The Child Quantity-Quality Trade-Off and Endogenous R&D in Economic Growth

Junjian Yi* Zikun Zhou†

December 2025

Abstract

This paper develops a growth model integrating price-determined child quantity-quality trade-off with endogenous R&D to examine the long-term relationship between population dynamics and economic growth. Technological progress affects the wage premium for human capital relative to wage of unskilled labor, which parents interpret as a price signal when choosing between fertility and child human capital investment. These decisions shape population growth and human capital supply, creating feedback effects on R&D incentive and thus technological progress. The resulting equilibrium features four development phases: Malthusian, prosperous, aging, and potentially stagnating. Our model uniquely predicts that the interaction between price-determined quantity-quality trade-offs and endogenous R&D inevitably generates population aging through a negative feedback loop: Population growth boosts R&D, raises human capital premiums, reduces fertility, and slows future population growth until population size ultimately begins to shrink. The framework yields distinctive policy implications: Well-designed population policies can prevent aging and sustain growth, while pro-R&D industrial policies may unintentionally depress fertility and undermine long-run growth.

JEL classification: J11, J13, J18, O11, O40

Key words: Child quantity-quality trade-off; endogenous R&D; human capital; population aging; economic growth

*China Center of Economic Research, Peking University. Email: junjian.yi@gmail.com

†China Center of Economic Research, Peking University. Email: zkzhou2018@nsd.pku.edu.cn

1 Introduction

Over the past half century, total fertility rates in many developed and emerging economies have steadily declined and remained persistently below the replacement level, resulting in widespread population aging. Accompanying this demographic shift are several distinctive patterns: Economic growth has decelerated markedly; human-capital accumulation has continued but at a slower pace; and the correlation between income and fertility (the income-fertility gradient) has shifted from negative to zero or even modestly positive (Doepke et al., 2023). Yet theoretical studies have rarely examined the central question: Is population aging inevitable? Moreover, how can we account for the distinctive patterns observed alongside population aging? And what is the long-run future of economic growth in an aging society? This paper seeks to address these questions.

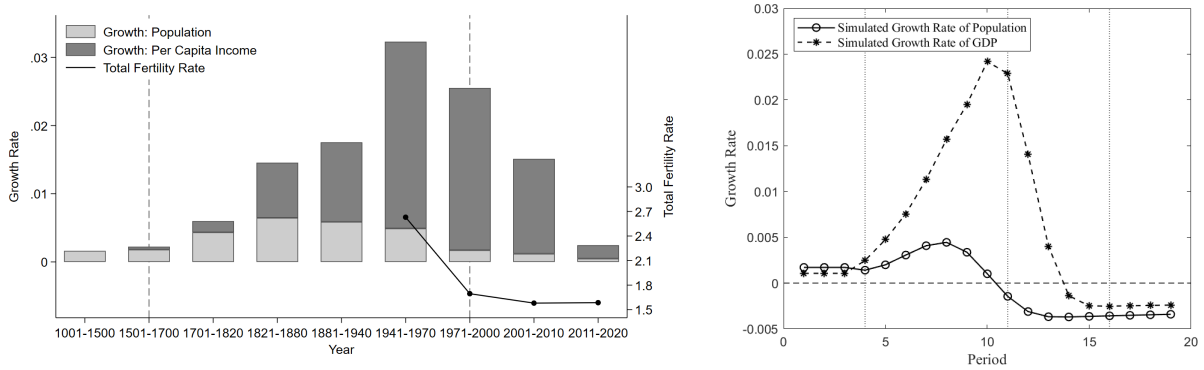
The following stylized facts motivate our study. Figure 1(a) decomposes historical GDP growth in Western Europe into population and per capita income growth. From 1000 to 1700 AD, GDP growth was driven almost entirely by population, with negligible gains in income per capita. Modern growth began around 1700, as per capita income started to rise. During the 19th and early 20th centuries, income growth accelerated, while population growth first increased and then declined. Around 1970, a distinct pattern linked to population aging emerged: Fertility fell below the replacement level and persisted at such a low level.[1] During the same time, per capita income growth, which had peaked in the mid-20th century, began to slow—falling to about 1% by the 2000s and below 0.5% in the 2010s.

The aging pattern spans developed economies globally. Figure 2 documents consistent trends across OECD Europe, OECD America, and the Asian Four (Japan, Singapore, Hong Kong, South Korea). Per capita GDP growth declined dramatically (Figure 2(a)), while fertility rates fell below the replacement level (Figure 2(b)). Consequently, the share of the population aged 65 or above rose steadily (Figure 2(c)).

The growth of households' investment in child human capital has also decelerated. Figure 3(a) shows that while the human-capital index has continued to rise in recent decades, its growth has slowed markedly and reached around zero in the 2010s.[2]

[1] Population growth remains slightly positive because life expectancy continues to rise.

[2] See Appendix A for details on the construction of the human-capital index, in which we exclude the effects of public educational investment to isolate parental investment in child human capital.



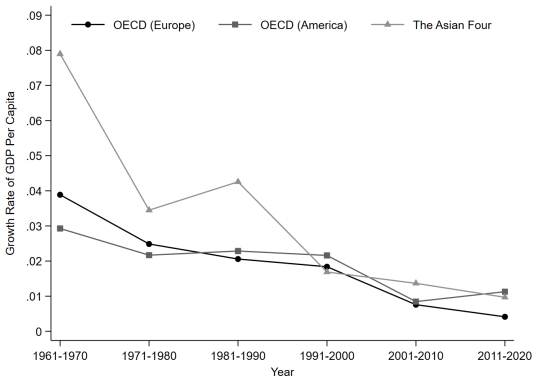
(a) Observed Growth in Western Europe (b) Simulated Growth

Figure 1: Growth of Population and Per Capita Income

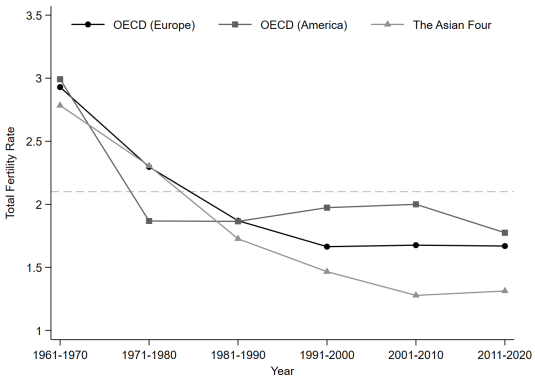
Data Source: Maddison Project Database 2020 and World Bank Open Data. Population after WWII excludes international immigrants. Western Europe refers to 19 economies labeled "Western Europe" in Maddison Project Database 2020, aggregated as a whole. Data on immigrants are from United Nations Population Division (2012) Trends in Total Migrant Stock and United Nations Population Division (2020) International Migrant Stock.

The fertility-income gradient has reversed from negative to zero or slightly positive (Doepke et al., 2023). Figure 4(a) shows that in the 1960s and 1970s, higher-income countries had lower fertility rates. By the 1990s the gradient had moved toward zero, and by the 2000s and 2010s it had turned slightly positive.

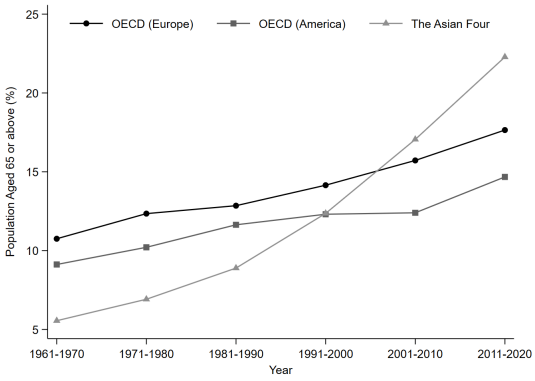
This paper develops a growth model to explain these stylized facts. We integrate the *price-determined child quantity-quality trade-off* with *endogenous R&D* to study the long-run joint dynamics of population and economic growth. In the model, R&D firms employ human capital to develop new products and pursue monopoly profits. R&D activities and the resulting technological progress alter the relative price of human capital—wage premium for human capital relative to wage of unskilled labor—which parents treat as a signal when making trade-offs between child quantity (fertility) and quality (human capital investment). These choices shape subsequent population growth and human capital supply, which in turn influence the benefits and costs of R&D. The resulting equilibrium generates four distinct phases of development: Malthusian, prosperous, aging, and potentially stagnating, as shown by Figure 5. Our model predicts that population aging is inevitable and further accounts for the stylized facts documented above. Distinct implications for the long-run dynamics of population and economic growth are derived when comparing population policies with pro-R&D industrial policies.



(a) Growth in Per Capita GDP



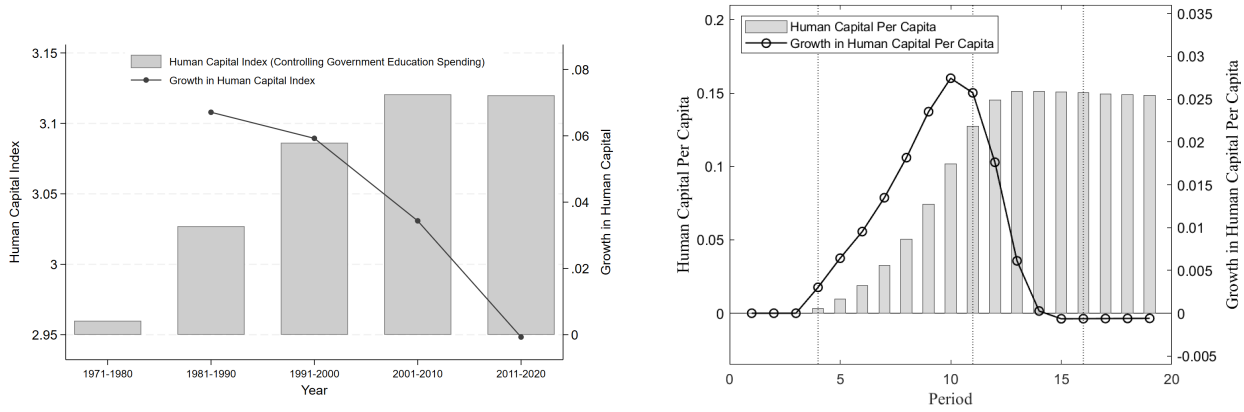
(b) Total Fertility Rate



(c) Population Aged 65 or above

Figure 2: Aging Transition by Economy Clubs

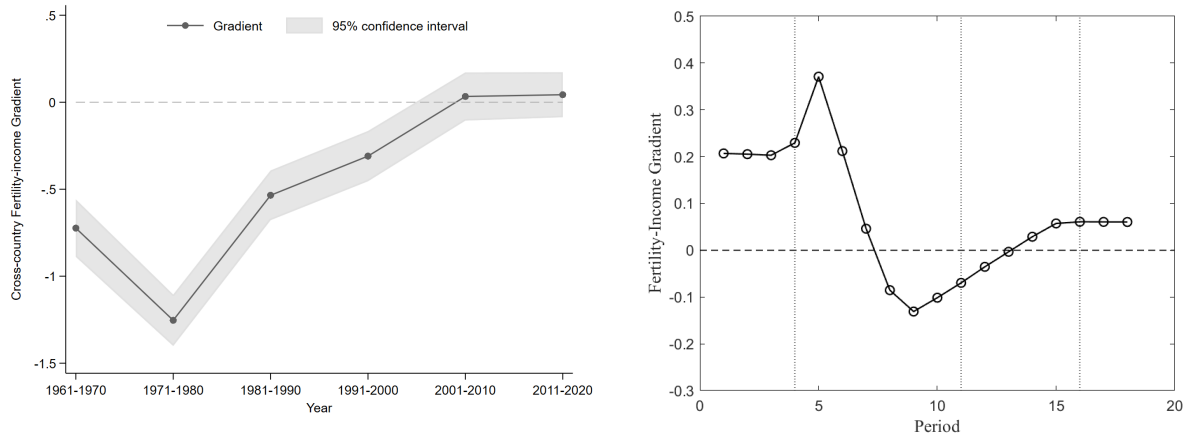
Data Source: World Bank Open Data and Penn World Table 10.01. OECD (Europe) refers to 18 initial members of OECD in Europe. OECD (America) refers to Canada and United States. The Asian Four refers to Japan, Singapore, Hong Kong, and South Korea. The average total fertility rate is weighted according to population size. The dotted line in Panel (b) illustrates the replacement level of the total fertility rate.



(a) Observed in OECD and the Asian Four (b) Simulated Human Capital

Figure 3: Change in Human Capital Index

Data Source: World Bank Open Data and Penn World Table 10.01. The average human capital index is weighted according to population size. We exclude variation caused by change in government expenditure on eduction per capita to capture the stylized facts in households' decisions. See Appendix A for details.



(a) Observed in OECD and the Asian Four (b) Simulated Fertility-Income Gradient

Figure 4: Change in Fertility-Income Gradient

Data Source: World Bank Open Data and Penn World Table 10.01. The fertility-income gradient is the OLS slope derived by regressing the total fertility rate of an economy on its logged GDP per capita for each decade. See Appendix A for details.

