The New Keynesian Term Structure of Interest Rates – work in progress –

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Abstract

This paper examines the term structure of interest rates and the key drivers of the risk premium, identifying the main factors that influence the premium investors require for bearing economic risks within a micro-founded New Keynesian model featuring Calvo pricing. We address two central questions: What are the key factors shaping the term premium and the inflation risk premium, and how can fundamental economic shocks be classified as "good" (reducing the premium) or "bad" (increasing it)? We present a novel global solution that improves our understanding of how investors price these macro risks in Treasury markets. We show that sticky prices are key for explaining the nominal term premium. By solving the representative investor's fundamental pricing equation and applying the Feynman-Kac formula, we compute moments of key variables for direct comparison with empirical data. Additionally, we connect our findings to affine term structure models and provide identifying restrictions for economic shocks in empirical work.

Keywords: Monetary economics, Calvo pricing, Term structure of interest rates
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1. Introduction

A series of macro-financial shocks—along with rising interest rates and high inflation—has sharply increased economic uncertainty and renewed focus on how investors price risks in financial markets. The term structure of interest rates, which maps bond yields to their maturities, has long been central to both macroeconomic theory and financial economics (see Gürkaynak and Wright, 2012; Borovička and Hansen, 2016). A key component is the term premium on sovereign bonds, which compensates investors for bearing inflation and consumption risk over the bond's lifetime. Identifying its determinants is challenging, as the yield curve is also shaped by monetary and fiscal policy. Monetary policy directly affects short-term interest rates, while forward guidance influences expectations across the entire yield curve (Rudebusch and Wu, 2008).[1] Fiscal policy impacts bond markets by altering borrowing needs and shaping expectations about long-run growth.[2] These policies shape investor expectations and saving behavior, and thereby influence term premia across maturities. A large body of empirical research shows that these premia are both significant and time-varying (Campbell and Shiller, 1991; Cochrane and Piazzesi, 2005; Cochrane, 2007; Gürkaynak, Sack, and Wright, 2007; Bekaert, Engstrom, and Ermolov, 2021), with inflation uncertainty as a key driver of bond market risk (Wright, 2011, 2012).

Understanding the determinants of the term premium requires a framework in which macroeconomic dynamics are aligned with a general equilibrium asset pricing kernel, thereby ensuring that the term structure reflects aggregate risk consistent with the joint behavior of consumption and inflation. By incorporating nominal rigidities (cf. Clarida, Galí, and Gertler, 2000; Woodford, 2001, 2003; Smets and Wouters, 2003), the New Keynesian (NK) framework provides a structural environment to analyze how aggregate and policy-induced shocks affect the risk premium on bonds in Treasury markets. Against this backdrop, our central research questions are: What are the key drivers of the term premium and the inflation risk premium? And how can we classify economic shocks as "good" (reducing the premium) or "bad" (increasing it)?

Despite substantial progress, most macro-finance models still rely on local approximations (Rudebusch and Swanson, 2008, 2012; Hördahl and Tristani, 2008; van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez, 2012; Andreasen, Fernández-Villaverde, and Rubio-Ramírez, 2018), limiting their ability to capture important nonlinearities and risk dynamics.[3] As a consequence, much of the literature abstracts from

[1] Gürkaynak, Sack, and Swanson (2005) and argue that the Fed has increasingly relied on unconventional policies to shape market perceptions of the expected path of short-term rates (cf. Backus and Wright, 2007; Christensen and Rudebusch, 2012). Backus, Chernov, Zin, and Zviadadze (2022) find that (discretionary) monetary policy is an important contributor to aggregate risk.

[2] Bansal and Shaliastovich (2013) show that bond risk premia rise with uncertainty about expected inflation and fall with uncertainty about expected growth. Corhay, Kind, Kung, and Morales (2023) argue that fiscal theory allows sovereign debt to influence expected inflation, even in a frictionless economy.

[3] If the underlying model exhibits economically meaningful nonlinearities, the Campbell–Shiller log-linearization can generate large numerical errors (cf. Pohl, Schmedders, and Wilms, 2018).

providing a structural account of the compensation investors demand for bearing fundamental economic risks, focusing instead on reduced-form indicators such as yield curve slopes or excess returns (Rudebusch and Swanson, 2012; Christiano, Motto, and Rostagno, 2014). To address this gap, we adopt a global solution approach capturing the full nonlinear structure of the model. This allows us to establish clear and direct link between model-implied risk premia and their empirical counterparts.

This paper aims to identify the key factors driving the term premium and the inflation risk premium, particularly in settings where core macroeconomic variables such as consumption and inflation are endogenous. Building on seminal contributions based on exogenous pricing kernels (Vasicek, 1977) and general equilibrium production models (Cox, Ingersoll, and Ross, 1985a,b), we adopt a framework in which consumption and inflation are integral components of the economic model (Rudebusch and Swanson, 2008; Bekaert, Cho, and Moreno, 2010; Kaltenbrunner and Lochstoer, 2010). We show how aggregate shocks, and their interaction with monetary and fiscal policy, shape the term premium—thus addressing our core question concerning "good" and "bad" fundamental shocks. The inflation risk premium is obtained as the difference between the term premia on nominal and real Treasury bonds.[4] For instance, we show that monetary policy shocks, while generating "good" inflation (i.e., reducing the inflation risk premium), actually *increase* the riskiness of nominal bonds by raising the term premium. This occurs because the term premium on long-term real bonds rises more than that on nominal bonds. In contrast, technology shocks (supply-side shocks, which move output and inflation in opposite directions), preference shocks, and discount rate shocks (both demand-side) increase the risk premium for both asset types, *and* the inflation risk premium. Our modular approach allows for scalability with respect to the number of shocks, and our illustrative examples demonstrate that the qualitative patterns remain robust—even when extending the model with additional shocks or a more complex correlation structure.

This work contributes to the literature in several ways. First, it develops a novel global solution for a macro-finance term structure model that incorporates recursive preferences (Kreps and Porteus, 1978; Duffie and Epstein, 1992) and is embedded in a NK framework with Calvo pricing (Calvo, 1983), thereby overcoming the limitations of local approximations. Our analysis identifies the sources of economic risk that amplify or mitigate the premium demanded by investors in Treasury markets. By solving the fundamental pricing equation (PDE), we compute moments of key macroeconomic and financial variables using the Feynman-Kac formula, allowing for direct comparison with empirical data.

Second, the paper highlights how endogenous consumption and inflation dynamics shape the yield curve by quantifying the sensitivity of bond prices to fundamental shocks. We classify these shocks according to whether they induce favorable or unfavorable shifts in

[4]Cieslak and Pflueger (2023) study the inflation risk premium in a reduced-form NK model and argue that "good" inflation can be linked to demand shocks, and "bad" inflation to cost-push shocks.

the term structure and compute their price elasticities. This approach links the behavior of the stochastic discount factor (SDF) to variations in the term premium across maturities, offering structural insights into the pricing of fundamental risks and providing identifying restrictions for empirical applications.

Financial market data offers a powerful lens through which to discipline and guide macroeconomic modeling. By focusing on shocks and preferences that are reflected in asset prices, we can better identify the risks and premia investors are willing to pay to hedge uncertainty in consumption and portfolio decisions. Haubrich, Pennacchi, and Ritchken (2012) decompose Treasury yields and analyze the term structures of real rates, expected inflation, and inflation risk premia. Building on existing macro-finance models, our framework incorporates long-run risks (Bansal and Yaron, 2004), a time-varying inflation objective (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2012), and shocks to preferences and the discount rate (see also Gürkaynak, Sack, and Wright, 2007; Gürkaynak and Wright, 2012). We view the representative agent model as a useful benchmark: it reveals how uncertainty about structural parameters, policy instruments, and risk attitudes is encoded in market prices, while still permitting analytical solutions in the frictionless limit—for instance, under long-run risk or monetary policy shocks, or when assuming unit intertemporal elasticity or non-recursive preferences. Our analysis focuses on equilibria with active monetary policy, where the instrument set includes long-end rate targets. The liquidity trap scenario has been studied in related work (Werning, 2012; Cochrane, 2017; Wieland, 2019). Further research should include financial frictions in the NK framework (cf. He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014), and extend the analysis to heterogeneous-agent New Keynesian (HANK) models, in which the asset pricing kernel is endogenously determined by joint saving and portfolio choices (Gabaix, Lasry, Lions, and Moll, 2016; Achdou, Han, Lasry, Lions, and Moll, 2021).

The remainder of the paper is organized as follows. Section 2 outlines the nonlinear NK model with shocks and its flexible-price counterpart. Section 3 presents the main results, discussing the role of uncertainty, the classification of shocks, and the asset pricing implications for the term structure of interest rates. Section 4 concludes.

2. The Nonlinear New Keynesian Model with Shocks

We describe now the environment for our investigation. It is the continuous-time version of the standard NK model (cf. Woodford, 2003). We summarize the equilibrium dynamics, show how to compute impulse responses, compute the effects of uncertainty, and how to solve the model in the policy function space. Throughout the paper, we keep the nonlinear structure of the model, which turns out to be quite relevant for non-normal times when considering large shocks and/or large deviations from the point of approximation.

2.1. The Model

The basic structure of the model is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, who uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good, and face the constraint that they can only adjust the price following Calvo's pricing rule (Calvo, 1983). Finally, there is a monetary authority that fixes the short-term nominal interest rate through open market operations with public debt, and a fiscal authority that taxes and consumes. We introduce four stochastic shocks, one to preferences (which can be loosely interpreted as a shock to aggregate demand, temporarily affecting the real interest rate), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. For simplicity, we do not explicitly model a shock to the inflation target, which is considered a policy instrument under the discretion of the central bank.

2.1.1. Households

Preferences. Consider an economy with a single consumer, interpreted as a representative of a large number of identical consumers. The representative consumer maximizes lifetime recursive utility (cf. Svensson, 1989; Duffie and Epstein, 1992),

$$U_0 \equiv \mathbb{E}_0 \int_0^\infty f(\mathbb{Z}_t, \mathbb{X}_t, U_t)\,\mathrm{d}t \tag{1}$$

in which $\mathbb{Z}_t \in \mathbb{U}_z$ denotes the n-vector of states, $\mathbb{X}_t \in \mathbb{U}_x$ denotes the m-vector of controls, and $f : \mathbb{U}_z \times \mathbb{U}_x \times \mathbb{R} \to \mathbb{R}$ is a normalized aggregator.

In what follows, we introduce preferences shocks d_t and variation in subjective discount rates r_t. Suppose that $\log d_t$ follows an Ornstein-Uhlenbeck (OU) process,

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{2}$$

It allows analytical tractability and ensures that d_t cannot become negative. The process $B_{d,t}$ is a standard Brownian motion for $\log d_t$, or equivalently,

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other exogenous stochastic processes, we will use both the formulation in level and in logs depending on the context and ease of notation. In addition to preference shocks, we allow for time-variation in discount rates r_t,

$$\mathrm{d}r_t = \rho_r(\rho - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} \tag{3}$$

with $\rho > 0$ denoting the subjective rate of time preference, with small but positive mean reversion ρ_r (near random walk) such that $r_{t+\Delta} \approx r_t + \sigma_r(B_{r,t+\Delta} - B_{r,t})$ has the ability to capture more persistent changes in the subjective time preference.

Suppose $\mathbb{X}_t = (c_t, l_t)^\top$ are the controls, and the *normalized aggregator* reads

$$f(\mathbb{Z}_t, \mathbb{X}_t, U_t) = d_t \left(\frac{c_t^{1-1/\eta}}{1-1/\eta} - \psi \frac{z_t^{1-1/\eta} l_t^{1+\vartheta}}{1+\vartheta} \right) ((1-\gamma) r_t U_t)^{1-\frac{1-1/\eta}{1-\gamma}} - \frac{1-\gamma}{1-1/\eta} r_t U_t, \quad (4)$$

for $\gamma \neq 1$ and $\eta \neq 1$,

$$\lim_{1/\eta \to 1} f(\mathbb{Z}_t, \mathbb{X}_t, U_t) = d_t \left(\log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right) (1-\gamma) r_t U_t - \log((1-\gamma) r_t U_t) r_t U_t, \quad (5)$$

or

$$f(\mathbb{Z}_t, \mathbb{X}_t, U_t) = d_t \left(\log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right) - r_t U_t, \quad \text{for } \gamma = 1/\eta = 1, \quad (6)$$

where z_t captures a stochastic trend, and $\gamma > 0$ reflects risk aversion, and $\eta > 0$ is the elasticity of intertemporal substitution.[5] Our normalized aggregator in (4) preserves intertemporal consistency, but can relax the assumption of time-additivity (Kreps and Porteus, 1978; Duffie and Epstein, 1992; Rudebusch and Swanson, 2012).[6] Moreover, our specification in (4) follows additively separable utility kernel, in line with little empirical evidence of nonseparability between consumption and leisure. We assume that the consumption level enters the utility kernel and hence introduce a productivity trend z_t to restore the balanced-growth property (cf. Rudebusch and Swanson, 2012).

Let a_t denote real financial wealth, the household's real wealth evolves according to

$$\mathrm{d}a_t = \left(r_t^b a_t - c_t + w_t l_t + T_t + \digamma_t \right) \mathrm{d}t, \quad (7)$$

in which $r_t^b \equiv i_t - \pi_t$ where i_t is the nominal interest rate on government bonds, π_t the rate of inflation of the price level p_t (or price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

2.1.2. The final good producer

There is one final good produced using intermediate goods with the production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} \, \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad (8)$$

[5] Note that the coefficient of relative risk aversion is different in a utility function that includes labor. This results extends to the case of recursive preferences and is used to define the coefficient of relative risk aversion in our setting (cf. Swanson, 2012, 2018).

