A Theory of Public Debt as a Macro-Financial Stability Tool *

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March 26, 2026

Abstract

In an overlapping generation model of secular stagnation, characterized by a low real interest rate and zero nominal interest rate, and where public bonds can carry a safety or risk premium, bubbles can arise in equilibrium. These bubbles can be both leveraged, issued by banks, and unleveraged, issued by investors. In this framework, public debt can serve as a macro-financial stability tool to maximize output and minimize bubbly and sovereign risk. First, it can lift the real interest rate and thus bring the economy out of the zero lower bound, leading output to the potential. Second, it can eliminate the incentive to create bubbles by satisfying the demand for overall assets in general and safe assets in particular. Third, it can keep a positive safety premium or, at least, a zero risk premium, if it does not exceed the maximum "safe" level. We investigate whether and under which conditions public debt can achieve these three objectives. While a public debt level considered "safe" can bring the economy out of the ZLB, it can only prevent leveraged bubbles from emerging, but not unleveraged ones. Indeed, preventing any bubble from occurring ex ante requires a level of debt so high that sovereign risk becomes positive and current/potential output falls because of lower capital accumulation. Therefore, there exists a trade-off between output maximization and the minimization of sovereign risk, on the one hand, and the minimization of bubbly risk, on the other.

Keywords: Zero lower bound, Asset Bubbles, Public Debt.
JEL classification: E21, E43, E44, E60, E62.

*We are grateful to Federica Romei, Carlo Galli, Antonella Trigari, Tito Cordella, Alberto Martin, Gino Gancia, Daniele Siena and Piergiovanna Natale and the participants to Liverpool Workshop on Macroeconomics, Bank of Finland and CEPR Joint Conference on Frontiers of Monetary Economics in 21st century, and New Challenges for Monetary-Fiscal Policy Interactions. The views expressed are those of the authors and do not necessarily reflect those of De Nederlandsche Bank.

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1 Introduction

Demand and supply developments have forced central banks to put an end to fifteen years of near-zero nominal interest rates. Against this background, the "natural" interest rate, the so-called r^*, which crucially affects the risk of hitting the zero lower bound (ZLB henceforth), seems to have inverted its historical declining trend in advanced countries. The key driver of the rebound in r^* is presumably the large post-pandemic expansion of public debt (Carvalho et al., 2025b), which may have at least partially closed the "saving glut" that dragged r^* down in the pre-pandemic era.[1]

Despite this recent development, the future relevance of the ZLB on the policy rate will depend on the interaction of fiscal policies and long-run phenomena. In particular, with secular trends in demographics and inequality continuing to exert downward pressure on r^* (see Carvalho et al., 2025a), ZLB episodes accompanied by large output losses could remain highly likely, unless the public debt increase is sufficiently large to counteract these forces, stabilizing r^*. In other words, the ZLB remains a risk for the economy in the medium run.[2] Or, as more emphatically pointed out by Blanchard (2023), *"[s]ecular stagnation is not over."*[3]

We investigate, from a theoretical perspective, whether the increase in public debt required to stabilize r^* and output – thereby reducing the frequency of ZLB episodes – is also compatible with two financial-stability objectives: zero sovereign risk, namely a zero probability of sovereign default, and the absence of asset bubbles.

Besides a low, and possibly negative, natural real interest rate, another defining feature of a secular stagnation environment is the possibility that asset price bubbles may arise and burst, with potentially detrimental consequences for financial stability (Caballero and Farhi, 2014). Therefore, when r^* is low and the ZLB risk is amplified, public debt can not only support output stabilization but may also enhance financial stability, preventing speculative behaviors through higher interest rates. This, however, may come at the cost of excessively high sovereign yields.[4] We aim to establish whether, and under which conditions, public debt can serve as a macro-financial stability tool to maximize output and minimize bubbly and sovereign risk in a low interest rate environment.

To that aim, we build a two-period overlapping-generations (OLG) model with investors, entrepreneurs and banks. The OLG structure breaks Ricardian Equivalence so that the level of government debt, which is the key determinant of safety/risk premium, matters also for the equi-

[1] The equilibrium (also called "neutral" or "natural") real interest rate is the real interest rate that prevails when the economy operates at full capacity (Greenspan, 1993). Woodford (2003) formalizes the concept in the context of the baseline New Keynesian model. Pre-pandemic estimates pointed to a very low natural interest rate in the main advanced economies (Holston et al., 2017). Instead, more recent estimates (e.g., Christensen and Mouabbi, 2024; Carvalho et al., 2025b) highlight an rebound in r^* in US and EU, starting from the post-pandemic period.

[2] See Cho et al. (2025).

[3] The notion of "secular stagnation" is used in the paper, consistently with the recent literature (see Summers, 2014, 2015), to refer to a low interest rate environment originating from an excess of saving over desired investment, which pushes r^* downward and makes ZLB episodes more frequent and longer.

[4] "*This decline in the long-run neutral real interest rate increases the future likelihood that the FOMC will be unable to achieve its objectives because of financial instability or because of a binding lower bound on the nominal interest rate.... the fiscal authority can mitigate this problem by issuing more public debt, although such issuance is not without cost. It is, of course, the province of the fiscal authority to determine whether those costs are worth the benefits...* " (Kocherlakota, 2015)

librium allocation. In addition, downward nominal wage rigidities ensure that monetary policy has real effects.

Since we normalize the growth rate of output to zero, a secular stagnation equilibrium can arise in this economy when the real interest rate is negative. In this situation, the nominal interest rate reaches the ZLB and output is below its full employment level. The secular stagnation equilibrium represents fertile terrain for bubbles to grow, as a real interest rate lower than the growth rate of output can drive some asset prices above their fundamental values (Samuelson, 1958; Tirole, 1985).

Bubbles are intrinsically worthless assets, which can be priced positively because they have the potentially desirable feature of allowing agents to transfer wealth over time. However, not all bubbles are born equal. In the model, both investors and banks can buy bubbly assets. While bubbles are only an investment vehicle for investors, banks can use bubbles as collateral to issue safe assets, which carry a safety premium similar to government bonds. Our model distinguishes leveraged bubbles, held by banks, from unleveraged bubbles, held by investors, because we view leveraged bubbles as a more prominent source of financial instability due to the credit boom that occurs before the bubble bursts (Jordá et al., 2015).[5]

Our main contribution is to characterize the equilibria that can emerge for different levels of government debt. Starting from a secular stagnation equilibrium, as the supply of government debt increases, so does the natural interest rate, and output moves closer and closer to its full-employment level. In line with the argument in Kocherlakota (2015), more government debt can reduce, and indeed fully eliminate, the risk of a binding ZLB by raising the natural interest rate.

While this unique full-employment equilibrium can be achieved with a "safe" public debt carrying a positive safety premium and zero sovereign risk, preventing ex-ante any bubble from occurring may require a level of debt so high that, in practice, sovereign risk considerations would become a first-order issue. Indeed, the government's capacity to issue safe public bonds can at most be exploited to shrink the bubble and prevent it from being (partially) leveraged, namely held also by banks, in the unique full-employment equilibrium, not to avoid any bubble. More precisely, increasing the supply of safe government debt all the way to its "safe" upper bound makes leveraged bubbles disappear, but not unleveraged ones held by investors. The latter, though they would be smaller than the (partially) leveraged ones in the final equilibrium, would still pose a threat to financial stability, and they can be avoided only for public debt levels above the maximum "safe", which are therefore associated with a positive sovereign risk.

Even in the absence of sovereign debt risk, higher government debt is not without costs. When capital is endogenous, higher debt crowds out private capital. Hence, to bring the economy out of the ZLB, public debt has to increase so much to crowd out substantially capital, delivering so a current/potential output below the maximum possible level. Likewise, any further increase in the public debt to shrink or prevent the bubble generates output losses. Since higher public debt is also associated with positive sovereign risk, when its level goes above the maximum "safe" one, there exists a trade-off between output maximization and the minimization of sovereign risk, on the one

[5] Schularick and Taylor (2012) and Jordá et al. (2013) document that credit booms typically precede financial crises and deep recessions.

hand, and the minimization of bubbly risk, on the other.

2 Related Literature

Our paper relates to two strands of the economic literature. Firstly, our paper belongs to a theoretical literature, which studies how a low "natural" interest rate, the so-called r^*, can lead to a secular-stagnation environment and explores the policy options to escape persistent ZLB episodes.[6]

In Michau (2022), a "trilemma" between full employment, low inflation and no Ponzi scheme on public debt arises, because only an unbacked public debt, working via wealth effects, can bring the economy out of a ZLB equilibrium. Instead, we show that sufficiently high public debt, by raising r^*, can achieve the same outcome, while simultaneously either preventing asset bubbles or sustaining a zero risk premium – another form of "trilemma". Debt-financed public spending delivers large output gains in our model, unlike Reis (2022), due to a "crowding-in" effect of public debt on capital,[7] which prevails over the traditional "crowding-out" effect at the ZLB. However, away from the ZLB, the crowding-in effect shuts down. As a consequence, current/potential output is lower in a full-employment equilibrium than the maximum attainable level due to the crowding-out effect. This result resembles the one in Acharya and Dogra (2022), where a key role is played by the riskiness of capital, which is absent in our setting.

Caballero and Farhi (2018), Rachel and Summers (2019), Eggertsson et al. (2019) also highlight the role of public debt in offsetting the r^* decline and reducing ZLB risk. Yet Rachel and Summers (2019) and Eggertsson et al. (2019) overlook the financial-stability benefits of a higher natural rate, along with the implication of higher government debt for sovereign yields, as their models abstract from both asset bubbles and sovereign risk. Caballero and Farhi (2018) incorporate bubbles to show how risky and safe ones can mitigate ZLB pressures arising from scarce safe assets. In contrast, we study how public debt issuance can prevent the formation of risky bubbles by expanding the supply of assets in general and safe assets in particular.

Secondly, our paper is linked to the broad literature on rational bubbles, in particular to the contributions of Kraay and Ventura (2007), Aoki and Nikolov (2015) and Bengui and Phan (2018). We share with Kraay and Ventura (2007) the idea that public debt can crowd out asset bubbles, but, while their model focuses only on an unleveraged bubble held by savers/investors, our analysis extends to effect of public debt on the emergence of leveraged bubbles held by the banking sector. The modelization of these two bubbles relates our work to the ones of Aoki and Nikolov (2015) and Bengui and Phan (2018). However, Bengui and Phan (2018) do not characterize the leveraged bubble as the one held by banks, which issue assets carrying a safety premium, but as the one own by general borrowers, who can default on their obligations if the bubble bursts. Furthermore, they do not model a production economy at the ZLB where the level of public debt affects the equilibrium allocation but an endowment economy without monetary and fiscal policies. On the

[6] Several policy solutions to secular stagnation, different from raising public debt, have been studied theoretically by Michau (2018). Ascari and Bonchi (2022) focus in particular on labor income policies.

[7] This effect is similar to the one studied by Woodford (1990).

other hand, Aoki and Nikolov (2015) characterize the leveraged bubble as the one held by banks, but then they emphasize how the identity of the bubble owner matters for the detrimental effects of bubble bursting, not for the emergence of different bubbles at the ZLB as we do.

3 The Model

We study a two-period overlapping-generation (OLG) economy consisting of investors (i), entrepreneurs (e), banks (b), firms, a government (g) in charge of fiscal policy and a central bank in charge of monetary policy. While the OLG structure is particularly suitable for our analysis, the best interpretation of the "generations" in our framework chimes with the entry and exit of agents from financial markets, as in Bernanke and Gertler (1989). Consequently, we think of a "period" as corresponding to the duration of a financial contract.

Investors and entrepreneurs have each of unit mass. Similar to Gertler and Karadi (2011), the representative investor consists of workers and bankers who pool their resources. In the first period, workers supply labor while the household as a whole allocates labor income between deposits, government bonds, loans to entrepreneurs, an intrinsically worthless ("bubbly") asset, and some resources for the fraction of members running the banks. In the second period, all members consume the financial income, including any bank profits, in an equal amount. In addition to the equity received from the investor household, bankers issue deposits to make loans to entrepreneurs and acquire bubbly assets subject to a collateral constraint. Young entrepreneurs receive a fraction of the capital income of old entrepreneurs, which serves as collateral to purchase new capital. This capital is then used as an input for production in the next period, together with the labor supplied by workers. The government levies taxes and issues debt to fund wasteful spending. The central bank seeks to stabilize inflation. While the goods market is perfectly competitive, a downward nominal wage rigidity (DNWR), similar to the one in Schmitt-Grohé and Uribe (2016), can prevent the full employment of resources in the labor market, thus making monetary policy non-neutral.

3.1 Investors

Investors are risk neutral. Their preferences are

$$U_t^i = \mathbb{E}_t \left[v \left(\frac{D_t^i + B_t^i}{Y_t} \right) Y_t + C_{t+1}^i \right], \tag{1}$$

where the term $\mathbb{E}_t(\cdot)$ denotes the expectation operator conditional on the information available at time t. Investors derive utility from second-period consumption, C_{t+1}^i, and from holdings of safe assets, which are deposits issued by banks, D_t^i, and can be bonds issued by the government, B_t^i, if the stock of public debt is not excessively high. The preferences for safe assets are increasing and

convex up to a satiation point $\overline{q}$[8]

$$v\left(\frac{D_t^i + B_t^i}{Y_t}\right) = \begin{cases} -\frac{1}{2}\left(\overline{q} - \frac{D_t^i + B_t^i}{Y_t}\right)^2 & \text{if} \quad 0 \leq \frac{D_t^i + B_t^i}{Y_t} \leq \overline{q} \\ 0 & \text{if} \quad \frac{D_t^i + B_t^i}{Y_t} > \overline{q}. \end{cases} \tag{2}$$

In their optimal decisions, investors take the aggregate output, Y_t, as given. Below we discuss some restrictions on the satiation point depending on the nature of the equilibrium.

In the first period, labor is the only source of income. The budget constraint is

$$D_t^i + B_t^i + L_t^i + \widehat{P}_t^B Q_t^{B,i} + N_t^b = (1-\tau)\frac{W_t}{P_t} h_t, \tag{3}$$

where $\tau \in (0,1)$ is a constant tax rate, W_t is the nominal wage, P_t is the price of the consumption good. While workers supply one unit of labor inelastically ($\bar{h}$), DNWR may imply some involuntary unemployment ($h_t < \bar{h}$). Investors make loans to entrepreneurs, L_t^i, and can purchase bubbly assets $Q_t^{B,i}$ at the price $\widehat{P}_t^B$, as well as investing in deposits and public bonds. Finally, investors provide some equity, $N_t^b = \gamma(1-\tau)W_t h_t / P_t$, to the fraction $\gamma \in (0,1)$ of household members who become bankers.