We begin in Section 2 by laying out the model environment. The economy features overlapping generations in discrete time and three life stages for each single-person household: childhood, young adulthood, and old age. In childhood, individuals receive human capital investment from their parents. In young adulthood, they choose consumption, allocate time between market work, fertility, and child human capital investment, and save through holding fixed resources (such as land). Parents' investment in each child's human capital responds

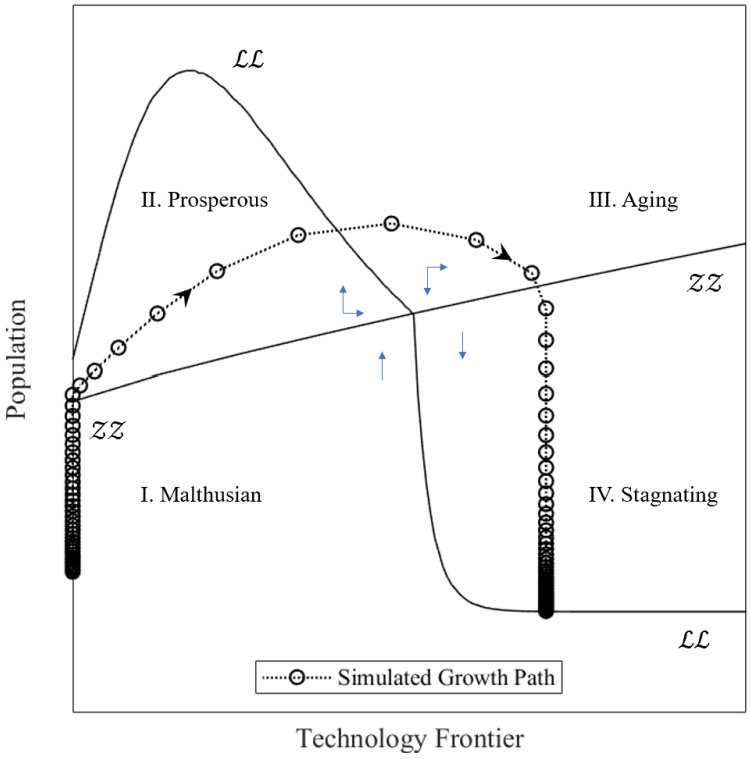


Figure 5: Phase Diagram

to the expected future relative price of human capital in the labor market. Because there is a trade-off between child quantity and quality—parents who have more children have less time to invest in each child's human capital—this relative price also shapes fertility decisions and, in turn, aggregate population growth. We refer to this mechanism as *price-determined child quantity-quality trade-off*. In old age, individuals consume savings and the returns to fixed resources.

Production occurs in two sectors—traditional and modern. The traditional sector combines unskilled labor with fixed resources to produce a perfectly competitive traditional good. The modern sector uses human capital to produce a variety of modern goods. Newly developed goods are protected by one-period patents and sold monopolistically, while previously developed goods are traded competitively.

Technological progress occurs in the R&D sector. In each period, potential entrants decide whether to undertake R&D, using human capital to invent new modern varieties and pursuing monopoly profits. The volume of R&D activity is determined by expected monopoly profits and the cost of employing human capital; in turn, R&D activity itself is a

key determinant of the demand for human capital. We refer to this mechanism as *endogenous R&D*.

Section 3 characterizes the equilibrium. Two corner solutions may arise: Parents do not invest in children's human capital, or no firm engages in R&D. We first pin down the conditions under which these corner solutions occur, and then determine the relative price of human capital. We subsequently establish the existence and uniqueness of the equilibrium and show that the economy's dynamics are fully described by two state variables: population and the technological frontier.

In Section 4 we analyze the economy's dynamics, as illustrated in Figure 5. A typical economy transitions through four distinct phases of development. Starting with a small population and no modern technology, the economy begins in a Malthusian phase, characterized by population growth without technological progress. Once population surpasses a critical threshold, it enters a prosperous phase with both population growth and technological progress. Over time fertility declines, and when it falls below the replacement rate, an aging phase emerges, marked by population decline and slowing technological progress. Finally, depending on how technological progress relates to the current technology frontier, the economy may enter a stagnating phase with both population decline and technological stagnation, converging to a steady state.

Section 5 explores the causes for inevitable population aging. This outcome arises from a negative feedback loop driven by the interaction between price-determined child quantity-quality trade-off and endogenous R&D. In the prosperous phase, population growth both increases aggregate demand for new products, thereby raising monopoly profits, and expands the aggregate supply of human capital, thereby lowering the cost of R&D. The resulting expansion of R&D activity increases its demand for human capital, inducing a rise in the relative price of human capital and a shift in parental choices away from child quantity toward child quality. Consequently, population growth induces a lower fertility and suppresses further population growth. This negative feedback loop does not cease until the population size begins to shrink, implying a below-replace fertility rate and population aging.

We conduct two exercises to confirm that inevitable population aging is indeed rooted in price-determined quantity-quality trade-off and endogenous R&D, and to compare our model with representative models from two related strands of the economic growth literature.

The first exercise analyzes the role of endogenous R&D. In our framework, the relative price of human capital is raised through two distinct channels: (i) Expanding R&D activity directly increases the demand for human capital in the R&D sector, and (ii) R&D-induced technological progress raises the demand for human capital relative to unskilled labor in the production sector. While channel (ii) is familiar with growth models with a price-determined child quantity-quality trade-off (e.g., Galor and Mountford, 2008), channel (i) is novel and stems from our introduction of endogenous R&D. Earlier models without channel (i) generate population aging only under particular assumptions about how production technology affects the demand for human capital. In our model, by contrast, the inevitability of population aging does not rely on particular assumptions about production technology: Once channel (i) is present, population aging becomes an inevitable consequence of the innovation process, even if channel (ii) is shut down. The first exercise demonstrates this point.

The second exercise analyzes the role of price-determined quantity-quality trade-off. Parents treat the relative price of human capital as a signal when choosing along the child quantity-quality trade-off: As the relative price rises, they allocate more resources to each child's human capital and reduce fertility. This price-determined trade-off creates an externality, because the relative price reflects only the scarcity of skilled relative to unskilled labor, but fails to capture the broader equilibrium effects of fertility on (i) the aggregate demand (market size) and hence R&D profits; (ii) the total supply of human capital, given per-capita human capital, and thus R&D costs; (iii) and the future population, which sustains growth. This externality, which is emphasized by Jones (2022), is amplified in our setting because it distorts not only parent's fertility-consumption but also child quantity-quality trade-offs. The second exercise demonstrates that, while preserving all other model features except that the quantity-quality trade-off is no longer determined by the relative price, our predictions return to those of standard endogenous R&D models.

Section 6 reviews in brief how our model's predictions align with the stylized facts on population, economic growth, human capital investment, and fertility-income gradient. Section 7 examines model implications for population and industrial policies. Well designed population policy can prevent population aging and sustain balanced growth in the long run. In contrast, industrial policy through R&D subsidies accelerates technological progress but, by raising the relative price of human capital, depresses fertility. This undermines future

population that sustains long-run technological progress.

1.1 Related Literature

Our study bridges two strands of the economic growth literature. The first strand, rooted in the seminal contribution of Barro and Becker (1989), seeks to provide a micro-foundation for endogenous fertility and derives its implications for economic growth.[3] Considering the close relationship between human capital and economic growth, Becker et al. (1990) introduce the trade-off between child quantity (fertility) and quality (human capital investment) to the growth model.[4] Introduction of the quantity-quality trade-off greatly enriches the growth theory (e.g., Greenwood and Seshadri, 2002; De la Croix and Doepke, 2003; De la Croix, 2013; Guo et al., 2022; Baudin and De la Croix, 2024). In these models, fertility outcomes and their growth implications stem from the demand for human capital, which itself depends endogenously on technological progress. Some later models thus explicitly connect population dynamics with technological progress (Tamura, 2002; Doepke, 2004).[5] In particular, unified growth theory, whose representative contribution is Galor and Weil (2000), captures the entire course of human history from the Malthusian stagnation to modern growth as a continuum in which the reinforcing interaction between technology and population gradually increases the return to human capital.

Although many growth models incorporate the child quantity-quality trade-off, they do not all treat it as *price-determined*. In models where the trade-off is not price-determined, population aging arises only contingently, depending on the parameters and functional forms that govern the trade-off. For example, Galor and Weil (2000) let technological progress affect the trade-off by entering directly into the human capital production function, rather than through market prices; the properties of this function then determine whether aging occurs.[6]

[3]In these models, parents are assumed to be either altruistic and care about the number (Baudin et al., 2015), income(Galor and Weil, 2000; Greenwood and Seshadri, 2002; Galor and Mountford, 2008), or utility (Becker and Barro, 1988; Barro and Becker, 1989) of their children; or selfish and care about the transfers from their children when the parents are old (Morand, 1999; Becker et al., 2016).

[4]Introduction of the child quantity-quality trade-off also seeks to explain the negative fertility-income gradient in the post-Malthusian era (Becker and Lewis, 1973; Doepke, 2015).

[5]Some other models, such as Goodfriend and McDermott (1995) and Hansen and Prescott (2002), link technological progress to human capital or population without relying on the quantity-quality trade-off.

[6]Specifically, Galor and Weil (2000) assume that child's human capital is a function of both parents' time investment and the growth rate of total factor productivity. Population aging hinges on the properties of this function. It occurs if the marginal return to parental time, under the steady-state rate of technological

By contrast, in models with a price-determined quantity-quality trade-off, population aging hinges on assumptions about production technology: How technological progress alters the demand for human capital in the production sector, thus the relative price of human capital. For instance, Galor and Mountford (2008) tie the trade-off to the relative wage of skilled to unskilled workers and assumes that, once technology is sufficiently advanced, further progress becomes Hicks neutral across skill types. In that case, technological progress no longer affects the relative price of human capital and therefore does not incur population aging.[7]

The second strand of the literature has sought to build a micro-foundation for technological progress for decades, such as the endogenous R&D models. Closely related to this paper, the predictions of these R&D models hinge crucially on two assumptions: One concerns the functional forms for R&D and the other population dynamics. By assuming that R&D exhibits a "scale effect"—whereby the growth rate of technology is independent of the current technology frontier, given the total resources devoted to R&D—early endogenous R&D models predict permanent economic growth as long as the population remains non-decreasing (Romer, 1990; Grossman and Helpman, 1993; Aghion and Howitt, 1992).[8] Relaxing the assumption on the scale effect, some later studies predict permanent economic growth only when the population grows continuously (Jones, 1995a; Kortum, 1997; Segerstrom, 1998).[9] Despite different predictions, both frameworks assume exogenous non-negative population growth. Some recent work—for example, Christiaans (2011), Sasaki and Hoshida (2017) and Jones (2022)—examines negative population growth in the endogenous R&D framework.

In this strand of literature, the study closest to ours is Jones (2022). He shows that,

progress, becomes so high that families optimally raise their time investment per child to the point that fertility falls below the replacement level.

[7]Specifically, Galor and Mountford (2008) assume that, at sufficiently advanced stages, the traditional sector does not employ labor and the industrial sector uses skilled and unskilled labor in a Cobb-Douglas form. Whether population aging occurs then depends on whether the steady-state ratio of numbers of skilled to unskilled workers implies below-replacement fertility through the quantity-quality trade-off. Similarly, Doepke (2004) assumes that both traditional and industrial sectors use skilled and unskilled labor in a Cobb-Douglas way.

[8]From now on, as in Jones (1995a), we use the term "scale effect" in its narrow sense: Conditional on total R&D resources, the technology growth rate is independent of the current technology frontier.

[9]Jones (1995b) provides empirical evidence that is inconsistent with the scale effect; Jones (1995a) proposes that the growth level of technology, rather than the growth rate, depends on R&D resources. Other notable models that eliminate the scale effect include Dinopoulos and Thompson (1998) and Howitt (1999).

under exogenous population decline, endogenous growth models with or without scale effects converge to steady states with stagnant technology. With endogenous fertility, population aging becomes parameter-dependent in competitive equilibrium and is one of multiple steady states in the social optimum. Jones (2022) argues that an externality occurs when comparing the competitive equilibrium with social optimum: Households do not internalize all positive effect of population on technological progress, which distorts their trade-off between fertility and current consumption. The paper differs from ours by endogenizing population in the Barro and Becker (1989) framework and thus not incorporating the child quantity-quality trade-off.

Our model links price-determined child quantity-quality trade-off with endogenous R&D, yielding three distinctive contributions.

First, this connection delivers robust predictions on population aging. Relative to models where the quantity-quality trade-off is not price-determined, our framework features additional equilibrium effects on quantity-quality trade-off operating through the relative price of human capital, which push fertility toward below-replacement levels. Relative to models with a price-determined trade-off but without endogenous R&D, we show that expanding R&D activity itself increases demand for human capital in the R&D sector and raises its relative price, beyond and above the effect of technological progress in the production sector on demand for human capital. Hence, population aging in our model does not hinge on assumptions about how technology affects human capital demand in the production sector, as shown by our analysis in Section 5.1.[10]

Compared to models with endogenous R&D, our framework endogenizes population dynamics through a price-determined quantity-quality trade-off. In this setting, the externality highlighted by Jones (2022) not only persists but is also amplified: It distorts both the fertility-consumption trade-off and the trade-off between fertility and child human capital investment. Fertility falls even more sharply, making population aging inevitable. As shown by our analysis in Section 5.2, once the quantity-quality trade-off is not price-determined, a balanced-growth steady state without aging becomes possible, as in the social optimum of

[10]One can modify the assumptions on production technology in Galor and Mountford (2008) or Doepke (2004) to generate population aging—say, by letting technological progress cease to be Hicks neutral and instead augment the relative demand for skilled versus unskilled workers. Our analysis in Section 5.1 shows that although our benchmark model adopts particular production assumptions to match the stylized facts, the inevitability of population aging in our framework does not depend on those assumptions.

Jones (2022).

Second, this model aligns with broader stylized facts. The model accounts for both the long-run coevolution of population and economic growth and, crucially, the salient features of contemporary population aging: decelerating GDP per capita growth, slowing growth in household human-capital investment, and reversal of the fertility-income gradient.

Third, this study yields distinctive policy implications. Our model provides a general framework for comparing population policies with industrial policies. We demonstrate that well-designed, budget-neutral population policies can prevent population aging while sustaining growth. In contrast, pro-R&D industrial policies, by raising the relative price of human capital, depress fertility and can ultimately weaken long-run innovation by reducing future population.

This paper also relates to the literature on population aging, technological progress, and economic growth (Cutler et al., 1990; Weil, 1997; Sheiner, 2014). While most studies find that aging depresses technological progress and economic growth (e.g., Favero et al., 2016; Aksoy et al., 2019; Maestas et al., 2023), some suggest the opposite (Acemoglu and Restrepo, 2017). In particular, the recent work by Liang et al. (2018) highlights the role of human capital in shaping this relationship. Distinct from this literature, in our model population aging and technological progress are jointly endogenously determined in equilibrium.

2 Environment

This section presents a framework that integrates the price-determined child quantity-quality trade-off and endogenous R&D. The economy, consisting of households, producers, and developers of new technology, features overlapping generations and discrete time.

2.1 Household

We model a couple as a single individual who lives through three periods. In the first period, as a child, she acquires human capital through her parents' time investment. In the second period, as a young adult, she makes decisions regarding her (1) consumption of traditional and modern goods; (2) fertility; (3) time allocation between working and investing in her children's human capital; and (4) financial investment in fixed resources. Her decisions

on fertility and time investment in children are motivated by pure altruism. In the final period, as an older adult, she consumes returns from her fixed resource investment, with no intergenerational transfers assumed.

