Resilience *vs.* Fragility: Global Properties of Economies with Endogenous Production Networks

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Abstract

We study a canonical static production-network economy in which sectors combine labor and intermediate inputs using constant-returns-to-scale technologies and adjust their supplier mix endogenously. First, we show that any such economy can be represented as one in which firms choose among Leontief technologies with heterogeneous productivities. Second, we leverage this to characterize equilibrium prices: the log-price vector is a min-max over pairs of networks of an affine function of log-productivity shocks. Economically, the two operators capture the forces of resilience and fragility: the "min" reflects gains from changing suppliers or technologies when prices move, while the "max" reflects amplification created by rigid input requirements. Third, specifications such as endogenous Cobb-Douglas and task-based technologies impose resilience or fragility, ruling out non-monotone effects of volatility on GDP. Fourth, we generalize Hulten's theorem to environments with technology switching and non-differentiability. Fifth, we derive upper and lower bounds on the contribution of productivity shocks to GDP volatility. Methodologically, these results provide a tractable global theory of endogenous production networks beyond local approximations and specific functional forms.

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1 Introduction

Modern production is characterized by the presence of long and complex supply chains. There are two broad perspectives on the economic implications of these interconnections. On the one hand, there is the perspective that rigid supply chains involved in networked production make the economy *fragile* (see *e.g.,* Elliott, Golub, and Leduc, 2022; Acemoglu and Tahbaz-Salehi, 2025), which we define as the property that (in logarithms) increases in volatility in microeconomic productivity raise prices and depress GDP on average. On the other hand, there is an influential and competing perspective that the ability to distribute production and change suppliers (see *e.g.,* Dew-Becker, 2023; Kopytov, Mishra, Nimark, and Taschereau-Dumouchel, 2024) may make the economy *resilient* or anti-fragile (Taleb, 2013).

At a theoretical level, analyzing the question of when the economy is resilient and when it is fragile requires going beyond canonical settings in which firms' prices are log-linear in their input prices, which obtains under log-linearizations of general models (Hulten, 1978; Liu, 2019; Carvalho, Nirei, Saito, and Tahbaz-Salehi, 2021) or equivalently when all firms operate Cobb-Douglas production functions (Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi, 2012). This is because, in such economies, changes in the microeconomic volatility of log-productivity have no effect on expected log-prices and log-GDP.

However, outside of the log-linear case, there is no known general solution for equilibrium prices and output. To overcome this challenge, the literature has proceeded in three directions, each yielding valuable insights: (i) specializing to specific production technologies (Oberfield, 2018; Acemoglu and Azar, 2020; Elliott et al., 2022; Kopytov et al., 2024), (ii) considering higher-order approximations (Baqaee and Farhi, 2019), or (iii) studying the asymptotic effects of shocks (Dew-Becker, 2023). Thus, while our understanding of endogenous production networks has advanced enormously, we lack a general theory that is valid outside of important but special cases (*cf.* (i)) or for shocks that are neither very small (*cf.* (ii)) nor very large (*cf.* (iii)).

For this reason, we study the global properties of the canonical static single-factor production network model (as in Acemoglu et al., 2012; Carvalho and Tahbaz-Salehi, 2019). We maintain the core assumption at the heart of input-output analysis (Leontief, 1941): production is constant returns to scale. We otherwise make no parametric assumptions on the nature of production technology at the microeconomic level.

Our main result provides globally valid and primitive formulae for equilibrium prices and output that hold for any such economy. We exploit these formulae to understand conditions on technology at the microeconomic level such that resilience and/or fragility emerge as endogenous macroeconomic properties. In proving these results, we also derive a number of new results in producer theory that may be of independent interest. Most importantly among these, we show that any cost function that can be generated by a CRS production function can also

be generated by an *endogenous Leontief* production function, where a producer chooses between Leontief production functions with heterogeneous productivities. We further apply the general results to provide a generalization of Hulten's (1978) theorem compatible with non-differentiable production functions—which arise naturally from endogeneity of the production network. Moreover, we provide robust bounds that describe when microeconomic productivity shocks can generate sizable GDP fluctuations, without making any additional assumptions on production technology.

Model. The model follows the canonical static single-factor production network setting (see Carvalho and Tahbaz-Salehi, 2019, for a review). There are many sectors, each perfectly competitive, that use as inputs the outputs of other sectors and a single primary factor (such as labor). A representative household inelastically supplies the primary factor. Within this context, we allow for an arbitrary CRS production function. Thus, our analysis allows us to nest: (i) the Cobb-Douglas benchmark (Acemoglu et al., 2012), (ii) nested constant elasticity of substitution (CES) technology (Baqaee and Farhi, 2019; Dew-Becker, 2023), (iii) heterogeneous CES technology (Hanoch, 1971; Chodorow-Reich, Gabaix, and Viviano, 2025), (iv) Cobb-Douglas production technology with endogenous supplier choice at the extensive and intensive margins (Acemoglu and Azar, 2020; Kopytov et al., 2024), (v) Leontief production technology with and without endogenous supplier choice (Acemoglu and Tahbaz-Salehi, 2025), and (vi) task-based production functions (Acemoglu and Restrepo, 2018).

Main Characterization Results. Our main results provide globally valid and primitive formulae for equilibrium prices and GDP for these economies. Our formula for equilibrium prices asserts that the equilibrium log-price vector is the "min-max" over pairs of input-output networks of the Leontief inverse of the maximizing network applied to the vector of log-productivities plus a *production wedge*. This wedge depends on the primitive production functions evaluated at the minimizing network and the relative entropies between the rows of the minimizing and maximizing networks. Intuitively, equilibrium prices are always the "min-max" of what prices would be in Cobb-Douglas economies with a certain kind of productivity distortion. The resulting formula for log-GDP is simply the inner product of the household's expenditure shares and this equilibrium log-price vector.

To derive these results, we first express the fixed-point equation for equilibrium prices that holds in such economies in terms of log-prices. In this system of equations, each sector's log-price is affine in its log-productivity and the log of its unit cost function. Next, we show that any unit cost function that arises from a CRS production function also results from a seemingly more specific endogenous Leontief production function. Accordingly, the cost function is simply the minimum over vectors of Leontief input weights of the inner product of these input weights and input prices, divided by the *as if* productivity of operating that Leontief technology. From

this, we show that the log-cost function is simply a "min-max" over pairs of vectors of input weights of an *affine* function of the input log-prices. Finally, we show that the operations of finding the fixed point and computing the "min-max" commute. That is, if we naively ignore the "min-max," compute the fixed point, and only then apply the "min-max" to the solution, then we obtain the true solution to the fixed point system. This allows us to obtain the primitive formula because, if we ignore the "min-max," then the fixed point system is linear and can be solved analytically.

Resilience *vs.* Fragility. We then apply this general representation to understand when the economy is resilient (fragile), *i.e.,* when equilibrium log-prices are concave (convex) functions of the log-productivity vector. The general representation theorem makes clear that neither property should generally be expected globally: the "min" will serve to induce concavity and the "max" will lead to convexity. Thus, the general representation suggests that prices will generally be concave in some regions and convex in others, depending on whether the force of the "min" or the "max" dominates.

Indeed, in a leading example, we show how switching between technologies can naturally give rise to an "S-shaped" response of average log-GDP to increases in the volatility of microeconomic log-productivity: small levels of volatility lower GDP on average, intermediate levels raise it, and high levels lower it again. Intuitively, if production technology is always locally rigid (*e.g.,* Leontief), then small levels of volatility are always harmful. However, the ability to switch between different technologies with different input intensities creates option value and a love of volatility. Ultimately, though, if production technology is always locally rigid, then very high volatility is asymptotically harmful.

This notwithstanding, many standard specifications of production functions ensure that the economy is globally resilient or fragile. This occurs when each sector's log-cost function is always a concave or convex function of log-prices, respectively. When this occurs, such specifications necessarily rule out qualitative phenomena: increases in productivity volatility are always either good or bad on average. Indeed, for this reason, second-order approximations in logarithms (as in Baqaee and Farhi, 2019) or a focus on asymptotics (as in Dew-Becker, 2023) obscure the presence of "S-shaped" responses.

To understand when assumptions on technology can rule out these qualitative phenomena, we characterize the property that a log-cost function is globally concave or convex. As a concrete example, a CES production function with elasticity of substitution greater (lesser) than unity has a globally concave (convex) log-cost function. More generally, we show that any concave log-cost function arises from an endogenous Cobb-Douglas production function, which has become a particularly popular production function in the study of endogenous production networks (see *e.g.,* Acemoglu and Azar, 2020; Kopytov et al., 2024). This shows both that

endogenous Cobb-Douglas technology is more general than it may appear: it allows the analyst to capture *any* concave log-cost function. However, by the same token, by ruling out convexity, such a specification ensures global resilience. On the other side, we show that any convex log-cost function arises from a particular form of task-based production (see *e.g.,* Acemoglu and Restrepo, 2018) where a set of tasks, each of which has Cobb-Douglas technology, must be completed to produce output.

Further Applications: Hulten's (1978) Theorem and Robust Variance Bounds. We next apply our general representation to derive the effects of small shocks of productivity on equilibrium prices and output. We show that equilibrium prices are always directionally differentiable, but not necessarily Gateaux differentiable, *i.e.,* the derivative depends on the direction of shock considered. The failure of Gateaux differentiability is economically important because it corresponds to the presence of technology switching, the core phenomenon that the endogenous production networks literature sets out to study: if a sector is ever indifferent at equilibrium prices between producing using different input mixes, then different directions of shocks will lead to different input mixes and different shock propagation. This is likely to occur when a sector can use multiple different production technologies: as relative prices change, there will be a critical point at which they become indifferent between a given pair of technologies, inducing non-differentiability. While such indifferences are non-generic (*i.e.,* they hold at a zero measure set of prices), ignoring the non-differentiabilities they induce can lead to inaccurate quantitative guidance, as we show in a leading application. Intuitively, when such a switch happens, the correct coefficients in any order of Taylor expansion jump and the previously accurate approximation breaks down.

We use this formula for the derivative of prices to derive a corresponding formula for the derivative of GDP, providing a generalization of Hulten's theorem: the effect on GDP is the *minimum* over the set of Domar weights (the vector of sales shares of nominal GDP across sectors) that may prevail at equilibrium of the inner product of these Domar weights and the log-productivity shocks. The presence of the minimum generalizes Hulten's theorem beyond the differentiable case and allows economically for the presence of technology switching. This provides nuance to the typical interpretation of Hulten's theorem as implying that, for small shocks, input-output structure can be ignored. This is because, even for small shocks, it is important to account for the possibility that changes in input mix lead to rapid changes in Domar weights and the direction of shock propagation.

Finally, we apply this generalization of Hulten's theorem to derive robust upper and lower bounds on the variance of log-GDP that hold without restricting technology. This allows us to understand when it is that microeconomic fluctuations in productivity can generate sizable fluctuations in aggregate output without making strong and untestable assumptions on produc-

tion technology. We define, for each sector, the smallest and largest Domar weight that arises in equilibrium across all productivity realizations. The squared largest (across sectors) smallest (across productivity realizations) Domar weight multiplied by the microeconomic volatility of log-productivity provides a lower bound, while the Herfindahl index of the biggest (across productivity realizations) Domar weights multiplied by the microeconomic volatility of log-productivity provides an upper bound. This result highlights that microeconomic fluctuations can generate large aggregate fluctuations when at least one sector maintains a non-negligible sales share of GDP even when it is very unproductive. Intuitively, this corresponds to the presence of sectors that cannot be substituted even when their products are very expensive, of which oil is perhaps a classic example.

Outline. The rest of the paper proceeds as follows. Section 2 provides a simple example to illustrate why understanding the global properties of models with endogenous production networks is important. Section 3 introduces the model. Section 4 provides the main characterization results. Section 5 studies when economic outcomes feature resilience or fragility. Section 6 studies the effects of small shocks and provides a generalization of Hulten's (1978) theorem. Section 7 provides robust upper and lower bounds for the variance of GDP. Section 8 concludes.

2 Why Global Properties Are Important: An Example

We begin with a simple and analytically solvable production network economy. We use this example to motivate the usefulness of a global solution by highlighting the potentially inaccurate quantitative and qualitative conclusions that arise from the use of approximations.

Setting. Suppose that there are two sectors in the economy, an upstream sector U and a downstream sector D. The upstream sector uses labor and upstream inputs to produce output:

$$Y_U = 2Z_U Q_{U,0}^{\frac{1}{2}} Q_{U,U}^{\frac{1}{2}} \tag{1}$$

where Z_U is the productivity of sector U, $Q_{U,0}$ is the input of a primary factor, and $Q_{U,U}$ is sector U's usage of goods produced in sector U. The downstream sector uses labor, upstream inputs, and downstream inputs to produce output and can do so using two different production technologies indexed by $\omega \in \{\underline{\omega}, \overline{\omega}\} \subset (0,1)$:

$$Y_D^{\omega} = 2Z_D Q_{D,0}^{\frac{1}{2}} \left(\min \left\{ \frac{Q_{D,\omega}}{\omega}, \frac{Q_{D,U}}{1-\omega} \right\} \right)^{\frac{1}{2}} \tag{2}$$

where Z_D is the productivity of sector D, $Q_{D,0}$ is the input of a primary factor, $Q_{D,\omega}$ is D's usage of goods produced using technology ω, and $Q_{D,U}$ is D's usage of upstream inputs.

Intuitively, upstream and downstream goods are always perfect complements in the production of downstream goods, but the downstream sector can choose between two distinct production technologies. The primary factor has an aggregate supply of one and has a price that we normalize to one. Finally, a household has Cobb-Douglas preferences:

$$U = 2C_D^{\frac{1}{2}}C_U^{\frac{1}{2}} \tag{3}$$

where C_i is final consumption from sector $i \in \{D, U\}$. We will examine the outcome of a competitive equilibrium in this simple setting, in which both sectors minimize costs and make no profits, the household maximizes its utility, and all markets clear.

Solving for Equilibrium Outcomes. By solving the cost-minimization problem of the upstream sector and applying the zero-profit condition, the price of the upstream sector follows:

$$P_U = \frac{1}{Z_U}P_U^{\frac{1}{2}} \implies P_U = \frac{1}{Z_U^2}. \tag{4}$$

We now perform the same steps for the downstream sector and find that its price must solve:

$$P_D = \frac{1}{Z_D}\left(\min_{\omega \in \{\underline{\omega}, \overline{\omega}\}} \{\omega P_D + (1-\omega)P_U\}\right)^{\frac{1}{2}}. \tag{5}$$

Intuitively, the cost of producing one unit of the composite intermediate input is just the weighted sum of input prices, and the sector will always choose the cheapest technology ω. By re-arranging this condition, it is immediate that the price of the downstream sector solves a piecewise quadratic equation. By solving these equations, it follows that $P_D \geq P_U$ if and only if $Z_D \leq Z_U$ and so we have that:

$$P_D = \mathbb{I}[Z_D \leq Z_U]\frac{\underline{\omega} + \sqrt{\underline{\omega}^2 + 4(1-\underline{\omega})\frac{Z_D^2}{Z_U^2}}}{2Z_D^2} + \mathbb{I}[Z_D > Z_U]\frac{\overline{\omega} + \sqrt{\overline{\omega}^2 + 4(1-\overline{\omega})\frac{Z_D^2}{Z_U^2}}}{2Z_D^2}. \tag{6}$$

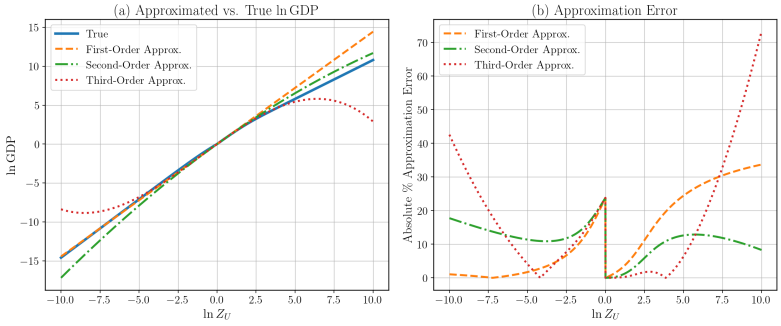
By solving the household's utility maximization problem, we obtain that the natural logarithm of GDP is given by:

$$\ln \text{GDP} = -\frac{1}{2}\ln P_D - \frac{1}{2}\ln P_U. \tag{7}$$

By using the expressions for equilibrium prices, we obtain:

$$\ln \text{GDP} = \frac{1}{2}\ln 2 + \ln Z_U + \ln Z_D - \frac{1}{2}\begin{cases} \ln\left(\underline{\omega} + \sqrt{\underline{\omega}^2 + 4(1-\underline{\omega})\frac{Z_D^2}{Z_U^2}}\right), Z_D \leq Z_U, \\ \ln\left(\overline{\omega} + \sqrt{\overline{\omega}^2 + 4(1-\overline{\omega})\frac{Z_D^2}{Z_U^2}}\right), Z_D > Z_U. \end{cases} \tag{8}$$

Figure 1: Local Approximations Can Provide Inaccurate Quantitative Guidance



Notes: Panel (a): Global solution for $\ln \text{GDP}$ (blue) vs. first- (orange), second- (green), and third-order (red) Taylor approximations to $\ln \text{GDP}$ in $\ln Z_U$ around the point $\ln Z_U = 1 \times 10^{-4}$. Panel (b): The percent absolute approximation error, *i.e.,* $|(\text{True} - \text{Approx.})/\text{True}|$, from each of these three approximations relative to the true, global solution. Parameters: $\underline{\omega} = 0.2$, $\overline{\omega} = 0.8$, $\ln Z_D = 0$.

We now proceed to study the global properties of equilibrium prices and GDP. The standard approach to studying the properties of this economy would be to take a first-order approximation (Hulten, 1978) or a second-order approximation (Baqaee and Farhi, 2019) of $\ln \text{GDP}$ in $\ln Z = (\ln Z_U, \ln Z_D)$. At all points $\ln Z_0$ *except* those such that $Z_U = Z_D$, we can perform these approximations to obtain:

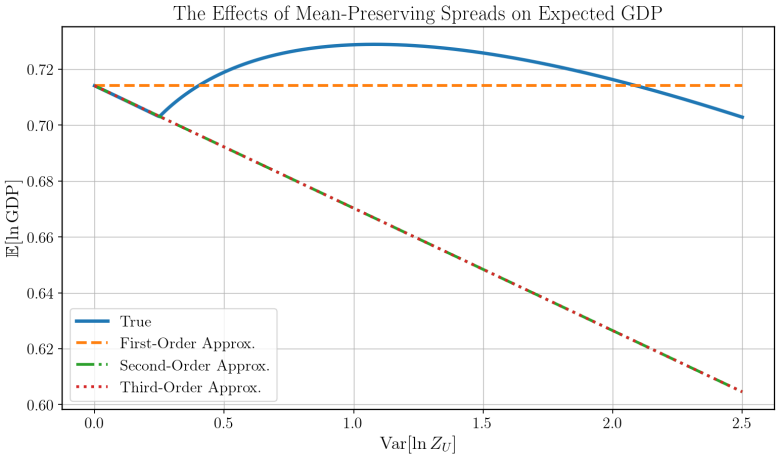
$$\ln \text{GDP} = \overline{\ln \text{GDP}} + D^T \Delta \ln Z + \frac{1}{2} (\Delta \ln Z)^T H \Delta \ln Z \tag{9}$$

where $\Delta \ln Z = \ln Z - \ln Z_0$, $\overline{\ln \text{GDP}}$ is a constant, and D is the gradient and H is the Hessian of $\ln \text{GDP}$ evaluated at the initial $\ln Z_0$.

Local Approximations Can Give Inaccurate Quantitative Guidance. We first examine the accuracy of these approximations, considering approximations of $\ln \text{GDP}$ in $\ln Z_U$ at a value that is positive but very close to zero, holding fixed $\ln Z_D = 0$. In Figure 1, we plot the resulting approximations (Panel (a)) and percentage absolute approximation errors (Panel (b)). For positive shocks, the approximations are highly accurate. Even a 100% positive shock gives rise to only a small error under the first-order approximation and an utterly negligible error under the second-order approximation. For negative shocks, however, the approximations break down, even locally. In particular, negative shocks lead to an approximation error of about 20% and the first-order approximation performs substantially better than the second-order approximation for even vanishingly small negative shocks.

One natural reaction to these results is to posit that second-order approximations are not enough and higher-order series expansions must be considered. However, as can be seen in Figure 1, this does not solve the problem that the local approximation breaks down for small negative shocks, and it can lead to larger approximation errors for non-negligible positive shocks.

Figure 2: Local Approximations Can Rule Out Qualitative Phenomena



Notes: This figure plots expected ln GDP when $\ln Z_U = k \pm x$ (each with probability 0.5) against $\text{Var}[\ln Z_U] = x^2$ under the global solution (blue) and under first- and second-order Taylor approximations around the point $\ln Z_U = k$. Parameters: $\underline{\omega} = 0.2$, $\overline{\omega} = 0.8$, $\ln Z_D = 1$, $k = 0.5$.

Indeed, the issue could not be remedied by any series expansion. The problem is that the Taylor approximation *functional* that returns an n-th order series approximation at a given value of $\ln Z$ is discontinuous at the kink at which $\ln Z_D = \ln Z_U$. For this reason, a global approach is necessary to provide accurate guidance as to the properties of this economy.

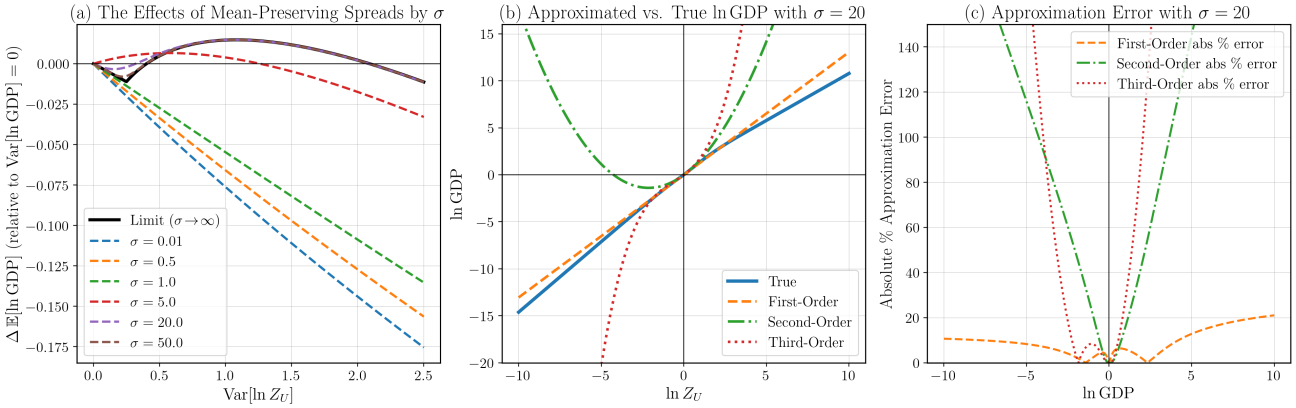
Local Approximations Can Rule Out Qualitative Phenomena. Putting aside the issue of the accuracy of approximations, we also argue that first-order and second-order approximations rule out qualitatively interesting and potentially important economic phenomena. One of the most important motivations in the modern production network literature is understanding how microeconomic shocks at the sector-level may affect the first and second moments of ln GDP. Consider now that $\ln Z$ is an IID random vector with mean 0 and variance-covariance matrix $\sigma^2 I$. Under the second-order approximation of Baqaee and Farhi (2019) in an arbitrary economy with n sectors, we have that:

$$\mathbb{E}[\ln \text{GDP}] = \overline{\ln \text{GDP}} + \frac{1}{2}\sigma^2 \sum_{i=1}^{n} H_{ii} \quad \text{Var}[\ln \text{GDP}] = 2\sigma^4 \sum_{i=1}^{n} H_{ii}^2 + \sigma^2 \sum_{i=1}^{n} D_i^2. \tag{10}$$

How does increasing the variance of microeconomic shocks affect the mean and variance of GDP growth? Thus, second-order methods imply that as we increase the variance of shocks, the effect on output must be (i) unambiguously positive or negative, and (ii) convex.

In our example economy, suppose that $\ln Z_U = k \pm x$, each with probability 0.5, while $\ln Z_D = 0$. Figure 2 plots the effects of increasing the volatility of upstream productivity on expected output. The global solution reveals an S-shaped response in which initial increases

Figure 3: Errors Arise From High (Local) Substitution Elasticities, Not Kinks



Notes: Panel (a): Effects of upstream productivity variance on expected ln GDP as in Figure 2 for varying σ. Panel (b): Approximated vs. True GDP for Taylor approximations as in Figure 1 with $\sigma = 20$. Panel (c): Absolute percent approximation errors as in Figure 1 with $\sigma = 20$.

in variance lower expected output, intermediate levels of variance increase aggregate output, and high levels of variance decrease output again. Intuitively, at low levels of variance, the economy mostly operates one Leontief technology downstream, and so variance in input prices is harmful. However, at intermediate levels of variance, the downstream sector can switch across technologies and the option value of switching technologies makes variance beneficial. Ultimately though, when variance becomes very large, the fact that they always operate a Leontief technology (even if they can choose the best one) makes variance lower average output.

