*. Suppose wages satisfy equation* (16) *and* $\alpha\sigma < 1$ *holds. Then the efficient city size is strictly positive, finite, and increasing in A, B, and L if and only if* $\varepsilon, \nu \in (0, 1)$*.*

Proof. In Appendix B. □

Breaking either homotheticity in income or unit price elasticity is sufficient to ensure that congestion forces eventually dominate agglomeration forces, thereby restoring a finite efficient city size. However, when only one margin departs from the Cobb–Douglas benchmark, productivity improvements are largely capitalized into housing prices rather than population. If housing demand is homothetic in income, higher productivity scales housing demand proportionally, so that rising housing costs offset the gains from agglomeration. If instead demand is unit-elastic with respect to prices, households adjust strongly to price changes, again allowing productivity gains to be absorbed through housing costs. Only when housing demand is inelastic with respect to both income and prices do increases in productivity translate into larger cities rather than higher housing costs.

This distinction clarifies the difference between PIGL preferences and Stone–Geary or fixed-housing specifications commonly used in urban models. Stone–Geary preferences in-

troduce income effects in housing demand and are therefore sufficient to restore a finite efficient city size. However, because Engel curves are linear in income and prices, productivity improvements are largely capitalized into housing prices rather than population. By contrast, PIGL preferences generate empirically relevant income and price effects simultaneously, ensuring not only the existence of finite cities but also the empirically consistent prediction that more productive cities are larger.

4 Urban systems

We now consider urban systems made of heterogeneous cities.

4.1 Heterogeneous cities and identical households

We maintain the assumption of identical households, and consider a large set of potential cities, endowed with heterogeneous fundamentals $\mathbf{X} \equiv (A, B, L)$. Let also

$$X \equiv \left(1 - \frac{1 - \alpha\sigma}{1 + \alpha\varepsilon}\right)^{\frac{\varepsilon+\sigma}{\sigma(1-\varepsilon)}} A\,(BL)^{\frac{1}{\sigma}} \tag{19}$$

collect the productivity and housing affordability advantages of the city, and rewrite the efficient size in equation (18) as

$$N = X^{\psi}, \qquad \psi \equiv \frac{1 + \alpha\varepsilon}{1 - \alpha\sigma} \frac{\sigma}{\varepsilon + \sigma}.$$

Assume that the equilibrium population size of individual cities is their efficient size, as would be the case if the local government maximizes the well-being of its inhabitants, as in Albouy, Behrens, Robert-Nicoud, and Seegert (2019) and Duranton and Puga (2023).

Let m_j denote the mean of some random variable j and η_{ij} denote the covariance between variables i, j, and define $\mathbf{M} \equiv \{m_j\}$ and $\mathbf{\Sigma} \equiv \{\eta_{ij}\}$ for $i, j = A, B, L$. We obtain:

Proposition 2. ***City size distribution.*** *(i) If the city scalar X is Pareto distributed with shape parameter $k > \psi$, then efficient city sizes are Pareto distributed with shape parameter $\zeta \equiv \frac{k}{\psi}$. (ii) If the components of the vector of city fundamentals $\mathbf{X}$ jointly follow a log normal distribution as $(\ln A, \ln B, \ln L)^{\top} \sim \mathcal{N}(\mathbf{M}, \mathbf{\Sigma})$, then efficient city sizes follow*

a log-normal distribution as $\ln N \sim \mathcal{N}(m_N, \eta_N)$*, where*

$$m_N \equiv \ln\left\{\left(\alpha\frac{\varepsilon+\sigma}{1+\alpha\varepsilon}\right)^{\frac{\varepsilon+\sigma}{\sigma(1-\varepsilon)}}\right\} + \psi m_A + \frac{\Psi}{\sigma}(m_B + m_L), \quad \eta_N \equiv \left(1, \frac{1}{\sigma}, \frac{1}{\sigma}\right)\mathbf{\Sigma}\left(1, \frac{1}{\sigma}, \frac{1}{\sigma}\right)^{\top}.$$

Proof. In Appendix B. □

Consistent with empirical evidence discussed in section 2, city size distributions are well approximated by Pareto or log-normal forms. This proposition notes that the model can both account for Zipf's law if the scalar X in equation (19) is Pareto distributed with shape parameter k close to ψ, or a log-normal distribution if the vector $\mathbf{X}$ is log-normally distributed.[5]

4.2 Heterogeneous cities and heterogeneous preferences

We now allow for idiosyncratic location preferences (Ahlfeldt, Redding, Sturm, and Wolf 2015; Allen and Arkolakis 2025; Diamond 2016; Fajgelbaum, Morales, Suárez Serrato, and Zidar 2019; Monte, Redding, and Rossi-Hansberg 2018; Redding 2016; Redding 2025), and let the size of cities be determined by the free mobility of households. Here we set $B = L = 1$ for simplicity, hence cities are indexed by the scalar $A \in \mathbb{R}_{++}$. Denote the mass of potential sites to host cities by $M > 0$, the equilibrium mass of inhabited cities by $M_+ < M$, the equilibrium set of inhabited cities by $\mathbb{A}_+ \subset \mathbb{R}_{++}$, and the associated distribution function by $F_{\mathbb{A}_+}$. Finally, let N_{Tot} denote the exogenous total urban population. Assume households draw idiosyncratic preference shocks that multiply the common component of utility $V(A)$ and that these shocks are Frechet distributed with shape parameter $\kappa > 1$. Then the fraction of households choosing to live in city A is

$$\frac{N(A)}{N_{\text{Tot}}} = \left(\frac{V(A)}{\mathbb{V}}\right)^{\kappa}, \qquad \mathbb{V} \equiv \left[M_+ \int_{x\in\mathbb{A}_+} (V(x))^{\kappa}\, \mathrm{d}F_{\mathbb{A}_+}(x)\right]^{\frac{1}{\kappa}}. \tag{20}$$