2.1.1 Preference, Budget Constraints, and Human Capital Production

For a young adult in period t, her utility is

$$u_t = v\left(c^y_{T,t}, \{c^y_{M,t}(z)\}_0^{\bar{z}_t}\right) + \beta \cdot v\left(c^o_{T,t+1}, \{c^o_{M,t+1}(z)\}_0^{\bar{z}_{t+1}}\right) + \beta\gamma \cdot \ln\left[n_t\left(w_{t+1} + r_{t+1}h_{t+1}\right)\right], \quad (1)$$

where

$$v\left(c_T, \{c_M(z)\}_0^{\bar{z}}\right) = \frac{1}{\rho}\ln\left((c_T)^\rho + \int_0^{\bar{z}} (c_M(z))^\rho \, \mathrm{d}z\right);\ \rho \in (0,1).$$

In this setup, $c^y_{T,t}$ and $c^y_{M,t}(z)$ represent her consumption of traditional and modern goods in period t when she is young, where z denotes the variety of modern goods and $\bar{z}_t$ is the modern technological frontier. $c^o_{T,t+1}$ and $c^o_{M,t+1}(z)$ denote her consumption in period $t+1$ when she is old. n_t is her fertility, and h_{t+1} the human capital for each of her children. w_{t+1} and r_{t+1} are, respectively, the wage rate for unskilled labor and the wage premium for (per efficient unit of) human capital in period $t+1$. The parameters $\beta \in (0,1)$ and $\gamma \in (0,\infty)$ represent the discount factor and the degree of altruism, while $\rho \in (0,1)$ governs the elasticity of substitution between different goods.

Her budget constraint in period t is given by

$$c^y_{T,t} + \int_0^{\bar{z}_t} P_t(z)c^y_{M,t}(z)\mathrm{d}z + q_t x_{t+1} = \left(1 - n_t(\tau(\bar{z}_t) + e_t)\right)\left(w_t + r_t h_t\right), \quad (2)$$

where $P_t(z)$ and q_t are, respectively, the price of modern good z and the price of fixed resources in period t. The price of the traditional good is normalized to 1. x_{t+1} denotes her fixed resource investment for period $t+1$ and e_t represents her time invested in each child's human capital. $\tau(\bar{z}_t)$ captures the fixed time cost of raising a child, which is assumed to be influenced solely by the modern technological frontier. In young adulthood, the individual is endowed with one unit of time, which she allocates between the household and labor market. Her labor income, consisting of the wages of unskilled labor (w_t) and skilled labor ($r_t h_t$), is divided between current consumption and investment in fixed resources.

Her budget constraint in period $t+1$, when she reaches old age, is given by

$$c^o_{T,t+1} + \int_0^{\bar{z}_{t+1}} P_{t+1}(z) c^o_{M,t+1}(z)\, dz = (d_{t+1} + q_{t+1}) x_{t+1}, \tag{3}$$

where d_{t+1} represents the return on her fixed resources in period $t+1$. In old age, she no longer participates in the labor market. Instead, she rents out her fixed resources to earn a return, and eventually sells these resources to the next generation.

Finally, the human capital production function, $\eta : [0, +\infty) \to [0, +\infty)$, is

$$h_{t+1} = \eta(e_t). \tag{4}$$

Remark. We adopt a framework similar to Galor and Weil (2000) and Galor and Mountford (2008), in which parents care about their children's total income, in contrast to the approach of Becker et al. (1990), which focuses on children's utilities. However, our results are robust to the latter model choice. Specifically, in Appendix B we demonstrate that household behavior remains qualitatively unchanged if the young adult's utility function is instead given by

$$\ln(u_t) = v\left(c^y_{T,t}, \{c^y_{M,t}(z)\}_0^{\bar{z}_t}\right) + \beta \cdot \left[v\left(c^o_{T,t+1}, \{c^o_{M,t+1}(z)\}_0^{\bar{z}_{t+1}}\right) + \ln\left(n_t^{\gamma_n} \cdot u_{t+1}^{\gamma_u}\right)\right]. \tag{5}$$

Here u_t is the cardinal utility of the young adult in period t. This equivalence arises because parents in our model affect their children's utility exclusively through improving their labor income by investing time in their human capital. This equivalence serves as a basis for the constrained planner's problem analysis in Section 5.2 below.

2.1.2 Optimization

In period t, the young adult maximizes her utility (Eq.(1)) by choosing her consumption, financial investment in fixed resources, fertility, and time investment in her children's human capital, subject to the budget constraints (Eqs.(2)-(3)), the human capital production function (Eq.(4)), and the non-negativity of all choice variables.

Assumption 1 (Home Production Technology)**.** *We assume the standard regularity conditions for the human capital production function $\eta(\cdot)$ and time cost function of child-rearing*

$\tau(\cdot)$*:*

i. $\eta(\cdot)$ *is twice continuously differentiable.* $0 < \eta'(\cdot) < \infty$, $\eta''(\cdot) < 0$, *and* $\eta(0) = 0$.

ii. $\tau(\cdot)$ *is continuously differentiable.* $\tau'(\cdot) \leq 0$, *and* $\tau(0) = \bar{\tau} \geq \underline{\tau} = \lim_{\bar{z}\to\infty} \tau(\bar{z}) > 0$.

Assumption 2 (Ruling out Two Degenerate Cases)**.** *We rule out two degenerate cases: Irrespective of the economic environment, households always choose fertility above the replacement level, such that* $\beta\gamma/(1+\beta+\beta\gamma) > \bar{\tau}$*; or they always choose fertility below the replacement level, such that* $\beta\gamma/(1+\beta+\beta\gamma) < \underline{\tau} + \bar{e}$*, if there exists* $\bar{e}$ *such that* $\eta'(\bar{e})(\underline{\tau} + \bar{e}) - \eta(\bar{e}) = 0$.

These two degenerate cases are inconsistent with empirical observations. Historically, in many economies fertility has shifted from below to above replacement levels or vice versa. Excluding these degenerate cases renders the interplay between economic growth and population dynamics an intriguing question.

Since the utility function is concave and the budget set is convex, Kuhn-Tucker conditions are sufficient to solve the optimization problem. Rearranging these conditions, we first observe

$$n_t(\tau(\bar{z}_t) + e_t) = \frac{\beta\gamma}{1+\beta+\beta\gamma},$$

which indicates that the parent allocates a portion of her time to rearing children, which is further divided between childbirth and investment in the child's human capital. This explicitly reflects the trade-off between having more children (quantity) and investing in their human capital (quality).

We then find

$$\eta'(e_t)(\tau(\bar{z}_t) + e_t) - \eta(e_t) + \left[\frac{\tau(\bar{z}_t)}{\beta\gamma}\lambda_t - 1\right]\frac{w_{t+1}}{r_{t+1}} = 0,\ \lambda_t \geq 0,\ e_t \geq 0,\ \lambda_t e_t = 0,$$

where λ_t is the Lagrangian multiplier for the non-negativity constraint on e_t. From this condition, we derive the following lemma.

Lemma 1. *Under Assumptions 1 and 2, both parental time investment in child human capital* e_t *and fertility* n_t *are uniquely determined by the relative price of human capital in the next period* $s_{t+1} \equiv r_{t+1}/w_{t+1}$ *and current technology frontier* $\bar{z}_t$*. Specifically:*

i. *Parental time investment* e_t *follows*

$$e_t = \varepsilon\left(s_{t+1}, \bar{z}_t\right).$$

where $\varepsilon(s,z) = 0$ *for* $\forall s \leq [\tau(z)\eta'(0)]^{-1}$, *and* $\varepsilon_1(s,z) > 0$ *for* $\forall s \geq [\tau(z)\eta'(0)]^{-1}$.

ii. *Fertility* n_t *follows*

$$n_t = \frac{\beta\gamma}{1+\beta+\beta\gamma} \cdot \frac{1}{\tau(\bar{z}_t) + \varepsilon\left(s_{t+1}, \bar{z}_t\right)}.$$

For $\forall \bar{z}_t \in [0,\infty)$, *when* $s_{t+1} \leq [\tau(\bar{z}_t)\eta'(0)]^{-1}$, $n_t > 1$; *when* $s_{t+1} \to \infty$, $n_t < 1$.[11]

Lemma 1 establishes that, conditional on the current technology frontier $\bar{z}_{t-1}$, both parents' human capital investment and fertility decision are determined by future relative price of human capital s_{t+1}, which is defined as the ratio between the wage premium for human capital r_{t+1} and the wage rate for unskilled labor w_{t+1}. [12] Parents choose how much to invest in their children's human capital according to how valuable they expect human capital to be in the future labor market—that is, the relative price of human capital. Because they trade-off between child quantity and quality, this expected price also shapes their fertility choices. We call this mechanism a *price-determined quantity-quality trade-off*. Specifically, when this relative price is sufficiently low, such that $s \leq [\tau(z)\eta'(0)]^{-1}$, parents allocate resources exclusively to fertility rather than to human capital investment, consistent with family behavior in Malthusian economies. Once this relative price exceeds the critical threshold, parents begin to invest in children's human capital, with higher prices inducing greater investment. Accordingly, low prices of human capital are associated with above-replacement fertility, whereas sufficiently high prices yield below-replacement fertility in the post-Malthusian era.

Finally, we derive a trade-off between the consumption of traditional and modern goods,

$$c^i_{M,s}(z) = P_s(z)^{-\frac{1}{1-\rho}} c^i_{T,s}, \ \forall z \in [0, \bar{z}_s], \ (s,i) \in \{(t,y),(t+1,o)\}, \tag{6}$$

and a trade-off between consumption in periods t and $t+1$,

$$q_t x_{t+1} = \frac{\beta}{1+\beta+\beta\gamma}(w_t + r_t h_t).$$

[11] Proofs for all lemmas and propositions are provided in Appendix C.

[12] Consider the commonly estimated Mincerian earnings equation, $\ln y_i = \beta_0 + \beta_1 h_i + \varepsilon_i$, where y_i denotes earnings and h_i denotes human capital. The wage rate for unskilled labor, w, in our model corresponds to the coefficient β_0, and the wage premium for (per efficient unit of) human capital, r, corresponds to the coefficient β_1. In empirical studies, β_1 is often referred to as the "return to human capital," whereas in theoretical studies, the "return to human capital" typically refers to the marginal utility gain from human capital investment, $\frac{\partial u}{\partial e}$. To avoid confusion, we use the term "the wage premium for human capital" to refer to r, and "the relative price of human capital" to refer to $s = \frac{r}{w}$ throughout.

2.2 Production

The production sector consists of a representative traditional goods producer and a continuum of modern goods producers. The market for traditional goods is perfectly competitive, while the market structure for modern goods depends on whether the good is newly developed.

2.2.1 Production of Traditional Goods

Traditional goods are produced using fixed resources and unskilled labor with a Cobb-Douglas technology. In a perfectly competitive market, a representative traditional producer in period t maximizes its profit:

$$\max_{X_t^p, L_t^p} \; A(X_t^p)^{1-\alpha}(L_t^p)^{\alpha} - d_t X_t^p - w_t L_t^p; \; \alpha \in (0,1),$$

where X_t^p and L_t^p are, respectively, the employed fixed resource and unskilled labor and A is the total factor productivity. By solving the problem, we have

$$w_t = A\left(\frac{X_t^p}{L_t^p}\right)^{1-\alpha}; \; d_t = A\left(\frac{L_t^p}{X_t^p}\right)^{\alpha}.$$

2.2.2 Production of Modern Goods

Modern goods are manufactured using human capital with a linear technology,[13] with productivity normalized to 1. For all previously developed goods $z \in [0, \bar{z}_{t-1}]$, producers operate in a perfectly competitive market. They maximize their profit:

$$\max_{H_t^p(z)} \; P_t(z)H_t^p(z) - r_t H_t^p(z), \tag{7}$$

where $H_t^p(z)$ is the human capital employed to produce good z.

For any newly developed good $z \in (\bar{z}_{t-1}, \bar{z}_t]$, a one-period patent is granted to the firm

[13] We make this simplification without loss of generality, as the concavity of the production function of modern goods can be captured by the concavity of the production function of human capital (η).

that develops the good. The firm thus operates as a monopolist in this period by

$$\max_{H_t^p(z),P_t(z)} \pi_t(z) = P_t(z)H_t^p(z) - r_t H_t^p(z), \tag{8}$$

subject to the demand for modern goods, which is derived by aggregating Eq.(6) across all consumers:

$$H_t^p(z) = P_t(z)^{-\frac{1}{1-\rho}} \left(L_t c_{T,t}^y + L_{t-1} c_{T,t}^o \right), \tag{9}$$

where $\pi_t(z)$ is the profit of the newly developed good z and L_t (L_{t-1}) is the population of the young (old) cohort in period t. Note that $\pi_t(z) = \pi_t$; that is, the monopoly profit is identical among all newly developed goods, as the solution of problem (8) subject to (9) is independent of z.[14]

By solving problems (7) and (8), we have

$$P_t(z) = \begin{cases} r_t, \text{ if } z \in [0, \bar{z}_{t-1}], \\ \dfrac{r_t}{\rho}, \text{ if } z \in (\bar{z}_{t-1}, \bar{z}]. \end{cases}$$

2.3 R&D

Prospective R&D firms decide whether to enter the R&D sector each period. Firms who enter employ human capital to develop new goods, striving to outpace competitors and secure patents. The probability of successful R&D depends on the existing technological frontier and the intensity of competition. Once a new good is developed, the R&D firm earns the monopoly profit π_t for one period.

The aggregate R&D outcome in period t is characterized by the measure of new goods developed: $\bar{z}_t - \bar{z}_{t-1} = \Delta(\bar{z}_{t-1}, N_t)$, where $N_t = \int_0^\infty m_{it} \mathrm{d}i$ represents the measure of firms engaged in R&D in period t with $m_{it} = 1$ if firm i participates and $m_{it} = 0$ otherwise. Each participating firm requires the human capital $\omega(\bar{z}_{t-1}) > 0$ as input and the success probability is $\Delta(\bar{z}_{t-1}, N_t)/N_t \equiv \delta(\bar{z}_{t-1}, N_t)$. A potential R&D firm's decision problem is then

$$\max_{m_{it} \in \{0,1\}} m_{it} \left[\delta(\bar{z}_{t-1}, N_t)\pi_t - \omega(\bar{z}_{t-1}) r_t \right]. \tag{10}$$

[14] Symmetry of newly developed products is a standard result in the literature on endogenous R&D models with variety expanding (e.g., Romer, 1990).

