A Macro-Finance Model of Credit Spreads*

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Abstract

We argue that fluctuations in corporate credit spreads are driven primarily by the changing risk-bearing capacity of financial intermediaries. Empirically, broker-dealer leverage tracks the excess bond premium closely, suggesting that spreads reflect intermediary balance sheet conditions more than firm-level default risk. We develop a continuous-time model in which experts finance capital using defaultable debt, subject to leverage constraints and state-contingent default penalties. The model accounts for the credit spread puzzle, establishes the non-neutrality of default, and uncovers a non-monotonic welfare effect of default stringency. Optimal policy takes the form of a penalty schedule that discourages excessive leverage in normal times while preventing fire sales during downturns.

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1 Introduction

Corporate credit spreads, the yield differential between private-sector debt and risk-free government securities, are among the most closely watched indicators of financial conditions and real economic activity. During the 2007–2009 financial crisis, sharply widening spreads reflected not only rising default risk among non-financial borrowers but also severe funding strains within the financial sector itself. Yet standard theory struggles to account for the level and variation of these spreads. Empirical work has long documented a "credit spread puzzle": observable firm-level default risk explains only a small fraction of the total yield spread (Collin-Dufresne et al. 2001).

Gilchrist and Zakrajsek (2012) sharpen this puzzle by decomposing corporate bond spreads into an expected default component and a residual "excess bond premium" (EBP), representing the compensation investors demand over and above expected losses. They show that it is the EBP, not the default component, that drives the predictive power of credit spreads for output and unemployment. This finding points away from the fundamental risk of non-financial firms and toward the time-varying risk appetite of the financial intermediaries that price and hold corporate debt (Adrian and Shin 2010a, He and Krishnamurthy 2013).

In this paper, we provide a microfoundation for the excess bond premium rooted in the risk-bearing capacity of the financial sector. We argue that movements in the EBP reflect shifts in the effective risk aversion of intermediaries, induced by binding balance sheet constraints. When intermediary net worth deteriorates, the capacity to bear risk contracts, raising the compensation required for holding corporate credit even when the underlying default probability moves little.

We make three contributions. First, we document a tight empirical link between intermediary leverage and the excess bond premium. Using data on primary dealer balance sheets, we show that intermediary leverage strongly predicts movements in the EBP. Consistent with the financial accelerator mechanism (Bernanke and Gertler 1986, Bernanke et al. 1999), leverage ratios also forecast future real activity: high intermediary leverage predicts a rising EBP and deteriorating macroeconomic conditions, supporting the view that supply-side financial frictions are a

primary driver of credit pricing.

Second, we develop a tractable continuous-time macro-finance model that accounts for these facts by embedding *endogenous default* into a heterogeneous-agent framework with leverage constraints. Building on Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2013), the economy is populated by productive "experts" (intermediaries) who face aggregate risk and leverage limits. Unlike standard models in which default is exogenous or absent (Basak and Cuoco 1998), we allow experts to issue defaultable debt to less productive households, where default is a strategic choice: experts weigh the marginal benefit of reduced debt service against a non-pecuniary penalty, following Shubik and Wilson (1977) and Dubey et al. (1989). This mechanism generates a financial accelerator for credit spreads. Negative shocks erode intermediary net worth, tighten leverage constraints, and raise effective risk aversion, causing the EBP to spike even when fundamental default risk increases only modestly, a pattern consistent with the data (Gilchrist and Zakrajsek 2012). Default is non-neutral in our setting: the option to default drives a wedge in the user cost of capital, suppressing investment and amplifying aggregate volatility. The model thereby resolves the credit spread puzzle by mapping the residual EBP to the shadow price of the intermediary's leverage constraint (Dubey et al. 2005, He et al. 2017).

Third, we analyze optimal policy in an economy with endogenous default. Default penalty stringency has non-monotonic welfare effects. Lenient penalties encourage excessive ex-ante leverage and systemic fragility, while excessively harsh penalties force rapid deleveraging during downturns, amplifying fire sales and deepening recessions (Goodhart et al. 2018, Peiris et al. 2024). We characterize the social planner's problem and show that optimal policy takes the form of a state-contingent penalty schedule that balances the gains from credit intermediation against the costs of financial instability.

Related Literature Our paper contributes to the literature on continuous-time macro-finance, building on He and Krishnamurthy (2011, 2013) and Basak and Cuoco (1998). Within this literature, our central concern is the transmission of financial shocks through intermediary balance sheet constraints. Di Tella (2017)

examines how exogenous uncertainty shocks contract intermediary balance sheets, while Adrian and Shin (2010b) and Phelan (2016) characterize conditions under which procyclical leverage arises and banks issue equity. Adrian and Boyarchenko (2012) analyze the dynamic interaction between bank and non-bank intermediaries and its implications for aggregate financial conditions. Anderson and Cesa-Bianchi (2024) study how monetary policy transmits to credit spreads via firm leverage. Our paper complements this work by providing an explicit treatment of endogenous default as the mechanism linking intermediary constraints to the excess bond premium.

A related strand examines capital misallocation from financial frictions. Moll (2014) shows that collateral constraints prevent productive agents from attaining optimal leverage, generating persistent misallocation and aggregate productivity losses. Our model shares this concern with the distributional consequences of financial frictions but focuses on the pricing implications for credit markets rather than the real allocation of capital.

We work in continuous time, following Romanidis (2019). This choice offers three advantages. It delivers analytical tractability, enabling a near closed-form characterization of equilibrium dynamics ahead of numerical solution. It permits a rigorous treatment of the default mechanism: consistent with Tsomocos (2003), Goodhart et al. (2006) and Dubey et al. (2005), the equilibrium is constrained-inefficient, which underscores the real consequences of regulatory intervention. And it facilitates the pricing of credit derivatives and the evaluation of macroprudential policies. In this spirit, Elenev et al. (2021) show that stricter leverage constraints may reduce systemic fragility but simultaneously curtail the intermediary sector's risk-bearing capacity, amplifying aggregate volatility, a trade-off that lies at the heart of our policy analysis.

Outline Section 2 presents the empirical evidence. Section 3 introduces the model. Section 4 derives equilibrium conditions and characterizes the key results. Section 5 describes the numerical solution and interprets the simulations. Section 6 analyzes the social planner's problem. Section 7 concludes.

2 Empirical Evidence

We examine the relationship between intermediary leverage and the excess bond premium, beginning with the construction of each series.

Excess Bond Premium. Gilchrist and Zakrajsek (2012) construct a credit spread index using a large panel of corporate bonds. For each bond, they compute a spread by subtracting the yield of a synthetic risk-free security with matching cash flows from the bond's actual yield, then average these individual spreads across firms to form the aggregate GZ spread.

They decompose this aggregate spread into two orthogonal components: an expected default component capturing firms' fundamental credit risk, and a residual *excess bond premium* (EBP). The EBP measures the variation in credit spreads that cannot be attributed to expected default losses, and thus reflects the time-varying compensation investors require for bearing credit risk beyond what default probabilities alone would justify.

Empirically, the GZ spread substantially outperforms traditional indicators, such as the Baa-Aaa spread, in forecasting economic activity. Crucially, this predictive power is driven almost entirely by the EBP rather than the default component. Shocks to the EBP are associated with large subsequent declines in consumption, investment, and output, suggesting that the premium captures supply-side financial frictions rather than borrower fundamentals.

2.1 Intermediary Leverage and the Excess Bond Premium

We argue that financial intermediary leverage offers a natural explanation for fluctuations in the excess bond premium. Our conception of leverage differs from that of Geanakoplos (2010), in which agents face *debt*-oriented constraints such as margin limits or value-at-risk requirements. Under that framework, leverage is procyclical, contracting during downturns, and commands a positive risk price. Our

approach instead imposes an *equity* constraint, akin to a skin-in-the-game requirement. This friction implies that leverage must rise when conditions deteriorate in order to support asset holdings, generating a countercyclical leverage profile and a negative price of leverage risk, consistent with He et al. (2017).

Following He et al. (2017), we construct intermediary leverage from the aggregate balance sheets of primary dealer holding companies. Each quarter, the intermediary capital ratio is defined as

$$\text{Capital Ratio}_t = \frac{\sum_i \text{Market Equity}_{i,t}}{\sum_i \left(\text{Market Equity}_{i,t} + \text{Book Debt}_{i,t}\right)},$$

where i indexes all active primary dealers in quarter t. Market equity is the quarter-end share price multiplied by shares outstanding; book debt, defined as total assets less common equity, proxies for intermediaries' largely short-term and collateralized liabilities. This ratio approximates a market-valued balance sheet and provides a forward-looking measure of capital adequacy. We define leverage as its inverse:

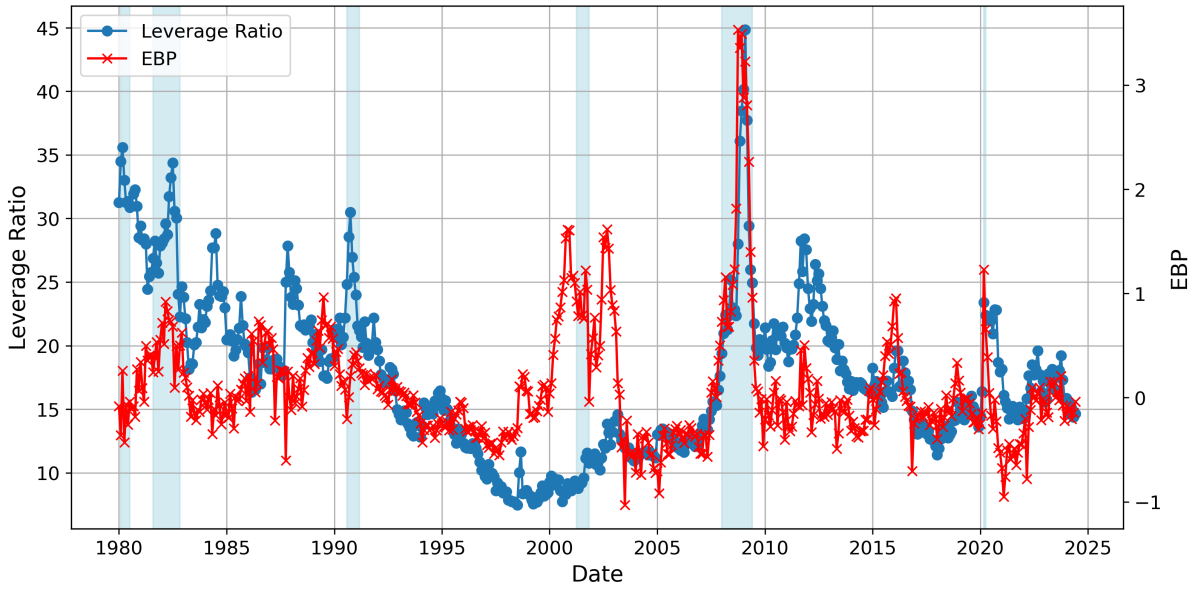
$$\eta_t \equiv \frac{1}{\text{Capital Ratio}_t}.$$

Figure 1a plots the excess bond premium alongside the primary dealer leverage ratio. The two series co-move strongly. During periods of elevated excess bond premiums, leverage tends to rise, driven primarily by declines in market equity. This pattern was especially pronounced during the 2007–2008 financial crisis, when a sharp spike in the EBP coincided with collapsing equity valuations and a jump in leverage. A similar dynamic occurred in early 2009 amid renewed stress in the financial sector, and again in March 2020 at the onset of the COVID-19 pandemic. In both episodes, subsequent Federal Reserve interventions helped stabilize markets, and both series retraced.

To confirm this relationship statistically, we estimate a series of predictive regressions of the form

$$\text{EBP}_{t+h} = \beta_0 + \beta_1 \,\text{TS}_t + \beta_2 \,\text{RFF}_t + \beta_3 \, D_t + \beta_4 \, x_t + \epsilon_{t+h},$$

where TS_t is the term spread, RFF_t is the real federal funds rate, D_t denotes CDS spreads on financial intermediaries—capturing the market's direct assessment of



(a) Excess bond premium against leverage ratio. Shaded bars indicate NBER recession dates.

Figure 1: Intermediary leverage ratio as calculated in He et al. (2017) against the excess bond premium as computed in Gilchrist and Zakrajsek (2012).

intermediary default risk—and x_t is the intermediary leverage ratio. The inclusion of D_t is important: by controlling for the price of intermediary default risk as perceived by the market, we isolate the independent contribution of balance sheet capacity, measured by x_t, to excess bond premia. Any residual predictive power of leverage therefore reflects the tightness of funding constraints rather than elevated default probabilities per se. We consider four nested specifications—a baseline with only TS_t and RFF_t, a default-only model, a leverage-only model, and a horse-race specification including all regressors jointly—estimated over the post-1985 sample at horizons $h \in \{0, 1, 3\}$ months. Table 1 reports the results.

Several results stand out. First, the baseline specification—comprising only the term spread and the real federal funds rate—has essentially no explanatory power, with adjusted R^2 values below 0.025 across all horizons. This underscores the inadequacy of conventional macro-financial controls for capturing the dynamics of the excess bond premium and motivates the introduction of balance sheet variables.

Second, and most importantly, the leverage ratio is a strong and robust predictor of the EBP, both contemporaneously and at horizons of up to three months. At $h = 0$, the leverage-only specification delivers an adjusted R^2 of 0.195, with a coefficient

Table 1: Post-1985 Regression Results for the Level of the Excess Bond Premium

(a) $h = 0$

Financial Indicators	Baseline	Default Only	Leverage Only	Horse Race
Term spread	-0.0468	-0.0178	0.0027	0.0249
	(0.0480)	(0.0436)	(0.0457)	(0.0472)
Real FFR	0.0297	0.0796***	0.0471**	0.0946***
	(0.0257)	(0.0292)	(0.0229)	(0.0287)
Default component	–	0.4504*	–	0.4308**
		(0.2520)		(0.1845)
Leverage Ratio	–	–	0.0504**	0.0489***
			(0.0253)	(0.0185)
Adjusted R^2	0.012	0.146	0.195	0.317
Observations	481	481	474	474

(b) $h = 1$

Financial Indicators	Baseline	Default Only	Leverage Only	Horse Race
Term spread	-0.0466	-0.0167	-0.0001	0.0233
	(0.0474)	(0.0449)	(0.0463)	(0.0486)
Real FFR	0.0343	0.0865***	0.0503**	0.1002***
	(0.0253)	(0.0291)	(0.0223)	(0.0286)
Default component	–	0.4713**	–	0.4523**
		(0.2385)		(0.1790)
Leverage Ratio	–	–	0.0466*	0.0450***
			(0.0242)	(0.0174)
Adjusted R^2	0.015	0.162	0.171	0.306
Observations	480	480	474	474

(c) $h = 3$

Financial Indicators	Baseline	Default Only	Leverage Only	Horse Race
Term spread	-0.0428	-0.0190	-0.0053	0.0139
	(0.0534)	(0.0491)	(0.0552)	(0.0547)
Real FFR	0.0417	0.0850***	0.0540**	0.0951***
	(0.0258)	(0.0319)	(0.0237)	(0.0320)
Default component	–	0.3892*	–	0.3728*
		(0.2351)		(0.1917)
Leverage Ratio	–	–	0.0358*	0.0344**
			(0.0203)	(0.0170)
Adjusted R^2	0.021	0.120	0.112	0.203
Observations	478	478	474	474

Note: Dependent variable is the level of the excess bond premium (EBP). All regressions use the post-1985 sample only. The baseline specification includes the term spread and the real federal funds rate. The "Default Only" specification adds the fitted default component of Gilchrist and Zakrajsek (2012). The "Leverage Only" specification adds intermediary leverage of He et al. (2017). The "Horse Race" specification includes all controls jointly. Standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.1$.

that is significant at the 5% level. The predictive power is fully maintained at $h = 1$, and persists meaningfully at $h = 3$. This pattern of forward-looking significance is consistent with the view that elevated intermediary leverage is not merely a coincident indicator of financial stress but reflects balance sheet conditions that take time to unwind, transmitting persistently into credit risk premia.

Third, the horse-race specification reveals that leverage retains its significance even after controlling for the default component. At all three horizons, the leverage coefficient remains statistically significant at the 1% level in the horse-race column, while the adjusted R^2 rises to 0.317, 0.306, and 0.203 at $h = 0$, $h = 1$, and $h = 3$ respectively—well above what either variable achieves individually. This suggests that leverage and default risk capture distinct and complementary sources of variation in the EBP: the default component reflects the fundamental creditworthiness of borrowers, while leverage captures the risk-bearing capacity of the intermediaries who price and hold corporate debt. Together, these results provide strong empirical support for the central mechanism of our model: it is the tightness of intermediary balance sheet constraints, not the default risk of non-financial firms, that is the dominant driver of excess compensation in corporate bond markets.

2.2 Intermediary Leverage and Economic Activity

We next assess whether the leverage ratio contains independent information about future real activity. Following Gilchrist and Zakrajsek (2012), we estimate the forecasting specification

$$\nabla^h Y_{t+h} = \alpha + \sum_{i=1}^{p} \beta_i \nabla Y_{t-i} + \gamma_1 TS_t + \gamma_2 RFF_t + \gamma_3 \eta_t + \epsilon_{t+h},$$

where ∇Y_{t-i} captures serial correlation in the outcome series, TS_t is the slope of the Treasury curve (10-year minus 3-month yield), RFF_t is the real federal funds rate, x_t is the leverage ratio, and $\nabla^h Y_{t+h}$ is the annualized growth rate of an economic activity measure over the next h periods.[1] By comparing specifications with and

[1] Formally, $\nabla^h Y_{t+h} = \frac{c}{h+1} \ln(Y_{t+h}/Y_{t-1})$, where c is a scaling constant (1,200 at monthly frequency or 400 at quarterly frequency). This measures the annualized log growth of Y from $t-1$ to $t+h$. When $h = 0$, the specification becomes a nowcast, using financial variables observed at t to predict a contemporaneous outcome not yet published.

without x_t, we test the marginal predictive content of intermediary leverage beyond the information already contained in the yield curve and monetary policy stance.

We obtain data on U.S. industrial production and employment from FRED. Table 2a presents results for the 3-month horizon, omitting the lagged activity terms for brevity. The leverage ratio enters with a negative and statistically significant coefficient in both specifications: higher intermediary leverage predicts lower industrial production and higher unemployment three months ahead. Including the leverage ratio also raises the adjusted R^2 meaningfully relative to the baseline.

Table 2b extends the horizon to 12 months. The leverage ratio remains statistically significant, and the improvement in fit is sustained. The negative relationship between leverage and future activity at both horizons is consistent with the countercyclical nature of intermediary leverage and supports the view that balance sheet constraints transmit to the real economy.

2.3 Macroeconomic Propagation of Leverage Shocks

We assess the macroeconomic propagation of intermediary balance sheet shocks, benchmarking against the EBP disturbance identified by Gilchrist and Zakrajsek (2012). That study shows that a one-standard-deviation increase in the EBP, approximately 20 basis points, generates a rapid contraction in real activity, with GDP falling roughly 0.5 percentage points below trend within five quarters.

To isolate the role of intermediary constraints directly, we replicate the eight-variable quarterly VAR of Gilchrist and Zakrajsek (2012) but replace the EBP with the primary dealer leverage ratio of He et al. (2017). We retain the recursive identification strategy, ordering financial variables last so that they respond contemporaneously to real shocks. The identified innovation is a surprise reduction in intermediary leverage, which we interpret as an exogenous tightening of financial intermediation capacity.

Figure 2a presents the impulse responses. Consumption, investment, and output all decline sharply on impact and reach a trough four to six quarters after the shock, closely tracking the responses to an EBP shock reported by Gilchrist and Zakrajsek

Table 2: Regression Results for Economic Activity: 3-Month and 12-Month Horizons

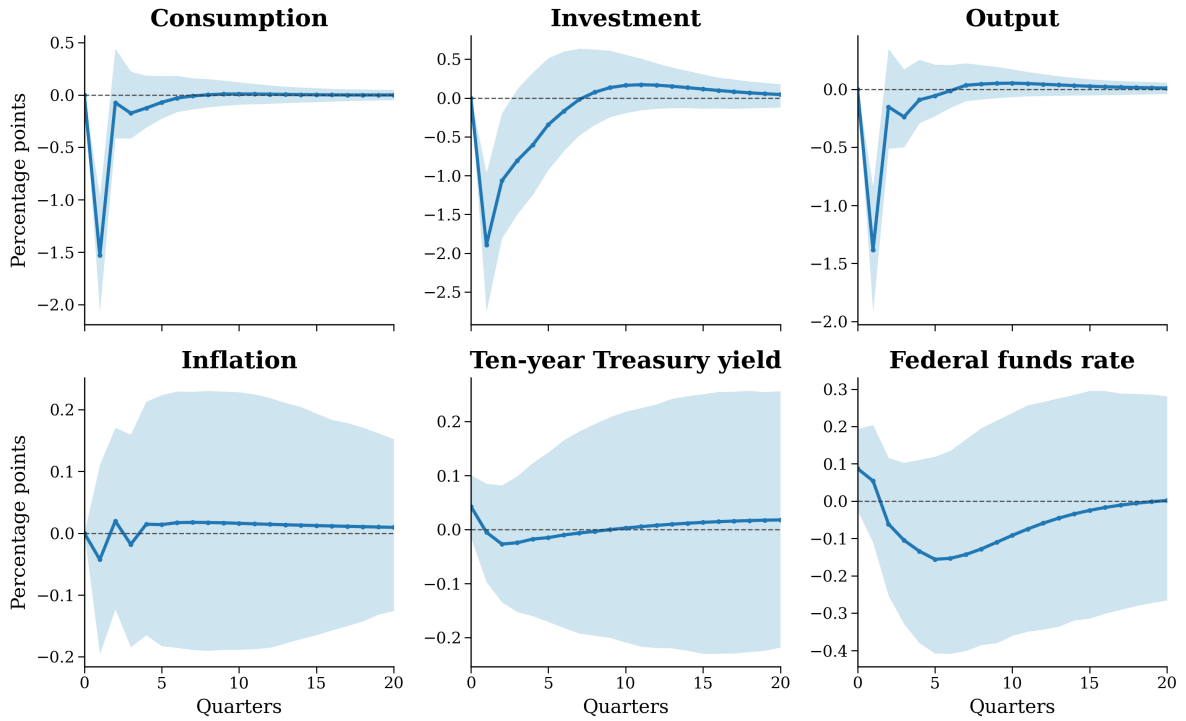
(a) 3-Month Horizon

Financial Indicators	Industrial Production		Unemployment Rate	
Term spread	-0.2631	-0.1901	-0.1208	0.0059
	(0.123)	(0.134)	(0.068)	(0.080)
Real FFR	0.2525	0.0678	0.0918	-0.0884
	(0.127)	(0.146)	(0.082)	(0.087)
Leverage Ratio	–	-0.2499***	–	-0.2806***
		(0.130)		(0.115)
Adjusted R^2	0.458	0.477	0.772	0.796

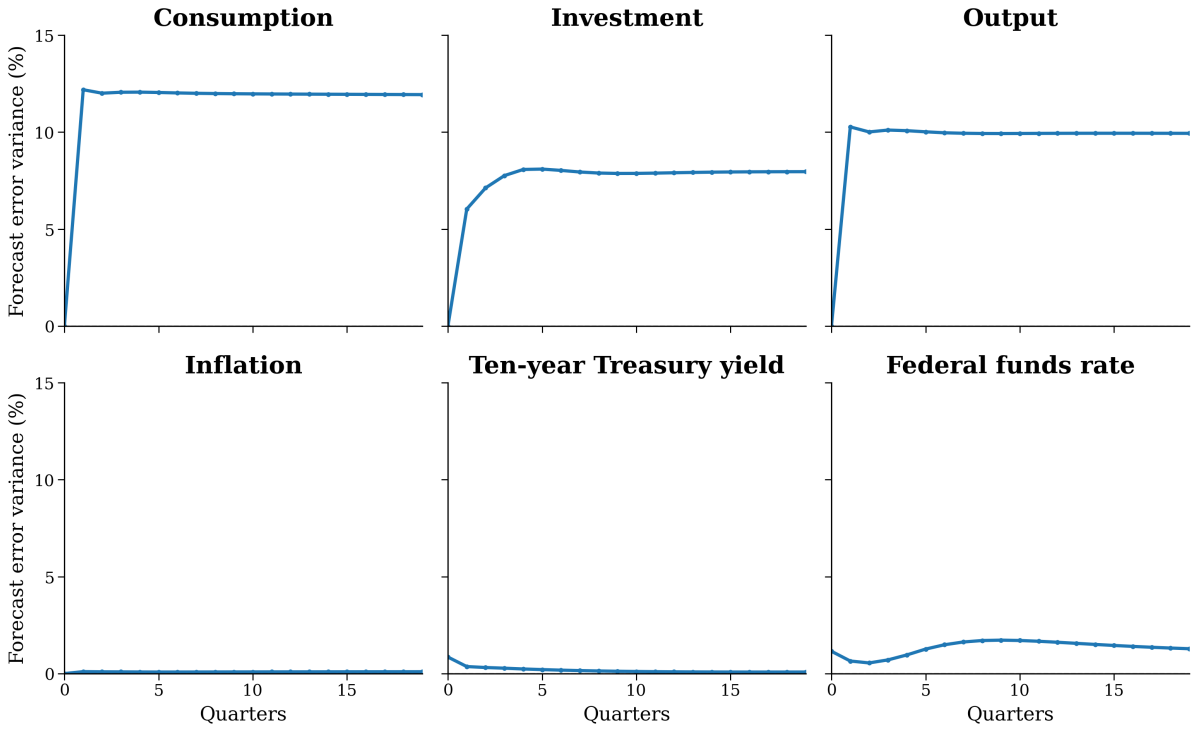
(b) 12-Month Horizon

Financial Indicators	Industrial Production		Unemployment Rate	
Term spread	-0.4045	-0.3040	-0.3115	-0.1661
	(0.240)	(0.240)	(0.150)	(0.162)
Real FFR	0.3181	0.1021	0.1720	-0.0065
	(0.296)	(0.285)	(0.230)	(0.201)
Leverage Ratio	–	-0.3375***	–	-0.3117***
		(0.351)		(0.259)
Adjusted R^2	0.157	0.190	0.508	0.535

Note: OLS coefficients with standard errors in parentheses. *** p<0.01, ** p<0.05, * p<0.1.



(a) Impulse response functions to an intermediary leverage ratio shock.



(b) Forecast error variance decomposition of macro variables attributable to the intermediary leverage ratio shock.

Figure 2: Macroeconomic dynamics following a leverage shock.

(2012). Monetary policy responds counter-cyclically and long-term Treasury yields fall, suggesting that transmission operates through a contraction in credit supply rather than a revision in expected risk-free rates.

Figure 2b reports the forecast error variance decomposition. Leverage shocks account for up to 35 percent of the variation in investment and approximately 20 percent of the variation in output, magnitudes broadly comparable to the explanatory power of the EBP in the existing literature.

To verify that these results are not driven by the extreme volatility of the 2007–2009 crisis, we re-estimate the VAR excluding 2007Q3–2009Q2. The impulse responses remain structurally and quantitatively stable, confirming that the intermediary leverage channel is a pervasive feature of the business cycle rather than an artifact of a single episode.

2.4 Default Conditions and the Excess Bond Premium

We next document the reduced-form relationship between the excess bond premium and aggregate default conditions, proxied by CDS spreads. The aim is not causal identification but rather to establish a transparent empirical regularity: the relationship between default conditions and the EBP is strongly *nonlinear*.

Baseline linear specification. We begin with the bivariate regression

$$\text{ebp}_t = \alpha + \beta d_t + \varepsilon_t, \tag{1}$$

estimated by OLS with heteroskedasticity-robust (HC1) standard errors, where d_t denotes CDS spreads. Column (1) of Table 3 reports a positive and statistically significant slope: on average, the EBP is higher when the default premium is elevated. The fit is limited, however ($R^2 = 0.045$), indicating that a single linear relationship does not adequately capture the joint distribution.

Regime dependence. Figure 3 plots the data with separate linear fits below and above the sample median of the default premium, $\tilde{d} = 1.586$. The scatter reveals a

pronounced convexity: the EBP is weakly declining in d at low levels of the default premium, but rises sharply once default conditions deteriorate.

Columns (2) and (3) of Table 3 quantify this pattern. Below the median, the slope is negative and precisely estimated. Above it, the slope reverses and becomes economically large, with a substantially higher R^2 of 0.239. The positive full-sample coefficient in column (1) thus masks a sharp asymmetry: the EBP is most sensitive to default conditions precisely when those conditions are most adverse.

Column (4) captures the same pattern in a single continuous specification by estimating a piecewise-linear model:

$$\text{ebp}_t = \alpha + \beta_L d_t + \beta_\Delta \big(d_t - \tilde{d}\big)\mathbf{1}\{d_t > \tilde{d}\} + \varepsilon_t, \tag{2}$$

where β_L is the slope below the median and $\beta_L + \beta_\Delta$ the slope above. The estimates imply a steep change in slope at the cutoff: $\widehat{\beta}_L = -0.729$ (s.e. 0.099) below, and $\widehat{\beta}_L + \widehat{\beta}_\Delta = 0.630$ above. The improvement in fit ($R^2 = 0.187$ versus 0.045) confirms that allowing for state dependence is quantitatively important.