In the second period, investors receive income from all their financial assets, including profits from the banks, Z_{t+1}^b, and consume. The budget constraint in real terms is

$$C_{t+1}^i = R_t^D D_t^i + \left(1 - \vartheta_{t+1}^g\right) R_t^g B_t^i + R_t^L L_t^i + \widehat{P}_{t+1}^B Q_t^{B,i} + Z_{t+1}^b, \tag{4}$$

where R_t^D is the gross return on deposits, R_t^g is the gross return on public bonds, and R_t^L is the gross return on loans to entrepreneurs. Bubbly assets are risky. In particular, their price at $t+1$ can collapse to zero with probability $\rho \in (0,1)$, while conditional on the bubble surviving, the price is positive ($\widehat{P}_{t+1}^B = P_{t+1}^B > 0$).[9] Instead, public bonds can be safe or risky, depending on whether the stock of public debt is below or above a certain threshold, as shown below. If government bonds are risky, there is a positive probability of sovereign default, and thus a positive fraction of their value, $\vartheta_{t+1}^g > 0$, is not repaid in the event of sovereign default. Instead, the haircut is zero, $\vartheta_{t+1}^g = 0$, if sovereign default does not occur or public bonds are safe.

The investors' optimization problem consists in maximizing (1) subject to (3) and (4). The no-arbitrage condition between deposits and loans to entrepreneurs is

$$R_t^D = R_t^L - \mu_t, \tag{5}$$

[8]The preference for safe assets in (2) allows us to capture, in reduced form, the evidence in Krishnamurthy and Vissing-Jorgensen (2012) that the safety premium is decreasing in the ratio of government debt to GDP and becomes zero after a certain threshold, captured by $\overline{q}$. Gorton et al. (2012) document the remarkable stability of the safe-asset share over time in US data, lending support to the existence of a satiation point for the demand of safe assets.

[9]We assume that a bubble that bursts does not re-emerge again.

where

$$\mu_t = \left(\overline{q} - \frac{D_t^i + B_t^i}{Y_t} \right) \geq 0, \tag{6}$$

while the no-arbitrage condition between deposits and public bonds is

$$R_t^D = \left(1 - E_t \vartheta_{t+1}^g\right) R_t^g. \tag{7}$$

Although loans to entrepreneurs are risk-free, bank deposits command a safety premium because of investors' preferences.[10] Instead, if government bonds are risky, they carry a positive risk premium, $\frac{1}{\left(1-E_t\vartheta_{t+1}^g\right)}$, which generates a spread between their return and that of deposits. However, if public debt is safe, the return on government bonds is the same as on deposits, $R_t^D = R_t^g$ for $E_t\vartheta_{t+1}^g = 0$. In other terms, equations (5) and (7) correspond to two distinct regimes. As will become clear below, when sovereign default probability is zero, government bonds are perceived as safe as deposits and command a positive safety premium through (5). When sovereign risk becomes positive, public bonds no longer provide safety services and therefore trade with a strictly positive risk premium, relative to deposits, in (7).

When the time-t price of the bubble is positive ($\widehat{P}_t^B = P_t^B > 0$), the second no-arbitrage condition that emerges from the investors' optimization problem is between the return on bubbles and the return on loans

$$(1-\rho)\frac{P_{t+1}^B}{P_t^B} \leq R_t^L. \tag{8}$$

If the expected return on bubbles is lower than the return on loans, investors do not hold any bubbles ($Q_t^{B,i} = 0$). Vice versa, if the two returns are the same, bubble holdings are positive ($Q_t^{B,i} > 0$).

3.2 Entrepreneurs

Entrepreneurs are also risk neutral and maximize their expected second-period consumption

$$U_t^e = \mathbb{E}_t C_{t+1}^e. \tag{9}$$

Young entrepreneurs do not get income but they receive a fraction $\phi^e \in (0,1)$ of capital income from old entrepreneurs, $\phi^e r_t^k K_{t-1}$, and can obtain loans from banks, L_t^e, to buy undepreciated capital, $(1-\delta)K_{t-1}$ where $\delta \in [0,1]$ is the depreciation rate, and to invest in new capital, I_t, so that their first-period budget constraint is

$$\phi^e r_t^k K_{t-1} + L_t^e = P_t^k[I_t + (1-\delta)K_{t-1}], \tag{10}$$

where P_t^k is the relative price of capital and investment obeys

$$I_t = K_t - (1-\delta)K_{t-1}. \tag{11}$$

[10] An alternative formulation would be to model loans to entrepreneurs as risky, in which case μ_t would represent the risk premium associated with those assets.

Borrowing is subject to the constraint

$$R_t^L L_t^e \leq \phi^e r_t^k K_{t-1}, \tag{12}$$

where young entrepreneurs pledge the income received/inherited from the previous generation to obtain loans.[11]

In the second period, old entrepreneurs receive capital income net of taxes, sell capital to young entrepreneurs, and pay back their debts. Therefore, their budget constraint in real terms is

$$C_{t+1}^e = [(1-\tau-\phi^e)\, r_{t+1}^k + (1-\delta)P_{t+1}^k]K_t - R_t^L L_t^e. \tag{13}$$

Entrepreneurs invest in capital up to the point where

$$\mathbb{E}_t\left[\frac{(1-\tau-\phi^e)r_{t+1}^k + (1-\delta)P_{t+1}^k}{P_t^k}\right] = (1+\theta_t)R_t^L, \tag{14}$$

where $\theta_t \geq 0$ is the Lagrange multiplier on the collateral constraint (12). The left-hand side of the last expression is the expected return on capital. The right-hand side is the cost of borrowing augmented by the its shadow cost.

3.3 Banks

Banks maximize their second-period expected real profits

$$Z_{t+1}^b = \mathbb{E}_t(R_t^L L_t^b + \widehat{P}_{t+1}^B Q_t^{B,b} - R_t^D D_t^b), \tag{15}$$

which are stochastic because the price of bubbles can collapse to zero.

In addition to receiving capital from investors, banks issue deposits to fund investment in loans to entrepreneurs and bubbles.[12] Therefore, their balance sheet at time t in real terms is

$$L_t^b + \widehat{P}_t^B Q_t^{B,b} = N_t^b + D_t^b. \tag{16}$$

A collateral constraint limits banks' deposit issuance according to

$$\phi^D D_t^b = \phi^L L_t^b + \phi^B \widehat{P}_t^B Q_t^{B,b}. \tag{17}$$

Specifically, to issue safe deposits, banks have to guarantee a share ϕ^D of their value by pledging assets, namely loans and bubbles, which, however, are not fully pledgeable. The pledgeability of loans and bubbles is measured by, respectively, $\phi^B \in [0,1]$ and $\phi^L \in [0, \phi^B)$.[13]

[11] Drechsel (2023) documents the prevalence of earnings-based borrowing constraints, like the one in equation (12), in US data.

[12] We interpret bank deposits in a broad sense, potentially including other bank liabilities, such commercial paper and other asset-backed securities that carry the same interest rate.

[13] Dividing both sides of (17) by N_t^b, we can reinterpret the constraint as a maximum (risk-weighted) leverage ratio.

Banks maximize (15) subject to the balance sheet constraint (16) and the collateral constraint (17). When the price of the bubbly asset at time t is positive, profit optimization yields a no-arbitrage condition between bubbles and loans

$$(1-\rho)\frac{P_{t+1}^B}{P_t^B}+\left(\frac{\phi^B-\phi^L}{\phi^D-\phi^L}\right)(R_t^L-R_t^D)\leq R_t^L. \tag{18}$$

As for investors, when the condition holds with a strict inequality sign, banks do not purchase any bubble ($Q_t^{B,b}=0$), while when the condition holds with equality, banks' bubble holdings are positive ($Q_t^{B,b}>0$). However, differently from investors, banks value the bubble not only for its return but also as a collateral instrument. The second term on the left-hand side of (18) measures the "collateral premium" of bubbles for banks, that is, the additional value, relative to investors, from holding the bubbly asset because of its potential pledgeability to raise more deposits.[14]

3.4 Firms

Firms operate for two periods but produce only in the second, when they hire labor from worker-investors and rent capital from entrepreneurs in perfectly competitive markets to produce final goods using a constant-returns-to-scale technology:

$$Y_t=h_t^{\alpha}K_{t-1}^{1-\alpha}, \tag{20}$$

where $\alpha\in(0,1)$ is the labor share. The first-order conditions are standard and imply that factor prices equal their marginal product

$$\frac{W_t}{P_t}=\alpha\frac{Y_t}{h_t}\qquad\text{and}\qquad r_t^k=(1-\alpha)\frac{Y_t}{K_{t-1}}. \tag{21}$$

Even though the labor market is perfectly competitive, involuntary unemployment is possible because of a DNWR, which we assume to be of the form

$$W_t=\max\left\{\overline{W}_t,W_t^{flex}\right\}, \tag{22}$$

where $\overline{W}_t\equiv\beta\Pi^*W_{t-1}$ is a "wage norm" and $W_t^{flex}\equiv\alpha P_tK_{t-1}^{1-\alpha}\bar{h}^{\alpha-1}$ is the full-employment competitive (or flexible) nominal wage.[15] As in Schmitt-Grohé and Uribe (2016), workers do not accept a nominal wage below the wage norm, given by a constant fraction $\beta\in(0,1]$ of the past wage indexed to the gross inflation target $\Pi^*>1$. When the actual nominal wage falls below the wage

[14] Alternatively, we can write equation (18) as

$$(1-\rho)\frac{P_{t+1}^B}{P_t^B}\leq\left(\frac{\phi^D-\phi^B}{\phi^D-\phi^L}\right)R_t^L+\left(\frac{\phi^B-\phi^L}{\phi^D-\phi^L}\right)R_t^D, \tag{19}$$

which highlights that banks hold bubbles if their return is a linear combination on the returns on loans and deposits, where the weights are a function of the collateral constraint parameters.

[15] Schmitt-Grohé and Uribe (2016) and Eggertsson et al. (2019) discuss the empirical relevance of downward nominal wage rigidities.

norm, the DNWR becomes binding. In this case, the labor market does not clear and some workers remain unemployed ($h_t < \bar{h}$).

3.5 Government and Central Bank

The government budget constraint in real terms is

$$B_t^g = (1 - \vartheta_t^g) R_{t-1}^g B_{t-1}^g + G_t - \tau Y_t. \tag{23}$$

The government collects taxes and issues one-period bonds, B_t^g, to finance public expenditure, G_t, and repay the bonds that mature inclusive of interest rate spending. The government can partially default on its outstanding debt B_{t-1}^g with probability ρ_t^g, which is governed by the following rule:[16]

$$\rho_t^g = \begin{cases} 0 & if \quad B_{t-1}^g \leq \bar{B}_{safe,t-1}^g \left(D_{t-1}^i, Y_{t-1}\right) \\ \rho^g & if \quad B_{t-1}^g > \bar{B}_{safe,t-1}^g \left(D_{t-1}^i, Y_{t-1}\right) \end{cases} \tag{24}$$

where $\rho^g \in (0,1)$ and

$$\bar{B}_{safe,t-1}^g \left(D_{t-1}^i, Y_{t-1}\right) \equiv \left(\bar{q} - \frac{D_{t-1}^i}{Y_{t-1}}\right) Y_{t-1}. \tag{25}$$

(24) establishes that there is a maximum level for the public debt, given by (25), below which the latter is considered "safe", with consequent zero probability of sovereign default, and above which sovereign default can occur with a constant positive probability.[17] The maximum safe public debt level depends on output and the supply of the alternative safe assets provided by banks.

Since we want to study the role of public debt as a policy tool to pursue three different goals – maximizing output, minimizing the bubbly and sovereign risk – we assume that public debt is exogenously set by the government ($B_t^g = \bar{B}^g$). However, if sovereign defaults, a fraction of past public debt, $\vartheta_t^g = \vartheta^g$ where $\vartheta^g \in (0,1)$, loses value. Instead, if sovereign does not default, there is no haircut on the value of outstanding public debt ($\vartheta_t^g = 0$). In other terms, the haircut on public debt is given by

$$\vartheta_t^g = \begin{cases} 0 & if \quad no \quad default \\ \vartheta^g & if \quad default. \end{cases} \tag{26}$$

In the event of sovereign default, there is also a partial market exclusion, that is a fraction of new public debt $\psi \in (0,1)$ cannot be financed in the period of default, while all the desired public debt can be financed if sovereign does not default. This can be formally expressed as

$$B_t^g = \begin{cases} \bar{B}^g & if \quad no \quad default \\ (1-\psi)\bar{B}^g & if \quad default. \end{cases} \tag{27}$$

[16] We specifically assume that sovereign defaults on its outstanding debt, at time t, before issuing new public debt.

[17] This assumption is consistent with the evidence of Ardagna et al. (2007), who show that only countries with above-average levels of debt experience a rise in sovereign yields as public debt increases. On the other hand, the form of the threshold (25), together with the assumed functional form of investors' preferences for safety (2), implies that a strictly positive safety premium on public bonds is always associated with zero sovereign risk, while a positive risk premium is always associated with a zero safety premium. We view this case as the standard one.

The central bank sets the nominal interest rate $\mathcal{R}_t$ according to the Taylor-type rule that takes into explicit account the ZLB

$$\mathcal{R}_t = \max\left\{1, R_t^* \Pi^* \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right\}, \tag{28}$$

where R_t^* is the natural real interest rate (that is, the real interest rate consistent with output being at the full-employment or potential level, $Y_t^* = \bar{h}^\alpha K_{t-1}^{1-\alpha}$) and $\phi_\pi > 1$.[18]

The Fisher equation links the nominal and the actual real interest rate

$$R_t = \frac{\mathcal{R}_t}{\mathbb{E}_t \Pi_{t+1}}. \tag{29}$$

If output is at full employment and inflation is on target, the actual real interest rate equals its natural value.

3.6 Equilibrium

Equilibrium in the goods and labor markets requires

$$Y_t = C_t^i + C_t^e + I_t + G_t \qquad \text{and} \qquad h_t \leq \bar{h}. \tag{30}$$

The market clearing conditions for loans, deposits and government debt are

$$L_t^e = L_t^i + L_t^b \tag{31}$$

$$D_t^i = D_t^b \tag{32}$$

$$B_t^i = B_t^g \tag{33}$$

Finally, we assume that, in each period, bubbles are in fixed supply and normalize the quantity to one

$$Q_t^{B,b} + Q_t^{B,i} = 1, \tag{34}$$

if and only if $\widehat{P}_t^B = P_t^B > 0$.