These qualitative patterns cannot be captured by first-, second-, and even third-order approximations to GDP. Thus, while approximation methods are powerful, they do rule out certain interesting and potentially economically reasonable phenomena.

These Problems Arise From (Locally) Fast Substitution. Of course, one might think that we are presenting a pathological example, and no reasonable model would display kinks like this. However, we will later show that such kinks are a *general property* of models with endogenous production networks. Intuitively, when relative prices cross critical values, firms may switch across technologies or input suppliers, leading input demands to jump and generating a kinked cost function. Thus, even if inputs are strongly complementary within a technology, switching across technologies generates substantial nonlinearities in output.

The general importance of kinks notwithstanding, the points we are making do not hinge on their presence. To see this, suppose that downstream goods are aggregated following:

$$Y_F = \left(\left(\frac{1}{2}\right)^{\frac{1}{\sigma}} Q_{F,\underline{\omega}}^{\frac{1-\sigma}{\sigma}} + \left(\frac{1}{2}\right)^{\frac{1}{\sigma}} Q_{F,\overline{\omega}}^{\frac{1-\sigma}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{11}$$

for $\sigma \geq 0$ and that households consume equal shares of upstream and final goods, $U = 2C_F^{\frac{1}{2}} C_U^{\frac{1}{2}}$. In the limit as $\sigma \to \infty$, prices and GDP in this economy are exactly as in the baseline economy. Moreover, for every $\sigma < \infty$, output does not display a kink and is instead an infinitely differentiable function of productivity shocks. Thus, approximations will always work locally.

We now examine how the accuracy of approximations depends on σ. As a benchmark, we examine how well approximations perform when the value of σ is chosen so that the model is capable of generating the qualitative property that there is an S-shaped response of expected GDP to upstream productivity volatility. Generating this phenomenon requires a σ value of approximately 20 (see panel (a) of Figure 3). In panels (b) and (c) of Figure 3, we show what happens when you approximate this economy. Notably, local approximations do *much* worse than the economy with the kink. Intuitively, to replicate technology switching in a smooth economy, very high substitution elasticities are required in one nest of the approximating CES economy. But having very high substitution elasticities gives rise to very large higher-order derivatives and Taylor approximations that behave erratically.

Thus, to capture technology switching, which is at the heart of studying endogenous production networks, it is qualitatively and perhaps quantitatively insufficient to study approximations; understanding the global properties of such economies is required.

3 Model

3.1 Primitives

Supply. We consider a canonical static model of a production network with $n \in \mathbb{N}$ sectors, following Acemoglu et al. (2012), Baqaee and Farhi (2019), Acemoglu and Azar (2020), Dew-Becker (2023), and Kopytov et al. (2024). Each sector $i \in [n] = \{1, \ldots, n\}$ produces a single output by combining a primary factor in amounts Q_{i0} and intermediate goods in amounts $Q_i = (Q_{ij})_{j=1}^n$ according to a production function $F_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$:

$$F_i(Q_{i0}, Q_{i1}, \ldots, Q_{in}) = Z_i \left(\frac{Q_{i0}}{1-\beta} \right)^{1-\beta} \left(\frac{G_i(Q_{i1}, \ldots, Q_{in})}{\beta} \right)^{\beta} \tag{12}$$

where $Z_i \in \mathbb{R}_{++}$ is the Hicks-neutral productivity, $\beta \in (0,1)$ is the intermediate input share of costs, and $G_i : \mathbb{R}_+^n \to \mathbb{R}_+$ is an intermediate input production function.

The key object that determines the network of input-output relationships is the intermediate input production function G_i. The current literature that provides analytical results for the equilibrium properties of production networks has considered various assumptions on G_i: Acemoglu et al. (2012) and a large subsequent literature (see Carvalho and Tahbaz-Salehi, 2019, for a review) assume a Cobb-Douglas production function, Dew-Becker (2023) assumes

a constant elasticity of substitution (CES) production function, Kopytov et al. (2024) assume a Cobb-Douglas production function with endogenous weights. The goal of our analysis is to study the *robust* properties of equilibrium outcomes in production networks, *i.e.,* those properties that obtain without functional form restrictions on the nature of the production network. Thus, we impose minimal regularity assumptions, a normalization that allows us to interpret the Z_i as Hicks-neutral productivity shocks, and the core economic assumption at the heart of input-output analysis: production is constant returns-to-scale.

Assumption 1. *For all sectors $i \in [n]$, the production function $G_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ is positive homogeneous, non-decreasing, and continuous.*[1] *Moreover,* $\max_{Q_i \in \Delta} G_i(Q_{i1}, \ldots, Q_{in}) = 1$*, where* $\Delta = \left\{ w \in \mathbb{R}^n_+ : \sum_{j=1}^n w_j = 1 \right\}$.

Our formulation is sufficiently general to encompass the universe of intermediate input production functions considered in this previous work. As we demonstrate in Section 3.4, this allows us to consider models in which the technologies, suppliers, and substitution elasticities of firms are chosen endogenously.

Remark 1. The assumption that the production function is Cobb-Douglas in the primary factor serves to maximize comparability to existing literature which often makes this assumption (see *e.g.,* Dew-Becker, 2023; Chodorow-Reich et al., 2025) and simplify expressions. However, it is not essential for our analysis. In Appendix C, we show that our main text results can be applied *as written* to a much larger class of economies that feature: (i) heterogeneous labor shares across sectors and (ii) endogenous technology choice between technologies with different labor shares. Moreover, we show that this expanded class of production functions is sufficiently rich to approximate *arbitrarily well* any CRS and monotone production function so long as the expenditure share on the primary factor is bounded away from zero. △

Demand. As the focus of our analysis is on the production network, we intentionally do not innovate on the household side of the model and follow Acemoglu et al. (2012), Dew-Becker (2023), and Kopytov et al. (2024). This is justified because, as is well known and we will shortly see, demand has no role in determining equilibrium prices. Instead, demand matters only for computing GDP given prices. A representative household inelastically supplies a unit of primary factor and consumes a bundle of goods $C \in \mathbb{R}^n_+$ with a Cobb-Douglas utility function:

$$U(C) = \prod_{i=1}^n \left(\frac{C_i}{\xi_i} \right)^{\xi_i} \qquad \forall C \in \mathbb{R}^n_+ \tag{13}$$

[1] It would be equivalent to assume that each G_i is only upper semicontinuous since we restrict to intermediate production functions that are positive homogeneous(see, e.g., Moore, 1999, Proposition 5.95).

where each $\xi_i \geq 0$ captures the importance of good i for the final consumer and $\sum_{i=1}^n \xi_i = 1$. Normalizing the wage in the economy to one and denoting the price of good i as P_i, the household faces the standard budget constraint $\sum_{i=1}^n C_i P_i \leq 1$. So long as U is positive homogeneous, our results characterizing GDP extend.[2]

3.2 Equilibrium

Given this structure for production and demand, the competitive equilibrium is defined by requiring that firms and consumers are price takers and that competition drives prices at the marginal cost.

Definition 1. *A competitive equilibrium is a vector of prices, $P^* \in \mathbb{R}^n_{++}$, a matrix of unitary input quantities $Q^* \in \mathbb{R}_+^{n\times(n+1)}$, a vector of production levels $Y^* \in \mathbb{R}^n_+$ and a final consumption vector $C^* \in \mathbb{R}^n_+$ such that*

1. *Firms choose a cost-minimizing vector of unitary inputs:*

$$(Q^*_{i0}, Q^*_i) \in \underset{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}}{\arg\min} \sum_{j=1}^n Q_{ij}P^*_j + Q_{i0} \quad s.t. \quad F_i(Q_{i0}, Q_i) \geq 1 \qquad \forall i \in [n]\,. \tag{14}$$

2. *Firms price at their marginal cost:*

$$P^*_i = \sum_{j=1}^n Q^*_{ij}P^*_j + Q^*_{i0} \qquad \forall i \in [n]\,. \tag{15}$$

3. *The labor market and all product markets clear:*

$$\sum_{i=1}^n Y^*_i Q^*_{i0} = 1\,, \quad \sum_{i=1}^n Y^*_i Q^*_{ij} + C^*_j = Y^*_j \qquad \forall j \in [n]\,. \tag{16}$$

4. *The household chooses consumption to maximize its utility:*

$$C^* \in \underset{C\in\mathbb{R}^n_+}{\arg\max}\, U(C) \quad s.t. \quad \sum_{i=1}^n C_i P^*_i \leq 1. \tag{17}$$

We will be concerned with characterizing equilibrium properties of prices and GDP. We call a solution to Equation 15 an equilibrium price vector.

[2]Intuitively, we can treat U as if it were a production function and apply Proposition 1. In Theorem 1, for the GDP characterization this induces an additional min-max layer over the household's expenditure shares on different goods.

3.3 Preliminaries: The Log-Cost Operator and Equilibrium Prices

Our characterization of the equilibrium leverages the properties of a particular transformation of the sectors' cost functions. By solving each sector's cost minimization problem, we obtain the following:

$$\min_{\substack{(Q_{i0},Q_i)\in\mathbb{R}^{n+1}_+:\\ F_i(Q_{i0},Q_i)\geq 1}} \sum_{j=1}^{n} Q_{ij}P_j + Q_{i0} = \frac{1}{Z_i}K_i\left(P\right)^{\beta} \qquad \forall i \in [n]\,, \forall P \in \mathbb{R}^n_{++} \tag{18}$$

where $K_i(P)$ is defined as:[3]

$$K_i(P) = \min_{Q_i\in\mathbb{R}^n_+} \sum_{j=1}^{n} Q_{ij}P_j \quad \text{s.t.} \quad G_i(Q_i) \geq 1. \tag{19}$$

We define the associated map from log-prices to log-costs as:

$$T_i(p) = \ln(K_i(\exp(p))) \quad \forall p \in \mathbb{R}^n. \tag{20}$$

The cost operator $K : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ and the *log-cost operator* $T : \mathbb{R}^n \to \mathbb{R}^n$ are the operators whose i-th row are respectively equal to K_i and T_i.

By defining the log-productivity of sector $i \in [n]$ as $x_i = -\frac{1}{1-\beta}\ln Z_i$, we can rewrite the fixed point equation that defines equilibrium price vectors Equation 15 as:

$$p_i = (1-\beta)x_i + \beta T_i(p) \qquad \forall i \in [n]. \tag{21}$$

We call a solution to this fixed point equation $\Pi(x)$. This represents a vector of equilibrium log-prices at a given vector of log-productivity shocks $x \in \mathbb{R}^n$. We denote the equilibrium aggregate log-output at such a vector of log-prices by

$$\Upsilon(x) = \max_{C\in\mathbb{R}^n_+} \ln U(C) \quad \text{s.t.} \quad \sum_{i=1}^{n} C_i \exp(\Pi_i(x)) \leq 1. \tag{22}$$

It is standard to show that:

$$\Upsilon(x) = -\sum_{i=1}^{n} \xi_i \Pi_i(x). \tag{23}$$

Because of this observation, it suffices to characterize equilibrium prices. The following easy lemma guarantees the existence and uniqueness of these equilibrium prices.

[3] See Lemma 18 in the Online Appendix for a proof of Equation 18 and that the minima in Equations 18 and 19 are attained.

Lemma 1. *For every vector of log-productivities $x \in \mathbb{R}^n$, there exists a unique equilibrium log-price vector $\Pi(x)$.*

The uniqueness of equilibrium prices follows from the fact that Equation 21 generates a contractive structure. While equilibrium prices and *aggregate* output are unique, there is not necessarily a unique equilibrium matrix of *input* quantities Q. We will characterize this stronger uniqueness property in Proposition 5.

3.4 Example Economies

Before embarking on our analysis, we give four classes of example economies that fall in our general framework and to which we will often turn to exemplify our general results.

Example 1 (Nested CES)**.** A standard non-linear specification of production in which the input-output network depends endogenously on equilibrium prices is the nested CES structure, in which sector $i \in [n]$ has the following technology:

$$G_i\left(Q_i\right) = \left(\sum_{j=1}^{n} \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i - 1}{\sigma_i}}\right)^{\frac{\sigma_i}{\sigma_i - 1}} \tag{24}$$

for some vector of weights $\alpha_i \in \Delta$ and elasticity of substitution $\sigma_i \in \bar{\mathbb{R}}_+$. As is well-known, $\sigma_i = 0$ corresponds to the Leontief production function $G_i(Q_i) = \min\left\{\frac{Q_{ij}}{\alpha_{ij}}\right\}$, $\sigma_i = \infty$ corresponds to the perfect substitutes production function $G_i(Q_i) = \sum_{i=1}^{n} Q_{ij}$, and $\sigma_i = 1$ corresponds to a Cobb-Douglas production function $G_i(Q_i) = \prod_{i=1}^{n} \left(Q_{ij}/\alpha_{ij}\right)^{\alpha_{ij}}$. For the nested CES structure, the log-cost operator is given in general by:

$$T_i(p) = \frac{1}{1-\sigma_i} \sum_{j=1}^{n} \alpha_{ij} \exp\left((1-\sigma_i)p_j\right). \tag{25}$$

Existing work has characterized the second-order implications of shocks with nested CES (Baqaee and Farhi, 2019) and the asymptotic effects with infinitely large shocks (Dew-Becker, 2023). However, the global equilibrium properties of nested CES economies have not been characterized. △

The following three examples feature a productivity shifter $A_i(w_i)$ associated with the technologies $w_i \in C_i \subseteq \Delta$ of firm i, where C_i is a convex set. First, consider a function $A_i : C_i \to (0, 1]$. We say that A_i is upper coercive if all its upper contour sets are relatively compact. We say that A_i is (strictly) log-concave if $\ln A_i$ is (strictly) concave. Second, consider a function $A_i : C_i \to [1, \infty)$. We say that A_i is lower coercive if its lower contour sets are compact

in the relative topology. We say that A_i is entropy log-convex if $w_i \mapsto \ln A_i(w_i) - \sum_{j=1}^n w_{ij} \ln w_{ij}$ is convex.[4]

Example 2 (Endogenous Cobb-Douglas)**.** Recent work that explores the endogenous determination of supply linkages has proposed models in which each sector $i \in [n]$ chooses their Cobb-Douglas input shares to economize on costs at the opportunity cost of perhaps suffering reduced productivity:

$$G_i(Q_i) = \max_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}} \tag{26}$$

for some upper coercive and log-concave production shifter $A_i : C_i \to (0, 1]$ where $C_i \subseteq \Delta$ is convex and $\max_{\alpha_i \in C_i} A_i(\alpha_i) = 1$. The interpretation of $A_i(\alpha_i)$ is that it represents the productivity of the sector when it operates using some feasible input shares $\alpha_i \in C_i$. The closer $A_i(\alpha_i)$ is to 1, the cheaper it is (in terms of lost productivity) for sector i to use technology α_i. The technologies $\alpha_i \notin C_i$ are the infeasible ones for sector i.

For the endogenous Cobb-Douglas structure, the log-cost operator is given by:

$$T_i(p) = \min_{\alpha_i \in C_i} \left\{ \sum_{j=1}^n \alpha_{ij} p_{ij} + \ln \left(\frac{1}{A_i(\alpha_i)} \right) \right\} \tag{27}$$

Influential papers in the literature have studied particular cases of this class. Acemoglu and Azar (2020) study a specification in which productivity depends only on the identity of the suppliers chosen by each sector, but their input shares are fixed. Kopytov et al. (2024) study a specification in which $A_i(\alpha_i)$ is smooth and strictly log-concave. △

Example 3 (Task-Based Production)**.** Recent work that studies the substitutability of different factors of production has studied task-based models in which output is produced by completing a set of different tasks and each task has its own production function (see *e.g.,* Acemoglu and Restrepo, 2018). Inspired by this, we consider an economy in which the set of tasks corresponds to the nonempty and convex set $C_i \subseteq \Delta$. Output is only produced if each task in this set is completed. That is, each sector has a Leontief production function over tasks. The output of each task is itself given by a Cobb-Douglas production with weights $\alpha_i \in C_i$ and some lower coercive and entropy log-convex productivity shifter $A_i : C_i \to [1, \infty)$ with $\min_{\alpha_i \in C_i} A_i(\alpha_i) = 1$. That is, the production function is given by:

$$G_i(Q_i) = \min_{\alpha_i \in C_i} A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}}. \tag{28}$$

[4]The name is motivated by the fact that $-\sum_{j=1}^n w_{ij} \ln w_{ij}$ is the entropy of the distribution w_i. Therefore, the property requires that the logarithm of A_i is sufficiently convex to prevail over the concavity of entropy. It is immediate to see that if A_i is entropy log-convex, then it is strictly log-convex.

The interpretation of this production function becomes clear when we plug it into the unitary cost-minimization problem defined in Equation 19:

$$K_i(P) = \min_{Q_i \in \mathbb{R}^n_+} \sum_{j=1}^n Q_{ij} P_j \quad \text{s.t.} \quad A_i(\alpha_i) \prod_{j=1}^n \left(\frac{Q_{ij}}{\alpha_{ij}} \right)^{\alpha_{ij}} \geq 1 \quad \forall \alpha_i \in C_i. \tag{29}$$

That is, the firm must acquire a vector of inputs Q_i that guarantees the production of a unit of output for all the tasks in C_i. With this, the tasks such that $A_i(\alpha_i)$ is high are those for which the constraint is less binding, while the tasks such that $A_i(\alpha_i)$ is close to 1 are bottlenecks. In this case, the log-cost operator is given by that of Equation 27, with the min replaced by a max. △

Example 4 (Endogenous CES)**.** Our general framework allows us to study a novel class of nested CES economies in which the input weights and elasticities of substitution are chosen endogenously. Formally, suppose for all $i \in [n]$ that:

$$G_i(Q_i) = \max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) \left(\sum_{j=1}^n \alpha_{ij}^{\frac{1}{\sigma_i}} Q_{ij}^{\frac{\sigma_i - 1}{\sigma_i}} \right)^{\frac{\sigma_i}{\sigma_i - 1}} \tag{30}$$

for some $\bar{\sigma}_i \in \overline{\mathbb{R}}_+$ indexing the maximum possible degree of substitution and an upper-semicontinuous $A_i : \Delta \times \overline{\mathbb{R}}_+ \to [0, 1]$ such that $\max_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} A_i(\alpha_i, \sigma_i) = 1$, representing the productivity of producing using different input weights and with different elasticities of substitution. In this case, the log-cost operator is given by:

$$T_i(p) = \min_{\alpha_i \in \Delta, \sigma_i \in [0, \bar{\sigma}_i]} \left\{ \frac{1}{1 - \sigma_i} \ln \left(\sum_{j=1}^n \alpha_{ij} \exp((1 - \sigma_i) p_j) \right) + \ln \left(\frac{1}{A_i(\alpha_i, \sigma_i)} \right) \right\}$$

A natural sub-class of the endogenous CES family emerges if we assume that sector $i \in [n]$ is endowed with some initial input weights α_i^0 and elasticity of substitution σ_i^0 and it can suffer an "adjustment cost" in productivity-equivalent units that is given by:

$$A_i(\alpha_i, \sigma_i) = \exp\left(-D(\alpha_i \| \alpha_i^0) - d(\sigma_i \| \sigma_i^0) \right) \tag{31}$$

where $D : \Delta^2 \to \overline{\mathbb{R}}_+$ is a divergence between α_i and α_i^0 and $d : \overline{\mathbb{R}}_+^2 \to \overline{\mathbb{R}}_+$ is a divergence between σ_i and σ_i^0. This specification captures the intuitive idea that if a sector tries to produce using a technology that is different from where it started, then there will be an opportunity cost. △

4 A General Formula for Equilibrium Prices and Output

In this section, we derive a general formula for equilibrium prices and output in terms of the primitive, non-parametric production structure. In so doing, we provide the first primitive formula for the global solution for equilibrium prices in a production network economy beyond the Cobb-Douglas case. A key step in proving this result is of independent interest: an arbitrary constant returns to scale economy can be represented "as if" it were an economy in which each sector can choose among a set of Leontief production technologies with different input weights and corresponding productivities.

4.1 The Sufficiency of Endogenous Leontief Technology

We define the relative entropy $R : \Delta \times \Delta \to [0, \infty]$ as:

$$R\left(w \| \tilde{w}\right) = \begin{cases} \sum_{j=1}^{n} w_j \ln\left(\frac{w_j}{\tilde{w}_j}\right) & \text{if} \quad w \ll \tilde{w}, \\ \infty & \text{otherwise,} \end{cases} \tag{32}$$

where $w \ll \tilde{w}$ means that w is absolutely continuous with respect to $\tilde{w}$, that is $w_j > 0$ implies that $\tilde{w}_j > 0$. With this, we have that:

Proposition 1 (Endogenous Leontief Representation)**.** *A cost operator K and a log-cost operator T are generated by a profile of intermediate input production functions G if and only if for every $i \in [n]$:*

$$K_i(P) = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^{n} \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}, \tag{33}$$

and

$$T_i(p) = \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \left\{ \sum_{j=1}^{n} w_{ij} p_j - R(w_i || \tilde{w}_i) + \ln\left(\frac{1}{G_i(\tilde{w}_i)}\right) \right\} \tag{34}$$

The first part of Proposition 1 follows from the constant returns to scale assumption on the production functions. Yet, to our knowledge, it is a novel representation of the cost function. Moreover, this representation lends itself to an important interpretation and, as we show next, it has far-reaching consequences for equilibrium prices and output. Specifically, Proposition 1 implies that the cost function of any sector is the same as the cost function of a sector with an *endogenous* Leontief production technology. Formally, by defining the endogenous Leontief production function as:

$$G_i^*(Q_i) = \max_{\tilde{w}_i \in \Delta} \left\{ G_i(\tilde{w}_i) \min_{j \in [n]} \frac{Q_{ij}}{\tilde{w}_{ij}} \right\} \qquad \forall i \in [n] \tag{35}$$

we observe that the cost functions derived from these production functions are the same as the ones derived from the original profile of production functions G_i. The interpretation of G_i^* is clear: each sector i has a rigid, Leontief production function given a production technology $\tilde{w}_i$, but they can flexibly choose their technology at the expense of a productivity opportunity cost $G_i(\tilde{w}_i)$. In other words, the original production function, once restricted to the simplex, coincides with the productivity shifter of the endogenous Leontief representation.

In turn, this representation can be directly connected with the models of endogenous production technologies recently analyzed in the literature (Acemoglu and Azar, 2020; Kopytov et al., 2024). These papers consider CRS production functions that, as in Equation 35, can be obtained as upper envelopes over technologies of smooth production functions, usually Cobb-Douglas, with technology-dependent productivity shifters.

Perhaps surprisingly, Proposition 1 asserts that, by accurately choosing the form of the productivity shifter, it is enough to consider upper envelopes of Leontief production functions to obtain *all* CRS cost operators. Since the profile of sectors' cost operators is sufficient to pin down the equilibrium outcomes, we find that any equilibrium of a CRS economy is *equivalent* to an equilibrium of an economy with endogenous Leontief production functions.

The second part of Proposition 1 simply rewrites the first part in terms of log-prices and leverages the Donsker-Varadhan formula. In turn, this allows us to interpret Leontief technology with weights $\tilde{w}_i$ as the worst possible Cobb-Douglas technology with a productivity gain that depends on the relative entropy between the chosen Cobb-Douglas weights w_i and the Leontief weights $\tilde{w}_i$. Putting together this observation with the endogeneity of $\tilde{w}_i$—the chosen Leontief technology is the best possible one—reveals that T_i is the min-max of an *affine* function of log-prices. This is the key implication of the CRS assumption and it is the one that we leverage to derive our representation of equilibrium prices.

4.2 Formulae for Equilibrium Prices and Output

We now use this representation to derive a representation for equilibrium log-prices. To this end, we define the following objects. We define $\mathcal{W} = \left\{W \in \mathbb{R}_+^{n\times n} : \forall i \in [n], w_i \in \Delta\right\}$ as the set of $n \times n$ row-stochastic matrices. Given an input-output matrix $W \in \mathcal{W}$, we define the Leontief-inverse as in Acemoglu et al. (2012):

$$\Omega_W = (1-\beta)(I-\beta W)^{-1} \tag{36}$$

We define the production-wedge operator $\tau^G : \mathcal{W}^2 \to \overline{\mathbb{R}}^n$ as:

$$\tau_i^G(W\|\tilde{W}) = \frac{\beta}{1-\beta}\left[\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right) - R\left(w_i\|\tilde{w}_i\right)\right] \qquad \forall i \in [n] \tag{37}$$

which is simply a rescaling of the additive productivity distortion term in the affine min-max representation of T_i (see Equation 34).