Smooth nonlinearity. As a complementary check, we estimate the quadratic approximation

$$\text{ebp}_t = \alpha + \beta_1 d_t + \beta_2 d_t^2 + \varepsilon_t, \tag{3}$$

reported in Table 4. The negative linear term and positive quadratic term ($\widehat{\beta}_1 = -1.574$, s.e. 0.192; $\widehat{\beta}_2 = 0.455$, s.e. 0.057) imply a U-shaped relationship with turning point

$$d^\star = -\frac{\widehat{\beta}_1}{2\widehat{\beta}_2} \approx 1.73, \tag{4}$$

close to the median split used above. Figure 4 overlays the fitted quadratic with a 95% confidence band. The band widens in the right tail, reflecting the relative scarcity of observations with very high default premia.

Interpretation. Across specifications, a consistent pattern emerges: the mapping from default conditions to the excess bond premium is *convex*. In benign credit environments, marginal increases in the default premium are not associated with a

higher EBP and may coincide with modest compression. Once default conditions become sufficiently adverse, however, the EBP rises sharply with d. This non-linearity is consistent with an interpretation in which the non-default component of corporate borrowing costs is particularly sensitive to credit-market stress, binding intermediary constraints, and heightened effective risk aversion in bad states, precisely the mechanism our model is designed to capture.

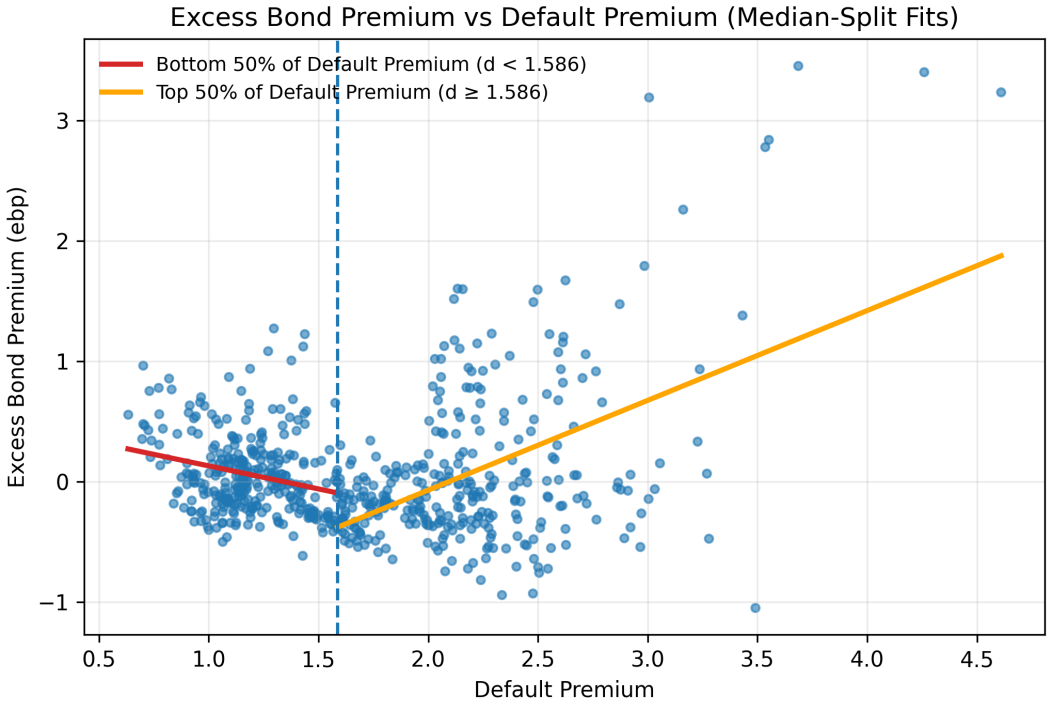


Figure 3: Excess bond premium against the default premium: median-split linear fits. Separate OLS fits are shown below and above the sample median $\tilde{d} = 1.586$ (vertical dashed line).

Table 3: OLS of excess bond premium on default premium (robust s.e.)

	(1) All data	(2) Bottom 50% of DP	(3) Top 50% of DP	(4) Piecewise slope
Constant	-0.252**	0.512***	-1.565***	0.885***
	(0.108)	(0.106)	(0.279)	(0.115)
Default premium	0.187***	-0.383***	0.746***	-0.729***
	(0.071)	(0.088)	(0.136)	(0.099)
$(d-\tilde{d})\,\mathbf{1}(d>\tilde{d})$				1.359***
				(0.207)
Observations	630	315	315	630
R^2	0.045	0.061	0.239	0.187

Dependent variable: ebp. Regressor: default premium d. Robust (HC1) standard errors in parentheses. Models (2) and (3) split at the median $\tilde{d} = 1.586$. Model (4): continuous kink at $\tilde{d}$; slope below is β_L, above is $\beta_L + \beta_\Delta$. $^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$.

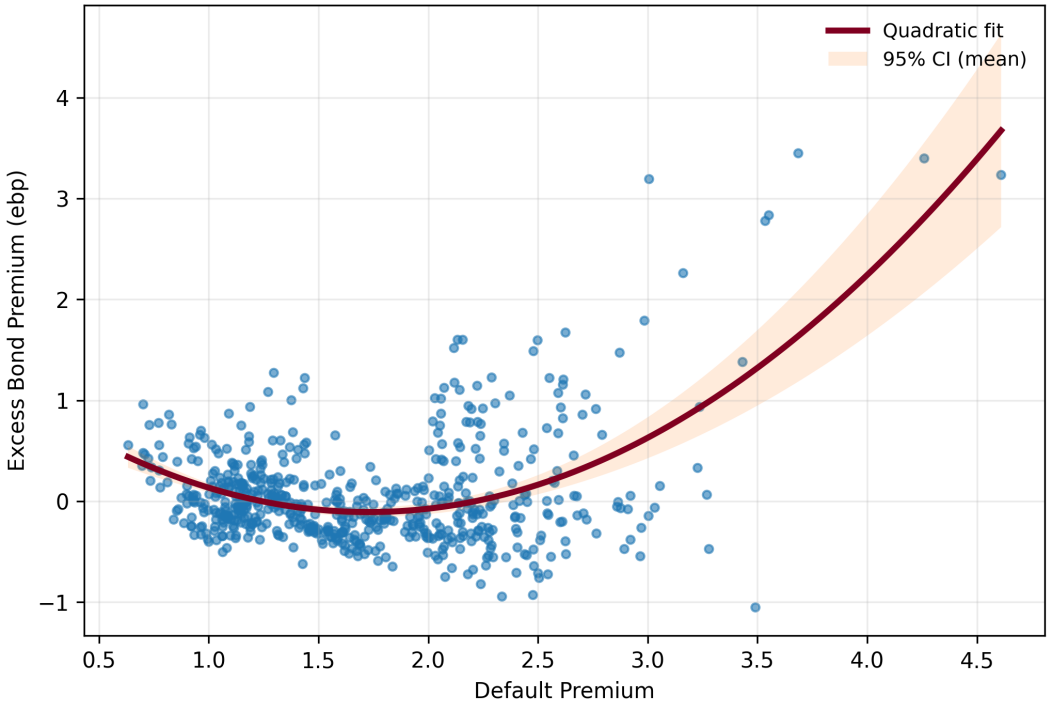


Figure 4: Excess bond premium against the default premium: quadratic fit. The figure overlays the fitted relation from (3) with a 95% confidence band for the conditional mean.

Table 4: Quadratic regression of excess bond premium on default premium (robust s.e.)

	(1) Quadratic
Constant	1.2523***
	(0.1508)
Default premium (d)	-1.5736***
	(0.1918)
d^2	0.4553***
	(0.0565)
Observations	630
R^2	0.2548
Adj. R^2	0.2524

Dependent variable: excess bond premium. Regressors: default premium d and d^2. OLS with HC1 robust standard errors in parentheses. $^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$.

3 The Model

We build on the continuous-time framework of Brunnermeier and Sannikov (2014), extending it to allow for *endogenous default*. The economy is populated by two types of agents: households and experts. Both own capital, but experts operate a superior production technology and finance their projects by issuing defaultable debt to households. We describe the technology and each agent's problem in full, highlighting the novel elements; standard derivations follow Brunnermeier and Sannikov (2014) and are collected in Appendix A.

3.1 Technology

The economy evolves over $[0, \infty)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ supporting a standard Brownian motion Z_t. There are two production technologies. Experts operate a high-productivity AK technology,

$$y_t = A k_t,$$

while households operate a less productive technology,

$$\underline{y}_t = \underline{A}\, \underline{k}_t, \qquad \underline{A} \leq A.$$

Capital held by experts evolves as

$$\frac{dk_t}{k_t} = \big(\phi(\iota_t) - \delta\big)\, dt + \sigma\, dZ_t, \tag{5}$$

where δ is the depreciation rate, $\sigma > 0$ is the exogenous volatility of aggregate productivity shocks, and $\phi(\iota_t) = \frac{1}{\theta}(\sqrt{1 + 2\theta\iota_t} - 1)$ is a strictly concave investment efficiency function that captures convex adjustment costs (Hayashi 1982). Household capital evolves analogously with depreciation rate $\underline{\delta}$.

We conjecture that the equilibrium price of capital satisfies the Ito process

$$\frac{dq_t}{q_t} = \mu_t^q\, dt + \sigma_t^q\, dZ_t, \tag{6}$$

where μ_t^q and σ_t^q are endogenous drift and diffusion coefficients to be determined in equilibrium. Applying Ito's product rule to $q_t k_t$ (see Appendix A.1), the total return on expert capital is

$$dr_t^k = \underbrace{\frac{A - \iota_t}{q_t}\, dt}_{\text{dividend yield}} + \underbrace{\big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\big)\, dt}_{\text{expected capital gain}} + \underbrace{\big(\sigma + \sigma_t^q\big)\, dZ_t}_{\text{capital risk}}, \tag{7}$$

and analogously for households with $(\underline{A}, \underline{\delta})$ in place of (A, δ). The total risk faced by experts has two components: fundamental capital risk σ and endogenous price risk σ_t^q, which reflects general equilibrium feedback through the capital price.

3.2 Experts

A representative expert holds capital k_t financed by a combination of internal net worth n_t and external debt d_t. The balance sheet identity is

$$\underbrace{q_t k_t}_{\text{assets}} = \underbrace{d_t}_{\text{debt}} + \underbrace{n_t}_{\text{equity}}, \tag{8}$$

where $q_t k_t$ is the market value of the capital stock. Experts cannot issue equity; all external finance takes the form of defaultable bonds issued to households.

Investment. Experts reinvest earnings at rate ι_t per unit of capital. The convex adjustment cost technology $\phi(\cdot)$ implies that the optimal investment rate satisfies the standard Tobin's q condition for both agents:

$$\iota_t = \underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1). \tag{9}$$

This equates the marginal cost of investment with the shadow price of installed capital q_t.

Default. The key departure from Brunnermeier and Sannikov (2014) is that default is a strategic, continuous choice. Over the interval $(t, t + dt)$, the expert owes debt service $(q_t k_t - n_t) r_t \, dt$ on outstanding obligations but chooses to repay only the fraction $\nu_t \in [0, 1]$, applying the same repayment rate to both interest and principal. Full repayment corresponds to $\nu_t = 1$; complete default to $\nu_t = 0$; partial default to interior values. Default is costly: the expert incurs a non-pecuniary penalty subtracted from the objective, following Shubik and Wilson (1977) and Dubey et al. (2005). In equilibrium, the expert equates the marginal benefit of debt relief to the marginal penalty cost, so partial default arises endogenously at interior values of ν_t.

Net worth dynamics. Defining leverage as $x_t \triangleq q_t k_t / n_t$, the expert's net worth evolves as (see Appendix A.2)

$$\frac{dn_t}{n_t} = \underbrace{x_t \, dr_t^k}_{\text{leveraged return}} - \underbrace{\nu_t (x_t - 1)(1 + r_t) \, dt}_{\text{actual debt service}} + \underbrace{(x_t - 1) \, dt}_{\text{debt relief from default}}. \tag{10}$$

The first term is the leveraged return on the capital portfolio. The second is the fraction ν_t of total debt service $(x_t - 1)(1 + r_t)$ actually paid. The third captures the fraction $(1 - \nu_t)$ of debt service retained through default. When $\nu_t = 1$, the net worth equation reduces to the standard leveraged return minus full debt service.

Leverage constraint. In addition to the equity issuance constraint, experts face the collateral constraint of Kiyotaki and Moore (1997):

$$x_t \leq U_t \triangleq \min\left\{\frac{1}{m}, \frac{q_t K_t}{n_t}\right\}, \tag{11}$$

where $1/m$ is the regulatory maximum leverage ratio, $m \in (0, 1]$ is the margin requirement, and K_t is aggregate capital. The first term is a regulatory ceiling; the second prevents an expert from holding more than the entire capital stock. The constraint binds when experts are undercapitalized relative to the aggregate economy and is the key source of financial amplification in the model.

Default penalty. The non-pecuniary cost of default is

$$\Upsilon(\nu_t, x_t, r_t) \triangleq \max\{0, 1-\nu_t\}^2 \mathbf{1}(x_t \geq 1)\,(x_t - 1)^2\,(1 + r_t)^2. \tag{12}$$

The penalty is zero when $\nu_t = 1$ (no default) and increasing in the repayment shortfall $(1 - \nu_t)$, the level of leverage $(x_t - 1)$, and the cost of debt service $(1 + r_t)$. The convexity in the repayment shortfall means that even risk-neutral experts behave in a risk-averse manner regarding default. Consistent with Goodhart et al. (2018) and Peiris et al. (2024), the penalty parameter $\lambda > 0$ is procyclical: higher in normal times, lower in recessions, reflecting the principle that default due to aggregate bad luck is treated more leniently than strategic default during good times.[2]

Optimization problem. The expert chooses investment ι_t, leverage x_t, and the repayment rate ν_t to maximize the expected growth rate of net worth net of the default penalty:

$$\max_{\iota_t,\, x_t,\, \nu_t \in [0,1]} \left\{\frac{\mathbb{E}_t[dn_t]}{n_t\, dt} - \lambda\, \Upsilon(\nu_t, x_t, r_t)\right\}, \tag{13}$$

subject to the net worth law of motion (100), the leverage constraint (82), and $\nu_t \in [0, 1]$. Experts reinvest all wealth and do not consume, which is without loss

[2] Appendix C.3 provides a Nash-bargaining microfoundation in which the non-pecuniary cost is proportional to the collateral-to-debt ratio. Since collateral values are procyclical, this delivers a countercyclical effective penalty, consistent with evidence on recovery rates in Goodhart et al. (2018).

of generality given log-linear preferences and preserves analytical tractability.[3]

Taking first-order conditions with respect to ν_t and x_t respectively yields two key optimality conditions. The default condition equates the marginal benefit of reducing debt service to the marginal penalty cost:

$$\underbrace{(x_t - 1)(1 + r_t)}_{\text{marginal benefit}} = \underbrace{2\lambda(1 - \nu_t)(x_t - 1)^2(1 + r_t)^2}_{\text{marginal cost}}, \tag{14}$$

which simplifies to the on-the-verge condition

$$(1 - \nu_t)(1 + r_t)(x_t - 1) = \frac{1}{2\lambda}. \tag{15}$$

The leverage condition equates the expected marginal return on capital to its marginal financing cost:

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = r_t. \tag{16}$$

Together, these conditions determine the equilibrium leverage x_t and repayment rate ν_t as functions of the state variable η_t.

3.3 Households

Households are indexed by $j \in (1, 2]$ and form a continuum of unit mass. They operate the less productive technology ($\underline{A} < A$) and act as net lenders to experts by holding both physical capital and bonds.

Balance sheet. The household holds capital $\underline{k}_t$ and bonds $\underline{d}_t$, with net worth

$$\underline{n}_t = q_t \underline{k}_t + \underline{d}_t. \tag{17}$$

The portfolio share allocated to capital is $\underline{x}_t \equiv q_t \underline{k}_t / \underline{n}_t$, with the remainder $1 - \underline{x}_t$ lent to experts. In equilibrium, $\underline{x}_t \leq 1$ since households are net lenders.

[3] Brunnermeier and Sannikov (2014) make the same assumption. The expert's objective can be microfounded by assuming a continuation value linear in net worth, which together with Bellman's principle delivers wealth maximization as the reduced-form objective.

Net worth dynamics. Over $(t, t+dt)$, the household earns output net of investment, realizes capital gains, receives interest income on bonds, and consumes. When experts choose repayment rate ν_t, the household receives $\nu_t \underline{d}_t r_t \, dt$ in interest but suffers the capital loss $(1-\nu_t)\underline{d}_t$ on the face value of defaulted debt. The dynamic budget constraint is (see Appendix A.3)

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t \, dt + d(q_t \underline{k}_t) + \nu_t \underline{d}_t r_t \, dt - (1-\nu_t)\underline{d}_t \, dt - \underline{c}_t \, dt. \tag{18}$$

Dividing through by $\underline{n}_t$ and using $1 - \underline{x}_t = \underline{d}_t / \underline{n}_t$, the net worth evolution in portfolio form is

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t \, d\underline{r}_t^k + \nu_t(1-\underline{x}_t)(1+r_t) \, dt - (1-\underline{x}_t) \, dt - \frac{\underline{c}_t}{\underline{n}_t} \, dt, \tag{19}$$

where $d\underline{r}_t^k$ is the total return on household capital analogous to (79) with $(\underline{A}, \underline{\delta})$. The second and third terms together give the effective return on lending: $\nu_t(1+r_t) - 1$, which is below the promised return r_t whenever default occurs ($\nu_t < 1$).

Optimization problem. Households take prices $\{q_t, r_t\}$ and the expert repayment rate ν_t as given, and choose consumption $\underline{c}_t$, investment $\underline{\iota}_t$, and portfolio share $\underline{x}_t$ to maximize expected discounted logarithmic utility:

$$\max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} \mathbb{E}\left[\int_0^\infty e^{-\gamma t} \log(\underline{c}_t) \, dt\right], \tag{20}$$

subject to the dynamic budget constraint (47) and the no-short-selling constraint $\underline{x}_t \geq 0$.

The Hamilton-Jacobi-Bellman equation for the household's problem, given that the value function takes the log-affine form $V(\underline{n}_t, t) = \frac{1}{\gamma}(\log \underline{n}_t + f(\eta_t))$ for some function $f(\cdot)$, yields three optimality conditions. First, the consumption-wealth ratio is constant:

$$\underline{c}_t = \gamma \underline{n}_t. \tag{21}$$

Second, investment follows the same Tobin's q rule as experts:

$$\underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1). \tag{22}$$

Third, the optimal portfolio share is the standard mean-variance expression, adjusted for default risk:

$$\underline{x}_t = \frac{\frac{\mathbb{E}_t[d\underline{r}_t^k]}{dt} - \nu_t(1+r_t) + 1}{(\sigma + \sigma_t^q)^2}. \tag{23}$$

The numerator is the expected excess return on capital over the effective lending return $\nu_t(1+r_t)-1$, adjusted for the certain return on wealth. The denominator is the variance of the capital return. As default risk rises (lower ν_t), the return on lending falls and households shift their portfolio toward capital.

4 Equilibrium

4.1 Market Clearing and the State Variable

The capital, debt, and goods markets must clear simultaneously. The key aggregation result, derived in Appendix A.4, is that total wealth equals the value of the aggregate capital stock:

$$N_t + \underline{N}_t = q_t K_t, \tag{24}$$

where N_t and $\underline{N}_t$ are aggregate expert and household net worth. The entire equilibrium is summarized by the experts' share of aggregate wealth,

$$\eta_t \triangleq \frac{N_t}{q_t K_t}, \tag{25}$$

which follows the Itô process

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)\, dt + \beta(\eta_t)\, dZ_t. \tag{26}$$

All equilibrium objects are functions of η_t alone. The leverage constraint expressed in terms of the state variable becomes

$$x_t \leq \min\left\{\frac{1}{m}, \frac{1}{\eta_t}\right\}. \tag{27}$$

The formal equilibrium definition is given in Appendix A.5. The model features two frictions: an occasionally binding leverage constraint and strategic default.

Both imply that the equilibrium is constrained Pareto inefficient, consistent with Geanakoplos and Polemarchakis (1985).

4.2 Equilibrium Characterization

Let $\psi_t \triangleq \int_0^1 k_t^i \, di \,/\, K_t$ denote the fraction of aggregate capital held by experts.

Proposition 1 (Lending direction and leverage shares)**.** *In equilibrium, experts are strictly borrowers and households strictly lenders:*

$$0 \leq \underline{x}_t \leq 1 \leq x_t \leq U_t,$$

with leverage ratios satisfying

$$x_t = \frac{\psi_t}{\eta_t}, \qquad \underline{x}_t = \frac{1-\psi_t}{1-\eta_t}. \tag{28}$$

This follows from the productivity gap $A > \underline{A}$: the return on expert capital always exceeds the bond rate in equilibrium, so intermediation is mutually beneficial and the direction of credit flow is unambiguous.

Proposition 2 (Optimal default threshold)**.** *The expert's optimal repayment rate satisfies*

$$(1-\nu_t)(1+r_t)(x_t-1) = \frac{1}{2\lambda}, \qquad 0 < \nu_t \leq 1. \tag{29}$$

Equation (98) equates the marginal benefit of default, the debt service saved $(1+r_t)(x_t-1)$, to the marginal penalty cost $\lambda(1-\nu_t)$. Two implications follow. First, the incentive to default is increasing in leverage: as debt obligations rise relative to equity, ν_t falls. Second, default is a continuous choice, so credit spreads are determined jointly with capital accumulation and never jump discontinuously.

Proposition 3 (Expert wealth dynamics)**.** *The law of motion for the expert wealth*

share is

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - r_t\,dt + \frac{dt}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\,dt\right) + \left[\frac{A-\iota_t}{q_t} + (1-\psi_t)(\underline{\delta}-\delta)\right]dt, \tag{30}$$

with drift and diffusion coefficients

$$\alpha(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t + \frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) + \frac{A-\iota_t}{q_t} + (1-\psi_t)(\underline{\delta}-\delta), \tag{31}$$

$$\beta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}(\sigma+\sigma_t^q). \tag{32}$$

The drift captures the leverage gap $(\psi_t - \eta_t)/\eta_t$, the risk-adjusted excess return, and the value of the default option. The diffusion confirms that expert wealth is amplified relative to the underlying assets whenever experts are levered ($\psi_t > \eta_t$).

Proposition 4 (Equilibrium price of capital)**.**

$$q_t(\eta_t) = -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[x_t(\eta_t)\eta_t A + (1-x_t(\eta_t)\eta_t)\underline{A}\right] + 1}. \tag{33}$$

The first term is the risk discount demanded by households; it vanishes as $\eta_t \to 1$ when the marginal investor becomes the risk-neutral expert. The second term is the productivity-weighted value of capital, which rises toward A as experts hold more of the capital stock.

Proposition 5 (Equilibrium interest rate)**.**

$$r_t = \frac{\underline{A}-\underline{\iota}_t}{q_t} + \phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - (\sigma+\sigma_t^q)^2\frac{1-x_t\eta_t}{1-\eta_t} + \frac{1}{2\lambda(x_t-1)}. \tag{34}$$

The rate decomposes into the household's expected return on capital, a precautionary savings term that depresses the rate when households bear more aggregate risk, and the default premium $1/[2\lambda(x_t-1)]$.

Proposition 6 (Excess bond premium)**.** *The default premium equals the spread between the equilibrium interest rate and the counterfactual rate absent default risk ($\lambda \to \infty$):*

$$r_t - \text{EBP}_t = \underbrace{\frac{1}{2\lambda(x_t - 1)}}_{\textit{default premium}}. \tag{35}$$

This decomposition isolates the excess bond premium as the component of the credit spread driven by intermediary risk-bearing capacity rather than direct default compensation. As η_t falls and the leverage constraint tightens, x_t rises and the default premium expands, even if the underlying default probability rises only modestly. This is the central mechanism through which the model resolves the credit spread puzzle.

Proposition 7 (Capital price volatility)**.**

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}\,\sigma}{1 - (x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}}. \tag{36}$$

Proposition 8 (Equilibrium leverage condition)**.** *The optimal leverage $x_t \in (1, U_t)$ solves*

$$\frac{A - \underline{A}}{q_t} + (\underline{\delta} - \delta) + (\sigma + \sigma_t^q)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} = \frac{1}{2\lambda(x_t - 1)}, \tag{37}$$

where $q_t = q_t(x_t, \eta_t)$ and $\sigma_t^q = \sigma_t^q(x_t, \eta_t)$ are as given above.

The left-hand side of (41) is the marginal benefit of leverage: the productivity wedge $(A - \underline{A})$ plus the risk premium households require to absorb more capital risk. The right-hand side is the marginal cost of leverage through the default channel. Solving this equation yields the equilibrium leverage function $x_t(\eta_t)$, from which all remaining equilibrium objects follow.

Proposition 9 (Household optimum)**.** *The household's optimal consumption, in-*

vestment, and portfolio share are

$$\underline{c}_t = \gamma \underline{n}_t, \tag{38}$$

$$\underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1), \tag{39}$$

$$\underline{x}_t = \frac{\frac{\mathbb{E}[d\underline{r}_t^k]}{dt} - \nu_t(1 + r_t) + 1}{(\sigma + \sigma_t^q)^2}. \tag{40}$$

Logarithmic utility delivers the familiar result that households consume a constant fraction γ of net worth. The portfolio share $\underline{x}_t$ is a mean-variance expression: as default risk rises (lower ν_t), lending becomes less attractive and households shift toward holding capital directly.

NEW

5 Simulations and Results

5.1 Calibration

The equilibrium dynamics reduce to a coupled system of three equations in $\{q_t, \psi_t, \sigma_t^q\}$.

Proposition 10. *The equilibrium triplet $\{q_t, \psi_t, \sigma_t^q\}$ satisfies*

$$q_t^2 + 2\theta\gamma(1 - \eta_t)q_t - 2\theta\big[\psi_t(A - \underline{A}) + \underline{A}\big] - 1 = 0, \tag{41}$$

$$(\sigma + \sigma_t^q)q'(\eta_t)(\psi_t - \eta_t) - \sigma_t^q q_t = 0, \tag{42}$$

$$\begin{aligned} &2\lambda(1 - \psi_t)(\psi_t - \eta_t)(\sigma + \sigma_t^q)^2 q_t \\ &+2\lambda(\psi_t - \eta_t)(1 - \eta_t)\big[A - \underline{A} - q_t(\delta - \underline{\delta})\big] - \eta_t(1 - \eta_t)q_t = 0. \end{aligned} \tag{43}$$

We solve this system using a shooting algorithm; details are provided in Appendix C.2. The benchmark calibration is summarized in Table 5. We set $\delta = \underline{\delta} = 0$ without loss of generality.

Table 5: Benchmark Calibration

Parameter	Value
Default penalty λ	1 and 3
Expert productivity A	3
Household productivity $\underline{A}$	1
Household discount rate γ	1
Exogenous volatility σ	0.1
Leverage constraint m	0.2

We normalize $\underline{A} = 1$ and set $A = 3$. The productivity wedge $A - \underline{A} = 2$ is large enough to make financial intermediation socially valuable without degenerating into a trivial one-sector economy. We set $\gamma = 1$, eliminating a redundant intertemporal parameter; the leverage constraint and default penalty are sufficient to drive the model's dynamics. Exogenous volatility $\sigma = 0.1$ implies an annualized unconditional standard deviation of output close to 2 percent, keeping the contribution of the default channel distinct from mechanical amplification of capital-gain risk. The margin requirement $m = 0.2$ imposes a maximum leverage ratio of 5 for experts, proxying the leverage of U.S. broker-dealers and ensuring the constraint binds in bad states while remaining slack in good states, consistent with the countercyclical spread dynamics documented by Gilchrist and Zakrajsek (2012).

To capture procyclical default penalties, following Goodhart et al. (2019) we set the cost of default to be three times higher in favorable states than in unfavorable ones. A microfoundation is provided in Appendix C.3. As $\lambda \to \infty$, the marginal cost of default dominates its benefit and default vanishes in equilibrium; a low λ encourages strategic default and amplifies credit spreads. Varying λ thus allows us to assess the sensitivity of real allocations to the institutional framework governing debt renegotiation.

5.2 Equilibrium Dynamics

Figure 5 plots four key equilibrium objects as functions of the expert wealth share η_t. At low η_t, households finance most of the capital stock ($\psi_t \approx 0$). As experts rebuild equity, ψ_t rises until the leverage constraint binds at the vertical red dashed line. Beyond this threshold ($\eta_t \gtrsim 0.55$), experts gradually acquire the entire capital

stock and ψ_t is pinned at unity. This kink marks the regime switch where the nonlinearities induced by leverage and default become active.

Leverage is highest just below the kink and falls sharply as experts become better capitalized. This countercyclical pattern aligns with the empirical evidence in Section 2 and with the equity-based balance sheet constraints of He and Krishnamurthy (2013), contrasting with the procyclical leverage implied by value-at-risk models.