The market for loans plays a central role in the financial system because it determines the risk-free real interest rate:

$$R_t = \frac{\phi^e \left(1-\alpha\right) Y_t}{\left(1-\tau\right)\alpha Y_t - B_t^g - P_t^B}. \tag{35}$$

The latter, which amounts to the natural interest rate R_t^* for $Y_t = Y_t^*$, is the real rate on loans that

[18] Relative to a standard interest rate rule (Taylor, 1993), for analytical tractability the monetary policy rule (28) does not include a response to the output gap. Our results would be qualitatively unchanged if we relaxed this assumption. Although the "natural and "real" rates refer to the interest rate charged on loans, $R_t^* \equiv R_t^{L,*}$ and $R_t \equiv R_t^L$, we decide to use the general and simpler notation R_t^* and R_t (see also (29) and (35)) because, in most of equilibria considered, $R_t^* \equiv R_t^{L,*} = R_t^{D,*}$ and $R_t \equiv R_t^L = R_t^D$.

equates in the demand from entrepreneurs L_t^e and the total supply of investors and banks,

$$L_t^i + L_t^b = (1-\tau)\,\alpha Y_t - B_t^g - P_t^B, \tag{36}$$

where L_t^i and L_t^b are derived, respectively, from (3) and (16) after some manipulations and taking $\widehat{P}_t^B = P_t^B > 0$. The equation above, rewritten as

$$(1-\tau)\,\alpha Y_t = L_t^e + B_t^g + P_t^B \tag{37}$$

by using the market clearing condition for loans, defines also the equilibrium condition in the overall financial market, in which the total demand for assets, or total savings, $(1-\tau)\,\alpha Y_t$, meets the total supply, given by loans, public bonds and bubbles.[19] Once explained the equilibrium conditions in the single markets, we can define a competitive equilibrium in our economy.

Given initial conditions on wages and government debt (W_{-1} and B_{-1}^g), a competitive equilibrium with DWNR consists of a sequence of prices $\left\{P_t, W_t, R_t^D, R_t^g, R_t, \mathcal{R}_t, P_t^B, P_t^k, r_t^k\right\}_{t=0}^{\infty}$ and quantities $\left\{L_t^i, D_t^i, B_t^i, Q_t^{B,i}, C_t^i, L_t^e, C_t^e, L_t^b, D_t^b, Q_t^{B,b}, Z_t^b, Y_t, h_t, K_t, G_t, I_t, B_t^g\right\}_{t=0}^{\infty}$ such that investors and entrepreneurs maximize their utility; banks and firms maximize their profit; the government satisfies its budget constraint (23); the probability of sovereign default ρ_t^g is governed by (24); haircut ϑ_t^g and capital market exclusion ψ_t^g are governed by (26) and (27); the central bank follows its policy rule (28); the Fisher equation (29) holds; the goods, loans, deposits, government bonds and bubbles markets clear; and the labor market either clears ($h_t = \bar{h}$ for $W_t = W_t^{flex}$) or labor rationing occurs ($h_t < \bar{h}$ for $W_t = \beta\Pi^* W_{t-1}$).

4 Steady State Equilibrium

Since variables are constant in steady state, we can drop the time subscript. In steady state, sovereign does not default and thus there is no haircut and capital market exclusion ($\vartheta^g = 0$ and $B^g = \bar{B}^g$), though public debt can be considered risky ($\rho^g \geq 0$). We focus on steady state equilibria in which the borrowing constraint for entrepreneurs (12) is binding, there is a fixed capital endowment, $K = \bar{K}$, and thus capital does not depreciate ($\delta = 0$) with investment being zero. We also take $\beta = 1$ in the DNWR (22), so that the inflation rate always equals the targeted level.[20] To prevent the demand for safety assets from being always satiated in steady state, we need to impose

$$\frac{\gamma\,(1-\tau)\,\alpha\phi^L}{(\phi^D - \phi^L)} < \overline{q} < (1-\gamma)\,(1-\tau)\,\alpha,$$

[19] Deposits issued by banks do not appear in 37 because it is in zero net supply. Although loans are formally liabilities, they can be thought as associated with the issuance of some assets (e.g., bonds and promissory notes) by entrepreneurs. For this reason, we report the loan demand L_t^e, instead of the supply, $L_t^i + L_t^b$, in equation 37.

[20] These equilibria are meaningful if entrepreneurs' consumption and banks profits are non-negative ($C^e \geq 0$ and $Z^b \geq 0$). We check that these conditions hold in all results discussed below. The Appendix considers two extensions in which we study, respectively, the general case of $\beta < 1$ and endogenous capital. In both cases, the substance of our main results remains unchanged.

which also implies a strictly positive supply of loans from investors if the demand for safe assets is satiated. Finally, to replicate the credit boom characterizing a leveraged bubble, positive bank's bubble holdings need to increase the amount of loans supplied by banking sector, and this requires

$$0 < \phi^L < (1-\gamma)\phi^D < \phi^D < \phi^B < 1.$$

None of our results, of course, depend on these parametric restrictions.

Steady state equilibria differ along three dimensions. The first is related to the level of output and of the nominal interest rate. The second is whether bubbles have a positive price.[21] The third is whether the sovereign risk, expressed by ρ^g in (24), is positive. In what follows, we first consider bubble-less equilibria to investigate how government debt affects the determination of output relative to its potential level and of the nominal interest rate.[22] Second, we derive conditions for the emergence of different bubbly equilibria depending on the level of government debt, taking output as given and assuming public bonds are safe. Finally, we investigate how the level of government debt simultaneously determines the bubble, economic activity and sovereign risk premium.

4.1 Bubbleless Equilibria: Output, Nominal and Real Rates

In a bubble-less equilibrium, denoted by NB that stands for "no-bubble", we can study the determination of the key macroeconomic variables, through aggregate demand and supply, which are both characterized by two regimes, governed respectively by ZLB and DNWR.

The DNWR (22) establishes the regime of aggregate supply (AS). For $\Pi \geq \beta\Pi^* = \Pi^*$, the full-employment nominal wage, $W_t^{flex} \equiv \alpha P_t \bar{K}^{1-\alpha}\bar{h}^{\alpha-1}$, is at least equal to the wage norm, $\Pi^* W_{t-1}$. Hence, the labor market clears and the AS is given by the potential output, $Y_{AS} = \bar{K}^{1-\alpha}\bar{L}^{\alpha} = Y^*$. On the other hand, if the flexible nominal wage level falls below the wage norm, wage cannot fall enough to clear the labor market. As a consequence of binding DNWR, price/wage inflation is given, $\Pi = \beta\Pi^* = \Pi^*$, while output and employment are demand-driven. Figure 1 depicts the two components of the AS curve as a vertical (AS^{FE}) and a horizontal segment (AS^{DNWR}).

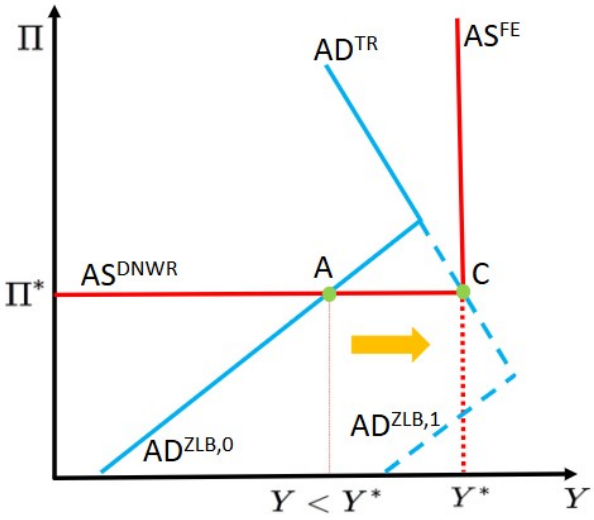
The interest rate rule (28) determines the regime of aggregate demand (AD).[23] If the ZLB does not bind ($\mathcal{R} > 1$), as inflation increases the central bank raises the nominal rate more than proportionally ($\phi_\pi > 1$), so that the real rate also raises, shrinking demand and stabilizing inflation. Therefore, AD relates negatively to inflation for a positive policy rate, as depicted by the downward-sloping curve AD^{TR} in Figure 1. In contrast, AD relates positively to inflation at the ZLB ($\mathcal{R} = 1$), as depicted by the upward-sloping curve AD^{ZLB} in the same figure. At the ZLB, the real interest

[21] As standard in OLG models (see, e.g., Martin and Ventura, 2011), bubbly equilibria require expectations to be "coordinated," that is, agents (young investors and/or banks) buy the bubble in the first period only under the expectation of being able to re-sell the asset in the second period to the next generation of agents at a higher price. In turn, the next generation should have the same expectation, and so on for the indefinite future. In this respect, the burst of the bubble can be interpreted as a revision in agents' expectations sustaining the bubbly equilibrium.

[22] When $\beta = 1$, inflation is always on target. As we show in the Appendix, for $\beta < 1$, inflation can be below target and government debt can also affect its equilibrium level.

[23] For the sake of exposition, we report the equations for AD in the two regimes in Appendix A.1.1.

Figure 1: TR-FE and ZLB-U equilibria



rate depends only on inflation via the Fisher equation, and a surge in inflation reduces the real rate, expanding demand.

The inflation rate at which the ZLB becomes binding, Π^{kink}, is represented graphically as a kink in the AD curve, and derived by equating the two arguments in the max operator of (28):

$$\Pi^{kink} = \left(\frac{1}{R^*_{NB}}\right)^{\frac{1}{\phi_\pi}} (\Pi^*)^{\frac{\phi_\pi - 1}{\phi_\pi}}.$$

For a given Π^*, a high Π^{kink} causes a high risk of hitting the ZLB since a low R^*_{NB} provides the central bank with little room to cut the policy rate. In other terms, the location of the AD kink reflects the natural rate level and the associated ZLB risk, affecting the intersection point between the AD and AS curves and thus the resulting the steady state equilibrium. Under the condition

$$R^*_{NB} < \frac{1}{\Pi^*} < 1, \tag{38}$$

the negative natural interest rate, $R^*_{NB} < 1$, is so low to make the ZLB binding even for a positive inflation target, $\Pi^* > 1$. Graphically, the AD kink lies above the inflation target, $\Pi^{kink,0} > \Pi^*$, and the upward-sloping curve $AD^{ZLB,0}$ crosses AS^{DNWR} at point A, in which both ZLB and DNWR bind. This point is a ZLB equilibrium featuring (negative) real rate above the natural level, binding ZLB, and output below the potential, as well as inflation at the minimum level imposed by the DNWR, which here coincides with the inflation target:[24] $R^*_{NB} < R_{NB,ZLB} = \frac{1}{\beta\Pi^*} = \frac{1}{\Pi^*} < 1$,

[24]The bubble-less steady state equilibria in our model share the same features and the associated conditions of the ones in Ascari and Bonchi (2022) (see Appendix A.1.2). However, Ascari and Bonchi (2022) express the conditions in terms of Π^*, taking R^* as given, while we express the same conditions in terms of R^*, or $\bar{B}^g$, taking Π^* as given. For $\beta = 1$, inflation is always at the target. Instead, for $\beta < 1$, inflation is positive but below the target, $1 < \Pi = \beta\Pi^* < \Pi^*$, in the ZLB equilibrium.

$\mathcal{R} = 1$, $Y = Y_{ZLB} < Y^*$, $\Pi = \beta\Pi^* = \Pi^*$. Instead, under the condition

$$R^*_{NB} > \frac{1}{\Pi^*} = \frac{1}{\beta\Pi^*}, \tag{39}$$

even if inflation reaches its minimum level imposed by the DNWR, namely the inflation target, $\beta\Pi^* = \Pi^*$, the natural rate is sufficiently high to prevent the ZLB from binding. In graphical terms, this translates into a very low AD kink, $\Pi^{kink,1}$, and thus no intersection points between $AD^{ZLB,1}$ and AS^{DNWR}. Indeed, the unique FE equilibrium, denoted by point C, occurs at the intersection of AD^{TR} and AS^{FE}, and it features real rate at the natural level, no binding ZLB, output at the potential (= full employment, FE) and inflation at the target: $R^*_{NB} = R_{NB,FE}$, $\mathcal{R} > 1$, $Y = Y^*$, $\Pi = \Pi^*$.

Since the natural interest rate in a bubble-less economy is pin down by the public debt level, as shown by (35) for $P^B = 0$ and $Y = Y^*$, condition (38) can be rewritten in terms of $\bar{B}^g$ as

$$\bar{B}^g < \bar{B}^g_{ZLB}, \tag{40}$$

while condition (39) can be alternatively written as

$$\bar{B}^g > \bar{B}^g_{ZLB}, \tag{41}$$

where

$$\bar{B}^g_{ZLB} \equiv \left[(1-\tau)\,\alpha - \Pi^*\phi^e\,(1-\alpha)\right] Y^*. \tag{42}$$

Put differently, the government, by setting the level of public debt, can set the natural interest rate and choose the steady state equilibrium and the associated output level in a bubble-less economy.[25]

4.2 Bubbly Equilibria: Bubble Price and Holdings

The abovementioned ZLB and FE equilibra can be bubble-less, as assumed before, or bubbly. We now focus on the latter case, deriving the conditions under which different bubbly steady state equilibria can emerge, starting from a bubble-less environment. The existence of asset bubbles crucially depends on the features of the pre-existing bubble-less steady state equilibrium, in particular on the associated real interest rate, R_{NB}. In this section, we assume public bonds are safe, because public debt is below the threshold (25), and thus $\rho^g = 0$ in (24). Therefore, $R^D = R^g = R - \mu$ results from (5) and (7), and public bonds can carry a positive safety premium $\mu \geq 0$.