The expected utility of households is equal to $\mathbb{E}U = \Gamma\left(1 - \frac{1}{\kappa}\right)\mathbb{V}$, where Γ is the gamma function, and it increases in the mass of cities M_+. This property can be interpreted as

[5] We can further extend the model by allowing each component of the vector of city fundamentals to be the product of multiple random underlying factors — such as amenities, topographic features, and productivity characteristics, following Lee and Li (2013). When city fundamentals aggregate many such independent or weakly correlated components, the central limit theorem implies that their logarithm is approximately normally distributed. As a result, the efficient size of cities follows a log-normal distribution.

reflecting a taste for variety of local amenities.

A useful property of Frechet distributed multiplicative preference shock is that the expected utility in inhabited city A is proportional to $\left(\frac{N(A)}{N_{\text{Tot}}}\right)^{-\frac{1}{\kappa}} V(A)$, which is equal to $\mathbb{V}$ by inversion of the first expression in equation (20). Thus, in spatial equilibrium, the expected utility is equalized across inhabited cities. We henceforth normalize $N_{\text{Tot}} = 1$ to reduce equation cluttering and, using equation (20), we obtain $(N(A))^{-\frac{1}{\kappa}} V(A) = \mathbb{V}$ whenever $N(A) > 0$. Preference heterogeneity across locations thus acts as an additional dispersion force.[6] Assuming that highly productive cities are inhabited first, we obtain:

Proposition 3. ***Spatial equilibrium with heterogeneous location preferences.*** *Assume*

$$1 < \alpha(1-\varepsilon)\kappa. \tag{21}$$

Then (i) the efficient size of cities is positive and finite; (ii) there exists a threshold urban productivity $\underline{A} > 0$ *and a scalar* $\mathbb{V} > 0$ *such that the populations of cities with a productivity larger than* $\underline{A}$ *are strictly larger than their efficient size, (iii) the population of the city with urban productivity* $\underline{A}$ *is at its efficient size, (iv) the populations of cities with a productivity below* $\underline{A}$ *are zero, and (v) expected utility in any inhabited city is equal to* $\Gamma\left(1-\frac{1}{\kappa}\right)\mathbb{V}$.

Proof. In Appendix B. □

With heterogeneous location preferences, households trade off agglomeration benefits against housing-market crowding and idiosyncratic tastes for locations. For any given city productivity A, expected utility is increasing in population at very small city sizes, because agglomeration effects dominate both housing costs and the dispersion induced by taste heterogeneity. At larger sizes, housing-market crowding and preference dispersion dominate, so expected utility eventually decreases with population. The condition in equation (21) ensures that this trade-off generates a unique interior population level at which expected utility is maximized.[7]

In spatial equilibrium with free mobility, households sort across cities until expected utility is equalized across all inhabited locations. Highly productive cities therefore attract residents beyond their efficient size (this condition implies that the equilibrium is robust to small perturbations), while less productive cities remain uninhabited. The marginal

[6] As $\kappa \to 1$, heterogeneity in the preference for amenities of households rises, increasing the ex-ante value of variety under idiosyncratic taste uncertainty and reinforcing this dispersion force.

[7] If it is violated, $N^{-1/\kappa} V(N)$ decreases monotonically from $+\infty$ for all $N > 0$.

inhabited city is exactly at its efficient size, as it is just attractive enough to draw residents. Preference heterogeneity thus acts as an additional dispersion force that truncates the urban system and prevents arbitrarily small or large cities from being populated in equilibrium.

4.3 Sorting across cities

We now embed our urban model into an economy with many cities with identical city fundamentals but heterogeneous workers. Here also we let the size of cities be determined by the free mobility of households. We characterize sorting across cities in the spirit of Behrens, Duranton, and Robert-Nicoud (2014). They work with the model featuring unitary housing consumption and a production function linear in land use, which is the special case $\varepsilon = \sigma = 0$ of our PIGL-UM model. For simplicity we abstract from selection into occupations, which is an important focus of their paper, but we add human capital spillovers, which are absent in theirs.

Without further loss of generality, we normalize the value of all components of city fundamental to one, $A = B = L = 1$. Workers are endowed with heterogeneous amounts of effective units of labor, or "talent" for short, $s, t \in \mathcal{T} \subset \mathbb{R}_{++}$; we assume that the cumulative distribution function of talent $F_{\mathcal{T}}$ is twice continuously differentiable. Consider for now that each city is inhabited by identical households, so with some abuse of notation we may refer to t as the type of the city as well. Denote the size, wage per unit of effective labor, and housing price of city t by $N(t)$, $w(t)$, and $p(t)$, respectively. Wages per unit effective labor reflect both agglomeration economies and human capital spillovers :

$$w(t) = (N(t))^{\alpha} t^{\lambda}.$$

If $\lambda \geq 0$ then the productivity of a worker is (weakly) increasing in that of the workforce. Using equations (14) and (16), the nominal earnings of the representative worker in that city is $w(t)\, t = (N(t))^{\alpha} t^{1+\lambda}$, and the housing price is:

$$p(t) = (N(t))^{\gamma} t^{(1+\lambda)\frac{\varepsilon}{\varepsilon+\sigma}}, \qquad \gamma \equiv \alpha + \frac{1-\alpha\sigma}{\varepsilon+\sigma}, \tag{22}$$

where $\gamma > \alpha$ under $\alpha\sigma < 1$. The nominal earnings a worker of type s living in a city of type t is $w(t)\, s = (N(t))^{\alpha} t^{\lambda} s$. Thus, the utility of a worker of type s living in a city of

type t, which we denote by $V(s,t)$, is

$$V(s,t) = \frac{1}{1-\varepsilon}\left\{\left[\left(N(t)\right)^{\alpha} t^{\lambda} s\right]^{1-\varepsilon} - \left[\left(N(t)\right)^{\gamma} t^{(1+\lambda)\frac{\varepsilon}{\varepsilon+\sigma}}\right]^{1-\varepsilon}\right\}.$$