The lending rate r_t and the excess bond premium both widen as η_t falls. The model reproduces the key finding of Gilchrist and Zakrajsek (2012): the majority of time-variation in credit spreads is driven by the non-default component. In our framework this non-default component is the shadow price of the leverage constraint, which rises sharply as expert equity is eroded. The hump-shaped default rate $1 - \nu_t$ reflects the optimality condition (98): at very low η_t, the high interest rate makes default costly; at intermediate values, high leverage raises the marginal benefit of default; and as experts recapitalize, default incentives dissipate.

Taken together, the four panels illustrate the balance-sheet amplification mechanism. A negative shock that erodes expert equity forces deleveraging, depresses capital prices, widens the EBP, and raises strategic default, all of which feed back into deeper contractions in investment and output. This endogenous loop is absent in models without default or equity constraints.

Figure 6 displays investment ι_t, the shadow price of capital q_t, and its conditional volatility σ_t^q.

Investment follows Tobin's q below the kink, rising one-for-one with the shadow price. Once the constraint binds, additional equity no longer raises leverage and q_t plateaus. In the far left tail, where experts are severely undercapitalized, q_t and ι_t collapse despite abundant high-return investment opportunities, a classic underinvestment externality.

The volatility panel is particularly striking. Return variance can be written as

$$(\sigma + \sigma_t^q)^2 = \sigma^2 \left[1 - (x_t - 1)\frac{q'(\eta_t)}{q_t/\eta_t}\right]^{-2},$$

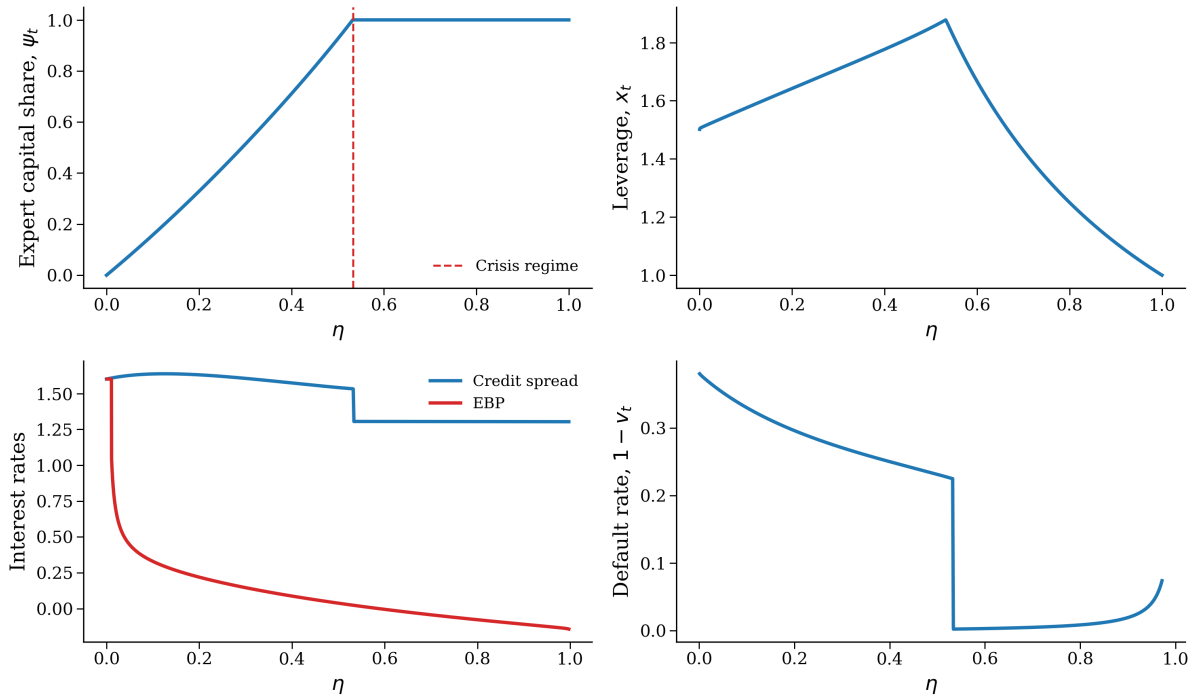


Figure 5: Equilibrium dynamics of key variables as functions of the expert wealth share η_t. The vertical red dashed line marks the leverage constraint kink.

where the multiplier is governed by leverage x_t and the price-elasticity $q'(\eta_t)/(q_t/\eta_t)$, a measure of market depth. Both terms surge as experts approach their leverage ceiling, generating the volatility spike just before the kink. Once experts are fully capitalized, $q' \approx 0$ and the amplification deactivates, returning volatility to the fundamental level σ.

5.3 Stationary Distribution

The law of motion for η_t implies a unique stationary (ergodic) density $d(\eta_t)$ satisfying the Kolmogorov Forward Equation:

$$0 = -\frac{\partial}{\partial \eta}\left[\mu_t^\eta \, \eta_t \, d(\eta_t)\right] + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}\left[(\sigma_t^\eta \eta_t)^2 \, d(\eta_t)\right].$$

Integrating once and imposing reflecting boundaries at $\eta = 0$ and $\eta = 1$ yields the following closed-form expression.

Proposition 11. *The stationary density is*

$$d(\eta_t, t) = \frac{\mathcal{A}}{\sigma_t^\eta \eta_t} \exp\left(\int_\eta \frac{2\mu_t^\eta \eta_t}{(\sigma_t^\eta \eta_t)^2}\, dt\right), \tag{44}$$

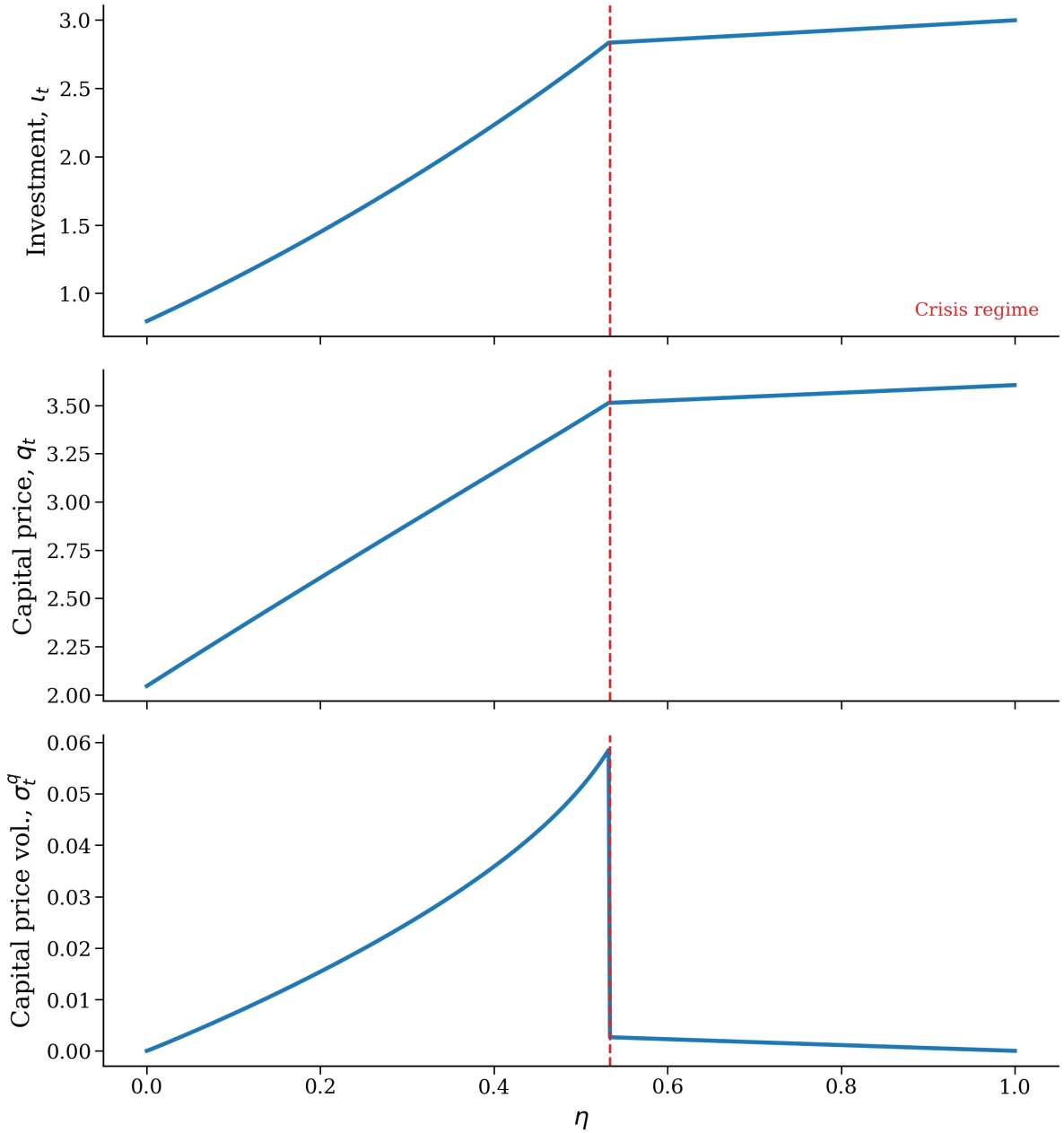


Figure 6: Investment, capital price, and price volatility as functions of η_t. The vertical red line marks the leverage constraint kink.

where $\mathcal{A}$ is the normalization constant satisfying $\int_0^1 d(\eta_t, t)\, d\eta = 1$.

Figure 7 plots the ergodic density under a lenient penalty ($\lambda = 1$, green) and a harsh penalty ($\lambda = 10$, blue). Two findings stand out.

Under harsh penalties, the density is heavily left-skewed, indicating that the economy spends considerable time at low η_t. Strict enforcement forces experts to deleverage slowly through retained earnings, concentrating probability mass in the crisis region where the leverage constraint binds and capital is misallocated to low-productivity households.

Under moderate penalties, the density shifts sharply rightward, peaking near the stochastic steady state $\eta^* \simeq 0.8$. Limited default allows experts to shed debt burdens, recapitalize faster, and resume intermediating capital. The economy therefore spends less time in the constrained region and more time near the first-best allocation. In this sense, some tolerance for default acts as an automatic stabilizer: it accepts a discrete loss for creditors today in exchange for a faster return to a high-output equilibrium, consistent with Goodhart et al. (2019).

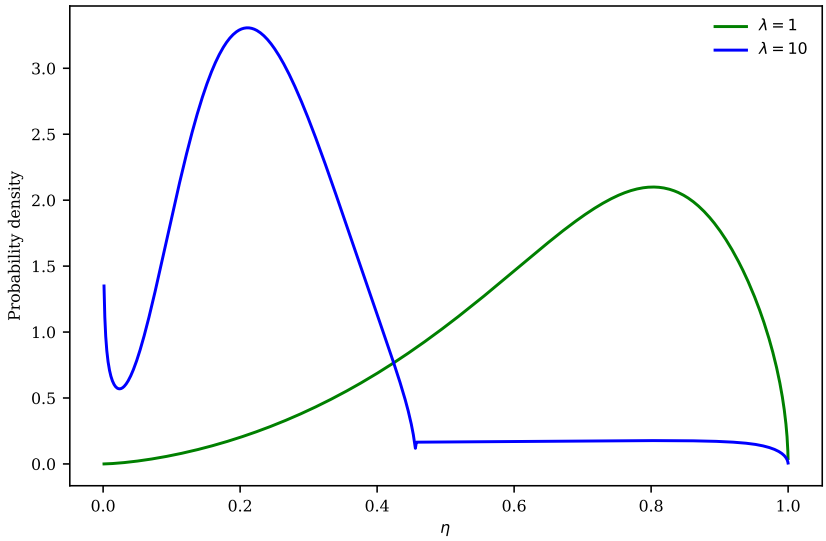


Figure 7: Ergodic density of the expert wealth share η_t under lenient ($\lambda = 1$, green) and harsh ($\lambda = 10$, blue) default penalties. Higher penalties shift probability mass toward low η_t, implying longer crisis spells.

6 Welfare Analysis

6.1 Welfare Decomposition

Substituting goods market clearing into the household utility function yields an analytical welfare decomposition.

Proposition 12. *Household welfare admits the decomposition*

$$\mathbb{E}\left[\int_0^\infty e^{-\gamma t}\log(\underline{C}_t)\,dt\right] = \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\gamma t}\log(K_t)\,dt\right]}_{\textit{Capital effect (extensive margin)}} + \underbrace{\mathbb{E}\left[\int_0^\infty e^{-\gamma t}\log\big((A-\underline{A})\psi_t + \underline{A} - \iota_t\big)\,dt\right]}_{\textit{Reallocation effect (intensive margin)}}. \quad (45)$$

The capital effect captures the aggregate size of the economy. Financial contractions that depress investment or accelerate capital destruction reduce K_t and lower welfare, with the concavity of log utility implying disproportionate harm in capital-scarce states.

The reallocation effect captures the efficiency of capital deployment. The term $(A-\underline{A})\psi_t+\underline{A}$ is the productivity-weighted average return: a higher ψ_t shifts capital toward experts, raising aggregate output even holding K_t fixed. However, this reallocation is costly in resources (ι_t), so policies that facilitate leverage improve welfare only when productivity gains exceed investment costs.

6.2 Non-Monotonic Welfare Effects of Default Stringency

Figure 8 plots the welfare surface over (λ, η). Welfare is maximized neither at zero penalties nor at strict enforcement, but at an intermediate value $\lambda^* \simeq 4$.

Lenient penalties ($\lambda \to 0$) encourage pervasive default, generate excessive leverage, and amplify volatility, all of which depress capital accumulation through the

extensive margin. Draconian penalties ($\lambda \to \infty$) cause experts to deleverage aggressively ex ante to avoid default, leaving capital with low-productivity households and depressing welfare through the intensive margin. The optimal penalty balances moral hazard discipline against crisis flexibility.

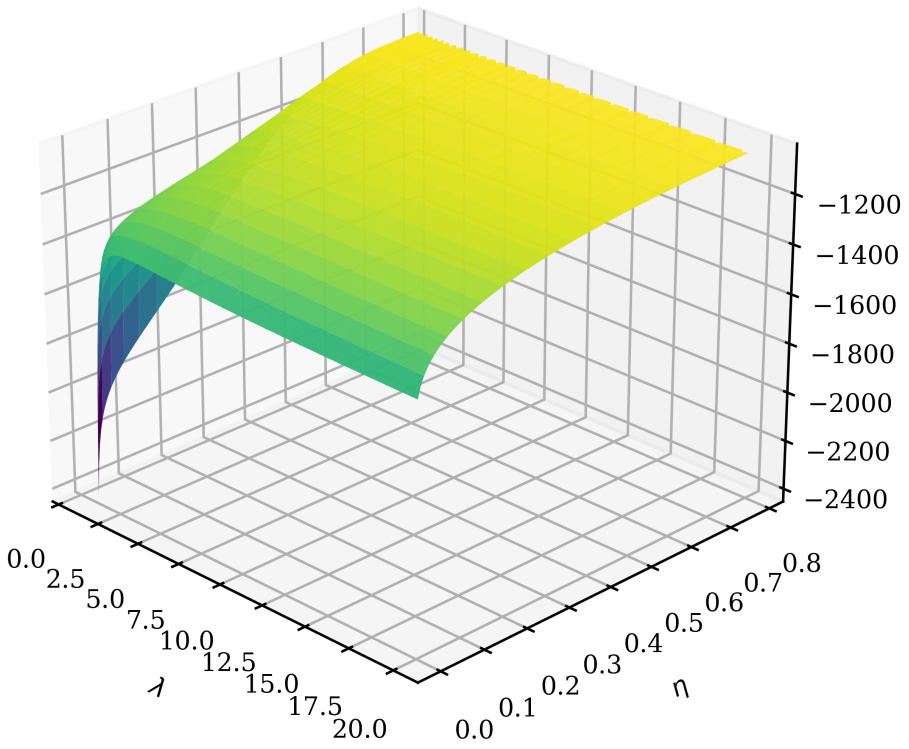


Figure 8: Lifetime welfare as a function of the default penalty λ and expert wealth share η. Welfare peaks at an intermediate penalty $\lambda^* \simeq 4$.

Figure 9 confirms that leverage is highest precisely where welfare is lowest: in low-η states with lenient penalties. As λ rises, the option value of default diminishes and the equilibrium leverage locus shifts downward.

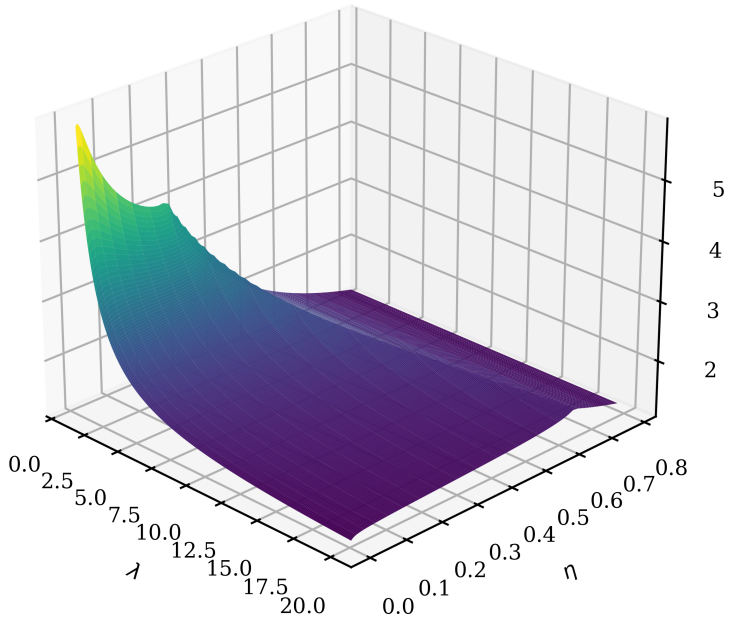


Figure 9: Expert leverage as a function of λ and η. Leverage is highest when penalties are lenient and experts are undercapitalized.

Figures 10 and 11 show the level and volatility of the capital price. The price q_t increases nearly linearly with η_t, driven by the reallocation of capital to high-productivity experts, and is relatively insensitive to λ. Volatility, by contrast, is strongly affected by the penalty regime.

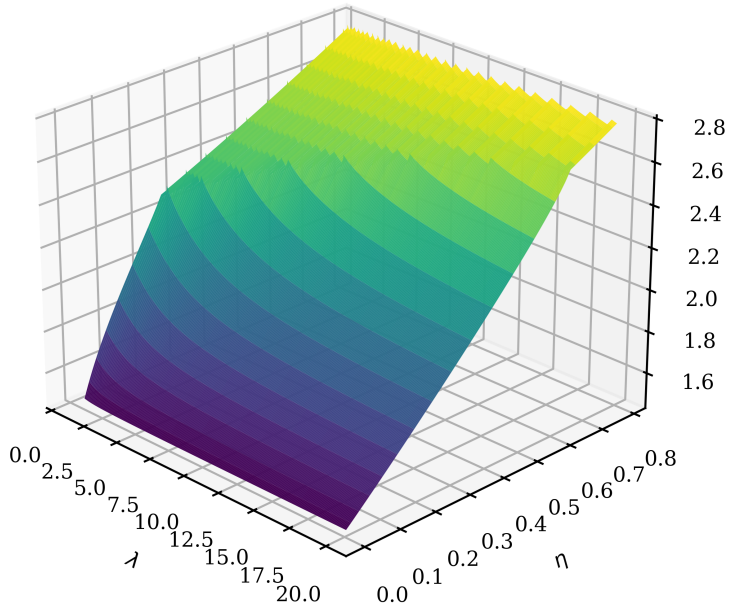


Figure 10: Shadow price of capital as a function of λ and η. The price is primarily driven by η; default penalties have a modest level effect.

Figure 11 reveals a deleveraging cone: volatility spikes sharply at low η_t due to the simultaneous action of high leverage and price illiquidity. Under lenient penalties the spike is severe; raising λ dampens the peak and shifts the crisis region rightward, as experts deleverage earlier in response to stronger default incentives. Figure 12 makes this comparison explicit, plotting volatility slices at $\lambda = 1$ and $\lambda = 3$ and highlighting the gap in the crisis region ($\eta \approx 0.3$).

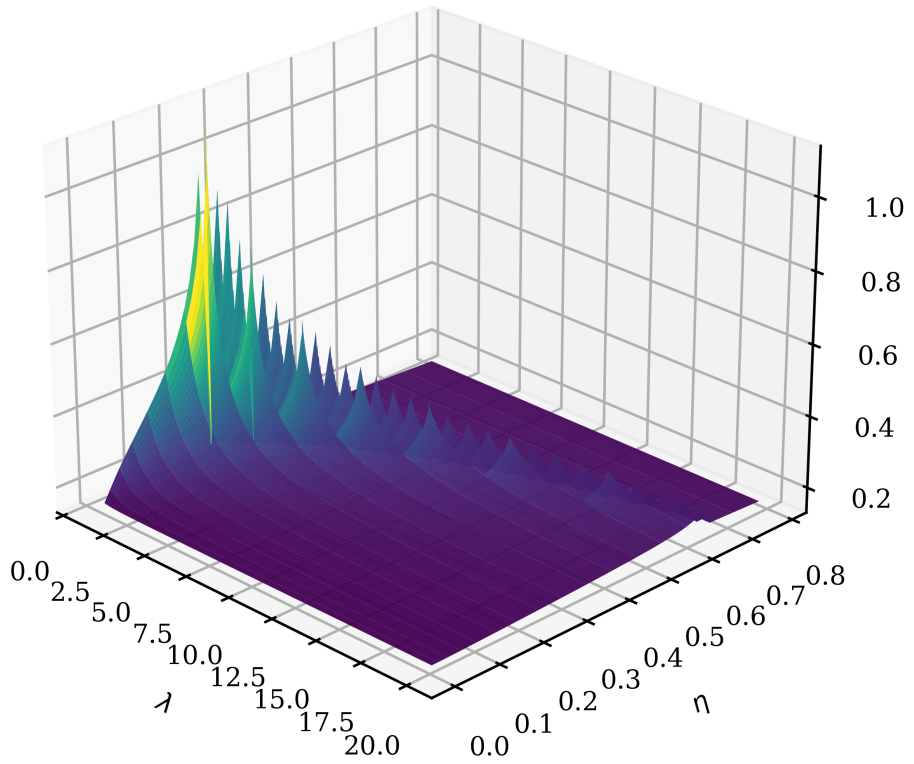


Figure 11: Capital price volatility as a function of λ and η. The deleveraging spike at low η is most severe under lenient penalties.

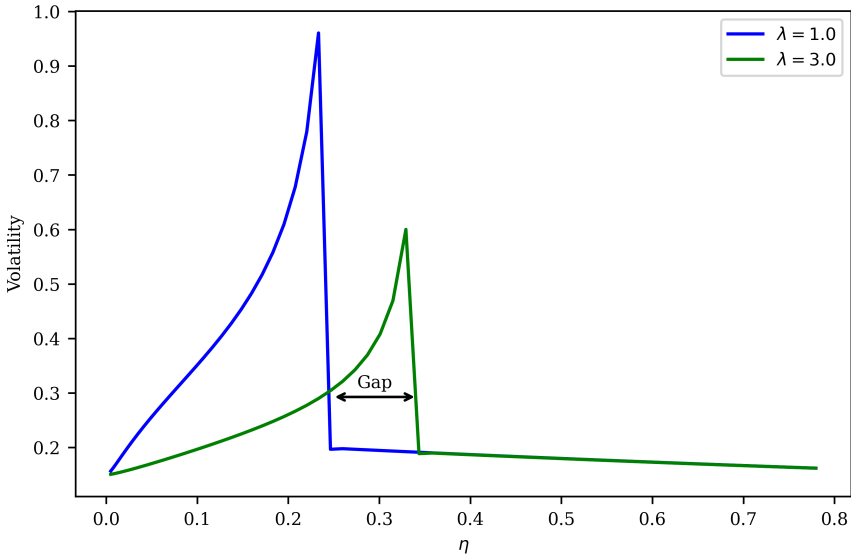


Figure 12: Volatility slices for $\lambda = 1$ (blue) and $\lambda = 3$ (green). The gap in the crisis region illustrates the reduction in systemic risk from moving to a moderate penalty regime.

Policy implication. Welfare is concave in λ while volatility is convex, so a moderate penalty is optimal. It preserves the intermediation benefits of the financial sector

while limiting the tail risk of forced deleveraging. Strict enforcement eliminates strategic default but amplifies systemic risk by removing the debt-restructuring flexibility that speeds recovery from crises. A calibrated intermediate penalty provides superior insurance for the aggregate economy.

7 Conclusion

This paper argues that fluctuations in the excess bond premium are driven by the time-varying risk-bearing capacity of the financial sector. We document a robust empirical link between intermediary leverage and the excess bond premium, showing that balance sheet constraints are a primary determinant of credit spreads and, through them, of real economic activity. Motivated by this evidence, we develop a tractable continuous-time general equilibrium model that integrates endogenous default with leverage constraints to account for these dynamics.

The model resolves the credit spread puzzle. As intermediary net worth and leverage evolve endogenously, equilibrium bond yields contain a significant spread component unrelated to fundamental default risk, mirroring the behavior of the excess bond premium in the data. Default is non-neutral: the option to renegotiate debt creates a wedge in the user cost of capital, suppressing investment and amplifying aggregate volatility, extending the financial accelerator mechanism of Bernanke et al. (1999) to a setting with strategic default.

Our normative analysis uncovers a non-monotonicity in the optimal design of bankruptcy institutions. Lenient penalties encourage excessive ex-ante leverage and heighten systemic fragility, while overly harsh penalties hinder balance sheet repair and prolong recessions by forcing fire sales. An intermediate level of enforcement is welfare-maximizing: it balances the disciplinary benefits of debt enforcement against the stabilizing benefits of debt restructuring, minimizing both the severity and duration of crises.

These results underscore the central role of financial intermediaries in pricing corporate credit and propagating shocks to the real economy. Policies that protect

intermediary net worth, whether through countercyclical capital requirements, liquidity backstops, or optimally calibrated default penalties, can effectively dampen credit spread volatility and stabilize macroeconomic aggregates. A natural direction for future work is to embed this mechanism in a richer policy environment incorporating monetary and fiscal instruments, and to use granular balance sheet data to quantify the welfare gains from alternative macroprudential regimes.

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A Model Derivations

A.1 Returns on Capital

We derive the total return on capital for experts; the household case is analogous. Applying Itô's product rule to $q_t k_t$,

$$d(q_t k_t) = q_t\, dk_t + k_t\, dq_t + dq_t\, dk_t.$$

Substituting the capital evolution (5) and the price process (77),

$$\begin{aligned}\frac{d(q_t k_t)}{q_t k_t} &= \frac{dk_t}{k_t} + \frac{dq_t}{q_t} + \frac{dk_t}{k_t}\cdot\frac{dq_t}{q_t} \\ &= \big(\phi(\iota_t) - \delta\big)\, dt + \sigma\, dZ_t + \mu_t^q\, dt + \sigma_t^q\, dZ_t + \sigma\sigma_t^q\, dt \\ &= \big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\big)\, dt + (\sigma + \sigma_t^q)\, dZ_t.\end{aligned}$$

Adding the dividend yield $(A-\iota_t)/q_t$ gives the total return on expert capital stated in equation (79):

$$dr_t^k = \frac{A-\iota_t}{q_t}\, dt + \big(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\big)\, dt + (\sigma + \sigma_t^q)\, dZ_t.$$

For households, replacing (A, δ) with $(\underline{A}, \underline{\delta})$ yields

$$d\underline{r}_t^k = \frac{\underline{A}-\underline{\iota}_t}{q_t}\, dt + \big(\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\big)\, dt + (\sigma + \sigma_t^q)\, dZ_t.$$

A.2 Expert Net Worth Evolution

Over $(t, t+dt)$ the expert earns output net of investment $(A-\iota_t)k_t\, dt$, realizes capital gains $d(q_t k_t)$, retains the fraction $(1-\nu_t)$ of the interest obligation through default, and pays the fraction ν_t of debt service. Collecting terms,

$$dn_t = (A-\iota_t)k_t\, dt + d(q_t k_t) - \nu_t d_t r_t\, dt + (1-\nu_t) d_t\, dt.$$

Dividing through by n_t and using $d_t = (x_t - 1)n_t$ together with (79),

$$\frac{dn_t}{n_t} = x_t\, dr_t^k - \nu_t(x_t-1)(1+r_t)\, dt + (x_t - 1)\, dt,$$

which is equation (100).