With these objects in hand, we state our representation for equilibrium prices and output:

Theorem 1 (Equilibrium Price and Output Representation)**.** *For every vector of log-productivities $x \in \mathbb{R}^n$, the unique equilibrium log-prices are given by*

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \Omega_W \left(x + \tau^G(W \| \tilde{W}) \right) \right\} \tag{38}$$

Moreover, the corresponding equilibrium log-output is given by:

$$\Upsilon(x) = \max_{\tilde{W} \in \tilde{\mathcal{W}}} \min_{W \in \mathcal{W}} \left\{ -\xi' \Omega_W \left(x + \tau^G(W \| \tilde{W}) \right) \right\} \tag{39}$$

Before unpacking the intuition behind these formulae, we remark that the right-hand side of Equation 38 is entirely *primitive*: it depends only on the production functions and not on any equilibrium objects.

To understand the formula for equilibrium prices, first fix $\tilde{W}$ and W that respectively are minimizers and maximizers in Equation 38. Observe that $\Pi(x) = \Omega_W(x + \tau^G(W \| \tilde{W}))$. That is, it is as if the initial productivity shock x is increased by the production wedge $\tau^G(W \| \tilde{W})$ and then this composite productivity shock propagates according to the Leontief-inverse of the input-output matrix W. This is the formula for equilibrium prices in a Cobb-Douglas economy with input-output matrix W and log-productivity vector $(x + \tau^G(W \| \tilde{W}))$.

Given that the economy we are studying is non-linear, the fact that the solution for equilibrium prices inherits the linear form corresponding to a distorted Cobb-Douglas form is surprising. We can use the endogenous Leontief representation of Proposition 1 to understand where this comes from. In particular, this representation guarantees that the log-cost operator—while nonlinear—is nonlinear in a restricted way: it must be the min-max of an affine function. That is, if we ignore the min and the max, the fixed point equation characterizing equilibrium prices reduces to the standard Cobb-Douglas system when the productivity is distorted. The content of this result is that the operations of taking the min-max at the micro level (for firms' costs) and at the macro level (for equilibrium prices) can be interchanged.

To understand the vectorial nature of this result, Theorem 1 says that if one maximizes every row of $\Omega_W(x - \tau^G(W \| \tilde{W}))$ over $\tilde{W}$ for every W and then chooses W to minimize every row of the resulting expression, then the vector of values corresponds to the equilibrium prices. In fact, the result guarantees the existence of a pair of input-output networks W and $\tilde{W}$ that respectively and *simultaneously* maximize and minimize the price of *every* sector. In particular, the result implies that the following algorithm can be used to solve for equilibrium prices. First,

define $\mathcal{W}^0 = \mathcal{W}$ and then recursively define

$$\mathcal{W}^i = \arg\min_{\tilde{W} \in \mathcal{W}^{i-1}} \max_{W \in \mathcal{W}} \left\{ \omega'_{W,i}(x + \tau^G(W \| \tilde{W})) \right\} \qquad \forall i \geq 1 \tag{40}$$

where $\omega_{W,i}$ is the i-th row of Ω_W. We then have that $\mathcal{W}^n$ is the set of minimizers in Equation 38. Given any element $\tilde{W} \in \mathcal{W}^n$, the *unique* corresponding W that attains the maximum simply solves the inner maximization problem. This algorithm requires the solution of no more than n *sequential* one-dimensional problems to find the equilibrium: it is not essential to *simultaneously* optimize over the elements of the price vector.

To understand what the production wedge looks like, we observe that if G_i is a Cobb-Douglas production function with *fixed* weights $\alpha_i \in \Delta$, then the wedge is proportional to the difference in the relative entropy between α_i and $\tilde{w}_i$ and the relative entropy between w_i and $\tilde{w}_i$:

$$\tau_i^G(W \| \tilde{W}) = \frac{\beta}{1-\beta} \left[R\left(\alpha_i \| \tilde{w}_i\right) - R\left(w_i \| \tilde{w}_i\right) \right] \tag{41}$$

Given this form, solving the min-max problem in Equation 38 yields the standard Cobb-Douglas solution that $\Pi(x) = \Omega_W x$, where $W = (\alpha_i)_{i=1}^n$ is the matrix of Cobb-Douglas weights. Similarly, if G_i is a CES production function with weights $\alpha_i \in \Delta$ and $\sigma_i \in \overline{\mathbb{R}}_+$, then the wedge is proportional to the Renyi divergence with parameter $\frac{1}{\sigma_i}$ between α_i and $\tilde{w}_i$ and the relative entropy between w_i and $\tilde{w}_i$:

$$\tau_i^G(W \| \tilde{W}) = \frac{\beta}{1-\beta} \left[D_{\sigma_i^{-1}}\left(\alpha_i \| \tilde{w}_i\right) - R\left(w_i \| \tilde{w}_i\right) \right] \tag{42}$$

where D_t is the Renyi divergence with parameter $t \in \overline{\mathbb{R}}$. In this case, our result provides the first formula for equilibrium prices under the nested CES structure.

5 Application I: Resilience *vs.* Fragility

We now apply our general representation of equilibrium prices to derive how properties of production functions translate into general equilibrium properties of prices. In the process of so doing, we will formalize the mathematical sense in which the general min-max formulation of equilibrium prices encodes the dueling forces of resilience and fragility.

5.1 Defining Resilience and Fragility

To define the notions of resilience and fragility, we ask: when is it true that making shocks to productivity more dispersed has adverse economic consequences? Informally, if the consequences are adverse, we call the economy fragile, and if the consequences are beneficial, we

call the economy resilient. While this is far from the only reasonable definition of fragility and resilience, this corresponds to the popular notion advanced by Taleb (2013): "Fragility can be defined as an accelerating sensitivity to a harmful stressor: this response plots as a concave curve and mathematically culminates in more harm than benefit from random events. Antifragility is the opposite, producing a convex response that leads to more benefit than harm." We follow this definition:

Definition 2 (Resilience and Fragility)**.** *Sector i is resilient (fragile) if T_i is concave (convex). The economy is resilient (fragile) if Π is concave (convex).*

To understand the motivation for this definition, assume that the log-prices faced by sector i have an initial distribution π over $\mathbb{R}^n$. The expected unitary log-cost of sector i is then $\mathbb{E}_\pi[T_i]$. We are interested in the effect on the expected log-cost of a mean-preserving spread of the distribution of log-prices, that is, the new distribution π' dominates π in the convex order. We have that sector i is resilient *if and only if* $\mathbb{E}_{\pi'}[T_i] \leq \mathbb{E}_\pi[T_i]$ for any such pair of distributions. Conversely, sector i is fragile *if and only if* $\mathbb{E}_{\pi'}[T_i] \geq \mathbb{E}_\pi[T_i]$ for any such pair of distributions. That is, resilient sectors have lower log-costs in expectation when the volatility of log-prices rises, while the opposite happens for fragile sectors. We can interpret resilience and fragility of the economy analogously by looking at the expected log-price effect of mean-preserving spreads in the distributions of log-shocks.

This notion of resilience and fragility is consistent with the intuitive idea that CES technologies are fragile (resilient) when the elasticity of substitution is smaller (larger) than one:

Example 1 (continuing from p. 14)**.** *A sector $i \in [n]$ with CES production technology with elasticity of substitution σ_i is resilient if and only if $\sigma_i \geq 1$ and fragile if and only if $\sigma_i \leq 1$.*

5.2 Resilience

We now characterize resilience at the sector level and show that equilibrium prices preserve resilience. The next result shows that a sector is resilient if and only if its cost operator is generated by an endogenous Cobb-Douglas production function (as in Example 2).

Proposition 2. *Fix $i \in [n]$ with log-cost function T_i. The following are equivalent:*

(i) Sector i is resilient.

(ii) T_i is generated by an endogenous Cobb-Douglas production function G_i^*:

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \tag{43}$$

for some nonempty convex set $C_i \subseteq \Delta$ *and production shifter* $A_i : C_i \to (0,1]$ *that is upper coercive, log-concave, and such that* $\max_{w_i \in C_i} A_i(w_i) = 1$.

Moreover, the productivity shifter A_i *is uniquely defined given* T_i.

One important implication of this result is that endogenous Cobb-Douglas production functions span the full scope of possibilities under resilience. Thus, insofar as one is interested in resilient technologies, this result implies that the approach of imposing endogenous Cobb-Douglas technology (as done by Acemoglu and Azar, 2020; Kopytov et al., 2024) is not a restriction. However, this formulation rules out the possibility of fragility.

Example 1 (continuing from p. 14)**.** *Interestingly, because CES production functions with elasticity* $\sigma_i > 1$ *induce a concave log-cost operator, it follows that they are cost-equivalent to some endogenous Cobb-Douglas production function. By inspection of the proof of Proposition 2, it possible to show that the productivity shifter recovering the CES log-cost operator is* $A_i(w_i) = \exp\left(-\frac{1}{1-\sigma_i} R(w_i||\alpha_i)\right)$ *and* $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$*, where* $\alpha \in \Delta$ *are the weights in the original CES production function.*

The next result uses this representation and Theorem 1 to derive the formula for equilibrium prices in economies in which all sectors are resilient:

Corollary 1. *If every sector is resilient, then the economy is resilient and such that*

$$\Pi(x) = \min_{W \in \mathcal{W}_C} \left\{ \Omega_W(x + \tau^A(W)) \right\} \tag{44}$$

and

$$\Upsilon(x) = \max_{W \in \mathcal{W}_C} \left\{ -\xi' \Omega_W(x + \tau^A(W)) \right\} \tag{45}$$

where $\mathcal{W}_C = \{W \in \mathcal{W} : w_i \in C_i, \forall i \in [n]\}$ *and* $\tau_i^A(W) = \frac{\beta}{1-\beta} \ln\left(\frac{1}{A_i(w_i)}\right)$ *for all* $i \in [n]$.

Thus, in resilient economies, the max in the general min-max representation of Theorem 1 disappears. This justifies our earlier informal interpretation of the min as inducing resilience. In this case, the production wedge also simplifies: G_i is replaced by A_i from the endogenous Cobb-Douglas representation.

5.3 Fragility

We now turn to the opposite question of when sectors and the aggregate economy are fragile. The following result shows that fragility is equivalent to a sector having a task-based production function with an entropy log-convex shifter A_i (as in Example 3). To understand this property, recall that cost operators K are always concave. Thus, while log-cost operators T can be convex,

this comes with the additional restriction that A_i is sufficiently convex that its logarithm plus a concave function (i.e., the entropy) remains convex. This is exactly the extra restriction needed to offset the resilience coming from cost minimization.

Proposition 3. *Fix $i \in [n]$ with log-cost function T_i. The following are equivalent:*

(i) Sector i is fragile

(ii) T_i is generated by a task-based production function G_i^*:

$$G_i^*(Q_i) = \min_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \tag{46}$$

for some nonempty convex set $C_i \subseteq \Delta$ and a production shifter $A_i : C_i \to [1, \infty)$ that is lower coercive, entropy log-convex, and such that $\min_{w_i \in C_i} A_i(w_i) = 1$.

Moreover, the productivity shifter A_i is uniquely defined given T_i.

While the endogenous Cobb-Douglas production function that characterizes resilience is familiar in the literature, this task-based mirror is (to the best of our knowledge) new. Moreover, the shifter in the task-based production function is required to be entropy log-convex, a stronger property than log-convexity. This stronger requirement comes from the need to be consistent with the co-occurrence of T_i being convex while K_i is concave. Indeed, it can be shown that this requirement is so strong that convexity of T_i implies differentiability of T_i.[5] Intuitively, under convexity, firms are not able to change their production technique fast enough to generate kinks in T_i.

Example 1 (continuing from p. 14)**.** *Similarly to the resilient case, because CES production functions with elasticity $\sigma_i < 1$ induce a convex log-cost operator, it follows that they are cost-equivalent to some task-based production function. By inspection of the proof of Proposition 3, it possible to show that the productivity shifter recovering the CES log-cost operator is $A_i(w_i) = \exp\left(\frac{1}{1-\sigma_i} R(w_i||\alpha_i)\right)$ and $C_i = \{w_i \in \Delta : w_i \ll \alpha_i\}$.*

The next result leverages this representation and Theorem 1 to provide the equilibrium prices in economies in which all sectors are fragile:

Corollary 2. *If every sector is fragile, then the economy is fragile and such that*

$$\Pi(x) = \max_{W \in \mathcal{W}_C} \left\{ \Omega_W(x + \tau^A(W)) \right\} \tag{47}$$

and

$$\Upsilon(x) = \min_{W \in \mathcal{W}_C} \left\{ -\xi' \Omega_W(x - \tau^A(W)) \right\} \tag{48}$$

[5] Formally, if T_i is a convex log-cost functional, then T_i is continuously Frechet differentiable.

Thus, just as resilience eliminated the max in the general min-max representation of Theorem 1, fragility eliminates the min in the representation. This, completes our justification of the general max as corresponding to fragility and the min as corresponding to resilience. Moreover, as in the resilient case, the production wedge is substantially simpler than the general representation.

5.4 Ranking Economies' Fragility

We have so far considered absolute notions of resilience and fragility. Of course, even if economies are neither resilient nor fragile in a global sense, we may nevertheless be interested in comparing their fragility. To do so, we adopt a comparative fragility definition measuring that of comparative risk or ambiguity aversion in decision theory (Ghirardato and Marinacci, 2002):

Definition 3 (Comparative Fragility)**.** *An economy with price operator* Π *is more fragile than an economy with price operator* Π' *if and only if* $\Pi \geq \Pi'$*.*

We obtain the following sufficient condition for one economy to be more fragile than another:

Corollary 3. *Consider two economies with vectors of intermediate goods production functions* G *and* G'*. If* $G' \leq G$*, then the economy with production functions given by* G *is more fragile than the economy with production functions given by* G'*.*

This result is immediate from Theorem 1: a pointwise increase in G leads to a pointwise reduction in the production wedge τ^G and thereby a lower value of the min-max problem that defines equilibrium prices. However, observe that the new profile of production functions G' is subject to all our assumptions on production functions, including $\max_{Q_i \in \Delta} G_i(Q_{i1}, \ldots, Q_{in}) = 1$. In other words, a change from G to G' cannot represent a uniform increase in productivity. Indeed, as the following examples show, the correct interpretation of this result is that an increase in production flexibility always leads to lower equilibrium prices and higher aggregate output.

5.5 Further Properties: Big Shocks and Elasticity

So far, we have considered a notion of fragility that corresponds to more dispersed shocks leading to worse outcomes. While this is a standard notion, there are other reasonable natural-language notions of fragility. Two potentially natural additional notions are that: (i) as shocks become bigger, their marginal adverse effect is larger, and (ii) the elasticity of substitution is greater or smaller than unitary. In this section, we show how these notions are related.

Formally, we define resilience and fragility to big shocks as:

Definition 4 (Resilience and Fragility to Big Shocks)**.** *Sector i is resilient (fragile) to big shocks if T_i is concave (convex) at zero. The economy is resilient (fragile) to big shocks if Π is concave (convex) at zero.*

To understand this, observe that T_i is convex at zero if for all $\lambda \in (0,1)$ and $p \in \mathbb{R}^n$:

$$\underbrace{T_i(p)}_{\text{Total Effect}} \geq \underbrace{\frac{T_i(\lambda p)}{\lambda}}_{\text{Average Effect}} \tag{49}$$

That is, the *total effect* of a shock in direction p—or $T_i(p)$—is larger than the *average effect* for all scaled down versions of that shock— or $T_i(\lambda p)/\lambda$. This is a mathematical definition of the intuitive notion the marginal effect of a shock is increasing in the size of the shock.

Moreover, we define elasticity as:

Definition 5 (Elasticity)**.** *Sector i is elastic (inelastic) if T_i is supermodular (submodular).*

To understand why we use this definition of elasticity, we observe that when T_i is twice continuously differentiable that it collapses to the standard notion of elasticity from producer theory. More technically, given the corresponding cost function K_i, we define the Allan-Uzawa elasticity of substitution as

$$\sigma_{jk}^{K_i}(P) = \frac{K_i(P)\frac{\partial^2 K_i(P)}{\partial P_j \partial P_k}}{\frac{\partial K_i(P)}{\partial P_j}\frac{\partial K_i(P)}{\partial P_k}} = 1 + \frac{\frac{\partial^2 T_i(p)}{\partial p_j \partial p_k}}{\frac{\partial T_i(p)}{\partial p_j}\frac{\partial T_i(p)}{\partial p_k}} \tag{50}$$

where the second equality follows by simple algebra. Here, we can see that supermodularity (submodularity) is equivalent to the elasticity of substitution being globally greater than one or smaller than one—the standard notion of elastic and inelastic. Indeed, for a CES production function, this notion of elasticity corresponds to the standard condition that $\sigma \lessgtr 1$. Our definition is more general in that it can be specified without assuming any differentiability property of T_i.

From this, we can see that elasticity encodes a notion of fragility to correlation. To understand this, suppose as before that the log-prices faced by sector i have distribution $\pi \in \mathbb{R}^n$. A standard notion of a random vector with distribution π' being more correlated than another random vector with distribution π is that the distribution π' dominates π in the supermodular stochastic order (see Shaked and Shanthikumar, 2007, for a discussion). That is, for all supermodular $f : \mathbb{R}^n \to \mathbb{R}$, we have that $\mathbb{E}_{\pi'}[f(p)] \geq \mathbb{E}_{\pi}[f(p)]$. Intuitively, if π' is more correlated than π, the expected value of a supermodular function will be larger under π' than π because the arguments are more likely to be *jointly* large and small. Thus, a sector being elastic implies that that sector is fragile to correlation.

Ranking Notions of Resilience and Fragility. The next result shows that these different notions of resilience and fragility can be ranked in order of stringency.

Proposition 4. *The following statements are true:*

(i) If a sector is inelastic (elastic), then it is fragile (resilient)
(ii) If a sector is fragile (resilient), then it is fragile (resilient) to big shocks

Thus, while all sectors being elastic or inelastic is sufficient for the conclusion that the economy is resilient or fragile, it is far from necessary. While these notions are equivalent with CES production technologies, away from this special case our analysis makes clear the distinct roles played by elasticity and resilience/fragility: elasticity determines the extent to which correlation is good or bad, while resilience and fragility determine the extent to which volatility is good or bad. As Theorem 1 and Corollaries 1 and 2 have shown that resilience and fragility characterize equilibrium outcomes, our results indicate that elasticity (while more conventional) is less reflective of the general structure of equilibrium outcomes.

6 Application II: A Generalized Hulten's Theorem

So far our results have described global properties of equilibrium outcomes. While that has been our main focus, we argue that this approach yields insights even for understanding the effects of small shocks.

6.1 Shock Propagation in the Small

To understand the effects of small shocks, we denote the starting point from which the economy is shocked by $x \in \mathbb{R}^n$, the direction of a productivity shock by $d \in \mathbb{R}^n$, and the size of the shock by $t \in \mathbb{R}_{++}$. With this, we define the average effect of a shock in direction d of size t starting from x as:

$$\Pi_t(x;d) = \frac{\Pi(x+td) - \Pi(x)}{t}. \tag{51}$$

To study small shocks, we characterize the directional derivative $\Pi_0(x;d) = \lim_{t\to 0} \Pi_t(x;d)$. We say that Π is *directionally differentiable* at x if this limit exists for every d and that Π is *Gateaux differentiable* at x if this limit is a linear function of d.[6]

We define the optimal technology choice profile across sectors from the endogenous Leontief

[6] All these properties and derived objects are analogously defined for the log-cost operator T and the real GDP function Υ.

representation (Proposition 1) as:

$$\Sigma(p) = \left\{ \tilde{W} \in \mathcal{W} : \tilde{w}_i \in \arg\min_{\hat{w}_i \in \Delta} \frac{\sum_{j=1}^n \hat{w}_{ij} \exp(p_j)}{G_i(\hat{w}_i)}, \forall i \in [n] \right\} \qquad \forall p \in \mathbb{R}^n \tag{52}$$

Intuitively, $\Sigma(p)$ corresponds to the set of cost-minimizing technology choices across firms when they face the log-price vector p. We also define the operator $L : \mathcal{W} \times \mathbb{R}^n \to \mathcal{W}$:

$$L_{ij}(\tilde{W}, p) = \frac{\tilde{w}_{ij} \exp(p_j)}{\sum_{k=1}^n \tilde{w}_{ik} \exp(p_k)} \qquad \forall i, j \in [n] \tag{53}$$

which returns the Cobb-Douglas expenditure shares given a technology choice matrix $\tilde{W}$. With these definitions in hand, we arrive at the following formula for the directional derivative of equilibrium log-prices:

Theorem 2. *For all $x \in \mathbb{R}^n$, the equilibrium log-price operator Π is directionally differentiable with:*

$$\Pi_0(x; d) = \min_{\tilde{W} \in \Sigma(\Pi(x))} \Omega_{L(\tilde{W}, \Pi(x))} d \qquad \forall d \in \mathbb{R}^n \tag{54}$$

Intuitively, this formula says that for every direction d, shocks propagate according to the Leontief inverse of a fictitious Cobb-Douglas economy with expenditure shares given by L. Moreover, the result says that among all of the cost-minimizing technologies that prevail at the equilibrium log-prices $\Pi(x)$, the fictitious Cobb-Douglas is the one that minimizes equilibrium prices following the shock. This implies that, for every x, the small-shock equilibrium operator $\Pi_0(x; d)$ is a concave function of the direction of productivity change. In other words, every CRS economy is *resilient* when we restrict to small shocks.

It turns out that a primitive formula for the directional derivative is available if we consider shocks starting from a symmetric steady state in which all sectors have the same productivity. In this case, we have that:

Corollary 4. *The directional derivative of Π at the symmetric steady state is given by:*

$$\Pi_0(0; d) = \min_{\tilde{W} \in \mathcal{W}_N} \Omega_{\tilde{W}} d \qquad \forall d \in \mathbb{R}^n \tag{55}$$

where $\mathcal{W}_N = \left\{ \tilde{W} \in \mathcal{W} : G_i(w_i) = 1, \forall i \in [n] \right\}$.

That is, starting from symmetry, shock propagation is governed by the set of "free" production networks at which all sectors have "as-if" unitary productivity in the endogenous Leontief interpretation.

Technology Switches and the Insufficiency of Local Changes. The following four equivalent statements characterize the points at which technology switching does not introduce jumps.

Proposition 5 (When Jumps Do Not Feature)**.** *Fix $x \in \mathbb{R}^n$. The following statements are equivalent:*

(i) T is Gateaux differentiable at $\Pi(x)$
(ii) Π is Gateaux differentiable at x
(iii) $\Sigma(\Pi(x))$ is a singleton
(iv) There exists a unique competitive equilibrium at x.

Moreover, for almost all $x \in \mathbb{R}^n$, these four equivalent conditions are true. Furthermore, they are true for all $x \in \mathbb{R}^n$ provided that all the sectors are fragile.

Thus, whenever there are multiple competitive equilibria or T is not Gateaux differentiable at equilibrium, jumps occur and the fundamental theorem of calculus may serve as a poor guide. While this result clarifies that these jump points comprise a zero measure subset of $\mathbb{R}^n$, their presence can nonetheless lead to arbitrarily large errors if the analyst ignores them.

6.2 The Generalized Hulten's Theorem

We now use Theorem 2 to derive the directional derivative of equilibrium output $\Upsilon_0(x;d) = \lim_{t\to 0} \frac{\Upsilon(x+td)-\Upsilon(x)}{t}$ and provide a generalization of Hulten's theorem that the change in aggregate labor productivity induced by infinitesimal productivity shocks is given by the inner product of the sales share vector (or Domar weights) with the vector of productivity shocks. To this end, let $\mathcal{E}(x)$ be the set of competitive equilibria at $x \in \mathbb{R}^n$. For any competitive equilibrium E, we define the Domar weight vector as $\lambda_E = (P_i^E Y_i^E)_{i=1}^n$.

Proposition 6 (Radial Hulten's Theorem)**.** *For all $x, d \in \mathbb{R}^n$, we have that Υ is directionally differentiable with*

$$\Upsilon_0(x;d) = \max_{E\in\mathcal{E}(x)} \lambda_E' d_Z \tag{56}$$

where $d_Z = -(1-\beta)d$ is the direction expressed in units of log-TFP.

This generalizes Hulten's theorem beyond the case in which firms' cost functions are differentiable. The technology switching that previously lead to discontinuity in the response of prices to shocks also leads to discontinuity in the response of output to shocks. This discontinuity manifests because switches in technology lead to jumps in the economy's Domar weights. The radial nature of this theorem highlights that scaling d by any $\eta > 0$ yields the same Domar weights but rotations of d may lead different Domar weights to govern shock propagation.

Intuitively, when the economy is at a technology switching point, different directions of shocks will propagate in different ways.

Corollary 5. *Consider two vectors of intermediate goods production functions G' and G. If $G' \geq G$, then*

$$co\left\{\lambda_E \in \Delta : E \in \mathcal{E}(0)\right\} \subseteq co\left\{\lambda_E \in \Delta : E \in \mathcal{E}'(0)\right\}. \tag{57}$$

Proof. By Corollary 3, we have that $\Upsilon'(x) \geq \Upsilon(x)$ for all $x \in \mathbb{R}^n$. In turn, this implies that, for every direction $d \in \mathbb{R}^n$,

$$-(1-\beta)\min_{E\in\mathcal{E}'(0)} \lambda_E d = \Upsilon'_0(0;d) \geq \Upsilon_0(0;d) = -(1-\beta)\min_{E\in\mathcal{E}(0)} \lambda_E d.$$

This yields that $co\left\{\lambda_E \in \Delta : E \in \mathcal{E}(0)\right\} \subseteq co\left\{\lambda_E \in \Delta : E \in \mathcal{E}'(0)\right\}$ as desired. □

This corollary implies an interesting comparative static. Suppose that at a given profile of intermediate production functions G' there exists a unique competitive equilibrium at symmetry $x = 0$ (or any $ke \in \mathbb{R}^n$, with $k \in \mathbb{R}$). Then any decrease in productivity of all sectors that yields a new profile of intermediate production functions $G \leq G'$ will preserve equilibrium uniqueness at symmetry. That is, equilibrium multiplicity can arise only if at least one sector becomes more productive for at least one combination of inputs.