[6] Versions with ordinal-equivalent aggregator functions can be found in Chacko and Viceira (2005), Wachter (2013), and more recently Parra-Alvarez, Posch, and Schrimpf (2022). This formulation better compares with standard time-additive preferences and allows to apply L'Hôpital's rule for $1/\eta \to 1$.

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (8), taking as given all intermediate goods prices p_{it} and the final good price p_t. Hence, the input demand functions are $y_{it} = (p_{it}/p_t)^{-\varepsilon} y_t$ for all i and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} \mathrm{d}i \right)^{\frac{1}{1-\varepsilon}} \tag{9}$$

is the (aggregate) price level.

2.1.3. Intermediate good producers

Each intermediate firm produces differentiated goods out of labor using

$$y_{it} = A_t z_t l_{it}, \tag{10}$$

where l_{it} is the amount of the labor input rented by the firm and where A_t follows

$$\mathrm{d} \log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}, \tag{11}$$

and z_t, following

$$\begin{aligned} \mathrm{d}z_t &= x_t z_t \mathrm{d}t + \sigma_t z_t \mathrm{d}B_{z,t}, & (12) \\ \mathrm{d}x_t &= \rho_x(\mu - x_t)\mathrm{d}t + \sigma_t \sigma_x \mathrm{d}B_{x,t}, & (13) \\ \mathrm{d}\sigma_t &= \rho_\sigma(\sigma_z - \sigma_t)\mathrm{d}t + \sigma_\sigma \mathrm{d}B_{\sigma,t}, & (14) \end{aligned}$$

is a non-stationary component of technology, a geometric Brownian motion with drift x_t, which induces persistent changes in the growth rate of z_t, and captures long-run risk (LRR) as in Bansal and Yaron (2004), and $\sigma_t \geq 0$ for fluctuating economic uncertainty around σ_z, or stochastic volatility (SV) following Heston (1993).[7]

Most importantly, marginal costs of the intermediate good producers are the same across firms,

$$mc_t = w_t/(A_t z_t). \tag{15}$$

The monopolistic firms engage in infrequent price setting à la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and given by $\delta e^{-\delta h}$,

[7]For comparison with Heston (1993), Cox, Ingersoll, and Ross (1985b), and Andreasen and Jørgensen (2020), we assume $\sigma_t \geq 0$, but do not mechanically enforce the restriction as in (2) or (11).

$\delta > 0$. Thus $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)}\,\mathrm{d}h = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{16}$$

A fraction of firms will receive the price-change signal per unit of time. All other firms cannot reoptimize their price, but (partially) index their price to π_t^* with $\mathrm{d}\bar{P}_{it} = \chi\pi_t^*\bar{P}_{it}\,\mathrm{d}t$, where the degree of indexation is $\chi \in [0,1]$. This implies that if a firm cannot reoptimize its price at t for a period length of $\tau - t$, its price at τ is $p_{i\tau} = p_{it}e^{\int_t^\tau \chi\pi_s^*\,\mathrm{d}s}$. Note that the higher the parameter δ, the lower price rigidities, in the frictionless case $\delta \to \infty$. Hence, for $\delta < \infty$ prices are set to maximize the expected discounted profits,

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{m_\tau}{m_t} e^{-\delta(\tau-t)} \left(\frac{p_{i\tau}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau}\right) \mathrm{d}\tau \quad \text{s.t.} \quad y_{i\tau} = \left(\frac{p_{i\tau}}{p_\tau}\right)^{-\varepsilon} y_\tau,$$

where m_t denotes the equilibrium state-price density $m_\tau \equiv \exp(\int_t^\tau f_U(c_u, l_u, V)\,\mathrm{d}u)\lambda_\tau$.

After dropping constants, we may write the first-order condition as

$$p_{it}x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1}\frac{x_{2,t}}{x_{1,t}}, \tag{17}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we define the auxiliary variables (interpreted as expected discounted marginal revenue and marginal costs),

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty m_\tau e^{-\delta(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau,$$
$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty m_\tau e^{-\delta(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau.$$

Both variables are forward looking (or jump variables) and determined in equilibrium.

Differentiating $x_{1,t}$ with respect to time gives

$$\mathrm{d}x_{1,t} = ((\delta - f_U + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t, \tag{18}$$

in which we define inflation $\pi_t \equiv (1/\,\mathrm{d}t)\mathbb{E}_t(\mathrm{d}p_t/p_t)$. Similarly we obtain:

$$\mathrm{d}x_{2,t} = ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - \lambda_t mc_t y_t)\,\mathrm{d}t. \tag{19}$$

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-\tau)} \left(p_{i\tau} e^{\int_\tau^t \chi\pi_s^*\,\mathrm{d}s}\right)^{1-\varepsilon} \mathrm{d}\tau,$$

making p_t a predetermined variable (Calvo's insight), because its level is given by past price quotations. Differentiating with respect to time and using $\mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\, p_t^{-\varepsilon}\mathrm{d}p_t$, we arrive at $(1-\varepsilon)\ \mathrm{d}p_t = \left(\delta p_{it}^{1-\varepsilon} p_t^{\varepsilon-1} - (\delta - (1-\varepsilon)\chi\pi_t^*)\right) p_t \mathrm{d}t$, which implies

$$\pi_t - \chi\pi_t^* = \frac{\delta}{1-\varepsilon}\left((\Pi_t^*)^{1-\varepsilon} - 1\right). \tag{20}$$

Differentiating (20) with respect to time, we obtain the inflation dynamics as

$$\mathrm{d}(\pi_t - \chi\pi_t^*) = -(\delta + (1-\varepsilon)(\pi_t - \chi\pi_t^*))\left(\pi_t - \chi\pi_t^* + (mc_t/x_{2,t} - 1/x_{1,t})\lambda_t y_t\right)\ \mathrm{d}t, \tag{21}$$

which is the NK Phillips-curve.

2.1.4. The government problem

In normal times, we assume that the government sets the nominal interest rate i_t through open market operations according to two alternative setups, i.e., the feedback model

$$i_t - i_{ss} = \phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/(y_{ss}z_t) - 1), \quad \phi_\pi > 0,\ \phi_y \geq 0, \tag{22a}$$

or the partial adjustment model

$$\mathrm{d}i_t = \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/(y_{ss}z_t) - 1) - (i_t - i_{ss}))\mathrm{d}t + \sigma_i \mathrm{d}B_{i,t}, \quad \theta > 0, \tag{22b}$$

which includes a response to inflation and output, and a desire to smooth interest rates. The partial adjustment model in (22b) specifies a time lag between the inflation rate and the interest rate, and allows for an output response and monetary policy shocks.

We assume similar to Ireland (2007) that

$$\begin{aligned}\mathrm{d}\pi_t^* \;=\;& \rho_\pi(\pi_{ss} - \pi_t^*)\,\mathrm{d}t + \sigma_\pi\,\mathrm{d}B_{\pi,t} + \sigma_{\pi,r}\,\mathrm{d}B_{r,t} + \sigma_{\pi,i}\,\mathrm{d}B_{i,t} + \sigma_{\pi,d}\,\mathrm{d}B_{d,t} + \sigma_{\pi,A}\,\mathrm{d}B_{A,t} \\ & + \sigma_{\pi,z}\,\mathrm{d}B_{z,t} + \sigma_{\pi,x}\,\mathrm{d}B_{x,t} + \sigma_{\pi,\sigma}\,\mathrm{d}B_{\sigma,t} + \sigma_{\pi,g}\,\mathrm{d}B_{g,t}\end{aligned} \tag{23}$$

where the coefficients $\sigma_{\pi,r}$, $\sigma_{\pi,i}$, $\sigma_{\pi,d}$, $\sigma_{\pi,A}$, $\sigma_{\pi,z}$, $\sigma_{\pi,x}$, $\sigma_{\pi,\sigma}$, $\sigma_{\pi,g}$ are preferences of the central bank, which allows to systematically adjust its inflation target in response to other shocks. Backus, Chernov, Zin, and Zviadadze (2022) refer to such shocks as discretionary monetary policy. Similar to (3), a small parameter ρ_π captures rather persistent changes in the inflation target. According the definition used in Ireland (2007), an adverse supply shock (corresponding to negative realizations of $B_{z,t}$ or $B_{A,t}$) tends to increase prices. To translate the short-run price pressures into more persistent movements in the inflation rate, e.g., $\sigma_{\pi,A} < 0$ helps us to formalize the stories of post-war monetary policy.

Following the traditional macro literature, we keep the fiscal block simple and assume zero net supply of government bonds such that $a_t = 0$ for all t. Transfers finance a given

stream of government consumption g_t, expressed as a constant share of output, $s_g s_{g,t}$, with a mean s_g and a stochastic component $s_{g,t}$ that follows

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{24}$$

such that

$$g_t = s_g s_{g,t} y_t \equiv -T_t. \tag{25}$$

For a more elaborate fiscal block involving public debt and its transmission channels to inflation dynamics and the term structure, see Liemen and Posch (2025).

2.1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t. \tag{26}$$

As in endowment economies, there is no possibility of transferring output intertemporally, meaning that the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i. \tag{27}$$

From (10), we aggregate the demand of all firms $A_t z_t l_{it} = (c_t + g_t)\left(p_{it}/p_t\right)^{-\varepsilon}$, and get $A_t z_t \int_0^1 l_{it}\mathrm{d}i = (c_t + g_t)\int_0^1 (p_{it}/p_t)^{-\varepsilon}\mathrm{d}i$, which can be written as

$$c_t + g_t = y_t = \frac{A_t z_t}{v_t} l_t, \tag{28}$$

by defining $v_t \equiv \int_0^1 (p_{it}/p_t)^{-\varepsilon}\mathrm{d}i$ as the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, v_t is a predetermined variable:

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} e^{-\varepsilon\int_\tau^t \chi\pi_s^* \,\mathrm{d}s}\mathrm{d}\tau. \tag{29}$$

Differentiating this expression with respect to time gives:

$$\mathrm{d}v_t \;=\; \left(\delta\left(\Pi_t^*\right)^{-\varepsilon} + \left(\varepsilon(\pi_t - \chi\pi_t^*) - \delta\right)v_t\right)\mathrm{d}t. \tag{30}$$

Similarly, we obtain an expression for aggregate profits,

$$F_t = (1 - mc_t v_t) y_t. \tag{31}$$

The profits of the firms in the economy are distributed to households as shown in (7).

2.2. Equilibria with price stickiness

For the case of $\delta < \infty$, we obtain the first-order conditions by defining the state space $\mathbb{U}_z \subseteq \mathbb{R}^{n_d}$ and the control region $\mathbb{U}_x \subseteq \mathbb{R}^m$, the *normalized aggregator* $f : \mathbb{U}_z \times \mathbb{U}_x \times \mathbb{R} \to \mathbb{R}$, the *drift function* $g : \mathbb{U}_z \times \mathbb{U}_x \to \mathbb{R}^{n_d}$, and the *diffusion function* $\sigma : \mathbb{U}_z \to \mathbb{R}^{n_d \times n_d}$. Given our description of the problem, we define the household's *value function*:

$$V(\mathbb{Z}_0; \mathbb{Y}_0) \equiv \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} U_0, \tag{32}$$

in which $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is a vector taken as parametric by the representative household and being determined in equilibrium being as functions of the state variables, subject to

$$\mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t, \tag{33}$$

where B_t is a given vector of independent standard Brownian motions. The instantaneous covariance matrix of $\mathbb{Z}_t$ is $\Sigma(\mathbb{Z}_t) \equiv \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top$, which may be less than full rank n_d.

By choosing the control $\mathbb{X}_t \in \mathbb{R}^2_+$, the Hamilton-Jacobi-Bellman (HJB) equation reads:

$$0 = \max_{\mathbb{X}_t} \left\{ f(\mathbb{Z}_t, \mathbb{X}_t, V(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \right\}, \tag{34}$$

where $V_{\mathbb{Z}}$ is the n-vector of partial derivatives $\partial V / \partial \mathbb{Z}_t$, $V_{\mathbb{Z}\mathbb{Z}}$ is the $n_d \times n_d$ Hessian matrix, and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. A neat result about our formulation is that the HJB equation (34) is a deterministic functional equation. In the discrete-time version, we would need to approximate expectations (or the n_d-dimensional integral).