An equilibrium features talent-homogeneous cities if type-s workers choose to live in type-s cities, that is, if $s = \text{argmax}_t V(s,t)$. We also contrast the equilibrium size of each talent-homogeneous city with that chosen by a utilitarian planner. We obtain:

Proposition 4. ***Sorting across cities.*** *In an equilibrium with talent-homogeneous cities, (i) city population is equal to:*

$$N^*(t) = \left\{\left(\frac{\alpha\sigma+\lambda}{1+\lambda}\right)^{\frac{\varepsilon+\sigma}{1-\varepsilon}} t^{\sigma(1+\lambda)}\right\}^{\frac{1}{1-\alpha\sigma}}. \tag{23}$$

(ii) Cities hosting more talented workers are larger, pay higher wages, and feature higher housing costs. (iii) If $\lambda(1-\alpha\sigma) \leq \alpha\varepsilon$, then the equilibrium size of talent-homogeneous cities $N^*(t)$ is smaller than the optimal size of talent-homogeneous cities $N^0(t)$:

$$\lambda < \alpha\varepsilon \;\Rightarrow\; N^*(t) < N^0(t) \equiv \left[\left(\frac{\alpha\sigma+\alpha\varepsilon}{1+\alpha\varepsilon}\right)^{\frac{\varepsilon+\sigma}{1-\varepsilon}} t^{\sigma(1+\lambda)}\right]^{\frac{1}{1-\alpha\sigma}}. \tag{24}$$

(iv) The size distribution of cities converges to Zipf's law as $\alpha\sigma \to 1$.

Proof. In Appendix B. □

These equilibrium properties are in line with empirical evidence (Combes, Duranton, and Gobillon 2008; Diamond 2016; Moretti 2013). Some properties of this equilibrium deserve special emphasis. First, the elasticity of city size with respect to worker talent is increasing in agglomeration elasticity α and housing supply elasticity σ, as a higher α strengthens agglomeration forces and a higher σ relaxes dispersion forces.

Second, this equilibrium is degenerate in the limiting cases of homothetic Cobb-Douglas preferences ($\lim_{\varepsilon\to 1} N^*(t) = 0$), or if dispersion forces dominate agglomeration forces at any city size, which arises if housing supply is inelastic or in the absence of agglomeration economies ($\lim_{\alpha\sigma\to 0} N^*(t) = 0$).

Third, as in the model of Behrens, Duranton, and Robert-Nicoud (2014), the equilibrium location decision of workers is inefficient. Consider first the case $\lambda = 0$. Housing

market crowding dominates agglomeration forces at the margin by $\alpha\sigma < 1$, and the worker does not internalize the full productivity gain of their location decisions. Conversely, if knowledge spillovers are sufficiently large in the sense $\lambda > \alpha\varepsilon$, then the pecuniary externality a worker imposes on fellow city dwellers is negative at the margin. Hence, upon making location decisions, the marginal urban dweller imposes a net cost on her fellow citizens and talent-homogeneous cities are over-sized in equilibrium.

Fourth, productivity is embedded in people rather than in places: if $\lambda = 0$, then the expression for $N^0(t)$ in equation (24) above is isomorphic to the efficient city size $N(A)$ in equation (18).

Finally, Behrens, Duranton, and Robert-Nicoud (2014) show that the size distribution of cities follows Zipf's law regardless of the distribution $F_{\mathcal{T}}$ if returns to city size are close to constant, under mild additional regularity conditions (their method of proof applies to our model, too). This case is not a mere intellectual curiosity: it is also a good representation of extant empirical evidence, which shows that the magnitude of the elasticity of urban costs with respect to city size γ, when estimated directly, is close to that of the elasticity of wages with respect to city size α (Combes, Duranton, and Gobillon 2019; Duranton and Puga 2023).

5 PIGL-AMM: Monocentric city model

The PIGL-UM model abstracts from within-city spatial structure. In this section, we introduce urban space, commuting costs, and endogenous land use, and show that the space-free equilibrium characterized in section 3 arises as the exact reduced form of a monocentric city model. We label this spatial model PIGL-AMM, in reference to Alonso (1964), Mills (1967), and Muth (1969). See Duranton and Puga (2015) for a recent treatment.

5.1 Preferences and housing demand

Households work and consume in a central business district (CBD) and reside at distance $x \geq 0$ from the CBD. Commuting is costly. Space extends in all directions on the Euclidean space. Let $p(x)$ denote the unit price of housing at location x.

Preferences extend the PIGL structure in equation (9) by allowing commuting costs

and housing consumption to enter utility jointly. Indirect utility is given by:

$$V(w,x) = \frac{1}{1-\varepsilon}\left[w^{1-\varepsilon} - \mu\left(p(x)\,\tau x^{\theta}\right)^{1-\varepsilon} - (1-\mu)\right], \tag{25}$$

where $\tau > 0$ is a commuting cost shifter and $\theta > 0$ is the elasticity of commuting cost with respect to distance. When $\theta = 0$ and $\tau = 1$, this formulation collapses to the space-free PIGL-UM preferences of section 3.