7 Application III: Robust Variance Bounds

In this section, we derive lower and upper bounds on the variance of real GDP under standard assumptions about the distribution of productivity shocks.

Assume now that X, the profile of the sectors' inverse log-productivities, is a random vector with independent components. Importantly, we are not making any assumptions about the boundedness of the distributions' supports or the existence of densities. Let Υ denote the random variable describing the equilibrium real GDP of the economy at every possible realization of the random vector X. In fact, we let X be the only source of randomness for Υ. Our goal is to find bounds on $\mathrm{Var}[\Upsilon]$ given the original distribution of X and the equilibrium operator yielding Υ.

By Proposition 5, we know that there exists a full-measure set $\mathcal{D} \subseteq \mathbb{R}^n$ such that Υ is (Gateaux) differentiable at every $x \in \mathcal{D}$. For every $x \in \mathcal{D}$, let $\lambda(x) \in \Delta$ denote the Domar weight vector corresponding to the *unique* competitive equilibrium at x.[7] For every sector $i \in [n]$, define its smallest and largest Domar weights respectively as

$$\underline{\lambda}_i = \inf_{x\in\mathcal{D}} \lambda_i(x) \qquad \text{and} \qquad \overline{\lambda}_i = \sup_{x\in\mathcal{D}} \lambda_i(x). \tag{58}$$

[7] This is well-defined again owing to Proposition 5.

Observe that, differently from any fixed Domar weight vector λ_E, we have that $\sum_{i=1}^n \underline{\lambda}_i \leq 1$ and $\sum_{i=1}^n \overline{\lambda}_i \geq 1$, usually with strict inequalities.[8]

We say that X_i is a Poincaré's random variable with constant $\eta_i > 0$ if for every bounded Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\text{Var}[f(X_i)] \leq \eta_i \mathbb{E}[|f'|(X_i)^2] \tag{59}$$

where $|f'|(t) = \limsup_{s \to t} \left| \frac{f(s)-f(t)}{s-t} \right|$ for all $t \in \mathbb{R}$. This is the absolute derivative of f and is always finite since f is Lipschitz continuous. Observe that Poincaré's random variables do not necessarily need to have a positive density or bounded support.[9] In particular, Poincaré's random variables are more general than the class of Gaussian distributions while maintaining a thin-tail assumption.

Proposition 7. *Under our maintained assumptions, we have*

$$(1-\beta)^2 \max_{i \in [n]} \underline{\lambda}_i^2 \text{Var}[X_i] \leq \text{Var}[\Upsilon] \leq (1-\beta)^2 \sum_{i=1}^n \overline{\lambda}_i^2 \text{Var}[X_i]. \tag{60}$$

Moreover, if each X_i is a Poincaré's random variable with constant $\eta_i > 0$, then

$$\text{Var}[\Upsilon] \leq \eta (1-\beta)^2 \sum_{i \in [n]} \mathbb{E}[\lambda_i(X)^2] \tag{61}$$

where $\eta = \max_{i \in [n]} \eta_i$.

Observe that we can get the bounds for the (fixed) Cobb-Douglas production economy directly from Equation 60. In this case, there exists a unique Domar weight vector $\lambda \in \Delta$ regardless of the realization of X. Therefore, we have

$$(1-\beta)^2 \max_{i \in [n]} \lambda_i^2 \text{Var}[X_i] \leq \text{Var}[\Upsilon] \leq (1-\beta) \max_{i \in [n]} \lambda_i \text{Var}[X_i], \tag{62}$$

yielding that $\max_{i \in [n]} \lambda_i \text{Var}[X_i]$ completely controls the variance of the real GDP.

We remark that the bounds on the real GDP's variance of Equation 60 are independent of the sectors' productivity distributions, with the exception of the microeconomic variance. That is, Equation 60 provides robust bounds on the volatility of the aggregate output. If we add the minimal assumption that the productivities' distributions are Poincaré, then we further refine

[8] In fact, the main case where the previous conditions hold with equality is that of a production economy where all sectors have Cobb-Douglas production functions as in Acemoglu et al. (2012). In that case, there exists a single Domar weight vector that is independent of the realization of the sectors' productivities.

[9] However, they always have finite variance, as it can be seen by just letting f be the identity function.

the bounds, which now depend on the maximum Domar weight respectively in the worst and best case.

8 Conclusion

In this paper, we have provided a global analysis of the canonical production network model with arbitrary constant returns to scale technology. Our main result provides a primitive formula for equilibrium prices that applies in any such economy. As a formula was previously known only in the Cobb-Douglas case (Acemoglu et al., 2012), this formula represents a substantial generalization and applies to popular special cases, such as nested CES economies (Baqaee and Farhi, 2019; Dew-Becker, 2023), endogenous Cobb-Douglas economies (Acemoglu and Azar, 2020; Kopytov et al., 2024), and many others (including those studied by Chodorow-Reich et al., 2025; Acemoglu and Tahbaz-Salehi, 2025).

Beyond the technical contribution of this paper, to demonstrate the general utility of this analysis, we used it in three applications. First, we studied the question of when the economy is resilient and when it is fragile. We found that the "min" and the "max" in the general representation correspond in a natural way to the competing forces of resilience and fragility. We also characterized the microeconomic properties of resilience and fragility at the sector level, revealing that standard specifications and approximation-based methods allow only for global resilience or fragility. Thus, such models cannot capture the qualitative possibility that changing input mix can lead volatility to having "S-shaped" effects on the economy, whereby small and large levels of volatility are bad, but intermediate levels are good.

Second, we provided a generalization of Hulten's (1978) theorem to cover the case of general (*i.e.,* non-differentiable) production technology. As we showed, such non-differentiabilities are a natural consequence of allowing for endogenous production networks. Moreover, the generalized theorem places important economic nuance on the received wisdom following Hulten (1978) that, for small shocks, it is not important to know anything about technology or the input-output network. Indeed, the presence of technology switching—which cannot be inferred from mere observation of the current Domar weights—leads current Domar weights to provide an inaccurate description of shock propagation, even for small shocks.

Finally, we further applied the generalized Hulten's (1978) theorem to provide robust upper and lower bounds on the variance of aggregate output that results from microeconomic productivity disturbances that hold regardless of the underlying production technology in the economy. These general bounds reveal that, in order to guarantee that microeconomic productivity disturbances matter in aggregate, there must generally be a sector whose sales share of GDP is substantial even when they are very unproductive and their goods are very expensive; *i.e.,* there must be a sector whose output is very hard to substitute.

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Appendices

A Preliminaries: nonexpansive operators and fixed points

In this appendix, with an abuse of notation, we let $x \in \mathbb{R}^n$ denote an arbitrary n-th vector, which does not need to be related to the vector of log-productivity shocks considered in the main text. We endow $\mathbb{R}^n$ with the supnorm $||x||_\infty = \max_{i\in[n]} |x_i|$. We say that an operator $S : \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive if $||S(x) - S(x')||_\infty \leq ||x - x'||_\infty$. Also, denote by D the set of constant vectors of $\mathbb{R}^n$.

We now recall a few definitions of operator properties that will be important in our analysis. We say that a selfmap S on $\mathbb{R}^n$ is normalized if $S(he) = he$ for all $h \in \mathbb{R}$, where $e = (1, \ldots, 1)$ is the n-dimensional unit vector. We say that such a selfmap is translation invariant if $S(p+he) = S(p) + he$ for all $p \in \mathbb{R}^n$ and $h \in \mathbb{R}$.

Given a nonexpansive operator $S : \mathbb{R}^n \to \mathbb{R}^n$, denote as $E(S)$ its fixed points, that is those points x such that $S(x) = x$. Fix $\beta \in (0, 1)$. We introduce the new operator $\tilde{S}_\beta : \mathbb{R}^n \to \mathbb{R}^n$ that maps each $x \in \mathbb{R}^n$ to the solution of the following equation (in $\tilde{x}$):

$$\tilde{x} = (1 - \beta)\, x + \beta S\left(\tilde{x}\right). \tag{63}$$

Lemma 3 in Cerreia-Vioglio, Corrao, and Lanzani (2025b) establishes that $\tilde{S}_\beta$ is well defined. In what follows, to lighten notation, we often denote $\tilde{S}_\beta(x) = \tilde{x}_\beta$. We define an additional ancillary operator $S_{\beta,x} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$S_{\beta,x}\left(x'\right) = (1 - \beta)\, x + \beta S\left(x'\right) \quad \forall x' \in \mathbb{R}^n, \tag{64}$$

and, for every $m \in \mathbb{N}$, let $S^m_{\beta,x}(x') = S_{\beta,x} \circ \ldots \circ S_{\beta,x}(x')$ denote its m-th fold iteration evaluated at x'. It is standard to show that $\lim_m S^m_{\beta,x}(x') = \tilde{S}_\beta(x)$ for all $x', x \in \mathbb{R}^n$, see, e.g., Lemma 3 in Cerreia-Vioglio et al. (2025b).

We next state without proof a simple lemma which characterizes $\tilde{S}_\beta$ when S is affine, non-decreasing, and translation invariant.

Lemma 2. *Let $S(x) = Wx + u$ for some $W \in \mathcal{W}$ and $u \in \mathbb{R}^n$. We have*

$$\tilde{S}_\beta(x) = \Omega_W\left(x + \frac{\beta}{(1-\beta)}u\right) \qquad \forall x \in \mathbb{R}^n. \tag{65}$$

In this case, $\tilde{S}_\beta$ is affine, non-decreasing, and translation invariant.

Lemma 3. *Let S and R be nonexpansive. If either S or R is non-decreasing and $S \geq R$, then $\tilde{S}_\beta \geq \tilde{R}'_\beta$.*

Proof. Assume S is non-decreasing and fix $x \in \mathbb{R}^n$. We prove by induction that $S^t_{\beta,x}(x) \geq R^t_{\beta,x}(x)$ for all $t \in \mathbb{N}$. For $t = 1$, note that $S^1_{\beta,x}(x) = (1-\beta)x + \beta S(x) \geq (1-\beta)x + \beta R(x) = R^1_{\beta,x}(x)$. Next, we assume the statement is true for t and we prove it holds for $t+1$. Since S is non-decreasing and $S \geq R$, we have that

$$S^{t+1}_{\beta,x}(x) = S_{\beta,x}\left(S^t_{\beta,x}(x)\right) = (1-\beta)x + \beta S(S^t_{\beta,x}(x)) \geq (1-\beta)x + \beta S(R^t_{\beta,x}(x)) \tag{66}$$

$$\geq (1-\beta)x + \beta R(R^t_{\beta,x}(x)) = R_{\beta,x}\left(R^t_{\beta,x}(x)\right) = R^{t+1}_{\beta,x}(x), \tag{67}$$

proving the statement. By passing to the limit, $\tilde{S}_\beta(x) = \lim_t S^t_{\beta,x}(x) \geq \lim_t R^t_{\beta,x}(x) = \tilde{R}_\beta(x)$. If, instead, R were the one to be non-decreasing, it would then be enough to change in the inductive step $(1-\beta)x + \beta S(R^t_{\beta,x}(x))$ with $(1-\beta)x + \beta R(S^t_{\beta,x}(x))$, to reach the same conclusion. □

Before stating and proving the next lemma we recall here the notion of gamma convergence. We say that a sequence of functions $f_n : \Delta \to \mathbb{R} \cup \{\infty\}$ gamma converges to some function $f : \Delta \to \mathbb{R} \cup \{\infty\}$, written $f_n \to_\Gamma f$, if and only if[10]

1. For every sequence $w_n \to w \in \Delta$, we have

$$f(w) \leq \liminf_n f_n(w_n),$$

2. For every $w \in \Delta$, there exists a sequence $w_n \to w$ such that

$$f(w) \geq \limsup_n f_n(w_n).$$

Proposition 8. *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be such that, for every $i \in [n]$, there exists a lower semicontinuous function $c_i : \Delta \to [0, \infty]$ such that $\min_{w_i \in \Delta} c_i(w_i) = 0$ and*

$$S_i(x) = \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \{\langle w_i, x\rangle - R(w_i||\tilde{w}_i) + c_i(\tilde{w}_i)\} \qquad \forall x \in \mathbb{R}^n.$$

We have that

$$\lim_{t \to 0} \frac{S_i(tx)}{t} = \min_{\tilde{w}_i \in \Delta : c_i(\tilde{w}_i) = 0} \langle \tilde{w}_i, x\rangle \qquad \forall i \in [n], \forall x \in \mathbb{R}^n. \tag{68}$$

Proof. We first show that Equation (68) holds for all $x \in \mathbb{R}^n_+$. This immediately implies that the equation holds for all $x \in \mathbb{R}^n$ since both operators in the equations are translation invariant. Fix $i \in [n]$, $x \in \mathbb{R}^n_+$, a sequence $t_n \downarrow 0$, and define $f_n : \Delta \to \mathbb{R} \cup \{\infty\}$ and $f : \Delta \to \mathbb{R} \cup \{\infty\}$ as

$$f_n(\tilde{w}_i) = \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x\rangle - R(w_i||\tilde{w}_i)\} + \frac{1}{t_n} c_i(\tilde{w}_i)$$

[10] This is not exactly the classic definition of gamma convergence. But, in our setting, it is equivalent to it (see Proposition 8.1 in Dal Maso (1993)).

and

$$f(\tilde{w}_i) = \begin{cases} \langle \tilde{w}_i, x \rangle & \text{if} \quad c_i(\tilde{w}_i) = 0 \\ \infty & \text{else} \end{cases}.$$

We next show that $f_n \to_\Gamma f$. Fix $\tilde{w}_i \in \Delta$. There are two cases. First assume that $c_i(\tilde{w}_i) > 0$. For every sequence $\tilde{w}_{i,n} \to \tilde{w}_i$ we have

$$\begin{aligned} \infty &\geq \limsup_n f_n(\tilde{w}_{i,n}) \geq \liminf_n f_n(\tilde{w}_{i,n}) = \liminf_n \left\{ \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_{i,n})\} + \frac{1}{t_n} c_i(\tilde{w}_{i,n}) \right\} \\ &\geq \liminf_n \left\{ \frac{1}{t_n} c_i(\tilde{w}_{i,n}) \right\} = \infty = f(\tilde{w}_i), \end{aligned}$$

where the last inequality follows because $x \in \mathbb{R}^n_+$ and $\frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_{i,n})\} \geq \min_j x_j$ for all $n \in \mathbb{N}$. Second, assume that $c_i(\tilde{w}_i) = 0$. For every sequence $\tilde{w}_{i,n} \to \tilde{w}_i$ we have

$$\begin{aligned} \liminf_n f_n(\tilde{w}_{i,n}) &= \liminf_n \left\{ \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_{i,n})\} + \frac{1}{t_n} c_i(\tilde{w}_{i,n}) \right\} \\ &\geq \liminf_n \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_{i,n})\} \\ &\geq \liminf_n \left\{ \frac{1}{t_n} \langle \tilde{w}_{i,n}, t_n x \rangle \right\} = \liminf_n \langle \tilde{w}_{i,n}, x \rangle = \langle \tilde{w}_i, x \rangle = f(\tilde{w}_i), \end{aligned}$$

where the last inequality follows by substituting $w_i = \tilde{w}_{i,n}$ in the objective function of the maximization. Moreover, by letting $\tilde{w}_{i,n} = \tilde{w}_i$ for all n, we have

$$\begin{aligned} \limsup_n f_n(\tilde{w}_{i,n}) &= \limsup_n \left\{ \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_{i,n})\} + \frac{1}{t_n} c_i(\tilde{w}_{i,n}) \right\} \\ &= \limsup_n \left\{ \frac{1}{t_n} \max_{w_i \in \Delta} \{\langle w_i, t_n x \rangle - R(w_i || \tilde{w}_i)\} \right\} = \langle \tilde{w}_i, x \rangle = f(\tilde{w}_i), \end{aligned}$$

where the third equality follows from Theorem 18 in Maccheroni, Marinacci, and Rustichini (2006). With this, we have

$$\begin{aligned} \lim_{t \to 0} \frac{S_i(tx)}{t} &= \lim_n \frac{S_i(t_n p)}{t_n} = \lim_n \min_{\tilde{w}_i \in \Delta} f_n(\tilde{w}_i) = \min_{\tilde{w}_i \in \Delta} f(\tilde{w}_i) \\ &= \min_{\tilde{w}_i \in \Delta : c_i(\tilde{w}_i) = 0} \langle \tilde{w}_i, x \rangle, \end{aligned}$$

where the second equality follows by definition of f_n and the third equality follows from Theorem 7.4 in Dal Maso (1993) since Δ is compact and $f_n \to_\Gamma f$. Because i and x were arbitrarily chosen, the statement follows. □

Given a nonexpansive operator S define $S_t(x) = \frac{S(tx)}{t}$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_{++}$.

Lemma 4. *Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be normalized, non-decreasing, translation invariant, and $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$. Then $\{S_{t_n}\}_{n\in\mathbb{N}}$ is equicontinuous and $\{S_{t_n}(x) : n \in \mathbb{N}\}$ is bounded for all $n \in \mathbb{N}$. In particular, there exists a subsequence $\{S_{t_{n_m}}\}_{m\in\mathbb{N}} \subseteq \{S_{t_n}\}_{n\in\mathbb{N}}$ which converges uniformly over compacta.*

Proof. Consider a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$. By definition of S_t we have that

$$\begin{aligned}\|S_{t_n}(x) - S_{t_n}(y)\|_\infty &= \frac{\|S(t_n x) - S(t_n y)\|_\infty}{t_n} \\ &\leq \frac{\|t_n x - t_n y\|_\infty}{t_n} = \|x - y\|_\infty \quad \forall x, y \in \mathbb{R}^n, \forall n \in \mathbb{N}.\end{aligned}$$

Since S is normalized and non-decreasing,

$$\|S_{t_n}(x)\|_\infty \leq \|x\|_\infty \qquad \forall x \in \mathbb{R}^n, \forall n \in \mathbb{N}.$$

It follows that $\{S_{t_n}\}_{n\in\mathbb{N}}$ is equicontinuous and $\{S_{t_n}(x) : n \in \mathbb{N}\}$ is bounded for all $n \in \mathbb{N}$. By Arzela-Ascoli Theorem (see, e.g., Munkres, 2014, p. 293), we have that there exists a subsequence $\{S_{t_{n_m}}\}_{m\in\mathbb{N}} \subseteq \{S_{t_n}\}_{n\in\mathbb{N}}$ which converges uniformly over compacta. □

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. If f is normalized, non-decreasing, translation invariant, and concave, we have that

$$f(x) = \min_{w\in\Delta} \{\langle w, x\rangle + c_f(w)\} \qquad \forall x \in \mathbb{R}^n$$

where $c_f : \Delta \to [0,\infty]$ is lower semicontinuous, convex, and grounded. In particular, c_f is the Fenchel conjugate of f and is defined as

$$c_f(w) = \sup_{x\in\mathbb{R}^n} \{f(x) - \langle w, x\rangle\} \qquad \forall w \in \Delta.$$

Similarly, if f is normalized, non-decreasing, translation invariant, and convex, we have that

$$f(x) = \max_{w\in\Delta} \{\langle w, x\rangle - c_f(w)\} \qquad \forall x \in \mathbb{R}^n$$

where $c_f : \Delta \to [0,\infty]$ is lower semicontinuous, convex, and grounded. In particular, c_f is the Fenchel conjugate of f and is defined as

$$c_f(w) = -\inf_{x\in\mathbb{R}^n} \{f(x) - \langle w, x\rangle\} \qquad \forall w \in \Delta.$$

For every function $c : \Delta \to [0,\infty]$ that is lower semicontinuous, convex, and grounded, define $\operatorname{dom} c = \{w \in \Delta : c(w) < \infty\}$.

Lemma 5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be normalized, non-decreasing, and translation invariant. If f is concave, then*

$$\lim_{t\to\infty} \frac{f(tx)}{t} = \min_{w\in\overline{\operatorname{dom} c_f}} \langle x, w\rangle \qquad \forall x \in \mathbb{R}^n.$$

If f is convex, then

$$\lim_{t\to\infty} \frac{f(tx)}{t} = \max_{w\in\overline{\operatorname{dom} c_f}} \langle x, w\rangle \qquad \forall x \in \mathbb{R}^n.$$

Proof. In both cases, the result is standard and follows by combining Definition 3.1, Proposition 5.7, and Theorem 7.4 in Dal Maso (1993). □

Lemma 6. *Let $f : \mathbb{R}^n \to \mathbb{R}$.*

(i) If there exists a compact metric space Θ, a function $\kappa : \Theta \times \mathbb{R}^n \to \mathbb{R}$, and a function $\gamma : \Theta \to \mathbb{R}_+$ such that

$$f(x) = \max_{\theta\in\Theta} \{\kappa(\theta, x) - \gamma(\theta)\} \qquad \forall x \in \mathbb{R}^n,$$

where: i) for every $\theta \in \Theta$, $\kappa(\theta, \cdot)$ is normalized, non-decreasing, translation invariant, and concave; ii) for every $x \in \mathbb{R}^n$, $\kappa(\cdot, x)$ is continuous; iii) γ is lower semicontinuous with $\sup_{\theta\in\Theta} \gamma(\theta) < \infty$, then

$$\lim_{t\to\infty} \frac{f(tx)}{t} = \max_{\theta\in\Theta} \min_{w\in\overline{\operatorname{dom} c_{\kappa(\theta,\cdot)}}} \langle x, w\rangle \qquad \forall x \in \mathbb{R}^n.$$

(ii) If there exists a compact metric space Θ, a function $\kappa : \Theta \times \mathbb{R}^n \to \mathbb{R}$, and a function $\gamma : \Theta \to \mathbb{R}_+$ such that

$$f(x) = \min_{\theta\in\Theta} \{\kappa(\theta, x) + \gamma(\theta)\} \qquad \forall x \in \mathbb{R}^n,$$

where: i) for every $\theta \in \Theta$, $\kappa(\theta, \cdot)$ is normalized, non-decreasing, translation invariant, and convex; ii) for every $x \in \mathbb{R}^n$, $\kappa(\cdot, x)$ is continuous; iii) γ is lower semicontinuous with $\sup_{\theta\in\Theta} \gamma(\theta) < \infty$; then

$$\lim_{t\to\infty} \frac{f(tx)}{t} = \min_{\theta\in\Theta} \max_{w\in\overline{\operatorname{dom} c_{\kappa(\theta,\cdot)}}} \langle x, w\rangle \qquad \forall x \in \mathbb{R}^n.$$

Proof. We only prove point (i) since point (ii) follows by entirely symmetric arguments.