In this setup, the R&D production function is

$$\bar{z}_t - \bar{z}_{t-1} = \Delta\left(\bar{z}_{t-1}, \frac{H_t^R}{\omega(\bar{z}_{t-1})}\right),$$

where H_t^R represents the total human capital devoted to R&D and $H_t^R/\omega(\bar{z}_{t-1})$ is the measure of firms engaged in R&D (N_t). This function states that the aggregate level of technological progress is determined by the total resources devoted to R&D and the current technology frontier. The properties of this function are characterized by Δ and ω. Regarding the regularity conditions of the R&D production function, we make the following assumptions.

Assumption 3 (R&D Production Technology)**.** *Δ and ω are both continuously differentiable and for any $\bar{z} \geq 0$ satisfy*

i. $\Delta(\bar{z}, 0) = 0$ *and* $\Delta(\bar{z}, N) > 0$ *for* $\forall N > 0$*;* $\Delta_2(\bar{z}, N) > 0$*,* $\forall N \geq 0$*.*

ii. $\delta_2(\bar{z}, N) < 0$*,* $\forall N \geq 0$*.*

iii. $\lim_{N\to 0} \delta(\bar{z}, N) > 0$*,* $\lim_{N\to\infty} \delta(\bar{z}, N) = 0$*.*

Assumption 3 states that technological progress requires R&D participation and accelerates with the number of participants. However, as participation increases, the success probability of each individual participant decreases. This probability is positive when there is a minimal number of participants, but converges to zero as the number of participants approaches infinity.

Remark. We emphasize that Assumption 3 does not impose any restrictions on the relationship between technological progress (Δ) and the current technological frontier($\bar{z}$). That is, our main result does not depend on whether R&D becomes easier or harder as the technological frontier advances.

This generalization renders our framework distinct from the endogenous growth literature. Early models (Romer, 1990; Grossman and Helpman, 1993; Aghion and Howitt, 1992), despite different micro-foundations, consistently exhibit a scale effect (Jones, 1995a): Technological growth rates are proportional to total R&D resources, but independent of the current technological frontier. In our notation, this corresponds to $\Delta/\bar{z}_{t-1} = f(H_t^R)$. However, later empirical studies do not provide sufficient evidence to support this scale effect (e.g. Jones, 1995b). Subsequent studies (Kortum, 1997; Segerstrom, 1998; Jones, 2002) propose that technological progress becomes increasingly difficult as the frontier advances. For

instance, Jones (2002) specifies $\Delta/\bar{z}_{t-1} = f(H_t^R) \cdot \bar{z}_{t-1}^{\psi-1},\ \psi < 1$.

Our setup places no restrictions on the relationship between Δ and $\bar{z}$; accordingly, our main result does not hinge on whether R&D becomes easier or harder as the frontier advances. To illustrate this, consider a parametric example:

$$\Delta(\bar{z}, N) = [(N+a)^\kappa - a^\kappa](\bar{z}+b)^\psi,\ \omega(\bar{z}) = \omega_0, \tag{11}$$

which yields the aggregate R&D production function:

$$\bar{z}_t - \bar{z}_{t-1} = \left[\left(\frac{H_t^R}{\omega_0} + a\right)^\kappa - a^\kappa\right](\bar{z}_{t-1}+b)^\psi.$$

Setting $\psi = 1$ in this example represents the scale effect in early endogenous growth models (Romer, 1990; Grossman and Helpman, 1993; Aghion and Howitt, 1992), while $\psi < 1$ eliminates this effect as proposed by later studies (Kortum, 1997; Segerstrom, 1998; Jones, 2002). Crucially, Assumption 3 above does not impose any restriction on ψ.

In the example, Assumption 3 requires $\kappa \in (0,1)$, reflecting contemporaneous spillovers in the R&D sector whereby additional participants reduce each individual's success probability. It further requires $a > 0$ and $b > 0$ to ensure that the functions Δ and δ are well defined in an absence of modern technology or human capital supply. When $a = b = 0$, this R&D production function reduces to the standard one used in the literature (e.g. Jones, 1995a):

$$\bar{z}_t - \bar{z}_{t-1} \propto \left(H_t^R\right)^\kappa \bar{z}_{t-1}^\psi.$$

We'll use specification (11) to conduct simulation in Section 4 and analyze the constrained planner's problem in Section 5.2.

We derive the following lemma to characterize R&D firms' decision.

Lemma 2. *Under Assumption 3, the measure of firms entering the R&D sector is uniquely determined by the monopoly profit π_t, the wage premium for human capital r_t, and the current technology frontier $\bar{z}_{t-1}$:*

$$N_t = \Phi\left(\frac{\pi_t}{r_t}, \bar{z}_{t-1}\right),$$

where Φ is a continuous function. Furthermore, $\Phi(\pi_t/r_t, \bar{z}_{t-1}) > 0$ and $\Phi_1(\pi_t/r_t, \bar{z}_{t-1}) > 0$ if

and only if

$$\frac{\pi_t}{r_t} > \frac{\omega(\bar{z}_{t-1})}{\delta_0(\bar{z}_{t-1})};$$

and $\Phi(\pi_t/r_t, \bar{z}_{t-1}) = 0$ *otherwise. Here* $\delta_0(\bar{z}_{t-1}) \equiv \delta(\bar{z}_{t-1}, 0) = \lim_{N\to 0} \Delta(\bar{z}_{t-1}, N)/N$ *is the success probability of R&D without competitors.*

Lemma 2 shows that the relative profit of R&D to the cost of human capital (π_t/r_t) determines the size of the R&D sector. The threshold ($\omega(\bar{z}_{t-1})/\delta_0(\bar{z}_{t-1})$) represents the expected human capital input per successful R&D without any competitors. If the relative profit falls below this threshold, no firms engage in R&D, resulting in technological stagnation. When the relative profit exceeds this threshold, the R&D sector emerges and expands as the relative profit rises.

3 Equilibrium

This section defines and analyzes the equilibrium of the economy. We start by deriving market-clearing conditions and the laws of motion. In period t, the market-clearing conditions for traditional and modern goods are

$$\begin{aligned}
&L_t c^y_{T,t} + L_{t-1} c^o_{T,t} = A(X^p_t)^{1-\alpha}(L^p_t)^{\alpha}, \\
&L_t c^y_{M,t}(z) + L_{t-1} c^o_{M,t}(z) = H^p_t(z);\ \forall z \in [0, \bar{z}_t],
\end{aligned}$$

and those for fixed resources, unskilled labor, and human capital are

$$\begin{aligned}
&x_t L_{t-1} = X^p_t = X, \\
&L^p_t = (1 - n_t(\tau(\bar{z}_t) + e_t))L_t, \\
&\int_0^{\bar{z}_t} H^p_t(z)\mathrm{d}z + N_t \omega(\bar{z}_{t-1}) = (1 - n_t(\tau(\bar{z}_t) + e_t))L_t h_t,
\end{aligned}$$

where X is a sufficiently large total endowment of fixed resources in the economy.

The dynamics of the economy are governed by two laws of motion. First, the young adult population evolves according to

$$L_t = n_{t-1} L_{t-1}. \tag{12}$$

For notational convenience, we hereafter use "population" to refer to young adult population L_t. Second, the technological frontier advances through R&D:

$$\bar{z}_t = \bar{z}_{t-1} + \Delta(\bar{z}_{t-1}, N_t). \tag{13}$$

An equilibrium growth path consists of the optimal decisions of all agents, a price system, and a path of state variables that together satisfy these market-clearing conditions and laws of motion, given the initial state. The formal definition is provided in Appendix D.

In equilibrium, there are two possible corner solutions: First, no firm engages in R&D; second, parents do not invest in child human capital. Lemma 3 characterizes the conditions under which these corner solutions arise.

Lemma 3. *Under Assumptions 1-3, there exists a continuous function* $\mathcal{Z} : [0, \infty) \to (0, \infty)$ *such that*

i. $N_t > 0$ *iff* $L_{t-1} > \mathcal{Z}(\bar{z}_{t-1})$*; otherwise* $N_t = 0$*;*

ii. *When* $\bar{z}_{t-1} = 0$*,* $h_t > 0$ *iff* $L_{t-1} > \mathcal{Z}(0)$*; when* $\bar{z}_{t-1} > 0$*,* $h_t > 0$*.*

Here $\mathcal{Z}$ *denotes the threshold population of the previous generation for R&D participation, which is a function of the current technological frontier.*

Lemma 3.i states that firms engage R&D participants if and only if the population in the previous generation (L_{t-1}) exceeds the threshold $\mathcal{Z}(\bar{z}_{t-1})$. This is for two reasons. First, a larger population increases total demand, augments monopoly profits π_t, and incentivizes R&D. This prediction is consistent with empirical evidence in the literature (Acemoglu and Linn, 2004; Peters, 2022; Aghion et al., 2024). Second, a larger population augments the total supply of human capital, reduces R&D costs r_t, and further incentivizes R&D.

Lemma 3.ii states that in the absence of modern technology, parents invest in child human capital if and only if the population exceeds the threshold $\mathcal{Z}(0)$. Without modern technology, human capital is used only for R&D activities, so investment occurs only when R&D participation exists. By contrast, once modern technology exists, human capital is also used to produce previously developed goods. As a result, investment in human capital occurs regardless of whether R&D takes place.

Lemma 3 has two significant implications for economic dynamics. First, there exists a Malthusian phase in equilibrium, characterized by no modern technology ($\bar{z}_{t-1} = 0$), no

technological progress ($N_t = 0$), and no parental investment in human capital ($e_{t-1} = h_t = 0$). In this phase, parents focus solely on childbirth, leading to a high fertility rate and continuous population growth. The Malthusian phase persists until the population reaches a threshold where both R&D and human capital investment become beneficial. At this point, modern growth begins. This result is consistent with Goodfriend and McDermott (1995) and Galor and Weil (2000).

Second, technological stagnation may occur even after the economy exits the Malthusian phase. If the population remains below the threshold $\mathcal{Z}(\bar{z}_{t-1})$, total demand for new products declines, which reduces monopoly profits, while the limited supply of human capital raises R&D costs. As a result, no firm finds it profitable to undertake R&D.

The following lemma pins down the relative price of human capital in equilibrium.

Lemma 4. *Under Assumptions 1-3, the current relative price of human capital $s_t = r_t/w_t$ can be uniquely determined by previous population L_{t-1} and technology frontier $\bar{z}_{t-1}$:*

$$\frac{r_t}{w_t} = s(L_{t-1}, \bar{z}_{t-1}),$$

where s is a continuous function. Furthermore,

i. *If $\bar{z} = 0$ and $L \leq \mathcal{Z}(0)$, $s(L, \bar{z}) = [\bar{\tau}\eta'(0)]^{-1}$; otherwise $s(L, \bar{z}) > [\bar{\tau}\eta'(0)]^{-1}$.*

ii. *If $\bar{z} = 0$ and $L \leq \mathcal{Z}(0)$, $s_1(L, \bar{z}) = 0$; otherwise $s_1(L, \bar{z}) > 0$.*

iii. *If $L < \mathcal{Z}(\bar{z})$, $s_2(L, \bar{z}) > 0$; if $L > \mathcal{Z}(\bar{z})$, $s_2(L, \bar{z})$ is undetermined.*

Lemma 4.i states that, throughout the Malthusian phase, the relative price of human capital remains at $[\bar{\tau}\eta'(0)]^{-1}$. Once the economy leaves the Malthusian phase, the relative price increases above $[\bar{\tau}\eta'(0)]^{-1}$ to encourage positive parental time investment in child human capital.

Lemma 4.ii states that after leaving the Malthusian phase, the relative price of human capital increases with the previous population. A larger population affects $s = r/w$ through two channels. On the one hand, it enlarges the market size and the supply of human capital, which expands R&D activity and boosts the demand for human capital in both the production and R&D sectors, thereby increasing r. On the other hand, it expands the supply of both unskilled labor and human capital, thereby decreasing both r and w. The first channel raises s, while the second has an ambiguous effect on s. In net, however, the

first channel dominates the second. Due to the strong substitutability of households' demand across goods ($\rho \in (0,1)$),[15] changes in the supply of unskilled labor or human capital result in only small variations in the relative price of modern goods. The equilibrium relative price of human capital therefore responds only modestly to supply changes, so the second channel is dominated by the first. In Section 5 we will further discuss the relationship between population size and the relative price of human capital by focusing on the first channel.

This result has an important implication: The population dynamics are self-regulating. A larger population raises the relative price of human capital, which renders investments in child human capital more attractive and fertility less appealing. Higher population levels thus lead to lower population growth rates. This prediction aligns with empirical evidence in the literature (Lutz et al., 2006; De la Croix and Gobbi, 2017).

Lemma 4.iii states that the relationship between the relative price of human capital and the technological frontier depends on whether R&D is occurring. With technology progressing, the demand for human capital arises from both the production and R&D sectors. While a more advanced technological frontier raises demand for human capital in the production sector, its effect on demand for human capital in the R&D sector is ambiguous, since we do not make any assumption on the relationship between the current technological frontier and R&D ($\partial\Delta/\partial\bar{z}_{t-1}$, Assumption 3). Consequently, $\partial s_t/\partial\bar{z}_{t-1}$ is undetermined. In contrast, with technology stagnating, the demand for human capital is solely determined by the production sector, which leads to a positive relationship between the technological frontier and the relative price of human capital, i.e., $\partial s_t/\partial\bar{z}_{t-1} > 0$.

The following proposition establishes the existence and uniqueness of equilibrium.