This formulation assumes that commuting costs and housing costs enter indirect utility in a complementary way. The direct utility, that is, the dual of V in equation (25), is:

$$U(c,h) = \frac{1}{1-\varepsilon}\left\{c^{1-\varepsilon}\left[1 - \left(\frac{h_{\min}(x)}{h}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{\varepsilon} - (1-\mu)\right\},$$

where $h_{\min}(x) \equiv \mu^{\frac{1}{1-\varepsilon}}\tau x^{\theta}$: the "subsistence" amount of housing increases with distance to the CBD. Thus, the utility cost of distance is incurred in units of housing (rather than in units of the numeraire good, as is often assumed).

From Roy's identity, Marshallian demand and housing expenditure share at location x are:

$$h^{D}(w,x) = \mu\left(\tau x^{\theta}\right)^{1-\varepsilon}\left(\frac{w}{p(x)}\right)^{\varepsilon}, \qquad e(w,x) = \mu\left(\frac{\tau x^{\theta} p(x)}{w}\right)^{1-\varepsilon}. \tag{26}$$

We normalize the reference location within each city to $x = 1$ to simplify notation; all results are invariant to this choice. In a within city spatial equilibrium, residents must be indifferent across all city locations, so $V(w,x) = V(w,1)$ for all $x \leq \overline{x}$, where $\overline{x}$ denotes the endogenous urban fringe. Then, the expression in equation (25) implies

$$\forall x \leq \overline{x}: \qquad p(x)\,\tau x^{\theta} = P, \tag{27}$$

where, like p in the model of section 3, P is a sufficient statistic for the urban *cost of living* faced by all residents. Specifically, $P \equiv p(1)\,\tau$ combines the housing price and the commuting cost at the reference location.

Substituting the spatial equilibrium condition of equation (27) into equations (25) and (26) yield

$$V(P,w,x) = \frac{1}{1-\varepsilon}\left[w^{1-\varepsilon} - \mu P^{1-\varepsilon} - (1-\mu)\right] \tag{28}$$

and

$$h^D(P,w,x) = \mu\tau x^\theta \left(\frac{w}{P}\right)^\varepsilon, \qquad e(P,w,x) = \mu\left(\frac{P}{w}\right)^{1-\varepsilon}, \tag{29}$$

respectively. Hence, the housing expenditure share is constant across space and is isomorphic to that in the space-free PIGL-UM model. Also, housing price $p(x)$ declines and housing consumption $h^D(x)$ increases with distance to the CBD. Empirical estimates reported in section 2 imply small but positive values of θ. As in section 4, we henceforth set $\mu = 1$.

5.2 Housing supply

Housing is produced competitively using land and a numeraire input with the same Cobb–Douglas technology as in section 3. Because housing prices vary with distance, housing supply per unit of land at location x is:

$$h^S(x) = B\left(p(x)\right)^\sigma = B\left(\frac{P}{\tau x^\theta}\right)^\sigma. \tag{30}$$

Thus, floor space density declines monotonically with distance from the CBD.

With free-entry, builders earn zero profits. The equilibrium land rent at location x is therefore

$$r(x) = \frac{B}{1+\sigma}\left(p(x)\right)^{1+\sigma}.$$

Just like the price of housing, the price of land declines with distance to the center. Let $\frac{\overline{r}}{1+\sigma} > 0$ denote the opportunity cost of land, with $\overline{r} > 0$. The urban fringe $\overline{x}$ is defined by the no-arbitrage condition $r(\overline{x}) = \frac{\overline{r}}{1+\sigma}$. Let

$$\overline{p} \equiv p(\overline{x}) = \left(\frac{\overline{r}}{B}\right)^{\frac{1}{1+\sigma}}$$

denote the housing price at the city boundary. Using equation (27), the city radius and total urbanized area are

$$\overline{x} = \left(\frac{P}{\overline{p}\tau}\right)^{\frac{1}{\theta}}, \qquad S = \pi\left(\frac{P}{\overline{p}\tau}\right)^{\frac{2}{\theta}},$$

where π denotes the mathematical constant, $\pi \approx 3.1416$.

5.3 Equilibrium and the cost of living

Following Combes, Duranton, Gobillon, Gorin, and Robert-Nicoud (2026), assume that the share of land convertible to urban use at distance x is equal to $(1-\psi)\,x^{-\xi}$ to account for topographical and institutional features of the city (Harari 2020; Hilber and Robert-Nicoud 2013; Saiz 2010). Here, ψ captures the typical fraction of land within the city boundaries that cannot be used for urban development (e.g., $\psi = \frac{1}{2}$ for a city that grows along a linear coast and $\psi = 0$ for a city that can expand freely in all directions), and ξ captures the extent to which this fraction evolves with the distance to the CBD: for instance, $\xi > 0$ if terrain becomes steeper and more difficult to build and/or if land use regulations become more stringent as distance to the CBD increases. We show in Online Appendix OA.2 that if the physical density gradient is steeper than the population gradient in equilibrium, then $\xi < 0$. The standard model is the special case $\xi = \psi = 0$.