From the discussion above, we define the *drift function* (with partial adjustment):

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t) = \begin{bmatrix} (i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t & (7) \\ \theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/(y_{ss}z_t) - 1) - \theta(i_t - i_{ss}) & (22b) \\ \delta\left(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\, v_t & (30) \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2)d_t & (2) \\ -(\rho_A \log A_t - \frac{1}{2}\sigma_A^2)A_t & (11) \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2)s_{g,t} & (24) \\ x_t z_t & (12) \\ \rho_x(\mu - x_t) & (13) \\ \rho_\sigma(\sigma_z - \sigma_t) & (14) \\ \rho_r(\rho - r_t) & (3) \\ \rho_\pi(\pi_{ss} - \pi_t^*) & (23) \end{bmatrix}, \tag{}$$

and the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_d d_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_A A_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_t z_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_t \sigma_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_r & 0 \\ 0 & \sigma_{\pi,i} & 0 & \sigma_{\pi,d} & \sigma_{\pi,A} & \sigma_{\pi,g} & \sigma_{\pi,z} & \sigma_{\pi,x} & \sigma_{\pi,\sigma} & \sigma_{\pi,r} & \sigma_\pi \end{bmatrix} \begin{matrix} (7) \\ (22b) \\ (30) \\ (2) \\ (11) \\ (24) \\ (12) \\ (13) \\ (14) \\ (3) \\ (23) \end{matrix} .$$

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$\begin{aligned} f_c(\mathbb{Z}_t, \mathbb{X}_t, V) &= V_a, \\ -f_l(\mathbb{Z}_t, \mathbb{X}_t, V) &= V_a w_t, \end{aligned}$$

which means for $\gamma \neq 1$

$$d_t c_t^{-1/\eta} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} = V_a, \tag{35}$$

$$d_t \psi z_t^{1-1/\eta} l_t^{\vartheta} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} = V_a w_t, \tag{36}$$

and for $\gamma = 1/\eta = 1$

$$d_t c_t^{-1} = V_a, \tag{37}$$

$$d_t \psi l_t^{\vartheta} = V_a w_t, \tag{38}$$

or, by eliminating the costate variable we obtain $-f_l/f_c = \psi l_t^{\vartheta} c_t^{1/\eta} z_t^{1-1/\eta} = w_t$, which is the standard static optimality condition between labor and consumption.

Most notably, the first-order conditions (35) and (36) yield *optimal controls*:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \\ l(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \end{bmatrix}$$

or

$$\mathbb{X}_t = \begin{bmatrix} \left((V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} \right)^{\eta} \\ \left((V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t)^{-1} d_t \psi z_t^{-(1-1/\eta)} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} \right)^{-1/\vartheta} \end{bmatrix}.$$

In particular, the first-order conditions (35) and (36) make the optimal controls functions

of the states, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Hence, the concentrated HJB equation reads:

$$0 = f(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t), V(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t))^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right), \quad (39)$$

and

$$0 = (f_U + i_t - \pi_t)\lambda_t + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t))^\top \lambda_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right), \quad (40)$$

or

$$\begin{aligned} 0 &= f_U + i_t - \pi_t + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t))^\top \tilde{\lambda}_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}\right) \\ &= f_U + i_t - \pi_t + g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}(\hat{\mathbb{Z}}_t; \hat{\mathbb{Y}}_t); \hat{\mathbb{Y}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}} - \gamma x_t + \tfrac{1}{2}\gamma(1+\gamma)\sigma_t^2 - \gamma\sigma_t\sigma_{\pi,z}\tilde{\lambda}_\pi \\ &\quad + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\right) \end{aligned}$$

for the costate equation by using the envelope theorem, where $\lambda_{\mathbb{Z}}$ is the n-vector of partial derivatives $\partial\lambda_t/\partial\mathbb{Z}_t$, and $\lambda_{\mathbb{Z}\mathbb{Z}}$ is the $n \times n$ Hessian matrix of the costate variable.[8]

Note that $V_a(\mathbb{Z}_t; \mathbb{Y}_t) \equiv \lambda_t$ in (35), (36), and (40) is readily interpreted as the marginal value of wealth or the current value of a unit of consumption in period t, and determines the asset pricing kernel in this economy. In what follows, we provide the asset pricing kernel or the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy.

As we show in the appendix, the marginal value of wealth evolves according to:

$$\mathrm{d}\lambda_t = -\left(f_U + i_t - \pi_t\right)\lambda_t \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \quad (41)$$

and the value function follows:

$$\mathrm{d}V = -f(\mathbb{Z}_t, \mathbb{X}_t, V)\,\mathrm{d}t + V_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t. \quad (42)$$

The implicit Euler equation (41) has a simple interpretation: the change in the marginal value of wealth depends on the effective rate of time preference $-f_U$ minus the real interest rate and the shocks to the economy. Note that the effective rate of time preference $-f_U$ depends on the preference for the resolution of uncertainty

$$-f_U = \frac{1-\gamma}{1-1/\eta}\left(f_{UU}V + r_t\right), \quad \gamma \neq 1,\ 1/\eta \neq 1$$

It shows that for the case of $V < 0$, an early resolution of uncertainty, i.e., $f_{UU} > 0$, the convex normalized aggregator effectively decreases the discount rate r_t.

Definition 1. *The recursive-competitive equilibrium of the NK model is defined by the sequence* $\{\mathbb{Z}_t, \mathbb{Y}_t, \lambda_t, V_t\}_{t=0}^{\infty}$, *which satisfies the dynamic and static equations in (33), (15),*

[8] Judd (1992) proposes to define (numerical) Euler equation errors based on (40) to gauge accuracy.

(17), (18), (19), (20), (25), (26), (31), (35), (41), and (42).

Following Duffie and Skiadas (1994), unlike in the case of time-additive utility, the case of recursive utility implies that the state-price density depends on the value function. In particular, we define (cf. Wachter, 2013, p.996)

$$m_s/m_t = \exp\left(\int_t^s f_U(\mathbb{Z}_\tau, \mathbb{X}_\tau, V)\,\mathrm{d}\tau\right)\frac{f_c(\mathbb{Z}_s, \mathbb{X}_s, V_s)}{f_c(\mathbb{Z}_t, \mathbb{X}_t, V_t)}, \tag{43}$$

where f_c, and f_U denote partial derivatives.[9] Using $\lambda_t = f_c$, we may also write

$$\begin{aligned}\mathrm{d}m_t &= f_U(\mathbb{Z}_t, \mathbb{X}_t, V)m_t\,\mathrm{d}t + m_t/\lambda_t\,\mathrm{d}\lambda_t \\ &= -\left(i_t - \pi_t\right)m_t\mathrm{d}t + m_t\lambda_{\mathbb{Z}}^\top/\lambda_t\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t\end{aligned} \tag{44}$$

For the case of $1/\eta = \gamma$ simplifies to $f_U = -r_t$ and $f_c = d_t c_t^{-1/\eta}$ and the SDF is:

$$m_s/m_t = e^{-\int_t^s r_\tau\,\mathrm{d}\tau}\frac{V_a(\mathbb{Z}_s; \mathbb{Y}_s)}{V_a(\mathbb{Z}_t; \mathbb{Y}_t)}, \quad \text{and} \quad m_s \equiv e^{-\int_t^s r_\tau\,\mathrm{d}\tau}\lambda_s, \tag{45}$$

or, equivalently, the present value shadow price (cf. Hansen and Scheinkman, 2009).

2.2.1. Equilibrium dynamics

In what follows, we restrict our attention to solutions leading the economy towards a (stochastic) balanced growth path, in which $\pi_t \to \pi_t^*$. Because government bonds are in net zero supply, we arrive at a system of 6 endogenous processes, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion v_t, the Taylor rule i_t, the Euler equation λ_t, the value function V, and the 6 exogenous processes, which summarize equilibrium dynamics:

$$\mathrm{d}\lambda_t = -\left(f_U + i_t - \pi_t\right)\lambda_t\mathrm{d}t + \lambda_{\mathbb{Z}}^\top\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \tag{46a}$$

$$\mathrm{d}V = -f(\mathbb{Z}_t, \mathbb{X}_t, V(\mathbb{Z}_t; \mathbb{Y}_t))\,\mathrm{d}t + V_{\mathbb{Z}}^\top\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \tag{46b}$$

$$\mathrm{d}x_{1,t} = ((\delta - f_U + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t \tag{46c}$$

$$\mathrm{d}x_{2,t} = ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - \lambda_t mc_t y_t)\,\mathrm{d}t \tag{46d}$$

$$\mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t. \tag{46e}$$

In the NK model, the auxiliary variables $x_{2,t}/x_{1,t}$ determine the optimal relative price Π_t^* in (17) for the price setting firms, and thus inflation π_t in (20) or $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$, and

$$w_t = -\frac{f_l(\mathbb{Z}_t, \mathbb{X}_t, V)}{f_c(\mathbb{Z}_t, \mathbb{X}_t, V)} = \psi l_t^\vartheta c_t^{1/\eta} z_t^{1-1/\eta} \tag{47}$$

[9] Because the exponential term is (locally) deterministic, the covariances of the state-price density with fundamentals, are determined by the marginal value of wealth $f_c(c_t, V) = \lambda_t$ (Wachter, 2013, p.1022).

from the first-order conditions (35) and (36) pins down marginal costs in (15).

Proposition 2. *Defining stationary variables* $\hat{\mathbb{Z}}_t \equiv (\hat{a}_t, i_t, v_t, d_t, A_t, s_{g,t}, \hat{z}_t, x_t, \sigma_t, r_t, \pi_t^*)^\top$, $\hat{z}_t \equiv z_t/z_t = 1$, $\hat{a}_t \equiv a_t/z_t$, *aggregates* $\hat{\mathbb{Y}}_t \equiv (mc_t, \Pi_t^*, \hat{x}_{1,t}, \hat{x}_{2,t}, \pi_t, \hat{g}_t, \hat{T}_t, \hat{y}_t, \hat{F}_t, \hat{w}_t)^\top$ *where* $\hat{x}_{1,t} \equiv x_{1,t}/z_t^{1-\gamma}$, $\hat{x}_{2,t} \equiv x_{2,t}/z_t^{1-\gamma}$, $\hat{g}_t \equiv g_t/z_t$, $\hat{T}_t \equiv T_t/z_t$, $\hat{y}_t \equiv y_t/z_t$, $\hat{F}_t \equiv F_t/z_t$, $\hat{w}_t \equiv w_t/z_t$, *controls* $\hat{\mathbb{X}}_t \equiv (\hat{c}_t, l_t)^\top$ *where* $\hat{c}_t \equiv c_t/z_t$, *the costate* $\hat{\lambda}_t \equiv \lambda_t/z_t^{-\gamma}$, *and thus the value function (1) is homogeneous of degree* $1-\gamma$ *under the scaling* $\mathbb{Z}_t \mapsto \hat{\mathbb{Z}}_t \equiv \mathbb{Z}_t \cdot [1/z_t, 1, 1, 1, 1, 1, 1/z_t, 1, 1, 1, 1]$, *and can be written as*

$$V(\mathbb{Z}_t) = \hat{V}(\hat{\mathbb{Z}}_t) z_t^{1-\gamma}, \quad \gamma \neq 1$$

where $\hat{V}(\hat{\mathbb{Z}}_t)$ *is the stationary value function, and* $z_t^{1-\gamma}$ *for* $\gamma \neq 1$ *denotes the stochastic trend for the value function with* $\hat{V}_z = \hat{V}_{\hat{z}} = 0$, $\hat{V}_{\hat{a}}(\hat{\mathbb{Z}}_t) \equiv \hat{\lambda}_t$, *and* $z_t\lambda_z/\lambda_t = -\gamma$, *and* $z_t V_z(\mathbb{Z}) = (1-\gamma)V - aV_a$, *which at* $a_t = 0$ *simplifies to* $z_t V_z(\mathbb{Z}) = (1-\gamma)V$.[10]

Proof. Online Appendix H.5 ■

Proposition 3. *By eliminating trends of all non-stationary variables using* $\hat{y}_t = y_t/z_t$, $\hat{c}_t = c_t/z_t$, $\hat{z}_t = 1$, $\hat{x}_{1,t} = x_{1,t}/z_t^{1-\gamma}$, $\hat{x}_{2,t} = x_{2,t}/z_t^{1-\gamma}$, *such that the ratio* $\hat{x}_{2,t}/\hat{x}_{1,t} = x_{2,t}/x_{1,t}$ *pins down inflation, for the value function* $\hat{V} = V/z_t^{1-\gamma}$, *and for the costate* $\hat{\lambda}_t = \lambda_t/z_t^{-\gamma}$,

$$\begin{aligned}
d\hat{\lambda}_t &= -(f_U + i_t - \pi_t - \gamma(x_t - \tfrac{1}{2}(\gamma+1)\sigma_t^2 + \sigma_t\tilde{\lambda}_\pi\sigma_{\pi,z}))\hat{\lambda}_t\,dt + \hat{\lambda}_{\hat{\mathbb{Z}}}^\top\sigma(\hat{\mathbb{Z}}_t)\,dB_t && \text{(48a)}\\
d\hat{V} &= -(\hat{f}(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t, \hat{V}) + (1-\gamma)(x_t - \tfrac{1}{2}\gamma\sigma_t^2 + \tilde{V}_\pi\sigma_{\pi,z}\sigma_t)\hat{V})\,dt + \hat{V}_{\hat{\mathbb{Z}}}^\top\sigma(\hat{\mathbb{Z}}_t)\,dB_t && \text{(48b)}\\
d\hat{x}_{1,t} &= ((\delta - f_U - (1-\gamma)(x_t - \sigma_t^2 + \tfrac{1}{2}\gamma\sigma_t^2) - (\varepsilon-1)(\pi_t - \chi\pi_t^*))\hat{x}_{1,t} - \hat{\lambda}_t\hat{y}_t)dt\\
&\quad -(1-\gamma)\hat{x}_{1,t}\sigma_t\,dB_{z,t} && \text{(48c)}\\
d\hat{x}_{2,t} &= ((\delta - f_U - (1-\gamma)(x_t - \sigma_t^2 + \tfrac{1}{2}\gamma\sigma_t^2) - \varepsilon(\pi_t - \chi\pi_t^*))\,\hat{x}_{2,t} - mc_t\hat{\lambda}_t\hat{y}_t)dt\\
&\quad -(1-\gamma)\hat{x}_{2,t}\sigma_t\,dB_{z,t} && \text{(48d)}\\
d\hat{\mathbb{Z}}_t &= g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)\,dt + \sigma(\hat{\mathbb{Z}}_t)\,dB_t && \text{(48e)}
\end{aligned}$$

summarizes the stationary system for $\gamma \neq 1$ *in which* z_t *no longer appears as a state.*

Proof. Online Appendix H.6 ■

Note that a stationary system is not required for our numerical solution, we may rather use the (necessary) condition that the system approaches a balanced growth path.