A.3 Household Net Worth Evolution

The household receives output $(\underline{A}-\underline{\iota}_t)\underline{k}_t\,dt$, realizes capital gains $d(q_t\underline{k}_t)$, receives interest $\nu_t\underline{d}_t r_t\,dt$ from experts, suffers the default loss $(1-\nu_t)\underline{d}_t$ on the face value of bonds, and pays for consumption $\underline{c}_t\,dt$. The dynamic budget constraint is therefore

$$d\underline{n}_t = (\underline{A}-\underline{\iota}_t)\underline{k}_t\,dt + d(q_t\underline{k}_t) - \underline{c}_t\,dt + \nu_t\underline{d}_t r_t\,dt - (1-\nu_t)\underline{d}_t\,dt. \tag{46}$$

Dividing by $\underline{n}_t$ and using $\underline{x}_t \equiv q_t\underline{k}_t/\underline{n}_t$ and $1-\underline{x}_t = \underline{d}_t/\underline{n}_t$,

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t\,d\underline{r}_t^k + \nu_t(1-\underline{x}_t)(1+r_t)\,dt - (1-\underline{x}_t)\,dt - \frac{\underline{c}_t}{\underline{n}_t}\,dt. \tag{47}$$

A.4 Market Clearing

Capital market. The total capital held by experts and households equals the aggregate supply:

$$\int_0^1 k_t^i\,di + \int_1^2 \underline{k}_t^j\,dj = K_t, \tag{48}$$

where aggregate capital evolves as

$$dK_t = \left(\int_0^1 \big(\phi(\iota_t^i)-\delta\big)k_t^i\,di + \int_1^2 \big(\phi(\underline{\iota}_t^j)-\underline{\delta}\big)\underline{k}_t^j\,dj\right)dt + \sigma K_t\,dZ_t. \tag{49}$$

Debt market. The face value of expert debt equals the face value of household bond holdings:

$$\int_0^1 d_t^i\,di = \int_1^2 \underline{d}_t^j\,dj. \tag{50}$$

Using the balance sheet identities $d_t^i = q_t k_t^i - n_t^i$ and $\underline{d}_t^j = \underline{n}_t^j - q_t\underline{k}_t^j$ and integrating, debt market clearing implies

$$N_t + \underline{N}_t = q_t K_t, \tag{51}$$

which is equation (24) in the main text.

Goods market. Output produced by firms and households equals household consumption:

$$\int_0^1 k_t^i(A - \iota_t^i)\, di + \int_1^2 \underline{k}_t^j(\underline{A} - \underline{\iota}_t^j)\, dj = \int_1^2 \underline{c}_t^j\, dj. \tag{52}$$

Using $\underline{c}_t^j = \gamma \underline{n}_t^j$ and the leverage definitions, this simplifies to the condition that net output equals aggregate household consumption, pinning down the investment rates jointly with (9).

Aggregate repayment rate. The macro repayment rate is the debt-weighted average of individual repayment choices:

$$\nu_t = \begin{cases} \dfrac{\int_0^1 \nu_t^i \mu_t^i\, di}{\int_0^1 \mu_t^i\, di}, & \text{if } \int_0^1 \mu_t^i\, di \neq 0, \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

where ν_t^i is the repayment rate of expert i and μ_t^i is the amount owed by expert i to households. In a symmetric equilibrium, $\nu_t^i = \nu_t$ for all i.

A.5 Formal Equilibrium Definition

Definition. A stochastic equilibrium with leverage and default consists of stochastic processes for the capital price $\{q_t, t \geq 0\}$, the interest rate $\{r_t, t \geq 0\}$, expert decisions $\{(k_t^i, \iota_t^i, \nu_t^i), t \geq 0\}$ for $i \in (0, 1]$, and household decisions $\{(\underline{k}_t^j, \underline{\iota}_t^j, \underline{c}_t^j), t \geq 0\}$ for $j \in (1, 2]$, such that:

1. Given $\{q_t, r_t\}$, each expert i maximizes (99) subject to (100) and (82), and each household j maximizes (105) subject to (47).
2. The capital market (48), debt market (50), and goods market (52) all clear.

The equilibrium is constrained Pareto inefficient owing to the two financial frictions: the occasionally binding leverage constraint prevents the unconstrained optimum from being attained, and strategic default introduces a further wedge between private and social incentives, consistent with Geanakoplos and Polemarchakis (1985).

B Solution Algorithm

B.1 Overview

The equilibrium system (41)–(43) consists of three coupled nonlinear equations in $\{q_t, \psi_t, \sigma_t^q\}$ as functions of the state variable $\eta_t \in [0, 1]$. We solve it numerically using a variant of the shooting algorithm. The steps are as follows.

B.2 Grid and Initialization

Discretize the state space into a uniform grid $\{\eta_0, \eta_1, \ldots, \eta_N\}$ with $N = 1000$ points on $[0, 1]$. At each grid point, the system (41)–(43) must hold simultaneously.

Boundary conditions. At $\eta_t = 1$, experts own the entire capital stock: $\psi_t = 1$, $\sigma_t^q = 0$, and $q_t = \sqrt{2\theta A + 1}$ from (41).

At $\eta_t = 0$, households own all capital: $\psi_t = 0$, $\sigma_t^q = 0$, and $q_t = \sqrt{2\theta \underline{A} + 1}$.

B.3 Shooting Procedure

1. **Initial guess.** At the right boundary $\eta_t = 1$, set $q_1 = \sqrt{2\theta A + 1}$, $\psi_1 = 1$, $\sigma_1^q = 0$, and compute $q'(1)$ from a first-order finite difference.

2. **Interior solution.** At each interior point η_j, solve the nonlinear system (41)–(43) using Newton's method, initializing at the solution from the adjacent point η_{j+1}.

3. **Regime check.** At each point, verify whether the leverage constraint (97) binds. If $x_t(\eta_j) < U_t(\eta_j)$, the interior optimality condition (43) applies. If $x_t(\eta_j) = U_t(\eta_j)$, set $x_t = U_t$ and resolve (41)–(42) for $\{q_t, \sigma_t^q\}$ with $\psi_t = x_t \eta_t$.

4. **Boundary matching.** Integrate from $\eta_t = 1$ toward $\eta_t = 0$ and verify that the left boundary condition $q(0) = \sqrt{2\theta \underline{A} + 1}$ is satisfied. If not, update the initial guess for $q'(1)$ using a bisection step and repeat.

5. **Convergence.** Iterate until the sup-norm error in the boundary residual falls below 10^{-8}.

B.4 Stationary Density

Given the converged functions $\{\alpha(\eta), \beta(\eta)\}$, the stationary density (44) is evaluated numerically using Simpson's rule on the same grid. The exponential integral is computed cumulatively from $\eta = 0$, and the result is normalized so that $\int_0^1 d(\eta)\, d\eta = 1$.

B.5 Welfare

The welfare decomposition (45) is evaluated by first computing the stationary density $d(\eta)$, then integrating the capital and reallocation terms against the density using the trapezoidal rule. For the surface plots in Figures 8–11, the algorithm is run independently for each value of λ on a grid of 50 points in $[0.5, 15]$.

C Proofs

C.1 Derivatives of the price of capital

Later we shall calculate the drift and volatility of the price of capital, and for that purpose, we need to calculate the first and second derivative of the price of capital with respect to η_t.

Corollary 1. *The first derivative of the price of capital, q_t, with respect to η_t, is given by*

$$q'(\eta_t) \triangleq \frac{\partial q_t(\eta_t)}{\partial \eta_t} \;=\; \theta\,\gamma \;+\; \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\,\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}, \tag{53}$$

where we defined

$$M(\eta_t) \triangleq \theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1. \tag{54}$$

The second derivative of the price of capital, q_t, with respect to η_t, is given by

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{\theta^2\gamma^2 + \theta(A-\underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)]}{[M(\eta_t)]^{1/2}} - \frac{\left[-\theta^2\gamma^2(1-\eta_t) + \theta(A-\underline{A})\left[x_t'(\eta_t)\eta_t + x_t(\eta_t)\right]\right]^2}{[M(\eta_t)]^{3/2}}. \tag{55}$$

Taking the first derivative of $q_t(\eta_t)$ asks: by how much does the shadow price of a unit of capital move when one more dollar of aggregate net worth is shifted from households to experts? Equation (53) decomposes the answer into three intuitive ingredients.

- Risk-sharing margin. The standalone term $\theta\gamma$ is always positive. Transferring wealth from risk-averse households to risk-neutral experts lowers the risk discount applied to capital and therefore raises its price, independent of productivity considerations.
- Curvature of the risk discount. The fraction's numerator contains $-\theta^2\gamma^2(1-\eta_t)$, which is negative when households still hold some wealth. This term captures diminishing marginal benefits of shifting risk: the first dollars that move out of household hands leads to a large fall in risk premium, but subsequent transfers do less and less. As $\eta_t \to 1$ the term vanishes, so the slope of q_t flattens.
- Productivity-leverage margin. The remaining piece,

  $$\theta(A-\underline{A})\left[x_t'(\eta_t)\eta_t + x_t(\eta_t)\right],$$

  reflects how a wealth transfer changes who finances the marginal unit of capital. If leverage rises with expert wealth $\left(x_t'(\eta_t) > 0\right)$, the expert sector not only owns more equity but can lever that equity into an even larger share of the capital stock, boosting the weight on the high productivity A. When the leverage ceiling binds, $x_t'(\eta_t) = 0$ and this amplification disappears.

Dividing by $\sqrt{M(\eta_t)}$ rescales the two adjustments by the level of the price itself. Early in a crisis, i.e. a low η_t, the productivity-leverage term is small, so the curvature term dominates and the price of capital is quite sensitive to further expert losses. Once experts rebuild equity and approach the constraint, the amplification term offsets the curvature term and q_t becomes less sensitive to additional transfers. The derivative therefore traces the slope of a fire-sale: steep when experts are scarce, flatter when they have regained balance-sheet capacity.

C.2 Solution Algorithm Details

We provide an algorithm to solve the system of equations in proposition 10, which is based on a variation of the Newton method. First, we denote the nonlinear system in equations (41)-(43) as $F(\eta_t)$ and the triplet $z_t \triangleq [q_t, \psi_t, \sigma_t^q]$. Then, we denote the Jacobian of the nonlinear system to be

$$J \equiv \nabla F(\eta_t).$$

We place a uniform grid $\{\eta_0, \eta_1, \ldots, \eta_N\}$ on the interval $[0, 1]$ with $\varepsilon \approx 10^{-5}$ to avoid division-by-zero terms in (42) and (43). For $\eta_0 = \varepsilon$ we set an initial guess $q_0 = q(\varepsilon)$ via bisection on (41) with $\psi_0 = 0$ and $\sigma_0^q = 0$, which pins down the left boundary where experts hold no equity.

At each interior knot η_{k+1} we take the converged triplet from η_k as the initial guess z_0 for the Newton iteration. After computing $J_n^{-1} F(z_n)$ we apply a standard back-tracking line search to guarantee

$$0 < q_{n+1}, \quad 0 \leq \psi_{n+1} \leq 1, \quad -\sigma < \sigma_{n+1}^q < 0,$$

and to ensure that the nonlinear residual $\|F(z_{n+1})\|_\infty$ decreases monotonically. For $\psi_t < 1$, use Newton's method to iteratively update

$$z_{n+1} = z_n - J_n^{-1} F(z_n),$$

to solve for z_t. For $\psi_t = 1$, we can solve for q_t from equation (41) and σ_t^q from equation (43). The tolerance for Newton convergence is 10^{-10}; the Jacobian is

recomputed every step by analytic differentiation of (41)-(43).

When ψ_{k+1} hits unity within numerical tolerance ($|\psi_{k+1}-1| < 10^{-8}$), the economy enters the region where experts finance the entire capital stock. This is a regime shift. For all subsequent grid points we solve (41) and (43) in closed form:

$$q_t = \underline{A} \quad \text{and} \quad \sigma_t^q = -\sigma,$$

which forces (42) to hold identically.

Once $z_t \triangle [q_t, \psi_t, \sigma_t^q]$ are known on the full grid, all other equilibrium quantities - investment, leverage, repayment rate, interest rate, and the drift-volatility pair $(\mu_t^\eta, \sigma_t^\eta)$ - follow by direct substitution into the closed-form expressions derived analytically in previous propositions.

Because $g(x_t)$ in Proposition 8 is strictly decreasing on $(1, U_t)$ and the mapping $\eta_t \mapsto \psi_t$ is monotone, the coupled system (41)-(43) features a unique continuously differentiable solution. Empirically, the Newton iteration converges in quite quickly on all interior grid points, confirming the local contraction property of $J^{-1}\nabla F$.

C.3 Procyclical shadow cost of renegotiation

In the main paper, we define the default penalty cost λ as:

$$\lambda = \begin{cases} \lambda_r . \text{in recessions} \\ \lambda_b, \text{in booms} \end{cases}$$

Here we sketch out a potential micro-foundation for a pro-cyclical default penalty. We show that the non-pecuniary default cost is proportional to the collateral-to-debt ratio. Because that collateral-to-debt ratio is pro-cyclical, the implied penalty is high in expansions and low in recessions, mirroring the evidence on recovery rates. We present a two-state Nash bargaining model by linking it to the pro-cyclical cost of renegotiation discussed in Goodhart et al. (2018).

Consider a firm that must repay face value D next period and owns collateral whose

market price P_s depends on the state $s \in \{b \text{ (bust)}, r \text{ (boom)}\}$ with $P_b < P_r$. The expert's options are

1. Repayment: the firm transfers D to creditors.
2. Default: the parties Nash-bargain over the collateral. If each side has weight $1/2$, creditors obtain $P_s/2$ while the firm saves $D - P_s/2$ relative to full repayment.

Define the expert's utility-equivalent renegotiation cost

$$\Lambda_s \;=\; \varphi \, \frac{P_s}{D},$$

where the positive scalar φ converts the loss into the non-pecuniary units that scale the quadratic penalty term $\lambda \, \Upsilon(\cdot)$ as in the main text. Because $P_r > P_b$ we have $\Lambda_r > \Lambda_b$: the penalty is pro-cyclical. We can map the renegotiation cost to the default penalty by simply setting

$$\lambda_b = \Lambda_b, \qquad \lambda_r = \Lambda_r.$$

Note that this formulation is a reduced form of Goodhart et al. (2018) pro-cyclical macro variable which governs the severity of the punishment enforced

$$\Omega_t = \phi \, \frac{\text{capital}_{t-1}}{(1 + r_{t-1}) \, \text{debt}_{t-1}} \, d_t^{-\gamma},$$

The price-to-debt ratio P_s/D in our microfoundation plays the same role as the capital-to-debt term in Goodhart et al. (2018). Now if we differentiate the renegotiation cost:

$$\frac{\partial \Lambda_s}{\partial P_s} \;=\; \frac{\varphi}{D} \;>\; 0.$$

A standard business-cycle model implies that for an aggregate state variable Z_t (such as TFP, demand, or liquidity shocks), $\partial P_s / \partial Z_t > 0$ for the aggregate state Z_t. Hence, we have

$$\frac{\partial \Lambda_s}{\partial Z_t} > 0.$$

That is, $\lambda = \Lambda_s$ inherits pro-cyclicality from the collateral price.

C.4 Equivalence to Brunnermeier-Sannikov (2014)

We establish that the setting of Brunnermeier and Sannikov (2014) is a limiting case of our model, but with a leverage constraint and an endogenous interest rate.

Proposition 13. *(Equivalence theorem) Let $\lambda = +\infty$. The equilibrium value of x_t is always equal to its upper limit i.e. U_t.*

C.5 Proof of Theorem 1

From proposition 15, the optimal choices of the expert for the investment and repayment rate are given by

$$\iota_t = \frac{q_t^2 - 1}{2\theta} \qquad \text{and} \qquad \nu_t = 1 - \frac{1}{2\lambda\left(x_t - 1\right)\left(1 + r_t\right)}.$$

We then substitute these into the objective function of the expert given by equation (99). We know from proposition 14 that $x_t \geq 1$. Simplifying the expression, by substituting the drift of the expert's net worth from equation (100), we get

$$\begin{aligned}
&\mathbb{E}\left[\frac{dn_t}{n_t}\right] - \lambda max\{0,\, (1-\nu_t)\}^2 I\left(x_t \geq 1\right)\left(x_t - 1\right)^2\left(1 + r_t\right)^2 dt \\
&= x_t \mathbb{E}[dr_t^k] - \nu_t\left(x_t - 1\right)\left(1 + r_t\right)dt + \left(x_t - 1\right)dt - \lambda\left(1 - \nu_t\right)^2\left(x_t - 1\right)^2\left(1 + r_t\right)^2 dt.
\end{aligned}$$

Take first order conditions with respect to leverage x_t to get

$$\mathbb{E}[dr_t^k] - \nu_t(1 + r_t) + 1 - 2\lambda(1 - \nu_t)^2(x_t - 1)(1 + r_t)^2 = 0.$$

Take first order condition with respect to default ν_t

$$\begin{aligned}
& -(x_t - 1)(1 + r_t) + 2\lambda(1 - \nu_t)(x_t - 1)^2(1 + r_t)^2 = 0 \\
\leftrightarrow \quad & -1 + 2\lambda(1 - \nu_t)(x_t - 1)(1 + r_t) = 0 \\
\leftrightarrow \quad & 2\lambda(1 - \nu_t)(x_t - 1)(1 + r_t) = 1 \\
\leftrightarrow \quad & (1 - \nu_t) = \frac{1}{2\lambda(x_t - 1)(1 + r_t)} \\
\Rightarrow \quad & \nu_t = 1 - \frac{1}{2\lambda(x_t - 1)(1 + r_t)}.
\end{aligned}$$

Combine the two first order conditions together

$$\begin{aligned}
& \mathbb{E}[dr_t^k] - \nu_t(1+r_t) + 1 - 2\lambda(1-\nu_t)^2(x_t-1)(1+r_t)^2 = 0 \\
\leftrightarrow \quad & \mathbb{E}[dr_t^k] - \left(1 - \frac{1}{2\lambda(x_t-1)(1+r_t)}\right)(1+r_t) + 1 \\
& - 2\lambda\left(\frac{1}{2\lambda(x_t-1)(1+r_t)}\right)^2 (x_t-1)(1+r_t)^2 = 0 \\
\leftrightarrow \quad & \mathbb{E}[dr_t^k] - \left(r_t - \frac{1}{2\lambda(x_t-1)}\right) - \left(\frac{1}{2\lambda(x_t-1)}\right) = 0 \\
\Rightarrow \quad & \mathbb{E}[dr_t^k] = r_t.
\end{aligned}$$

Therefore, the expected return on capital for experts is the risk-free rate

$$\mathbb{E}[dr_t^k] = r_t.$$

C.6 Proof of Theorem ??

The value function of the household is given by equation (105)

$$\max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(\underline{c}_t\right) dt\right].$$

The net worth evolution is given by equation (47)

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + \nu_t(1-\underline{x}_t)(1+r_t)dt - (1-\underline{x}_t)\,dt - \frac{\underline{c}_t}{\underline{n}_t}dt.$$

Recall that the net worth postulate is given by equation (95)

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t$$

and the consumption to net worth ratio is $\underline{\rho}_t = \frac{\underline{c}_t}{\underline{n}_t}$. Let the value function of the household be

$$V\left(t,\, \underline{n}_t,\, \eta_t\right) = \sup_{\underline{x}_t \geq 0,\, \underline{\iota}_t \geq 0,\, \underline{\rho}_t \geq 0} \mathbb{E}_t\left[\int_t^{+\infty} e^{-\gamma s}\left[log\left(\underline{\rho}_t \underline{n}_t\right)\right] ds\right] \tag{56}$$

The HJB equation that the value function V must satisfy to be optimized is given

by

$$\gamma V\left(t,\ \underline{n}_t,\ \eta_t\right) = \sup_{\underline{x}_t \geq 0,\ \underline{\iota}_t \geq 0,\ \underline{\rho}_t \geq 0} \left\{ e^{-\gamma t} \log(\underline{c}_t) + \mathbb{E}_t\big[dV\left(t,\ \underline{n}_t,\ \eta_t\right) \big] \right\}.$$

This is an implicit backward stochastic differential equation for the value process. We can apply Ito's lemma to the value function and take its drift term

$$\mathbb{E}_t\big[dV\left(t,\ \underline{n}_t,\ \eta_t\right) \big] = \frac{\partial V}{\partial \underline{n}_t} \mu_t^{\underline{n}} \underline{n}_t + \frac{1}{2} \frac{\partial^2 V}{\partial \underline{n}_t^2} (\sigma_t^{\underline{n}} \underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t} \mu_t^{\eta} \eta_t + \frac{1}{2} \frac{\partial^2 V}{\partial \eta_t^2} (\sigma_t^{\eta} \eta_t)^2.$$

Therefore, the HJB equation is

$$\gamma V(t, \underline{n}_t, \eta_t) = \sup_{\underline{x}_t \geq 0,\ \underline{\iota}_t \geq 0,\ \underline{\rho}_t \geq 0} \left\{ e^{-\gamma t} log\left(\underline{c}_t\right) + \frac{\partial V}{\partial \underline{n}_t} \mu_t^{\underline{n}} \underline{n}_t + \frac{1}{2} \frac{\partial^2 V}{\partial \underline{n}_t^2} (\sigma_t^{\underline{n}} \underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t} \mu_t^{\eta} \eta_t + \frac{1}{2} \frac{\partial^2 V}{\partial \eta_t^2} (\sigma_t^{\eta} \eta_t)^2 \right. \tag{57}$$

where the drifts and the volatilities of $\underline{n}_t$ and η_t are given by (47) and (95). We conjecture that the value function takes the form

$$V \triangleq V(t, \underline{n}_t, \eta_t) = e^{-\gamma t} \left(f\left(\eta_t\right) + \frac{log\left(\underline{n}_t\right)}{\gamma} \right),$$

for some function f. The derivatives of this value function guess are

$$\begin{aligned}
\frac{\partial V}{\partial \underline{n}_t} &= e^{-\gamma t} \frac{1}{\gamma \underline{n}_t}, \\
\frac{\partial^2 V}{\partial \underline{n}_t^2} &= -e^{-\gamma t} \frac{1}{\gamma \underline{n}_t^2}, \\
\frac{\partial V}{\partial \eta_t} &= e^{-\gamma t} f'(\eta_t), \\
\frac{\partial^2 V}{\partial \eta_t^2} &= e^{-\gamma t} f''(\eta_t).
\end{aligned}$$

Plug this into the HJB equation

$$\gamma V = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \left\{ e^{-\gamma t} log\left(\underline{c}_t\right) + \frac{\partial V}{\partial \underline{n}_t} \mu_t^{\underline{n}} \underline{n}_t + \frac{1}{2} \frac{\partial^2 V}{\partial \underline{n}_t^2} (\sigma_t^{\underline{n}} \underline{n}_t)^2 + \frac{\partial V}{\partial \eta_t} \mu_t^{\eta} \eta_t + \frac{1}{2} \frac{\partial^2 V}{\partial \eta_t^2} (\sigma_t^{\eta} \eta_t)^2 \right\}$$

$$\leftrightarrow \quad \gamma f\left(\eta_t\right) + log\left(\underline{n}_t\right) = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \left\{ log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t} \mu_t^{\underline{n}} \underline{n}_t - \frac{1}{2} \frac{1}{\gamma \underline{n}_t^2} (\sigma_t^{\underline{n}} \underline{n}_t)^2 + f'(\eta_t) \mu_t^{\eta} \eta_t + \frac{1}{2} f''(\eta_t) (\sigma_t^{\eta} \right.$$

Look at the right-hand side for now

$$
\begin{aligned}
& log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + f'(\eta_t)\mu_t^{\eta}\eta_t + \frac{1}{2}f''(\eta_t)(\sigma_t^{\eta}\eta_t)^2 \\
\leftrightarrow \quad & log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)(\beta(\eta_t)\eta_t)^2.
\end{aligned}
$$

The net worth of household in equation (47) can be written as

$$
\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t d\underline{r}_t^k + \left[\nu_t(1-\underline{x}_t)(1+r_t) - (1-\underline{x}_t) - \frac{\underline{c}_t}{\underline{n}_t}\right]dt,
$$

and take the return on capital in equation (88)

$$
d\underline{r}_t^k = \left[\frac{\underline{A}-\underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right]dt + \left(\sigma + \sigma_t^q\right)dZ_t
$$

to get that the drift of net worth is

$$
\begin{aligned}
& \mu_t^{\underline{n}} = \left[\nu_t(1-\underline{x}_t)(1+r_t) - (1-\underline{x}_t) - \frac{\underline{c}_t}{\underline{n}_t}\right] + \underline{x}_t\left[\frac{\underline{A}-\underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right], \\
\leftrightarrow \quad & \mu_t^{\underline{n}} = \left[\nu_t(1-\underline{x}_t)(1+r_t) - (1-\underline{x}_t) - \frac{\underline{c}_t}{\underline{n}_t}\right] + \underline{x}_t E\left[\frac{d\underline{r}_t^k}{dt}\right], \\
\Rightarrow \quad & \mu_t^{\underline{n}} = (1-x_t)[\nu_t(1+r_t) - 1] - \frac{\underline{c}_t}{\underline{n}_t} + \underline{x}_t E\left[\frac{d\underline{r}_t^k}{dt}\right].
\end{aligned}
$$

The volatility of net worth is

$$
\sigma_t^{\underline{n}} = \underline{x}_t(\sigma + \sigma_t^q).
$$

Plug these into the right-hand side of the HJB

$$
\begin{aligned}
& log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\mu_t^{\underline{n}}\underline{n}_t - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}(\sigma_t^{\underline{n}}\underline{n}_t)^2 + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)(\beta(\eta_t)\eta_t)^2 \\
\leftrightarrow \quad & log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\left[\underline{n}_t(1-x_t)[\nu_t(1+r_t)-1] - \underline{c}_t + \underline{n}_t\underline{x}_t E\left[\frac{d\underline{r}_t^k}{dt}\right]\right] - \frac{1}{2}\frac{1}{\gamma \underline{n}_t^2}\left(\underline{n}_t\underline{x}_t(\sigma+\sigma_t^q)\right)^2 \\
& + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 \\
\Rightarrow \quad & log\left(\underline{c}_t\right) + \frac{1}{\gamma \underline{n}_t}\left[\underline{x}_t\underline{n}_t\left(E\left[\frac{d\underline{r}_t^k}{dt}\right] - \nu_t(1+r_t) + 1\right) + \underline{n}_t\left[\nu_t(1+r_t)-1\right] - \underline{c}_t\right] - \frac{\left(\underline{x}_t(\sigma+\sigma_t^q\right.} \\
& + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 \qquad 2\gamma
\end{aligned}
$$

We therefore have that the HJB equation after substituting in the value function V, the drift and the volatility of the household's wealth $\underline{n}_t$ and state variable η_t is given by

$$\gamma f(\eta_t) + log(\underline{n}_t) = \sup_{\substack{\underline{x}_t \geq 0 \\ \underline{\iota}_t \geq 0 \\ \underline{\rho}_t \geq 0}} \left\{ \frac{1}{\gamma \underline{n}_t} \left[\underline{x}_t \underline{n}_t \left(E\left[\frac{d\underline{r}_t^k}{dt}\right] - \nu_t(1+r_t) + 1 \right) + \underline{n}_t \left[\nu_t(1+r_t) - 1\right] - \underline{c}_t \right] \right.$$
$$\left. - \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 + \log(\underline{c}_t) \right\}. \tag{58}$$

We take first order conditions. First, when maximizing with respect to the investment rate $\underline{\iota}_t$, we need only look at the total return on capital for household which gives us

$$\phi'(\underline{\iota}_t) = \frac{1}{q_t}.$$

We plug in the functional form for adjustment costs to conclude that investment rate of households is governed by

$$\underline{\iota}_t = \frac{1}{2\theta}\left(q_t^2 - 1\right).$$

The solution of experts follows *mutatis mutandis*. Now, set

$$\underline{b}_t \triangleq E\left[\frac{d\underline{r}_t^k}{dt}\right] - \nu_t(1+r_t) + 1.$$

Re-arranging equation (58), we obtain f which satisfies

$$\gamma f(\eta_t) + log(\underline{n}_t) = \frac{1}{\gamma \underline{n}_t}\left[\underline{x}_t \underline{n}_t \underline{b}_t + \underline{n}_t\left[\nu_t(1+r_t) - 1\right] - \underline{c}_t\right]$$
$$- \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} + f'(\eta_t)\alpha(\eta_t)\eta_t + \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 + \log(\underline{c}_t).$$

Re-arrange

$$\gamma f(\eta_t) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\left(\beta(\eta_t)\eta_t\right)^2 = \frac{\underline{x}_t \underline{b}_t}{\gamma} + \frac{\nu_t(1+r_t) - 1}{\gamma} - \frac{\underline{c}_t}{\gamma \underline{n}_t}$$
$$- \frac{\left(\underline{x}_t(\sigma + \sigma_t^q)\right)^2}{2\gamma} + \log(\underline{c}_t) - log(\underline{n}_t).$$

Recall that $\underline{\rho}_t \triangleq \frac{\underline{c}_t}{\underline{n}_t}$ so then

$$\gamma f\left(\eta_t\right) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\big(\beta(\eta_t)\eta_t\big)^2 - \frac{\nu_t(1+r_t)-1}{\gamma} = \frac{\underline{x}_t\underline{b}_t}{\gamma} - \frac{\big(\underline{x}_t(\sigma+\sigma_t^q)\big)^2}{2\gamma} + \log\left(\underline{\rho}_t\right) - \frac{\underline{\rho}_t}{\gamma}.$$

Therefore, the function f satisfies

$$\Lambda - \frac{1}{\gamma}\sup_{\underline{x}_t \geq 0}\left\{\underline{x}_t\underline{b}_t - \frac{1}{2}\big(\underline{x}_t(\sigma+\sigma_t^q)\big)^2\right\} - \sup_{\underline{\rho}_t \geq 0}\left\{\log\left(\underline{\rho}_t\right) - \frac{\underline{\rho}_t}{\gamma}\right\} = 0, \tag{59}$$

where we define

$$\Lambda \triangleq \gamma f\left(\eta_t\right) - f'(\eta_t)\alpha(\eta_t)\eta_t - \frac{1}{2}f''(\eta_t)\big(\beta(\eta_t)\eta_t\big)^2 - \frac{\nu_t(1+r_t)-1}{\gamma}$$

We now take the first order conditions in equation (59). First, taking the first order condition with respect to net worth $\underline{\rho}_t$ gives us

$$\frac{1}{\underline{\rho}_t} - \frac{1}{\gamma} = 0.$$

Therefore, we conclude

$$\underline{c}_t = \gamma\underline{n}_t. \tag{60}$$

Next, we take first order condition with respect to leverage $\underline{x}_t$ which gives us

$$\underline{b}_t - \underline{x}_t(\sigma+\sigma_t^q)^2 = 0.$$

Simplifying, we get that the optimal leverage for the household is

$$\underline{x}_t = \frac{E\left[\frac{dr_t^k}{dt}\right] - \nu_t(1+r_t) + 1}{(\sigma+\sigma_t^q)^2}. \tag{61}$$

This concludes the proof.