Population density at location x is defined as the number of residents per square meter of land, which is equal to the fraction of land converted to urban use times housing supply, divided by housing demand. Using equations (29) and (30) , we obtain:

$$D(x) \equiv (1-\psi)\,x^{-\xi}\frac{h^S(x)}{h^D(x)} = (1-\psi)\,\frac{B}{\mu\tau^{1+\sigma}}w^{-\varepsilon}P^{\varepsilon+\sigma}x^{-\xi-\theta(1+\sigma)}. \tag{31}$$

Total urban population satisfies

$$N = 2\pi\int_0^{\overline{x}} xD(x)\,\mathrm{d}x.$$

The integral above converges if and only if $\xi < 2-\theta(1+\sigma)$, which we henceforth impose, and diverges to $-\infty$ otherwise. Using equations (27) and (31), this condition yields a closed-form expression for the equilibrium cost of living:

$$P = \left(\frac{N}{\Lambda}\right)^{\frac{1}{\varepsilon+\eta}} w^{\frac{\varepsilon}{\varepsilon+\eta}}, \tag{32}$$

where

$$\eta \equiv \frac{2-\xi}{\theta} - 1, \qquad \Lambda \equiv \frac{2\pi(1-\psi)}{\eta-\sigma}\theta^{-1}\tau^{-1-\eta}\overline{r}^{-\frac{\eta-\sigma}{1+\sigma}}B^{\frac{1+\eta}{1+\sigma}}.$$

Crucially, equations (32) and (14) are formally isomorphic, with P (the cost of living combining housing prices and commuting costs) replacing p (the equilibrium housing price).

The housing supply elasticity σ and land endowment L of the space-free PIGL-UM model are replaced here by η and Λ, which summarize commuting technology and land availability. From this point onward, the monocentric city PIGL-AMM model collapses to PIGL-UM, and hence all the results of propositions 1 (efficient city size), 2 (city size distribution), 3 (heterogeneous location preferences), and 4 (sorting across cities) apply to PIGL-AMM, too. Summarizing:

Proposition 5. ***PIGL-UM and PIGL-AMM.*** *The space-free PIGL-UM model of section 3 arises as the exact reduced form of the PIGL-AMM model of this section.*

Proof. In Appendix B. □

5.4 Gradients

Equations (27) and (30) give us the housing price gradient and physical density gradient (also known as the floor to area ratio):

$$\mathcal{E}(p,x) \equiv \frac{\partial \ln p(x)}{\partial \ln x} = -\theta, \qquad \mathcal{E}\left(h^S,x\right) \equiv \frac{\partial \ln h^S(x)}{\partial \ln x} = -\sigma\theta.$$

Summing the two, we obtain the gradient of the land price $r(x) \equiv \frac{\sigma}{1+\sigma} p(x) h^S(x)$:

$$\mathcal{E}(r,x) \equiv \frac{\partial \ln r(x)}{\partial \ln x} = -(1+\sigma)\theta.$$

Using equations (26), (27), and (30), the gradient of population density is equal to

$$\mathcal{E}(D,x) \equiv \frac{\partial \ln D(x)}{\partial \ln x} = -\xi - \theta(1+\sigma).$$

Empirical estimates of price and land gradients reported in section 2 imply values of θ and σ consistent with the model.

From these we conclude that the housing price, land prices, built-up density, and population density all decay with distance to the CBD. Furthermore, a higher θ makes proximity to the center more desirable, resulting in steeper price and density gradients. These properties are consistent with the large body of literature reviewed in Duranton and Puga (2015). Online Appendix OA.2 provides the details.

6 Summary and conclusion

This paper develops PIGL-UM, a unifying and analytically tractable framework for urban analysis that incorporates empirically relevant income and price effects in housing demand. By adopting non-homothetic PIGL preferences, the model nests the two polar cases dominating the literature and restores a non-degenerate interior efficient city size under weak and realistic conditions. The framework delivers closed-form solutions for housing markets, land rents, and city scale, and extends naturally to urban systems with heterogeneous fundamentals, preferences, and worker talent.

A central contribution of the paper is to show that PIGL-UM is not a reduced-form shortcut, but the exact equilibrium representation of a fully specified monocentric city model with commuting costs, endogenous land use, and spatial price gradients. This spatial extension, labeled PIGL-AMM, collapses to a single sufficient statistic — the urban cost of living — so that all equilibrium conditions governing city size and welfare carry over unchanged. Together, PIGL-UM and PIGL-AMM provide a unified foundation for analyzing city size, housing markets, sorting, and internal urban structure within a single framework.

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Table

Table 1: Empirical discipline and implied parameter values.

Object	Parameter(s)	Value	Sources
Income elasticity of housing demand	ε	0.7–0.8	CDG 2019, 2021 (implied); FW 2025.
Price elasticity of housing demand	ν	0.6–0.9	CDG 2019, 2021 (implied); FW 2025.
Housing supply elasticity	σ	$0 - 10$	BSD 2025; DW 1994; FW 2025; GGS 2006; S 2010; WGD 2016.
Land share in housing supply	$\frac{1}{1+\sigma}$	$0.\bar{3} \to \sigma = 2$	AM 2014; BSD 2025; CDG 2021
Labor productivity elasticity wrt population	α	0.02–0.05	CDG 2008, CDGR 2010.
Urban cost elasticity wrt population	$\frac{1-\nu}{\varepsilon+\sigma}$	0.03–0.08	CDG 2019, 2021.
Pareto shape of city size distribution	ζ	$0.84 - 1.43$	CDGGRN 2026.
Commuting cost elasticity wrt distance to CBD	θ	0.04–0.08	CDG 2019; DP 2023.

Notes: Reported elasticities from existing work (Ahlfeldt and McMillen 2014; Baum-Snow and Duranton 2025; Combes, Duranton, and Gobillon 2008; Combes, Duranton, and Gobillon 2019; Combes, Duranton, and Gobillon 2021; Combes, Duranton, Gobillon, Gorin, and Robert-Nicoud 2026; Combes, Duranton, Gobillon, and Roux 2010; DiPasquale and Wheaton 1994; Duranton and Puga 2023; Finlay and Williams 2025; Glaeser, Gyourko, and Saks 2006; Saiz 2010; Wu, Gyourko, and Deng 2016) map into the equilibrium relationships of the model and jointly discipline the key preference and technology parameters.