Considering the general case in which both investors and banks can hold bubbly assets ($Q^{B,i} \geq 0$

[25] Figure 1 is purely illustrative. Indeed, we plot the downward-sloping AD curve as linear, while it is in general non-linear. Furthermore, an increase in $\bar{B}^g$ moves not only the AD kink, and thus AD^{ZLB}, but also AD^{TR}. However, the effect of public debt on AD that is relevant for the determination of the equilibrium is that on AD^{ZLB}, not on AD^{TR}. More generally, our results are unchanged with the general non-linear representation of the AD^{TR} that moves with $\bar{B}^g$.

and $Q^{B,b} \geq 0$) and taking Y as given, the bubble price is positive under the condition[26]

$$P^B = \left[(1-\tau)\,\alpha - \frac{\phi^e\,(1-\alpha)}{(1-\rho)}\right] Y - \bar{B}^g > 0 \tag{43}$$

which can be re-written as

$$\frac{\phi^e\,(1-\alpha)\,Y}{(1-\tau)\,\alpha Y - \bar{B}^g} = R_{NB} < (1-\rho)\,. \tag{44}$$

Condition (44) is standard in the literature (see, e.g, Weil, 1987). In a bubble-less environment, the real rate falls below the risk-adjusted return of the bubble, namely its survival probability $1-\rho$, when there is a large amount of savings but few assets to allocate them. This creates an incentive to invest in bubbly assets despite they are worthless. However, as shown by (44), public debt can affect the bubble existence. Indeed, a higher $\bar{B}^g$ increases the overall asset supply in (37), decreasing the residual amount of savings can be invested in the bubble. This raises R_{NB} and reduces the profitability of holding bubbly assets. Therefore, only if the public debt is not extremely large, specifically

$$\bar{B}^g < \bar{B}^g_U \tag{45}$$

where

$$\bar{B}^g_U\,(Y) \equiv \left[(1-\tau)\,\alpha - \frac{\phi^e\,(1-\alpha)}{(1-\rho)}\right] Y, \tag{46}$$

condition (44) holds and a bubble can emerge because there are still residual savings to be absorbed. When public debt is safe, it can affect the existence of the bubble but also its nature as expressed by the bubble holdings of investors and banks,

$$0 \leq Q^{B,b} = \frac{\Phi \mu_{NB} Y}{P^B} \leq 1 \qquad \text{and} \qquad 0 \leq Q^{B,i} = 1 - \frac{\Phi \mu_{NB} Y}{P^B} \leq 1, \tag{47}$$

which range between 0 and 1 because the bubble is in fixed unit supply and where $\Phi \equiv \frac{\phi^D}{\phi^B + \phi^L\left(\frac{\phi^B - \phi^D}{\phi^D - \phi^L}\right)} < 1$ and

$$\mu_{NB} \equiv \overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right) \gamma\,(1-\tau)\,\alpha - \frac{\bar{B}^g}{Y} \geq 0. \tag{48}$$

Bubbly investment yields not only the risk-adjusted return $1-\rho$, as for investors, but also a collateral premium for banks. Indeed, banks can also use bubbles as collateral to issue additional liabilities and finance lending. The resulting increase in bank profits depends on the additional liabilities generated to finance lending, Φ, and on the bubble-less safety premium μ_{NB}, which captures the spread between loan returns and funding costs. Both therefore affect banks' incentives to hold bubbly assets, as shown by $Q^{B,b}$ in (47). Since $Q^{B,b}$ depends on μ_{NB}, public debt can influence the banks' bubble holdings through the safety premium, for which the government competes with banks in the market for safe assets, and which depends negatively on the supply of public bonds in (48). More precisely, only if public debt is low enough to keep the safety premium positive ($\mu_{NB} > 0$),

[26] The equations for the bubble price and holdings can be obtained by solving a system of equations, which comprises (3), (5), (6), (7), (8), (12), (16), (17), (31), (32), (33) and (34).

that is

$$\bar{B}^g < \bar{B}^g_L, \tag{49}$$

where

$$\bar{B}^g_L(Y) \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma(1-\tau)\alpha\right] Y \tag{50}$$

there is still incentive for banks to collateralize the bubble and thus to hold bubbly assets ($Q^{B,b} > 0$). Putting together equations (45) and (49), we get the condition

$$\bar{B}^g < \bar{B}^g_L(Y) < \bar{B}^g_U(Y) \tag{51}$$

for which a bubbly equilibrium exists, and it is *partially leveraged* as both investors and banks hold bubbly assets, $Q^{B,i} > 0$ and $Q^{B,b} > 0$ in (47).[27] Instead, for

$$\bar{B}^g_L(Y) \leq \bar{B}^g < \bar{B}^g_U(Y), \tag{52}$$

A bubbly steady state equilibrium still emerges, but it is *unleveraged*, as bubbly assets are held exclusively by investors ($Q^{B,i} = 1$). Indeed, $\bar{B}^g_{NB,L}(Y) \leq \bar{B}^g$ drives the safety premium to zero, banks no longer benefit from using bubbles as collateral and exit the bubble market ($Q^{B,b} = 0$).[28]

4.3 Public Debt as a Macro-financial Stability Tool

Since public debt affects output, bubble price and sovereign risk in our framework, we study whether it can prevent the ZLB from binding and the emergence of asset bubbles in steady state, while carrying a positive safety or a zero risk premium. We focus here on the feasibility, rather than the desirability, of such an outcome, in order to assess the effectiveness of public debt as macro-financial stability tool to maximize output and minimize bubbly and sovereign risk. Specifically, if public debt $\bar{B}^g$ is increased to move the economy from an initial bubble-less ZLB equilibrium (point A, Figure 1) to a final FE equilibrium (point C, Figure 1), we study whether a bubble may emerge in the final equilibrium, and whether sovereign risk is positive or zero.[29]

Our main analytical result can be encapsulated in the form of a proposition, which is based on two main sets of parametric assumptions. The first set of assumptions characterizes the nature of the initial equilibrium, which is assumed to be a bubble-less ZLB equilibrium, where a partially

[27] To be precise, a partially leveraged bubble to exist also requires $\bar{B}^g \leq \bar{B}^g_{upper}$, where $\bar{B}^g_{upper}$ is an upper bound on the public debt value that guarantees $0 \leq Q^{B,b} \leq 1$ and $0 \leq Q^{B,i} \leq 1$ in (47), as explained in Appendix A.2.2

[28] Conditions (51) and (52) require specific parametric assumptions implying $\bar{B}^g_{NB,U}(Y) > \bar{B}^g_{NB,L}(Y)$ (see Appendix A.2.2). Under opposite assumptions delivering $\bar{B}^g_{NB,U}(Y) < \bar{B}^g_{NB,L}(Y)$, the condition $\bar{B}^g < \bar{B}^g_{NB,U}(Y) < \bar{B}^g_{NB,L}(Y)$ would be compatible with (45) and (49), allowing for a positive bubble price and a positive safety premium. However, the public debt could reach values higher than the upper bound, $\bar{B}^g > \bar{B}^g_{upper}$ in this case, violating the constraints $0 \leq Q^{B,b} \leq 1$ and $0 \leq Q^{B,i} \leq 1$.

[29] For $\beta = 1$, a mixed $ZLB - FE$ equilibrium arises if $\bar{B}^g = \bar{B}^g_{ZLB}$. It shares with the ZLB equilibrium the binding ZLB ($\mathcal{R} = 1$) and $\Pi = \beta\Pi^* = \Pi^*$, and with the FE equilibrium it shares $R = R^*$ and $Y = Y^*$. We study this intermediate case, along with others emerging for $\beta < 1$, in Appendix A.2.2.

leveraged bubble can emerge.[30] Instead, the second set of assumptions,

$$\frac{\bar{B}^g}{Y_{ZLB}} < \left(\frac{\bar{B}^g}{Y}\right)_{safe} \tag{53}$$

and

$$\bar{B}^g_{safe}\left(Y_{ZLB}\right) < \bar{B}^g_{ZLB}, \tag{54}$$

where

$$\left(\frac{\bar{B}^g}{Y}\right)_{safe} \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha\right] \qquad \bar{B}^g_{safe}\left(Y_{ZLB}\right) \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha\right] Y_{ZLB},$$

implies, on the one hand, that public debt is initially safe, because the public debt-to-GDP *ratio* in the initial ZLB equilibrium falls below its maximum safe level $\left(\frac{\bar{B}^g}{Y}\right)_{safe}$ in (53). This leaves some safe fiscal space to exploited by the government. On the other hand, however, the corresponding maximum safe public debt *level* $\bar{B}^g_{safe}\left(Y_{ZLB}\right)$ could be not sufficiently high in the original steady state to stimulate output up to the potential level, as established by (54). It is worth noting that $\left(\frac{\bar{B}^g}{Y}\right)_{safe}$ is associated with both zero safety and risk premium on public bonds ($\mu = \rho^g = 0$). Analogously, for any given Y, $\bar{B}^g_{safe}\left(Y\right)$ is associated with $\mu = \rho^g = 0$. Instead, for $\bar{B}^g \leq \bar{B}^g_{safe}\left(Y\right)$, $\mu \geq 0$ and $\rho^g = 0$, while $\mu = 0$ and $\rho^g > 0$ for $\bar{B}^g > \bar{B}^g_{safe}\left(Y\right)$.[31] A proof of the proposition below is given in Appendix A.2.

Proposition 1. ***Public debt cannot simultaneously deliver output at the potential, zero bubble size and zero sovereign risk in steady state.*** *For $\bar{B}^g \leq \bar{B}^g_{safe}\left(Y^*\right)$ implying a zero sovereign risk ($\rho^g = 0$), the economy can move from the ZLB equilibrium to the FE equilibrium, reaching the potential output level ($Y = Y^*$), but partially leveraged or unleveraged bubbles can still occur in the final equilibrium ($P^B > 0$ with $Q^{B,b} \geq 0$ and $Q^{B,i} \leq 1$). Only if $\bar{B}^g > \bar{B}^g_{safe}\left(Y^*\right)$ and sovereign risk is positive ($\rho^g > 0$), bubbles can be prevented in the final FE equilibrium ($P^B = 0$).*

Proposition 1 highlights the limits of using public debt for macro-financial stability purposes, pointing to the existence of a trade-off between the minimization of bubbly risk and that of sovereign risk. In a ZLB equilibrium, the real rate is very low because negative, but its natural level is even lower, $R^* < R_{ZLB} < 1$, due to anemic demand. A large increase in $\bar{B}^g$ is required to boost demand and raise R^* so to lead the economy to the FE equilibrium. Equally, the associated rise in output, from Y_{ZLB} to Y^*, is also substantial thereby expanding the available safe fiscal space – higher Y raises $\bar{B}^g_{safe}(Y)$. Therefore, the public debt-to-GDP ratio $\frac{\bar{B}^g}{Y}$ can remain below its maximum safe level, in the FE equilibrium. This ensures that a public debt level corresponding to potential output, $\bar{B}^g_{ZLB} < \bar{B}^g$, can carry a positive or zero safety premium ($\mu_{FE} \geq 0$) and zero risk premium

[30] All assumptions are detailed in Appendix A.2.1. Since we refer exclusively in this section to variables in a bubble-less steady state equilibrium, we omit the subscript NB denoting such equilibrium.

[31] Given (2), (5), (7), (24), (25), (33), as well as $D^b\left(Y\right) = D^i\left(Y\right) = \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha Y$, we obtain $R^D = R^g = R$ for $\bar{B}^g = \bar{B}^g_{safe}\left(Y\right)$, because $\mu = 0$ and $\rho^g = 0$. For $\bar{B}^g \leq \bar{B}^g_{safe}\left(Y\right)$, we obtain $R^D = R^g = R - \mu$ with $\mu \geq 0$ and $\rho^g = 0$, while $R^D = R = \left(1 - \vartheta^g \rho^g\right) R^g$, with $\mu = 0$ and $\rho^g > 0$, arises for $\bar{B}^g > \bar{B}^g_{safe}\left(Y\right)$.

($\rho^g = 0$), $\bar{B}^g \leq \bar{B}^g_{safe}(Y^*)$, though it cannot also rule out the emergence of asset bubbles. Indeed, the resulting condition $\bar{B}^g_{ZLB} < \bar{B}^g \leq \bar{B}^g_{safe}(Y^*) = \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*)$ allows for the existence of partially leveraged and unleveraged bubbles, consistently with (51) and (52).

In the new equilibrium, higher output increases savings and thus the price of the bubble that could emerge (see (43)).[32] In particular, the bubble size is the largest possible for $Y = Y^*$, and only further increases in $\bar{B}^g$, above the maximum safe level $\bar{B}^g_{safe}(Y^*)$, can prevent bubbles from arising in the FE equilibrium. Avoiding bubbles therefore requires $\bar{B}^g_{ZLB} < \bar{B}^g_{safe}(Y^*) = \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) \leq \bar{B}^g$, which entails zero safety premium and positive sovereign risk in the final FE equilibrium ($\mu_{FE} = 0$ and $\rho^g > 0$) – though it is incompatible with both (51) and (52).

To be more precise, the government capacity of issuing safe public bonds can at most be exploited to shrink, not to eliminate, the bubble and prevent it from being partially leveraged. Indeed, the maximum "safe" public debt corresponding to zero sovereign risk coincides with the debt level preventing the bubble from being held by banks in (50),

$$\bar{B}^g_{safe}(Y) = \bar{B}^g_L(Y).$$

This is because $\bar{B}^g_{safe}(Y)$ is not only associated with $\rho^g = 0$ but also with a zero safety premium ($\mu = 0$). When public debt is initially safe and its level increases, the safety premium on bank liabilities shrinks. As a result, for $\mu = 0$, banks no longer benefit from using bubbles as collateral to issue liabilities, and exit the bubble market. Thus, if public debt raises precisely up to $\bar{B}^g_{safe}(Y^*)$ but below the threshold $\bar{B}^g_U(Y^*)$ – namely $\bar{B}^g_{ZLB} < \bar{B}^g_{safe}(Y^*) = \bar{B}^g_L(Y^*) = \bar{B}^g < \bar{B}^g_U(Y^*)$ – the bubble can still emerge in the FE equilibrium but can be only unleveraged, as $Q^{B,b} = 0$ and $Q^{B,i} = 1$ for $\mu = 0$ in (47). Furthermore, the unleveraged bubble is also smaller than the partially leveraged bubble that would otherwise emerged for a slightly lower and still safe public debt level – namely $\bar{B}^g_{ZLB} < \bar{B}^g < \bar{B}^g_{safe}(Y^*) = \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*)$. Indeed, in the FE equilibrium where output is fixed at the potential level, $Y = Y^*$, any public debt increase reduces the amount of savings that would otherwise be invested in the bubble through (43).

4.3.1 Endogenous Capital

In the previous analysis, we assumed exogenous capital. Proposition 1 still holds when capital is endogenous, as shown in Appendix A.3. However, when capital accumulation is endogenous, a more general trade-off arises between output maximization and the minimization of sovereign risk, on the one hand, and the minimization of bubbly risk, on the other.