Proposition 1. *Under Assumptions 1-3, given any initial state* $\{L_0, \bar{z}_0, X\}$ *such that* $L_0 > 0$, $X > 0$, $\bar{z}_0 \geq 0$, *there exists a unique equilibrium growth path. Specifically, the economy is entirely described by two state variables, the population* L_t *and the technology frontier* $\bar{z}_t$. *The equilibrium growth paths of the state variables are, for* $\forall t \geq 1$*:*

$$L_t = \frac{\beta\gamma}{1+\beta+\beta\gamma} \cdot \frac{L_{t-1}}{\tau(\bar{z}_{t-1}) + \varepsilon(r_t/w_t, \bar{z}_{t-1})}, \tag{14}$$

[15] $\rho \in (0,1)$ means that the elasticity of substitution between households' consumption of different goods $(1/(1-\rho))$ lies in $(1,\infty)$. This is a standard and necessary assumption for solving the monopolist's problem faced by new technology developers (e.g., Romer, 1990).

$$\bar{z}_t = \Delta\left(\bar{z}_{t-1}, \Phi\left(C \cdot \left(\frac{L_{t-1}^{1-\alpha\rho}}{(r_t/w_t)\cdot(\tau(\bar{z}_{t-1})+\varepsilon(r_t/w_t,\bar{z}_{t-1}))^{1-\alpha\rho}}\right)^{\frac{1}{1-\rho}}, \bar{z}_{t-1}\right)\right) + \bar{z}_{t-1}, \qquad (15)$$

where C is a positive constant and s_t is determined by Lemma 4; the path $\{L_t, \bar{z}_t\}_{t=0}^{\infty}$ uniquely determines all other decisions and prices in equilibrium.

Eq.(14) arises from the law of motion for the population Eq.(12), where $[\beta\gamma/(1+\beta+\beta\gamma)]/(\tau(\bar{z}_{t-1})+\varepsilon(r_t/w_t,\bar{z}_{t-1}))$ determines the fertility rate n_t according to Lemma 1.ii. Eq.(15) arises from the law of motion for the technological frontier Eq.(13), where $L_{t-1}/(\tau+\varepsilon(r_t/w_t,\bar{z}_{t-1}))$ is proportional to current population L_t according to Eq.(14), and the current population L_t together with the prices r_t/w_t determine the relative profit of a new good π_t/r_t. According to Lemma 2, $\Phi(\pi_t/r_t,\bar{z}_{t-1}) = N_t$ represents the measure of developers who participate in R&D and $\Delta(\bar{z}_{t-1}, N_t)$ represents the variety of new goods developed.

4 The Dynamics

This section characterizes, analyzes, and simulates the dynamics of the economy.

4.1 Two Critical Loci

We first introduce two critical loci in the $(\bar{z}, L)$ state space that determine the economic evolution.

Definition 1. *The population balance locus $\mathcal{LL}$ is the set of state variables $(\bar{z}, L)$ for which the fertility rate lies at the replacement level—i.e. $n = 1$—in equilibrium:*

$$\mathcal{LL} = \{(\bar{z}, L) : \tau(\bar{z}) + \varepsilon(s(L,\bar{z}),\bar{z}) = \beta\gamma/(1+\beta+\beta\gamma)\}.$$

Definition 2. *The technology threshold locus $\mathcal{ZZ}$ is the set of state variables $(\bar{z}, L)$ for which the population lies at the threshold size for technological progress:*

$$\mathcal{ZZ} = \{(\bar{z}, L) : L = \mathcal{Z}(\bar{z})\},$$

where $\mathcal{Z}(\cdot)$ is defined in Lemma 3.

Before fully characterizing the dynamics, we introduce the fourth assumption, which only relates to the economic prospects after population aging.

Assumption 4 (No Technology Singularity)**.**

$$\lim_{\bar{z}\to\infty} \frac{\omega(\bar{z})}{\delta_0(\bar{z})} \cdot \bar{z} > 0.$$

Assumption 4 ensures that the expected human capital input per successful R&D either remains strictly positive as technology approaches infinity or decays to zero at a rate no faster than $\bar{z}^{-1}$. Intuitively, Assumption 4 excludes technology singularity—a scenario in which human capital input per successful innovation diminishes too rapidly, which would naturally lead to explosive growth. In our parametric example (Eq.(11)), Assumption 4 requires $\psi \leq 1$, thereby accommodating both the scale effect specification ($\psi = 1$) and specifications that eliminate the scale effect ($\psi < 1$).

The following proposition establishes the properties of the two loci that characterize the economy's dynamics.

Proposition 2. ***Under Assumptions 1-3****:*

i. *There exists a continuous function* $\mathcal{L} : [0, \infty) \to (0, \infty)$ *such that* $\mathcal{LL} = \{(\bar{z}, \mathcal{L}(\bar{z}))\}$.

ii. $L_t > L_{t-1}$, $\bar{z}_t = \bar{z}_{t-1}$ *if* $L_{t-1} \in (0, \min\{\mathcal{Z}(\bar{z}_{t-1}), \mathcal{L}(\bar{z}_{t-1})\})$;
$L_t > L_{t-1}$, $\bar{z}_t > \bar{z}_{t-1}$ *if* $L_{t-1} \in (\mathcal{Z}(\bar{z}_{t-1}), \mathcal{L}(\bar{z}_{t-1}))$;
$L_t < L_{t-1}$, $\bar{z}_t > \bar{z}_{t-1}$ *if* $L_{t-1} \in (\max\{\mathcal{Z}(\bar{z}_{t-1}), \mathcal{L}(\bar{z}_{t-1})\}, \infty)$;
$L_t < L_{t-1}$, $\bar{z}_t = \bar{z}_{t-1}$ *if* $L_{t-1} \in (\mathcal{L}(\bar{z}_{t-1}), \mathcal{Z}(\bar{z}_{t-1}))$; *for* $\forall \bar{z}_{t-1} \geq 0$.

iii. $\mathcal{L}(0) > \mathcal{Z}(0)$.

iv. $\lim_{\bar{z}\to\infty} \mathcal{L}(\bar{z}) = 0$.

v. ***Further under Assumption 4***, $\lim_{\bar{z}\to\infty} \mathcal{Z}(\bar{z}) > 0$.

Here $\mathcal{L}$ *denotes the threshold population of the previous generation for replacement-level fertility, which is a function of the current technological frontier.*

Proposition 2 establishes the properties of the two loci. Proposition 2.i states that $\mathcal{LL}$ can be characterized by a function $\mathcal{L}(\bar{z})$ representing the population level that sustains replacement fertility at each technology level. Proposition 2.ii reveals how these loci partition the state space into four regions with distinct dynamics. Given the current technological

frontier $\bar{z}_t$, a state below $\mathcal{LL}$ ($L_t < \mathcal{L}(\bar{z}_t)$) indicates that the relative price of human capital remains low. This renders quantity (fertility) more attractive than quality (human capital investment), and thus drives population growth. Conversely, for states above $\mathcal{LL}$, the population contracts. A state above $\mathcal{ZZ}$ ($L_t > \mathcal{Z}(\bar{z}_t)$) indicates that the population is larger than the threshold size for technological progress, which encourages R&D. Conversely, for states below $\mathcal{ZZ}$, the technology stagnates.

Propositions 2.iii-2.v compare $\mathcal{L}$ with $\mathcal{Z}$. In the absence of modern technology, the threshold population for replacement-level fertility exceeds that for R&D participation ($\mathcal{L}(0) > \mathcal{Z}(0)$), which ensures the possibility of simultaneous population growth and technological progress. When modern technology tends to infinity, $\mathcal{L}(\bar{z}) \to 0$ due to the child quantity-quality trade-off. The behavior of $\mathcal{Z}(\bar{z})$ is more subtle, since no assumption has been imposed on the relationship between Δ and $\bar{z}$; the addition of Assumption 4 is required to guarantee that $\lim_{\bar{z}\to\infty} \mathcal{Z}(\bar{z}) > 0$.

4.2 Four Phases of Economic Development

Proposition 2 indicates that the economy typically transitions through four distinct phases—Malthusian, prosperous, aging, and stagnating—as shown in Figure 5 in the Introduction section, where the dotted line with circles plots the growth path of a typical economy and the solid line plots the two loci simulated with parametric specification (11). Appendix E provides details on the simulation.

The following Corollary describes the developmental trajectory.

Corollary 1. ***Under Assumptions 1-3****, commencing from an initial state with a low population ($L_0 < \mathcal{Z}(0)$) and the absence of modern technology ($\bar{z}_0 = 0$), the economy evolves sequentially through a Malthusian phase ($L_{t+1} > L_t$, $\bar{z}_{t+1} = \bar{z}_t$), a prosperous phase ($L_{t+1} > L_t$, $\bar{z}_{t+1} > \bar{z}_t$), and an aging phase ($L_{t+1} < L_t$, $\bar{z}_{t+1} > \bar{z}_t$). Furthermore,* ***under Assumption 4****, the economy transitions into a stagnating phase ($L_{t+1} < L_t$, $\bar{z}_{t+1} = \bar{z}_t$) before ultimately converging to a steady state ($L_{t+1} = L_t$, $\bar{z}_{t+1} = \bar{z}_t$).*

The economy begins in the Malthusian phase, in which the state lies below both $\mathcal{LL}$ and $\mathcal{ZZ}$ ($L \in (0, \min\{\mathcal{Z}(\bar{z}), \mathcal{L}(\bar{z})\})$) with no modern technology ($\bar{z} = 0$). In this phase, there exists no technological advancement or human capital investment. The fertility rate is high,

leading to continuous population growth.[16]

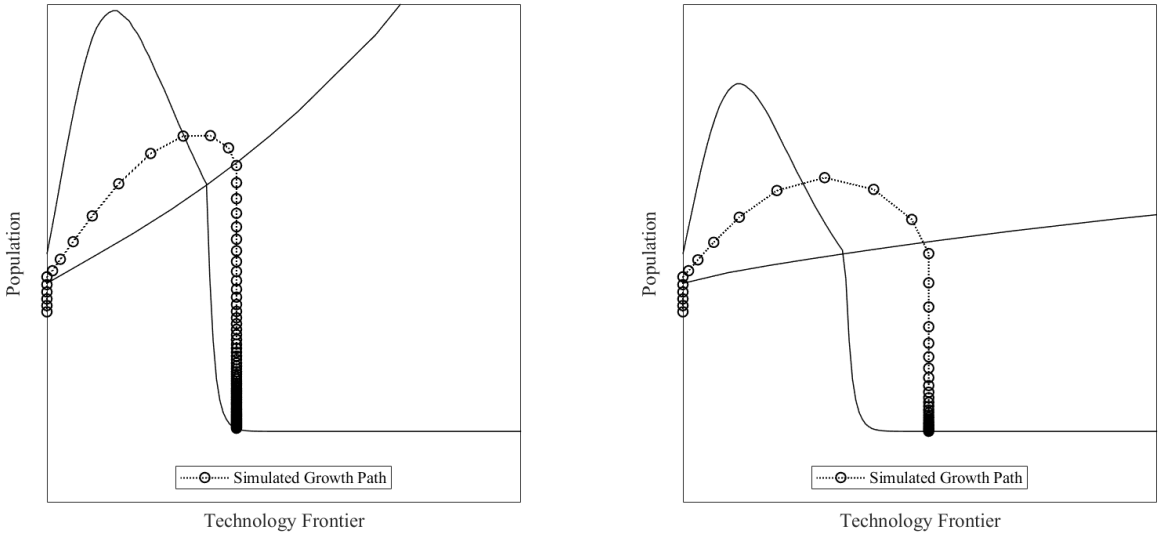
Once the population exceeds the threshold for R&D participation, the economy shifts to the prosperous phase, in which the state lies below $\mathcal{LL}$ but above $\mathcal{ZZ}$ ($L_t \in (\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t))$). During this phase, both technology and population grow, with human capital rising alongside increasing relative prices $s = r/w$. The fertility rate rises at first because the developed modern technology brings the cost of child-rearing $\tau(\bar{z})$ down, but declines afterward because of the increasing relative price of human capital.

As the fertility rate drops below the replacement level, the economy shifts to the aging phase, in which the state is above both $\mathcal{LL}$ and $\mathcal{ZZ}$ ($L_t \in (\max\{\mathcal{Z}(\bar{z}_t), \mathcal{L}(\bar{z}_t)\}, \infty)$). In this phase, technological progress continues, but the population starts to shrink. The age structure becomes skewed, with the number of older adults exceeding that of young adults in each period, which leads to the typical characteristics of an aging society. The pace of technological advances and growth in per capita income tends to slow, because the aging population fails to provide an attractive market size and a sufficient total human capital supply. The rise in the relative price of human capital then decelerates, stalls, and even reverses, bringing the decline in fertility to an end.

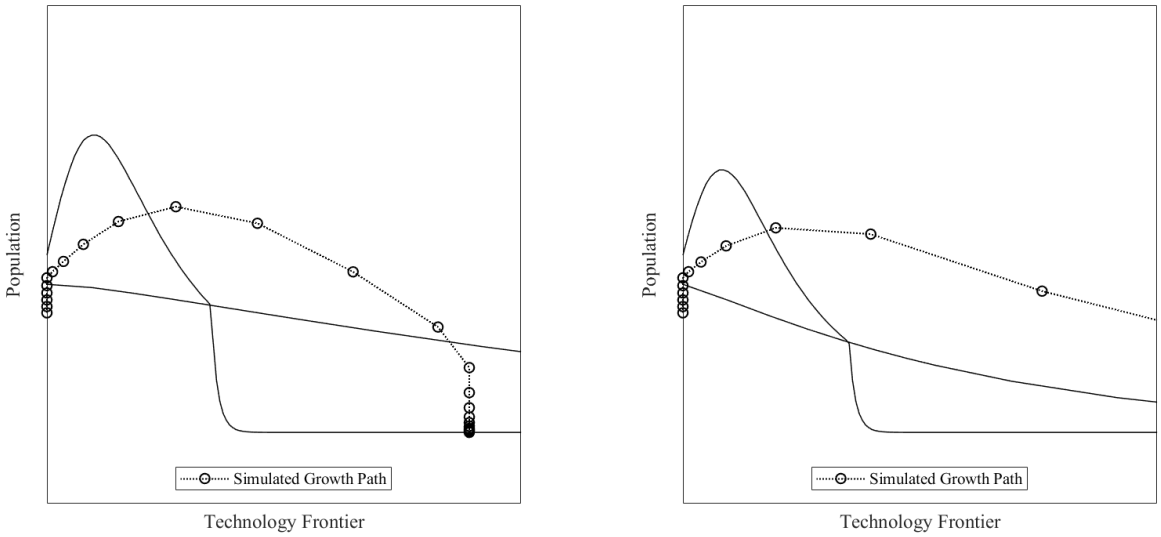
With the addition of Assumption 4, the population eventually falls below the threshold for technological progress, halting R&D. The economy then enters the stagnating phase, with the state above $\mathcal{LL}$ but below $\mathcal{ZZ}$ ($L_t \in (\mathcal{L}(\bar{z}_t), \mathcal{Z}(\bar{z}_t))$). In this phase, technological stagnation sets in while the population continues to decline. As the relative price of human capital decreases, the fertility rate slowly rebounds. Once it returns to the replacement level, the economy reaches a steady state.