2.2.2. Numerical solution

We solve the HJB as a deterministic PDE, leveraging that in continuous time the HJB itself is deterministic (cf. Chang, 2004).[11] Our algorithm is a risk-corrected Picard iteration:

[10] A similar decomposition is available for the logarithmic case $\gamma = 1$ (cf. Online Appendix H.5).

[11] In discrete time, solving the HJB equation requires evaluating expectations; by contrast, in continuous time these expectations are embedded in differential operators, so no numerical quadrature is needed.

we start from the perfect-foresight problem (no shocks); at each outer step, we freeze the diffusion curvature terms from the previous iterate, solve the resulting deterministic boundary value problem in the time domain, update the policies from the new gradients, and repeat until HJB residuals and Euler equation errors are below tolerance. The novelty for macro-finance models is (i) recasting the SDE system as a time-domain PDE boundary value problem and (ii) solving it via successive approximation (Picard). At the fixed point, the procedure recovers the policy functions of the original stochastic model.

We start with obtaining the time-domain PDEs. Consider $\lambda_t = \lambda(\mathbb{Z}_t)$ and $V = V(\mathbb{Z}_t)$, and use Itô's formula to arrive at

$$\begin{aligned} \mathrm{d}\lambda_t &= \mathrm{d}\mathbb{Z}_t^\top \lambda_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t, \qquad (49) \\ \mathrm{d}V &= \mathrm{d}\mathbb{Z}_t^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t, \qquad (50) \end{aligned}$$

which leads us to the following proposition.[12]

Proposition 4. *By subtracting the quadratic variation terms from the dynamics for the costate $d\lambda_t$ and the value function $dV(\mathbb{Z}_t;\mathbb{Y}_t)$, respectively, yields the drift identities*

$$\begin{aligned} d\lambda_t - \tfrac{1}{2}tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)dt &= d\mathbb{Z}_t^\top \lambda_{\mathbb{Z}}, \\ dV - \tfrac{1}{2}tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right)dt &= d\mathbb{Z}_t^\top V_{\mathbb{Z}}, \end{aligned}$$

and inserting $d\lambda_t$ in (41), dV in (42), and the state dynamics $d\mathbb{Z}_t$ from (33) we eliminate time (and shocks) obtain the HJB and the costate equations in (39) and (40).

Proof. Insert $\mathrm{d}\lambda_t$ from (41) into the state dynamics in (33) to obtain

$$\begin{aligned} -\left(f_U + i_t - \pi_t\right)\lambda_t \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t = \\ g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \lambda_{\mathbb{Z}}\,\mathrm{d}t + \mathrm{d}B_t^\top \sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}} \end{aligned}$$

which, by collecting terms noting that $\lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t = \mathrm{d}B_t^\top \sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}}$ is a scalar, implies (40). Follow the same steps for the concentrated HJB in (39). ■

A system of PDEs which implies the same policy function is constructed using (41) and (42) and Proposition 4 by subtracting the quadratic variation terms from the costate variable and the value function (accounting for risk) and apply expectations such that

$$\begin{aligned} \mathbb{E}_t(\mathrm{d}\lambda_t) - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t &= \mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top \lambda_{\mathbb{Z}}\right), \\ \mathbb{E}_t\left(\mathrm{d}V\right) - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t &= \mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top V_{\mathbb{Z}}\right), \end{aligned}$$

jointly solves (40) and (39). In fact, taking conditional expectations eliminates the mar-

[12] Recall that $\mathrm{tr}\left(\sigma\sigma^\top \cdot\right)$ includes mixed second derivatives, so the result covers non-diagonal σ.

tingale terms and we refer to the system

$$\begin{aligned}
\mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top \lambda_{\mathbb{Z}}\right) &= -(f_U + i_t - \pi_t)\lambda_t \mathrm{d}t - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}})\,\mathrm{d}t, && \text{(51a)}\\
\mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top V_{\mathbb{Z}}\right) &= -f(\mathbb{Z}_t, \mathbb{X}_t, V)\,\mathrm{d}t - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}})\mathrm{d}t, && \text{(51b)}\\
\mathrm{d}x_{1,t} &= ((\delta - f_U - (\varepsilon - 1)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t, && \text{(51c)}\\
\mathrm{d}x_{2,t} &= ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))\,x_{2,t} - mc_t\lambda_t y_t)\,\mathrm{d}t, && \text{(51d)}\\
\mathbb{E}_t(\mathrm{d}\mathbb{Z}_t) &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t, && \text{(51e)}
\end{aligned}$$

as the system of time-domain PDEs, where the first two equations refer to the curvature adjusted drift terms of the costate and the value function, respectively. Inflation π_t is determined by the ratio of auxiliary variables $x_{2,t}$ and $x_{1,t}$. The deterministic system of PDEs reflects "risk-adjusted" dynamics, which we solve for the unknown policy functions. Observe that the system coincides with the perfect-foresight solution if $\|\sigma(\mathbb{Z}_t)\| = 0$.

Alternatively, we use the stationary system (48) and subtract the quadratic variation terms from the dynamics for the differentials $\mathrm{d}\hat{\lambda}_t$ and $\mathrm{d}\hat{V}$,

$$\begin{aligned}
\mathbb{E}_t(\mathrm{d}\hat{\lambda}_t) - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\,\mathrm{d}t &= \mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top \hat{\lambda}_{\hat{\mathbb{Z}}}),\\
\mathbb{E}_t(\mathrm{d}\hat{V}) - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\mathrm{d}t &= \mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top \hat{V}_{\hat{\mathbb{Z}}}),
\end{aligned}$$

such that we solve the stationary system of PDEs,

$$\begin{aligned}
\mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top \hat{\lambda}_{\hat{\mathbb{Z}}}) &= -(f_U + i_t - \pi_t - \gamma(x_t - \tfrac{1}{2}(\gamma+1)\sigma_t^2 + \sigma_t\tilde{\lambda}_\pi\sigma_{\pi,z}))\hat{\lambda}_t\,\mathrm{d}t\\
&\quad -\tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\,\mathrm{d}t,\\
\mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top \hat{V}_{\hat{\mathbb{Z}}}) &= -(\hat{f}(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t, \hat{V}) + (1-\gamma)(x_t - \tfrac{1}{2}\gamma\sigma_t^2 + \tilde{V}_\pi\sigma_{\pi,z}\sigma_t)\hat{V})\,\mathrm{d}t\\
&\quad -\tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\mathrm{d}t \quad \text{for} \quad \gamma \neq 1,\\
\mathbb{E}_t(\,\mathrm{d}\hat{x}_{1,t}) &= ((\delta - f_U - (1-\gamma)(x_t - \sigma_t^2 + \tfrac{1}{2}\gamma\sigma_t^2) - (\varepsilon - 1)(\pi_t - \chi\pi_t^*))\hat{x}_{1,t} - \hat{\lambda}_t\hat{y}_t)\mathrm{d}t\\
\mathbb{E}_t(\,\mathrm{d}\hat{x}_{2,t}) &= (\left(\delta - f_U - (1-\gamma)(x_t - \sigma_t^2 + \tfrac{1}{2}\gamma\sigma_t^2) - \varepsilon(\pi_t - \chi\pi_t^*)\right)\hat{x}_{2,t} - mc_t\hat{\lambda}_t\hat{y}_t)\mathrm{d}t\\
\mathbb{E}_t(\,\mathrm{d}\hat{\mathbb{Z}}_t) &= g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)\,\mathrm{d}t.
\end{aligned}$$

and π_t is determined by the ratio of auxiliary variables $\hat{x}_{2,t}$ and $\hat{x}_{1,t}$. While $V(\mathbb{Z}_t)$ and $\lambda(\mathbb{Z}_t)$ approach a balanced growth path, $\hat{V}(\hat{\mathbb{Z}}_t)$ and $\hat{\lambda}(\hat{\mathbb{Z}}_t)$ are stationary.

With either approach, the steady-state interest rate, which is computed from the deterministic system reads (for details see the Online Appendix L):

$$i_{ss} = -f_U + \pi_{ss} - \tfrac{1}{2}\gamma(\gamma+1)\sigma_z^2 - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_{ss})\sigma(\hat{\mathbb{Z}}_{ss})^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}) + \gamma\sigma_z\tilde{\lambda}_\pi\sigma_{\pi,z} + \gamma\mu$$

Intuitively, the time-domain PDEs capture the full risk-adjusted dynamics. They include the drifts of the state variables, and through curvature terms they also include

the drifts of the value and the costate that modify the Euler equations and thus the consumption-saving rule. The optimal decision rule is a stationary Markov feedback policy. The control at each instant is given by a fixed mapping from the current state. Along the stable manifold the stationary variables remain at the stochastic steady state, and in the nonstationary representation the path reflects balanced-growth dynamics. Away from the manifold the feedback rule yields controls that steer the system back.[13]

We solve the time-domain PDEs with a risk-corrected Picard iteration. The trick is to regard the PDEs as functional ODEs, thereby separating the time dimension from the solution in the state space (cf. Posch and Trimborn, 2013, for models with rare events). This separation permits straightforward parallelization across nodes. We initialize the algorithm starting from the perfect-foresight solution, $\sigma(\mathbb{Z}_t) = 0$, and solve the system of ODEs subject to transversality conditions for given set of initial states. We obtain the optimal values of the controls and jump variables on a tensor-product grid by function approximation (collocation). We then fix curvature terms $\tilde{\lambda}_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}/\lambda_t$, $\tilde{V}_{\mathbb{Z}\mathbb{Z}} \equiv V_{\mathbb{Z}\mathbb{Z}}/V$, or in the stationary system, $\tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{\lambda}_t$, $\tilde{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{V}$, $\tilde{\lambda}_\pi \equiv \hat{\lambda}_\pi/\hat{\lambda}_t$, $\tilde{V}_\pi \equiv \hat{V}_\pi/\hat{V}$, respectively, and again solve the resulting system of ODEs as boundary value problems. After each solve we update the value function, the costate variable, and their derivatives on the grid. We then refresh the curvature terms from the updated functions and iterate until HJB residuals and Euler equation errors are below tolerance (cf. Table 1).

The transitional dynamics account for risk and the policy functions are solved globally, so optimal controls may differ considerably along the transition toward the steady state. We compare our solution with a linear approximation (for example, the sequence-space Jacobians in Auclert, Bardóczy, Rognlie, and Straub, 2021), and with a two-step approach that fixes the policy functions at the certainty equivalent (nonlinear) solution and computes the risk effects from that approximation. These comparisons suggest variants that balance computational cost and accuracy, depending on the specific model.

In the boundary value problem (BVP) we seek a function $x : [0, T] \mapsto \mathbb{R}^k$ that satisfies the (conditional) deterministic system for given initial conditions for the states $\mathbb{Z}_0$ and the TVC assuming that variables approach their (stochastic) balanced growth path. Because the time horizon is infinite, we may use a time transformation:

$$t = \frac{\tau}{\nu(1-\tau)} \quad \text{for} \quad \tau \in [0, 1),$$

where ν is a positive (nuisance) parameter, such that for $t \to \infty$ we have that $\tau \to 1$. Alternatively, we may set T sufficiently large but finite number.[14]

[13]The economy generally admits a stationary (steady-state) distribution. Following common usage, we use the term "stochastic steady state" to denote the fixed point of the risk-adjusted deterministic system. This object is distinct from the stationary distribution.

[14]Trimborn, Koch, and Steger (2008) introduced the relaxation algorithm to applications in economics. In contrast to their approach, we use projection methods to solve the boundary value problem, which

2.2.3. Sequence-space Jacobians and two-step approximate solutions

Note that the recursion as set out in Table 1 is only required if we are interested in the solution of the stochastic model where $\|\sigma(\mathbb{Z}_t)\| \neq 0$. In what follows we elaborate on approximate solutions based on the perfect-foresight equilibrium mappings of aggregate variables around the steady state (cf. Auclert, Bardóczy, Rognlie, and Straub, 2021). Those ideas allow us to efficiently compute the transitional dynamics by rapid computation of sequence-space Jacobians.[15] Alternatively, we may also fix the derivatives $\lambda_{\mathbb{ZZ}}$ and $V_{\mathbb{ZZ}}$ at the steady state and compute the effects of risk for $\|\sigma(\mathbb{Z}_t)\| > 0$, either using the perfect-foresight solution (henceforth two-step approximate solution) or iteratively from the risk-adjusted solution. The latter approaches keep the nonlinear solution for aggregate state variables, but assume constant effects of uncertainty.[16] Another approach, closer to sequence-space Jacobians, is to use first-order perturbation, which keeps second-order effects *and* a linear model (cf. Parra-Alvarez, Polattimur, and Posch, 2021).

Below, we compare the three approaches: (i) a first-order (linear) approximation, (ii) a two-step approximation, and (iii) an iterative approximation to the full nonlinear PDE system (39) and (40). The first reduces the problem to a linear system, whereas the other two keep the aggregate-state dynamics nonlinear.

2.2.4. Numerical solution in the policy function space

We also solve the HJB equation (39) directly by collocation using the Matlab `CompEcon` toolbox (Judd, 1992; Miranda and Fackler, 2002).