Macroeconomic outcomes in the bubble-less equilibria (Section 4.1) change with endogenous capital, as potential output is no longer constant but depends on the steady-state capital level. In the ZLB equilibrium, capital is given by

$$K_{ZLB} = [\phi^e(1-\alpha) + (1-\tau)\alpha] Y_{ZLB} - \bar{B}^g = (\Theta - 1)\bar{B}^g,$$

[32] As will become clear in the next section, public debt has a positive, indirect "crowding-in" effect on the bubble size, which works through output and can be stronger than the direct, negative "crowding-out" effect.

where $Y_{ZLB} = \frac{\bar{B}^g}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)}$ and $\Theta \equiv \frac{(1-\tau)\alpha + \phi^e(1-\alpha)}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)} > 1$, and in turn affects the potential output

$$Y^*_{ZLB} = K^{1-\alpha}_{ZLB} \bar{h}^\alpha = \left[(\Theta - 1)\bar{B}^g\right]^{1-\alpha} \bar{h}^\alpha.$$

Public debt diverts savings away from capital through a direct ("crowding-out") effect on K_{ZLB}, given by $-\bar{B}^g$. Equally, it sustains Y_{ZLB} because output is demand-driven in the ZLB equilibrium, thereby generating an indirect ("crowding-in") effect on K_{ZLB}, $[\phi^e(1-\alpha) + (1-\tau)\alpha] Y_{ZLB} = \Theta \bar{B}^g$, through which the resulting larger savings are channelled – via intergenerational transfers and loans – towards capital accumulation. Given $\Theta > 1$, the indirect effect is stronger than the direct one. Thus, public debt $\bar{B}^g$ positively affects capital accumulation and potential output, as well as current output, as long as there is a negative output gap at the ZLB, i.e. $Y_{ZLB} < Y^*_{ZLB}$, and thus output is demand-determined.[33] However, as the economy runs at the potential, output is no longer demand-driven. Therefore, away from the ZLB, the positive crowding-in effect of public debt on capital disappears, and the negative crowding-out effect survives, so that any increase in public debt reduces capital and potential/current output ($Y_{FE} = Y^*_{FE}$) in a FE equilibrium.[34]

The different effect of public debt on current output, depending on whether the ZLB binds or not, translates into different effects on bubble size across the two regimes. This can be understood through equation (43), that is

$$P^B = \left[(1-\tau)\alpha - \frac{\phi^e(1-\alpha)}{(1-\rho)}\right] Y - \bar{B}^g.$$

Public debt has also a negative direct crowding-out effect on the bubble, $-\bar{B}^g$, and an indirect crowding-in effect, $\left[(1-\tau)\alpha - \frac{\phi^e(1-\alpha)}{(1-\rho)}\right] Y$, which can be positive *or* negative according to the impact of $\bar{B}^g$ on Y. In the ZLB equilibrium, public debt raises demand/output and, as for capital, the *positive* crowding-in effect is stronger than the negative crowding-out effect. Thus, a higher public debt raises both steady-state output and the size of the bubble that can emerge.[35] By contrast, in a FE equilibrium, higher public debt reduces output and the potential bubble size. Away from the ZLB, a higher public debt decreases current/potential output via lower capital accumulation. Therefore, the crowding-in effect of public debt on the bubble turns *negative* and reinforces the crowding-out effect, lowering further overall savings available for bubbly investment.

Since an increase in public debt affects steady-state output and bubble size in the same direction

[33] Although public debt can expand both supply and demand at the ZLB, it raises demand more than supply, thereby ultimately closing the initial negative output gap.

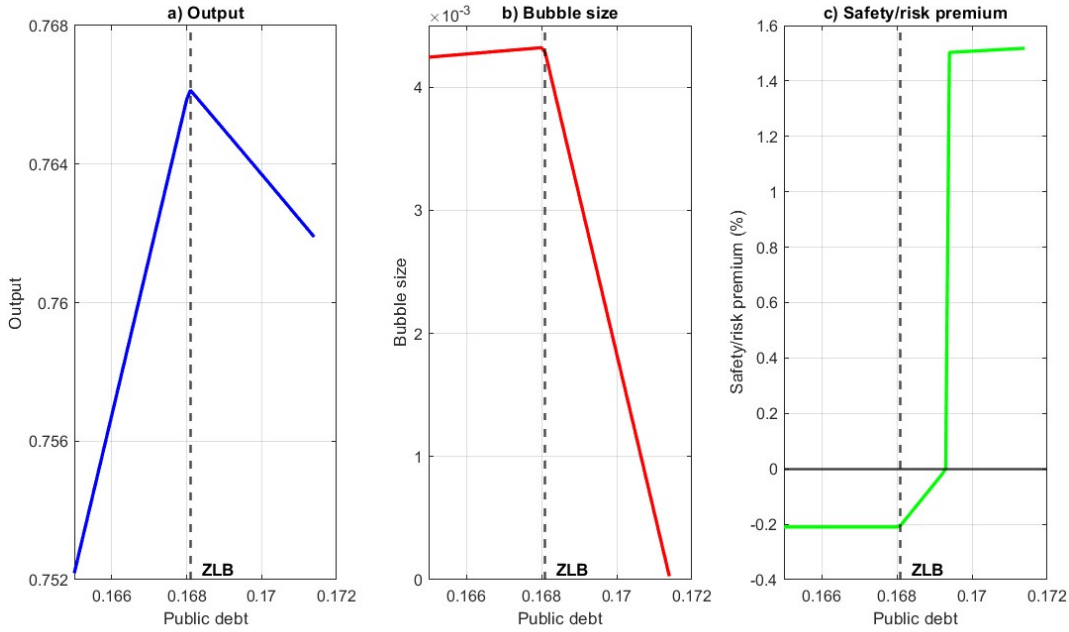
[34] In graphical terms, a public debt increase always shifts the upward-sloping AD curve to the right, as in Figure 1. However, a higher $\bar{B}^g$ now shifts also the vertical AS curve – which is no longer fixed but moves with the steady-state capital – to the right when the economy stays in a ZLB equilibrium, and to the left when it stays in a FE equilibrium. These and other general implications of endogenous capital are detailed in Appendix A.3.

[35] Plugging $Y_{ZLB} = \frac{\bar{B}^g}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)}$ in (43) yields the bubble price in a ZLB equilibrium,

$$P^B = \frac{\phi^e(1-\alpha)}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)} \left[\Pi^* - \frac{1}{(1-\rho)}\right] \bar{B}^g > 0,$$

where $\frac{1}{\beta \Pi^*} = \frac{1}{\Pi^*} = R_{ZLB} < (1-\rho)$, as assumed to derive Proposition 1.

Figure 2: Public debt and trade-off between the macro-financial objectives



– though with opposite signs depending on whether the ZLB binds or not – there exists a trade-off between output maximization and bubbly risk minimization, as illustrated in Figure 2.[36]

In the left and center panels of Figure 2, we plot respectively output and bubble size, on the vertical axis, against the associated public debt level in steady state, on the horizontal axis. The relationship between output and public debt is plotted in blue in the left panel, while, in the center panel, the relationship between output and the bubble size in red. Finally, the dashed vertical line, corresponding to $\bar{B}^g = \bar{B}^g_{ZLB}$, separates steady-state outcomes under a ZLB equilibrium (to the left) from those under a FE equilibrium (to the right), consistently with the conditions (40) and (41) associated with the two equilibria.

Starting from a ZLB equilibrium, to the left of the dashed line, a higher public debt raises output, and in fact the blue curve representing their relationship is initially upward-sloping (left panel). This holds up to $\bar{B}^g = \bar{B}^g_{ZLB}$, where the blue curve intersects the dashed line and production is maximized. To be precise, output reaches its maximum level in a $ZLB - FE$ equilibrium, which is a mixed equilibrium between the ZLB and FE equilibria featuring $R = R^*$, $\mathcal{R} = 1$, $Y = Y^*$ and $\Pi = \beta\Pi^* = \Pi^*$. Intuitively, output is maximized in this mixed equilibrium because the crowding-in and crowding-out effects of public debt on capital perfectly compensate. Equally, the bubble size is the largest possible for $\bar{B}^g = \bar{B}^g_{ZLB}$, because it grows with public debt and output in the ZLB equilibrium. In the center panel, this is depicted by the red curve that is also upward-sloping

[36] The numerical simulations in Figure 2 are obtained using the following calibration of the main parameters: $\bar{h} = 1$, $\delta = 0.02$, $\alpha = 0.6$, $\bar{q} = 0.3$, $\gamma = 0.27$, $\tau = 0.07$, $\phi^B = 0.82$, $\phi^D = 0.8133$, $\phi^L = 0.2787$, $\phi^e = 0.83$, $\rho = 0.003$, $\beta = 1$, $\pi^* = 1.02$, $\phi_\pi = 2$, $\rho^g = 0.03$, $\theta^g = 0.5$, and $\psi = 0.04$.

up to the maximum point, corresponding again to the intersection with the dashed line where a $ZLB - FE$ equilibrium arises. Therefore, maximizing output comes at the cost of maximizing the size of the bubble that can emerge in the steady state equilibrium. However, if public debt increases further, $\bar{B}^g > \bar{B}^g_{ZLB}$, moving the economy to the FE equilibrium – to the right of the dashed line – any further increases in public debt reduces output via lower capital accumulation, and thus the potential bubble size. This is depicted by the downward-sloping segment of the blue and red curves in the left and center panels. As a new feature arising from endogenous capital, shrinking or preventing the bubble comes at the cost of lower potential/current output in the FE equilibrium relative to its maximum attainable level, as higher public debt crowds out capital.

On the other hand, government can simultaneously maximize output and minimize the sovereign risk by setting the public debt level, while it cannot minimize the bubble size alongside the sovereign risk. This has been already discussed above, but it is now illustrated by right panel of Figure 2, where the spread between the return on public bonds and the real rate, $R^g - R$, is plotted against the corresponding steady-state public debt level. The spread is negative (or zero), for $B^g \leq \bar{B}^g_{safe}$, because of a positive (or zero) safety premium and a zero risk premium on public bonds – $\mu \geq 0$ and $\rho^g = 0$ in (5) and (7). On the contrary, it is positive, for $B^g > \bar{B}^g_{safe}$, due to a zero safety premium and a positive risk premium – $\mu = 0$ and $\rho^g > 0$ in (5) and (7). As can be seen by confronting the left and right panels, a negative spread, and thus zero sovereign risk, is associated with the maximum attainable output, which can be achieved for $B^g \leq \bar{B}^g_{safe}$. Indeed, in the right panel, the intersection between the green line and the dashed line, corresponding to $\bar{B}^g = \bar{B}^g_{ZLB}$ at which output is maximized, occurs at $R^g - R = -2\%$. By contrast, the bubble size takes a zero value in the center panel only for a public debt level such that $B^g > \bar{B}^g_{safe}$. At this level, the associated spread in the right panel is positive – approximately 1.5% – implying positive sovereign risk.

5 Conclusions

While the possibility of using public debt to maximize output and minimize bubbly and sovereign risk appears appealing and intuitive, it remains unclear whether, and under which conditions, public debt can actually be used for these purposes. Our paper provides key theoretical insights into the trade-off between different objectives when public debt is used for macro-financial stability purposes in a low interest rate environment.

We build a two-period OLG model with investors, banks and entrepreneurs. The OLG structure breaks Ricardian Equivalence so that the level of government debt, which is the key determinant of safety/risk premium, matters also for the equilibrium allocation. When the natural interest rate is extremely low, the economy gets stuck in a steady state ZLB equilibrium, where output is below the full-employment level and unleveraged and leveraged bubbly assets, bought respectively by investors and banks, can emerge.

Our results show that the government can bring the economy out of the ZLB equilibrium for "safe" public debt levels corresponding to a positive safety premium and zero sovereign risk. How-

ever, preventing any bubble from emerging ex ante requires a public debt above the maximum safe level and a consequent positive risk premium on government bonds. Indeed, the government cannot issue enough safe bonds to prevent any asset bubble. Rather, since bubbles can be initially bought by both banks and investors, raising the public debt up to the maximum safe level can make the bubble smaller and fully unleveraged, discouraging banks to invest in bubbly assets. Equally, away from the ZLB, any further increase in the public debt to shrink or prevent the bubble generates output losses by crowding out productive capital. Therefore, there exists a trade-off between output maximization and the minimization of sovereign risk, on the one hand, and the minimization of bubbly risk, on the other.

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A Steady State Equilibrium

A.1 Bubbleless Equilibria: Output, Nominal and Real Rates

A.1.1 AD

If the ZLB is not binding, $\mathcal{R} > 1$, the equation for AD is

$$Y_{AD}^{TR} = \left[\frac{1}{(1-\tau)\,\alpha - \frac{\phi^e(1-\alpha)\Gamma}{\Pi^{\phi_\pi - 1}}}\right]\bar{B}^g, \tag{55}$$

where $\Gamma \equiv R^{*-1}\left(\Pi^*\right)^{\phi_\pi - 1}$. Equation (55) can be computed by combining (28), (29) and (35). By using the same equations, AD becomes

$$Y_{ZLB}^{AD} = \left[\frac{1}{(1-\tau)\,\alpha - \phi^e\,(1-\alpha)\,\Pi}\right]\bar{B}^g, \tag{56}$$

if the ZLB binds, $\mathcal{R} = 1$.

A.1.2 Steady State Equilibria: A Taxonomy

There are three general types of steady state equilibrium in our model according to the main macro variables (R, $\mathcal{R}$, Y, Π), which are:

(A) ZLB occurring at the intersection of the AD^{ZLB} and the AS^{DNWR} and featuring

$$R^* < R_{ZLB} = \frac{1}{\beta\Pi^*} < 1$$
$$\mathcal{R} = 1$$
$$Y = Y_{ZLB} \equiv \left[\frac{1}{(1-\tau)\,\alpha - \beta\Pi^*\phi^e\,(1-\alpha)}\right]\bar{B}^g < Y^*$$
$$\Pi = \beta\Pi^* \leq \Pi^*;$$

(B) $ZLB - FE$ occurring at the intersection of the AD^{ZLB} and the AS^{FE} and featuring

$$R^* = R_{ZLB-FE}$$
$$\mathcal{R} = 1$$
$$Y = Y^*$$
$$\beta\Pi^* \leq \Pi = \frac{1}{R^*} \leq \Pi^*;$$

(C) FE occurring at the intersection of the AD^{TR} and the AS^{FE} and featuring

$$R^* = R_{FE}$$

$$\mathcal{R} > 1$$

$$Y = Y^*$$

$$\Pi = \Pi^*.$$

These equilibra can be both bubbleless and bubbly. While these are the three general categories of equilibria, the number and the nature of the equilibria that actually emerge depend on the public debt level, which determines the natural interest rate, and the degree of wage indexation to the inflation target (β), which determines the minimum inflation level corresponding to binding DNWR. As regards the degree of indexation, we can distinguish the cases of $\beta < 1$ and $\beta = 1$. Equally, we can identify two key thresholds for public debt:

$$\bar{B}^g_{ZLB} \equiv [(1-\tau)\,\alpha - \Pi^* \phi^e\,(1-\alpha)]\,Y^* \tag{57}$$

$$\bar{B}^g_{FE} \equiv [(1-\tau)\,\alpha - \beta\Pi^* \phi^e\,(1-\alpha)]\,Y^*, \tag{58}$$