For each $t \in (0, \infty)$, define $\kappa_t : \Theta \times \mathbb{R}^n \to \mathbb{R}$ by $\kappa_t(\theta, x) = \kappa(\theta, tx)/t$ for all $x \in \mathbb{R}^n$ and for all $\theta \in \Theta$. We also define $\kappa_\infty : \Theta \times \mathbb{R}^n \to \mathbb{R}$ by $\kappa_\infty(\theta, x) = \lim_{t\to\infty} \kappa_t(\theta, x)$ for all $x \in \mathbb{R}^n$

and for all $\theta \in \Theta$. Since $\kappa : \Theta \times \mathbb{R}^n \to \mathbb{R}$ is concave with respect to the second component, we have that κ_∞ is well defined by Lemma 5. Fix $x \in \mathbb{R}^n$ and an increasing divergent sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$. For each $n \in \mathbb{N}$ define $J_n : \Theta \to \mathbb{R}$ and $J : \Theta \to \mathbb{R}$ by $J_n(\theta) = -\kappa_{t_n}(\theta, x)$ and $J(\theta) = -\kappa_\infty(\theta, x)$ for all $\theta \in \Theta$. Since $\theta \mapsto \kappa(x, \theta)$ is continuous for all $x \in \mathbb{R}^n$, we have that each J_n is continuous. By Proposition 5.4 and Remark 5.5 in Dal Maso (1993) and since $\kappa : \Theta \times \mathbb{R}^n \to \mathbb{R}$ is concave with respect to the second component, we have that $J_n(\theta) = -\kappa_{t_n}(\theta, x) \uparrow -\kappa_\infty(\theta, x) = J(\theta)$ for all $\theta \in \Theta$ and, in particular, $J_n \to_\Gamma J$ and J is lower semicontinuous. For each $n \in \mathbb{N}$ define $L_n : \Theta \to \mathbb{R}$ and $L : \Theta \to \mathbb{R}$ by $L_n(\theta) = \gamma(\theta)/t_n$ and $L(\theta) = 0$ for all $\theta \in \Theta$. By Proposition 5.2 in Dal Maso (1993) and since $\sup_{\theta\in\Theta} \gamma(\theta) < \infty$ and L is continuous, we have that L_n converges to L uniformly and, in particular, $L_n \to_\Gamma L$. By Remark 4.8 and Proposition 6.20 in Dal Maso (1993) and since each L_n is finite and L is finite and continuous, this implies that $(J_n + L_n) \to_\Gamma J + L$. By Theorem 7.4 in Dal Maso (1993) and since Θ is compact, we can conclude that

$$\begin{aligned}\lim_{n\to\infty} -\frac{f(t_n x)}{t_n} &= \lim_{n\to\infty}\left[-\max_{\theta\in\Theta}\left\{\frac{\kappa(\theta, t_n x)}{t_n} - \frac{\gamma(\theta)}{t_n}\right\}\right] = \lim_{n\to\infty}\min_{\theta\in\Theta}\left\{-\frac{\kappa(\theta, t_n x)}{t_n} + \frac{\gamma(\theta)}{t_n}\right\} \\ &= \lim_{n\to\infty}\min_{\theta\in\Theta}\{J_n(\theta) + L_n(\theta)\} = \min_{\theta\in\Theta}\{J(\theta) + L(\theta)\} = \min_{\theta\in\Theta} J(\theta) = \min_{\theta\in\Theta}\{-\kappa_\infty(\theta, x)\}.\end{aligned}$$

Since x and $\{t_n\}_{n\in\mathbb{N}} \subseteq (0,\infty)$ were arbitrarily chosen, this implies that

$$\lim_{t\to\infty}\frac{f(tx)}{t} = \max_{\theta\in\Theta}\kappa_\infty(\theta, x) = \max_{\theta\in\Theta}\min_{w\in\overline{\operatorname{dom} c_{\kappa(\theta,\cdot)}}}\langle x, w\rangle \qquad \forall x \in \mathbb{R}^n,$$

where the last equality follows from Lemma 5. □

B Proofs of the Results in the Main Text

Proof of Lemma 1. Observe that: (i) the normalization that $\max_{Q_i\in\Delta} G_i(Q_{i1}, \ldots, Q_{in}) = 1$ immediately implies that $K_i(e) = \min_{Q_i\in\mathbb{R}^n_+ : G_i(Q_i)\geq 1}\sum_{j=1}^n Q_{ij} \leq 1$ and (ii) by positive homogeneity of G if there existed $\hat{Q}_i \in \mathbb{R}^n_+$ with $\sum_{j=1}^n \hat{Q}_{ij} = \lambda < 1$ and $G_i(\hat{Q}_i) = 1$, we would also have $G_i(\frac{1}{\lambda}\hat{Q}_i) = 1/\lambda > 1$ and $\frac{1}{\lambda}\hat{Q}_i \in \Delta$, a contradiction with the normalization that $\max_{Q_i\in\Delta} G_i(Q_{i1}, \ldots, Q_{in}) = 1$. Thus, $K_i(e) = 1$. With this, the positive homogeneity of K readily implies that T is normalized and translation invariant. Since T is normalized, nondecreasing, and translation invariant, T is nonexpansive; see, e.g., Lemma 2 in Cerreia-Vioglio, Corrao, and Lanzani (2024). With this, the existence and uniqueness of the equilibrium prices follow by, e.g., Lemma 3 in Cerreia-Vioglio et al. (2025b). □

Proof of Proposition 1. Fix $P \in \mathbb{R}^n_{++}$. Fix also $i \in [n]$ and $\tilde{w}_i \in \Delta$ such that $G_i(\tilde{w}_i) \neq 0$. Define $Q_i = \frac{\tilde{w}_i}{G_i(\tilde{w}_i)} \in \mathbb{R}^n_+$. Since G_i is positive homogeneous, we have that $G_i(Q_i) = 1$. By definition

of Q_i, it follows that

$$\frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \sum_{j=1}^n Q_{ij} P_j \geq K_i(P). \tag{69}$$

Because $\tilde{w}_i$ was arbitrarily chosen, it follows that

$$K_i(P) \leq \min_{\tilde{w}_i \in \Delta : G_i(\tilde{w}_i) \neq 0} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} = \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}. \tag{70}$$

Next, fix Q_i such that $G_i(Q_i) \geq 1$ and define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i$. It follows that

$$\sum_{j=1}^n Q_{ij} P_j \geq \frac{\sum_{j=1}^n Q_{ij} P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij} P_j}{G_i(w_i)} \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)} \tag{71}$$

where the equality follows from positive homogeneity of G_i. Because Q_i was arbitrarily chosen, it follows that $K_i(P) \geq \min_{\tilde{w}_i \in \Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij} P_j}{G_i(\tilde{w}_i)}$, proving the first part of the statement.

By definition of T, and the first part of the statement, we have

$$T_i(p) = \min_{\tilde{w}_i \in \Delta} \ln \left(\sum_{j=1}^n \tilde{w}_{ij} \exp(p_j) \right) + \ln \left(\frac{1}{G_i(\tilde{w}_i)} \right). \tag{72}$$

Equation 34 then follows by, e.g., Proposition 1.4.2 in Dupuis and Ellis (2011).[11] □

Define the function $H : [0, \infty] \times \Delta \times \Delta \to [0, \infty]$ as

$$H(\lambda, w_i, \tilde{w}_i) = \begin{cases} \lambda \sum_{j=1}^k w_{ij} \log \left(\frac{w_{ij}}{\tilde{w}_{ij}} \right) & \text{if} \quad w_i \ll \tilde{w}_i \quad \text{and} \quad \lambda < \infty, \\ 0 & \text{if} \quad w_i = \tilde{w}_i \quad \text{and} \quad \lambda = \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{73}$$

Lemma 7. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be such that, for every $i \in [n]$, there exists $\lambda_i \in [0, \infty]$ and a lower semicontinuous function $c_i : \Delta \to [0, \infty]$ with*

$$T_i(x') = \min_{\tilde{w}_i \in \Delta} \max_{w_i \in \Delta} \{ \langle w_i, x' \rangle - H(\lambda_i, w_i, \tilde{w}_i) + c_i(\tilde{w}_i) \} \qquad \forall x' \in \mathbb{R}^n. \tag{74}$$

We have that

$$\Pi(x) = \min_{\tilde{W} \in \mathcal{W}} \max_{W \in \mathcal{W}} \left\{ \Omega_W \left(x - \tau^c(W || \tilde{W}) \right) \right\} \qquad \forall x \in \mathbb{R}^n \tag{75}$$

where $\tau_i^c(W||\tilde{W}) = \frac{\beta}{1-\beta} (H(\lambda_i, w_i, \tilde{w}_i) - c_i(\tilde{w}_i))$ for all $i \in [n]$.

[11] The steps of the proof of that result need to be adapted in the obvious way given that we have $\ln\left(\sum_{j=1}^n \tilde{w}_{ij} \exp(p_j)\right)$ in place of $-\ln\left(\sum_{j=1}^n \tilde{w}_{ij} \exp(-p_j)\right)$.

Proof. For every $i \in [n]$, fix $\lambda_i \in [0, \infty]$ and $c_i : \Delta \to [0, \infty]$ as in the statement. For all $\tilde{W}, W \in \mathcal{W}$, define $\alpha\left(W, \tilde{W}\right) \in \mathbb{R}^n$ as

$$\alpha\left(W, \tilde{W}\right)_i = -H\left(\lambda_i, w_i, \tilde{w}_i\right) + c_i\left(\tilde{w}_i\right) \qquad \forall i \in [n] \tag{76}$$

and the affine operator $T_{\tilde{W},W} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T_{\tilde{W},W}\left(x'\right) = Wx' + \alpha\left(W, \tilde{W}\right). \tag{77}$$

Also, for every $\tilde{W} \in \mathcal{W}$, define the operator $T^*_{\tilde{W}} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$T^*_{\tilde{W}}\left(x'\right) = \max_{W \in \mathcal{W}} T_{\tilde{W},W}\left(x'\right). \tag{78}$$

Observe that the operator in Equation (78) is well defined since

$$\left(\max_{W \in \mathcal{W}} T_{\tilde{W},W}\left(x'\right)\right)_i = \max_{w_i \in \Delta} \left\{\left\langle w_i, x'\right\rangle - H\left(\lambda_i, w_i, \tilde{w}_i\right) + c_i\left(\tilde{w}_i\right)\right\} \qquad \forall i \in [n], \forall x' \in \mathbb{R}^n. \tag{79}$$

Clearly, each $T^*_{\tilde{W}}$ is non-decreasing and translation invariant, hence nonexpansive. With this, we have

$$T\left(x'\right) = \min_{\tilde{W} \in \mathcal{W}} T^*_{\tilde{W}}\left(x'\right) \qquad \forall x' \in \mathbb{R}^n. \tag{80}$$

By Lemma 2, for all $\tilde{W}, W \in \mathcal{W}$, we have

$$\tilde{T}_{\tilde{W},W,\beta}\left(x\right) = \Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W, \tilde{W}\right)\right) \qquad \forall x \in \mathbb{R}^n. \tag{81}$$

Next, fix $\tilde{W} \in \mathcal{W}$ and observe that for all $W \in \mathcal{W}$ and $x' \in \mathbb{R}^n$, we have $T_{\tilde{W},W}\left(x'\right) \leq T^*_{\tilde{W}}\left(x'\right)$. By Lemma 3, we also have that

$$\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W, \tilde{W}\right)\right) = \tilde{T}_{\tilde{W},W,\beta}\left(x\right) \leq \tilde{T}^*_{\tilde{W},\beta}\left(x\right) \tag{82}$$

for all $W \in \mathcal{W}$ and $x \in \mathbb{R}^n$. By Equation (78), for all $x \in \mathbb{R}^n$, there exists $\hat{W} \in \mathcal{W}$ such that

$$\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \left(1-\beta\right)x + \beta T^*_{\tilde{W}}\left(\tilde{T}^*_{\tilde{W},\beta}\left(x\right)\right) = \left(1-\beta\right)x + \beta T_{\tilde{W},\hat{W}}\left(\tilde{T}^*_{\tilde{W},\beta}\left(x\right)\right) \tag{83}$$

yielding that $\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \tilde{T}_{\tilde{W},\hat{W},\beta}\left(x\right)$, hence that

$$\tilde{T}^*_{\tilde{W},\beta}\left(x\right) = \max_{W \in \mathcal{W}}\left\{\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W, \tilde{W}\right)\right)\right\}. \tag{84}$$

Because $\tilde{W} \in \mathcal{W}$ was arbitrarily chosen, it follows that Equation 84 holds for all $\tilde{W} \in \mathcal{W}$.

Next, fix $x \in \mathbb{R}^n$. Because $T \leq T^*_{\tilde{W}}$ for all $\tilde{W} \in \mathcal{W}$, Lemma 3 implies that

$$\Pi(x) \leq \tilde{T}^*_{\tilde{W},\beta}(x) \tag{85}$$

for all $\tilde{W} \in \mathcal{W}$. By (80), there exists $\tilde{W}^* \in \mathcal{W}$ such that

$$\Pi(x) = (1-\beta)x + \beta T(\Pi(x)) = (1-\beta)x + \beta T^*_{\tilde{W}^*}(\Pi(x)) \tag{86}$$

yielding that $\Pi(x) = \tilde{T}^*_{\tilde{W}^*,\beta}(x)$. This in turn yields that

$$\min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}(x) \leq \tilde{T}^*_{\tilde{W}^*,\beta}(x) = \Pi(x) \leq \min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}(x), \tag{87}$$

hence that

$$\Pi(x) = \min_{\tilde{W}\in\mathcal{W}} \tilde{T}^*_{\tilde{W},\beta}(x) = \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{\Omega_W\left(x + \frac{\beta}{1-\beta}\alpha\left(W,\tilde{W}\right)\right)\right\} \tag{88}$$

$$= \min_{\tilde{W}\in\mathcal{W}} \max_{W\in\mathcal{W}} \left\{\Omega_W\left(x - \tau^c\left(W||\tilde{W}\right)\right)\right\}, \tag{89}$$

as desired. □

Proof of Theorem 1. By Proposition 1, observe that for every $i \in [n]$, T_i can be written as in Equation 74 with $\lambda_i = 1$ and $c_i(\tilde{w}_i) = \ln\left(\frac{1}{G_i(\tilde{w})}\right)$ for all $\tilde{w} \in \Delta$, which is clearly lower semicontinuous. Therefore, the result immediately follows by Lemma 7 and Equation 23. □

Proof of Corollary 3. By Proposition 1, we have that $T' \leq T$, and that they are both nonexpansive and non-decreasing. Therefore, by Lemma 3 we obtain $\Pi'(x) \leq \Pi(x)$. By Equation 23, it follows that $\Upsilon'(x) \geq \Upsilon(x)$. □

Proof of Proposition 2. Fix $i \in [n]$ and a log-cost operator T_i.

(i) implies (ii). Because sector i is resilient and T_i is a log-cost operator, by Proposition 1 we have that T_i is normalized, non-decreasing, translation invariant, and concave. By Lemma 26 in Maccheroni et al. (2006), it follows that

$$T_i(p) = \min_{w_i\in\Delta} \{\langle w_i, p\rangle + d_i(w_i)\} \qquad \forall p \in \mathbb{R}^n, \tag{90}$$

where $d_i : \Delta \to [0,\infty]$ is lower semicontinuous, convex, and such that $\min_{w_i\in\Delta} d_i(w_i) = 0$. In particular, if T_i is positive homogeneous $d_i(w_i) \in \{0,\infty\}$ for all $w_i \in \Delta$. With this, we have

$$K_i(P) = \exp(T_i(\ln(P))) = \min_{w_i\in\Delta} \left\{\frac{\prod_{j=1}^n P_j^{w_{ij}}}{\exp(-d_i(w_i))}\right\} \qquad \forall P \in \mathbb{R}^n_{++}. \tag{91}$$

Next, define

$$G_i^*(Q_i) = \max_{w_i \in \Delta} \left\{ \exp\left(-d_i(w_i)\right) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \qquad \forall Q_i \in \mathbb{R}_+^n. \tag{92}$$

Observe that G_i is non-decreasing and such that

$$\max_{\tilde{w}_i \in \Delta} G_i^*(\tilde{w}_i) = \max_{w_i \in \Delta} \left\{ \exp\left(-d_i(w_i)\right) \max_{\tilde{w}_i \in \Delta} \prod_{j=1}^{n} \left(\frac{\tilde{w}_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} = 1. \tag{93}$$

Moreover, since the objective function in the maximization of Equation 92 is the product of two upper semicontinuous functions, it is upper semicontinuous in (Q_i, w_i). In turn, Lemma 17.30 in Aliprantis and Border (2006) implies that G_i is upper semicontinuous, and Proposition 5.94 in Moore (1999) implies that it is, in fact, continuous.

By letting $C_i = \{w_i \in \Delta : d_i(w_i) < \infty\}$ and $A_i(w_i) = \exp\left(-d_i(w_i)\right)$ for all $w_i \in C_i$, we obtain

$$G_i^*(Q_i) = \max_{w_i \in C_i} \left\{ A_i(w_i) \prod_{j=1}^{n} \left(\frac{Q_{ij}}{w_{ij}} \right)^{w_{ij}} \right\} \qquad \forall Q_i \in \mathbb{R}_+^n. \tag{94}$$

Observe that, given the properties of d_i, we have that C_i is nonempty and convex A_i is upper coercive, log-concave, and such that $\max_{w_i \in C_i} A_i(w_i) = 1$. Moreover, if T_i is positive homogeneous, and thus $d(w_i) \in \{0, \infty\}$ for all $w_i \in \Delta$, A_i is equal to 1 on its domain. Next, it is easy to see that, for all $P \in \mathbb{R}_{++}^n$,

$$\min_{Q_i \in \mathbb{R}_+^n : G_i^*(Q_i) \geq 1} \sum_{j=1}^{n} Q_{ij} P_j = \min_{w_i \in C_i} \left\{ \frac{\prod_{j=1}^{n} P_j^{w_{ij}}}{A_i(w_i)} \right\} = K_i(P), \tag{95}$$

yielding that T_i is generated by G_i^*.

(ii) implies (i) The cost operator generated by G_i is

$$K_i(P) = \min_{w_i \in C_i} \frac{\prod_{j=1}^{n} P_j^{w_{ij}}}{A_i(w_i)} \qquad \forall P \in \mathbb{R}_{++}^n. \tag{96}$$

In turn, this yields that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln\left(\frac{1}{A_i(w_i)} \right) \right\} \qquad \forall p \in \mathbb{R}^n, \tag{97}$$

which is a concave function, as desired. Moreover, if $A_i(w_i)$ is identically equal to 1 on its domain, $\ln\left(\frac{1}{A_i(w_i)}\right) \in \{0, \infty\}$ so T_i is positive homogeneous. □

Proof of Corollary 1. For every sector $i \in [n]$, Proposition 2 and Equation 97 give that

$$T_i(p) = \min_{w_i \in C_i} \left\{ \langle w_i, p \rangle + \ln\left(\frac{1}{A_i(w_i)}\right) \right\} \qquad \forall p \in \mathbb{R}^n. \tag{98}$$

By Equation 98, observe that for every $i \in [n]$, T_i can be written as in Equation 74 with $\lambda_i = \infty$ and

$$c_i(\tilde{w}_i) = \begin{cases} \ln\left(\frac{1}{A_i(\tilde{w}_i)}\right) & \text{if} \quad w \in C_i, \\ \infty & \text{otherwise.} \end{cases} \tag{99}$$

which is clearly lower semicontinuous. Therefore, the result immediately follows by Lemma 7. □

Lemma 8. *Suppose that T_i is generated by a production function G_i. If T_i is convex then there exists a grounded, lower semicontinuous, and convex $d : \Delta \to [0, \infty]$ such that*

$$T_i(x) = \max_{w_i \in \Delta} \{ \langle w_i, x \rangle - d(w_i) \} \qquad \forall x \in \mathbb{R}^n \tag{100}$$

and

$$d(w_i) = \sup_{\tilde{w}_i : G_i(\tilde{w}_i) > 0} \left\{ R(w_i || \tilde{w}_i) - \ln\left(\frac{1}{G_i(\tilde{w}_i)}\right) \right\} \qquad \forall w_i \in \Delta. \tag{101}$$

Proof. Define $\bar{T}_i : \mathbb{R}^n \to \mathbb{R}$ by $\bar{T}_i(x) = -T_i(-x)$ for all $x \in \mathbb{R}^n$. By Proposition 1, we have that $\bar{T}_i$ is normalized, non-decreasing, translation invariant, concave, and such that

$$\bar{T}_i(x) = \max_{\tilde{w}_i \in \Delta} \left\{ -\ln\left(\sum_{j=1}^{n} e^{-x_j} \tilde{w}_{ij}\right) - \ln\left(\frac{1}{G_i(\tilde{w}_i)}\right) \right\} \qquad \forall x \in \mathbb{R}^n. \tag{102}$$

By Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2014) and since $\bar{g}$ is normalized, non-decreasing, translation invariant, and concave, we have that $d : \Delta \to [0, \infty]$, defined by $d(w_i) = \sup_{x \in \mathbb{R}^n} \left\{ \bar{T}_i(x) - \langle w_i, x \rangle \right\}$ for all $w_i \in \Delta$, is such that

$$\bar{T}_i(x) = \min_{\tilde{w}_i \in \Delta} \{ \langle \tilde{w}_i, x \rangle + d(\tilde{w}_i) \} \qquad \forall x \in \mathbb{R}^n. \tag{103}$$

By definition of d and since $\bar{T}_i$ is normalized, we have that d is grounded, lower semicontinuous,

and convex. By definition of d and Equation 102, we also have that

$$d(w_i) = \sup_{x\in\mathbb{R}^n}\left\{\max_{\tilde{w}_i\in\Delta}\left\{-\ln\left(\sum_{i=1}^n e^{-x_i}\tilde{w}_{ij}\right)-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)\right\}-\langle w_i,x\rangle\right\} \tag{104}$$

$$= \sup_{x\in\mathbb{R}^n}\sup_{\tilde{w}_i\in\Delta}\left\{-\ln\left(\sum_{i=1}^n e^{-x_i}\tilde{w}_{ij}\right)-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)-\langle w_i,x\rangle\right\} \tag{105}$$

$$= \sup_{x\in\mathbb{R}^n}\sup_{\tilde{w}_i:G_i(\tilde{w}_{ij})>0}\left\{-\ln\left(\sum_{i=1}^n e^{-x_i}\tilde{w}_{ij}\right)-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)-\langle w,x\rangle\right\} \tag{106}$$

$$= \sup_{\tilde{w}_i:G_i(\tilde{w}_i)>0}\sup_{x\in\mathbb{R}^n}\left\{-\ln\left(\sum_{i=1}^n e^{-x_i}\tilde{w}_i\right)-\langle w_i,x\rangle-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)\right\} \tag{107}$$

$$= \sup_{\tilde{w}_i:G_i(\tilde{w}_i)>0}\left\{\sup_{x\in\mathbb{R}^n}\left\{-\ln\left(\sum_{i=1}^n e^{-x_i}\tilde{w}_i\right)-\langle w_i,x\rangle\right\}-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)\right\} \tag{108}$$

$$= \sup_{\tilde{w}_i:G_i(\tilde{w}_i)>0}\left\{R(w_i||\tilde{w}_i)-\ln\left(\frac{1}{G_i(\tilde{w}_i)}\right)\right\} \quad \forall w_i\in\Delta. \tag{109}$$

Since $T_i(x) = -\bar{T}_i(-x)$ for all $x\in\mathbb{R}^n$, the statement follows. □

Lemma 9. *Suppose that T_i and K_i are generated by a production function G_i. If T_i is convex, then*

$$K_i(y) = \max_{\tilde{w}_i\in\Delta}\frac{\Pi_{j=1}^n y_j^{\tilde{w}_{ij}}}{e^{d(\tilde{w}_i)}} \qquad \forall y\in\mathbb{R}^n_{++} \tag{110}$$

where $d:\Delta\to[0,\infty]$ is defined as in Equation 101.

Proof. Observe that $e^{\sum_{j=1}^n \tilde{w}_{ij}\ln y_j} = \Pi_{j=1}^n e^{\tilde{w}_{ij}\ln y_j} = \Pi_{j=1}^n y_j^{\tilde{w}_{ij}}$ for all $\tilde{w}_i\in\Delta$ and for all $y\in\mathbb{R}^n_{++}$. By Equation 100, this implies that

$$K_i(y) = \exp\left(\max_{\tilde{w}_i\in\Delta}\{\langle\tilde{w}_i,\ln(y)\rangle - d(\tilde{w}_i)\}\right) = \max_{\tilde{w}_i\in\Delta} e^{\langle\tilde{w}_i,\ln(y)\rangle - d(\tilde{w}_i)}$$
$$= \max_{\tilde{w}_i\in\Delta}\frac{e^{\langle\tilde{w}_i,\ln(y)\rangle}}{e^{d(\tilde{w}_i)}} = \max_{\tilde{w}_i\in\Delta}\frac{\Pi_{j=1}^n y_j^{\tilde{w}_{ij}}}{e^{d(\tilde{w}_i)}} \quad \forall y\in\mathbb{R}^n_{++},$$

proving the statement. □

Lemma 10. *If $K_i:\mathbb{R}^n_{++}\to\mathbb{R}_{++}$ is non-decreasing, continuous, positive homogeneous, and such that $K_i(e)=1$, then there exists an upper semicontinuous and concave function $\Gamma:\Delta\to[0,\infty]$ such that*

$$K_i(y) = \min_{\tilde{w}\in\Delta}\frac{\langle\tilde{w}_i,y\rangle}{\Gamma(\tilde{w}_i)} \qquad \forall y\in\mathbb{R}^n_{++} \tag{111}$$

and $\max_{\tilde{w}_i\in\Delta}\Gamma(\tilde{w}_i)=1$. Moreover, $\Gamma:\Delta\to[0,\infty]$ is the unique upper semicontinuous and concave function such that $\max_{\tilde{w}_i\in\Delta}\Gamma(\tilde{w}_i)=1$ satisfying (111).