Since the functional form of the solution is unknown, a strategy for solving the HJB equation is to approximate $V(\mathbb{Z}_t; \mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t)v$, in which v is an n-vector of coefficients and ϕ is the $n \times n$ interpolation matrix, the tensor product of the univariate bases at n_d nodes for dimension $d = 1, ..., \dim(\mathbb{Z}_t)$. The computational burden is reduced when replacing the tensor product by sparse grids (Winschel and Krätzig, 2010). Starting from the HJB equation (39), we approximate the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t)$:

$$0 = f(\mathbb{Z}_t, \mathbb{X}_t, \phi(\mathbb{Z}_t)v) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t))^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t)v + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{ZZ}}(\mathbb{Z}_t)v\right),$$

where $\phi_{\mathbb{Z}}$ and $\phi_{\mathbb{ZZ}}$ are the known derivatives (multidimensional arrays) of the interpolation matrix evaluated at the specified nodes. Although this approach typically requires a good initial guess, it is very useful for verification whether a solution, e.g., obtained from the system of time-domain PDEs indeed solves the HJB equation.

turns out to be relatively efficient and (even for a few approximation points) highly accurate.

[15] We fix the first-order derivatives $\lambda_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ at the steady state, which then implies that $\lambda_{\mathbb{ZZ}} = V_{\mathbb{ZZ}} = 0$.

[16] It still reflects the solution of the stochastic model and hence accounts for precautionary savings.

2.3. The flexible-price limit

An important benchmark solution to the NK model is the case where prices become more flexible. In the frictionless limit, $\delta \to \infty$, the household's *value function* still is defined as in (32), and the firms set prices to maximize profits

$$\max_{p_{it}} \left(\frac{p_{it}}{p_t} y_{it} - mc_t y_{it} \right) \quad \text{s.t.} \quad y_{it} = \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} y_t.$$

After dropping constants, we may write the first-order condition as:

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} mc_t$$

Hence, in the flexible-price case, all firms set prices $p_{it} = p_t$ such that $v_t \equiv 1$. Given the value of marginal costs, we compute consumption and hours from (47) and (15):

$$c_t = (1 - s_g s_{g,t})^{\frac{1/\eta+\vartheta-1}{1/\eta+\vartheta}} \left((\varepsilon - 1)/(\psi\varepsilon) \right)^{\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1+\vartheta}{1/\eta+\vartheta}} z_t \tag{52}$$

such that $l_t = \left((\varepsilon - 1)/((1 - s_g s_{g,t})\psi\varepsilon) \right)^{\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1-1/\eta}{1/\eta+\vartheta}}$ and the (natural) output:

$$y_t^n = (1 - s_g s_{g,t})^{-\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1+\vartheta}{1/\eta+\vartheta}} \left((\varepsilon - 1)/(\psi\varepsilon) \right)^{\frac{1}{1/\eta+\vartheta}} z_t. \tag{53}$$

Based on the definition, the output gap is readily available from $x_t \equiv y_t / y_t^n - 1$.

Proposition 5 (Value function). *In the frictionless limit, $\delta \to \infty$, the value function of the representative consumer with recursive preferences in (1) reads*

$$V(\mathbb{Z}_t) = \frac{1}{1-\gamma} \left(\frac{1 - 1/\eta}{1-\gamma} d_t r_t^{1 - \frac{1-1/\eta}{1-\gamma}} (1-\gamma) c_t^{-1/\eta} a_t + \frac{1 - 1/\eta}{1-\gamma} h \right)^{\frac{1-\gamma}{1-1/\eta}}, \tag{54}$$

for $\gamma \neq 1$ and $\eta \neq 1$, which for $\gamma = 1/\eta$ reads $V(\mathbb{Z}_t) = d_t c_t^{-1/\eta} a_t + h/(1-\gamma)$, and

$$\lim_{1/\eta \to 1} V(\mathbb{Z}_t) = \frac{1}{1-\gamma} \exp\left(d_t (1-\gamma) r_t c_t^{-1} a_t + h(d_t, A_t, s_{g,t}, z_t, x_t, \sigma_t, r_t) \right), \quad \gamma \neq 1, \tag{55}$$

or

$$\lim_{1/\eta \to 1} V(\mathbb{Z}_t) = d_t c_t^{-1} a_t + h(d_t, A_t, s_{g,t}, z_t, x_t, \sigma_t, r_t), \quad \gamma = 1, \tag{56}$$

where c_t is given by (52) and $h = h(d_t, A_t, s_{g,t}, z_t, x_t, \sigma_t, r_t)$ satisfies the PDE

$$\begin{aligned} 0 \;=\; & \left((1-\gamma)V \right)^{\frac{1-1/\eta}{1-\gamma}} V^{-1} f(\mathbb{Z}_t, \mathbb{X}_t, V) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t))^\top h_{\mathbb{Z}} + \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t) \sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}\mathbb{Z}} \right) \\ & + \left(1 - \frac{1 - 1/\eta}{1-\gamma} \right) \left((1-\gamma)V \right)^{-\frac{1-1/\eta}{1-\gamma}} \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t) \sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}} h_{\mathbb{Z}}^\top \right), \quad \gamma \neq 1 \end{aligned} \tag{57}$$

or $-f(\mathbb{Z}_t, \mathbb{X}_t, V) = g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t))^\top h_{\mathbb{Z}} + \frac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}\mathbb{Z}}\right)$ *for* $\gamma = 1/\eta = 1$.

Proof. Online Appendix H.7 ■

3. Results

Our results confirm previous findings that the effects of uncertainty in the NK model on the policy functions and macroeconomic aggregates are small, and primarily a level shift. Both the deterministic and the stochastic steady states are very close. In what follows, we consider the implications of price rigidities and fundamental shocks for the term structure of interest rates and the premium investors require for bearing economic risks.

3.1. Effects of uncertainty

3.1.1. Equilibrium interest rates, inflation, and the natural rate

After having obtained the effects of uncertainty (either approach) on the costate and value functions and/or the consumption function, we use Itô's formula

$$\begin{aligned} \mathrm{d}\lambda_t &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \lambda_{\mathbb{Z}}\, \mathrm{d}t + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\, \mathrm{d}B_t \\ &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} \lambda_t\, \mathrm{d}t + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}\right) \lambda_t \mathrm{d}t + \tilde{\lambda}_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\lambda_t\, \mathrm{d}B_t, \end{aligned}$$

such that the implicit Euler equation (41), pins down the equilibrium inflation rate as

$$\pi_t = f_U + i_t + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}), \tag{58}$$

i.e., the costate equation that defines Euler equation errors, or equivalently,

$$\begin{aligned} \pi_t &= f_U + i_t + g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}(\hat{\mathbb{Z}}_t; \hat{\mathbb{Y}}_t); \hat{\mathbb{Y}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}} - \gamma x_t + \tfrac{1}{2}\gamma(1+\gamma)\sigma_t^2 - \gamma\sigma_t\sigma_{\pi,z}\tilde{\lambda}_\pi \\ &\quad + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\right) \end{aligned}$$

to study the effects of uncertainty on the consumption-saving decision and interest rates, compute impulse response functions, and the implied term structure of interest rates.

In general equilibrium, the risk-free rate is defined as (cf. Posch, 2011)

$$-\frac{1}{\mathrm{d}t}\mathbb{E}_t\left[\frac{\mathrm{d}m_t}{m_t}\right] = -f_U(\mathbb{Z}_t, \mathbb{X}_t, V) - \frac{1}{\mathrm{d}t}\mathbb{E}_t\left[\frac{\mathrm{d}\lambda_t}{\lambda_t}\right] \equiv r_t^f.$$

Although the instantaneous real return on the risk-free asset is riskless and carries no premium, a default-free zero-coupon bond is risky over finite holding horizons: its price varies with the state and can covary with the household's marginal utility (the SDF), so the expected holding-period return can differ from the return earned by rolling over short-term risk-free bills (e.g., Rudebusch and Swanson, 2008, p. 115).

In the short-run, the nominal policy rate, $i_t \equiv r_t^f + \pi_t$, is under the control of the monetary authority.[17] Our result in (58) can be used to show how the equilibrium *real* rate, r_t^f, is affected by the different state variables. Using (58) we conclude:

$$r_t^f = -f_U(\mathbb{Z}_t, \mathbb{X}_t) - g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} - \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}). \tag{59}$$

or

$$\begin{aligned} r_t^f &= -f_U(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t) - g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}} - \tfrac{1}{2}\text{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}) \\ &\quad +\gamma x_t - \tfrac{1}{2}\gamma(1+\gamma)\sigma_t^2 + \gamma\sigma_t\sigma_{\pi,z}\tilde{\lambda}_\pi \end{aligned}$$

Following Barsky, Justiniano, and Melosi (2014), we define the natural rate as the real interest rate prevailing in an economy with flexible prices, or the second-best equilibrium (cf. Blanchard and Galí, 2007), i.e., defining λ_t^n as the frictionless limits:

$$r_t^n = -f_U(\mathbb{Z}_t, \mathbb{X}_t) - g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}}^n - \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}^n) \tag{60}$$

or

$$r_t^n = -f_U(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t) - g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}}^n - \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^n) + \gamma x_t - \tfrac{1}{2}\gamma(1+\gamma)\sigma_t^2$$

where $f_U(\cdot)$ and $g(\cdot)$ are evaluated at the flexible price values such that $\tilde{\lambda}_i = \tilde{\lambda}_\pi = 0$. Defining the (Wicksellian) natural rate $r_t^n \to r_{ss}^n$ as the interest rate once transitory shocks have abated (variables approach their stochastic steady state), we find

$$\begin{aligned} r_{ss}^n &= -f_U(\hat{\mathbb{Z}}_{ss}, \hat{\mathbb{X}}_{ss}) - \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_{ss})\sigma(\mathbb{Z}_{ss})^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^n) + \gamma\mu - \tfrac{1}{2}\gamma(1+\gamma)\sigma_z^2 \\ &= -\frac{1/\eta - \gamma}{1-\gamma}\left(-(1-\gamma)(\mu - \tfrac{1}{2}\gamma\sigma_z^2) - \tfrac{1}{2}\text{tr}(\sigma(\hat{\mathbb{Z}}_{ss})\sigma(\hat{\mathbb{Z}}_{ss})^\top \tilde{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\right) + \rho \\ &\quad -\tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_{ss})\sigma(\mathbb{Z}_{ss})^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^n) + \gamma\mu - \tfrac{1}{2}\gamma(1+\gamma)\sigma_z^2, \quad \gamma \neq 1 \end{aligned}$$

and

$$r_{ss}^n = -\rho - \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_{ss})\sigma(\mathbb{Z}_{ss})^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^n) + \mu - \sigma_z^2, \quad \gamma = 1$$

Below we compare the effects of uncertainty for the different solutions. We start with the first-order approximation where $\frac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) = 0$, and then discuss the nonlinear solution, when evaluated at the same reference point. By construction, there is

[17] Having the nominal rate i_t under the control of the monetary authority (henceforth policy rate), is a modelling assumption and reflects the idea of an overnight facility.

no effect on the steady-state interest rate in the first-order approximation, such that

$$r_{ss}^f = \rho + (1/\eta)\mu, \quad i_{ss} = \rho + (1/\eta)\mu + \pi_{ss}$$

and neither for the macroeconomic aggregates (because of certainty equivalence). In the nonlinear solution, $g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}}$ is not necessarily zero at the (deterministic) steady state for $\sigma(\mathbb{Z}_t) \neq 0$, which also implies that $-\frac{1}{2}\text{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}) \neq 0$. An important implication is that the equilibrium real interest rate (or the natural rate of interest) is different, even when evaluated at the same reference point (i.e., the steady state).

A key insight is that the shape of the term structure crucially depends on the chosen solution method. In what follows, we compute moments of the limiting distribution for the variables of interest and show how local approximations can be misleading in studying the model-implied term structure of interest rates and moments of the ergodic distribution. Most importantly, we are mainly interested in identifying which fundamental shocks create "unfavorable shifts" in the relevant state varibles. An unfavorable shift is a change to a state typically with lower consumption (higher costate, SDF or "hunger"), i.e.,

$$\tilde{\lambda}_{\mathbb{Z}_j} > 0, \quad j = 1, \ldots, n_d \tag{61}$$

or the market price of risk associated with state j.

For example, an increase in technology puts downward pressure on prices causing marginal costs and inflation to fall, and generates higher output and consumption, which in turn decreases the marginal value of wealth (households feel wealthier), $\tilde{\lambda}_A < 0$. In contrast, an adverse monetary policy shock, i.e., higher nominal interest rates, cause consumption to fall through intertemporal substitution, which in turn lowers inflation, but increases the marginal value of wealth, $\tilde{\lambda}_i > 0$. It implies a correlation between consumption and inflation that is exactly opposite (cf. Rudebusch and Swanson, 2012).