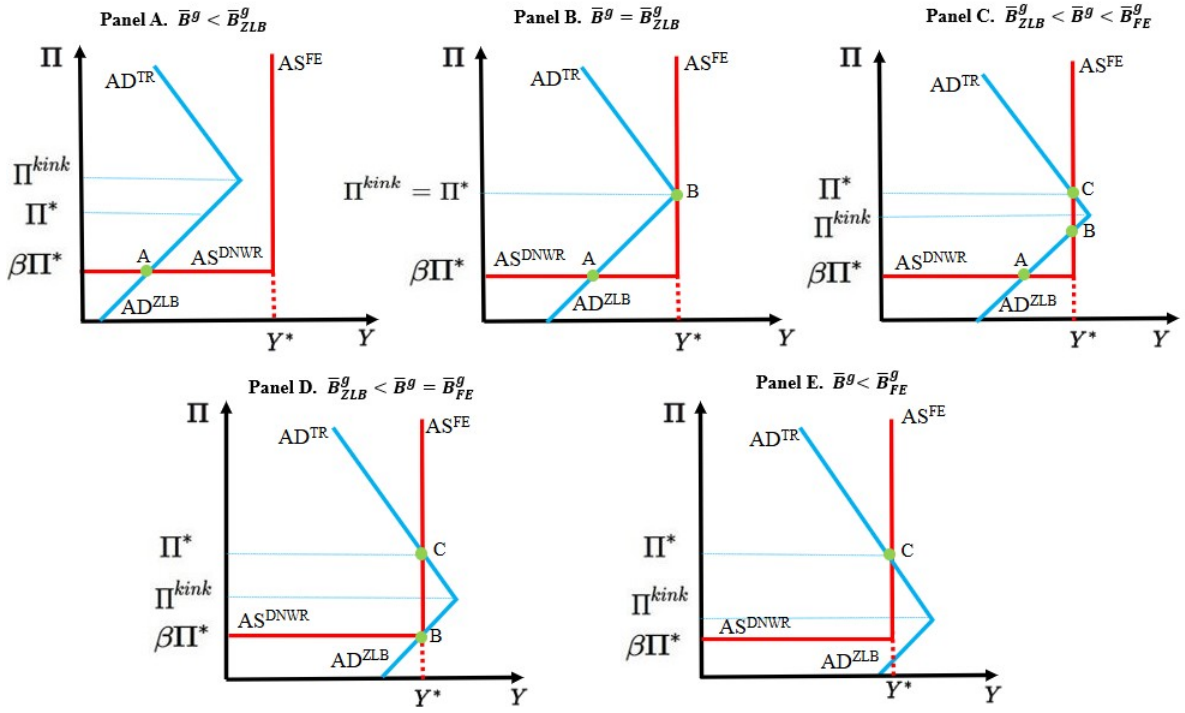
where $\bar{B}^g_{ZLB} < \bar{B}^g_{FE}$ for $\beta < 1$ and $\bar{B}^g_{ZLB} = \bar{B}^g_{FE}$ for $\beta = 1$. If $\bar{B}^g < \bar{B}^g_{ZLB}$, a unique ZLB equilibrium emerges, while a unique FE equilibrium appears if $\bar{B}^g > \bar{B}^g_{FE}$. It is worth noting that the equilibrium types described above share the same features and that the conditions characterizing the equilibria that emerge - discussed below - are identical to those in Ascari and Bonchi (2022). However, Ascari and Bonchi (2022) formulate these conditions in terms of Π^*, taking R^* as given, while we express them in terms of $\bar{B}^g$, which determines R^*, taking Π^* as given.[37]

- **Case** $\beta < 1$

  Taking the inflation target as given, as public debt, and thus the natural interest rate, raises, five different cases arise in Figure 3. We start from the two non-generic abovementioned cases, depicted respectively by Panel A and E. Under the condition $\bar{B}^g < \bar{B}^g_{ZLB}$, which delivers $R^* < 1/\Pi^* < 1$, a unique ZLB equilibrium (point A, Panel A) emerges. Instead, a unique FE equilibrium (point C, Panel E) exists under the condition $\bar{B}^g > \bar{B}^g_{FE}$, which delivers $R^* > 1/\beta\Pi^* > 1/\Pi^*$. Along with the non-generic cases just described, if $\bar{B}^g_{ZLB} \leq \bar{B}^g \leq \bar{B}^g_{FE}$ multiple steady state equilibria are possible as shown by the corresponding panels in Figure 3. In particular, for $\bar{B}^g$ levels equal to/higher than $\bar{B}^g_{ZLB}$ but lower than/equal to $\bar{B}^g_{FE}$ (Panels B, C, D), a ZLB equilibrium is still possible but not unique. To be more precise and summarize: Panel A: if $\bar{B}^g < \bar{B}^g_{ZLB}$, which corresponds to $R^* < 1/\Pi^* < 1$, only the ZLB equilibrium exists at point A; Panel B: if $\bar{B}^g = \bar{B}^g_{ZLB}$ implying $R^* = 1/\Pi^*$, two equilibria co-exist: a ZLB equilibrium at point A and a $ZLB - FE$ equilibrium at point B, where output is at the potential level, the nominal interest rate implied by the Taylor rule is precisely zero and the inflation rate is at the target; Panel C: if $\bar{B}^g_{ZLB} < \bar{B}^g < \bar{B}^g_{FE}$, that is $1/\Pi^* < R^* < 1/\beta\Pi^*$,

[37] For further details, we refer to Ascari and Bonchi (2022).

Figure 3: Steady state equilibria ($\beta < 1$)

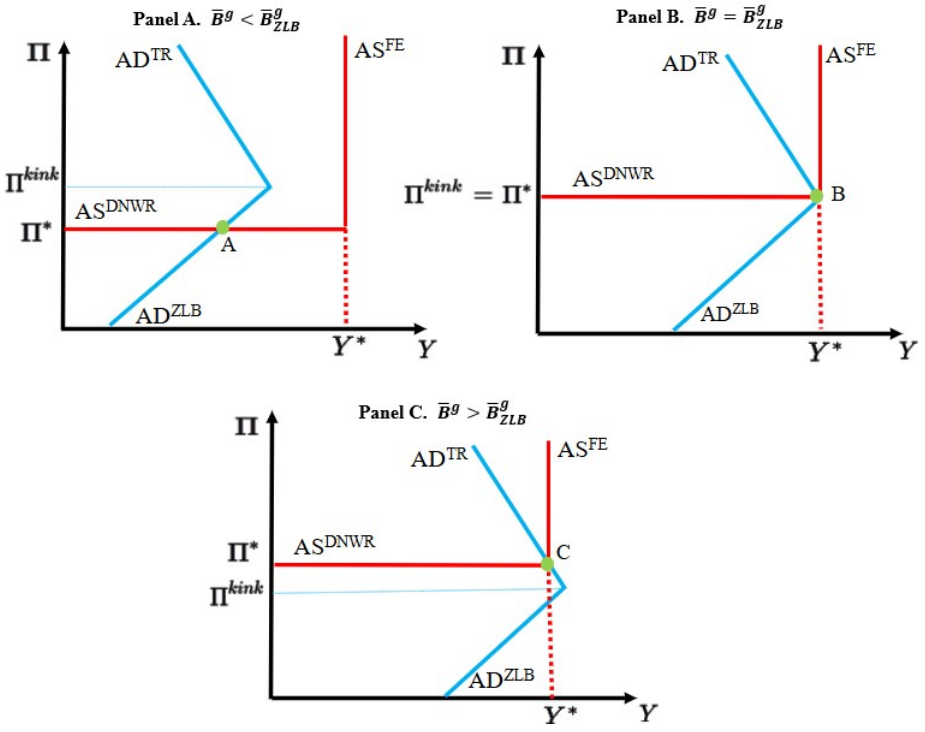


three equilibria co-exist: ZLB at point A, $ZLB - FE$ at point B and FE at point C; Panel D: if $\bar{B}^g_{ZLB} < \bar{B}^g = \bar{B}^g_{FE}$, which delivers $R^* = 1/\beta\Pi^*$, two equilibria co-exist: FE at point C and a $ZLB - FE$ equilibrium at point B, where output is at the potential, the ZLB is binding (and i is off the Taylor rule) and the inflation rate is below the target and equal to $\beta\Pi^*$; Panel E: if $\bar{B}^g > \bar{B}^g_{FE}$, and thus $R^* > 1/\beta\Pi^*$, only the FE equilibrium arises at point C.

- **Case** $\beta = 1$

  In this case, only three panels arise (Figure 4). Indeed, for $\beta = 1$, given (57) and (58), $\bar{B}^g_{FE} \equiv [(1-\tau)\,\alpha - \Pi^*\phi^e\,(1-\alpha)]\,Y^* = \bar{B}^g_{ZLB}$. Therefore, while Panel A is unchanged, Panel B and D of Figure 3 collapse to Panel B of Figure 4, where ZLB and FE equilibria do not emerge, and the unique equilibrium is the $ZLB - FE$ ($Y = Y^*$, $\mathcal{R} = 1$, and $\Pi = \Pi^* = \beta\Pi^* = 1/R^*$). Furthermore, Panel C of Figure 3 vanishes, and Panel E of the same figure becomes Panel C in Figure 4. The three panels can be summarized as follows: Panel A: if $\bar{B}^g < \bar{B}^g_{ZLB}$, and thus $R^* < 1/\Pi^* < 1$, only the ZLB equilibrium emerges at point A; Panel B: if $\bar{B}^g = \bar{B}^g_{ZLB}$ implying $R^* = 1/\Pi^* = 1/\beta\Pi^*$, only the $ZLB - FE$ equilibrium exists at point B, where output is at the potential level, the nominal interest rate implied by the Taylor rule is precisely zero and the inflation rate is at the target; Panel C: if $\bar{B}^g > \bar{B}^g_{ZLB}$, that is $R^* > 1/\beta\Pi^* = 1/\Pi^*$, only the FE equilibrium exists at point C.

Figure 4: Steady state equilibria ($\beta = 1$)



A.2 Public Debt as a Macro-Financial Stability Tool

To streamline the notation, since we refer exclusively to variables in a bubble-less steady state equilibrium, we omit the subscript NB. In what follows, we will use, among others, the definitions of $\bar{B}^g_{ZLB}$ and $\bar{B}^g_{FE}$, given by (57) and (58). Before providing a proof of Proposition 1, we need to state some parametric assumptions.

A.2.1 Proposition 1: Parametric Assumptions

First of all, we need to assume

$$R^* = \frac{\phi^e (1-\alpha) Y^*}{(1-\tau)\alpha Y^* - \bar{B}^g} < 1,$$

so that the condition (38) for the emergence of a ZLB equilibrium (point A, Panel A of Figures 3 and 4) can hold. Secondly, we impose a positive bubble price in the initial ZLB equilibrium by

$$\frac{1}{\beta \Pi^*} = R_{ZLB} < (1-\rho), \tag{59}$$

which is consistent with (44). We finally impose

$$(1-\tau)\alpha - \frac{\phi^e (1-\alpha)}{(1-\rho)} > \left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma (1-\tau)\alpha\right], \tag{60}$$

which can be easily shown, by using (46) and (50), to deliver

$$\bar{B}_L^g(Y) < \bar{B}_U^g(Y) < \bar{B}_{upper}^g(Y), \tag{61}$$

where

$$\bar{B}_{upper}^g \equiv \frac{1}{1-\Phi}\left\{(1-\tau)\alpha - \frac{\phi^e(1-\alpha)}{(1-\rho)} - \Phi\left[\overline{q} - \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha\right]\right\}Y > 0,$$

with $\Phi \equiv \frac{\phi^D}{\phi^B+\phi^L\left(\frac{\phi^B-\phi^D}{\phi^D-\phi^L}\right)} < 1$. Assumptions (60) through (61) imply, on the one hand, $\bar{B}_L^g(Y) < \bar{B}_U^g(Y)$ so that (51) and (52) can hold, and thus a partially leveraged bubble can arise in the initial equilibrium and may become unleveraged as public debt increases. On the other hand, bubble holdings in (47) lie between 0 and 1 only if $\bar{B}^g \leq \bar{B}_{upper}^g$. Therefore, assumptions (60) through (61) ensure that the conditions for the existence of bubbles, (51) and (52), are consistent with a fixed unit supply of the bubble, since $\bar{B}^g < \bar{B}_L^g < \bar{B}_U^g < \bar{B}_{upper}^g$ and $\bar{B}_L^g \leq \bar{B}^g < \bar{B}_U^g < \bar{B}_{upper}^g$.

We make some further assumptions, after having derived the maximum "safe" public debt level in a bubble-less steady state,

$$\bar{B}_{safe}^g(Y) = \bar{B}_L^g(Y) \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha\right]Y, \tag{62}$$

by using (25) and $D^b = D^i = \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha Y$[38]. It coincides with $\bar{B}_L^g(Y)$ and becomes

$$\bar{B}_{safe}^g(Y^*) = \bar{B}_L^g(Y^*) \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha\right]Y^* \tag{63}$$

for $Y = Y^*$. Given (2), (24), (25), (33), as well as $D^b = D^i = \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha Y$, $\bar{B}_{safe}^g(Y)$ denotes the public debt *level* corresponding to a zero safety premium and zero probability of sovereign default ($\mu = \rho^g = 0$). A public debt below this threshold implies $\mu > 0$ and $\rho^g = 0$, whereas above it $\mu = 0$ and $\rho^g > 0$. Analogously, the maximum "safe" public debt-to-GDP *ratio*,

$$\left(\frac{\bar{B}^g}{Y}\right)_{safe} \equiv \left[\overline{q} - \left(\frac{\phi^L}{\phi^D-\phi^L}\right)\gamma(1-\tau)\alpha\right], \tag{64}$$

is associated with $\mu = \rho^g = 0$. We assume that the public debt-to-GDP ratio in the initial ZLB

[38]This can be derived, for $P^B = 0$, by using $N^b = \gamma(1-\tau)Wh/P = \gamma(1-\tau)\alpha Y$ and combining (16) and (17).

equilibrium is lower than the maximum safe level:[39]

$$\frac{\bar{B}^g}{Y_{ZLB}} = (1-\tau)\,\alpha - \beta\Pi^*\phi^e\,(1-\alpha) < \left(\frac{\bar{B}^g}{Y}\right)_{safe}. \tag{65}$$

In turn, given (58) and (63), assumption (65) implies

$$\bar{B}^g_{FE} < \bar{B}^g_L\,(Y^*) = \bar{B}^g_{safe}\,(Y^*)\,. \tag{66}$$

We also assume

$$\bar{B}^g_{ZLB} < \bar{B}^g_{safe}\,(Y_{ZLB}) = \bar{B}^g_L\,(Y_{ZLB}) < \bar{B}^g_{FE}. \tag{67}$$

A.2.2 Proposition 1: A Proof

We distinguish the cases of $\beta < 1$ and $\beta = 1$ also in the proof of Proposition 1. A proof of Proposition 1 is presented next and structured in the following way. In the proof of Proposition 1, we show if the conditions associated with the different panels in Figure 3 and Figure 4 are compatible with the ones for the existence of partially leveraged or unleveraged bubbles, (51) and (52), and the one for a positive sovereign risk (24). For $\beta < 1$, a proof of Propositions 1 follows from the analysis of the case in Panel E (Panel C for $\beta = 1$).