C.7 Proof of Proposition 14

We first show that $1 \leq x_t \leq U_t$. The right-hand side of the second inequality follows directly from the leverage constraint. We only need to show the first inequality. In

equilibrium, at an instant t, the expected return on bonds (accounting for default) is r_t^d if the household is the lender and $\underline{r}_t^d$ if the expert is the lender. Then, the expert who is risk neutral will lend only if $\underline{r}_t^d - \mathbb{E}\left[dr_t^k\right] > \mathbb{E}\left[dr_t^k\right] - r_t^d$ where $\mathbb{E}\left[dr_t^k\right]$ is the expected return on capital. Likewise, households prefer to borrow only if $\mathbb{E}\left[d\underline{r}_t^k\right] - \underline{r}_t^d > r_t^d - \mathbb{E}\left[d\underline{r}_t^k\right]$. The fact that the household is risk averse implies that its excess return on capital should be higher than the excess return on lending, in order to borrow, since default is also strategic, and thus, anticipated due to rational expectations. Hence, both of these equations need to be satisfied, which leads to a contradiction since $\mathbb{E}\left[dr_t^k\right] > \mathbb{E}\left[d\underline{r}_t^k\right]$ or to $x_t = 1 = \underline{x}_t$[4], i.e., no lending takes place.

C.8 Proof of Proposition 15

The objective function of the firm is

$$\max_{\iota_t,\, x_t,\, \nu_t} \left\{ \mathbb{E}\left[\frac{dn_t}{n_t}\right] - \lambda \Upsilon(\nu_t, x_t, r_t) dt \right\},$$

Note that the investment rate ι_t, appears only in the drift term of the evolution of net worth $\mathbb{E}\left[\frac{dn_t}{n_t}\right]$. Recall that the net worth evolution of the firm from (100)

$$\frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \left(x_t - 1\right)\left(1 + r_t\right) dt + \left(x_t - 1\right) dt,$$

and the return on capital (79)

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t.$$

Thus,

$$\mathbb{E}\left[\frac{dn_t}{n_t}\right] = x_t\left(\frac{A - \iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt - \nu_t \left(x_t - 1\right)\left(1 + r_t\right) dt + \left(x_t - 1\right) dt.$$

To solve for the optimal value for the investment, we need to differentiate the above equation with respect to ι_t and set it equal to 0. Note that since ϕ is concave,

[4] We will later see that this cannot constitute an equilibrium.

this ensures we obtain the maximum. Recall the functional form is

$$\phi(x) \triangleq \frac{1}{\theta}\left(\sqrt{1+2\theta x}-1\right).$$

Hence, the optimal investment satisfies

$$\begin{aligned} & \phi'(\iota_t) = \frac{1}{q_t} \\ \leftrightarrow \quad & \frac{1}{\sqrt{1+2\theta\iota_t}} = \frac{1}{q_t} \\ \Rightarrow \quad & \iota_t = \frac{1}{2\theta}\left(q_t^2-1\right). \end{aligned}$$

Finally, we need to derive the optimal repayment rate. Note that the objective function is a quadratic function in ν_t with a negative coefficient on ν_t^2

$$\Upsilon(\nu_t, x_t, r_t) \triangleq max\{0,\ (1-\nu_t)\}^2 I\left(x_t \geq 1\right)\left(x_t-1\right)^2\left(1+r_t\right)^2.$$

Thus, its maximum is obtained where its derivative is equal to zero, i.e.,

$$\begin{aligned} & -\left(x_t-1\right)\left(1+r_t\right)dt + 2\lambda\left(1-\nu_t\right)\left(x_t-1\right)^2\left(1+r_t\right)^2 dt = 0 \\ \Rightarrow \quad & \nu_t = 1 - \frac{x_t-1}{2\lambda\left(x_t-1\right)^2\left(1+r_t\right)} \\ \Rightarrow \quad & \nu_t = 1 - \frac{1}{2\lambda\left(x_t-1\right)\left(1+r_t\right)}. \end{aligned}$$

This is clearly less than or equal to 1, since from proposition (14), $x_t \geq 1$, and for sufficiently high values of λ, and since x_t is bounded by $\frac{1}{m}$, it is greater than 0[5, 6].

C.9 Proof of Proposition 16

For an expert we have

$$x_t n_t = \frac{q_t k_t}{n_t} n_t = q_t k_t.$$

[5] In fact, in the simulations, it turns out, it is significantly greater than 0, for a great range of values for λ.

[6] Note here, that if the optimal choice for the expert, turned out to be 0 repayment, this would imply no borrowing as the household would prefer to maintain its capital rather than lending to the expert who would default completely. In addition, the repayment rate depends non-monotonically on on leverage.

Aggregating over all experts and dividing through by $q_t K_t$ we get

$$\begin{aligned}\int_0^1 q_t k_t^i di &= q_t \int_0^1 k_t^i di \\ &= q_t K_t \frac{\int_0^1 k_t^i di}{K_t} \\ &= q_t K_t \psi_t.\end{aligned}$$

Then, we have

$$\begin{aligned}&\int_0^1 x_t^i n_t^i di = q_t K_t \psi_t \\ \leftrightarrow \quad & x_t N_t = q_t K_t \psi_t \\ \leftrightarrow \quad & x_t \frac{N_t}{q_t K_t} = \psi_t \\ \leftrightarrow \quad & x_t \eta_t = \psi_t \\ \Rightarrow \quad & x_t = \frac{\psi_t}{\eta_t}.\end{aligned}$$

Finally from (91), we have

$$\begin{aligned}&\int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t \\ \leftrightarrow \quad & x_t N_t + \underline{x}_t \underline{N}_t = q_t K_t,\end{aligned}$$

where $\underline{N}_t = q_t K_t - N_t$ is the market clearing condition for the debt market. Now, dividing through by $q_t K_t$ we get

$$x_t \eta_t + \underline{x}_t (1 - \eta_t) = 1$$

which gives the desired result after substituting for $x_t = \frac{\psi_t}{\eta_t}$

$$\frac{\psi_t}{\eta_t} \eta_t + \frac{\psi_t}{\eta_t} (1 - \eta_t) = 1,$$

simplifies to

$$x_t = \frac{\psi_t}{\eta_t}.$$

Furthermore,

$$\underline{x}_t = \frac{1-\psi_t}{1-\eta_t}.$$

This concludes the proof.

C.10 Proof of Proposition 3

The evolution of wealth for a given expert i is from equation (100)

$$\frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \left(x_t - 1\right)\left(1 + r_t\right)dt + \left(x_t - 1\right)dt.$$

The definition of leverage is $x_t \triangleq \frac{q_t k_t}{n_t}$, which becomes

$$\frac{dn_t}{n_t} = \frac{q_t k_t}{n_t} dr_t^k - \nu_t \left(\frac{q_t k_t - n_t}{n_t}\right)(1 + r_t)dt + \left(\frac{q_t k_t - n_t}{n_t}\right)dt.$$

Integrate both sides

$$\int_0^1 \frac{dn_t^i}{n_t^i} di = dr_t^k \int_0^1 \frac{q_t k_t^i}{n_t^i} di - (1+r_t)dt \int_0^1 \nu_t^i \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right) di + \int_0^1 \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right) didt. \tag{62}$$

We work through each integral. First, recall that the aggregation of wealth over all experts is

$$N_t \triangleq \int_0^1 n_t^i di$$

and the fraction of capital held by firms is

$$\psi_t \triangleq \frac{\int_0^1 k_t^i di}{K_t}.$$

Therefore, the left-hand side

$$\int_0^1 \frac{dn_t^i}{n_t^i} di = \frac{dN_t}{N_t}.$$

Next,

$$\int_0^1 \frac{q_t k_t^i}{n_t^i} di = \frac{\psi_t q_t K_t}{N_t}.$$

Also

$$\int_0^1 \nu_t^i \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right) di = \nu_t \left(\frac{\psi_t q_t K_t}{N_t} - 1\right).$$

Finally

$$\int_0^1 \left(\frac{q_t k_t^i - n_t^i}{n_t^i}\right) di = \left(\frac{\psi_t q_t K_t}{N_t} - 1\right).$$

Plug the integrals into equation (62)

$$\frac{dN_t}{N_t} = dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)(1 + r_t)dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1\right)dt.$$

Therefore, we get that the aggregate wealth evolution over experts is

$$dN_t = \psi_t q_t K_t dr_t^k - \nu_t \psi_t q_t K_t \left(1 + r_t\right) dt + \nu_t N_t \left(1 + r_t\right) dt + \psi_t q_t K_t dt - N_t dt. \tag{63}$$

In equilibrium, the wealth share of firms from equation (96) is

$$\eta_t = \frac{N_t}{q_t K_t}.$$

By Ito's quotient rule, we have that

$$\frac{d\eta_t}{\eta_t} = \frac{dN_t}{N_t} - \frac{d\left(q_t K_t\right)}{q_t K_t} + \left(\frac{d\left(q_t K_t\right)}{q_t K_t}\right)^2 - \frac{dN_t}{N_t}\frac{d\left(q_t K_t\right)}{q_t K_t}. \tag{64}$$

Recall that the evolution of total supply of capital from equation (**??**) is

$$dK_t = \left(\int_0^1 \left(\phi\left(\iota_t^i\right) - \delta\right) k_t^i di + \int_1^2 \left(\phi\left(\underline{\iota}_t^j\right) - \underline{\delta}\right) \underline{k}_t^j dj\right) dt + \sigma K_t dZ_t.$$

The optimal investment by experts from proposition 15 and the optimal investment by households from theorem **??** are identical

$$\iota_t \triangleq \iota_t^i = \underline{\iota}_t^j = \frac{1}{2\theta}\left(q_t^2 - 1\right),$$

for $i \in [0,1]$ and $j \in [1,2]$.

$$\begin{aligned}
dK_t &= \left(\int_0^1 \left(\phi\left(\iota_t^i\right) - \delta \right) k_t^i di + \int_1^2 \left(\phi\left(\underline{\iota}_t^j\right) - \underline{\delta} \right) \underline{k}_t^j dj \right) dt + \sigma K_t dZ_t \\
&= \left(\int_0^1 \left(\phi\left(\iota_t\right) - \delta \right) k_t^i di + \int_1^2 \left(\phi\left(\iota_t\right) - \underline{\delta} \right) \underline{k}_t^j dj \right) dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) \left(\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj \right) - \left(\int_0^1 \delta k_t^i di + \int_1^2 \underline{\delta k}_t^j dj \right) \right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\int_0^1 \delta k_t^i di + \int_1^2 \underline{\delta k}_t^j dj \right) \right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta \int_0^1 k_t^i di + \underline{\delta} \int_1^2 \underline{k}_t^j dj \right) \right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta \psi_t K_t + \underline{\delta}(1-\psi_t) K_t \right) \right] dt + \sigma K_t dZ_t \\
&= \left[\phi\left(\iota_t\right) K_t - \left(\delta \psi_t + \underline{\delta}(1-\psi_t) \right) K_t \right] dt + \sigma K_t dZ_t \\
\Rightarrow \frac{dK_t}{K_t} &= \left[\phi\left(\iota_t\right) - \delta \psi_t - \underline{\delta}(1-\psi_t) \right] dt + \sigma dZ_t
\end{aligned}$$

The postulate for the price of capital from equation (77) is

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t.$$

Applying Ito's product rule

$$\begin{aligned}
d(q_t K_t) &= q_t dK_t + K_t dq_t + dq_t dK_t \\
&= q_t \left[\left[\phi\left(\iota_t\right) - \delta \psi_t - \underline{\delta}(1-\psi_t) \right] K_t dt + \sigma K_t dZ_t \right] + K_t \left[\mu_t^q q_t dt + \sigma_t^q q_t dZ_t \right] + (\sigma K \\
&= q_t K_t \left[\left[\phi\left(\iota_t\right) - \delta \psi_t - \underline{\delta}(1-\psi_t) \right] dt + \sigma dZ_t \right] + q_t K_t \left[\mu_t^q dt + \sigma_t^q dZ_t \right] + q_t K_t (\sigma \sigma_t^q \\
\Rightarrow \frac{d(q_t K_t)}{q_t K_t} &= \left[\phi\left(\iota_t\right) - \delta \psi_t - \underline{\delta}(1-\psi_t) + \mu_t^q + \sigma \sigma_t^q \right] dt + (\sigma + \sigma_t^q) dZ_t.
\end{aligned}$$

Now, compare this to the return on capital for firms in equation (79)

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma \sigma_t^q \right) dt + \left(\sigma + \sigma_t^q \right) dZ_t.$$

Re-arrange this expression

$$\left(\phi\left(\iota_t\right) + \mu_t^q + \sigma \sigma_t^q \right) dt + \left(\sigma + \sigma_t^q \right) dZ_t = dr_t^k - \frac{A - \iota_t}{q_t} dt + \delta dt.$$

Plug this into the capital gains on aggregate capital

$$\begin{aligned}\frac{d(q_tK_t)}{q_tK_t} &= \left[\phi(\iota_t) - \delta\psi_t - \underline{\delta}(1-\psi_t) + \mu_t^q + \sigma\sigma_t^q\right]dt + (\sigma+\sigma_t^q)dZ_t \\ &= \left[\phi(\iota_t) + \mu_t^q + \sigma\sigma_t^q\right]dt + (\sigma+\sigma_t^q)dZ_t + \left[-\delta\psi_t - \underline{\delta}(1-\psi_t)\right]dt \\ &= dr_t^k - \frac{A-\iota_t}{q_t}dt + \delta dt + \left[-\delta\psi_t - \underline{\delta}(1-\psi_t)\right]dt \\ &= dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt.\end{aligned}$$

Therefore, the capital gains on aggregate capital is

$$\frac{d(q_tK_t)}{q_tK_t} = dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt \tag{65}$$

where the first two terms of the right hand side are the expert capital gains and the third term is the adjustment for the household's capital.

Substitute equation (63) and equation (65) into equation (64)

$$\begin{aligned}\frac{d\eta_t}{\eta_t} &= \frac{dN_t}{N_t} - \frac{d(q_tK_t)}{q_tK_t} + \left(\frac{d(q_tK_t)}{q_tK_t}\right)^2 - \frac{dN_t}{N_t}\frac{d(q_tK_t)}{q_tK_t} \\ &= \left[dr_t^k\frac{\psi_tq_tK_t}{N_t} - \nu_t\left(\frac{\psi_tq_tK_t}{N_t}-1\right)(1+r_t)dt + \left(\frac{\psi_tq_tK_t}{N_t}-1\right)dt\right] \\ &- \left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt\right] + \left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt\right]^2 \\ &- \left[dr_t^k\frac{\psi_tq_tK_t}{N_t} - \nu_t\left(\frac{\psi_tq_tK_t}{N_t}-1\right)(1+r_t)dt + \left(\frac{\psi_tq_tK_t}{N_t}-1\right)dt\right] \\ &\times \left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt\right]\end{aligned}$$

The quadratic variation term only depends on the Brownian coefficient loading on the expert's return of capital

$$\left[dr_t^k - \frac{A-\iota_t}{q_t}dt - (1-\psi_t)(\underline{\delta}-\delta)dt\right]^2 = (\sigma+\sigma_t^q)^2dt.$$

The same fact applies to the cross-variation term

$$
\begin{aligned}
&- \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t \left(\frac{\psi_t q_t K_t}{N_t} - 1 \right) (1 + r_t) dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1 \right) dt \right] \\
&\times \left[dr_t^k - \frac{A - \iota_t}{q_t} dt - (1 - \psi_t) (\underline{\delta} - \delta) dt \right] \\
&= - \left[\frac{\psi_t q_t K_t}{N_t} \right] (\sigma + \sigma_t^q)^2 dt.
\end{aligned}
$$

We can group these terms

$$
(\sigma + \sigma_t^q)^2 dt - \left[\frac{\psi_t q_t K_t}{N_t} \right] (\sigma + \sigma_t^q)^2 dt = \left[1 - \frac{\psi_t}{\eta_t} \right] (\sigma + \sigma_t^q)^2 dt = \left[\frac{\eta_t - \psi_t}{\eta_t} \right] (\sigma + \sigma_t^q)^2 dt,
$$

where we used $\eta_t \triangleq \frac{N_t}{q_t K_t}$. Substitute this in

$$
\begin{aligned}
\frac{d\eta_t}{\eta_t} &= \left[dr_t^k \frac{\psi_t q_t K_t}{N_t} - \nu_t \left(\frac{\psi_t q_t K_t}{N_t} - 1 \right) (1 + r_t) dt + \left(\frac{\psi_t q_t K_t}{N_t} - 1 \right) dt \right] \\
&\quad - \left[dr_t^k - \frac{A - \iota_t}{q_t} dt - (1 - \psi_t) (\underline{\delta} - \delta) dt \right] + \left[\frac{\eta_t - \psi_t}{\eta_t} \right] (\sigma + \sigma_t^q)^2 dt \\
&= \left[dr_t^k \frac{\psi_t}{\eta_t} - \nu_t \left(\frac{\psi_t - \eta_t}{\eta_t} \right) (1 + r_t) dt + \left(\frac{\psi_t - \eta_t}{\eta_t} \right) dt \right] \\
&\quad - \left[dr_t^k - \frac{A - \iota_t}{q_t} dt - (1 - \psi_t) (\underline{\delta} - \delta) dt \right] + \left[\frac{\eta_t - \psi_t}{\eta_t} \right] (\sigma + \sigma_t^q)^2 dt \\
&= \frac{\psi_t - \eta_t}{\eta_t} dr_t^k + \left[-\nu_t \left(\frac{\psi_t - \eta_t}{\eta_t} \right) (1 + r_t) dt + \left(\frac{\psi_t - \eta_t}{\eta_t} \right) dt \right] \\
&\quad - \left[-\frac{A - \iota_t}{q_t} dt - (1 - \psi_t) (\underline{\delta} - \delta) dt \right] + \left[\frac{\eta_t - \psi_t}{\eta_t} \right] (\sigma + \sigma_t^q)^2 dt \\
&= \frac{\psi_t - \eta_t}{\eta_t} \left(dr_t^k - (\sigma + \sigma_t^q)^2 dt \right) - \nu_t \left(\frac{\psi_t - \eta_t}{\eta_t} \right) (1 + r_t) dt + \left(\frac{\psi_t - \eta_t}{\eta_t} \right) dt \\
&\quad + \frac{A - \iota_t}{q_t} dt + (1 - \psi_t) (\underline{\delta} - \delta) dt.
\end{aligned}
$$

Therefore, we obtain

$$
\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t} \left(dr_t^k + [1 - \nu_t (1 + r_t)] dt - (\sigma + \sigma_t^q)^2 dt \right) + \left[\frac{A - \iota_t}{q_t} + (1 - \psi_t) (\underline{\delta} - \delta) \right] dt.
$$

Finally, take the optimal choice for the default rate, ν_t given by equation (98)

$$
\nu_t = 1 - \frac{1}{2\lambda (x_t - 1) (1 + r_t)}.
$$

We have that

$$\begin{aligned}1-\nu_t(1+r_t) &= 1-\left[1-\frac{1}{2\lambda\left(x_t-1\right)\left(1+r_t\right)}\right](1+r_t)\\ &= 1-(1+r_t)+\frac{1}{2\lambda(x_t-1)}\\ &= -r_t+\frac{1}{2\lambda(x_t-1)}.\end{aligned}$$

Therefore, we conclude

$$\frac{d\eta_t}{\eta_t}=\frac{\psi_t-\eta_t}{\eta_t}\left(dr_t^k-r_t dt+\frac{1}{2\lambda(x_t-1)}dt-\left(\sigma+\sigma_t^q\right)^2 dt\right)+\left[\frac{A-\iota_t}{q_t}+(1-\psi_t)\left(\underline{\delta}-\delta\right)\right]dt.$$

C.11 Proof of Proposition 4

From the market clearing condition for consumption in equation (93),

$$\int_0^1 x_t^i n_t^i\left(A-\iota_t^i\right)di+\int_1^2 \underline{x}_t^j \underline{n}_t^j\left(\underline{A}-\underline{\iota}_t^j\right)dj=q_t\int_1^2 \underline{n}_t^j\underline{\rho}_t^j dj.$$

Plug in $\underline{c}_t=\underline{\rho}_t\underline{n}_t$ and aggregate to arrive at

$$x_t N_t\left(A-\iota_t\right)+\underline{x}_t\underline{N}_t\left(\underline{A}-\underline{\iota}_t\right)=q_t\underline{c}_t.$$

Using the definition of leverage $x_t N_t=\psi_t q_t K_t$ and $\underline{x}_t\underline{N}_t=\left(1-\psi_t\right)q_t K_t$,

$$\psi_t q_t K_t\left(A-\iota_t\right)+\left(1-\psi_t\right)q_t K_t\left(\underline{A}-\underline{\iota}_t\right)=q_t\underline{c}_t.$$

Now, use the fact that $\underline{c}_t=\gamma\underline{N}_t$, $\underline{N}_t=q_t K_t-N_t$ and $\eta_t=\frac{N_t}{q_t K_t}$

$$\underline{c}_t=\gamma\left[q_t K_t-N_t\right]=\gamma\left[q_t K_t-\eta_t q_t K_t\right]=\gamma q_t K_t\left[1-\eta_t\right].$$

Plug this into the goods market clearing

$$\psi_t q_t K_t\left(A-\iota_t\right)+\left(1-\psi_t\right)q_t K_t\left(\underline{A}-\underline{\iota}_t\right)=q_t\gamma q_t K_t\left[1-\eta_t\right].$$

This cancels out

$$\psi_t\left(A-\iota_t\right)+\left(1-\psi_t\right)\left(\underline{A}-\underline{\iota}_t\right)=q_t\gamma\left[1-\eta_t\right].$$

We further substitute for $\iota_t = \underline{\iota}_t = \frac{1}{2\theta}\left(q_t^2 - 1\right)$ into the previous equation and we get

$$
\begin{aligned}
& \psi_t\left(A - \iota_t\right) + \left(1 - \psi_t\right)\left(\underline{A} - \underline{\iota}_t\right) = q_t\gamma\left[1 - \eta_t\right] \\
\leftrightarrow \quad & \psi_t\left(A - \frac{1}{2\theta}\left(q_t^2 - 1\right)\right) + \left(1 - \psi_t\right)\left(\underline{A} - \frac{1}{2\theta}\left(q_t^2 - 1\right)\right) = q_t\gamma\left[1 - \eta_t\right] \\
\leftrightarrow \quad & \psi_t A + (1 - \psi_t)\underline{A} - \left(\frac{\psi_t}{2\theta}\left(q_t^2 - 1\right) + \frac{(1 - \psi_t)}{2\theta}\left(q_t^2 - 1\right)\right) = q_t\gamma\left[1 - \eta_t\right] \\
\leftrightarrow \quad & 2\theta\psi_t A + 2\theta(1 - \psi_t)\underline{A} - \left(\psi_t\left(q_t^2 - 1\right) + (1 - \psi_t)\left(q_t^2 - 1\right)\right) = 2\theta q_t\gamma\left[1 - \eta_t\right] \\
\leftrightarrow \quad & 2\theta\psi_t A + 2\theta(1 - \psi_t)\underline{A} - \left(q_t^2 - 1\right) = 2\theta q_t\gamma\left(1 - \eta_t\right).
\end{aligned}
$$

This gives us the quadratic equation

$$q_t^2 + 2\theta\gamma\left(1 - \eta_t\right)q_t - 2\theta\left[\psi_t A + (1 - \psi_t)\underline{A}\right] - 1 = 0.$$

Use the quadratic formula and keep the positive root

$$
\begin{aligned}
q_t &= \frac{-2\theta\gamma(1 - \eta_t) \pm \sqrt{\left[2\theta\gamma(1 - \eta_t)\right]^2 - 4\left(-2\theta\left[\psi_t A + (1 - \psi_t)\underline{A}\right] - 1\right)}}{2} \\
&= \frac{-2\theta\gamma(1 - \eta_t) \pm \sqrt{4\theta^2\gamma^2(1 - \eta_t)^2 + 2\theta\left[\psi_t A + (1 - \psi_t)\underline{A}\right] + 1}}{2} \\
&= \frac{-2\theta\gamma(1 - \eta_t) + 2\sqrt{\theta^2\gamma^2(1 - \eta_t)^2 + 2\theta\left[\psi_t A + (1 - \psi_t)\underline{A}\right] + 1}}{2} \\
&= -\theta\gamma(1 - \eta_t) + \sqrt{\theta^2\gamma^2(1 - \eta_t)^2 + 2\theta\left[\psi_t A + (1 - \psi_t)\underline{A}\right] + 1}.
\end{aligned}
$$

Substitute in $\psi_t = x_t\eta_t$ from the proof of proposition 16 to conclude

$$q_t = -\theta\gamma(1 - \eta_t) + \sqrt{\theta^2\gamma^2(1 - \eta_t)^2 + 2\theta\left[x_t\eta_t A + (1 - x_t\eta_t)\underline{A}\right] + 1}.$$

C.12 Proof of Corollary 1

We consider the price function

$$q_t(\eta_t) \;=\; -\,\theta\,\gamma\,[\,1{-}\eta_t\,] + \sqrt{\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A \;+\; \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1}\,.$$

We wish to compute the first derivative $\frac{\partial q_t(\eta_t)}{\partial \eta_t}$. First rewrite q_t as a sum of a linear and non-linear part

$$q_t(\eta_t) \;=\; g(\eta_t) \;+\; \sqrt{M(\eta_t)}\,,$$

where

$$g(\eta_t) \;=\; -\,\theta\,\gamma\,[\,1-\eta_t\,], \quad M(\eta_t) \;=\; \theta^2\,\gamma^2\,(1-\eta_t)^2 + 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \big(1-x_t(\eta_t)\,\eta_t\big)\,\underline{A}\right] + 1.$$

It follows that

$$\frac{\partial}{\partial\eta_t}\,q_t(\eta_t) \;=\; \frac{d}{d\eta_t}\,g(\eta_t) \;+\; \frac{d}{d\eta_t}\,\sqrt{M(\eta_t)}\,.$$

Differentiating the linear term $g(\eta_t)$ gives us

$$g'(\eta_t) \;=\; \theta\,\gamma$$

We differentiate the square-root term via the chain rule. First we write

$$\frac{d}{d\eta_t}\,\sqrt{M(\eta_t)} \;=\; \frac{1}{2\,\sqrt{M(\eta_t)}} \;\cdot\; f'(\eta_t).$$

So differentiating and using the product rule, we have

$$f'(\eta_t) = -\,2\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; 2\,\theta\,\big(A-\underline{A}\big)\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right].$$

Substitute $f'(\eta_t)$ back into the square-root derivative

We have

$$\frac{1}{2\,\sqrt{M(\eta_t)}}\,f'(\eta_t) = \frac{1}{2\,\sqrt{M(\eta_t)}}\left[-\,2\,\theta^2\,\gamma^2\,(1-\eta_t) + 2\,\theta\,(A-\underline{A})\,\big(\,x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\big)\right].$$

Factor out the 2 in the numerator and cancel with the 2 in the denominator:

$$\frac{d}{d\eta_t}\,\sqrt{M(\eta_t)} = \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\,\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}.$$

So now combine both terms for the final derivative:

$$\frac{\partial q_t(\eta_t)}{\partial\eta_t} \;=\; \theta\,\gamma + \frac{-\,\theta^2\,\gamma^2\,(1-\eta_t) \;+\; \theta\,(A-\underline{A})\,\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{\theta^2\,\gamma^2\,(1-\eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \big(1-x_t(\eta_t)\,\eta_t\big)\,\underline{A}\right] \;+\; 1}}\,.$$

Define

$$M(\eta_t) \triangleq \theta^2 \gamma^2 (1-\eta_t)^2 \; + \; 2\theta \left[x_t(\eta_t)\, \eta_t\, A + \left(1 - x_t(\eta_t)\, \eta_t\right) \underline{A} \right] \; + \; 1.$$

Therefore we have

$$\frac{\partial q_t(\eta_t)}{\partial \eta_t} \; = \; \theta \gamma \; + \; \frac{-\theta^2 \gamma^2 (1-\eta_t) \; + \; \theta (A - \underline{A}) \left[x_t'(\eta_t)\, \eta_t + x_t(\eta_t) \right]}{\sqrt{M(\eta_t)}}.$$

To get the second derivative, let

$$g(\eta_t) \triangleq -\theta^2 \gamma^2 (1-\eta_t) \; + \; \theta (A - \underline{A}) \left[x_t'(\eta_t)\, \eta_t + x_t(\eta_t) \right],$$

so that we have

$$\frac{\partial q_t(\eta_t)}{\partial \eta_t} = \theta\gamma + \frac{g(\eta_t)}{\sqrt{M(\eta_t)}}.$$

The second derivative is

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{1}{2} \left[M(\eta_t) \right]^{-\frac{3}{2}} g(\eta_t) M'(\eta_t).$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial M(\eta_t)}{\partial \eta_t} &= -2\theta^2\gamma^2(1-\eta_t) + 2\theta(A - \underline{A}) \left[x_t'(\eta_t)\eta_t + x_t(\eta_t) \right], \\
\frac{\partial g(\eta_t)}{\partial \eta_t} &= \theta^2\gamma^2 + \theta(A - \underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)].
\end{aligned}$$

Notice that

$$\frac{\partial M(\eta_t)}{\partial \eta_t} = 2g(\eta_t).$$

Then the second derivative is

$$\begin{aligned}
\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} &= \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{1}{2} \frac{g(\eta_t) M'(\eta_t)}{\left[M(\eta_t) \right]^{3/2}} \\
&= \frac{g'(\eta_t)}{\sqrt{M(\eta_t)}} - \frac{g(\eta_t)^2}{\left[M(\eta_t) \right]^{3/2}} \\
&= \frac{g'(\eta_t) M(\eta_t) - g(\eta_t)^2}{\left[M(\eta_t) \right]^{3/2}}.
\end{aligned}$$

Therefore, we have

$$\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = \frac{g'(\eta_t)M(\eta_t) - g(\eta_t)^2}{[M(\eta_t)]^{3/2}} = \frac{g'(\eta_t)}{M(\eta_t)^{1/2}} - \frac{g(\eta_t)^2}{[M(\eta_t)]^{3/2}}.$$

More explicitly, this is

$$\begin{aligned}\frac{\partial^2 q_t(\eta_t)}{\partial \eta_t^2} = & \frac{\theta^2\gamma^2 + \theta(A-\underline{A})[x_t''(\eta_t)\eta_t + 2x_t'(\eta_t)]}{\left[\theta^2\gamma^2(1-\eta_t)^2 \; + \; 2\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \; + \; 1\right]^{1/2}} \\ & - \frac{\left[-\theta^2\gamma^2(1-\eta_t) \; + \; \theta(A-\underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]\right]^2}{\left[\theta^2\gamma^2(1-\eta_t)^2 \; + \; 2\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \; + \; 1\right]^{3/2}}\end{aligned}$$

C.13 Proof of Proposition 5

From theorem **??**, we have that the optimal leverage by the household is

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(r_t+1) + 1}{\left(\sigma + \sigma_t^q\right)^2}. \tag{66}$$

We will solve for each term in this expression.