We emphasizes that the dynamics before the aging phase in our main results do not depend on Assumption 4, which only influences the long-run dynamics once the economy has entered the aging phase. This point is further illustrated by Figure 6, which plots phase diagrams under different values of ψ in our parametric example (11). We highlight three observations. First, population aging arises inevitably regardless of whether Assumption 4 holds ($\psi \leq 1$ or $\psi > 1$). Second, when Assumption 4 is satisfied ($\psi \leq 1$), the economy

[16]If an exogenous technology shock happens to a Malthusian economy, it shifts to a state below both $\mathcal{LL}$ and $\mathcal{ZZ}$ but possesses some modern technology (to the right of the y-axis in Figure 5). We call these states quasi-Malthusian. A quasi-Malthusian economy moves similarly to a Malthusian one, with expanding population and no technological progress.



(a) $\psi = 0$ (b) $\psi = 1$



(c) $\psi = 2$ (d) $\psi = 3$

Figure 6: Phase Diagram with Different R&D production Technology

necessarily converges to a stagnating phase, as stated in Corollary 1 and illustrated in Figures 6(a) and 6(b). Third, without Assumption 4 ($\psi > 1$), two distinct long-run outcomes are possible: The economy may either stagnate (Figure 6(c)) or exhibit explosive growth (Figure 6(d)).

5 Causes for Inevitable Aging

This section shows that the inevitability of population aging is rooted in the interaction between the price-determined child quantity-quality trade-off and endogenous R&D. Proposition 2 indicates that population aging is inevitable. The following corollary summarizes this result.

Corollary 2. ***Under Assumptions 1-3****, population aging is inevitable following the prosperous phase.*

Inevitable population aging is straightforward by Figure 5: While the slope of the growth path is upward during prosperity, the slope of population balance locus $\mathcal{LL}$ becomes downward as technology advances. Eventually, the growth path must intersect $\mathcal{LL}$ from below to above, triggering the transition to the aging phase.

The economic mechanism driving inevitable population aging is a self-reinforcing negative feedback loop generated by the interaction between endogenous R&D and a price-determined quantity-quality trade-off, as illustrated in Figure 7. In the prosperous phase, a larger population ((a) in Figure 7) raises aggregate demand for new products, thereby increasing monopoly profits for R&D (b1), and, holding per capita human capital fixed, expands the aggregate supply of human capital, thereby lowering the cost of R&D (b2). Both forces expand R&D activity (c). This expansion increases R&D's demand for human capital (d1) and thus generates a rise in its relative price (e). The price-determined quantity-quality trade-off then translates this rising relative price into lower fertility and lower future population growth (f). Consequently, any increase in population size triggers equilibrium responses that raise the relative price of human capital, reduce fertility, and dampen subsequent population growth. This negative feedback loop does not cease until population size starts to contract, at which point fertility falls below replacement. Population aging is then inevitable.

In our model, this negative feedback loop arises from the interaction of two building blocks, as shown by Figure 7. Endogenous R&D links a growing population to an increase in the relative price of human capital, and a price-determined quantity-quality trade-off links this higher relative price to a reduction in future population growth. In what follows, we conduct two exercises to show that inevitable population aging is indeed rooted in these two mechanisms and to compare our model with representative models from two related strands

of the literature on economic growth.

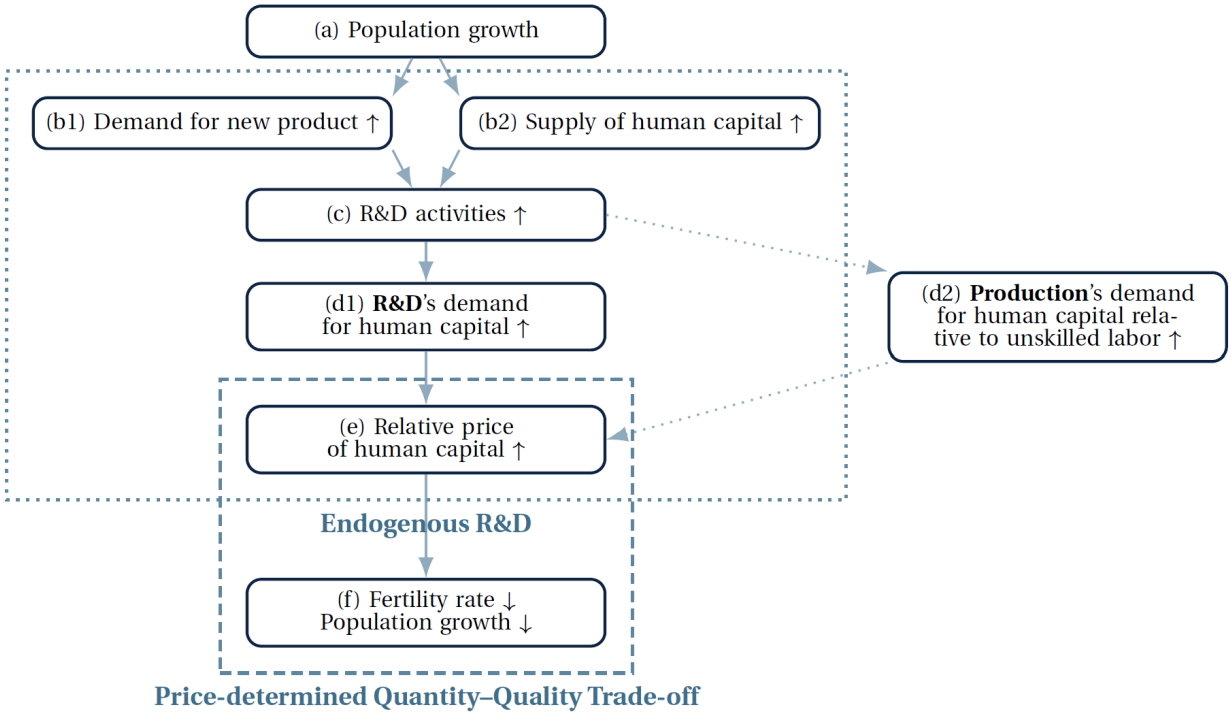


Figure 7: Negative Feedback Loop

5.1 Endogenous R&D

The first exercise focuses on how a growing population leads to a rising relative price of human capital within the feedback loop. In our model, the increase in relative price of human capital is driven by two distinct channels: (i) the expansion of R&D activity itself directly raises the demand for human capital ((d1) in Figure 7), and (ii) R&D-induced technological progress raises the demand for human capital relative to unskilled labor in the production sector (d2). While channel (ii) is already present in models with a price-determined quantity-quality trade-off in the literature (e.g., Galor and Mountford, 2008), channel (i) is novel and arises from our introduction of endogenous R&D. In the absence of channel (i), population aging in these earlier frameworks is a contingent outcome, depending on specific assumptions on production technology that affects the demand for human capital. By contrast, once channel (i) is operative in our model, population aging becomes inevitable even if channel (ii) is shut

down.[17] We now demonstrate this claim.

In the benchmark model, production takes place in two sectors: a traditional sector that employs unskilled labor and a modern sector that employs human capital. Technological progress, in the form of an expansion in the variety of modern goods, affects only the modern sector. Since the goods produced by the modern and traditional sectors are substitutes ($\rho \in (0,1)$), technological progress increases the demand for human capital compared to unskilled labor in the production sector, thereby raising the relative price of human capital. We call this effect of technological progress on the relative price through the production sector as a "production-sector" effect.

To mute this production-sector effect, we assume that the total factor productivity of the traditional sector (A_t) grows with the technology frontier of the modern sector:

$$A_t = A_0 + \tilde{A} \cdot \bar{z}_{t-1}^{(1-\rho)/\rho}, \tag{16}$$

where $A_0 > 0$ is the productivity level in the absence of modern technology, and $\tilde{A} > 0$ is a constant. Under this specification, technological progress also raises the demand for unskilled labor, thereby preventing technological progress from altering the relative price of human capital through the production sector, as demonstrated in the following lemma.

Lemma 4.iii'. *Under Assumptions 1-3, with the total factor productivity of the traditional sector following Eq.(16),* $\lim_{\bar{z}\to\infty} s_2(L,\bar{z}) = 0$*, for* $\forall L \leq \mathcal{Z}(\bar{z})$.

Lemma 4.iii' shows that, with the alternative production setting, when modern technology is sufficiently developed, the relative price of human capital is no longer affected by the technology frontier in the no-R&D regime.[18] This can be directly compared with Lemma 4.iii, which shows that in the benchmark model, the relative price of human capital always increases with the modern technology frontier in an absence of R&D activity. Our alternative production technology (Eq.(16)) is directly comparable to that in Galor and Mountford (2008) in the sense that, when technology is sufficiently advanced, technological progress no longer affects the relative price of human capital through the production sector.

[17] In the benchmark model, we retain channel (ii) to align our model predictions with other stylized facts.

[18] This result holds only asymptotically (i.e., for a sufficiently advanced frontier) because of the constant term $A_0 > 0$ in Eq.(16), which guarantees that traditional production is well defined in the Malthusian phase without modern technology.

Corollary 2'. *Under Assumptions 1-4, with the total factor productivity of the traditional sector following Eq.(16), population aging is inevitable following the prosperous phase.*

Corollary 2' shows that population aging remains a robust outcome in this alternative setting.[19] It establishes that inevitable aging is driven by the innovation process itself: Expanding R&D activity raises the demand for human capital in the R&D sector and thereby increases its relative price, independently of the production-sector effect. Consequently, inevitable population aging in our framework does not hinge on specific assumptions on production technology.

Growth-accounting analyses provide supportive evidence for our model, where expanding R&D activity raises demand for human capital and thus its relative price, independently of the production-sector effect. Jones (2002) documents a several-fold rise in the share of scientists and engineers in postwar U.S. employment (R&D intensity), showing that this rising research effort accounts for a large fraction of economic growth. Fernald and Jones (2014) update and broaden this picture, showing marked increases in R&D spending as a share of GDP not only in advanced economies but also in major emerging economies such as Japan, South Korea, and China.

5.2 Price-Determined Child Quantity-Quality Trade-off

The second exercise focuses on how a rising relative price of human capital leads to declining fertility within the feedback loop. Parents treat this relative price as a signal when choosing along the child quantity-quality trade-off: As the relative price rises, they invest more in each child's human capital and reduce fertility. This price-determined quantity-quality trade-off generates an externality: The relative price reflects the scarcity of skilled relative to unskilled labor in the labor market, but fails to capture three broader equilibrium effects of fertility on (1) the aggregate demand (market size) and thus R&D profits; (2) the total supply of human capital, given per capita human capital, and thus R&D costs; and (3) the future population, which sustains growth. Parents do not consider these broader effects when making the quantity-quality trade-off based on the relative price. This externality, emphasized by Jones

[19] After muting the production-sector effect, inevitable population aging requires slightly stronger assumptions on R&D—namely, additional Assumption 4. However, our R&D production function still accommodates both scale-effect specifications and those that eliminate scale effects even under Assumption 4.

(2022), is amplified in our setting because it distorts not only the fertility-consumption but also the quantity-quality trade-offs. We now demonstrate that, while preserving all other model features except that the quantity-quality trade-off is no longer determined by the relative price, our predictions return to those of standard endogenous R&D models.

Specifically, we formulate a constrained planner's problem as a dynastic optimization in which a representative family head chooses consumption, savings, fertility, and investment in child human capital for all future generations, fully aware of the implications of these decisions for prices, monopoly profits, and technological progress. Given initial conditions $(L_0, L_1, h_1, \bar{z}_0)$, the family head born in period 0[20]

$$\max_{\substack{\{c^y_{M,t}(z), c^o_{M,t}(z)\}_0^{\bar{z}_t}, \\ c^y_{T,t}, c^o_{T,t}, x_t, n_t, e_t, h_t, \pi_t, r_t, L_t, \bar{z}_t}} \sum_{t=1}^{\infty} \left(\frac{\beta\gamma}{1+\beta}\right)^t \left[v\left(c^y_{T,t}, \{c^y_{M,t}(z)\}_0^{\bar{z}_t}\right) + \frac{1+\beta}{\gamma} v\left(c^o_{T,t}, \{c^o_{M,t}(z)\}_0^{\bar{z}_t}\right) + \beta\gamma \ln(n_t) \right], \tag{17}$$

subject to five equilibrium conditions that preserve all other features in our baseline model:

1. The aggregate budget constraint: $\forall t \geq 1$,

$$r_t \int_0^{\bar{z}_{t-1}} \left(L_t c^y_{M,t}(\bar{z}) + L_{t-1} c^o_{M,t}(\bar{z})\right) \mathrm{d}\bar{z} + \frac{r_t}{\rho} \int_{\bar{z}_{t-1}}^{\bar{z}_t} \left(L_t c^y_{M,t}(\bar{z}) + L_{t-1} c^o_{M,t}(\bar{z})\right) \mathrm{d}\bar{z}$$
$$+ L_t c^y_{T,t} + L_{t-1} c^o_{T,t} = A X^{1-\alpha} \left[(1 - n_t(\tau(\bar{z}_t) + e_t)) L_t\right]^{\alpha} + r_t (1 - n_t(\tau(\bar{z}_t) + e_t)) L_t h_t,$$

   where the left-hand side represents total expenditure on previously developed modern goods, newly developed modern goods, and traditional goods. The right-hand side captures aggregate income from the traditional sector and modern/R&D sectors combined.

2. Solving the monopolists' profit maximization problem (8) and imposing market-clearing conditions for modern goods, we have: $\forall t \geq 1$,

$$\pi_t = C' \cdot \left[(1 - n_t(\tau(\bar{z}_t) + e_t)) L_t\right]^{\alpha} \cdot r_t^{-\frac{\rho}{1-\rho}},$$

   where C' is a positive constant.