Our general aim is to understand whether and which type of bubble may emerge in equilibrium, and whether the sovereign risk is positive, when public debt increases to move the economy from an initial ZLB equilibrium (point A, Panel A of Figures 3 and 4) to a final FE equilibrium (point C, Panel E of Figure 3 and C of Figure 4). To that aim, we assume that the economy, in the case of multiple equilibria (Panel B, C, D of Figure 3), remains at a ZLB equilibrium, which, however, moves/changes along the curve AS^{DNWR} as public debt increases, and it disappears only for $B^g > B^g_{FE}$ in Panel E, where the FE equilibrium is unique.

- **Case** $\beta < 1$

  Panel A. Given $\bar{B}^{g,1} < \bar{B}^g_{ZLB}$ and $Y^1_{ZLB} \equiv \left[\frac{1}{(1-\tau)\alpha - \beta\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^{g,1}$, as well as the assumptions (59), (61) and (67), the conditions

$$\bar{B}^{g,1} < \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_U\left(Y^1_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right) \leq \bar{B}^g_{FE}$$

$$\bar{B}^{g,1} < \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_U\left(Y^1_{ZLB}\right) \leq \bar{B}^g_{FE} < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right)$$

$$\bar{B}^{g,1} < \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_{FE} < \bar{B}^g_U\left(Y^1_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right),$$

[39] Although $\bar{B}^g$ is exogenously given, in the ZLB equilibrium, there is a one-to-one mapping from Y_{ZLB} to $\bar{B}^g$, $Y_{ZLB} = \left[\frac{1}{(1-\tau)\alpha - \beta\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^g$, and viceversa, $\bar{B}^g = [(1-\tau)\,\alpha - \beta\Pi^*\phi^e\,(1-\alpha)]\,Y_{ZLB}$. Given that and (62), condition (65) implies $\bar{B}^g < \bar{B}^g_{safe}\,(Y_{ZLB}) = \bar{B}^g_L\,(Y_{ZLB})$ for any $\bar{B}^g$ corresponding to a ZLB equilibrium. Since the previous condition is incompatible with (52), the bubble cannot be unleveraged in the ZLB equilibrium. Therefore, in the proof of Proposition 1, we can neglect this case.

where $\bar{B}^g_{FE} \gtreqless \bar{B}^g_{NB,U}\left(Y^1_{ZLB}\right)$ and $\bar{B}^g_{FE} \gtreqless \bar{B}^g_{upper}\left(Y^1_{ZLB}\right)$ for $\frac{1}{\beta\Pi^*} < (1-\rho)$, $Y^1_{ZLB} < Y^*$, $\left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha\right] > 0$ and $0 < \Phi < 1$, allow for a partially leveraged bubble in the ZLB equilibrium (point A) corresponding to Y^1_{ZLB} because compatible with (51). Furthermore, as we have

$$\bar{B}^{g,1} < \bar{B}^g_L\left(Y^1_{ZLB}\right) = \bar{B}^g_{safe}\left(Y^1_{ZLB}\right),$$

the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu^1_{ZLB} > 0$).

Panel B. Given $\bar{B}^{g,2} = \bar{B}^g_{ZLB} > \bar{B}^{g,1}$ and $Y^2_{ZLB} \equiv \left[\frac{1}{(1-\tau)\alpha - \beta\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^{g,2} > Y^1_{ZLB}$, as well as the assumptions (59), (61) and (67), the conditions

$$\bar{B}^{g,2} = \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^2_{ZLB}\right) < \bar{B}^g_U\left(Y^2_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^2_{ZLB}\right) \leq \bar{B}^g_{FE}$$

$$\bar{B}^{g,2} = \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^2_{ZLB}\right) < \bar{B}^g_U\left(Y^2_{ZLB}\right) \leq \bar{B}^g_{FE} < \bar{B}^g_{upper}\left(Y^2_{ZLB}\right)$$

$$\bar{B}^{g,2} = \bar{B}^g_{ZLB} < \bar{B}^g_L\left(Y^2_{ZLB}\right) < \bar{B}^g_{FE} < \bar{B}^g_U\left(Y^2_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^2_{ZLB}\right),$$

where $\bar{B}^g_{FE} \gtreqless \bar{B}^g_U\left(Y^2_{ZLB}\right)$ and $\bar{B}^g_{FE} \gtreqless \bar{B}^g_{upper}\left(Y^2_{ZLB}\right)$ for $\frac{1}{\beta\Pi^*} < (1-\rho)$, $Y^2_{ZLB} < Y^*$, $\left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha\right] > 0$ and $0 < \Phi < 1$, allow for a partially leveraged bubble in the ZLB equilibrium (point A) corresponding to Y^2_{ZLB} because compatible with (51). Furthermore, as we have

$$\bar{B}^{g,2} < \bar{B}^g_L\left(Y^2_{ZLB}\right) = \bar{B}^g_{safe}\left(Y^2_{ZLB}\right),$$

the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu^2_{ZLB} > 0$).

Panel C. Given $\bar{B}^g_{FE} > \bar{B}^{g,3} > \bar{B}^g_{ZLB} = \bar{B}^{g,2} > \bar{B}^{g,1}$ and $Y^3_{ZLB} \equiv \left[\frac{1}{(1-\tau)\alpha - \beta\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^{g,3} > Y^2_{ZLB} > Y^1_{ZLB}$, as well as the assumptions (59), (61) and (67), the conditions

$$\bar{B}^g_{ZLB} < \bar{B}^{g,3} < \bar{B}^g_L\left(Y^3_{ZLB}\right) < \bar{B}^g_U\left(Y^3_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^3_{ZLB}\right) \leq \bar{B}^g_{FE}$$

$$\bar{B}^g_{ZLB} < \bar{B}^{g,3} < \bar{B}^g_L\left(Y^3_{ZLB}\right) < \bar{B}^g_U\left(Y^3_{ZLB}\right) \leq \bar{B}^g_{FE} < \bar{B}^g_{upper}\left(Y^3_{ZLB}\right)$$

$$\bar{B}^g_{ZLB} < \bar{B}^{g,3} < \bar{B}^g_L\left(Y^3_{ZLB}\right) < \bar{B}^g_{FE} < \bar{B}^g_U\left(Y^3_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^3_{ZLB}\right),$$

where $\bar{B}^g_{FE} \gtreqless \bar{B}^g_U\left(Y^3_{ZLB}\right)$ and $\bar{B}^g_{FE} \gtreqless \bar{B}^g_{upper}\left(Y^3_{ZLB}\right)$ for $\frac{1}{\beta\Pi^*} < (1-\rho)$, $Y^3_{ZLB} < Y^*$, $\left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma\left(1-\tau\right)\alpha\right] > 0$ and $0 < \Phi < 1$, allow for a partially leveraged bubble in the ZLB equilibrium (point A) corresponding to Y^3_{ZLB} because compatible with (51). Furthermore, as we have

$$\bar{B}^{g,3} < \bar{B}^g_L\left(Y^3_{ZLB}\right) = \bar{B}^g_{safe}\left(Y^3_{ZLB}\right),$$

the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu^3_{ZLB} > 0$).

Panel D. Given $\bar{B}^{g,4} = \bar{B}^g_{FE} > \bar{B}^{g,3} > \bar{B}^g_{ZLB} = \bar{B}^{g,2} > \bar{B}^{g,1}$ and $Y_{ZLB-FE} = \left[\frac{1}{(1-\tau)\alpha-\beta\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^{g,4} = Y^* > Y^3_{ZLB} > Y^2_{ZLB} > Y^1_{ZLB}$, as well as the assumptions (59), (61) and (66), the condition

$$\bar{B}^{g,4} = \bar{B}^g_{FE} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*)$$

allows for a partially leveraged bubble in the $ZLB-FE$ equilibrium (point B) that corresponds to $Y_{ZLB-FE} = Y^*$ because compatible with (51). The partially leveraged bubble cannot collapse into an unleveraged bubble because (66) rules out

$$\bar{B}^g_L(Y^*) \leq \bar{B}^{g,4} = \bar{B}^g_{FE},$$

and thus (52) cannot hold in the $ZLB - FE$ equilibrium. Furthermore, as we have

$$\bar{B}^{g,4} = \bar{B}^g_{FE} < \bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*),$$

the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu_{ZLB-FE} > 0$).

Panel E. Given $B^5 > \bar{B}^{g,4} = \bar{B}^g_{FE} > \bar{B}^{g,3} > \bar{B}^g_{ZLB} = \bar{B}^{g,2} > \bar{B}^{g,1}$, as well as the assumptions (59), (61) and (66), three different cases are possible.

1. The condition
$$\bar{B}^g_{FE} < \bar{B}^{g,5} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*)$$
allows for a partially leveraged bubble in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because compatible with (51). Furthermore, as we have
$$\bar{B}^{g,5} < \bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*),$$
the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu_{FE} > 0$).

2. The condition
$$\bar{B}^g_{FE} < \bar{B}^g_L(Y^*) \leq \bar{B}^{g,5} < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*)$$
allows for an unleveraged bubble in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because compatible with (52). Furthermore, as we can have
$$\bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) < \bar{B}^{g,5},$$
the probability of sovereign default can be positive ($\rho_g > 0$) according to (24), while the

safety premium, given by (48), is necessarily zero ($\mu_{FE} = 0$).

3. The condition
$$\bar{B}^g_{FE} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^{g,5} < \bar{B}^g_{upper}(Y^*)$$
prevents any bubble from occurring in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because incompatible with (51) and (52). Furthermore, as we have
$$\bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) < \bar{B}^{g,5},$$
the probability of sovereign default is positive ($\rho_g > 0$) according to (24), while the safety premium, given by (48), is zero ($\mu_{FE} = 0$).

- **Case** $\beta = 1$

The assumptions outlined in Appendix A.2.1, along with (60) and (61), still hold in this case. However, we now have $\beta = 1$ and thus $\beta\Pi^* = \Pi^*$. Therefore, given (57) and (58), $\bar{B}^g_{ZLB} = \bar{B}^g_{FE}$. Therefore, assumption (66) becomes
$$\bar{B}^g_{ZLB} < \bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) \tag{68}$$
while assumption (67) takes the shape
$$\bar{B}^g_L(Y_{ZLB}) = \bar{B}^g_{safe}(Y_{ZLB}) < \bar{B}^g_{ZLB}. \tag{69}$$

Panel A. Given $\bar{B}^{g,1} < \bar{B}^g_{ZLB}$ and $Y^1_{ZLB-U} \equiv \left[\frac{1}{(1-\tau)\alpha - \Pi^*\phi^e(1-\alpha)}\right]\bar{B}^{g,1}$, as well as the assumptions (59), (61) and (69), the conditions
$$\bar{B}^{g,1} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_U\left(Y^1_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right) \leq \bar{B}^g_{ZLB}$$
$$\bar{B}^{g,1} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_U\left(Y^1_{ZLB}\right) \leq \bar{B}^g_{ZLB} < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right)$$
$$\bar{B}^{g,1} < \bar{B}^g_L\left(Y^1_{ZLB}\right) < \bar{B}^g_{ZLB} < \bar{B}^g_U\left(Y^1_{ZLB}\right) < \bar{B}^g_{upper}\left(Y^1_{ZLB}\right),$$
where $\bar{B}^g_{ZLB} \gtreqless \bar{B}^g_U\left(Y^1_{ZLB}\right)$ and $\bar{B}^g_{ZLB} \gtreqless \bar{B}^g_{upper}\left(Y^1_{ZLB}\right)$ for $\frac{1}{\beta\Pi^*} = \frac{1}{\Pi^*} < (1-\rho)$, $Y^1_{ZLB} < Y^*$, $\left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma(1-\tau)\alpha\right] > 0$ and $0 < \Phi < 1$, allow for a partially leveraged bubble in the ZLB equilibrium (point A) corresponding to Y^1_{ZLB} because compatible with (51). Furthermore, as we have
$$\bar{B}^{g,1} < \bar{B}^g_L\left(Y^1_{ZLB}\right) = \bar{B}^g_{safe}\left(Y^1_{ZLB}\right),$$
the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu^1_{ZLB} > 0$).

Panel B. Given $\bar{B}^{g,2} = \bar{B}^g_{ZLB} > \bar{B}^{g,1}$ and $Y_{ZLB-FE} = \left[\frac{1}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)}\right] \bar{B}^{g,2} = Y^* > Y^1_{ZLB}$, as well as the assumptions (59), (61) and (68), the condition

$$\bar{B}^{g,2} = \bar{B}^g_{ZLB} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*),$$

allows for a partially leveraged bubble in the equilibrium (point B) that corresponds to $Y_{ZLB-FE} = Y^*$ because compatible with (51). The partially leveraged bubble cannot collapse into an unleveraged bubble because (66) rules out

$$\bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) \leq \bar{B}^{g,2} = \bar{B}^g_{ZLB}$$

and thus (52) cannot hold in the $ZLB - FE$ equilibrium. Furthermore, as we have

$$\bar{B}^{g,2} = \bar{B}^g_{ZLB} < \bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*)$$

the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu_{ZLB-FE} > 0$).

Panel C. Given $\bar{B}^{g,3} > \bar{B}^{g,2} = \bar{B}^g_{ZLB} > \bar{B}^{g,1}$, as well as the assumptions (59), (61) and (68), three different cases are possible.

1. The condition
$$\bar{B}^g_{ZLB} < \bar{B}^{g,3} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*)$$
allows for a partially leveraged bubble in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because compatible with (51). Furthermore, as we have
$$\bar{B}^{g,3} < \bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*)$$
the probability of sovereign default is zero ($\rho_g = 0$) according to (24), while the safety premium, given by (48), is positive ($\mu_{FE} > 0$).

2. The condition
$$\bar{B}^g_{ZLB} < \bar{B}^g_L(Y^*) \leq \bar{B}^{g,3} < \bar{B}^g_U(Y^*) < \bar{B}^g_{upper}(Y^*)$$
allows for an unleveraged bubble in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because compatible with (52). Furthermore, as we can have
$$\bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) < \bar{B}^{g,3}$$
the probability of sovereign default can be positive ($\rho_g > 0$) according to (24), while the safety premium, given by (48), is necessarily zero ($\mu_{FE} = 0$).

3. The condition

$$\bar{B}^g_{ZLB} < \bar{B}^g_L(Y^*) < \bar{B}^g_U(Y^*) < \bar{B}^{g,3} < \bar{B}^g_{upper}(Y^*)$$

prevents any bubble from occurring in the FE equilibrium (point C) corresponding to $Y_{FE} = Y^*$ because incompatible with (51) and (52). Furthermore, as we have

$$\bar{B}^g_L(Y^*) = \bar{B}^g_{safe}(Y^*) < \bar{B}^{g,3}$$

the probability of sovereign default is positive ($\rho_g > 0$) according to (24), while the safety premium, given by (48), is zero ($\mu_{FE} = 0$).