Proof. Call $\bar{K}_i$ the extension of K_i to $\mathbb{R}^n$ such that $\bar{K}_i(y) = 0$ for all $y \notin \mathbb{R}^n_{++}$. Since $K_i > 0$ and K_i is non-decreasing and positive homogeneous, it is immediate to see that $\bar{K}_i$ is non-decreasing and positive homogeneous. Since $\mathbb{R}^n_{++}$ is open and K_i is continuous on $\mathbb{R}^n_{++}$ and $K_i > 0$, we have that

$$\left\{\bar{K}_i > t\right\} = \left\{y \in \mathbb{R}^n_{++} : K_i(y) > t\right\} \qquad \forall t \geq 0,$$

it follows that the latter set is relatively open in $\mathbb{R}^n_{++}$ and so the latter and the former are open. Moreover, since $\bar{K}_i \geq 0$, we have that $\left\{\bar{K}_i > t\right\} = \mathbb{R}^n$ for all $t < 0$, proving that $\bar{K}_i$ is lower semicontinuous. Since K_i is quasiconcave on $\mathbb{R}^n_{++}$ and

$$\left\{\bar{K}_i \geq t\right\} = \left\{y \in \mathbb{R}^n_{++} : K_i(y) \geq t\right\} \qquad \forall t > 0,$$

it follows that the former set is convex. Moreover, since $\bar{K}_i \geq 0$, we have that $\left\{\bar{K}_i \geq t\right\} = \mathbb{R}^n$ for all $t \leq 0$, proving that $\bar{K}_i$ is quasiconcave. By (Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011, Theorem 8.1 and Proposition 8.1) and its proof and since $\bar{K}_i$ is non-decreasing, lower semicontinuous, positive homogeneous, and such that $\bar{K}_i(e) = K_i(e) = 1$, we have that

$$K_i(y) = \min_{\tilde{w}_i \in \tilde{\Delta}} \left\{ \frac{\langle \tilde{w}_i, y\rangle^+}{c_1(\tilde{w}_i)} - \frac{\langle \tilde{w}_i, y\rangle^-}{c_2(\tilde{w}_i)} \right\} \qquad \forall y \in \mathbb{R}^n_+ \tag{112}$$

where $\tilde{\Delta}$ is a nonempty closed and convex subset of Δ, $c_1 : \tilde{\Delta} \to [0, \infty)$ is concave and upper semicontinuous, $c_2 : \tilde{\Delta} \to (0, \infty]$ is convex and lower semicontinuous. Moreover, $\tilde{\Delta} = \{\tilde{w}_i \in \Delta : \operatorname{dom} H(\cdot, \tilde{w}_i) \neq \emptyset\}$ where $\bar{H} : \mathbb{R} \times \Delta \to (-\infty, \infty]$ is defined by

$$\bar{H}(t, w_i) = \sup\left\{\bar{K}_i(y) : \langle w_i, y\rangle \leq t\right\} \qquad \forall (t, w_i) \in \mathbb{R} \times \Delta$$

and c_1 and c_2 are unique with $\bar{H}(1, w_i) = \frac{1}{c_1(w_i)}$ and $-\bar{H}(-1, w_i) = \frac{1}{c_2(w_i)}$ for all $w_i \in \tilde{\Delta}$. Since $\bar{K}_i(y) = 0$ for all $y \notin \mathbb{R}^n_{++}$, it is immediate to see that $\bar{H}(t, w_i) = 0$ for all $w_i \in \Delta$ and for all $t \leq 0$, that is, $(-\infty, 0] \in \operatorname{dom} \bar{H}(\cdot, w_i)$ for all $w_i \in \Delta$, proving that $\tilde{\Delta} = \Delta$ and $c_2 = \infty$. By (112) and since $\bar{K}_i(e) = 1$, we have that $1 = \min_{\tilde{w}_i \in \Delta} \frac{\langle \tilde{w}_i, e\rangle}{c_1(\tilde{w}_i)} = \frac{1}{\max_{\tilde{w}_i \in \Delta} c_1(\tilde{w}_i)}$, that is, $\max_{\tilde{w}_i \in \Delta} c_1(\tilde{w}_i) = 1$. By restricting (112) to $\mathbb{R}^n_{++}$, (111) follows with c_1 in place of Γ. Next, consider $\tilde{\Gamma} : \Delta \to [0, \infty]$ such that $\tilde{\Gamma}$ is upper semicontinuous, concave, and such that $\max_{\tilde{w}_i \in \Delta} \tilde{\Gamma}(\tilde{w}_i) = 1$ and $\tilde{\Gamma}$ satisfies (111). By (Cerreia-Vioglio et al., 2011, Proposition 8.1) and since $c_2 = \infty$ and $\bar{K}_i(y) = 0$ for all $y \notin \mathbb{R}^n_{++}$, this implies that

$$\bar{K}_i(y) = \min_{\tilde{w}_i \in \Delta} \left\{ \frac{\langle \tilde{w}_i, y\rangle^+}{\tilde{\Gamma}(\tilde{w}_i)} - \frac{\langle \tilde{w}_i, y\rangle^-}{c_2(\tilde{w}_i)} \right\} \quad \forall y \in \mathbb{R}^n,$$

proving that $c_1 = \tilde{\Gamma}$ and uniqueness. □

Proof of Proposition 4. 1. It follows by Theorem 14 in Marinacci and Montrucchio (2008).

2. Immediate. □

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be normalized, non-decreasing, and translation invariant. Define $\Pi^1 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Pi^1(x) = (1-\beta)\, x + \beta T(x) \qquad \forall x \in \mathbb{R}^n. \tag{113}$$

For all integers $t > 1$ define

$$\Pi^t(x) = (1-\beta)\, x + \beta T\left(\Pi^{t-1}(x)\right) \qquad \forall x \in \mathbb{R}^n. \tag{114}$$

Lemma 11. *For all $x \in \mathbb{R}^n$, we have $\lim_t \Pi^t(x) = \Pi(x)$.*

Proof. Fix $x \in \mathbb{R}^n$. Define $\hat{T}_{\beta,x} : \mathbb{R}^n \to \mathbb{R}^n$ as $\hat{T}_{\beta,x}(y) = (1-\beta)\, x + \beta T(y)$. Observe that $\Pi(x) = y^*$ where the latter is the unique vector solving the fixed-point equation $y^* = \hat{T}_{\beta,x}(y^*)$. Moreover, by the Banach contraction principle, for all $y \in \mathbb{R}^n$, we have $\Pi(x) = \lim_t \left(\hat{T}_{\beta,x}\right)^t (y)$. Next, fix $y = x$. We next show by induction that $\left(\hat{T}_{\beta,x}\right)^t (y) = \Pi^t(x)$ for all $t \in \mathbb{N}$. First, observe that $\hat{T}_{\beta,x}(y) = (1-\beta)\, x + \beta T(y) = (1-\beta)\, x + \beta T(x) = \Pi^1(x)$. Assume by way of induction that $\left(\hat{T}_{\beta,x}\right)^\tau (y) = \Pi^\tau(x)$ for all $\tau \leq t-1$. We have that

$$\begin{aligned} \left(\hat{T}_{\beta,x}\right)^t (y) &= \hat{T}_{\beta,x}\left(\left(\hat{T}_{\beta,x}\right)^{t-1}(y)\right) = \hat{T}_{\beta,x}\left(\Pi^{t-1}(x)\right) && (115)\\ &= (1-\beta)\, x + \beta T\left(\Pi^{t-1}(x)\right) = \Pi^t(x), && (116) \end{aligned}$$

as desired. It then follows by induction that $\left(\hat{T}_{\beta,x}\right)^t (y) = \Pi^t(x)$ for all $t \in \mathbb{N}$. Therefore, $\Pi(x) = \lim_t \left(\hat{T}_{\beta,x}\right)^t (y) = \lim_t \Pi^t(x)$. Because x was arbitrarily chosen, the statement follows. □

Lemma 12. *Suppose the economy is resilient (fragile) to large shocks. For all $t \in \mathbb{N}$, $\lambda \in [0,1]$, $x \in \mathbb{R}^n$, $\Pi^t(\lambda x) \geq \lambda \Pi^t(x)$ ($\Pi^t(\lambda x) \leq \lambda \Pi^t(x)$).*

Proof. We prove the case where T is resilient to big shocks by induction. The case of fragility to large shocks is analogous. First, recall that Π^1 is defined as

$$\Pi^1(x) = (1-\beta)\, x + \beta T(x) \qquad \forall x \in \mathbb{R}^n, \tag{117}$$

which clearly satisfies $\Pi^1(\lambda x) \geq \lambda \Pi^1(x)$. Next assume by way of induction that $\Pi^\tau(\lambda x) \geq$

$\lambda\Pi^\tau(x)$ for all $\tau \leq t-1$. Next, fix $x \in \mathbb{R}^n$ and $\lambda \in [0,1]$, and observe that

$$\begin{aligned}
\Pi^t(\lambda x) &= (1-\beta)\lambda x + \beta T\left(\Pi^{t-1}(\lambda x)\right) = \lambda(1-\beta)x + \beta T\left(\Pi^{t-1}(\lambda x)\right) && (118)\\
&\geq \lambda(1-\beta)x + \beta T\left(\lambda\Pi^{t-1}(x)\right) \geq \lambda(1-\beta)x + \lambda\beta T\left(\Pi^{t-1}(x)\right) && (119)\\
&= \lambda\left((1-\beta)x + \beta T\left(\Pi^{t-1}(x)\right)\right) = \lambda\Pi^t(x), && (120)
\end{aligned}$$

where the first inequality follows from the inductive hypothesis and monotonicity of T and the second by the assumption on T. By induction, $\Pi^t(\lambda x) \geq \lambda\Pi^t(x)$ for all $t \in \mathbb{N}$ as desired. □

We are going to say that $g : \mathbb{R}^n \to \mathbb{R}$ is *regular at* x if and only if

$$\lim_{h\downarrow 0} \frac{g(x+hd)-g(x)}{h} = \liminf_{x'\to x, h\downarrow 0} \frac{g(x'+hd)-g(x')}{h}.$$

We say that g is *regular* if and only if it is regular at each point of its domain. Define $M : \Delta \times \mathbb{R}^n \to \mathbb{R}$ by $M(\alpha, x) = \ln\left(\sum_{i=1}^n e^{x_i}\alpha_i\right)$ for all $\alpha \in \Delta$ and for all $x \in \mathbb{R}^n$. We define $\mathcal{M} : \mathcal{W} \times \mathbb{R}^n \to \mathbb{R}^n$ by $\mathcal{M}_i(W, x) = M(w_i, x)$ for all $W \in \mathcal{W}$, for all $x \in \mathbb{R}^n$, and for all $i \in [n]$. For each $g : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ define by $\sigma_g(x)$ the subset of αs in Δ such that $g(x) = M(\alpha, x) + c(\alpha)$. Note that if T_i is generated by a production function G_i, then it easily follows by Proposition 1 that $\sigma_{T_i}(x)$ is nonempty and compact for all $x \in \mathbb{R}^n$.

Remark 1. *We list here some notation and make four observations. Given an operator $S : \mathbb{R}^n \to \mathbb{R}^n$, we denote the Jacobian of S at x by $J_S(x)$. If S is normalized, non-decreasing, and translation invariant and admits Jacobian at x, then $J_S(x)$ is a stochastic matrix. We endow the space of $n \times n$ matrices with the norm $\|\ \|_* : M_n(\mathbb{R}) \to [0,\infty)$ defined by*

$$\|W\|_* = \max_{i\in[n]} \sum_{j=1}^n |w_{ij}| \qquad \forall W \in M_n(\mathbb{R}). \tag{121}$$

This is the operator norm when W is viewed as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathbb{R}^n$ is endowed with the supnorm $\|\ \|_\infty$. Observe that if W is a stochastic matrix, then $\|W\|_* = 1$. First, observe that when W and $\hat{W}$ are two stochastic matrices,[12] we have that

$$\left\|W^t - \hat{W}^t\right\|_* \leq t\left\|W - \hat{W}\right\|_* \qquad \forall t \in \mathbb{N}_0. \tag{122}$$

Second, consider the quantity $\kappa = (1-\beta)\sum_{t=0}^\infty \beta^t t$. Note that κ is well defined and finite. It is well defined because the partial sums are positive and increasing. Moreover, set $\gamma = \frac{1}{2}\frac{1}{\beta} + \frac{1}{2}$. Since $\beta \in (0,1)$, note that $\gamma \in (1, 1/\beta)$ and $\beta\gamma \in (0,1)$. It follows that eventually $\gamma^t \geq t$ and, in particular, eventually $(\beta\gamma)^t \geq \beta^t t$. This implies that κ is finite. Third, given $\beta \in (0,1)$,

[12] See footnote 32 of Cerreia-Vioglio et al. (2025b).

define $B : \mathcal{W} \to \mathcal{W}$ *by* $B(W) = \Omega_W$ *for all* $W \in \mathcal{W}$. *Note that*

$$\left\| B(W) - B\left(\hat{W}\right) \right\|_* = \left\| \Omega_W - \hat{W}_\beta \right\|_* = \left\| (1-\beta) \sum_{t=0}^{\infty} \beta^t \left(W^t - \hat{W}^t \right) \right\|_* \tag{123}$$

$$\leq (1-\beta) \sum_{t=0}^{\infty} \beta^t \left\| W^t - \hat{W}^t \right\|_* \leq (1-\beta) \sum_{t=0}^{\infty} \beta^t t \left\| W - \hat{W} \right\|_* \tag{124}$$

$$\leq \kappa \left\| W - \hat{W} \right\|_* \qquad \forall W, \hat{W} \in \mathcal{W}. \tag{125}$$

This proves that B *is a Lipschitz continuous operator. Fourth,* B *is injective. By Horn and Johnson (2012), Theorem 8.3.4 and Problem 5.6.P26, observe that for each* $W \in \mathcal{W}$ *the spectral radius of* βW *is* $\beta < 1$ *and, in particular,* $(I - \beta W)^{-1} = \sum_{t=0}^{\infty} \beta^t W^t$. *We can conclude that* $B(W) = B\left(\hat{W}\right)$ *implies that* $(I - \beta W)^{-1} = \left(I - \beta \hat{W}\right)^{-1}$, *yielding that* $I - \beta \hat{W} = I - \beta W$, *that is,* $W = \hat{W}$, *proving injectivity.*

Lemma 13. *Let* $i \in [n]$. *If* T_i *is generated by a production function* G_i *then* T_i *is regular.*

Proof. Fix $x \in \mathbb{R}^n$. Since K_i is concave and locally Lipschitz, we have that $-K_i$ is locally Lipschitz and regular a la Clarke at $\exp(x)$ (see Clarke, 1990, Definition 2.3.4 and Proposition 2.3.6). By Clarke (1990), p. 32 and Theorem 2.3.10, and since exp is continuously differentiable, it follows that $-K_i \circ \exp$ is regular a la Clarke at x. Define $\gamma : \mathbb{R}_{--} \to \mathbb{R}_{++}$ by $\gamma(s) = -\ln(-s)$ for all $s \in \mathbb{R}_{--}$. By Clarke (1990), p. 32 and Theorem 2.3.10, and since γ is continuously differentiable, we have that γ is regular a la Clarke. By Clarke (1990), point (i) of Theorem 2.3.9, and since γ is non-decreasing and $-K_i \circ \exp$ is regular a la Clarke at x, this implies that $\gamma \circ -K_i \circ \exp$ is regular a la Clarke at x. It is immediate to see that $g = -\gamma \circ -K_i \circ \exp$, proving that T_i is regular in our sense. □

Lemma 14. *Let* $i \in [n]$. *If* T_i *is generated by a production function* G_i *then for each* $x, d \in \mathbb{R}^n$

$$\partial_C T_i(x) = \operatorname{co}\{\nabla M(w_i, x) : w_i \in \sigma_{T_i}(x)\} \text{ and } \lim_{h \downarrow 0} \frac{T_i(x + hd) - T_i(x)}{h} = \min_{w_i \in \sigma_{T_i}(x)} \langle \nabla M(w_i, x), d \rangle. \tag{126}$$

Proof. The result follows from Clarke (2013), Theorem 10.22, whose assumptions are satisfied by Lemma 13. □

Corollary 6. *Fix* $i \in [n]$. *If* T_i *is generated by a production function* G_i, *then for each* $y, z \in \mathbb{R}^n$ *there exist* $\lambda_{y,z} \in [0,1]$ *and* $w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1 - \lambda_{y,z}) y)$ *such that*

$$T_i(z) - T_i(y) \geq \langle \nabla M(w_i, \lambda_{y,z} z + (1 - \lambda_{y,z}) y), z - y \rangle. \tag{127}$$

Proof. Fix $y, z \in \mathbb{R}^n$. By Lebourg's Mean Value Theorem (see, e.g., Clarke, 1990, Theorem 2.3.7) and since T_i is Lipschitz continuous, there exists $\lambda_{y,z} \in [0,1]$ such that

$$T_i(z) - T_i(y) \geq \min_{w_i \in \partial_C T_i(\lambda_{y,z} z + (1-\lambda_{y,z}) y)} \langle w_i, z - y \rangle . \tag{128}$$

By Lemma 14, we have that

$$T_i(z) - T_i(y) \geq \min_{w_i \in \partial_C T_i(\lambda_{y,z} z + (1-\lambda_{y,z}) y)} \langle w_i, z - y \rangle = \min_{w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1-\lambda_{y,z}) y)} \langle \nabla M(w_i, \lambda_{y,z} z + (1-\lambda_{y,z}) y), z - y \rangle . \tag{129}$$

By choosing $w_i \in \sigma_{T_i}(\lambda_{y,z} z + (1-\lambda_{y,z}) y)$ which achieves the min above, the statement follows. □

Lemma 15. *Let $W \in \mathcal{W}$, $\beta \in (0,1)$, and $d \in \mathbb{R}^n$. If $z \in \mathbb{R}^n$ is such that $z \leq (1-\beta) d + \beta W z$ (resp. $\geq$), then $z \leq \Omega_W d$ (resp. $\geq$).*

Proof. We prove by induction that

$$z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \text{ (resp. } \geq) \quad \forall \tau \in \mathbb{N}_0. \tag{130}$$

Initial Step. $\tau = 0$. By assumption, we have that

$$z \leq (1-\beta) d + \beta W z = (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \text{ (resp. } \geq), \tag{131}$$

proving the initial step.

Inductive Step. We assume the inequality is true for τ and prove it holds for $\tau + 1$. By assumption, we have that

$$z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} z \leq (1-\beta) \sum_{t=0}^{\tau} \beta^t W^t d + \beta^{\tau+1} W^{\tau+1} ((1-\beta) d + \beta W z) \tag{132}$$

$$= (1-\beta) \sum_{t=0}^{\tau+1} \beta^t W^t d + \beta^{\tau+2} W^{\tau+2} z, \tag{133}$$

proving the inductive step.

The inequality (130) follows by induction. By passing to the limit in τ and since W is a stochastic matrix, we have that $z \leq \Omega_W d$ (resp. $\geq$). □

Given $x, d \in \mathbb{R}^n$ and $h > 0$, define $q_{h,x,d} = \frac{\Pi(x+hd) - \Pi(x)}{h}$. Given $x, d \in \mathbb{R}^n$, since Π is Lipschitz continuous of order 1, we have that $\{q_{h,x,d}\}_{h>0} \subseteq [-\|d\|_\infty, \|d\|_\infty]^n$.

Lemma 16. *Let T be generated by a production function G and $\beta \in (0,1)$. If $x, d \in \mathbb{R}^n$ and $\tilde{W} \in \Sigma(\Pi(x))$ and $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k,x,d} = q$, then*

$$q \leq \Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))} d. \tag{134}$$

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d, we denote $q_{h,x,d}$ simply by q_h for all $h > 0$. Consider $\tilde{W} \in \Sigma(\Pi(x))$. Note that, by Proposition 1, for each $x \in \mathbb{R}^n$

$$T(y) - T(x) \leq \mathcal{M}\left(\tilde{W}, y\right) - \mathcal{M}\left(\tilde{W}, x\right) \qquad \forall \tilde{W} \in \Sigma(x), \forall y \in \mathbb{R}^n. \tag{135}$$

Therefore, we have that

$$\begin{aligned}
\Pi(x+hd) - \Pi(x) &= (1-\beta)(x+hd) + \beta T(\Pi(x+hd)) - (1-\beta)x - \beta T(\Pi(x)) && (136)\\
&= (1-\beta)hd + \beta\left[T(\Pi(x+hd)) - T(\Pi(x))\right] && (137)\\
&\leq (1-\beta)hd + \beta\left(\mathcal{M}\left(\tilde{W}, \Pi(x+hd)\right) - \mathcal{M}\left(\tilde{W}, \Pi(x)\right)\right) \qquad \forall h > 0. && (138)
\end{aligned}$$

By the Mean Value Theorem, we have that for each $h > 0$ and for each $i \in [n]$ there exists $\tau_{i,h} \in [0,1]$ such that

$$\begin{aligned}
&\mathcal{M}_i\left(\tilde{W}, \Pi(x+hd)\right) - \mathcal{M}_i\left(\tilde{W}, \Pi(x)\right) && (139)\\
&= M(\tilde{w}_i, \Pi(x+hd)) - M(\tilde{w}_i, \Pi(x)) && (140)\\
&= \left\langle \nabla M(\tilde{w}_i, \tau_{i,h}\Pi(x+hd) + (1-\tau_{i,h})\Pi(x)), \Pi(x+hd) - \Pi(x)\right\rangle. && (141)
\end{aligned}$$

For each $h > 0$ define W_h the matrix whose i-th entry is $\nabla M(\tilde{w}_i, \tau_{i,h}\Pi(x+hd) + (1-\tau_{i,h})\Pi(x))$. Note that

$$\mathcal{M}\left(\tilde{W}, \Pi(x+hd)\right) - \mathcal{M}\left(\tilde{W}, \Pi(x)\right) = W_h(\Pi(x+hd) - \Pi(x)) \quad \forall h > 0. \tag{142}$$

Define also W_0 as the matrix whose i-th entry is $\nabla M(\tilde{w}_i, \Pi(x))$, that is, $W_0 = J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))$. Since Π is Lipschitz continuous of order 1 and $(\omega, z) \mapsto \nabla M(\omega, z)$ is continuous on $\Delta \times \mathbb{R}^n$, we have that $\lim_{h\downarrow 0} W_h = W_0$. Since $h > 0$, it follows that

$$\begin{aligned}
\frac{\Pi(x+hd) - \Pi(x)}{h} &\leq (1-\beta)d + \beta\left(\frac{\mathcal{M}\left(\tilde{W}, \Pi(x+hd)\right) - \mathcal{M}\left(\tilde{W}, \Pi(x)\right)}{h}\right) && (143)\\
&= (1-\beta)d + \beta W_h \frac{\Pi(x+hd) - \Pi(x)}{h} \qquad \forall h > 0, && (144)
\end{aligned}$$

that is,

$$q_h \leq (1-\beta)\, d + \beta W_h q_h \qquad \forall h > 0. \tag{145}$$

By Lemma 15 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \leq \Omega_{W_h} d$ for all $h > 0$. By Remark 1 and since $\lim_k W_{h_k} = W_0$, $\lim_k q_{h_k} = q$, and $q_{h_k} \leq \Omega_{W_{h_k}} d$ for all $k \in \mathbb{N}$, it follows that $q \leq \Omega_{W_0} d$, proving (134) follows. □

Lemma 17. *Let T be generated by a production function G and $\beta \in (0,1)$. If $x, d \in \mathbb{R}^n$ and $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ is such that $\lim_k h_k = 0$ and $\lim_k q_{h_k,x,d} = q$, then there exists $\hat{W} \in \Sigma(\Pi(x))$*

$$q \geq \Omega_{J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))} d. \tag{146}$$

In particular, $q = \min_{\tilde{W}\in\Sigma(\Pi(x))} \Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))} d$.

Proof. Fix $x, d \in \mathbb{R}^n$. Since we fixed x and d, we denote $q_{h,x,d}$ by simply q_h for all $h > 0$. By Corollary 6, for each $h > 0$ and for each $i \in [n]$ there exist $\lambda_{i,h} \overset{\text{def}}{=} \lambda_{i,\Pi(x),\Pi(x+hd)} \in [0,1]$ and $\hat{w}_{i,h} \in \sigma_{T_i}(\lambda_{i,h}\Pi(x+hd) + (1-\lambda_{i,h})\Pi(x))$ such that

$$\begin{aligned} &T_i(\Pi(x+hd)) - T_i(\Pi(x)) &(147)\\ &\geq \langle \nabla M(\hat{w}_{i,h}, \lambda_{i,h}\Pi(x+hd) + (1-\lambda_{i,h})\Pi(x)), \Pi(x+hd) - \Pi(x)\rangle. &(148)\end{aligned}$$

Define $\hat{W}_h$ the matrix whose i-th row is $\hat{w}_{i,h}$ and define W_h the matrix whose i-th row is $\nabla M(\hat{w}_{i,h}, \lambda_{i,h}\Pi(x+hd) + (1-\lambda_{i,h})\Pi(x))$. This implies that

$$\begin{aligned} \Pi(x+hd) - \Pi(x) &= (1-\beta)(x+hd) + \beta T(\Pi(x+hd)) - (1-\beta)x - \beta T(\Pi(x)) &(149)\\ &= (1-\beta)hd + \beta[T(\Pi(x+hd)) - T(\Pi(x))] &(150)\\ &\geq (1-\beta)hd + \beta W_h(\Pi(x+hd) - \Pi(x)). &(151)\end{aligned}$$

By dividing by h, we obtain that

$$q_h \geq (1-\beta)\, d + \beta W_h q_h \qquad \forall h > 0. \tag{152}$$

By Lemma 15 and since h was arbitrarily chosen and W_h is a stochastic matrix, we have that $q_h \geq \Omega_{W_h} d$ for all $h > 0$. Consider now $\{h_k\}_{k\in\mathbb{N}} \subseteq (0,\infty)$ such that $\lim_k h_k = 0$ and $\lim_k q_{h_k} = q$. Since $\left\{\hat{W}_{h_k}\right\}_{k\in\mathbb{N}} \subseteq \mathcal{W}$, we have that there exists a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}} \subseteq \{h_k\}_{k\in\mathbb{N}}$ such that $\lim_l \hat{W}_{h_{k_l}} = \hat{W}$ and, in particular, $\lim_l \hat{w}_{i,h_{k_l}} = \hat{w}_i$ for all $i \in [n]$. Since Π is Lipschitz continuous, we have that $\hat{w}_i \in \sigma_{T_i}(\Pi(x))$ for all $i \in [n]$ and, in particular, $\hat{W} \in \Sigma(\Pi(x))$.[13] Since Π is

[13] Consider $\{x_l\}_{l\in\mathbb{N}} \subseteq \mathbb{R}^n$ and $\{\alpha_l\}_{l\in\mathbb{N}} \subseteq \Delta$ such that $\alpha_l \in \sigma_{T_i}(x_l)$ for all $l \in \mathbb{N}$, $\lim_l \alpha_l = \alpha$, and $\lim_l x_l = x$.