From corollary **??**, we have that the return on capital for households is

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t}dt + \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right)dt + \left(\sigma + \sigma_t^q\right)dZ_t. \tag{67}$$

Also, from proposition 15, we have that the on-the-verge condition for default is given by

$$\nu_t = 1 - \frac{1}{2\lambda\left(x_t - 1\right)\left(1 + r_t\right)}. \tag{68}$$

The debt market clearing condition from equation (92) is

$$N_t + \underline{N}_t = q_t K_t.$$

Divide by $q_t K_t$ and use the definition $\eta_t \triangleq \frac{N_t}{q_t K_t}$ to rewrite this as

$$\eta_t + \underline{\eta}_t = 1. \tag{69}$$

Take the capital market clearing condition in equation (91)

$$\int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t.$$

Divide by $q_t K_t$

$$\frac{x_t N_t}{q_t K_t} + \frac{\underline{x}_t \underline{N}_t}{q_t K_t} = 1.$$

Use $\eta_t \triangleq \frac{N_t}{q_t K_t}$ from equation (96)

$$x_t \eta_t + \underline{x}_t \underline{\eta}_t = 1. \tag{70}$$

Plug equation (69) into equation (70) to get

$$\underline{x}_t = \frac{1 - x_t \eta_t}{1 - \eta_t}. \tag{71}$$

We combine the results together. Starting off with the optimal leverage equation (66):

$$\begin{aligned}
& \underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(r_t + 1) + 1}{\left(\sigma + \sigma_t^q\right)^2} \\
\leftrightarrow_{(1)} \quad & \frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(r_t + 1) + 1}{\left(\sigma + \sigma_t^q\right)^2} \\
\leftrightarrow_{(2)} \quad & \frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] - \nu_t(r_t + 1) + 1}{\left(\sigma + \sigma_t^q\right)^2} \\
\leftrightarrow_{(3)} \quad & \frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] - \left[1 - \frac{1}{2\lambda(x_t - 1)(1 + r_t)}\right](r_t + 1) + 1}{\left(\sigma + \sigma_t^q\right)^2} \\
\leftrightarrow \quad & \frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] + \frac{1}{2\lambda(x_t - 1)(1 + r_t)}(1 + r_t) - (1 + r_t) + 1}{\left(\sigma + \sigma_t^q\right)^2} \\
\leftrightarrow \quad & \frac{1 - x_t \eta_t}{1 - \eta_t} = \frac{\left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right] + \frac{1}{2\lambda(x_t - 1)} - r_t}{\left(\sigma + \sigma_t^q\right)^2}.
\end{aligned}$$

$\leftrightarrow_{(1)}$ uses the capital market clearing condition in equation (71), $\leftrightarrow_{(2)}$ uses the return on capital equation (67), and $\leftrightarrow_{(3)}$ uses the repayment rate from on-the-

verge default condition in equation (68). Therefore, we have

$$\frac{1-x_t\eta_t}{1-\eta_t}=\frac{\frac{\underline{A}-\underline{\iota}_t}{q_t}+\phi\left(\underline{\iota}_t\right)-\underline{\delta}-r_t+\mu_t^q+\sigma\sigma_t^q+\frac{1}{2\lambda(x_t-1)}}{\left(\sigma+\sigma_t^q\right)^2}$$

Re-arranging the above equation yields the interest rate

$$r_t=\frac{\underline{A}-\underline{\iota}_t}{q_t}+\phi\left(\underline{\iota}_t\right)-\underline{\delta}+\mu_t^q+\sigma\sigma_t^q-\left(\sigma+\sigma_t^q\right)^2\frac{1-x_t\eta_t}{1-\eta_t}+\frac{1}{2\lambda(x_t-1)}.$$

C.14 Proof of Proposition 6

When $\lambda\to\infty$, no agent has incentive to default since she incurs an infinite penalty. As a result, the default premium in this case is zero. For some finite values of λ, default occurs. So, the corresponding interest rate is the sum of the default premium and the interest rate that arises from the model without default.

C.15 Proof of Proposition 7

We apply Ito's lemma to the function $q_t=q\left(\eta_t\right)$ and we obtain

$$dq_t=q'\left(\eta_t\right)d\eta_t+\frac{1}{2}q''\left(\eta_t\right)\left(d\eta_t\right)^2.$$

Recall that

$$\frac{d\eta_t}{\eta_t}=\alpha(\eta_t)dt+\beta(\eta_t)dZ_t.$$

Therefore, the quadratic variation of the wealth share is

$$\left(d\eta_t\right)^2=\eta_t^2\beta(\eta_t)^2dt.$$

We can then write the evolution of the price of capital as

$$dq_t=q'(\eta_t)\left[\alpha(\eta_t)\eta_t dt+\beta(\eta_t)\eta_t dZ_t\right]+\frac{1}{2}q''(\eta_t)\eta_t^2\beta(\eta_t)^2dt.$$

Divide by q_t to obtain

$$\frac{dq_t}{q_t}=\left(\frac{q'\left(\eta_t\right)}{q_t}\alpha\left(\eta_t\right)\eta_t+\frac{q''\left(\eta_t\right)}{q_t}\frac{\beta\left(\eta_t\right)^2}{2}\eta_t^2\right)dt+\frac{q'\left(\eta_t\right)}{q_t}\beta\left(\eta_t\right)\eta_t dZ_t. \quad (72)$$

Recall the drift and volatility of the expert's relative wealth from corollary **??** is

$$\alpha(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t dt + \frac{1}{2\lambda(x_t - 1)} - (\sigma + \sigma_t^q)^2\right) + \frac{A - \iota_t}{q_t} + (1 - \psi_t)(\underline{\delta} - \delta),$$
$$\beta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q).$$

From proposition 16, we have $x_t = \frac{\psi_t}{\eta_t}$, to get

$$\alpha(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t dt + \frac{1}{2\lambda(x_t - 1)} - (\sigma + \sigma_t^q)^2\right) + \frac{A - \iota_t}{q_t} + (1 - x_t\eta_t)(\underline{\delta} - \delta),$$
$$\beta(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q).$$

To derive the volatility of the price of capital, first recall from equation (72)

$$\frac{dq_t}{q_t} = \left(\frac{q'(\eta_t)}{q_t}\alpha(\eta_t)\eta_t + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2\right)dt + \frac{q'(\eta_t)}{q_t}\beta(\eta_t)\eta_t dZ_t.$$

We also know $\beta(\eta_t)$ from corollary **??** is equal to

$$\beta(\eta_t) = \frac{x_t\eta_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q).$$

Hence, from the price of capital postule in equation (77),

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t,$$

we get that from matching coefficients

$$\begin{aligned}\sigma_t^q &= \frac{q'(\eta_t)}{q_t}\beta(\eta_t)\eta_t \\ &= \frac{q'(\eta_t)}{q_t}\frac{x_t\eta_t - \eta_t}{\eta_t}(\sigma + \sigma_t^q)\eta_t \\ &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t - \eta_t)(\sigma + \sigma_t^q).\end{aligned}$$

Re-arrange to obtain

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\frac{q'(\eta_t)}{q_t}\sigma}{1 - (x_t\eta_t - \eta_t)\frac{q'(\eta_t)}{q_t}}.$$

To derive the drift of the price of capital, we again match coefficients where

$$\begin{aligned}\mu_t^q &= \frac{q'(\eta_t)}{q_t}\alpha(\eta_t)\eta_t + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2 \\ &= \frac{q'(\eta_t)}{q_t}\eta_t\left[\frac{x_t\eta_t-\eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t + \frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) + \frac{A-\iota_t}{q_t} + (1-x_t\eta_t)(\underline{\delta}-\right. \\ &+ \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2 \\ &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t + \frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) \\ &+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)(\underline{\delta}-\delta) + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.\end{aligned}$$

So in the region where $\frac{\mathbb{E}_t[dr_t^k]}{dt} = r_t$, we have

$$\begin{aligned}\mu_t^q &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) \\ &+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)(\underline{\delta}-\delta) + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.\end{aligned}$$

For completeness, we look at the case where $\frac{\mathbb{E}_t[dr_t^k]}{dt} \neq r_t$. Recall the drift of return on capital for firms from equation (79)

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = \frac{A-\iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q.$$

Plug this in

$$\begin{aligned}\mu_t^q &= \frac{q'(\eta_t)}{q_t}(x_t\eta_t-\eta_t)\left(\left[\frac{A-\iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q\right] - r_t + \frac{1}{2\lambda(x_t-1)} - (\sigma+\sigma_t^q)^2\right) \\ &+ \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t(1-x_t\eta_t)(\underline{\delta}-\delta) + \frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2.\end{aligned}$$

Rewrite this as

$$\mu_t^q = \chi(\xi + \mu_t^q) + R,$$

where

$$\chi \triangleq \frac{q'(\eta_t)}{q_t}\left(x_t\eta_t - \eta_t\right),$$
$$\xi \triangleq \left[\frac{A-\iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q\right] - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma + \sigma_t^q\right)^2,$$
$$R \triangleq \frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t\left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right) + \frac{q''\left(\eta_t\right)}{q_t}\frac{\beta\left(\eta_t\right)^2}{2}\eta_t^2.$$

Solve for μ_t^q to get

$$\mu_t^q = \frac{\xi\chi + R}{1-\chi}.$$

Therefore, we have

$$\mu_t^q = \frac{\left(\frac{q'(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)\right)\left(\frac{A-\iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma+\sigma_t^q\right)^2\right)}{1 - \frac{q'(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)}$$
$$+ \frac{\frac{q'(\eta_t)}{q_t}\eta_t\frac{A-\iota_t}{q_t} + \frac{q'(\eta_t)}{q_t}\eta_t\left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right)}{1-\frac{q'(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)} + \frac{\frac{q''(\eta_t)}{q_t}\frac{\beta(\eta_t)^2}{2}\eta_t^2}{1-\frac{q'(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)}.$$

This simplifies to

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t\left(1-\frac{q'(\eta_t)}{q_t}\left(x_t\eta_t-\eta_t\right)\right)}\right]\left\{\left(x_t-1\right)\left(\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma+\sigma_t^q\right)^2\right)\right.$$
$$\left. + \frac{A-\iota_t}{q_t}x_t + \left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right)\right\} + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t^2}{2\left[q_t - q'(\eta_t)\left(x_t\eta_t-\eta_t\right)\right]}.$$

Further simplify

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\left(x_t\eta_t-\eta_t\right)}\right]\left\{\left(x_t-1\right)\left(\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma+\sigma_t^q\right)^2\right)\right.$$
$$\left. + \frac{A-\iota_t}{q_t}x_t + \left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right)\right\} + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t^2}{2\left[q_t - q'(\eta_t)\left(x_t\eta_t-\eta_t\right)\right]} \times \frac{q'(\eta_t)}{q'(\eta_t)}.$$

We arrive at

$$\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\eta_t\left(x_t-1\right)}\right]\left\{\left(x_t-1\right)\left(\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t-1)} - \left(\sigma+\sigma_t^q\right)^2\right)\right.$$
$$\left. + \frac{A-\iota_t}{q_t}x_t + \left(1-x_t\eta_t\right)\left(\underline{\delta}-\delta\right) + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t}{2q'(\eta_t)}\right\}. \tag{73}$$

We can solve further to make the expression only in terms of exogenous variables.

From proposition 5, we want to plug in the interest rate

$$r_t = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}.$$

So the term multiplying $x_t - 1$ becomes

$$\begin{aligned}
&\phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - r_t + \frac{1}{2\lambda(x_t - 1)} - \left(\sigma + \sigma_t^q\right)^2 \\
&= \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}\right] \\
&+ \frac{1}{2\lambda(x_t - 1)} - \left(\sigma + \sigma_t^q\right)^2 \\
&= \phi\left(\iota_t\right) - \delta + \sigma\sigma_t^q - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t}\right] - \left(\sigma + \sigma_t^q\right)^2 \\
&= \left(\underline{\delta} - \delta\right) - \frac{\underline{A} - \underline{\iota}_t}{q_t} - \mu_t^q + \frac{\eta_t\left(1 - x_t\right)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2.
\end{aligned}$$

So we have

$$\begin{aligned}
\mu_t^q = \left[\frac{q'(\eta_t)\eta_t}{q_t - q'(\eta_t)\eta_t\left(x_t - 1\right)}\right] \Bigg\{ &\left(x_t - 1\right)\left(\left(\underline{\delta} - \delta\right) - \frac{\underline{A} - \underline{\iota}_t}{q_t} - \mu_t^q + \frac{\eta_t\left(1 - x_t\right)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2\right) \\
&+ \frac{A - \iota_t}{q_t}x_t + \left(1 - x_t\eta_t\right)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)\beta(\eta_t)^2\eta_t}{2q'(\eta_t)}\Bigg\}.
\end{aligned}$$

We now solve for μ_t^q. Let

$$F \triangleq \frac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\left(x_t - 1\right)}.$$

Denote by

$$\begin{aligned}
S \triangleq &\left(x_t - 1\right)\left[\left(\underline{\delta} - \delta\right) - \frac{\underline{A} - \underline{\iota}_t}{q_t} + \frac{\eta_t\left(1 - x_t\right)}{1 - \eta_t}\left(\sigma + \sigma_t^q\right)^2\right] \\
&+ \frac{A - \iota_t}{q_t}x_t + \left(1 - x_t\,\eta_t\right)\left(\underline{\delta} - \delta\right) + \frac{q''(\eta_t)\,\beta(\eta_t)^2\,\eta_t}{2\,q'(\eta_t)}.
\end{aligned}$$

Observe that μ_t^q appears with a minus sign inside the bracket multiplied by $(x_t - 1)$. Distributing F, we have

$$\mu_t^q = FS - F(x_t - 1)\mu_t^q.$$

Solve for μ_t^q

$$\mu_t^q = \frac{FS}{1 + F(x_t - 1)}.$$

Therefore the explicit form we need in implementing code

$$\mu_t^q = \frac{\dfrac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\,(x_t - 1)} \times S}{1 \;+\; \dfrac{q'(\eta_t)\,\eta_t}{q_t - q'(\eta_t)\,\eta_t\,(x_t - 1)}\,(x_t - 1)}.$$

C.16 Proof of Proposition 8

First, we prove the first part. First, recall that the expected return on capital for the firm from corollary **??** is

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} = \frac{A - \iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q.$$

The interest rate is given by proposition 5

$$r_t = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}.$$

From equation (**??**) in theorem 1, we have that the excess return on capital is

$$\begin{aligned} & \frac{\mathbb{E}\left[dr_t^k\right]}{dt} - r_t = 0 \\ \leftrightarrow \quad & \frac{A - \iota_t}{q_t} + \phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q \\ & - \left[\frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} + \frac{1}{2\lambda(x_t - 1)}\right] = 0. \end{aligned}$$

Since $\iota_t = \underline{\iota}_t$, we have

$$\frac{A - \underline{A}}{q_t(x_t, \eta_t)} + (\underline{\delta} - \delta) + \left(\sigma + \sigma_t^q(x_t, \eta_t)\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} - \frac{1}{2\lambda(x_t - 1)} = 0.$$

Therefore we have

$$\frac{A - \underline{A}}{q_t(x_t, \eta_t)} + (\underline{\delta} - \delta) + \left(\sigma + \sigma_t^q(x_t, \eta_t)\right)^2 \frac{1 - x_t\eta_t}{1 - \eta_t} - \frac{1}{2\lambda(x_t - 1)} = 0.$$

The fact that the parameters of the function $g\left(x_t\right)$ are functions of the exoge-

nous variables and the state variable η_t follows from proposition 4, corollary 1 and proposition 7.

C.17 Proof of Proposition 13

The proof follows immediately from observing that once default is removed, then the objective function of the expert becomes linear in x_t and should, therefore, be either U_t or 1. It can be easily shown that irrespective of the value of x_t, the expert's excess return over the interest rate r_t, in the no default case is always positive i.e.,

$$\frac{\mathbb{E}\left[dr_t^k\right]}{dt} - r_t > 0$$

and hence, the only equilibrium can be the one that satisfies $x_t = U_t$ for all t.

C.18 Proof of Proposition (10)

Proof. For (A), take goods market clearing condition from equation (**??**)

$$\int_0^1 k_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = \int_1^2 \underline{c}_t^j dj.$$

Recall the definition of capital share

$$\psi_t \triangleq \frac{\int_0^1 k_t^i di}{K_t}.$$

Also define aggregate consumption of households as

$$\underline{C}_t \triangleq \int_1^2 \underline{c}_t^j dj.$$

Finally, recall that optimal investment for the firm and household are equal. Divide by K_t in the goods market clearing condition to get

$$\psi_t(A - \iota_t) + (1 - \psi_t)\left(\underline{A} - \iota_t\right) = \frac{\underline{C}_t}{K_t}.$$

We can simplify this to get

$$\psi_t(A - \underline{A}) + \underline{A} - \iota_t = \frac{\underline{C}_t}{K_t}.$$

Plug in optimal investment from equation (101)

$$\iota_t = \frac{1}{2\theta}\left(q_t^2 - 1\right),$$

to get

$$\psi_t(A - \underline{A}) + \underline{A} - \frac{1}{2\theta}\left(q_t^2 - 1\right) = \frac{\underline{C}_t}{K_t}.$$

Re-arrange

$$q_t^2 - 2\theta\left[\psi_t(A - \underline{A}) + \underline{A}\right] - 1 = -2\theta\left(\frac{\underline{C}_t}{K_t}\right).$$

Optimal aggregate consumption is a fraction of household aggregate net worth

$$\underline{C}_t = \gamma \underline{N}_t.$$

Hence

$$\begin{aligned}\frac{\underline{C}_t}{K_t} &= \frac{\gamma \underline{N}_t}{K_t} \\ &= \frac{\gamma(1-\eta_t)q_t K_t}{K_t} \\ &= \gamma(1-\eta_t)q_t,\end{aligned}$$

where we used the fact that the expert's net worth share is defined as

$$\eta_t \triangleq \frac{N_t}{q_t K_t}.$$

Therefore, we conclude

$$q_t^2 + 2\theta\gamma(1-\eta_t)q_t - 2\theta\left[\psi_t(A - \underline{A}) + \underline{A}\right] - 1 = 0.$$

For (B), the volatility of the price of capital, σ_t^q, is from proposition (7)

$$\sigma_t^q = \frac{q'(\eta_t)}{q_t}\left(x_t\eta_t - \eta_t\right)\left(\sigma + \sigma_t^q\right).$$

Re-arrange

$$\sigma_t^q q_t = (\sigma + \sigma_t^q)\, q'(\eta_t)\,(x_t\eta_t - \eta_t)\,.$$

Finally, recall that optimal leverage is

$$x_t = \frac{\psi_t}{\eta_t}.$$

So plug this in to get

$$(\sigma + \sigma_t^q)\, q'(\eta_t)\,(\psi_t - \eta_t) - \sigma_t^q q_t = 0.$$

For (C), the optimal household leverage from equation (40) is

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \nu_t(1 + r_t) + 1}{(\sigma + \sigma_t^q)^2}.$$

From proposition 16, the optimal leverage is

$$x_t = \frac{\psi_t}{\eta_t} \qquad \text{and} \qquad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t}.$$

The optimal default from equation (98) is

$$\nu_t = 1 - \frac{1}{2\lambda\,(x_t - 1)\,(1 + r_t)}.$$

Plug both of these results in

$$\frac{1 - \psi_t}{1 - \eta_t} = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \left[1 - \frac{1}{2\lambda(x_t - 1)(1 + r_t)}\right](1 + r_t) + 1}{(\sigma + \sigma_t^q)^2}.$$

This simplifies to

$$\frac{1 - \psi_t}{1 - \eta_t}(\sigma + \sigma_t^q)^2 = \frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t + \frac{1}{2\lambda(x_t - 1)}.$$

Multiply by $2\lambda(1 - \eta_t)$

$$2\lambda\,(1 - \psi_t)\,(\sigma + \sigma_t^q)^2 = 2\lambda(1 - \eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t\right] + \frac{1 - \eta_t}{(x_t - 1)}.$$

Plug in $x_t = \frac{\psi_t}{\eta_t}$

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t\right] + \frac{1-\eta_t}{(\frac{\psi_t}{\eta_t}-1)}.$$

Rewrite

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t\right] + \left[\frac{1-\eta_t}{(\frac{\psi_t-\eta_t}{\eta_t})}\right].$$

Rewrite this into

$$2\lambda\left(1-\psi_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t\right] + \eta_t\left[\frac{1-\eta_t}{(\psi_t-\eta_t)}\right].$$

Multiply through

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t\right] + \eta_t\left[1-\eta_t\right].$$

By theorem (1), the expected return on capital for the firm is equal to the interest rate

$$\mathbb{E}[dr_t^k] = r_t.$$

Therefore we can plug this in

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - \frac{\mathbb{E}\left[dr_t^k\right]}{dt}\right] + \eta_t\left[1-\eta_t\right].$$

Plugging in the expected returns and cancelling terms out gives us

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 = 2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\frac{\underline{A}-A}{q_t} - (\underline{\delta}-\delta)\right] + \eta_t\left[1-\eta_t\right].$$

Multiply by q_t to get

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 q_t = 2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[\underline{A}-A-q_t(\underline{\delta}-\delta)\right] + \eta_t\left[1-\eta_t\right]q_t.$$

Re-arrange

$$2\lambda\left(1-\psi_t\right)\left(\psi_t-\eta_t\right)\left(\sigma+\sigma_t^q\right)^2 q_t + 2\lambda\left(\psi_t-\eta_t\right)(1-\eta_t)\left[A-\underline{A}-q_t(\delta-\underline{\delta})\right] - \eta_t\left[1-\eta_t\right]q_t = 0.$$

□

C.19 Numerical Algorithm

For a fixed η_t, we want to solve the equation (41) which is

$$g(x_t) = \frac{A - \underline{A}}{q_t(x_t, \eta_t)} + (\underline{\delta} - \delta) + [\sigma + \sigma_t^q(x_t, \eta_t)]^2 \left(\frac{1 - x_t\eta_t}{1 - \eta_t}\right) - \frac{1}{2\lambda(x_t - 1)} = 0.$$

Take the volatility of the price of capital σ_t^q from equation (7) as

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}\sigma}{1 - (x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}}.$$

The equilibrium price of capital from equation (4) is

$$q_t(\eta_t) = -\theta\gamma(1 - \eta_t) + \sqrt{\theta^2\gamma^2(1 - \eta_t)^2 + 2\theta\left[x_t(\eta_t)\eta_t A + (1 - x_t(\eta_t)\eta_t)\underline{A}\right] + 1}.$$

Recall the definition in equation (54) defined as

$$M(\eta_t) \triangleq \theta^2\,\gamma^2\,(1 - \eta_t)^2 \;+\; 2\,\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] \;+\; 1.$$

This becomes

$$q_t(\eta_t) = -\theta\gamma(1 - \eta_t) + \sqrt{M(\eta_t)}.$$

So then

$$g(x_t) = \frac{A - \underline{A}}{-\theta\gamma(1 - \eta_t) + \sqrt{M(\eta_t)}} + (\underline{\delta} - \delta) + [\sigma + \sigma_t^q(x_t, \eta_t)]^2 \left(\frac{1 - x_t\eta_t}{1 - \eta_t}\right) - \frac{1}{2\lambda(x_t - 1)} = 0.$$

Plug in equation (53)

$$q'(\eta_t) \triangleq \frac{\partial q_t(\eta_t)}{\partial \eta_t} \;=\; \theta\,\gamma \;+\; \frac{-\,\theta^2\,\gamma^2\,(1 - \eta_t) \;+\; \theta\,(A - \underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}.$$

Then we can evaluate

$$\frac{q'(\eta_t)}{q_t(\eta_t)} \;=\; \frac{\theta\,\gamma \;+\; \dfrac{-\,\theta^2\,\gamma^2\,(1 - \eta_t) \;+\; \theta\,(A - \underline{A})\left[x_t'(\eta_t)\,\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}}{-\,\theta\,\gamma\,(1 - \eta_t) \;+\; \sqrt{M(\eta_t)}}.$$

C.19.1 Boundary conditions

From equation (82), the leverage constraint is

$$x_t \leq min\left\{\frac{1}{m}, \frac{q_t K_t}{n_t}\right\}.$$

Since

$$\eta_t \triangleq \frac{N_t}{q_t K_t},$$

the leverage constraint becomes

$$x_t(\eta_t) \leq min\left\{\frac{1}{m}, \frac{1}{\eta_t}\right\}.$$

The state variable $\eta_t \in [0,1]$ and m is an exogenous margin constraint. So when $\eta_t \leq m$, we have

$$x_t(\eta_t) = \frac{1}{m},$$

and when $\eta_t > m$,

$$x_t(\eta_t) = \frac{1}{\eta_t}.$$

This gives us the boundary conditions that

$$x_t(0) = \frac{1}{m},$$
$$x_t(1) = 1.$$

C.19.2 Differential equation system

We have a first order nonlinear ODE

$$g(x_t;\eta_t) = \frac{A-\underline{A}}{-\theta\gamma(1-\eta_t)+\sqrt{M(\eta_t)}} + (\underline{\delta}-\delta) + [\sigma+\sigma_t^q(x_t,\eta_t)]^2\left(\frac{1-x_t\eta_t}{1-\eta_t}\right) - \frac{1}{2\lambda(x_t-1)} = 0$$

$$M(\eta_t) = \theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[x_t(\eta_t)\eta_t A + \left(1-x_t(\eta_t)\eta_t\right)\underline{A}\right] + 1.$$

$$\sigma_t^q = \frac{(x_t\eta_t-\eta_t)\frac{q'(\eta_t)}{q_t(\eta_t)}\sigma}{1-(x_t\eta_t-\eta_t)\frac{q'(\eta_t)}{q_t(\eta_t)}}$$

$$\frac{q'(\eta_t)}{q_t(\eta_t)} = \frac{\theta\gamma + \frac{-\theta^2\gamma^2(1-\eta_t) + \theta(A-\underline{A})\left[x_t'(\eta_t)\eta_t + x_t(\eta_t)\right]}{\sqrt{M(\eta_t)}}}{-\theta\gamma(1-\eta_t) + \sqrt{M(\eta_t)}}.$$