Appendix

A Proofs of lemmas

Lemma 1.

Proof. Taking the appropriate limits of equation (9), and using L'Hôpital's rule in the case $\varepsilon, \nu \to 1$, we get:

$$\lim_{\varepsilon,\nu\to 0} V(p,w) = w - \mu p - (1-\mu), \qquad \lim_{\varepsilon,\nu\to 1} V(p,w) = \ln w - \mu \ln p.$$

□

Using the direct form of utility in equation (11), using L'Hôpital's rule we get $\lim_{\varepsilon\to 1} U(c,h) = (1-\mu)\ln c + \mu \ln h$. Turning to $\varepsilon \to 0$, we find

$$h > \mu^{\frac{1}{1-\varepsilon}} \;\Rightarrow\; \lim_{\varepsilon\to 0} U(c,h) = c \lim_{\varepsilon\to 0} \left[1 - \left(\mu^{-\frac{1}{1-\varepsilon}} h\right)^{-\frac{1}{\varepsilon}+1}\right]^{\varepsilon} = c,$$

where the second equality holds for any h as long as $h > h_{\min} \equiv \mu^{\frac{1}{1-\varepsilon}}$ holds. Thus we may write $\lim_{\varepsilon\to 0} h^D(p,w) = \mu$, as is confirmed by inspection of equation (10), and $\lim_{\varepsilon\to 0} U(c,h) = c\mathbb{1}_{h_{\min}}(h)$, where $\mathbb{1}_{h_{\min}}(h) = 1$ if $h \geq \mu^{\frac{1}{1-\varepsilon}}$ and $\mathbb{1}_{h_{\min}}(h) = 0$ otherwise.

Lemma 2.

Proof. A land developer/builder chooses z to maximize profit per unit of land:

$$\pi(z) = p\left(\frac{1+\sigma}{\sigma}\right)^{\frac{\sigma}{1+\sigma}} B^{\frac{1}{1+\sigma}} z^{\frac{\sigma}{1+\sigma}} - z - r,$$

where r is the unit bid rent to be determined in equilibrium. The first-order condition is

$$p\left(\frac{\sigma}{1+\sigma}\frac{B}{z}\right)^{\frac{1}{1+\sigma}} = 1.$$

Solving for z gives

$$z = \frac{\sigma}{1+\sigma} B p^{1+\sigma}.$$

Substituting back into (12), the supply of housing per unit of land is $h^S(p) = Bp^{\sigma}$. Multiplying by the inelastic land supply L gives the result. □

Lemma 3.

Proof. Using equations (10) and (13), we obtain $Nw^{\varepsilon}p^{-\nu} = BLp^{\sigma}$. We obtain equation (14) by solving this expression for p. □

B Proofs of propositions

Proposition 1. Efficient city size.

Proof. (i) The function V is concave and increasing in N at zero whenever

$$\alpha(1-\varepsilon) < \frac{1+\alpha\varepsilon}{\varepsilon+\sigma}(1-\varepsilon) \;\Leftrightarrow\; \alpha\sigma < 1.$$

If this condition holds, then the efficient size is strictly positive and finite. (ii) Note that $\frac{1-\alpha\sigma}{1+\alpha\epsilon}$ is lower than 1 if $\alpha\sigma < 1$. Hence, taking the limit $\varepsilon \to 1$ from below, we have:

$$\lim_{\varepsilon\to 1} N = \left[A^{\frac{\sigma}{1+\sigma}}(BL)^{\frac{1}{1+\sigma}} \lim_{\varepsilon\to 1}\left\{\left(1-\frac{1-\alpha\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}}\right\}\right]^{\frac{1+\alpha}{1-\alpha\sigma}} = 0.$$

□

Corollary 1.

Proof. The efficient size N is strictly positive and finite if and only if this inequality holds:

$$(1+\sigma)\varepsilon > \nu + \sigma - \frac{1-\nu}{\alpha}.$$

Under this condition, the efficient size N is increasing in A if and only if $\sigma(1-\varepsilon)+\nu-\varepsilon > 0$ holds, which we rewrite as

$$\nu + \sigma > (1+\sigma)\varepsilon.$$

Thus, a necessary condition for both conditions to hold simultaneously is $\nu \in (0,1)$. In this case, the inequality above holds only if $\varepsilon < 1$. □

Proposition 2. City size distribution.

Proof. (i) If $X \sim \mathcal{P}(X_0, k)$, namely, if X follows a Pareto distribution with scale parameter X_0 and scale parameter k, then $G(X) = 1 - \left(\frac{X}{X_0}\right)^{-k}$ and $g(X) = k(X_0)^k X^{-k-1}$, where G and g denote the cumulative distribution and probability density functions of X,

respectively. Let F denote the cumulative distribution function of N. Using equations (18) and (19) and a change of variable, we have:

$$\begin{aligned} F(N) &= 1 - \int_N^\infty k \left(X\left(N_0\right)\right)^k \left(X\left(n\right)\right)^{-k-1} \mathrm{d}X\left(n\right) \\ &= 1 - \zeta \left(N_0\right)^\zeta \int_N^\infty n^{-\zeta-1} \mathrm{d}n \\ &= 1 - \left(\frac{N_0}{N}\right)^\zeta. \end{aligned}$$

The integral above converges if $\zeta \equiv \frac{k}{\theta} > 1$. (ii) From equation (35) and taking logs, we have:

$$\begin{aligned} \ln N &= \ln \left\{ \left[\left(1 - \frac{1-\alpha\sigma}{1+\alpha\varepsilon}\right)^{\frac{1}{1-\varepsilon}} A^{\frac{\sigma}{\varepsilon+\sigma}} \left(BL\right)^{\frac{1}{\varepsilon+\sigma}} \right]^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}} \right\} \\ &= \frac{\varepsilon+\sigma}{\left(1-\alpha\sigma\right)\left(1-\varepsilon\right)} \ln \left(1 - \frac{1-\alpha\sigma}{1+\alpha\varepsilon}\right) + \psi \ln A + \frac{\psi}{\sigma} \ln \left(BL\right). \end{aligned}$$

If $\ln A$, $\ln B$ and $\ln L$ are jointly normally distributed, then $\ln N$ is also normally distributed. □

Proposition 3. Spatial equilibrium with heterogeneous location preferences.