Lipschitz continuous and $(\omega, z) \mapsto \nabla M(\omega, z)$ is continuous on $\Delta \times \mathbb{R}^n$, we have that

$$\lim_l \nabla M\left(\hat{w}_{i,h_{k_l}}, \lambda_{i,h_{k_l}} \Pi(x + h_{k_l} d) + \left(1 - \lambda_{i,h_{k_l}}\right) \Pi(x)\right) = \nabla M(\hat{w}_i, \Pi(x)) \quad \forall i \in [n].$$

This implies that $\lim_l W_{h_{k_l}} = J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))$ with $\hat{W} \in \Sigma(\Pi(x))$. By Remark 1 and since $\lim_l W_{h_{k_l}} = J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))$, $\lim_l q_{h_{k_l}} = q$, and $q_{h_{k_l}} \geq W_{h_{k_l},\beta} d$ for all $l \in \mathbb{N}$, it follows that $q \geq J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))_\beta d$ with $\hat{W} \in \Sigma(\Pi(x))$, proving the first part of the statement. By Lemma 16, we have that $J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))_\beta d \geq q$ and, in particular,

$$q = J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(x))_\beta d \leq J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d \quad \forall \tilde{W} \in \Sigma(\Pi(x)),$$

proving that $q = \min_{\tilde{W} \in \Sigma(x)} J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d$. □

Proof of Theorem 2. Fix $x, d \in \mathbb{R}^n$. Consider a sequence $\{h_k\}_{k\in\mathbb{N}} \subseteq (0, \infty)$ such that $\lim_k h_k = 0$. Since we fixed x and d, we denote $q_{h,x,d}$ simply by q_h for all $h > 0$. As noted, $\{q_{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence. Consider a subsequence $\left\{q_{h_{k_l}}\right\}_{l\in\mathbb{N}} \subseteq \{q_{h_k}\}_{k\in\mathbb{N}}$. Since $\{q_{h_k}\}_{k\in\mathbb{N}}$ is a bounded sequence, $\left\{q_{h_{k_l}}\right\}_{l\in\mathbb{N}}$ has a further subsequence $\left\{q_{h_{k_{l(m)}}}\right\}_{m\in\mathbb{N}}$ that converges.

By Lemma 17, it converges to $\min_{\tilde{W}\in\Sigma(\Pi(x))} J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(x))_\beta d$. By (Carothers, 2000, p. 47, Exercise 39) we can conclude that

$$\lim_k \frac{\Pi(x + h_k d) - \Pi(x)}{h_k} = \min_{\tilde{W}\in\Sigma(\Pi(x))} \Omega_{L(\tilde{W},\Pi(x))} d.$$

Since $\{h_k\}_{k\in\mathbb{N}}$ and d were arbitrarily chosen, the statement follows. □

Proof of Corollary 4. First, observe that for every $W \in \mathcal{W}$, $L(W, 0) = W$. Second, by definition of Σ, we have

$$\Sigma(0) = \left\{\tilde{W} \in \mathcal{W} : \tilde{w}_i \in \arg\min_{\hat{w}_i \in \Delta} \frac{1}{G_i(\hat{w}_i)}, \forall i \in [n]\right\}, \tag{153}$$

which by Assumption 1 implies that $\Sigma(0) = \left\{\tilde{W} \in \mathcal{W} : G_i(w_i) = 1, \forall i \in [n]\right\}$. Given these two observations, the result immediately follows by Theorem 2. □

Corollary 7. *Fix $i \in N$ and let T_i be generated by a production function G_i. The following statements are equivalent:*

(i) T_i is Gateaux differentiable at $x \in \mathbb{R}^n$;

Since T_i and M are continuous and G_i is continuous, we have that

$$T_i(x) = \liminf_l T_i(x_l) = \liminf_l \left[M(\alpha_l, x_l) + \ln G_i(\alpha)\right] \geq M(\alpha, x) + \ln G_i(\alpha) \geq T_i(x),$$

proving that $T_i(x) = M(\alpha, x) + \ln G_i(\alpha)$ and, in particular, $\alpha \in \sigma_{T_i}(x)$.

(ii) $\sigma_{T_i}(x)$ *is a singleton.*

Proof. Fix $x \in \mathbb{R}^n$. Define $T_{i,\circ}(x;\cdot) : \mathbb{R}^n \to \mathbb{R}$ by $T_{i,\circ}(x;d) = \lim_{h \downarrow 0} \frac{T_i(x+hd)-T_i(x)}{h}$ for all $d \in \mathbb{R}^n$. By Lemma 14, it follows that $T_{i,\circ}(x;\cdot) = \min_{w \in \sigma_{T_i}(x)} \langle \nabla M(w_i, x), d \rangle$.

(i) implies (ii). Since T_i is Gateaux differentiable at $x \in \mathbb{R}^n$, $T_{i,\circ}(x;\cdot)$ is linear and, in particular, $\{\nabla M(w_i, x) : w_i \in \sigma_{T_i}(x)\}$ is a singleton. Since if $w_i \neq w_i'$ then $\nabla M(w_i, x) \neq \nabla M(w_i', x)$, we can conclude that $\sigma_{T_i}(x)$ is a singleton.

(ii) implies (i). If $\sigma_{T_i}(x)$ is a singleton, that is, $\sigma_{T_i}(x) = \{w_i\}$, then $T_{i,\circ}(x;d) = \langle \nabla M(w_i, x), \cdot \rangle$, proving the statement. □

Lemma 18. *If* $i \in [n]$ *and* $P \in \mathbb{R}^n_{++}$*, then Equation 18 holds. Moreover, we have that:*

(i) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}^{n+1}_+$ *solves (18), then* $\hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i (P)^\beta$ *and there exists* $\gamma > 0$ *such that* $\left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right)$ *solves (19).*

(ii) If $Q_i \in \mathbb{R}^n_+$ *solves (19), then there exist* $\gamma > 0$ *and* $\hat{Q}_{i0} > 0$ *such that* $\left(\hat{Q}_{i0}, \gamma Q_{i1}, ..., \gamma Q_{in}\right) \in \mathbb{R}^{n+1}_+$ *solves (18).*

(iii) If $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}^{n+1}_+$ *solves (18) and* $P_i = \min_{(Q_{i0}, Q_i) \in \mathbb{R}^{n+1}_+ : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^n Q_{ij} P_j + Q_{i0}$*, then*

$$\beta = \frac{\sum_{j=1}^n \hat{Q}_{ij} P_j}{P_i}. \tag{154}$$

Proof. Fix $P \in \mathbb{R}^n_{++}$ and $i \in [n]$. As a preliminary step, we show that the minimum in Equation 19 is attained. Observe first that the minimized function is continuous, and the set over which the minimization is performed is nonempty and includes some $\hat{Q}_i \in \Delta$. The set over which the minimization is performed is closed by the continuity of G_i. Moreover, for any $P \in \mathbb{R}^n_{++}$ the same value is obtained if we restrict the set over which the minimization is performed to $\{Q_i : G_i(Q_i) \geq 1 \text{ and } P_j Q_{ij} \leq \sum_{k=1}^n P_k \hat{Q}_{ik}, \forall j \in [n]\}$ which is also compact, and the minimum is thus attained there.

A similar argument shows that the minimum in Equation 18 is attained. We can thus now prove the proper lemma statement, assuming that the minima are attained.

Since G_i is positive homogeneous, note that

$$\min_{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}:F_i(Q_{i0},Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0}=\min_{Q_{i0}\in\mathbb{R}_+,Q_i\in\mathbb{R}_+^n:Z_i\left(\frac{Q_{i0}}{1-\beta}\right)^{1-\beta}\left(\frac{G_i(Q_i)}{\beta}\right)^{\beta}\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0} \tag{155}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\min_{Q_i\in\mathbb{R}_+^n:Q_{i0}^{1-\beta}G_i(Q_i)^{\beta}\geq\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0} \tag{156}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\min_{Q_i\in\mathbb{R}_+^n:G_i(Q_i)\geq\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}}\sum_{j=1}^{n}Q_{ij}P_j\right\} \tag{157}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}\min_{Q_i\in\mathbb{R}_+^n:G_i(Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j\right\} \tag{158}$$

$$=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right)\right\} \tag{159}$$

$$=\min_{x\in\mathbb{R}_{++}}\left\{x+\left(A\left(\frac{1}{x}\right)^{1-\beta}\right)^{\frac{1}{\beta}}B.\right\} \tag{160}$$

where $x=Q_{i0}$, $A=\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}>0$, and $B=K_i\left(P\right)$. Define $f:\mathbb{R}_{++}\to\mathbb{R}_{++}$ by $f\left(x\right)=x+A^{\frac{1}{\beta}}x^{-\frac{1-\beta}{\beta}}B$ for all $x>0$. Since $P\in\mathbb{R}_{++}^n$ and G_i is positive homogeneous, we have that $B>0$. It follows that

$$f'\left(x\right)=1-\frac{1-\beta}{\beta}A^{\frac{1}{\beta}}Bx^{-\frac{1-\beta}{\beta}-1}=1-\frac{1-\beta}{\beta}A^{\frac{1}{\beta}}B\left(\frac{1}{x}\right)^{\frac{1}{\beta}}\quad\forall x>0. \tag{161}$$

It follows that f' is increasing and takes value 0 at $x^{\star}=\left(\frac{1-\beta}{\beta}\right)^{\beta}AB^{\beta}>0$ and, in particular, f is convex and minimized at $\left(\frac{1-\beta}{\beta}\right)^{\beta}AB^{\beta}$. Since $A=\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}$ and $B=K_i\left(P\right)$, we have that

$x^\star = \left(\frac{1-\beta}{\beta}\right)^\beta AB^\beta = \frac{1-\beta}{Z_i} K_i (P)^\beta$. Plugging $x^\star$ in f, we have that

$$f(x^\star) = \frac{1-\beta}{Z_i} K_i (P)^\beta + A^{\frac{1}{\beta}} \left(\frac{1-\beta}{Z_i} K_i (P)^\beta\right)^{-\frac{1-\beta}{\beta}} K_i (P) \tag{162}$$

$$= \frac{1-\beta}{Z_i} K_i (P)^\beta + \frac{\beta (1-\beta)^{\frac{1-\beta}{\beta}}}{Z_i^{\frac{1}{\beta}}} \frac{(1-\beta)^{-\frac{1-\beta}{\beta}}}{Z_i^{-\frac{1-\beta}{\beta}}} K_i (P)^{-(1-\beta)} K_i (P) \tag{163}$$

$$= \frac{1-\beta}{Z_i} K_i (P)^\beta + \frac{\beta}{Z_i} K_i (P)^\beta = \frac{1}{Z_i} K_i (P)^\beta, \tag{164}$$

proving the first part of the statement.

(i). Consider $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ that solves (18). Since $F_i\left(\hat{Q}_{i0}, \hat{Q}_i\right) \geq 1$, we have that $\hat{Q}_{i0} > 0$. It follows that $G_i\left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right) \geq \left(\frac{\beta^\beta (1-\beta)^{1-\beta}}{Z_i} \left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$. Define $\gamma = \left(\frac{\beta^\beta (1-\beta)^{1-\beta}}{Z_i} \left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{-\frac{1}{\beta}} > 0$. Since G_i is positive homogeneous, note that $G_i\left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right) \geq 1$. By contradiction, assume that $\left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right)$ does not solve (19). It follows that there exists $\bar{Q}_i \in \mathbb{R}_+^n$ such that $G_i\left(\bar{Q}_{i1}, ..., \bar{Q}_{in}\right) \geq 1$ and $\sum_{j=1}^n \bar{Q}_{ij} P_j < \sum_{j=1}^n \gamma \hat{Q}_{ij} P_j$. Consider $\left(\hat{Q}_{i0}, \gamma^{-1} \bar{Q}_{i1}, ..., \gamma^{-1} \bar{Q}_{in}\right) \in \mathbb{R}_+^{n+1}$. By the initial part of the proof and since G_i is positive homogeneous, we have that

$$\min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^n Q_{ij} P_j + Q_{i0} = \hat{Q}_{i0} + \sum_{j=1}^n \hat{Q}_{ij} P_j > \hat{Q}_{i0} + \sum_{j=1}^n \gamma^{-1} \bar{Q}_{ij} P_j \tag{165}$$

$$\geq \hat{Q}_{i0} + \min_{Q_i \in \mathbb{R}_+^n : G_i(Q_i) \geq \left(\frac{\beta^\beta (1-\beta)^{1-\beta}}{Z_i} \left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}} \sum_{j=1}^n Q_{ij} P_j \tag{166}$$

$$\geq \min_{Q_{i0} \in \mathbb{R}_{++}} \left\{ Q_{i0} + \min_{Q_i \in \mathbb{R}_+^n : G_i(Q_i) \geq \left(\frac{\beta^\beta (1-\beta)^{1-\beta}}{Z_i} \left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}} \sum_{j=1}^n Q_{ij} P_j \right\} \tag{167}$$

$$= \min_{(Q_{i0}, Q_i) \in \mathbb{R}_+^{n+1} : F_i(Q_{i0}, Q_i) \geq 1} \sum_{j=1}^n Q_{ij} P_j + Q_{i0}, \tag{168}$$

a contradiction, proving the second part of the point. As for the first part, by the proof of the main statement and since $(\hat{Q}_{i0}, \hat{Q}_i) \in \mathbb{R}_+^{n+1}$ solves (18) and $\left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right)$ solves (19), we can

conclude that

$$\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right)\right\} \tag{169}$$

$$=\hat{Q}_{i0}+\sum_{j=1}^{n}\hat{Q}_{ij}P_j=\hat{Q}_{i0}+\gamma^{-1}\sum_{j=1}^{n}\gamma\hat{Q}_{ij}P_j \tag{170}$$

$$=\hat{Q}_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right) \tag{171}$$

proving that $\hat{Q}_{i0}$ minimizes the left-hand side of the expression. By the proof of the main statement, it follows that $\hat{Q}_{i0}=\frac{1-\beta}{Z_i}K_i\left(P\right)^{\beta}$.

(ii). Define $\hat{Q}_{i0}>0$ to be the unique point that minimizes $Q_{i0}+\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right)$. By the proof of the main statement, it follows that $\hat{Q}_{i0}=\frac{1-\beta}{Z_i}K_i\left(P\right)^{\beta}$. Define $\gamma=\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$. Define $(\hat{Q}_{i0},\hat{Q}_i)=\left(\hat{Q}_{i0},\gamma Q_{i1},...,\gamma Q_{in}\right)\in\mathbb{R}_{+}^{n+1}$. By the proof of the main statement and since $Q_i\in\mathbb{R}_{+}^{n}$ solves (19), we have that $K_i\left(P\right)=\sum_{j=1}^{n}Q_{ij}P_j$ and

$$\min_{(Q_{i0},Q_i)\in\mathbb{R}_{+}^{n+1}:F_i(Q_{i0},Q_i)\geq 1}\sum_{j=1}^{n}Q_{ij}P_j+Q_{i0}=\min_{Q_{i0}\in\mathbb{R}_{++}}\left\{Q_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{Q_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right)\right\} \tag{172}$$

$$=\hat{Q}_{i0}+\left(\frac{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}K_i\left(P\right) \tag{173}$$

$$=\hat{Q}_{i0}+\gamma\sum_{j=1}^{n}Q_{ij}P_j=\hat{Q}_{i0}+\sum_{j=1}^{n}\gamma Q_{ij}P_j \tag{174}$$

$$=\hat{Q}_{i0}+\sum_{j=1}^{n}\hat{Q}_{ij}P_j. \tag{175}$$

Since G_i is positive homogeneous and $G_i\left(Q_i\right)\geq 1$, we have that $G_i\left(\gamma Q_i\right)\geq\left(\frac{\beta^{\beta}(1-\beta)^{1-\beta}}{Z_i}\left(\frac{1}{\hat{Q}_{i0}}\right)^{1-\beta}\right)^{\frac{1}{\beta}}$, yielding that $F_i\left(\hat{Q}_{i0},\hat{Q}_i\right)\geq 1$ and proving the point.

(iii). By the proof of the main statement and point (i) and since $(\hat{Q}_{i0},\hat{Q}_i)\in\mathbb{R}_{+}^{n+1}$ solves

(18) and $P_i = \min_{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}:F_i(Q_{i0},Q_i)\geq 1} \sum_{j=1}^n Q_{ij}P_j + Q_{i0}$, we have that

$$\frac{\sum_{j=1}^n \hat{Q}_{ij}P_j}{P_i} = \frac{\sum_{j=1}^n \hat{Q}_{ij}P_j + \hat{Q}_{i0} - \hat{Q}_{i0}}{P_i} = \frac{\sum_{j=1}^n \hat{Q}_{ij}P_j + \hat{Q}_{i0}}{P_i} - \frac{\hat{Q}_{i0}}{P_i} \tag{176}$$

$$= \frac{\min_{(Q_{i0},Q_i)\in\mathbb{R}_+^{n+1}:F_i(Q_{i0},Q_i)\geq 1} \sum_{j=1}^n Q_{ij}P_j + Q_{i0}}{P_i} - \frac{\hat{Q}_{i0}}{P_i} \tag{177}$$

$$= 1 - \frac{\hat{Q}_{i0}}{P_i} = 1 - \frac{\frac{1-\beta}{Z_i}K_i\left(P\right)^\beta}{\frac{1}{Z_i}K_i\left(P\right)^\beta} = \beta, \tag{178}$$

proving the point. □

Lemma 19. *Let $i \in [n]$. If K_i is generated by a production function G_i, we have that:*

(i) If $Q_i \in \mathbb{R}_+^n$ solves (19), then $\frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$ solves the right-hand side problem of (33).

(ii) If $\tilde{w}_i \in \Delta$ solves the right-hand side problem of (33), then there exists $\gamma > 0$ such that $\gamma\tilde{w}_i \in \mathbb{R}_+^n$ solves (19).

Proof. (i). Assume $Q_i \in \mathbb{R}_+^n$ solves (19). Define $w_i = \frac{1}{\sum_{j=1}^n Q_{ij}} Q_i \in \Delta$. Since G_i is positive homogeneous and $G_i\left(Q_i\right) \geq 1$, observe that

$$K_i\left(P\right) = \sum_{j=1}^n Q_{ij}P_j \geq \frac{\sum_{j=1}^n Q_{ij}P_j}{G_i(Q_i)} = \frac{\sum_{j=1}^n w_{ij}P_j}{G_i(w_i)} \geq K_i\left(P\right), \tag{179}$$

proving that w_i solves the right-hand side problem of (33).

(ii). Assume that w_i solves the right-hand side problem of (33). It follows that $G_i\left(w_i\right) > 0$. Consider $\gamma = G_i\left(w_i\right)^{-1}$ and define $Q_i = \gamma w_i$. Since G_i is positive homogeneous, we have that $G_i\left(Q_i\right) = 1$ and

$$K_i\left(P\right) = \min_{\tilde{w}_i\in\Delta} \frac{\sum_{j=1}^n \tilde{w}_{ij}P_j}{G_i\left(\tilde{w}_i\right)} = \frac{\sum_{j=1}^n w_{ij}P_j}{G_i\left(w_i\right)} = \frac{\sum_{j=1}^n \gamma w_{ij}P_j}{G_i\left(\gamma w_i\right)} = \frac{\sum_{j=1}^n Q_{ij}P_j}{G_i\left(Q_i\right)} = \sum_{j=1}^n Q_{ij}P_j \geq K_i\left(P\right), \tag{180}$$

proving that $Q_i = \gamma w_i$ solves (19). □

Corollary 8. *Let $i \in [n]$ and $P \in \mathbb{R}_{++}^n$. The following statements are true:*

(i) If $\hat{Q}_i \in \mathbb{R}_+^{n+1}$ solves (18), then $\frac{1}{\sum_{j=1}^n \hat{Q}_{ij}} \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ solves the right-hand side problem of (33).

(ii) If $w_i \in \Delta$ solves the right-hand side problem of (33), then there exist $\gamma > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = \left(\hat{Q}_{i0}, \gamma w_{i1}, ..., \gamma w_{in}\right) \in \mathbb{R}_+^{n+1}$ solves (18).

Proof. (i). Fix $i \in [n]$ and $P \in \mathbb{R}^n_{++}$. By Lemma 18, we have that there exists $\gamma > 0$ such that $\left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right)$ solves (19). By Lemma 19, it follows that $\frac{1}{\sum_{j=1}^n \gamma \hat{Q}_{ij}} \left(\gamma \hat{Q}_{i1}, ..., \gamma \hat{Q}_{in}\right) = \frac{1}{\sum_{j=1}^n \hat{Q}_{ij}} \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ solves the right-hand side problem of (33).

(ii). By Lemma 19, if $w_i \in \Delta$ solves the right-hand side problem of (33), then there exists $\gamma_1 > 0$ such that $\gamma_1 \tilde{w}_i$ solves (19). By Lemma 18, there exist $\gamma_2 > 0$ and $\hat{Q}_{i0} > 0$ such that $\hat{Q}_i = \left(\hat{Q}_{i0}, \gamma_2 \gamma_1 w_{i1}, ..., \gamma_2 \gamma_1 w_{in}\right) \in \mathbb{R}^{n+1}_+$ solves (18). □

We denote the set of all competitive equilibria by $\mathfrak{E}$. Given $\mathcal{E} \in \mathfrak{E}$, that is a tuple $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$, we define $E_{\mathcal{E}}$ as the $n \times n$ matrices such that its ij-th entries is $\frac{\hat{Q}_{ij} \bar{P}_j}{F_i\left(\hat{Q}_i\right) \bar{P}_i}$Ẇe also define $\tilde{W}_{\mathcal{E}}$ in as the matrix whose ij-th entry is $\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}}$.

Proposition 9. *We have that:*

(i) $\mathfrak{E}$ *is nonempty.*

(ii) $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E} \in \mathfrak{E}} = \Sigma\left(\Pi\left(\bar{x}\right)\right)$ *and* $\left\{L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right\}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}} = \left\{J_{\mathcal{M}\left(\tilde{W}, \cdot\right)}\left(\Pi\left(\bar{x}\right)\right) : \tilde{W} \in \Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}$.

(iii) If $\mathcal{E} \in \mathfrak{E}$, *then* $\lambda^T_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)} = \xi^T \left(I - \beta L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right)^{-1}$.

(iv) If $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right), \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$, *then* $\tilde{P} = \bar{P}$.