The boundary condition

$$x_t(0) = g(0)$$

We propose the algorithm. Let

$$x_t'(\eta_t) \approx \frac{x_t(\eta_t) - x_t(\eta_t - h)}{h}.$$

Then note that

$$\begin{aligned} x_t'(\eta_t)\eta_t + x_t(\eta_t) &= \left[\frac{x_t(\eta_t) - x_t(\eta_t - h)}{h}\right]\eta_t + x_t(\eta_t) \\ &= \frac{\eta_t x_t(\eta_t) - \eta_t x_t(\eta_t - h) + h x_t(\eta_t)}{h} \\ &= h^{-1}\left[x_t(\eta_t)\left[\eta_t + h\right] - \eta_t x_t(\eta_t - h)\right]. \end{aligned}$$

Therefore, we get

$$\frac{q'(\eta_t)}{q_t(\eta_t)} = \frac{\theta\gamma + \frac{-\theta^2\gamma^2(1-\eta_t) + \theta(A-\underline{A})h^{-1}\left[x_t(\eta_t)\left[\eta_t + h\right] - \eta_t x_t(\eta_t - h)\right]}{\sqrt{M(\eta_t)}}}{-\theta\gamma(1-\eta_t) + \sqrt{M(\eta_t)}}$$

So we have the nonlinear system

$$g(x_t;\eta_t) = \frac{A-\underline{A}}{q_t(\eta_t)} + (\underline{\delta} - \delta) + \left[\sigma + \sigma_t^q(x_t,\eta_t)\right]^2\left(\frac{1 - x_t\eta_t}{1-\eta_t}\right) - \frac{1}{2\lambda(x_t - 1)} = 0$$

$$M(\eta_t) = \theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[x_t(\eta_t)\,\eta_t\,A + \left(1 - x_t(\eta_t)\,\eta_t\right)\underline{A}\right] + 1.$$

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\frac{q'(\eta_t)}{q_t(\eta_t)}\sigma}{1 - (x_t\eta_t - \eta_t)\frac{q'(\eta_t)}{q_t(\eta_t)}}$$

$$\frac{q'(\eta_t)}{q_t(\eta_t)} = \frac{\theta\gamma + \frac{-\theta^2\gamma^2(1-\eta_t) + \theta(A-\underline{A})h^{-1}\left[x_t(\eta_t)\left[\eta_t + h\right] - \eta_t x_t(\eta_t - h)\right]}{\sqrt{M(\eta_t)}}}{-\theta\gamma(1-\eta_t) + \sqrt{M(\eta_t)}},$$

with the initial condition

$$x_t(0) = g(0).$$

Note that when $\eta_t = 0$, we have

$$g(0) = \frac{A-\underline{A}}{q_t(0)} + \sigma^2 - \frac{1}{2\lambda(x_t - 1)}.$$

C.20 Proof of Proposition (12)

Investment rate is the same for experts and households

$$\iota_t = \iota_t^i = \underline{\iota}_t^i.$$

Aggregate consumption of households is

$$\underline{C}_t \triangleq \int_1^2 \underline{c}_t^j dj.$$

Therefore, rewrite the goods market clearing condition

$$\begin{aligned}
& \int_0^1 k_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = \int_1^2 \underline{c}_t^j dj \\
\leftrightarrow \quad & \int_0^1 k_t^i \left(A - \iota_t\right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \iota_t\right) dj = \underline{C}_t \\
\leftrightarrow \quad & A \int_0^1 k_t^i di + \underline{A} \int_1^2 \underline{k}_t^j dj - \iota_t \left[\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj\right] = \underline{C}_t \\
\leftrightarrow \quad & A \int_0^1 k_t^i di + \underline{A} \int_1^2 \underline{k}_t^j dj - \iota_t K_t = \underline{C}_t \\
\leftrightarrow \quad & A k_t + \underline{A} \underline{k}_t - \iota_t K_t = \underline{C}_t.
\end{aligned}$$

We can then plug this into the household problem from equation (105)

$$\begin{aligned}
\mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(\underline{C}_t\right) dt\right] &= \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(A k_t + \underline{A} \underline{k}_t - \iota_t K_t\right) dt\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(K_t \frac{A k_t + \underline{A} \underline{k}_t - \iota_t K_t}{K_t}\right) dt\right] \\
&= \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(K_t\right) dt\right] + \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(\frac{A k_t + \underline{A} \underline{k}_t}{K_t} - \iota_t\right) \right. \\
&= \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(K_t\right) dt\right] + \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(A \psi_t + \underline{A}(1 - \psi_t) - \right.\right.
\end{aligned}$$

where we used $\psi_t \triangleq \frac{\int_0^1 k_t^i di}{K_t}$. Therefore, we have the welfare decomposition

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(\underline{C}_t\right) dt\right] = \underbrace{\mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(K_t\right) dt\right]}_{\text{Capital effect}} + \underbrace{\mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} log\left(\left(A-\underline{A}\right)\psi_t + \underline{A} - \iota_t\right) dt\right]}_{\text{Reallocation + investment effect}}$$

D Additional Derivations

This section outlines the details of the derivation of the model.

D.1 Derivation of Equation (??)

Proof. We have

$$\begin{aligned} dn_t &= Ak_t dt - \iota_t k_t dt + d\left(q_t k_t\right) - \nu_t d_t dt - \nu_t \left(q_t k_t - n_t\right) dR_t + d_t dt \\ &= (A - \iota_t) k_t dt + d\left(q_t k_t\right) - \nu_t d_t dt - \nu_t d_t dR_t + d_t dt \\ &= (A - \iota_t) k_t dt + d\left(q_t k_t\right) - \nu_t d_t dt - \nu_t d_t r_t dt + d_t dt \\ &= (A - \iota_t) k_t dt + d\left(q_t k_t\right) - \nu_t d_t r_t dt + d_t dt - \nu_t d_t dt \\ &= (A - \iota_t) k_t dt + d\left(q_t k_t\right) - \nu_t d_t r_t dt + (1 - \nu_t) d_t dt. \end{aligned}$$

□

D.2 Derivation of Equation (78)

Proof.

$$\begin{aligned} d\left(q_t k_t\right) &= q_t dk_t + k_t dq_t = \left(\phi\left(\iota_t\right) - \delta\right) q_t k_t dt + k_t dq_t + \sigma q_t k_t dZ_t \\ &= q_t k_t \left[\left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t\right]. \end{aligned}$$

□

D.3 Derivation of Equation (79)

Proof. Total return on capital is dividends plus capital gains

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \frac{d(q_t k_t)}{q_t k_t}.$$

Plug in our expression for capital gains. □

D.4 Derivation of Equation (100)

Proof. Take the net worth evolution

$$dn_t = (A - \iota_t)k_t dt + d\left(q_t k_t\right) - \nu_t d_t r_t dt + (1 - \nu_t)d_t dt.$$

Re-arrange terms

$$dn_t = (A - \iota_t)k_t dt + d\left(q_t k_t\right) - \nu_t d_t (1 + r_t)dt + d_t dt.$$

Take the first two terms and rewrite it in terms of leverage ratio

$$\begin{aligned}(A - \iota_t)k_t dt + d\left(q_t k_t\right) &= (A - \iota_t)\frac{q_t}{q_t}k_t dt + d\left(q_t k_t\right)\frac{q_t k_t}{q_t k_t} \\ &= \Big(\frac{A - \iota_t}{q_t}\Big)x_t n_t dt + d\frac{(q_t k_t)}{q_t k_t}x_t n_t.\end{aligned}$$

So we have

$$\begin{aligned} & dn_t = \Big(\frac{A - \iota_t}{q_t}\Big)x_t n_t dt + d\frac{(q_t k_t)}{q_t k_t}x_t n_t - \nu_t d_t (1 + r_t)dt + d_t dt \\ \leftrightarrow \quad & \frac{dn_t}{n_t} = \Big(\frac{A - \iota_t}{q_t}\Big)x_t dt + d\frac{(q_t k_t)}{q_t k_t}x_t - \nu_t \frac{d_t}{n_t}(1 + r_t)dt + \frac{d_t}{n_t}dt \\ \leftrightarrow \quad & \frac{dn_t}{n_t} = x_t\left[\frac{A - \iota_t}{q_t}dt + d\frac{(q_t k_t)}{q_t k_t}\right] - \nu_t \frac{d_t}{n_t}(1 + r_t)dt + \frac{d_t}{n_t}dt. \end{aligned}$$

Recall that return on capital is

$$dr_t^k = \frac{A - \iota_t}{q_t}dt + \frac{d(q_t k_t)}{q_t k_t}.$$

Plug this in

$$\frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \frac{d_t}{n_t}(1 + r_t)dt + \frac{d_t}{n_t}dt$$

Plug in balance sheet $q_t k_t = d_t + n_t$

$$\begin{aligned} \leftrightarrow \quad & \frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \frac{(q_t k_t - n_t)}{n_t}(1 + r_t)dt + \frac{(q_t k_t - n_t)}{n_t}dt \\ \Rightarrow \quad & \frac{dn_t}{n_t} = x_t dr_t^k - \nu_t \big(x_t - 1\big)(1 + r_t)dt + \big(x_t - 1\big)dt. \end{aligned}$$

□

D.5 Derivation of Equation (82)

Proof. Take

$$mq_t k_t \leq min\{n_t,\ mq_t K_t\},$$

and divide by n_t and m

$$\frac{q_t k_t}{n_t} \leq min\left\{\frac{1}{m},\ \frac{q_t K_t}{n_t}\right\}.$$

Finally, plug in the definition of leverage ratio for the firm $x_t \triangleq \frac{q_t k_t}{n_t}$. □

D.6 Derivation of Equation (46)

Proof.

$$\begin{aligned}
d\underline{n}_t &= \underline{A}\underline{k}_t dt - \underline{\iota}_t \underline{k}_t dt + d(q_t \underline{k}_t) + \nu_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + \nu_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + \nu_t \underline{d}_t r_t dt + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) - \underline{c}_t dt + \nu_t \underline{d}_t r_t dt - (1 - \nu_t)\underline{d}_t dt.
\end{aligned}$$

□

D.7 Derivation of Equation (87)

Proof.

$$\begin{aligned}
d(q_t \underline{k}_t) &= q_t d\underline{k}_t + \underline{k}_t dq_t \\
&= \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta}\right) q_t \underline{k}_t dt + \underline{k}_t dq_t + \sigma q_t \underline{k}_t dZ_t \\
&= q_t \underline{k}_t \left[\left(\phi\left(\underline{\iota}_t\right) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t\right].
\end{aligned}$$

□

D.8 Derivation of Equation (88)

Proof. Total return on capital is dividends plus capital gains

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t} dt + \frac{d(q_t \underline{k}_t)}{q_t \underline{k}_t}.$$

Plug in our expression for capital gains. □

D.9 Derivation of Equation (47)

Proof. Take the net worth evolution

$$d\underline{n}_t = \underline{A}\underline{k}_t dt - \underline{\iota}_t \underline{k}_t dt + d(q_t \underline{k}_t) + \nu_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt.$$

Take the first two terms and rewrite it in terms of leverage ratio and the return on capital

$$\begin{aligned}
(\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d\left(q_t \underline{k}_t\right) &= (\underline{A} - \underline{\iota}_t)\frac{q_t}{q_t}\underline{k}_t dt + d\left(q_t \underline{k}_t\right)\frac{q_t \underline{k}_t}{q_t \underline{k}_t} \\
&= \left(\frac{\underline{A} - \underline{\iota}_t}{q_t}\right)\underline{x}_t \underline{n}_t dt + d\frac{(q_t \underline{k}_t)}{q_t \underline{k}_t}\underline{x}_t \underline{n}_t \\
&= \left[\left(\frac{\underline{A} - \underline{\iota}_t}{q_t}\right)dt + d\frac{(q_t \underline{k}_t)}{q_t \underline{k}_t}\right]\underline{x}_t \underline{n}_t \\
&= d\underline{r}_t^k \underline{x}_t \underline{n}_t.
\end{aligned}$$

Plug this back into the net worth evolution

$$\begin{aligned}
d\underline{n}_t &= \underline{A}\underline{k}_t dt - \underline{\iota}_t \underline{k}_t dt + d(q_t \underline{k}_t) + \nu_t \left(\underline{n}_t - q_t \underline{k}_t\right) dR_t + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= d\underline{r}_t^k \underline{x}_t \underline{n}_t + \nu_t \underline{d}_t r_t dt + \nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= d\underline{r}_t^k \underline{x}_t \underline{n}_t + (1 + r_t)\nu_t \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\
&= d\underline{r}_t^k \underline{x}_t \underline{n}_t + (1 + r_t)\nu_t (\underline{n}_t - q_t \underline{k}_t) dt - (\underline{n}_t - q_t \underline{k}_t) dt - \underline{c}_t dt.
\end{aligned}$$

Divide by $\underline{n}_t$

$$\begin{aligned}\frac{d\underline{n}_t}{\underline{n}_t} &= \underline{x}_t d\underline{r}_t^k + (1+r_t)\nu_t\left(\frac{\underline{n}_t - q_t\underline{k}_t}{\underline{n}_t}\right)dt - \left(\frac{\underline{n}_t - q_t\underline{k}_t}{\underline{n}_t}\right)dt - \underline{c}_t dt \\ &= \underline{x}_t d\underline{r}_t^k + (1+r_t)\nu_t\big(1-\underline{x}_t\big)dt - \big(1-\underline{x}_t\big)dt - \frac{\underline{c}_t}{\underline{n}_t}dt.\end{aligned}$$

□

D.10 Derivation of Equation (91)

Proof.

$$\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t.$$

Multiply both sides by q_t and recall that the definition of leverage is $x_t \triangleq \frac{q_t k_t}{n_t}$. Therefore

$$\begin{aligned} & \int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj = K_t \\ \leftrightarrow \quad & \int_0^1 q_t k_t^i di + \int_1^2 q_t \underline{k}_t^j dj = q_t K_t \\ \Rightarrow \quad & \int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t. \end{aligned}$$

□

D.11 Derivation of Equation (92)

Proof. Recall that the balance sheets for both the firm and the household are

$$\begin{aligned} q_t k_t &= d_t + n_t \\ \underline{n}_t &= \underline{d}_t + q_t \underline{k}_t \end{aligned}$$

Then from (50)

$$\begin{aligned}
& \int_0^1 d_t^i di = \int_1^2 \underline{d}_t^j dj \\
\leftrightarrow \quad & \int_0^1 (q_t k_t^i - n_t^i) di = \int_1^2 (\underline{n}_t^j - q_t \underline{k}_t^j) dj \\
\leftrightarrow \quad & q_t \left(\int_0^1 k_t^i di + \int_1^2 \underline{k}_t^j dj \right) = \int_0^1 n_t^i di + \int_1^2 \underline{n}_t^j dj \\
\Rightarrow \quad & q_t K_t = N_t + \underline{N}_t.
\end{aligned}$$

□

D.12 Derivation of Equation (93)

Proof. Take (**??**) and use the definition of leverage

$$\begin{aligned}
& \int_0^1 k_t^i \left(A - \iota_t^i \right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j \right) dj = \int_1^2 \underline{c}_t^j dj \\
\leftrightarrow \quad & \int_0^1 k_t^i \left(A - \iota_t^i \right) di + \int_1^2 \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j \right) dj = \int_1^2 \underline{n}_t^j \underline{\rho}_t^j dj \\
\leftrightarrow \quad & \int_0^1 q_t k_t^i \left(A - \iota_t^i \right) di + \int_1^2 q_t \underline{k}_t^j \left(\underline{A} - \underline{\iota}_t^j \right) dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j dj \\
\Rightarrow \quad & \int_0^1 x_t^i n_t^i \left(A - \iota_t^i \right) di + \int_1^2 \underline{x}_t^j \underline{n}_t^j \left(\underline{A} - \underline{\iota}_t^j \right) dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j dj.
\end{aligned}$$

□

D.13 Derivation of Equation (96)

Proof. From (92), we have

$$N_t + \underline{N}_t = q_t K_t.$$

Plug in (94)

$$\eta_t \triangleq \frac{N_t}{N_t + \underline{N}_t}.$$

We get

$$\begin{aligned} & N_t + \underline{N}_t = q_t K_t \\ \leftrightarrow \quad & \frac{N_t}{\eta_t} = q_t K_t \\ \Rightarrow \quad & \eta_t = \frac{N_t}{q_t K_t}. \end{aligned}$$

□

D.14 Derivation of Equation (82)

Proof. The leverage constraint (82) is

$$x_t \leq min\left\{ \frac{1}{m}, \frac{q_t K_t}{n_t} \right\}.$$

Plug in the expression for wealth share (96)

$$\eta_t = \frac{N_t}{q_t K_t}.$$

□

E Model without Default

We derive the model without default.

E.1 Experts (No Default)

Firms are managed by experts indexed by $i \in [0,1]$. For simplicity, we omit the index i in what follows. A representative firm holds capital k_t and produces output at rate Ak_t. Experts finance operations either internally, by reinvesting earnings, or externally, by issuing *default-free* debt to households.

A key financial friction in the model is that firms cannot issue equity. Let d_t denote the face value of debt and n_t the expert's net worth. The balance sheet identity is

$$q_t k_t = d_t + n_t, \tag{74}$$

where q_t is the market price of capital.

Debt service (no default). Debt is serviced in full. Over the interval $(t, t+dt)$ the firm owes (and repays) interest

$$d_t \, dR_t = (q_t k_t - n_t) \, dR_t, \qquad dR_t = r_t dt. \tag{75}$$

There is no repayment choice variable and no default penalty.

Net worth evolution. Expert net worth evolves as production net of investment plus capital gains, minus interest payments on debt:

$$dn_t = (A-\iota_t)k_t \, dt + d(q_t k_t) - r_t d_t \, dt = (A-\iota_t)k_t \, dt + d(q_t k_t) - r_t \, (q_t k_t - n_t) \, dt. \tag{76}$$

We postulate that the equilibrium price of capital follows

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t. \tag{77}$$

Applying Itô's product rule to $d(q_t k_t)$ and using the capital accumulation process yields the firm's capital gains rate

$$\frac{d(q_t k_t)}{q_t k_t} = (\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t. \tag{78}$$

Hence the total return on capital for firms is

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + (\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t. \tag{79}$$

Levered net worth dynamics. Define leverage $x_t \equiv \frac{q_t k_t}{n_t}$, so that $d_t = q_t k_t - n_t = (x_t - 1) n_t$. Dividing (76) by n_t implies

$$\frac{dn_t}{n_t} = x_t\, dr_t^k - (x_t - 1)\, r_t\, dt. \tag{80}$$

Leverage constraint. Experts face a margin (leverage) constraint:

$$m q_t k_t \leq \min\{n_t,\, m q_t K_t\}, \tag{81}$$

which is equivalent to

$$x_t \leq \min\left\{\frac{1}{m},\, \frac{q_t K_t}{n_t}\right\}. \tag{82}$$

Expert's problem (no default). Experts reinvest all wealth and do not consume. The expert chooses investment and leverage to maximize expected net worth growth:

$$\max_{\iota_t,\, x_t} \mathbb{E}\left[\frac{dn_t}{n_t}\right], \tag{83}$$

subject to (80) and (82).

E.2 Households (No Default)

Households, indexed by $j \in (1, 2]$, represent a continuum of unit mass. They act as credit providers in the economy. While households possess production capabilities, their technology is less efficient than that of experts ($\underline{A} < A$). Consequently, they

allocate wealth between physical capital and lending to experts via *default-free* debt.

Technology. Households produce output at rate $\underline{A}\underline{k}_t dt$ and reinvest $\underline{\iota}_t \underline{k}_t dt$ into physical capital. Capital evolves according to

$$\frac{d\underline{k}_t}{\underline{k}_t} = \left(\phi\left(\underline{\iota}_t\right) - \underline{\delta}\right) dt + \sigma dZ_t. \tag{84}$$

Balance sheet. Let $\underline{d}_t$ denote the face value of bonds held by the household. Net worth satisfies

$$\underline{n}_t = q_t \underline{k}_t + \underline{d}_t. \tag{85}$$

Net worth evolution (no default). Debt is serviced in full, so lending earns the risk-free rate r_t with no loss on principal. The household budget constraint becomes

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) - \underline{c}_t dt + r_t\, \underline{d}_t dt. \tag{86}$$

Return on household capital. Using the price process $\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t$ and Itô's product rule,

$$\frac{d(q_t \underline{k}_t)}{q_t \underline{k}_t} = \left(\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + (\sigma + \sigma_t^q) dZ_t. \tag{87}$$

Hence the total return on household capital is

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t} dt + \left(\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q\right) dt + (\sigma + \sigma_t^q) dZ_t. \tag{88}$$

Portfolio-weight form. Define the household portfolio weight on capital by

$$\underline{x}_t \equiv \frac{q_t \underline{k}_t}{\underline{n}_t}, \qquad 1 - \underline{x}_t = \frac{\underline{d}_t}{\underline{n}_t}.$$

Dividing (86) by $\underline{n}_t$ yields the wealth evolution in returns form:

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t\, d\underline{r}_t^k + (1 - \underline{x}_t)\, r_t\, dt - \frac{\underline{c}_t}{\underline{n}_t} dt. \tag{89}$$

In equilibrium, households are net lenders, so $\underline{x}_t \leq 1$.

Household problem (no default). Households choose consumption $\underline{c}_t$, investment $\underline{\iota}_t$, and portfolio weight $\underline{x}_t$ to maximize discounted expected log utility:

$$\max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} \mathbb{E}\left[\int_0^\infty e^{-\gamma t} \log\left(\underline{c}_t\right)\, dt\right], \tag{90}$$

subject to (89), taking $\{q_t, r_t\}$ as given.

F Additional Derivations (No Default)

This section outlines the details of the derivations for the *no-default* version of the model. Throughout, experts repay in full, so

$$\nu_t \equiv 1 \quad \text{for all } t,$$

and there are no default losses/transfers between experts and households.

F.1 Derivation of Equation (??)

Proof. Start from the expert net worth accounting over $(t, t+dt)$. With full repayment, the expert services interest on outstanding debt and rolls over principal. Using the same rollover convention as in the baseline derivation,

$$\begin{aligned} dn_t &= Ak_t dt - \iota_t k_t dt + d(q_t k_t) \;-\; d_t\, dt \;-\; d_t\, dR_t \;+\; d_t\, dt \\ &= (A - \iota_t)k_t dt + d(q_t k_t) - d_t\, dR_t. \end{aligned}$$

The principal repayment $-d_t dt$ cancels with new issuance $+d_t dt$ under full repayment. Finally, since $dR_t = r_t dt$,

$$dn_t = (A - \iota_t)k_t dt + d(q_t k_t) - d_t r_t dt,$$

which is Equation (**??**) in the no-default model. □

F.2 Derivation of Equation (78)

Proof. Apply Itô's product rule to the market value of expert capital:

$$d(q_t k_t) = q_t\, dk_t + k_t\, dq_t + dq_t\, dk_t.$$

Using

$$\frac{dk_t}{k_t} = (\phi(\iota_t) - \delta)\, dt + \sigma dZ_t, \qquad \frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t,$$

we have

$$\begin{aligned} q_t\, dk_t &= q_t k_t \left[(\phi(\iota_t) - \delta)\, dt + \sigma dZ_t\right], \\ k_t\, dq_t &= q_t k_t \left[\mu_t^q dt + \sigma_t^q dZ_t\right], \\ dq_t\, dk_t &= (q_t \sigma_t^q dZ_t)(k_t \sigma dZ_t) = q_t k_t\, \sigma \sigma_t^q\, dt. \end{aligned}$$

Summing terms yields

$$d(q_t k_t) = q_t k_t \left[(\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t\right],$$

or equivalently

$$\frac{d(q_t k_t)}{q_t k_t} = (\phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q)\, dZ_t,$$

which is Equation (78). □

F.3 Derivation of Equation (79)

Proof. Total return on capital equals dividend yield plus capital gains:

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \frac{d(q_t k_t)}{q_t k_t}.$$

Plugging Equation (78) into the capital-gains term delivers Equation (79). □

F.4 Derivation of Equation (100)

Proof. Under no default, Equation (**??**) is

$$dn_t = (A - \iota_t)k_t dt + d(q_t k_t) - d_t r_t dt.$$

By definition of the return on capital,

$$q_t k_t \, dr_t^k = (A - \iota_t)k_t dt + d(q_t k_t),$$

so the net worth evolution can be written compactly as

$$dn_t = q_t k_t \, dr_t^k - d_t r_t dt.$$

Divide by n_t and use leverage $x_t \equiv \frac{q_t k_t}{n_t}$:

$$\frac{dn_t}{n_t} = x_t \, dr_t^k - \frac{d_t}{n_t} r_t dt.$$

Using the balance sheet identity $q_t k_t = n_t + d_t$ implies $\frac{d_t}{n_t} = x_t - 1$. Therefore,

$$\frac{dn_t}{n_t} = x_t \, dr_t^k - (x_t - 1) r_t dt,$$

which is Equation (100) in the no-default model. □

F.5 Derivation of Equation (82)

Proof. Start from

$$mq_t k_t \leq \min\{n_t, \, mq_t K_t\}.$$

Divide both sides by mn_t:

$$\frac{q_t k_t}{n_t} \leq \min\left\{\frac{1}{m}, \, \frac{q_t K_t}{n_t}\right\}.$$

Using $x_t \triangleq \frac{q_t k_t}{n_t}$ gives Equation (82). □

F.6 Derivation of Equation (46)

Proof. With no expert default ($\nu_t \equiv 1$), households receive interest in full and principal is rolled over (principal inflow cancels new lending). Starting from the same rollover accounting as in the baseline appendix,

$$\begin{aligned} d\underline{n}_t &= \underline{A}\underline{k}_t dt - \underline{\iota}_t \underline{k}_t dt + d(q_t \underline{k}_t) + (\underline{n}_t - q_t \underline{k}_t)\, dR_t + \underline{d}_t dt - \underline{d}_t dt - \underline{c}_t dt \\ &= (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + (\underline{n}_t - q_t \underline{k}_t)\, dR_t - \underline{c}_t dt. \end{aligned}$$

Using the household balance sheet $\underline{n}_t = q_t \underline{k}_t + \underline{d}_t$ implies $\underline{n}_t - q_t \underline{k}_t = \underline{d}_t$, so

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + \underline{d}_t\, dR_t - \underline{c}_t dt.$$

Finally, with $dR_t = r_t dt$,

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + \underline{d}_t r_t dt - \underline{c}_t dt,$$

which is Equation (46) in the no-default model. □

F.7 Derivation of Equation (87)

Proof. Apply Itô's product rule:

$$d(q_t \underline{k}_t) = q_t\, d\underline{k}_t + \underline{k}_t\, dq_t + dq_t\, d\underline{k}_t.$$

Using

$$\frac{d\underline{k}_t}{\underline{k}_t} = (\phi(\underline{\iota}_t) - \underline{\delta})\, dt + \sigma dZ_t, \qquad \frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t,$$

we get

$$\frac{d(q_t \underline{k}_t)}{q_t \underline{k}_t} = (\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q)\, dt + (\sigma + \sigma_t^q) dZ_t,$$

which is Equation (87). □

F.8 Derivation of Equation (88)

Proof. Total return on household capital equals dividend yield plus capital gains:

$$d\underline{r}_t^k = \frac{\underline{A} - \underline{\iota}_t}{q_t} dt + \frac{d(q_t \underline{k}_t)}{q_t \underline{k}_t}.$$

Plug Equation (87) into the capital-gains term to obtain Equation (88). □

F.9 Derivation of Equation (47)

Proof. Under no default, the household net worth evolution (Equation 46) is

$$d\underline{n}_t = (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t) + \underline{d}_t r_t dt - \underline{c}_t dt.$$

Using the definition of the household return on capital,

$$q_t \underline{k}_t \, d\underline{r}_t^k = (\underline{A} - \underline{\iota}_t)\underline{k}_t dt + d(q_t \underline{k}_t),$$

we can rewrite

$$d\underline{n}_t = q_t \underline{k}_t \, d\underline{r}_t^k + \underline{d}_t r_t dt - \underline{c}_t dt.$$

Divide by $\underline{n}_t$ and define the household portfolio weight on capital

$$\underline{x}_t \equiv \frac{q_t \underline{k}_t}{\underline{n}_t}, \qquad 1 - \underline{x}_t = \frac{\underline{d}_t}{\underline{n}_t}.$$

Then

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t \, d\underline{r}_t^k + (1 - \underline{x}_t) \, r_t dt - \frac{\underline{c}_t}{\underline{n}_t} dt,$$

which is Equation (47) in the no-default model. □

F.10 Derivation of Equation (91)

Proof. Start from capital market clearing:

$$\int_0^1 k_t^i \, di + \int_1^2 \underline{k}_t^j \, dj = K_t.$$

Multiply both sides by q_t and use the definitions $x_t^i \equiv \frac{q_t k_t^i}{n_t^i}$ and $\underline{x}_t^j \equiv \frac{q_t \underline{k}_t^j}{\underline{n}_t^j}$:

$$q_t \int_0^1 k_t^i \, di + q_t \int_1^2 \underline{k}_t^j \, dj = q_t K_t,$$
$$\int_0^1 q_t k_t^i \, di + \int_1^2 q_t \underline{k}_t^j \, dj = q_t K_t,$$
$$\int_0^1 x_t^i n_t^i \, di + \int_1^2 \underline{x}_t^j \underline{n}_t^j \, dj = q_t K_t,$$

which is Equation (91). □

F.11 Derivation of Equation (92)

Proof. Using the balance sheets

$$q_t k_t = d_t + n_t, \qquad \underline{n}_t = q_t \underline{k}_t + \underline{d}_t,$$

and debt market clearing $\int_0^1 d_t^i di = \int_1^2 \underline{d}_t^j dj$, we have

$$\int_0^1 (q_t k_t^i - n_t^i) \, di = \int_1^2 (\underline{n}_t^j - q_t \underline{k}_t^j) \, dj,$$
$$q_t \left(\int_0^1 k_t^i \, di + \int_1^2 \underline{k}_t^j \, dj \right) = \int_0^1 n_t^i \, di + \int_1^2 \underline{n}_t^j \, dj.$$

Using $\int_0^1 k_t^i \, di + \int_1^2 \underline{k}_t^j \, dj = K_t$ and defining aggregates $N_t \equiv \int_0^1 n_t^i di$, $\underline{N}_t \equiv \int_1^2 \underline{n}_t^j dj$, we obtain

$$q_t K_t = N_t + \underline{N}_t,$$

which is Equation (92). □

F.12 Derivation of Equation (93)

Proof. Consumption market clearing is

$$\int_0^1 k_t^i (A - \iota_t^i) \, di + \int_1^2 \underline{k}_t^j (\underline{A} - \underline{\iota}_t^j) \, dj = \int_1^2 \underline{c}_t^j \, dj.$$

Using $\underline{c}_t^j = \underline{n}_t^j \underline{\rho}_t^j$ and multiplying by q_t:

$$q_t \int_0^1 k_t^i (A - \iota_t^i)\, di + q_t \int_1^2 \underline{k}_t^j (\underline{A} - \underline{\iota}_t^j)\, dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j\, dj.$$

Now use $q_t k_t^i = x_t^i n_t^i$ and $q_t \underline{k}_t^j = \underline{x}_t^j \underline{n}_t^j$ to get

$$\int_0^1 x_t^i n_t^i (A - \iota_t^i)\, di + \int_1^2 \underline{x}_t^j \underline{n}_t^j (\underline{A} - \underline{\iota}_t^j)\, dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j\, dj,$$

which is Equation (93). □

F.13 Derivation of Equation (96)

Proof. From Equation (92),

$$N_t + \underline{N}_t = q_t K_t.$$

Using the definition of the expert wealth share

$$\eta_t \triangleq \frac{N_t}{N_t + \underline{N}_t},$$

we obtain

$$\eta_t = \frac{N_t}{q_t K_t},$$

which is Equation (96). □

F.14 Derivation of Equation (82) in terms of wealth share

Proof. The leverage constraint is

$$x_t \leq \min\left\{\frac{1}{m}, \frac{q_t K_t}{n_t}\right\}.$$

In the representative-expert case with unit mass of experts, $n_t = N_t$, so using $q_t K_t = \frac{N_t}{\eta_t}$ from Equation (96) gives

$$\frac{q_t K_t}{n_t} = \frac{N_t/\eta_t}{N_t} = \frac{1}{\eta_t}.$$

Hence the constraint can be written as

$$x_t \leq \min\left\{\frac{1}{m}, \frac{1}{\eta_t}\right\}.$$

□

F.15 Equilibrium Analysis

In the *no-default* benchmark, the model features two primary frictions. First, experts cannot issue equity and therefore finance capital purchases using internal net worth and *risk-free* debt issued to households. Second, financial markets are incomplete because experts face a leverage (margin) constraint that binds when they are undercapitalized. Since experts always repay in full, household bond claims are riskless and there are *no credit losses and no credit spreads*; all amplification operates through balance sheets, collateral values, and the leverage constraint. As in collateral equilibrium models (e.g., Geanakoplos and Polemarchakis (1985)), the resulting equilibrium can be constrained inefficient due to pecuniary externalities operating through prices and borrowing limits.