3.1.2. Term structure of interest rates

Following Rudebusch and Swanson (2012), the term premium on long-term nominal bonds compensates investors for inflation and consumption risk over the lifetime of the bond. The term premium is defined by comparing the equilibrium prices under the physical and the risk-neutral probability measures. Consider a (zero-coupon) nominal bond with unity payoff at maturity N. Using the expectation approach, the equilibrium price reads:

$$P_t^{(N)} = \mathbb{E}_t \left(m_{t+N}/m_t e^{-\int_t^{t+N} \pi_s ds} \right), \tag{62}$$

which can be solved, by simulating for a given maturity N forward a few thousand times, and take the average. Similarly, we could obtain the hypothetical bond price under the

risk-neutral probability measure, i.e., using a risk-neutral discount factor (or equivalently the risk-free rate) rather than the household's SDF:

$$\bar{P}_t^{(N)} = \mathbb{E}_t^{\mathbb{Q}} \left(m_{t+N}/m_t e^{-\int_t^{t+N} \pi_s ds} \right) = \mathbb{E}_t^{\mathbb{Q}} \left(e^{-\int_t^{t+N} i_s \, \mathrm{d}s} \right) \tag{63}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation under the risk-neutral probability measure and thus the risk-neutral evaluation of the bond price (Rudebusch and Swanson, 2008, 2012). Then, the yield is the (fictional) interest rate that justifies the quoted price, such that the log price $p_t^{(N)} \equiv \log P_t^{(N)}$ satisfies $y_t^{(N)} = -(1/N)p_t^{(N)}$, and the yield curve is a plot of the yields as a function of their maturity. Hence, the expectation approach is easily adapted for the perfect-foresight solution (the term premium would be zero).[18]

An alternative solution offers the PDE approach (Cochrane, 2005, chap. 19.4), in which the basic pricing equation for the price $P_t^{(N)}$ reads

$$\mathbb{E}_t \left((\, \mathrm{d}P_t^{(N)})/P_t^{(N)} \right) - \left(1/P_t^{(N)} (\partial P_t^{(N)}/\partial N) + i_t \right) \mathrm{d}t = -\mathbb{E}_t \left((\, \mathrm{d}P_t^{(N)}/P_t^{(N)}) \, (\, \mathrm{d}\lambda_t/\lambda_t) \right).$$

where $-P_N^{(N)}/P_t^{(N)}$ is the instantaneous forward rate, and for the nominal zero-coupon bond we have that $\lim_{N\to 0} (\partial P_t^{(N)}/\partial N)/P_t^{(N)} = -i_t$. The pricing equation is an application of Feynman-Kac to compute the conditional expectation by PDEs (cf. Appendix A.1).

Proposition 6. *From the PDE approach we obtain the covariance with the SDF as*

$$-\mathbb{E}_t \left((\, dP_t^{(N)}/P_t^{(N)}) \, (\, d\lambda_t/\lambda_t) \right) \;\; = \;\; -\tilde{\lambda}_{\mathbb{Z}}^{\top} \sigma(\mathbb{Z}_t) \sigma(\mathbb{Z}_t)^{\top} \tilde{P}_{\mathbb{Z}}^{(N)} \, dt,$$

where $\tilde{P}_{\mathbb{Z}}^{(N)} \equiv P_{\mathbb{Z}}^{(N)}/P_t^{(N)}$.

Proof. Online Appendix H.8 ■

Following the PDE approach and Proposition 6, the fundamental PDE reads

$$\begin{aligned} g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^{\top} P_{\mathbb{Z}}^{(N)} + \tfrac{1}{2} \mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top} P_{\mathbb{Z}\mathbb{Z}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)} i_t \\ = -\tilde{\lambda}_{\mathbb{Z}}^{\top} \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top} P_{\mathbb{Z}}^{(N)} \end{aligned} \tag{64}$$

where for the hypothetical price $\bar{P}_t^{(N)}$ the covariance terms on the RHS are zero. Focusing on the covariance has a deeper motivation: Understanding whether fundamental shocks increase or decrease either the term premium or inflation risk premium crucially depends on the sensitivity of the bond price, $\tilde{P}_{\mathbb{Z}}$, because $\tilde{\lambda}_{\mathbb{Z}}$ is endogenously given in the NK model, and both sensitivities are largely independent from either the size of the shocks or the correlation structure summarized in the covariance matrix $\Sigma(\mathbb{Z}) \equiv \sigma(\mathbb{Z})\sigma(\mathbb{Z})^{\top}$.

[18] The expectation approach is used in macroeconomic models (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2008, 2012), but is challenging to compute the complete term structure.

Proposition 7 (Affine Term Structure Model). *Observe that in the ATSM the prices* $\ln P_t^{(N)}$ *(and* $\ln \bar{P}_t^{(N)}$*) are linear functions of the state variables,*

$$\ln P_t^{(N)} = A(N) - B(N)^\top \mathbb{Z}_t,$$

where using the denominator layout (vector of derivatives as column vector)

$$\partial(\ln P_t^{(N)})/\partial \mathbb{Z}_t = -B(N), \quad \tilde{P}_{\mathbb{Z}}^{(N)} = -B(N)$$

such that $\tilde{P}_{\mathbb{Z}\mathbb{Z}}$ *depends entirely on the outer product of* $B(N)$*,*

$$\tilde{P}_{\mathbb{Z}}^{(N)} = \partial(\ln P_t^{(N)})/\partial \mathbb{Z}_t, \quad \text{and} \quad \tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)} = -\tilde{P}_{\mathbb{Z}}^{(N)} B(N)^\top = B(N)B(N)^\top.$$

Hence

$$\tfrac{1}{2} tr(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)}) = \tfrac{1}{2} tr((\tilde{P}_{\mathbb{Z}}^{(N)})^\top \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{P}_{\mathbb{Z}}^{(N)})$$

where $\tilde{P}_{\mathbb{Z}}^{(N)} \equiv P_{\mathbb{Z}}^{(N)}/P_t^{(N)}$*, and* $\tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)} \equiv P_{\mathbb{Z}\mathbb{Z}}^{(N)}/P_t^{(N)}$*.*

Proof. Online Appendix H.9 ■

ATSMs are widely used due to their analytical tractability as bond yields are affine functions of underlying factors. Our framework such factors are state variables and thus we naturally imposes a no-arbitrage condition, which typically enhances the empirical performance. Most importantly, the covariance term in the fundamental pricing equation for the ATSM is the same as in the nonlinear version (cf. Online Appendix H.9)

$$\begin{aligned} g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top P_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\mathrm{tr}((P_{\mathbb{Z}}^{(N)})^\top \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{P}_{\mathbb{Z}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)} i_t = \\ -\lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top P_{\mathbb{Z}}^{(N)}, \end{aligned} \tag{65}$$

such that we do not have to rely on second-order derivatives,

$$P_{\mathbb{Z}\mathbb{Z}}^{(N)} = -P_{\mathbb{Z}}^{(N)} B(N)^\top = P_t^{(N)} B(N)B(N)^\top = P_{\mathbb{Z}}^{(N)} (P_{\mathbb{Z}}^{(N)})^\top / P_t^{(N)}$$

or

$$\tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)} = \tilde{P}_{\mathbb{Z}}^{(N)} (\tilde{P}_{\mathbb{Z}}^{(N)})^\top \tag{66}$$

The cost of analytical tractability is the need to linearize the dynamics in $g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)$ to preserve linearity in the ATSM. While valid under normal conditions, this approach fails to capture nonlinear dynamics during crises and overlooks feedback effects from monetary and fiscal policy changes, which affect sensitivities, $\tilde{\lambda}_{\mathbb{Z}}$ (and the macro variables) and thus the effects on price sensitivities $\tilde{P}_{\mathbb{Z}}^{(N)}$ when solving the PDE.

Even in normal times close to equilibrium values, the yield curve's implications can differ even qualitatively. For instance, suppose the nonlinear model is the data-generating

process. Initializing the (certainty-equivalent) ATSM at deterministic steady-state values would predict remarkably correct term premium, but counterfactually, an upward-sloping yield curve. In contrast, the nonlinear NK model would produce a negatively sloped yield curve as in the data, as the model's value exceeds the data's unconditional mean.[19] While the ATSM accurately predicts the term premium, it fails to capture the qualitative properties of the data when forecasting yields and forward rates. If initialized at the true data-generating process mean, the (certainty-equivalent) ATSM would again imply the correct term premium but predict a much steeper term structure, due to initialization at a lower value than the model's unconditional moment (cf. Figures 9 and 10).

In either case (ATSM or nonlinear), however, we use the same strategy as it allows us to solve a linear system based on the fundamental pricing equation. Our results confirm the link in (66). Note also that for the ATSM we need an approximate solution for the costate as an affine function of the state variables (such that $\lambda_{\mathbb{Z}}$ is constant).

Our strategy is to approximate $P^{(N)} \approx \Phi(N, \mathbb{Z})u$, where $\Phi = \Phi_0 \otimes \Phi_{n_d} \otimes \Phi_{n_d-1} \otimes \cdots \otimes \Phi_1$ is the $n \times n$ interpolation matrix, i.e., the tensor product of the univariate interpolation matrices Φ_j based on $1 \times n_j$ elementary vectors $\phi_j(Z_j)$ for $j = 0, ..., n_d$, each producing

$$\phi(N, \mathbb{Z}) \equiv \phi_N(N) \otimes \phi_{n_d}(\mathbb{Z}_{n_d}) \otimes \phi_{n_d-1}(\mathbb{Z}_{n_d-1}) \otimes \cdots \otimes \phi_1(\mathbb{Z}_1),$$

row vectors of $n \equiv \prod_{i=0}^{n_d} n_i$ columns, properly stacked, such that each row of Φ corresponds to one k of the n nodal values of $S \equiv \{N, \mathbb{Z}\}$, or $\phi(N_k, \mathbb{Z}_{1k}, \ldots, \mathbb{Z}_{n_d k})$. Hence, the $n_j \times n_j$ interpolation matrices Φ_j evaluate the $1 \times n_j$ elementary vectors at their univariate nodes, each row corresponding to one of the n_j nodal values of either N or $\mathbb{Z}_1, \ldots, \mathbb{Z}_{n_d}$.

Suppose that we construct a $n \times n_d + 1$ grid S from univariate nodes N and $\mathbb{Z}$,

$$S \equiv \{N, \mathbb{Z}\} = \begin{bmatrix} N_1 & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ N_2 & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ N_1 & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ N_2 & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ N_1 & \mathbb{Z}_{3,1} & \mathbb{Z}_{1,2} & \ldots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{n_1,1} & \mathbb{Z}_{n_2,2} & \ldots & \mathbb{Z}_{n_{n_d},n_d} \end{bmatrix}$$

[19] Note that in the nonlinear model, initializing state variables at their unconditional mean does not necessarily imply that yields are at their unconditional mean due to the nonlinear transformation.

in which the lth row represents one nodal point in the $n_d + 1$ dimensional space $\mathbb{R}^{n_d+1}$.

Similarly, we require $P^{(N)}_{\mathbb{Z}_j} \approx \Phi_{\mathbb{Z}_j} u$, $\partial P^{(N)}/\partial N \approx \Phi_N u$, $P^{(N)}_{\mathbb{Z}_j}$, and $P^{(N)}_{\mathbb{Z}_i\mathbb{Z}_j} \approx \Phi_{\mathbb{Z}_i\mathbb{Z}_j} u$, to approximate the first- and second-order derivatives, respectively,

$$\sum_{j=1}^{n_d} \mathrm{diag}(g_j(S))\Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \mathrm{diag}(\Sigma_{jj})\Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \mathrm{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_j\mathbb{Z}_k} u$$
$$-\Phi_N u - \mathrm{diag}(i)\Phi u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} \mathrm{diag}(\tilde{\lambda}_{\mathbb{Z}_j}(S))\mathrm{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_k} u,$$

where

$$\Sigma_{jk}(\mathbb{Z}) \equiv \sum_{i=1}^{n_d} \sigma_{ji}(\mathbb{Z})\sigma_{ki}(\mathbb{Z}),$$

or by defining $\circ$ (element-wise operation) of an n-vector and an $n \times k$ matrix as multiplying corresponding elements, operating along the k columns of the matrix, such that

$$\sum_{j=1}^{n_d} g_j(S) \circ \Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(\mathbb{Z}) \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} u$$
$$-\Phi_N u - i \circ \Phi u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} (\tilde{\lambda}_{\mathbb{Z}_j} \circ \Sigma_{jk}) \circ \Phi_{\mathbb{Z}_k} u$$

where component functions $g_j(\mathbb{Z}, \mathbb{X}; \mathbb{Y})$ are evaluated at the $n \times 1$ vector of n nodal values, and i is the n-vector along the grid S. Each of the matrices of partial derivatives $\Phi_{\mathbb{Z}_k}$ and $\Phi_{\mathbb{Z}_k\mathbb{Z}_l}$ is a $n \times n$, and each Σ_{jk} is $n \times 1$ and $\tilde{\lambda}_{\mathbb{Z}_j}(S)$ is $n \times 1$.