Proof. First, the conditions for a spatial equilibrium are:

$$\left(N\left(A\right)\right)^{-\frac{1}{\kappa}} V\left(A\right) \leq \mathbb{V}, \qquad 0 \leq N\left(A\right), \qquad 0 = N\left(A\right) \left[\left(N\left(A\right)\right)^{-\frac{1}{\kappa}} V\left(A\right) - \mathbb{V}\right]. \tag{33}$$

Second, using equation (17), we write the expected utility in a city as a function of its population as

$$\mathbb{V}\left(A,N\right) \equiv N^{-\frac{1}{\kappa}} V\left(A,N\right) = \frac{1}{1-\varepsilon} N^{-\frac{1}{\kappa}} \left[\left(AN^\alpha\right)^{1-\varepsilon} - \left(A^{\frac{\varepsilon}{\varepsilon+\sigma}} N^{\frac{1+\alpha\varepsilon}{\varepsilon+\sigma}}\right)^{1-\varepsilon}\right], \quad \varepsilon \in \left(0,1\right), \ 1 < \kappa. \tag{34}$$

$\mathbb{V}\left(A,N\right)$ is increasing in N at the origin and concave in N if and only if

$$\frac{1}{\kappa} < \alpha\left(1-\varepsilon\right) < 1 + \frac{1}{\kappa}.$$

The second inequality in the expression above is redundant by $\alpha, \varepsilon \in (0,1)$. If the condition

in equation (21) holds, then the efficient city size is positive and equal to

$$N^e(A) = \left\{ \left[1 - \frac{(1-\alpha\sigma)(1-\varepsilon)\kappa}{(1+\alpha\varepsilon)(1-\varepsilon)\kappa - (\varepsilon+\sigma)} \right]^{\frac{1}{1-\varepsilon}} A^{\frac{\sigma}{\varepsilon+\sigma}} \right\}^{\frac{\varepsilon+\sigma}{1-\alpha\sigma}},$$

which yields property (i). In turn, expected utility evaluated at the efficient size, is increasing in A. Together with equation (33), this result implies properties (ii) and (v). The proof of properties (iii) and (iv) is by contradiction, as shown by Albouy, Behrens, Robert-Nicoud, and Seegert (2019) in the limit case $\varepsilon \to 0$, $\kappa \to \infty$. □

Proposition 4. Sorting across cities.

Proof. (i) Assuming for now that $N(t)$ is continuously differentiable, type-s workers solve the program $\max_t V(s,t)$. The first order condition to this program is:

$$0 = \alpha \frac{N'(t)}{N(t)} \left[(N(t))^{\alpha} s \right]^{1-\varepsilon} - (N(t))^{\gamma(1-\varepsilon)} t^{\frac{\varepsilon}{\varepsilon+\sigma}(1-\varepsilon)-1} \left[\gamma \frac{N'(t)}{N(t)} t + \frac{\varepsilon}{\varepsilon+\sigma} \right].$$

The equilibrium features talent-homogeneous cities if type-t workers choose to live in type-t cities, that is, if $s = t$. Plugging $s = t$ into the first order condition above yields an ordinary differential equation:

$$0 = \frac{1}{t} \left\{ \left[\alpha \frac{N'(t)}{N(t)} t + \lambda \right] \left[(N(t))^{\alpha} t^{1+\lambda} \right]^{1-\varepsilon} - \left[(N(t))^{\gamma} t^{\frac{\varepsilon}{\varepsilon+\sigma}} \right]^{1-\varepsilon} \left[\gamma \frac{N'(t)}{N(t)} t + \frac{\varepsilon}{\varepsilon+\sigma} \right] \right\}.$$

This equation is homogeneous in logs, so that a natural solution to this differential equation is of the form $N(t) = ct^f$, some $c, f \in \mathbb{R}$. Plugging this conjecture into the expression above yields:

$$(\alpha f + \lambda) c^{\alpha(1-\varepsilon)} t^{(1+\lambda+\alpha f)(1-\varepsilon)} = c^{\gamma(1-\varepsilon)} t^{\left(\frac{\varepsilon}{\varepsilon+\sigma}+\gamma f\right)(1-\varepsilon)} \left(\gamma f + \frac{\varepsilon}{\varepsilon+\sigma} \right).$$

Thus c, f solve this two-equation system:

$$\begin{aligned} (\alpha f + \lambda) c^{\alpha(1-\varepsilon)} &= c^{\gamma(1-\varepsilon)} \left(\gamma f + \frac{\varepsilon}{\varepsilon+\sigma} \right) \\ 1 + \lambda + \alpha f &= \frac{\varepsilon}{\varepsilon+\sigma} + \gamma f. \end{aligned}$$