A.3 Endogenous Capital

A.3.1 Case $\beta = 1$

We consider here the case of endogenous capital, taking $\beta = 1$ and $P^k = 1$. Capital depreciates at the rate $\delta \in (0,1)$, and investment in new capital is positive. Our previous analysis of the bubbly equilibria goes unchanged because macroeconomic variables are taken as given in Section 4.2. Instead, the macroeconomic outcomes in the bubble-less equilibria (Section 4.1) are affected by endogenous capital, even though the AS and AD equations remain unchanged. Indeed, potential output is no longer constant, but it depends on the steady-state capital level. In the ZLB equilibrium, capital is given by

$$K_{ZLB} = [\phi^e(1-\alpha) + (1-\tau)\alpha] Y_{ZLB} - \bar{B}^g = (\Theta - 1)\bar{B}^g, \tag{70}$$

where $Y_{ZLB} = \frac{\bar{B}^g}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)}$ and $\Theta \equiv \frac{(1-\tau)\alpha + \phi^e(1-\alpha)}{(1-\tau)\alpha - \Pi^* \phi^e (1-\alpha)} > 1$, and in turn affects the potential output,

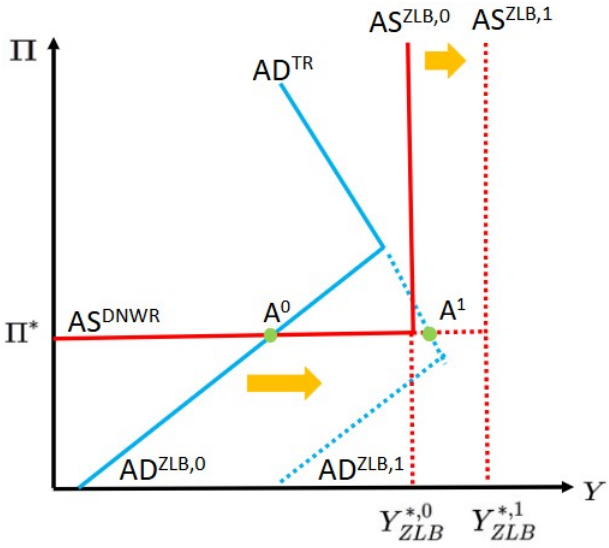
$$Y^*_{ZLB} = K^{1-\alpha}_{ZLB} \bar{h}^\alpha = \left[(\Theta - 1)\bar{B}^g\right]^{1-\alpha} \bar{h}^\alpha, \tag{71}$$

and thus the natural interest rate,

$$R^*_{ZLB} = \frac{\phi^e(1-\alpha) Y^*_{ZLB}}{\left[(1-\tau)\alpha Y^*_{ZLB} - \bar{B}^g\right]}.$$

To put differently, the vertical segment of the AS corresponding to the potential output, which was previously fixed, moves with the steady-state capital level in the ZLB equilibrium. In particular, it moves with public debt $\bar{B}^g$ that can positively affect capital accumulation at the ZLB, as depicted by Figure 5. Low output Y_{ZLB} depresses the amount of savings channelled to entrepreneurs – through intergenerational transfers and loans – for capital accumulation, as expressed by $[\phi^e(1-\alpha) + (1-\tau)\alpha] Y_{ZLB} - \bar{B}^g$ in (70). Although higher public debt has a negative *direct* ("crowding-out") effect on K_{ZLB}, equal to $-\bar{B}^g$, it also sustains demand/output Y_{ZLB}, thereby generating a positive *indirect* ("crowding-in") effect on K_{ZLB}, namely $[\phi^e(1-\alpha) + (1-\tau)\alpha] Y_{ZLB} = \Theta \bar{B}^g$. Because this indirect effect is stronger than the direct one at the ZLB – $\Theta > 1$ in (70) –

Figure 5: AD and AS curve with endogenous capital



public debt not only sustains demand/output, shifting the upward-sloping AD curve to the right,[40] but also fosters capital accumulation and potential output at the ZLB, as shown in equations (70) and (71), shifting the vertical AS curve to the right.

The innovation carried by endogenous capital and just explained does not alter the conditions for the existence and uniqueness of ZLB and FE equilibria, (38) and (39), but only the definition of $\bar{B}^g_{ZLB}$ that becomes

$$\bar{B}^g_{ZLB} \equiv \left[(1-\tau)\,\alpha - \Pi^* \phi^e\,(1-\alpha)\right] Y^*_{ZLB-FE} \tag{72}$$

where

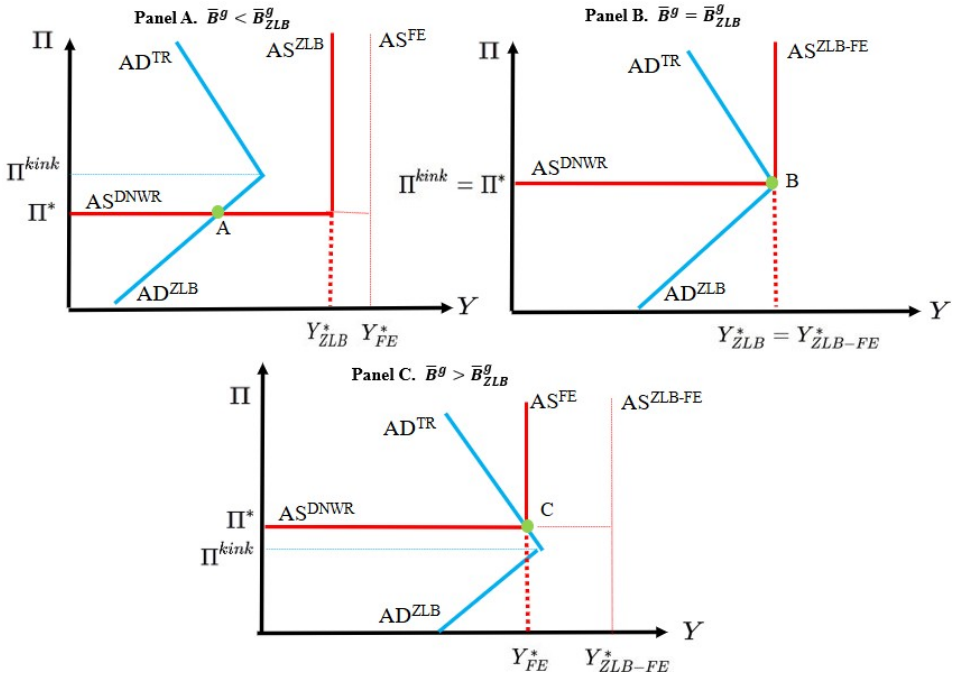
$$Y^*_{ZLB-FE} = \left[(1+\Pi^*)\phi^e\,(1-\alpha)\right]^{\frac{1-\alpha}{\alpha}}\,\bar{h}.$$

We can finally plot Figure 6, which depicts the three panels and the corresponding equilibria for different public debt levels. Since the conditions for the existence of asset bubbles remain unchanged with endogenous capital, as do those associated with the different panels in Figure 6, Proposition 1 still holds.[41] However, when capital is endogenous, a key difference arises relative to the case of exogenous capital. Public debt can expand both supply and demand at the ZLB as long as there is a negative output gap, i.e. $Y_{ZLB} < Y^*_{ZLB}$, because output is demand-driven. Thus, the positive indirect effect of $\bar{B}^g$ on K_{ZLB} – working through Y_{ZLB} – is stronger than the negative crowding out effect. However, as public debt reaches the level $\bar{B}^g_{ZLB}$ associated with the ZLB–FE equilibrium

[40] Although public debt now affects potential output through capital accumulation, an increase in public debt still has a positive overall effect on the natural rate, shifting the upward-sloping AD curve to the right, regardless of whether the ZLB binds.

[41] As regards the proof of Proposition 1 (Appendix A.2), however, we need to make the further parametric assumption $\frac{Y^*_{ZLB-FE}}{Y^*_{FE}} < \frac{\left[\overline{q} - \left(\frac{\phi^L}{\phi^D - \phi^L}\right)\gamma(1-\tau)\alpha\right]}{\left[(1-\tau)\alpha - \beta\Pi^*\phi^e(1-\alpha)\right]}$ regarding the FE equilibria considered.

Figure 6: Steady state equilibria (β=1) with endogenous capital



B (Figure 6, Panel B), the initial negative output gap closes, so that $Y_{ZLB-FE} = Y^*_{ZLB-FE}$.[42] At this point, output is no longer demand-determined, the positive indirect effect of public debt on capital disappears, and the standard crowding-out effect prevails. Hence, any further increase in public debt reduces capital accumulation and potential (and thus current) output, as illustrated in Panel C of Figure 6, where $Y^*_{FE} < Y^*_{ZLB-FE}$ for $\bar{B}^g > B_{ZLB}$.

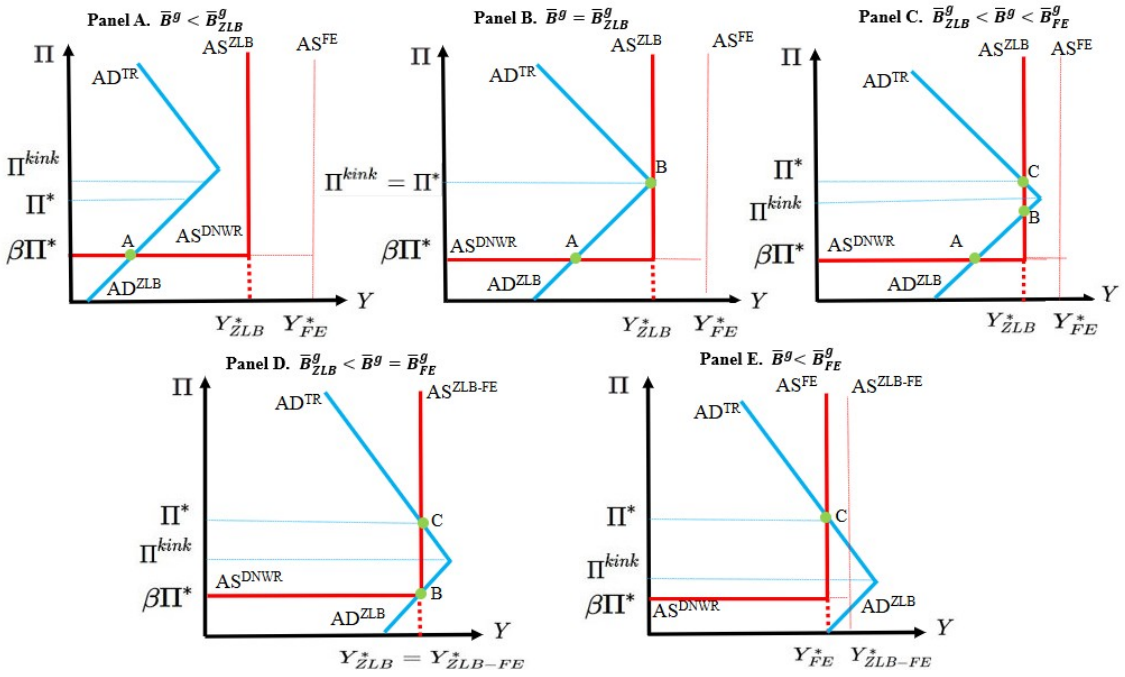
A.3.2 Case $\beta < 1$

We describe here the main features of the model with endogenous capital in the general case of $\beta < 1$, highlighting a few differences relative to $\beta = 1$. The steady-state levels of capital, potential output and natural interest rate in the ZLB equilibrium are unchanged for $\beta < 1$, with the only difference that $\beta\Pi^* < \Pi^*$ in the equations above and thus we need to re-define $\Theta \equiv \frac{(1-\tau)\alpha+\phi^e(1-\alpha)}{(1-\tau)\alpha-\beta\Pi^*\phi^e(1-\alpha)}$. As regards the existence and uniqueness of the ZLB and TFE equilibria, they depend respectively on the conditions (57) and (58) (see Appendix A.1.2), and thus there are two different thresholds associated with those conditions, $\bar{B}^g_{ZLB}$ and $\bar{B}^g_{FE}$, not only one, $\bar{B}^g_{ZLB}$, as for $\beta = 1$. Both those thresholds change with endogenous capital, and they become respectively

$$\bar{B}^g_{ZLB} = \Psi \bar{B}^g_{FE} \tag{73}$$

[42] Although, in principle, supply can increase initially more than demand for a higher $\bar{B}^g$, enlarging the output gap $Y_{ZLB} - Y^*_{ZLB} = \left[1 - \left(\frac{\bar{h}}{h_{ZLB}}\right)^{\alpha}\right]\left[\frac{1}{(1-\tau)\alpha-\Pi^*\phi^e(1-\alpha)}\right]\bar{B}^g$, at some point, during the transition from Panel A to Panel B, demand has to increase more than supply for a higher public debt. This precisely happens for $\bar{B}^g > [(1-\tau)\alpha - \Pi^*\phi^e(1-\alpha)]\,[(1+\Pi^*)\phi^e(1-\alpha)]^{\frac{1-\alpha}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}\bar{h}$.

Figure 7: Steady state equilibria ($\beta < 1$) with endogenous capital



$$\bar{B}^g_{FE} \equiv [(1-\tau)\,\alpha - \beta\Pi^*\phi^e\,(1-\alpha)]\,Y^*_{ZLB-FE}, \tag{74}$$

where $\Psi \equiv \left[\frac{(1-\tau)\alpha-\Pi^*\phi^e(1-\alpha)}{(1-\tau)\alpha-\beta\Pi^*\phi^e(1-\alpha)}\right]^{\frac{1}{\alpha}} < 1$ implies $\bar{B}^g_{FE} > \bar{B}^g_{ZLB}$, and

$$Y^*_{ZLB-FE} = [(1+\beta\Pi^*)\phi^e\,(1-\alpha)]^{\frac{1-\alpha}{\alpha}}\,\bar{h}.$$

Furthermore, for $\beta < 1$, there are five different panels (see Appendix A.2), corresponding to the different possible values of the public debt. These panels in the case of endogenous capital are depicted in Figure 7, which distinguishes from Figure 6 where the panels are three. In any case, since only the definitions of $\bar{B}^g_{ZLB}$ and $\bar{B}^g_{FE}$ modify because of endogenous capital, while the same does not happen to the conditions for the existence of the different bubbles, as well as the conditions associated with the five panels, the proof of Proposition 1 goes unchanged with endogenous capital also for $\beta < 1$. Finally, likewise the case of $\beta = 1$, reaching the FE equilibrium C (Figure 7, Panel E) comes at the cost of a lower current/potential output relative to the maximum attainable because $Y^*_{FE} < Y^*_{ZLB-FE}$.