Given the linearity of the Feynman-Kac equation and the boundary conditions, a collocation solution requires solving a linear matrix equation, $\mathbf{\Phi} u = 0_n$ together with the boundary condition $\Phi(0, \mathbb{Z})u = 1_n$, where

$$\mathbf{\Phi} \equiv \sum_{j=1}^{n_d} g_j \circ \Phi_{\mathbb{Z}_j} + \tfrac{1}{2}\Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} + \sum_{k=1+j}^{n_d} \Sigma_{jk} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} + \sum_{k=1}^{n_d} (\tilde{\lambda}_{\mathbb{Z}_j} \circ \Sigma_{jk}) \circ \Phi_{\mathbb{Z}_k} - \Phi_N - i \circ \Phi.$$

So we may concatenate the matrices and solve the linear $2n \times n$ equation

$$\begin{bmatrix} \mathbf{\Phi} \\ \Phi(0, \mathbb{Z}) \end{bmatrix} u = \begin{bmatrix} 0_n \\ 1_n \end{bmatrix} \quad \Leftrightarrow \quad u = \begin{bmatrix} \mathbf{\Phi} \\ \Phi(0, \mathbb{Z}) \end{bmatrix}^{-1} \begin{bmatrix} 0_n \\ 1_n \end{bmatrix}. \tag{67}$$

The solution to (64), or its approximation (67), implies the term structure of interest rates for *any* given state vector and maturity of (nominal) zero-coupon Treasury bonds,

$$y^{(N)}_t \equiv y^{(N)}(\mathbb{Z}_t) = -\log P^{(N)}(\mathbb{Z}_t)/N, \tag{68}$$

or its risk-neutral version

$$\bar{y}_t^{(N)} \equiv \bar{y}^{(N)}(\mathbb{Z}_t) = -\log \bar{P}^{(N)}(\mathbb{Z}_t)/N. \tag{69}$$

A similar approach with $\lim_{N\to 0}(\partial P_{R,t}^{(N)}/\partial N)/P_{R,t}^{(N)} = -r_t^f$ implies the term structure of interest rates $r^{(N)}(\mathbb{Z}_t)$ and its risk neutral version $\bar{r}^{(N)}(\mathbb{Z}_t)$ for Treasury inflation-protected securities (TIPS), or the price of the real zero-coupon Treasury bonds. Based on those general equilibrium prices, many asset pricing facts are simply computed.

3.1.3. Forward interest rates

The use of forward interest rates is quite common in central banks among other indicators to assess the market expectations about future monetary policy. Although they contain the same information as the standard yield curve, forward rates more easily allow a separation of expectations for the short, medium and long term (cf. Svensson, 1994).

For example, we may compute the *one-year* forward rate $f_t^{N\to N+1}$ as

$$f_t^{N\to N+1} = \log P_t^{(N)} - \log P_t^{(N+1)} \equiv p_t^{(N)} - p_t^{(N+1)}$$

which is a forward contract fixed today of investing a dollar in N years into a zero-coupon bond and holding that bond for the remaining time to maturity $N+1$.

We define the instantaneous forward rate $f_t^{(N)}$ as the rate for a forward contract with an infinitesimal investment period after maturity N. It can be approximated with an overnight forward rate, i.e., with maturity one day after settlement (cf. Svensson, 1994),

$$f_t^{(N)} = -\partial \log P_t^{(N)}/\partial N = -1/P_t^{(N)}\partial P_t^{(N)}/\partial N \tag{70}$$

such that the fundamental pricing equation has the following interpretation

$$\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)})/P_t^{(N)}\right) - (-f_t^{(N)} + i_t)\,\mathrm{d}t = -\mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)}/P_t^{(N)})\,(\,\mathrm{d}\lambda_t/\lambda_t)\right),$$

where we may define the instantaneous holding period return (hpr),

$$\begin{aligned}\mathrm{hpr}_t^{(N)} &= \mathrm{d}P_t^{(N)}/P_t^{(N)} - 1/P_t^{(N)}(\partial P_t^{(N)}/\partial N)\mathrm{d}t \\ &= \mathrm{d}P_t^{(N)}/P_t^{(N)} + f_t^{(N)}\mathrm{d}t,\end{aligned}$$

such that the expected $\mathrm{hpr}_t^{(N)}$ of the zero-coupon bond with maturity N to equal the spot rate of the short-term bond plus the premium of the long-term bond, which compensates the investor that i_t does not evolve as expected (related to its riskiness)

$$\mathbb{E}_t(\mathrm{hpr}_t^{(N)}) = i_t - \mathbb{E}_t\left((\,\mathrm{d}P_t^{(N)}/P_t^{(N)})\,(\,\mathrm{d}\lambda_t/\lambda_t)\right).$$

From our solution

$$\mathbb{E}_t(\text{hpr}_t^{(N)}) = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{P}_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\text{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)}) + f_t^{(N)}.$$

For example, consider the benchmark long-term bond in the US data, which is 10-year Treasury note (40-quarter zero-coupon bond) in Rudebusch and Swanson (2012). Even though the nominal bond is default-free, it is still risky in the sense that its price co-varies with the household's marginal utility of consumption. In the model, the covariance with the SDF is negative in the sense that the price generally falls for adverse shocks, i.e., when the SDF increases. Hence, households regard holding the bond as being riskier than rolling over short-term bonds and demand a premium over the short rate i_t.

3.1.4. Term premium and inflation risk premium

We define the (level of the) term premium

$$TP_{N,t}^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - \bar{y}^{(N)}(\mathbb{Z}_t), \tag{71}$$

and similarly

$$TP_{R,t}^{(N)} \equiv r^{(N)}(\mathbb{Z}_t) - \bar{r}^{(N)}(\mathbb{Z}_t), \tag{72}$$

which measures the 'riskiness' of the nominal and the real zero-coupon bonds, i.e., the compensation that investors require for bearing the risk that i_t or r_t^f does not evolve as expected, respectively. We may decompose nominal term premium into the term premium of the real bond and the inflation risk premium, i.e., the premium that investors require for bearing the risk that π_t does not evolve as expected. Hence

$$\begin{aligned} RP_{\pi,t}^{(N)} &= TP_{N,t}^{(N)} - TP_{R,t}^{(N)} \\ &= y^{(N)}(\mathbb{Z}_t) - r^{(N)}(\mathbb{Z}_t) - (\bar{y}^{(N)}(\mathbb{Z}_t) - \bar{r}^{(N)}(\mathbb{Z}_t)), \end{aligned} \tag{73}$$

which follows Haubrich, Pennacchi, and Ritchken (2012).

Defining the forward interest rate $\bar{f}_t^{(N)}$ under the risk-neutral probability measure as

$$\bar{f}_t^{(N)} = \partial \log \bar{P}_t^{(N)} / \partial N = -1/\bar{P}_t^{(N)} \partial P_t^{(N)} / \partial N$$

we may define the forward (nominal) term premium

$$TP_{f,t}^{(N)} \equiv f_t^{(N)} - \bar{f}_t^{(N)}, \tag{74}$$

which is another measure for the compensation for risk (excess return), and refers to the

covariance with the SDF in Proposition 6.[20] Note that we may thus approximate

$$\begin{aligned} TP_t^{(N)} \approx\ TP_{f,t}^{(N)} &\equiv \mathbb{E}_t(\mathrm{hpr}_t^{(N)}) - i_t \\ &= -\tilde{\lambda}_{\mathbb{Z}}^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}\tilde{P}_{\mathbb{Z}}^{(N)} \\ &= -\tilde{\lambda}_{\mathbb{Z}}^{\top}\Sigma(\mathbb{Z}_t)\tilde{P}_{\mathbb{Z}}^{(N)}, \end{aligned} \tag{75}$$

as the covariance measure of the term premium (or forward term premium).

Moreover, we may use (75) to approximate the inflation risk premium (73) by

$$RP_{\pi,t}^{(N)} \approx RP_{f,t}^{(N)} \equiv -\tilde{\lambda}_{\mathbb{Z}}^{\top}\Sigma(\mathbb{Z}_t)(\tilde{P}_{\mathbb{Z},N}^{(N)} - \tilde{P}_{\mathbb{Z},R}^{(N)}),$$

i.e., which shocks generate "good" versus "bad" inflation (cf. Cieslak and Pflueger, 2023). Any unfavorable shock $\lambda_{\mathbb{Z}_j} > 0$ with $j = 1, \ldots, n_d$ creates higher term premia for the Treasury bonds iff its covariance with the SDF is positive, i.e., the sensitivity of the bond price is negative, $P_{\mathbb{Z}_j}^{(N)} < 0$ (diagonal covariance matrix), or the total effect of the jth shock on the bond price is negative, $\Sigma_j(\mathbb{Z})P_{\mathbb{Z}}^{(N)} < 0$ (full covariance matrix).

Although the term premium is the most accurate measure of the riskiness of long-term bonds, it is not directly observable. Rudebusch and Swanson (2008, 2012) suggest that, for example, the average slope of the term structure serves as a useful empirical proxy for the term premium. However, the effectiveness of these measures depends on the type of shock. While the average slope of the term structure is a reliable indicator in the case of technology shocks, it does not appear to be a valid proxy for preference shocks. Hence, such measures are only applicable for specific types of shocks (see Tables 13 and 12). For illustration, we may compute the slope of the yield curve, e.g., as

$$TS_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - i_t \tag{76}$$

such that $\mathbb{E}_t(TS_{t+k}^{(N)}) = \mathbb{E}_t(y^{(N)}(\mathbb{Z}_{t+k})) - \mathbb{E}_t(i_{t+k})$ and

$$\mathbb{E}(TS_t^{(N)}) = \lim_{t\to\infty} \mathbb{E}_t(y^{(N)}(\mathbb{Z}_t)) - \mathbb{E}(i_t) = \mathbb{E}(y^{(N)}(\mathbb{Z})) - \mathbb{E}(i) \tag{77}$$

denotes the average slope of the term structure.

As argued in Backus, Chernov, Zin, and Zviadadze (2022), shocks to the monetary policy rule correlate with changes in the level (68), because the behavior is similar at both the long and short ends of the curve, whereas shocks to "discretion" correlate with changes in its slope (77), often measured by the 5-year forward spread, $f_t^{(5)} - i_t$.

[20] As argued in Svensson (1994), there are three related but distinct term premia, the forward term premium, the holding period term premium, and the rollover term premium.

3.1.5. Sign restrictions in VARs

When deriving sign restrictions for shock identification in VARs, we need to study the sensitivities of nominal or real yields with respect to shocks, $\partial TP_t^{(N)}/\partial \mathbb{Z}_t = y_{\mathbb{Z}}^{(N)} - \bar{y}_{\mathbb{Z}}^{(N)}$. For example, technology shocks imply

$$\partial y_t^{(N)}/\partial A_t \equiv y_A^{(N)} = -\frac{P_A^{(N)}}{P_t^{(N)}}/N = -\tilde{P}_A^{(N)}/N$$

and

$$\partial \bar{y}_t^{(N)}/\partial A_t \equiv \bar{y}_A^{(N)} = -\frac{P_A^{(N)}}{P_t^{(N)}}/N = -\tilde{\bar{P}}_A^{(N)}/N$$

for the risk neutral evaluation. Hence, we obtain

$$\partial TP_t^{(N)}/\partial A_t = -(\tilde{P}_A^{(N)} - \tilde{\bar{P}}_A^{(N)})/N$$

Two results are worth noting. First, the cyclicality of the term premium does not hinge on whether shocks are "good" or "bad", but on how the term premium responds. What matters are the elasticities of the bond price and the risk-neutral bond price with respect to the states. Cyclicality arises when shocks move these prices differently. For example, if yields fall after a technology shock and the bond price is more sensitive than under the risk-neutral measure, the term premium is countercyclical. Second, the term premium in the NK model is time-varying and depends on both preferences and policy parameters.[21]

3.1.6. Model-implied forecasts

In fact, the model-implied inflation forecast is a forward contract to inflation, which can be more informative than using forward rates (Gürkaynak, Sack, and Wright, 2007). Similar to the term structure of interest rates, we compute the π_{t+N} as a function of the current state variables $\mathbb{Z}_t$ and the forecasting horizon N. For the N-year ahead future expected inflation rate, we compute $\pi_t^{(N)}$ from (cf. Appendix A.1)

$$\partial \pi_t^{(N)}/\partial N \;=\; g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top \pi_{\hat{\mathbb{Z}}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \pi_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^{(N)})$$

together with the solution $\hat{\mathbb{X}}_t = \hat{\mathbb{X}}(\hat{\mathbb{Z}}_t)$ and $\hat{\mathbb{Y}}_t = \hat{\mathbb{Y}}(\hat{\mathbb{Z}}_t)$ and by imposing the boundary condition $\pi_t^{(0)} = \pi_t = \pi(\hat{\mathbb{Z}}_t)$. Similar to the term structure of interest rates, the solution to the PDE implies the N-years ahead inflation expectations for given state variables as

$$\pi_t^{(N)} = \pi^{(N)}(\hat{\mathbb{Z}}_t). \tag{78}$$

[21] Note that the time variability requires a solution method beyond the linear approximation.

Because the model time unit is years, we may relate the N-year ahead inflation forecast $\pi_t^{(N)}$ to the empirical NY1Y measure. Similarly, we obtain the model-implied forecast for any other variable, such as the expected prices/yields or the expected state variables, or even their limiting distribution and any moment $\mathbb{E}(\hat{\mathbb{Z}}_t^k)$ for $k \geq 0$.

Consider the N-year ahead forecast for the spot rate, i.e., the expected instantaneous forward rate in N years hence, $i_t^{(N)}$, which is a function of the forecasting horizon and the current states $\hat{\mathbb{Z}}_t$. We may solve Kolmogorov's backward equation, which reads

$$\partial i_t^{(N)}/\partial N \quad = \quad g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top i_{\hat{\mathbb{Z}}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top i_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^{(N)})$$

with boundary condition $i_t^{(0)} = i_t$. Moreover, as shown above, the expected spot rate

$$i_t^{(N)} \equiv \mathbb{E}_t(\mathrm{hpr}_t^{(N)}) - TP_{f,t}^{(N)},$$

or, equivalently, the holding period return under the risk-neutral evaluation.