Solving and using equation (22) yields the expression in equation 23. The second order condition is satisfied (the algebraic expression is too unwieldy to be revealing). (ii) N^* is increasing in t by inspection of equation (23). In turn, $p^*(t) = (N^*(t))^{\gamma} t^{\frac{\varepsilon}{\varepsilon+\sigma}}$ and $w^*(t) = (N^*(t))^{\alpha}$ are increasing in t by inspection. (iii) A planner aiming to create

talent-homogeneous cities would solve the following problem:

$$\max_{N(t)} \int_{t\in\mathcal{T}} V(t,t)\,\mathrm{d}F_{\mathcal{T}}(t) = \frac{1}{1-\varepsilon}\int_{t\in\mathcal{T}} \Big[t^{(1+\lambda)(1-\varepsilon)} \left(N(t)\right)^{\alpha(1-\varepsilon)} - t^{\frac{\varepsilon}{\varepsilon+\sigma}(1+\lambda)(1-\varepsilon)} \left(N(t)\right)^{\gamma(1-\varepsilon)} \Big]\,\mathrm{d}F_{\mathcal{T}}(t).$$

The solution to this problem gives the optimal talent-homogeneous city sizes in equation 24. Finally,

$$\left(\frac{N^*(t)}{N^0(t)}\right)^{\frac{(1-\alpha\sigma)(1-\varepsilon)}{\varepsilon+\sigma}} = \frac{\alpha\sigma+\lambda}{\alpha\sigma+\alpha\varepsilon}\frac{1+\alpha\varepsilon}{1+\lambda},$$

which is smaller than one if and only if $\lambda < \alpha\varepsilon$. (iv) Rewrite expression (23) as $N^*(t) \propto t^{\frac{\sigma(1+\lambda)}{1-\alpha\sigma}}$, which is isomorphic to the corresponding expression in Behrens, Duranton, and Robert-Nicoud (2014). They show that the size distribution of cities converges to Zipf's law as $\alpha\sigma \to 1$ regardless of the distribution $F_{\mathcal{T}}$ under mild regularity conditions. □

Proposition 5. PIGL-UM and PIGL-AMM.

Proof. The level of indirect utility in the PIGL-UM model of section 3, given by equation (9), and the equilibrium level of indirect utility in the PIGL-AMM model of section 5, given by equation (28), are isomorphic. In turn, the equilibrium expressions for the cost of living – equations (14) for PIGL-UM and (32) for PIGL-AMM – are isomorphic, too. Then propositions 1, 2, 3, and 4 all apply to PIGL-AMM as well. □

Online Appendix (not for publication)

OA.1 More general PIGL preferences

Assume preferences obey equation (9). Plugging the equilibrium price of equation (14) into the indirect utility function V yields, for $\mu = 1$:

$$V(N) = (1-\varepsilon)^{-1}\left[A^{1-\varepsilon}N^{\alpha(1-\varepsilon)} - 1\right] - (1-\nu)^{-1}\left[A^{\varepsilon\frac{1-\nu}{\nu+\sigma}}(BL)^{-\frac{1-\nu}{\nu+\sigma}}N^{\frac{(1+\alpha\varepsilon)(1-\nu)}{\nu+\sigma}} - 1\right],$$

which is concave in N if and only if $0 < \alpha(1-\varepsilon) < \frac{(1+\alpha\varepsilon)(1-\nu)}{\nu+\sigma}$, or, equivalently,

$$\alpha(1+\sigma)(\nu-\varepsilon) < (1-\alpha\sigma)(1-\nu) \tag{35}$$

for $\varepsilon, \nu > 0$. Note that this condition is implied by the combined assumptions $\nu = \varepsilon \in (0,1)$ and $\alpha\sigma < 1$ in the main text. In this case, the efficient population scale of the city is finite and positive, and equal to

$$N = \left[\left(\alpha\frac{\nu+\sigma}{1+\alpha\varepsilon}\right)^{\nu+\sigma}(BL)^{1-\nu}A^{\sigma(1-\varepsilon)+\nu-\epsilon}\right]^{\frac{1}{(1-\alpha\sigma)(1-\nu)-\alpha(1+\sigma)(\nu-\varepsilon)}}.$$

The efficient size is degenerate in the limit case, $\nu = \varepsilon = 1$, that is, if preferences are Cobb-Douglas. By contrast, it is positive and finite if housing demand is income-inelastic ($0 < \varepsilon < 1$) and $\nu = 1$, or if housing demand is price-inelastic ($0 < \nu < 1$) and homothetic ($\varepsilon = 1$). In both cases, however, it is decreasing in A. See corollary 1 for details.

OA.2 Gradients

Here we characterize the housing price, land price, physical density, and population density gradients – namely, how these four economic variables vary with distance to the CBD. Equations (27) and (30) give us the housing price gradient and physical density gradient (also known as the floor to area ratio):

$$\frac{\partial \ln p(x)}{\partial \ln x} = -\theta, \qquad \frac{\partial \ln h^S(x)}{\partial \ln x} = -\sigma\theta.$$

Summing the two, we obtain the gradient of the land price $r(x) \equiv \frac{\sigma}{1+\sigma}p(x)h^S(x)$:

$$\frac{\partial \ln r(x)}{\partial \ln x} = -(1+\sigma)\theta.$$

Using equations (26), (27), and (30), the gradient of population density is equal to

$$\frac{\partial \ln D(x)}{\partial \ln x} = -\xi - \theta(1+\sigma).$$

All gradients are negative, and a higher θ makes proximity to the center more desirable, resulting in steeper price and density gradients.

The population gradient is steeper than the physical density gradient if and only if

$$\xi + \theta > 0,$$

which holds if commuting costs are steep and/or land use regulations do not ease with distance to the CBD.