Definition 1. Let aggregate net worth for experts and households be defined, respectively, as:

$$N_t \triangleq \int_0^1 n_t^i di \qquad \text{and} \qquad \underline{N}_t \triangleq \int_1^2 \underline{n}_t^j dj.$$

Define the consumption-to-net-worth ratio for households as $\underline{\rho}_t^j \triangleq \underline{c}_t^j / \underline{n}_t^j$.

Using the definition of leverage $x_t^i = q_t k_t^i / n_t^i$, we can rewrite the market clearing conditions in terms of wealth and leverage. The capital market clearing condition (48) becomes:

$$\int_0^1 x_t^i n_t^i di + \int_1^2 \underline{x}_t^j \underline{n}_t^j dj = q_t K_t. \tag{91}$$

The debt market clearing condition (50) implies that the total value of capital in the economy must equal the sum of aggregate net worth:

$$N_t + \underline{N}_t = q_t K_t. \tag{92}$$

Finally, the goods market clearing condition (**??**) can be expressed as:

$$\int_0^1 x_t^i n_t^i \left(A - \iota_t^i\right) di + \int_1^2 \underline{x}_t^j \underline{n}_t^j \left(\underline{A} - \underline{\iota}_t^j\right) dj = q_t \int_1^2 \underline{n}_t^j \underline{\rho}_t^j dj. \tag{93}$$

To characterize the equilibrium dynamics, we define the state variable η_t as the experts' share of aggregate wealth. Since all other endogenous variables can be expressed as functions of η_t, its evolution summarizes the dynamics of the entire economy.

Definition 2. The experts' wealth share is defined as:

$$\eta_t \triangleq \frac{N_t}{N_t + \underline{N}_t}. \tag{94}$$

We postulate that η_t follows the Itô process:

$$\frac{d\eta_t}{\eta_t} = \alpha(\eta_t)dt + \beta(\eta_t)dZ_t. \tag{95}$$

Using (92) and (94), we can express the experts' wealth share as the ratio of their net worth to the total value of capital:

$$\eta_t = \frac{N_t}{q_t K_t}. \tag{96}$$

Substituting this into the expert's leverage constraint (82), the equilibrium constraint becomes a function of the state variable:

$$x_t \leq \min\left\{\frac{1}{m}, \frac{1}{\eta_t}\right\}. \tag{97}$$

F.16 Equilibrium Lending Constraint

Before solving the optimization problems, we establish a key property of the equilibrium financial structure.

Proposition 14 (No Expert-to-Household Lending)**.** *In equilibrium, experts strictly act as borrowers. Consequently, the expert's leverage choice x_t satisfies:*

$$0 \leq \underline{x}_t \leq 1, \qquad 1 \leq x_t \leq U_t,$$

where the maximum feasible leverage is:

$$U_t \;=\; \min\left\{\tfrac{1}{m},\; \tfrac{q_t K_t}{n_t}\right\}.$$

Proof. All proofs are deferred to Appendix C.4. □

Proposition 14 implies that the flow of funds is always from households to experts. This results from the productivity differential ($A > \underline{A}$) and the agents' risk preferences. Experts, who maximize the growth rate of wealth, operate the superior technology. Households, who are risk-averse and consume, operate the inferior technology.

If an expert were to lend to a household, the marginal unit of wealth would earn the bond rate r_t, which must be less than the expert's internal expected return on capital $\mathbb{E}_t[dr_t^k]$ for the expert to be willing to lend. Because $\mathbb{E}_t[dr_t^k] > \mathbb{E}_t[d\underline{r}_t^k]$ due to the productivity gap, there is always a gain from intermediation where the household lends to the expert. The only scenario where lending does not occur is if both agents are self-financed ($x_t = \underline{x}_t = 1$), which is not a competitive equilibrium given the gains from trade. Thus, the equilibrium is characterized by $x_t \geq 1$, with the upper bound U_t determined by the tighter of the regulatory margin requirement ($1/m$) or the market-imposed wealth constraint ($1/\eta_t$).

F.16.1 Expert's Problem (No Default)

In the no-default benchmark, experts always repay in full:

$$\nu_t \equiv 1. \tag{98}$$

Accordingly, default penalties and repayment choices drop out of the expert problem.

The expert chooses investment ι_t and leverage x_t to maximize the expected growth rate of net worth,

$$\max_{\iota_t,\, x_t} \left\{ \mathbb{E}\left[\frac{dn_t}{n_t}\right] \right\}, \tag{99}$$

subject to the net worth evolution (no default),

$$\frac{dn_t}{n_t} = x_t\, dr_t^k - (x_t - 1)\, r_t\, dt, \tag{100}$$

and the leverage constraint (97). Investment is subject to convex adjustment costs $\phi(\cdot)$.

Proposition 15 (Optimal Investment and Leverage (No Default))**.** *In equilibrium, the expert's optimal investment rate satisfies the Tobin's q condition:*

$$\iota_t = \frac{1}{2\theta}\big(q_t^2 - 1\big). \tag{101}$$

Moreover, let $dr_t^k = \mu_t^k dt + \sigma_t^k dZ_t$*. Given* (q_t, r_t) *and* $x_t \in [1, U_t]$*, the optimal leverage satisfies the Kuhn–Tucker conditions:*

$$\mu_t^k - r_t - \kappa_t + \chi_t = 0, \tag{102}$$

$$\kappa_t\,(U_t - x_t) = 0, \qquad \kappa_t \geq 0, \tag{103}$$

$$\chi_t\,(x_t - 1) = 0, \qquad \chi_t \geq 0. \tag{104}$$

In particular:

$$\begin{cases} 1 < x_t < U_t \;\Rightarrow\; \mu_t^k = r_t, \\ x_t = U_t \;\Rightarrow\; \mu_t^k \geq r_t, \\ x_t = 1 \;\Rightarrow\; \mu_t^k \leq r_t. \end{cases}$$

Equation (101) is the standard Tobin's q relation: experts invest until the marginal cost of installation equals the shadow price q_t. In contrast to the defaultable-debt economy, there is no endogenous repayment choice and no default threshold; household lending is risk-free and earns r_t.

Theorem 1 (Equilibrium Leverage and Returns (No Default))**.** *In equilibrium, experts choose leverage* $x_t \in [1, U_t]$ *satisfying the Kuhn–Tucker system (102)–(104). Hence, whenever the leverage constraint binds* $(x_t = U_t)$*, experts earn a (weakly) positive expected excess return on capital relative to the bond rate:*

$$\mu_t^k - r_t \;\geq\; 0, \quad \textit{with equality iff } 1 < x_t < U_t.$$

Furthermore, experts earn a weakly higher expected return on capital than households:

$$\mathbb{E}\big[dr_t^k\big] \geq \mathbb{E}\big[d\underline{r}_t^k\big].$$

Theorem 1 emphasizes the key mechanism in the no-default benchmark: any wedge between the expected return on capital and the risk-free lending rate is governed by the *shadow value* of relaxing the leverage constraint. When experts are undercapitalized, the constraint binds, experts cannot scale up high-productivity investment, and expected returns on expert capital can exceed r_t even though experts maximize the expected growth rate of wealth.

F.17 Proof of Theorem 1

With $\nu_t \equiv 1$, the expert's net worth evolution is

$$\frac{dn_t}{n_t} = x_t\, dr_t^k - (x_t - 1) r_t dt.$$

Write $dr_t^k = \mu_t^k dt + \sigma_t^k dZ_t$. Since the expert maximizes $\mathbb{E}[dn_t/n_t]$, the relevant (instantaneous) objective is

$$\max_{x_t \in [1, U_t]} \left\{ x_t \mu_t^k - (x_t - 1) r_t \right\}.$$

Form the Lagrangian with multipliers $\kappa_t \geq 0$ for the upper bound $x_t \leq U_t$ and $\chi_t \geq 0$ for the lower bound $x_t \geq 1$:

$$\mathcal{L} = x_t \mu_t^k - (x_t - 1) r_t + \kappa_t (U_t - x_t) + \chi_t (x_t - 1).$$

The first-order and complementarity conditions are:

$$\mu_t^k - r_t - \kappa_t + \chi_t = 0, \qquad \kappa_t (U_t - x_t) = 0, \qquad \chi_t (x_t - 1) = 0,$$

with $\kappa_t, \chi_t \geq 0$. This is exactly (102)–(104), implying: (i) if $1 < x_t < U_t$, then $\kappa_t = \chi_t = 0$ and $\mu_t^k = r_t$; (ii) if $x_t = U_t$, then $\chi_t = 0$ and $\mu_t^k - r_t = \kappa_t \geq 0$; (iii) if $x_t = 1$, then $\kappa_t = 0$ and $r_t - \mu_t^k = \chi_t \geq 0$. The return ranking $\mathbb{E}[dr_t^k] \geq \mathbb{E}[d\underline{r}_t^k]$

follows from the productivity gap $A > \underline{A}$ (and the fact that both capital stocks are priced by the same q_t), implying experts' marginal product and hence their expected return on capital is weakly higher in equilibrium. □

F.18 Proof of Proposition 15

The expert's objective is

$$\max_{\iota_t, x_t} \mathbb{E}\left[\frac{dn_t}{n_t}\right],$$

and ι_t enters only through the drift of dr_t^k. Using

$$dr_t^k = \frac{A - \iota_t}{q_t} dt + \left(\phi\left(\iota_t\right) - \delta + \mu_t^q + \sigma\sigma_t^q\right) dt + \left(\sigma + \sigma_t^q\right) dZ_t,$$

we have

$$\mu_t^k = \frac{A - \iota_t}{q_t} + \phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q.$$

Holding x_t fixed, the investment choice maximizes μ_t^k and satisfies

$$\frac{\partial \mu_t^k}{\partial \iota_t} = -\frac{1}{q_t} + \phi'(\iota_t) = 0 \quad \Longleftrightarrow \quad \phi'(\iota_t) = \frac{1}{q_t}.$$

With

$$\phi\left(x\right) \triangleq \frac{1}{\theta}\left(\sqrt{1 + 2\theta x} - 1\right) \quad \Rightarrow \quad \phi'(x) = \frac{1}{\sqrt{1 + 2\theta x}},$$

we obtain

$$\frac{1}{\sqrt{1 + 2\theta\iota_t}} = \frac{1}{q_t} \quad \Rightarrow \quad \iota_t = \frac{1}{2\theta}\left(q_t^2 - 1\right),$$

which is (101). Under no default, there is no repayment-rate choice and $\nu_t \equiv 1$ as in (98). □

F.19 Proof of Proposition 16

For an expert we have

$$x_t n_t = \frac{q_t k_t}{n_t} n_t = q_t k_t.$$

Aggregating over all experts and dividing through by $q_t K_t$ yields

$$\int_0^1 x_t^i n_t^i \, di = \int_0^1 q_t k_t^i \, di = q_t K_t \, \psi_t,$$

so

$$x_t N_t = q_t K_t \psi_t \quad \Longleftrightarrow \quad x_t \eta_t = \psi_t \quad \Rightarrow \quad x_t = \frac{\psi_t}{\eta_t}.$$

Finally, from (91),

$$x_t N_t + \underline{x}_t \underline{N}_t = q_t K_t,$$

divide by $q_t K_t$ and use $\eta_t = N_t/(q_t K_t)$ and $1 - \eta_t = \underline{N}_t/(q_t K_t)$ to get

$$x_t \eta_t + \underline{x}_t (1 - \eta_t) = 1.$$

Substituting $x_t = \psi_t / \eta_t$ implies

$$\psi_t + \underline{x}_t (1 - \eta_t) = 1 \quad \Rightarrow \quad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t}.$$

This concludes the proof. □

F.19.1 Household's Problem (No Default from Experts)

In the no-default benchmark, experts always repay in full:

$$\nu_t \equiv 1.$$

Hence household lending is risk-free and earns the short rate r_t. The household chooses consumption $\underline{c}_t$, investment $\underline{\iota}_t$, and portfolio weight on physical capital $\underline{x}_t \equiv \frac{q_t \underline{k}_t}{\underline{n}_t}$ to maximize

$$\max_{\underline{\iota}_t, \underline{x}_t, \underline{c}_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\gamma t} \log\left(\underline{c}_t\right) \, dt\right], \tag{105}$$

subject to the (default-free) net worth evolution:

$$\frac{d\underline{n}_t}{\underline{n}_t} = \underline{x}_t \, d\underline{r}_t^k + (1 - \underline{x}_t) \, r_t \, dt - \frac{\underline{c}_t}{\underline{n}_t} \, dt, \qquad 0 \leq \underline{x}_t \leq 1, \tag{106}$$

where $d\underline{r}_t^k$ is the household return on capital defined in (88):

$$d\underline{r}_t^k = \mu_t^{\underline{k}}\, dt + (\sigma + \sigma_t^q)\, dZ_t, \qquad \mu_t^{\underline{k}} \triangleq \frac{\underline{A} - \underline{\iota}_t}{q_t} + (\phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q)\,.$$

The solution is characterized by the Hamilton–Jacobi–Bellman (HJB) equation.

Theorem 2 (Household policies under no default)**.** *In the no-default benchmark, the household's optimal consumption, investment, and portfolio choice satisfy:*

$$\underline{c}_t = \gamma\, \underline{n}_t, \tag{107}$$

$$\underline{\iota}_t = \frac{1}{2\theta}\left(q_t^2 - 1\right), \tag{108}$$

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t}{(\sigma + \sigma_t^q)^2}, \quad \textit{(interior solution)}, \tag{109}$$

with the equilibrium allocation respecting the lending condition $0 \le \underline{x}_t \le 1$ *(so the implemented policy is the projection of (109) onto* $[0,1]$ *if needed).*

Equation (107) is the standard log-utility result: households consume a constant fraction γ of net worth. Equation (108) is the Tobin's q rule implied by the common adjustment-cost technology. Equation (109) is a Merton-style portfolio rule: $\underline{x}_t$ increases with the expected excess return on household capital, $\mu_t^{\underline{k}} - r_t$, and decreases with the variance $(\sigma + \sigma_t^q)^2$ of that return. Relative to the defaultable-debt case, the "effective lending return" is simply the risk-free rate r_t because $\nu_t \equiv 1$.

F.20 Proof of Theorem 2 (sketch)

The household value function is

$$V(t,\, \underline{n}_t,\, \eta_t) = \sup_{\underline{x}_t \in [0,1],\, \underline{\iota}_t \ge 0,\, \underline{c}_t \ge 0} \mathbb{E}_t\left[\int_t^{+\infty} e^{-\gamma s} \log(\underline{c}_s)\, ds\right].$$

Under no default, the budget constraint is (106). Writing $\underline{\rho}_t \equiv \underline{c}_t/\underline{n}_t$ and using $d\underline{r}_t^k = \mu_t^k dt + (\sigma + \sigma_t^q) dZ_t$, the drift and volatility of household net worth are

$$\mu_t^{\underline{n}} = \underline{x}_t\, \mu_t^k + (1 - \underline{x}_t)\, r_t - \underline{\rho}_t,$$
$$\sigma_t^{\underline{n}} = \underline{x}_t\, (\sigma + \sigma_t^q).$$

Conjecture the standard homothetic form

$$V(t, \underline{n}_t, \eta_t) = e^{-\gamma t} \left(f(\eta_t) + \frac{\log(\underline{n}_t)}{\gamma} \right),$$

and apply Itô to obtain the HJB. The controls $\underline{\rho}_t$, $\underline{\iota}_t$, and $\underline{x}_t$ enter the HJB only through: (i) $\log(\underline{\rho}_t)$ and $-\underline{\rho}_t/\gamma$, (ii) the drift μ_t^k (via $\underline{\iota}_t$), and (iii) the mean–variance term in $\underline{x}_t$,

$$\underline{x}_t(\mu_t^k - r_t) - \frac{1}{2}\underline{x}_t^2(\sigma + \sigma_t^q)^2.$$

Consumption. The FOC w.r.t. $\underline{\rho}_t$ yields

$$\frac{1}{\underline{\rho}_t} - \frac{1}{\gamma} = 0 \quad \Rightarrow \quad \underline{\rho}_t = \gamma \quad \Rightarrow \quad \underline{c}_t = \gamma \underline{n}_t,$$

proving (107).

Investment. The FOC w.r.t. $\underline{\iota}_t$ depends only on the dependence of μ_t^k on $\underline{\iota}_t$ and gives

$$\phi'(\underline{\iota}_t) = \frac{1}{q_t} \quad \Rightarrow \quad \underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1),$$

proving (108).

Portfolio share. Let the expected excess return be

$$\underline{b}_t \triangleq \mu_t^k - r_t = \frac{\mathbb{E}[d\underline{r}_t^k]}{dt} - r_t.$$

The FOC w.r.t. $\underline{x}_t$ for an interior solution is

$$\underline{b}_t - \underline{x}_t(\sigma + \sigma_t^q)^2 = 0 \quad \Rightarrow \quad \underline{x}_t = \frac{\underline{b}_t}{(\sigma + \sigma_t^q)^2} = \frac{\frac{\mathbb{E}[d\underline{r}_t^k]}{dt} - r_t}{(\sigma + \sigma_t^q)^2},$$

which is (109). Imposing $0 \le \underline{x}_t \le 1$ yields the constrained optimum as the projection onto $[0, 1]$. □

F.20.1 Equilibrium Characterization (No Default)

In this subsection we impose *no default* by experts:

$$\nu_t \equiv 1,$$

so household lending is risk-free and earns the short rate r_t. Relative to the baseline model, the only friction is the leverage (margin) constraint faced by experts; there is no default penalty and no default premium component in bond spreads.

Let ψ_t denote the fraction of aggregate capital held by experts:

$$\psi_t = \frac{\int_0^1 k_t^i di}{K_t}, \qquad 1 - \psi_t = \frac{\int_1^2 \underline{k}_t^j dj}{K_t}.$$

Proposition 16 (Leverage identities)**.** *In equilibrium, leverage ratios of experts and households are determined by their shares of aggregate wealth and capital:*

$$x_t = \frac{\psi_t}{\eta_t} \qquad \textit{and} \qquad \underline{x}_t = \frac{1 - \psi_t}{1 - \eta_t}. \tag{110}$$

The evolution of the economy is driven by the expert wealth share η_t.

Proposition 17 (Law of motion for expert wealth share (no default))**.** *The expert wealth share satisfies*

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - r_t\,dt - \left(\sigma + \sigma_t^q\right)^2 dt\right) + \left[\frac{A - \iota_t}{q_t} + (1 - \psi_t)\left(\underline{\delta} - \delta\right)\right] dt. \tag{111}$$

Corollary 2 (Drift and volatility of η_t (no default))**.** *The drift μ_t^η and volatility σ_t^η of η_t are*

$$\mu_t^\eta \triangleq \alpha(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t - \left(\sigma + \sigma_t^q\right)^2\right) + \frac{A - \iota_t}{q_t} + (1 - \psi_t)\left(\underline{\delta} - \delta\right), \tag{112}$$

$$\sigma_t^\eta \triangleq \beta(\eta_t) = \frac{\psi_t - \eta_t}{\eta_t}\left(\sigma + \sigma_t^q\right). \tag{113}$$

The drift term $\alpha(\eta_t)$ captures expected growth of expert wealth relative to the aggregate. Compared to the defaultable case, the only change is that the "default benefit" term disappears; relative wealth growth is driven purely by the leverage gap, the risk-adjusted excess return on capital over the risk-free rate, and net production.

We can solve for the equilibrium price of capital q_t as a function of the state η_t.

Proposition 18 (Equilibrium price of capital)**.** *The equilibrium price of capital solves:*

$$q_t(\eta_t) = -\theta\gamma(1-\eta_t) + \sqrt{\theta^2\gamma^2(1-\eta_t)^2 + 2\theta\left[x_t(\eta_t)\eta_t A + \left(1 - x_t(\eta_t)\eta_t\right)\underline{A}\right] + 1}. \tag{114}$$

This pricing equation is unchanged in form: households discount capital because they are risk averse and are the marginal investors when $1-\eta_t$ is large, while the productivity-weighted term reflects who operates the capital (experts versus households).

Finally, we derive the equilibrium interest rate and price dynamics.

Proposition 19 (Equilibrium interest rate (no default))**.** *The equilibrium interest rate is*

$$r_t = \frac{A - \underline{\iota}_t}{q_t} + \phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q - \left(\sigma + \sigma_t^q\right)^2 \frac{1 - x_t\eta_t}{1-\eta_t}. \tag{115}$$

Relative to the defaultable economy, equation (115) removes the default premium term. The remaining wedge is entirely an *intermediation/risk-bearing* component: when households are forced to hold more risky capital (higher $(1-x_t\eta_t)/(1-\eta_t)$), the precautionary term lowers the risk-free rate.

To close the model we still need the volatility of the capital price.

Proposition 20 (Volatility of the price of capital)**.** *The volatility of the price of capital is*

$$\sigma_t^q = \frac{(x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}\,\sigma}{1 - (x_t\eta_t - \eta_t)\,\frac{q'(\eta_t)}{q_t(\eta_t)}}. \tag{116}$$

The key difference relative to the defaultable model is the characterization of expert leverage.

Proposition 21 (Corner leverage choice under no default)**.** *In the no-default economy ($\nu_t \equiv 1$), the expert's leverage choice is a corner solution:*

$$x_t = U_t \;\equiv\; \min\left\{\tfrac{1}{m},\, \tfrac{1}{\eta_t}\right\}. \tag{117}$$

Equivalently, experts lever up to the tightest of the regulatory margin requirement or the market-imposed wealth constraint.

Interpretation. Without default, experts are locally risk-neutral in wealth and face a linear objective in x_t. Whenever the expected excess return on capital over the risk-free rate is positive, they strictly prefer higher leverage and hence choose the maximal feasible leverage.

F.21 Proof of Proposition 16

Unchanged. (It uses only market clearing and the definitions of ψ_t and η_t.)

F.22 Proof of Proposition 17

Under no default, the expert net worth evolution is

$$\frac{dn_t}{n_t} = x_t dr_t^k - (x_t - 1)\, r_t dt, \qquad x_t \equiv \frac{q_t k_t}{n_t}.$$

Proceeding exactly as in the baseline derivation, aggregation yields

$$\frac{dN_t}{N_t} = \frac{\psi_t}{\eta_t}\, dr_t^k - \frac{\psi_t - \eta_t}{\eta_t}\, r_t\, dt.$$

The evolution of $q_t K_t$ is unchanged and implies

$$\frac{d(q_t K_t)}{q_t K_t} = dr_t^k - \frac{A - \iota_t}{q_t} dt - (1 - \psi_t)(\underline{\delta} - \delta)\, dt.$$

Applying Itô's quotient rule to $\eta_t = N_t/(q_t K_t)$ delivers

$$\frac{d\eta_t}{\eta_t} = \frac{\psi_t - \eta_t}{\eta_t}\left(dr_t^k - r_t\, dt - (\sigma + \sigma_t^q)^2\, dt\right) + \left[\frac{A - \iota_t}{q_t} + (1 - \psi_t)(\underline{\delta} - \delta)\right] dt,$$

which is (111). □

F.23 Proof of Corollary 2

Immediate from Proposition 17 and the diffusion term of dr_t^k, which loads on $(\sigma + \sigma_t^q) dZ_t$.

F.24 Proof of Proposition 18

Unchanged. (It uses goods market clearing, $\underline{c}_t = \gamma \underline{N}_t$, and $\iota_t = \underline{\iota}_t = \frac{1}{2\theta}(q_t^2 - 1)$, none of which involve default.)

F.25 Proof of Proposition 19

Under no default, the household optimal portfolio share satisfies

$$\underline{x}_t = \frac{\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - r_t}{(\sigma + \sigma_t^q)^2}.$$

Market clearing implies

$$\underline{x}_t = \frac{1 - x_t \eta_t}{1 - \eta_t}.$$

Equating the two and solving for r_t yields

$$r_t = \frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} - (\sigma + \sigma_t^q)^2 \frac{1 - x_t \eta_t}{1 - \eta_t}.$$

Using

$$\frac{\mathbb{E}\left[d\underline{r}_t^k\right]}{dt} = \frac{\underline{A} - \underline{\iota}_t}{q_t} + \phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q,$$

gives (115). □

F.26 Proof of Proposition 21

With $\nu_t \equiv 1$, the expert's expected net-worth growth rate is

$$\frac{\mathbb{E}_t[dn_t]}{n_t\,dt} = x_t \frac{\mathbb{E}_t[dr_t^k]}{dt} - (x_t - 1)r_t.$$

Differentiating w.r.t. x_t gives the marginal value of leverage:

$$\frac{\partial}{\partial x_t}\left(x_t \frac{\mathbb{E}_t[dr_t^k]}{dt} - (x_t - 1)r_t\right) = \frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t.$$

Using the expressions for $\mathbb{E}[dr_t^k]/dt$ and r_t (from Proposition 19), one can write

$$\frac{\mathbb{E}_t[dr_t^k]}{dt} - r_t = \frac{A - \underline{A}}{q_t} + (\underline{\delta} - \delta) + (\sigma + \sigma_t^q)^2 \frac{1 - x_t\eta_t}{1 - \eta_t}.$$

Given $A > \underline{A}$ and (in equilibrium) $0 \leq \underline{x}_t \leq 1$ so that $1 - x_t\eta_t \geq 0$, this marginal value is nonnegative, and strictly positive whenever households hold some risky capital. Hence the expert strictly prefers the maximal feasible leverage, implying the corner solution

$$x_t = U_t = \min\left\{\tfrac{1}{m},\ \tfrac{1}{\eta_t}\right\}.$$

□