[20] We derive the dynastic utility function (Eq.(17)) from Eq.(5), which is equivalent to the parental utility function in our baseline model (Eq.(5)) when $\gamma_n = (1+\beta)\gamma_v = \gamma < 1/\beta$. Therefore, any differences between our baseline model and the constrained planner's problem do not stem from differences in the utility function.

3. Deriving optimal R&D entry decisions and combined with human capital market clearing, we have: $\forall t \geq 1$,

$$(1-\rho)^{-1} \cdot \left[\rho^{-\frac{\rho}{1-\rho}} \bar{z}_{t-1} + \Delta\left(\bar{z}_{t-1}, \Phi\left(\frac{\pi_t}{r_t}, \bar{z}_{t-1}\right)\right)\right] \frac{\pi_t}{r_t} = (1 - n_t(\tau(\bar{z}_t) + e_t))L_t h_t,$$

where the left-hand side represents aggregate demand for human capital (increasing in previously developed variety $\bar{z}_{t-1}$, newly developed variety Δ, and monopoly profit π_t, and decreasing in cost of human capital r_t), while the right-hand side shows aggregate supply. Function $\Phi(\cdot,\cdot)$, defined in Lemma 2, captures optimal R&D firm entry decisions.

4. The human capital production function (4).

5. The law of motion for population (12) and the law of motion for the technology frontier: $\forall t \geq 1$,

$$\bar{z}_t = \bar{z}_{t-1} + \Delta\left(\bar{z}_{t-1}, \Phi\left(\frac{\pi_t}{r_t}, \bar{z}_{t-1}\right)\right).$$

Remark. In both the baseline model and the constrained planner's problem, parents make quantity-quality trade-off decisions, but the underlying determinants differ. In the baseline model, parents take relative prices of human capital as given and base their decisions exclusively on them. By contrast, in the constrained planner's problem, parents are fully aware of how their choices affect the supply side for all subsequent generations and internalize these effects when making quantity-quality trade-off decisions.

To situate our analysis within the literature for comparison (e.g. Romer, 1990; Jones, 1995a), we use the parametric specification (11) (Section 2.3). As noted earlier, this specification is a variant of the standard one commonly used in the literature.

The following proposition characterizes the solution of the constrained planner's problem.

Proposition 3. *Under Assumptions 1-3, with Δ and ω specified as in (11), the following results hold:*

i. *For $\psi < 1$, if $\underline{\tau}$ is sufficiently small*[21]*, there exists an asymptotic balanced growth path*

[21] Specifically,

$$\underline{\tau} \leq \frac{\tilde{\beta}(\gamma_1 - 1)}{1 + \tilde{\beta}\gamma_1},$$

that solves the constrained planner's problem, characterized by sustained population growth and technological progress. Specifically, there exists $\bar{n} > 1$ *such that with some initial states*

$$\lim_{t\to\infty} L_t/L_{t-1} = \bar{n}, \ \lim_{t\to\infty} \bar{z}_t/\bar{z}_{t-1} = \bar{n}^{\frac{\kappa}{1-\psi}}.$$

ii. *For* $\psi = 1$*, if* $\underline{\tau}$ *is within a specific range,*[22] *there exists an asymptotic balanced growth path that solves the constrained planner's problem, characterized by a constant population and sustained technological progress. Specifically, there exists* $\bar{L} > 0$, $\bar{g} > 1$ *such that with some initial states*

$$\lim_{t\to\infty} L_t = \bar{L}, \ \lim_{t\to\infty} \bar{z}_t/\bar{z}_{t-1} = \bar{g}.$$

iii. *For* $\psi > 1$*, no asymptotic balanced growth path exists with both positive population and sustained technological progress.*

Proposition 3.i demonstrates that when $\psi < 1$—namely, holding resources devoted to R&D constant, the growth rate of technology declines as the technology frontier advances—the constrained planner's solution features an asymptotic balanced growth path with sustained growth in both population and technology. The technological progress rate is endogenously determined by the optimal population growth rate $\bar{n}$. Proposition 3.ii states that when $\psi = 1$—namely, R&D activities show the standard scale effect—the constrained planner's solution converges to a balanced growth path with population stabilization: Population converges to a constant level while technological progress continues at a sustained rate. In both cases, the balanced growth path exists only in an asymptotic sense because we require $a > 0$ and $b > 0$ in the R&D production function. Proposition 3.iii states that when $\psi > 1$—namely, there exists a technology singularity—no balanced growth path exists that simultaneously maintains positive long-run population and sustained technological progress. The solution may yield explosive technological growth coupled with population expansion,

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where $\tilde{\beta} = \beta\gamma/(1+\beta) \in (0,1)$, $\gamma_1 = (1+\beta)(1+\gamma)/(1+\beta+\gamma) > 1$.

[22] Specifically,

$$\frac{\tilde{\beta}\gamma_1}{1+\tilde{\beta}\gamma_1} - \epsilon^{-1}\left(\tilde{\beta}^{-1}+\gamma_1\right) < \underline{\tau} \leq \frac{\tilde{\beta}((\gamma_1+\zeta\gamma_2)/(1+\zeta)-1)}{1+\tilde{\beta}(\gamma_1+\zeta\gamma_2)/(1+\zeta)},$$

where $\gamma_2 = (1+\beta)/(1+\beta-\beta\gamma) > \gamma_1$, $\zeta = [(1-\rho)/\rho]\cdot[(1+\beta)/(1+\beta-\beta\gamma)] \in (0,\infty)$, and $\epsilon^{-1}(\cdot)$ is the inverse function of $\eta'(\cdot)/\eta(\cdot)$.

or stable technological progress coupled with population decline.

Proposition 3 yields two key implications. First, the solution to the constrained planner's problem differs fundamentally from the baseline equilibrium. In contrast to the baseline equilibrium, aging is no longer inevitable in the solution of constrained planner's problem, where long-run technological progress can be sustained without a technological singularity. This finding demonstrates that the price-determined quantity-quality trade-off and the resulting externality are fundamental for inevitable population aging in the baseline equilibrium.

Second, the solution to the constrained planner's problem is consistent with the standard endogenous R&D literature. When $\psi < 1$, the parametric specification (11) generates an R&D production function similar to Jones (1995a, 2002, 2022). Correspondingly, our solution mirrors their results: Sustained technological progress requires continued population growth, with the technological progress rate determined by the population growth rate. When $\psi = 1$, the specification resembles early endogenous growth models such as Romer (1990); Grossman and Helpman (1993); Aghion and Howitt (1992). Our results similarly show that constant population levels can sustain technological progress through the scale effect, with the technological progress rate determined by the population level rather than its growth rate. This result demonstrates that the difference between our results and those of existing endogenous R&D models stems from endogenizing population dynamics through a price-determined quantity-quality trade-off.

6 Alignment with Stylized Facts

This section briefly reviews how our model's predictions align with the stylized facts.

Population and economic growth. Our model prediction (Figure 1(b)) is closely aligned with the historical patterns of population and economic growth observed (Figure 1(a)). In Figure 1(b), the curve with circle markers plots the simulated growth rate of population, while the curve with star markers shows the simulated growth rate of GDP; the difference between these two curves thus represents the simulated growth rate of GDP per capita. In both panels, vertical dotted lines divide the distinct phases of economic development identified in our theoretical framework.

Figure 1(b) shows that during the Malthusian phase, simulated population growth re-

mains positive while per capita income growth stays near zero. In the prosperous phase, fertility first rises and then falls sharply, accompanied by accelerating growth in both GDP and per capita income. Once fertility falls below the replacement level—marking the transition to the aging phase—per capita income growth slows immediately, though it remains positive. In contrast to the model's prediction, the observed data show that population growth stays mildly positive even after fertility drops below replacement. This discrepancy arises from continuously rising life expectancy in reality, a factor not incorporated in our model.

Human capital. The model's prediction of human capital dynamics (Figure 3(b)) aligns well with the data (Figure 3(a)). In Figure 3(b), the gray bars depict the level of human capital per capita, while the line with circle markers indicates its growth. During the Malthusian phase without modern technology, human capital does not accumulate. In the prosperous phase, accumulation begins and accelerates as technological progress raises its relative price. However, once the economy enters the aging phase, human capital growth decelerates, stalls, or even reverses. This decline arises because a shrinking young population slows technological progress, reducing or reversing the rise in the relative price of human capital.

Fertility-income gradient. Population economics has long been interested in the relationship between fertility and per capita GDP (the fertility-income gradient). It is well established that this gradient shifts from positive in the Malthusian phase to negative in the prosperous phase. Strikingly, Doepke et al. (2023) document a further shift—from negative to slightly positive—in many developed economies in recent decades. Yet no theoretical study has thus far systematically examined this new reversal.

Interestingly, our model's prediction of the fertility-income gradient captures not only the historical shifts but also the recent reversal documented by Doepke et al. (2023). Figure 4 contrasts the simulated gradients (Figure 4(b)) with empirical ones over six decades for OECD and Asian Four economies (Figure 4(a)). In Figure 4(b), the solid line depicts the simulated fertility-income gradient, and shows that the gradient indeed turns positive in the later aging phase.

This reversal of the fertility-income gradient is straightforward in our model. Lemma 4 in Section 3 implies that both state variables—the population and the technology frontier—

affect the relative price of human capital. In the later phase of population aging, technological progress slows, making population dynamics the main driver of fertility changes. Population dynamics are self-regulating: When the young population shrinks, the relative price of human capital falls, causing fertility to rebound. With per capita income growth remaining positive, the fertility-income gradient therefore turns positive.

7 Policy Implications

This section explores the policy implications of the model. We examine two classes of policies widely discussed in the literature: Population policy, which aims to regulate population growth, and industrial policy, which aims to promote R&D and technological progress.

7.1 Population Policy

Our analysis of the constrained planner's problem in Section 5.2 has direct implications for population policy. Throughout Section 5.2, we assume that the household head born in period 0 fully internalizes the effects of fertility and investment in children's human capital. We emphasize, however, that even without such a household head, the solution of the constrained planner's problem can be implemented by a budget-neutral government that taxes wage premium for human capital and subsidizes wages of unskilled labor or fertility (or vice versa) through the following policy scheme:

$$n_t^* = \frac{\beta\gamma}{1+\beta+\beta\gamma} \cdot \frac{1}{\tau(\bar{z}_t) + \varepsilon(s_{t+1}^*, \bar{z}_t)}; \tag{18}$$

$$s_{t+1}^* = \frac{(1-\mathcal{T}_{t+1}^r) r_{t+1}}{(1-\mathcal{T}_{t+1}^w) w_{t+1}}; \tag{19}$$

$$\mathcal{T}_{t+1}^r r_{t+1} + \mathcal{T}_{t+1}^w w_{t+1} = 0. \tag{20}$$

Here, n_t^* denotes the fertility rate chosen by the constrained planner; s_{t+1}^* is the relative price of human capital that derives this fertility choice in equilibrium; and $\mathcal{T}_{t+1}^r$ and $\mathcal{T}_{t+1}^w$ denote tax rates on wage premium for human capital and wage rate for unskilled labor (or fertility), respectively (negative values indicate subsidies). Eq.(18) determines the target relative price needed to implement the constrained planner's allocation; Eq.(19) ensures that the tax-

subsidy scheme generates this target price; and Eq.(20) maintains budget balance. Thus, in our model, a properly designed population policy can at the same time avoid population aging and induce balanced growth in the long run.

7.2 Industrial Policy

To evaluate the effect of an industrial policy, we go back to the baseline model in Section 2. Suppose the government in period $t-1$ credibly announces an ad valorem subsidy on newly developed products in period t at rate $\mathcal{S}_t^p$, which will be financed by a lump-sum tax. In period t, a monopolist that introduces a new variety $z > \bar{z}_{t-1}$ then solves

$$\max_{H_t^p(z),P_t(z)} (1+\mathcal{S}_t^p)P_t(z)H_t^p(z) - r_t H_t^p(z),$$

subject to Eq.(9). From now on, superscript s denotes the equilibrium result under this subsidy. The following lemma characterizes the one-shot effect of such an industrial policy.

Lemma 5. *If $\mathcal{S}_t^p > 0$, under Assumptions 1-3, the following results hold:*

i. $\pi_t^s/r_t^s > \pi_t/r_t$.

ii. $r_t^s/w_t^s \geq r_t/w_t$, $L_t^s \leq L_t$, $h_t^s \geq h_t$, $\bar{z}_t^s \geq \bar{z}_t$, *with the inequalities holding strictly iff*

$$\frac{\pi_t^s}{r_t^s} > \frac{\omega(\bar{z}_{t-1})}{\delta_0(\bar{z}_{t-1})}.$$

Lemma 5.i is straightforward: An ad valorem subsidy on sales of newly developed products raises monopoly profits relative to the human-capital input cost, thereby strengthening incentives for R&D. Consequently, demand for human capital in both the R&D and production sectors rises. Lemma 5.ii states the relative price of human capital increases weakly and, provided there is positive R&D participation, strictly so. This yields three short-term effects: Population growth decelerates, human capital accumulation rises, and technological progress accelerates.

Lemma 5 indicates that industrial policy has dual effects on economic growth. On the one hand, it directly promotes R&D and accelerates economic growth; on the other hand, by reducing population—and given that population positively affects technological progress in our framework—it can hinder innovation in the long run. Consequently, when population is

endogenously determined by the child quantity-quality trade-off, the long-run effect of such industrial policy might be less positive than in other endogenous growth models (Romer, 1990).