Proof. (i). Consider the (unique) equilibrium price $\bar{P}$ and a matrix $\hat{Q}$ such that $(\hat{Q}_{i0}, \hat{Q}_i)$ solves Equation 18 for all $i \in [n]$ with $P = \bar{P}$. Compute $\bar{C} \in \operatorname{argmax}_{C \in \mathbb{R}^n_+ : \sum_{i=1}^n \bar{P}_i C_i \leq 1} U\left(C\right)$. Next, consider the equation $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}} E_{\bar{P}, \hat{Q}}$ with unknown $\lambda \in \mathbb{R}^n$ where $\bar{D}$ is the vector whose j-th entry is $\bar{C}_j \bar{P}_j$. By point (iii) of Lemma 18 and since $\bar{P}$ is an equilibrium price, we have that $\beta = \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j / \bar{P}_i$ for all $i \in [n]$. Since $F_i\left(\hat{Q}_i\right) = 1$ for all $i \in [n]$, this implies that

$$\frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} = \frac{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k}{\bar{P}_i} \frac{\hat{Q}_{ij} \bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k} = \beta \frac{\hat{Q}_{ij} \bar{P}_j}{\sum_{k=1}^n \hat{Q}_{ik} \bar{P}_k} \quad \forall i, j \in [n], \tag{181}$$

that is $E_{\bar{P}, \hat{Q}} = \beta L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)$, yielding that $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}} \beta L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)$. Since $L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)$ is a stochastic matrix, $\bar{D} \geq 0$, and $\beta \in (0, 1)$, it follows that this equation has a (unique) solution $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} \left(I - \beta L\left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)}, \ln\left(\bar{P}\right)\right)\right)^{-1} \geq 0$. Since $\bar{P} \in \mathbb{R}^n_{++}$ and $\lambda \geq 0$, define $\bar{Y} \in \mathbb{R}^n_+$ by $\bar{Y}_i = \lambda_i \backslash \bar{P}_i$ for all $i \in [n]$. Since F_i is positive homogeneous and $F_i\left(\hat{Q}_i\right) = 1$ for all $i \in [n]$ and $\lambda^{\mathrm{T}} = \bar{D}^{\mathrm{T}} + \lambda^{\mathrm{T}} E_{\bar{P}, \hat{Q}}$, it follows that for each $j \in [n]$

$$F_j\left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j\right) \bar{P}_j = F_j\left(\hat{Q}_j\right)\left(\bar{P}_j\right) = \lambda_j = \bar{D}_j + \sum_{i=1}^n \lambda_i \frac{\hat{Q}_{ij} \bar{P}_j}{\bar{P}_i} = \bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j \quad \forall j \in [n]. \tag{182}$$

By dividing by $\bar{P}_j$ and since $\bar{P}_j > 0$ for all $j \in [n]$, it follows that the market clearing conditions of the n sectors are satisfied. Since $\bar{C}_j \bar{P}_j + \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j = F_j \left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j \right) \bar{P}_j$ for all $j \in [n]$, we obtain that

$$\sum_{i=1}^n \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n \sum_{i=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j + \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{j=1}^n F_j \left(\bar{Y}_j \hat{Q}_{j0}, \bar{Y}_j \hat{Q}_j \right) \bar{P}_j. \quad (183)$$

Since the preferences of the representative agent are monotone and $\bar{P}_i = \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j + \hat{Q}_{i0}$ for all $i \in [n]$, we have that

$$1 = \sum_{j=1}^n \bar{C}_j \bar{P}_j = \sum_{i=1}^n \left(F_i \left(\bar{Y}_i \hat{Q}_{i0}, \bar{Y}_i \hat{Q}_i \right) \bar{P}_i - \sum_{j=1}^n \bar{Y}_i \hat{Q}_{ij} \bar{P}_j \right) = \sum_{i=1}^n \left(\bar{Y}_i \left[F_i \left(\hat{Q}_{i0}, \hat{Q}_i \right) \bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) \quad (184)$$

$$= \sum_{i=1}^n \left(\bar{Y}_i \left[\bar{P}_i - \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \right] \right) = \sum_{i=1}^n \bar{Y}_i \hat{Q}_{i0} \quad \forall i \in [n]. \quad (185)$$

This proves that $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right)$ satisfies condition 3 of a competitive equilibrium (cf. Definition 1). By construction of $\bar{P}$ and $\bar{C}$, conditions 1 and 4 of a competitive equilibrium are also satisfied. Finally, since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (18) for all $i \in [n]$ with $P = \bar{P}$, we have that condition 2 of a competitive equilibrium is satisfied, proving that $\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right) \in \mathfrak{E}$ and the latter set is nonempty.

(ii). Fix an equilibrium $\mathcal{E} = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right)$. It follows that $\bar{P} = \exp \Pi (\bar{x})$. By point (i) of Corollary 8 and since $(\hat{Q}_{i0}, \hat{Q}_i)$ solves (18) for all $i \in [n]$, then $\frac{1}{\sum_{k=1}^n \hat{Q}_{ik}} \left(\hat{Q}_{i1}, ..., \hat{Q}_{in} \right)$ solves the right-hand side problem of (33) with $\bar{p} = \Pi (\bar{x})$ for all $i \in [n]$, proving that $\tilde{W}_{\mathcal{E}} \in \Sigma (\Pi (\bar{x}))$. Define $W = J_{\mathcal{M}\left(\tilde{W}_{\mathcal{E}}, \cdot\right)} (\Pi (\bar{x}))$. It follows that

$$w_{ij} = \frac{\tilde{w}_{ij} e^{\Pi_j (\bar{x})}}{\sum_{k=1}^n \tilde{w}_{ik} e^{\Pi_k (\bar{x})}} = \frac{\tilde{w}_{ij} \bar{P}_j}{\sum_{k=1}^n \tilde{w}_{ik} \bar{P}_k} = \frac{\frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_j}{\sum_{k=1}^n \frac{\hat{Q}_{ik}}{\sum_{k=1}^n \hat{Q}_{ik}} \bar{P}_k} = \frac{\hat{Q}_{ij} P_j}{\sum_{k=1}^n \hat{Q}_{ik} P_k}, \quad (186)$$

proving that $W = L \left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right)}, \ln \left(\bar{P} \right) \right)$. We can conclude that $\left\{ \tilde{W}_{\mathcal{E}} \right\}_{\mathcal{E} \in \mathfrak{E}\mathfrak{E}} \subseteq \Sigma (\Pi (\bar{x}))$ and $\left\{ L \left(\tilde{W}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right)}, \ln \left(\bar{P} \right) \right) \right\}_{\left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C} \right) \in \mathfrak{E}} \subseteq \left\{ J_{\mathcal{M}\left(\tilde{W}, \cdot \right)} (\Pi (\bar{x})) : \tilde{W} \in \Sigma (\Pi (\bar{x})) \right\}$.

Consider $\tilde{W} \in \Sigma (\Pi (\bar{x}))$. By point (ii) of Corollary 8, it follows that there exist (unique) $\gamma > 0$ and (unique) $\hat{Q}_{i0} > 0$ such that $\left(\hat{Q}_{i0}, \gamma \tilde{w}_{i1}, ..., \gamma \tilde{w}_{in} \right) \in \mathbb{R}_+^{n+1}$ solves (18) for all $i \in [n]$ with $P = \bar{P} = \exp \Pi (\bar{x})$. By part 1 of this proof, there exist (unique) $\bar{Y} \in \mathbb{R}_+^n$ and $\bar{C} \in \mathbb{R}_+^n$

such that $\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)\in\mathfrak{E}$. Finally, since $\tilde{W}$ is a stochastic matrix, observe that

$$\frac{\hat{Q}_{ij}}{\sum_{k=1}^{n}\hat{Q}_{ik}}=\frac{\gamma\tilde{w}_{ij}}{\sum_{k=1}^{n}\gamma\tilde{w}_{ik}}=\tilde{w}_{ij}\quad\forall i,j\in[n]\,, \tag{187}$$

proving that $\tilde{W}\in\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$ and, in particular, $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}\supseteq\Sigma\left(\Pi\left(\bar{x}\right)\right)$. Finally, consider $W\in\left\{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right):\tilde{W}\in\Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}$, that is, $W=J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right)$ for some $\tilde{W}\in\Sigma\left(\Pi\left(\bar{x}\right)\right)$. By (186) and since $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}=\Sigma\left(\Pi\left(\bar{x}\right)\right)$, we have that $\tilde{W}=\tilde{W}_{\mathcal{E}}$ for some $\mathcal{E}=\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)\in\mathfrak{E}$ and $W=\tilde{E}_{L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)},\ln\left(\bar{P}\right)\right)}$, proving that

$$\left\{L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)},\ln\left(\bar{P}\right)\right)\right\}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)\in\mathfrak{E}}\supseteq\left\{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(\bar{x}\right)\right):\tilde{W}\in\Sigma\left(\Pi\left(\bar{x}\right)\right)\right\}. \tag{188}$$

(iii). Fix an equilibrium $\mathcal{E}=\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)$. It follows that

$$\bar{Y}_j\bar{P}_j=\bar{C}_j\bar{P}_j+\sum_{i=1}^{n}\bar{Y}_i\hat{Q}_{ij}\bar{P}_j=\bar{D}_j+\sum_{i=1}^{n}\left(\bar{Y}_i\bar{P}_i\right)\frac{\hat{Q}_{ij}\bar{P}_j}{\bar{P}_i}\quad\forall j\in[n]\,, \tag{189}$$

that is, $\lambda_{\mathcal{E}}^{\mathrm{T}}=\bar{D}^{\mathrm{T}}+\lambda_{\mathcal{E}}^{\mathrm{T}}E_{\mathcal{E}}$. By the same arguments leading to (181), we can conclude that $\lambda_{\mathcal{E}}^{\mathrm{T}}=\bar{D}^{\mathrm{T}}+\lambda_{\mathcal{E}}^{\mathrm{T}}\beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)},\ln\left(\bar{P}\right)\right)$, proving that $\lambda_{\mathcal{E}}^{\mathrm{T}}=\bar{D}^{\mathrm{T}}\left(I-\beta L\left(\tilde{W}_{\left(\bar{P},\hat{Q},\bar{Y},\bar{C}\right)},\ln\left(\bar{P}\right)\right)\right)^{-1}$. Since $\bar{D}^{\mathrm{T}}=\xi^{\mathrm{T}}$, the statement follows.

(iv). It is obvious. □

Corollary 9. *Let T be generated by a production function G. If $x\in\mathbb{R}^n$ and T is Gateaux differentiable at $\Pi\left(x\right)$, then $\Sigma\left(\Pi\left(x\right)\right)=\left\{\tilde{W}\right\}$ and*

$$\Pi_0(x;d)=\Omega_{J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi(x)\right)}d=\Omega_{J_T\left(\Pi(x)\right)}d\quad\forall d\in\mathbb{R}^n. \tag{190}$$

Proof. Since T is Gateaux differentiable at $\Pi\left(x\right)$, so is T_i for all $i\in[n]$. By Corollary 7, we have that $\sigma_{T_i}\left(\Pi\left(x\right)\right)$ is a singleton for all $i\in[n]$ and, in particular, $\Sigma\left(\Pi\left(x\right)\right)$ is a singleton. This implies that $J_T\left(\Pi\left(x\right)\right)=J_{\mathcal{M}\left(\tilde{W},\cdot\right)}\left(\Pi\left(x\right)\right)$. By Theorem 2, the statement follows. □

Remark 2. *Fix $x\in\mathbb{R}^n$. Consider two stochastic matrices W and $\hat{W}$ and assume that $\Omega_{J_{\mathcal{M}(W,\cdot)}(x)}=\Omega_{J_{\mathcal{M}\left(\hat{W},\cdot\right)}(x)}$. By the last part of Remark 1, it follows that $J_{\mathcal{M}(W,\cdot)}\left(x\right)=J_{\mathcal{M}\left(\hat{W},\cdot\right)}\left(x\right)$. By definition of $\mathcal{M}$, it follows that $\nabla M\left(w_i,x\right)=\nabla M\left(\hat{w}_i,x\right)$ for all $i\in[n]$, that is,*

$$\frac{e^{x_h}w_{ih}}{\sum_{j=1}^{n}e^{x_j}w_{ij}}=\frac{e^{x_h}\hat{w}_{ih}}{\sum_{j=1}^{n}e^{x_j}\hat{w}_{ij}}\qquad\forall i,h\in[n]\,.$$

For each $i \in [n]$ *set* $\lambda_i = \sum_{j=1}^n e^{x_j} w_{ij} / \sum_{j=1}^n e^{x_j} \hat{w}_{ij}$. *We obtain that* $w_{ih} = \lambda_i \hat{w}_{ih}$ *for all* $i, h \in [n]$ *and, in particular,* $1 = \sum_{h=1}^n w_{ih} = \lambda_i \sum_{h=1}^n \hat{w}_{ih} = \lambda_i$ *for all* $i \in [n]$. *We can conclude that* $w_{ih} = \hat{w}_{ih}$ *for all* $i, h \in [n]$, *that is,* $W = \hat{W}$. ▲

Proof of Proposition 5. (i) is equivalent to (iii). By Corollary 7, the statement follows.

(iv) implies (iii). By point (ii) of Proposition 9, if $\mathfrak{E}$ is a singleton, then $\Sigma(\Pi(\bar{x})) = \left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$ is a singleton.

(iii) implies (iv). By point (ii) of Proposition 9, if $\Sigma(\Pi(\bar{x}))$ is a singleton, so is $\left\{\tilde{W}_{\mathcal{E}}\right\}_{\mathcal{E}\in\mathfrak{E}}$. Assume there exist two equilibria $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right), \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right) \in \mathfrak{E}$ that generate the unique $\tilde{W}_{\mathcal{E}}$. It follows that

$$\tilde{w}_{ij} = \frac{\hat{Q}_{ij}}{\sum_{k=1}^n \hat{Q}_{ik}} = \frac{\tilde{Q}_{ij}}{\sum_{k=1}^n \tilde{Q}_{ik}} \qquad \forall i, j \in [n]. \tag{191}$$

For each $i \in [n]$ define $\mu_i = \sum_{k=1}^n \tilde{Q}_{ik} / \sum_{k=1}^n \hat{Q}_{ik}$. By Equation 191, we have that $\left(\tilde{Q}_{i1}, ..., \tilde{Q}_{in}\right) = \mu_i \left(\hat{Q}_{i1}, ..., \hat{Q}_{in}\right)$ for all $i \in [n]$. By point (iv) of Proposition 9, we have that $\bar{P} = \tilde{P}$. By point (i) of Lemma 18 and and since $(\hat{Q}_{i0}, \hat{Q}_i), (\tilde{Q}_{i0}, \tilde{Q}_i) \in \mathbb{R}^{n+1}_+$ solve (18) for all $i \in [n]$ with $P = \bar{P} = \tilde{P}$, then $\tilde{Q}_{i0} = \hat{Q}_{i0} = \frac{1-\beta}{Z_i} K_i (P)^\beta$ for all $i \in [n]$ and

$$\hat{Q}_{i0} + \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j = \tilde{Q}_{i0} + \sum_{j=1}^n \tilde{Q}_{ij} \bar{P}_j \implies \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j = \sum_{j=1}^n \tilde{Q}_{ij} \bar{P}_j = \mu_i \sum_{j=1}^n \hat{Q}_{ij} \bar{P}_j \tag{192}$$

$$\implies \mu_i = 1 \qquad \forall i \in [n]. \tag{193}$$

This implies that $\hat{Q}_{ij} = \tilde{Q}_{ij}$ for all $i, j \in [n]$, proving that $\hat{Q} = \tilde{Q}$. By the proof of existence of a competitive equilibrium (cf. Proposition 9), we have that $\bar{Y} = \tilde{Y}$ is uniquely pinned down by $\bar{P} = \tilde{P}$ and $\hat{Q} = \tilde{Q}$. Given the preferences of the agent and $\bar{P} = \tilde{P}$, we have that $\bar{C} = \tilde{C}$, proving $\left(\tilde{P}, \tilde{Q}, \tilde{Y}, \tilde{C}\right) = \left(\bar{P}, \hat{Q}, \bar{Y}, \bar{C}\right)$ and uniqueness of the equilibrium.

(i) implies (ii). By Corollary 9, the implication follows.

(ii) implies (i). By Theorem 2 and since Π is Gateaux differentiable at $\bar{x}$, we have that there exists $W \in \mathcal{W}$ such that

$$\Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x}))} d \geq \Pi_0(\bar{x}, d) = Wd \quad \forall d \in \mathbb{R}^n, \forall \tilde{W} \in \Sigma(\Pi(\bar{x})). \tag{194}$$

It follows that $\Omega_{J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x}))} = W$ for all $\tilde{W} \in \Sigma(\Pi(\bar{x}))$. By Remark 2, we can conclude that $J_{\mathcal{M}(\tilde{W},\cdot)}(\Pi(\bar{x})) = J_{\mathcal{M}(\hat{W},\cdot)}(\Pi(\bar{x}))$, that is $\tilde{W} = \hat{W}$, for all $\tilde{W}, \hat{W} \in \Sigma(\Pi(\bar{x}))$, proving that $\Sigma(\Pi(\bar{x}))$ is a singleton and that T is Gateaux differentiable at $\Pi(\bar{x})$. □

Define $\hat{\Upsilon} : \mathbb{R}^n \to \mathbb{R}$ as $\hat{\Upsilon}(x) = -\Upsilon(x)$. It is immediate to verify that $\hat{\Upsilon}$ is non-decreasing, normalized, and translation invariant, hence Lipschitz continuous of order 1. By Proposition 6,

for every $i \in [n]$ and $x \in \mathcal{D}$, we have $\frac{\partial}{\partial x_i}\hat{\Upsilon}(x) = (1-\beta)\lambda_i(x)$. Therefore, we have $(1-\beta)\underline{\lambda}_i = \inf_{x\in\mathcal{D}} \frac{\partial}{\partial x_i}\hat{\Upsilon}(x)$ and $(1-\beta)\overline{\lambda}_i = \sup_{x\in\mathcal{D}} \frac{\partial}{\partial x_i}\hat{\Upsilon}(x)$ for all $i \in [n]$.

Lemma 20. *For every $i \in [n]$, there exists a non-decreasing function $R_i : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\hat{\Upsilon}(x) = (1-\beta)\underline{\lambda}_i x_i + (1-(1-\beta)\underline{\lambda}_i)R_i(x) \qquad \forall x \in \mathbb{R}^n. \tag{195}$$

Proof. The proof is identical to that of the beginning of the proof of Part 2 of Theorem 3 in Cerreia-Vioglio et al. (2024). □

Proof of Proposition 7. Clearly, we have $\mathrm{Var}[\Upsilon] = \mathrm{Var}[\hat{\Upsilon}]$. We start with the lower bound. Consider any $j \in \arg\max_{i\in[n]} \mathrm{Var}[X_i]\underline{\lambda}_i^2$. We have

$$\begin{aligned}
\mathrm{Var}[\hat{\Upsilon}(X)] &= \mathrm{Var}[(1-\beta)\underline{\lambda}_j X_j + (1-(1-\beta)\underline{\lambda}_j)R_j(X)] && (196)\\
&= (1-\beta)^2\underline{\lambda}_j^2\mathrm{Var}[X_j] + (1-(1-\beta)^2\underline{\lambda}_j)^2\mathrm{Var}[R_j(X)] && (197)\\
&+ 2(1-\beta)\underline{\lambda}_j(1-(1-\beta)\underline{\lambda}_j)\mathrm{Cov}[X_j, R_j(X)] && (198)\\
&\geq (1-\beta)^2\underline{\lambda}_j^2\mathrm{Var}[X_j] && (199)
\end{aligned}$$

where the first equality follows from Lemma 20, the second equality is standard, the first inequality follows from the fact that $\mathrm{Var}[R_j(X)] \geq 0$ and that, by Harris' inequality, $\mathrm{Cov}[X_j, R_j(X)] \geq 0$, and the second inequality follows by assumption.

The proof of the upper bound is in Proposition 1 of Cerreia-Vioglio, Corrao, Flynn, and Lanzani (2025a). □

C Relaxing Cobb-Douglas in Labor

Suppose that the equilibrium log-price equation for each firm $i \in [n]$ can be written as

$$p_i = -\ln(Z_i) + T_i(p_0, p) \tag{200}$$

where $p_0 \in \mathbb{R}$ is the log-price of the primary input and $T_i : \mathbb{R}^{n+1} \to \mathbb{R}$. This generalizes Equation 21 by allowing T_i to also depend on the price of labor. Next, suppose that there exist $\bar{\beta} \in (0,1)$ and a normalized, non-decreasing, and translation invariant $S : \mathbb{R}^{n+1} \to \mathbb{R}^n$ such that

$$T_i(p_0, p) = \left(1-\bar{\beta}\right)p_0 + \bar{\beta}S_i(p_0, p) \tag{201}$$

for all $(p_0, p) \in \mathbb{R}^{n+1}$ and $i \in [n]$. Assume also that S is a valid log-cost operator.[14] The previous fixed-point equation then becomes

$$p_i = \left(1 - \bar{\beta}\right)(x_i + p_0) + \bar{\beta} S_i(p_0, p) \qquad \forall i \in [n] \tag{202}$$

where, as in the main text, $x_i = \frac{-\ln(Z_i)}{\left(1-\bar{\beta}\right)}$ for all $i \in [n]$. Let $\Pi(p_0, x) \in \mathbb{R}^n$ denote the unique solution of the previous fixed point equation indexed to $(p_0, x) \in \mathbb{R}^{n+1}$. This is well-defined because $y \mapsto \left(1 - \bar{\beta}\right)(x + p_0 e) + \bar{\beta} S(p_0, y)$ is a contraction, hence admits a unique fixed point. In our main analysis, we always focus on $\Pi(0, x)$ since it is always possible to normalize $p_0 = 0$.

Consider the cloned economy where, for each original sector $i \in [n]$, we generate two sectors given by the original one i and a new cloned sector $i + n$. With this, the new cloned economy has $2n$ sectors. Define $\tilde{S} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ as

$$\tilde{S}_j(\tilde{p}) = \begin{cases} S_j(\tilde{p}_{j+n}, \tilde{p}_1, ..., \tilde{p}_n) & \text{if} \quad j \in \{1, ..., n\}, \\ \tilde{p}_j & \text{if} \quad j \in \{n+1, ..., 2n\}. \end{cases} \tag{203}$$

Intuitively, to fit this more general dependence on labor in the baseline framework of the paper, for every sector i, we are adding an extra sector $i + n$ that transforms the household's labor supply in labor services sold to sector i (this is a fairly standard approach, see, e.g., La'O and Tahbaz-Salehi (2022)). It is immediate to see that $\tilde{S}$ is normalized, monotone, and translation invariant. Assume that $\tilde{S}$ represents the log-cost operator of our cloned economy.

The equilibrium price equation in the cloned economy parametrized to the log-cost of labor $p_0 \in \mathbb{R}$ and the cloned log-productivity $\tilde{x} \in \mathbb{R}^{2n}$ is

$$\tilde{p} = \left(1 - \bar{\beta}\right)(\tilde{x} + p_0 e) + \bar{\beta} \tilde{S}(\tilde{p}). \tag{204}$$

Call $\tilde{\Pi}(p_0, \tilde{x}) \in \mathbb{R}^{2n}$ the unique solution of the previous fixed-point equation.

Remark 3. Define

$$\tilde{x}_j = \begin{cases} x_j & \text{if} \quad j \in \{1, .., n\}, \\ 0 & \text{if} \quad j \in \{n+1, ...2n\}. \end{cases} \tag{205}$$

We have

$$\tilde{\Pi}_j(0, \tilde{x}) = \begin{cases} \Pi_j(0, x) & \text{if} \quad j \in \{1, .., n\}, \\ 0 & \text{if} \quad j \in \{n+1, ...2n\}. \end{cases} \tag{206}$$

[14] This is tantamount to assuming that the production function of sector i can be written as

$$F_i(Q_0, Q) = Z_i \left(\frac{Q_0}{1 - \bar{\beta}}\right)^{1-\bar{\beta}} \left(\frac{G_i(Q_0, Q)}{\bar{\beta}}\right)^{\bar{\beta}}$$

where $G_i : \mathbb{R}^{n+1} \to \mathbb{R}$ is positive homogeneous, non-decreasing, continuous, and with $\max_{\{Q_0, Q : Q_0 + \sum_{i=1}^n Q_i = 1\}} G_i(Q_0, Q) = 1$.

Proof. That $\tilde{\Pi}_j(0,\tilde{x}) = 0$ for all $j \in \{n+1,...2n\}$ follows from

$$\tilde{\Pi}_j(0,\tilde{x}) = \left(1-\bar{\beta}\right)(\tilde{x}_j + 0) + \bar{\beta}\tilde{S}_j\left(\tilde{\Pi}(0,\tilde{x})\right) = \bar{\beta}\tilde{\Pi}_j(0,\tilde{x}) \qquad \forall j \in \{n+1,...2n\}. \tag{207}$$

With this, we have that for all $j \in [n]$

$$\tilde{\Pi}_j(0,\tilde{x}) = \left(1-\bar{\beta}\right)(\tilde{x}_j + 0) + \bar{\beta}\tilde{S}_j\left(\tilde{\Pi}(0,\tilde{x})\right) = \left(1-\bar{\beta}\right)x_j + \bar{\beta}S_j\left(0,\tilde{\Pi}(0,\tilde{x})_1,...,\tilde{\Pi}(0,\tilde{x})_n\right). \tag{208}$$

Since this coincides with the system of equations in (202) that defines $\Pi_j(0,x)$ and it admits a unique solution, we obtain $\tilde{\Pi}_j(0,\tilde{x}) = \Pi_j(0,x)$ for all $j \in \{1,...n\}$. □

Crucially, the economy described by the log-cost operator of Equation (203) satisfies our Assumption 1, and therefore, all our results can be applied to it verbatim. Given the relation established between the prices of the cloned and original economy, these results are easily transferred to the latter, too.

Moreover, consider any production function of the form

$$F_i(Q_0,Q) = Z_i G_i(Q_0,Q) \tag{209}$$

where $G_i : \mathbb{R}^{n+1} \to \mathbb{R}$ is positive homogeneous, non-decreasing, continuous, and "normalized" with $\max_{\{Q_0,Q:Q_0+\sum_{i=1}^n Q_i=1\}} G_i(Q_0,Q) = 1$, i.e., a CRS, monotone, continuous, production function. Consider, a compact subset B of $\mathbb{R}^{n+2}$. For every $\varepsilon > 0$ there is a production function with a log-cost of the form in Equation 201 where S_i is a valid log-cost operator and such that the supnorm of the difference between the output of the original production and the one this one is smaller than ε for all productivity and input combinations $(Z_i,Q_0,Q_1,...,Q_N) \in B$. Therefore, *every* economy with production functions that satisfy Equation 209 can be approximated arbitrarily well by an economy that satisfies Assumption 1.

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