To formalize this idea, suppose that the government announces an industrial policy that subsidizes newly developed products in each period t according to the rule $\mathcal{S}^p(L_{t-1}, \bar{z}_{t-1})$. The next proposition characterizes its effect on the two critical loci, $\mathcal{LL}^s$ and $\mathcal{ZZ}^s$:

Proposition 4. *If $0 \leq \mathcal{S}^p(\cdot,\cdot) < \infty$, the following results hold:*

i. *Proposition 2 holds for both the loci $\mathcal{LL}^s$ and $\mathcal{ZZ}^s$.*

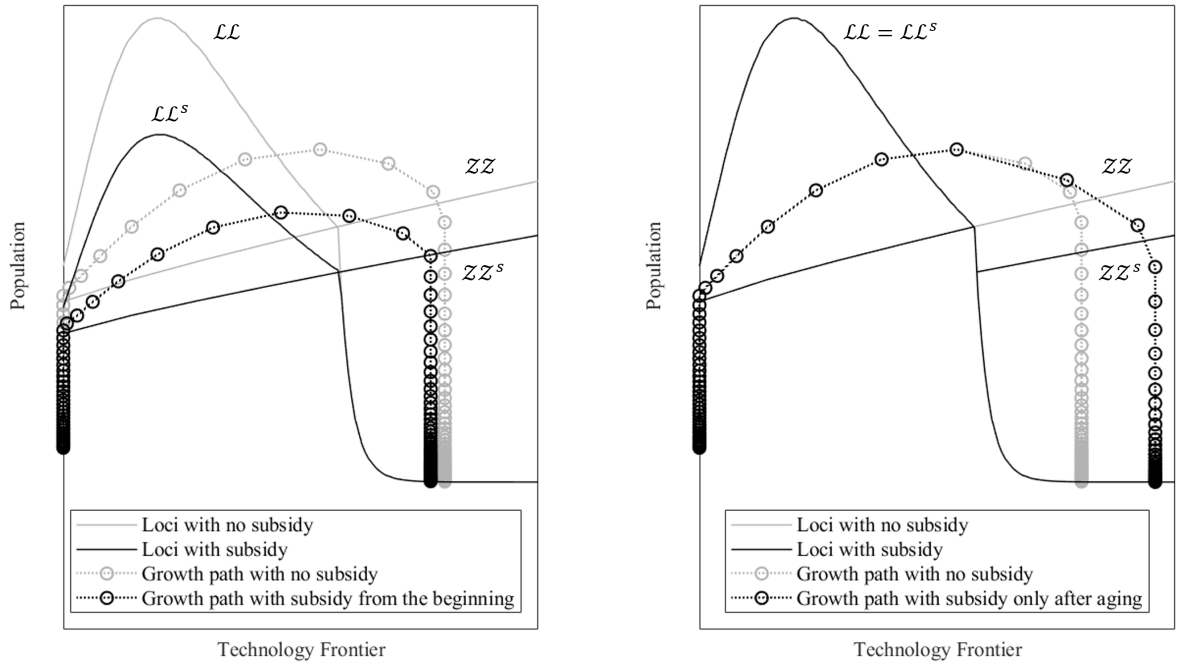
ii. *Under Assumptions 1-3, $\mathcal{Z}^s(\bar{z}_{t-1}) \leq \mathcal{Z}(\bar{z}_{t-1})$, with the inequality holding strictly iff $\mathcal{S}^p(\mathcal{Z}(\bar{z}_{t-1}), \bar{z}_{t-1}) > 0$.*

iii. *Under Assumptions 1-3, $\mathcal{L}^s(\bar{z}_{t-1}) \leq \mathcal{L}(\bar{z}_{t-1})$, with the inequality holding strictly iff $\mathcal{S}^p(\mathcal{L}(\bar{z}_{t-1}), \bar{z}_{t-1}) > 0$ and $\mathcal{L}^s(\bar{z}_{t-1}) > \mathcal{Z}^s(\bar{z}_{t-1})$.*

Proposition 4.i indicates that the industrial policy does not qualitatively change the dynamics of the economy: Starting from a Malthusian phase, the economy transitions through the prosperous, the aging, and probably the stagnating phases. In particular, unlike population policy, a pro-R&D industrial policy cannot prevent population aging.

Propositions 4.ii-4.iii compare the critical loci with and without the R&D subsidy. Based on simulation, Figure 8(a) plots the subsidized loci in black ($\mathcal{S}^p(L, \bar{z}) > 0$, $\forall L, \bar{z}$) and the baseline loci in gray ($\mathcal{S}^p(L, \bar{z}) = 0$, $\forall L, \bar{z}$). Proposition 4.ii states that $\mathcal{ZZ}^s$ lies below $\mathcal{ZZ}$, which implies that the R&D subsidy lowers the population threshold required for technological progress to occur. Proposition 4.iii states that the left arm of $\mathcal{LL}^s$—the portion that lies above $\mathcal{ZZ}^s$—shifts down relative to the left arm of $\mathcal{LL}$, while the right arms of $\mathcal{LL}^s$ and $\mathcal{LL}$ coincide. This indicates that the subsidy also lowers the population threshold for the onset of population aging.

We have four observations from Figure 8(a). First, the economic dynamics are qualitatively the same with or without the subsidy. Second, the economy exits the Malthusian phase earlier under industrial policy. Third, during the prosperous phase, the subsidized economy exhibits faster technological progress and slower population growth, which causes it to enter population aging at a lower population level. Fourth, in the long run, the negative population effect induced by the policy outweighs its positive technological effect, so the economy stagnates at a lower technology level.



(a) Subsidy from the Beginning

(b) Subsidy only after Aging

Figure 8: Phase Diagram with R&D Subsidy ($\mathcal{S}^p = 0.2$)

Different from Figure 8(a), Figure 8(b) considers an alternative scheme that subsidizes only after population aging sets in ($\mathcal{S}^p(L, \bar{z}) > 0$ only if $L > \mathcal{L}(\bar{z})$). By allowing population to reach its peak as in the no-subsidy economy, this scheme attenuates the negative population effect while still lowering the population threshold for R&D after entering the aging phase. Consequently, the economy attains a higher technology level before settling into stagnation.

As discussed above, industrial policy has two opposing effects on economic growth: It boosts R&D in the short run but reduces population and thereby hinders technological progress in the long run. The net impact depends on whether the technology effect or the population effect dominates, which in turn hinges on the specifics of the policy scheme. Nonetheless, we conclude that industrial policy cannot prevent population aging.

8 Conclusion

This paper develops a growth model that integrates the priced-determined child quantity-quality trade-off with endogenous R&D. We demonstrate that the economy evolves through four distinct phases, with population aging as an inevitable equilibrium outcome once an economy enters modern growth. The model predictions are consistent with the stylized facts observed across developed economies in recent decades, such as decelerating economic growth, persistently below-replacement fertility rate, slowing human capital accumulation, and the reversal of fertility-income gradients from negative to positive.

Our analysis shows that inevitable population aging in our framework is the outcome of a self-reinforcing negative feedback loop created by the interaction between a price-determined quantity-quality trade-off and endogenous R&D. A larger population stimulates R&D activity. This expansion increases R&D's demand for human capital and raises its relative price. Because the child quantity-quality trade-off is determined by this relative price, parents tilt towards fewer children and more human capital investment, pushing the fertility rate below the replacement level and slowing future population growth. Thus, the joint operation of endogenous R&D and the price-determined quantity-quality trade-off makes sustained growth with an ever-expanding population dynamically infeasible, rendering population aging inevitable.

Several extensions are worth pursuing. First, household decision-making could be enriched by incorporating marriage, old-age support, and bequests. Second, introducing physical capital accumulation would allow analysis of how savings behavior and capital-labor substitution interact with demographic transitions. Third, the innovation framework could be broadened to include alternative R&D approaches, such as quality improvements and skill-biased automation. While these extensions would add complexity to the model, they may also generate valuable insights.

References

Acemoglu, D. and Linn, J. (2004). Market Size in Innovation: Theory and Evidence from the Pharmaceutical Industry. *Quarterly Journal of Economics*, 119(3):1049–1090.

Acemoglu, D. and Restrepo, P. (2017). Secular Stagnation? The Effect of Aging on Economic Growth in the Age of Automation. *American Economic Review*, 107(5):174–179.

Aghion, P., Bergeaud, A., Lequien, M., and Melitz, M. J. (2024). The Heterogeneous Impact of Market Size on Innovation: Evidence from French Firm-level Exports. *Review of Economics and Statistics*, 106(3):608–626.

Aghion, P. and Howitt, P. (1992). A Model of Growth through Creative Destruction. *Econometrica*, 60(2):323–351.

Aksoy, Y., Basso, H. S., Smith, R. P., and Grasl, T. (2019). Demographic Structure and Macroeconomic Trends. *American Economic Journal: Macroeconomics*, 11(1):193–222.

Barro, R. J. and Becker, G. S. (1989). Fertility Choice in a Model of Economic Growth. *Econometrica*, 57(2):481–501.

Baudin, T. and De la Croix, D. (2024). *The Emergence of the Child Quantity-quality Tradeoff: Insights from Early Modern Academics*. Centre for Economic Policy Research.

Baudin, T., De la Croix, D., and Gobbi, P. E. (2015). Fertility and Childlessness in the United States. *American Economic Review*, 105(6):1852–1882.

Becker, G. S. and Barro, R. J. (1988). A Reformulation of the Economic Theory of Fertility. *Quarterly Journal of Economics*, 103(1):1–25.

Becker, G. S. and Lewis, H. G. (1973). On the Interaction between the Quantity and Quality of Children. *Journal of Political Economy*, 81(2, Part 2):S279–S288.

Becker, G. S., Murphy, K. M., and Spenkuch, J. L. (2016). The Manipulation of Children's Preferences, Old-Age Support, and Investment in Children's Human Capital. *Journal of Labor Economics*, 34(S2):S3–S30.

Becker, G. S., Murphy, K. M., and Tamura, R. (1990). Human Capital, Fertility, and Economic Growth. *Journal of Political Economy*, 98(5, Part 2):S12–S37.

Christiaans, T. (2011). Semi-endogenous growth when population is decreasing. *Economics Bulletin*, 31(3):2667–2673.

Cutler, D. M., Poterba, J. M., Sheiner, L. M., Summers, L. H., and Akerlof, G. A. (1990). An Aging Society: Opportunity or Challenge? *Brookings Papers on Economic Activity*, 1990(1):1–73.

De la Croix, D. (2013). *Fertility, Education, Growth, and Sustainability*. Cambridge University Press.

De la Croix, D. and Doepke, M. (2003). Inequality and Growth: Why Differential Fertility Matters. *American Economic Review*, 93(4):1091–1113.

De la Croix, D. and Gobbi, P. E. (2017). Population density, fertility, and demographic convergence

in developing countries. *Journal of Development Economics*, 127:13–24.

Dinopoulos, E. and Thompson, P. (1998). Schumpeterian Growth without Scale Effects. *Journal of Economic Growth*, 3:313–335.

Doepke, M. (2004). Accounting for Fertility Decline during the Transition to Growth. *Journal of Economic Growth*, 9:347–383.

Doepke, M. (2015). Gary Becker on the Quantity and Quality of Children. *Journal of Demographic Economics*, 81(1):59–66.

Doepke, M., Hannusch, A., Kindermann, F., and Tertilt, M. (2023). The economics of fertility: A new era. In *Handbook of the Economics of the Family*, volume 1, pages 151–254. Elsevier.

Favero, C. A., Galasso, V., et al. (2016). Demographics and the Secular Stagnation Hypothesis in Europe. In *After the Crisis. Reform, Recovery and Growth in Europe*, pages 109–146. Oxford University Press.

Fernald, J. G. and Jones, C. I. (2014). The future of US economic growth. *American Economic Review*, 104(5):44–49.

Galor, O. and Mountford, A. (2008). Trading Population for Productivity: Theory and Evidence. *Review of Economic Studies*, 75(4):1143–1179.

Galor, O. and Weil, D. N. (2000). Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond. *American Economic Review*, 90(4):806–828.

Goodfriend, M. and McDermott, J. (1995). Early Development. *American Economic Review*, pages 116–133.

Greenwood, J. and Seshadri, A. (2002). The US Demographic Transition. *American Economic Review*, 92(2):153–159.

Grossman, G. M. and Helpman, E. (1993). *Innovation and Growth in the Global Economy*. MIT Press.

Guo, R., Yi, J., and Zhang, J. (2022). The Child Quantity–Quality Trade-off. *Handbook of Labor, Human Resources and Population Economics*, pages 1–23.

Hansen, G. D. and Prescott, E. C. (2002). Malthus to Solow. *American Economic Review*, 92(4):1205–1217.

Howitt, P. (1999). Steady Endogenous Growth with Population and R&D Inputs Growing. *Journal of Political Economy*, 107(4):715–730.

Jones, C. I. (1995a). R&D-Based Models of Economic Growth. *Journal of Political Economy*, 103(4):759–784.

Jones, C. I. (1995b). Time Series Tests of Endogenous Growth Models. *Quarterly Journal of*

Economics, 110(2):495–525.

Jones, C. I. (2002). Sources of US Economic Growth in a World of Ideas. *American Economic Review*, 92(1):220–239.

Jones, C. I. (2022). The End of Economic Growth? Unintended Consequences of a Declining Population. *American Economic Review*, 112(11):3489–3527.

Kortum, S. S. (1997). Research, Patenting, and Technological Change. *Econometrica*, 65(6):1389–1419.

Liang, J., Wang, H., and Lazear, E. P. (2018). Demographics and Entrepreneurship. *Journal of Political Economy*, 126(S1):S140–S196.

Lutz, W., Testa, M. R., and Penn, D. J. (2006). Population density is a key factor in declining human fertility. *Population and Environment*, 28(2):69–81.

Maestas, N., Mullen, K. J., and Powell, D. (2023). The Effect of Population Aging on Economic Growth, the Labor Force, and Productivity. *American Economic Journal: Macroeconomics*, 15(2):306–332.

Morand, O. F. (1999). Endogenous Fertility, Income Distribution, and Growth. *Journal of Economic Growth*, 4:331–349.

Peters, M. (2022). Market Size and Spatial Growth—Evidence from Germany's Post-war Population Expulsions. *Econometrica*, 90(5):2357–2396.

Romer, P. M. (1990). Endogenous Technological Change. *Journal of Political Economy*, 98(5, Part 2):S71–S102.

Sasaki, H. and Hoshida, K. (2017). The Effects of Negative Population Growth: An Analysis Using a Semiendogenous R&D Growth Model. *Macroeconomic Dynamics*, 21(7):1545–1560.

Segerstrom, P. S. (1998). Endogenous Growth without Scale Effects. *American Economic Review*, 88(5):1290–1310.

Sheiner, L. (2014). The Determinants of the Macroeconomic Implications of Aging. *American Economic Review*, 104(5):218–223.

Tamura, R. (2002). Human Capital and the Switch from Agriculture to Industry. *Journal of Economic Dynamics and Control*, 27(2):207–242.

Weil, D. N. (1997). The economics of population aging. *Handbook of Population and Family Economics*, 1:967–1014.