Our strategy allows us to compute the coefficients from a *linear* equation where we approximate $i_t^{(N)} \approx \Phi(N, \hat{\mathbb{Z}}_t)u$ in which u is an n-vector of coefficients, and Φ denotes the known $n \times n$ basis matrix, the tensor product of the univariate interpolation matrices, and $\Phi_{\mathbb{Z}}$ and $\Phi_{\mathbb{Z}\mathbb{Z}}$ are the known multidimensional arrays of derivatives of the interpolation matrix, evaluated at the specified nodes (cf. Section 3.1.2). For example,

$$\sum_{j=1}^{n_d} g_j(\hat{\mathbb{Z}}, \hat{\mathbb{X}}; \hat{\mathbb{Y}}) \circ \Phi_{\hat{\mathbb{Z}}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj} \circ \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(\hat{\mathbb{Z}}) \circ \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k} u = \Phi_N u$$

or

$$\sum_{j=1}^{n_d} \mathrm{diag}(g_j(S))\Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \mathrm{diag}(\Sigma_{jj})\Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \mathrm{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_j\mathbb{Z}_k} u = \Phi_N u,$$

or more compactly, we require that at each collocation node $l \in 1, \ldots, n$,

$$g(\mathbb{Z}_l, \mathbb{X}_l; \mathbb{Y}_l)^\top \Phi_{\mathbb{Z}} u + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_l)\sigma(\mathbb{Z}_l)^\top \Phi_{\mathbb{Z}\mathbb{Z}} u) - \Phi_N u = 0$$

together with the n boundary conditions

$$\Phi(0, \hat{\mathbb{Z}}_l) = i_l 1_n, \quad l \in 1, \ldots, n$$

such that the linear $2n \times n$ equation for the unknown coefficients, defining

$$\mathbf{\Phi} \equiv \sum_{j=1}^{n_d} g_j \circ \Phi_{\mathbb{Z}_j} + \tfrac{1}{2}\Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} + \sum_{k=1+j}^{n_d} \Sigma_{jk} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} - \Phi_N,$$

and

$$\begin{bmatrix} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{bmatrix} u = \begin{bmatrix} 0_n \\ i \end{bmatrix} \quad \Leftrightarrow \quad u = \begin{bmatrix} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{bmatrix}^{-1} \begin{bmatrix} 0_n \\ i \end{bmatrix}, \tag{79}$$

where i denotes the $n \times 1$ vector in $S \in \mathbb{R}^{n_d}$. Similar to the term-structure of interest rates, the solution to (79) the N-period-ahead conditional expectation (forecast) for the variables of interest, e.g., the nominal interest rate i_t, given a particular state $\mathbb{Z}_0$. Under certain regularity conditions, the forecast converges to its unconditional mean.

3.1.7. Term structure of inflation

The term structure of inflation shows how expected inflation depends on the forecasting horizon. How is the measure related to empirical measure of break-even inflation?

Having computed nominal and TIPS yields of zero-coupon Treasury bonds, we can solve for the rates of inflation compensation. Those are the inflation rates which, if realized, would leave an investor indifferent between holding a real and a nominal Treasury security (Gürkaynak, Sack, and Wright, 2010, p.76). For the continuously compounded inflation compensation rate at maturity N, the breakeven inflation rate is,

$$BE_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - r^{(N)}(\mathbb{Z}_t), \tag{80}$$

The breakeven inflation rate is also used for computing the inflation risk premium in (73) by subtracting the risk-neutral breakeven inflation rate. Hence, this measure shows the model-implied expected inflation (only if inflation risk premium is negligible).

Below we show the implementation for models in order to shed light on the effects of structural shocks on term premia and the inflation risk premium. The latter compensates the investors of nominal bonds for the risk that inflation does not evolve as expected. If nominal bonds tend to provide higher returns in bad economic states, inflation would be good to hedge against unfavorable shifts (and would generate a negative premium).

3.2. Data

Our data is from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), and the Cleveland Fed's Center for Inflation Research from 1990 to 2025.[22] The dataset primarily consists of macroeconomic indicators, including monthly consumption (PCE), inflation based on consumer price index (CPI) and personal consumption expenditures (PCEPI). Additionally, it includes wages (compensation of employees), hours (average weekly hours), and the output gap derived from quarterly US Gross Domestic Product

[22]For details see the FRED: https://fred.stlouisfed.org and the Cleveland Fed's Center for Inflation Research: https://www.clevelandfed.org/indicators-and-data/inflation-expectations

(cf. Figure 1).[23] The financial market data encompasses the US Effective Federal Funds Rate (Fed Funds), the N-Year Treasury Constant Maturity Rate (Govt), and the N-Year Treasury Inflation Protected Securities Rate (TIPS). It also includes fitted data from Gürkaynak, Sack, and Wright (2007, 2010), Laubach and Williams (2003), and Haubrich, Pennacchi, and Ritchken (2012).[24]

3.3. "Good" and "bad" shocks in the NK Model

Assets with a negative covariance between its payoffs with consumption can be used as a vehicle to hedge against unfavorable shifts, because they tend to pay off when the SDF is high. The reason is that they tend to provide higher returns when marginal utility is higher (often coinciding with bad economic times, i.e., in adverse situations) when the investor most need it to. Hence, investors are usually willing to accept lower expected returns for assets that hedge against unfavorable shifts. In what follows, we categorize fundamental economic shocks generating a positive risk premium as "bad" shocks.

Cieslak and Pflueger (2023) classify inflation as "good" or "bad," depending on whether it reduces or increases bond risk premia. The authors define a demand shock as anything that can increase consumption and output at a given risk-free rate, whereas a supply shock captures any disturbance to the relationship between the output gap and marginal costs of production due to monopolistic competition.[25] Hence, demand shocks generally should lead to inflation of the "good" variety and result in a negative inflation risk premium, while supply shocks lower the real value of long-term nominal bonds, generate inflation of the "bad" variety, inducing a positive inflation risk premium. Based on our definition in (73), we categorize the shocks in supply shocks (which move output and inflation in opposite directions) and demand shocks (which move output and inflation in the same direction) and show that the conclusion is more nuanced. Specifically, monetary policy shocks, while producing "good" inflation, actually increase the riskiness of nominal bonds by raising the term premium. This is due to the fact that the term premium on long-term real bonds rises more than that on nominal bonds. We find that technology shocks (supply shocks), preference shocks, and shocks to the discount rate (both demand shocks) increase the nominal and real term premium and the inflation risk premium.

We ask which macro risks induce "unfavorable shifts" as in (61). For example, by

[23] Since our results are not sensitive to the choice of data, we simply use HP-filtered data as a proxy for the output gap (see Figure 2 for alternative measures, cf. Kamber, Morley, and Wong, 2018).

[24] Available at https://www.federalreserve.gov/data/yield-curve-models.htm for the nominal yield curve (feds200628.csv from Gürkaynak, Sack, and Wright, 2007), the TIPS yield curve and inflation compensation (feds200805.csv from Gürkaynak, Sack, and Wright, 2010), the three-factor nominal term structure model (feds200533.csv from Kim and Wright, 2005), the inflation expectations and real rates (expectations.xlsx from Haubrich, Pennacchi, and Ritchken, 2012), and estimates of the natural rate from Laubach and Williams (2003): https://www.newyorkfed.org/research/policy/rstar

[25] Cieslak and Pflueger (2023) define (reduced-form) demand shocks as anything that enters the Euler equation, whereas supply shocks as appearing in the Phillips curve equation.

raising interest rates, monetary policy shocks make it more attractive to save and less attractive to borrow, increasing the marginal value of wealth, $\tilde{\lambda}_i > 0$. Such shocks are inducing an adverse situation, typically associated with lower consumption. The same holds for preference shocks, $\tilde{\lambda}_d > 0$, discount rate shocks, $\tilde{\lambda}_r > 0$, and shocks to economic uncertainty, $\tilde{\lambda}_\sigma > 0$. In contrast, supply shocks (or so-called cost-push), however, decrease the marginal value of wealth by lowering the marginal costs of production, and we obtain for temporary shocks to technology $\tilde{\lambda}_A < 0$, permanent shocks to technology $\tilde{\lambda}_z < 0$, shocks to technology growth $\tilde{\lambda}_x < 0$, and shocks to the inflation target, $\tilde{\lambda}_\pi < 0$.

Having identified the sensitivities of the marginal value of wealth, $\lambda_{\mathbb{Z}_j}$, we are ready to categorize the fundamental shocks based on the sign of $\Sigma_j(\mathbb{Z})P_{\mathbb{Z}}^{(N)}$, i.e., the change of the bond price with respect to states $P_{\mathbb{Z}}^{(N)}$ (or percentage terms $\tilde{P}_{\mathbb{Z}}^{(N)}$) weighted by the jth line of the covariance matrix, hence from (75) the jth shock (risk) generates

$$TP_j^{(N)} \approx TP_{f,j}^{(N)} \equiv -\tilde{\lambda}_{\mathbb{Z}_j}\Sigma_j(\mathbb{Z})\tilde{P}_{\mathbb{Z}}^{(N)}. \tag{81}$$

The presence of risk associated with state j will increase the term premium as long as the effects on the discount factor and yields are qualitatively the same (same direction). This decomposition holds for the price of both the nominal and the real zero-coupon Treasury bond. "Bad" shocks may generate "good" inflation, if they generate a higher term premium for real bonds than for nominal bonds, generating a negative risk premium

$$RP_j^{(N)} \approx RP_{f,j}^{(N)} \equiv -\tilde{\lambda}_{\mathbb{Z}_j}\Sigma_j(\mathbb{Z})(\tilde{P}_{\mathbb{Z},N}^{(N)} - \tilde{P}_{\mathbb{Z},R}^{(N)}). \tag{82}$$

It is important to note that the decomposition is helpful for identifying the theoretical contribution and importance of shocks to the *level* of term premia and risk premia. This is complementary to sign restrictions of impulse responses for shock identification used in the literature, which sheds light on the cyclicality or variance decomposition in VARs.

Summarizing, "good" shocks are those who decrease the term premium, meaning they tend to cause yields and the discount factor to move in opposite directions. Fundamental shocks that result in unfavorable shifts in the state j, where $\tilde{\lambda}_{\mathbb{Z}_j} > 0$, can still be considered "good" as long as they decrease yields, shown by $-\tilde{P}_{\mathbb{Z}_j}^{(N)} < 0$. Notably, $\tilde{\lambda}_{\mathbb{Z}_j}$ can be seen as a semi-elasticity, indicating, say, that a one percentage point increase in i_t results in an approximately $\tilde{\lambda}_i$ percent change in λ_t. Furthermore, whether adverse shocks with $\tilde{\lambda}_{\mathbb{Z}_j} > 0$ generate "good" inflation (decrease the inflation risk premium) depends on whether they cause nominal and real yields, represented by their total percentage price changes, to move such that $-\tilde{P}_{\mathbb{Z}_j,N}^{(N)} \gtrless -\tilde{P}_{\mathbb{Z}_j,R}^{(N)}$. For shocks decreasing both term premia, the decrease in nominal yields should be more sensitive than the decrease in real yields. Hence, this relationship is quantitatively affected by how sensitive prices are to changes.[26]

[26]Rudebusch and Swanson (2012) study the term premium of long-term nominal bonds. From the IRFs of bond price and consumption, we may categorize their shocks into fundamental shocks of the

3.3.1. Technology shocks

For example, the model with technology shocks for $\delta < \infty$ has $n_d \equiv \dim(\mathbb{Z}) = 4$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_4$ for the relevant states $\mathbb{Z} = (i, v, A, z)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 \times n_4 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and a grid of $n = 5640$ ($n = 70 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \mathbb{Z})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(y(S_l))/(y_{ss}z_l) - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \\ \mu z_l \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 \\ 0 & 0 & 0 & \sigma_z z_l \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 \\ 0 & 0 & 0 & \sigma_z^2 z_l^2 \end{bmatrix}$$

such that

$$\Sigma_{jk}(\mathbb{Z}_l) \equiv \sigma_{j1}(\mathbb{Z}_l)\sigma_{k1}(\mathbb{Z}_l) + \sigma_{j2}(\mathbb{Z}_l)\sigma_{k2}(\mathbb{Z}_l) + \sigma_{j3}(\mathbb{Z}_l)\sigma_{k3}(\mathbb{Z}_l) + \sigma_{j4}(\mathbb{Z}_l)\sigma_{k4}(\mathbb{Z}_l),$$

denotes the lth element in the $n \times 1$ matrix Σ_{jk} for $j = 1, \ldots, 4$ and $k = 1, \ldots, 4$, i.e.,

$$\Sigma_{1k}(\mathbb{Z}_l) = \sigma_{11}(\mathbb{Z}_l)\sigma_{k1}(\mathbb{Z}_l) + \sigma_{12}(\mathbb{Z}_l)\sigma_{k2}(\mathbb{Z}_l) + \sigma_{13}(\mathbb{Z}_l)\sigma_{k3}(\mathbb{Z}_l) + \sigma_{14}(\mathbb{Z}_l)\sigma_{k4}(\mathbb{Z}_l)$$

or $\Sigma_{11}(\mathbb{Z}) = \sigma_i^2 \cdot 1_n$, $\Sigma_{12}(\mathbb{Z}) = 0_n$, $\Sigma_{13}(\mathbb{Z}) = 0_n$, $\Sigma_{14}(\mathbb{Z}) = 0_n$,

$$\Sigma_{2k}(\mathbb{Z}_l) = \sigma_{21}(\mathbb{Z}_l)\sigma_{k1}(\mathbb{Z}_l) + \sigma_{22}(\mathbb{Z}_l)\sigma_{k2}(\mathbb{Z}_l) + \sigma_{23}(\mathbb{Z}_l)\sigma_{k3}(\mathbb{Z}_l) + \sigma_{24}(\mathbb{Z}_l)\sigma_{k4}(\mathbb{Z}_l)$$

or $\Sigma_{21}(\mathbb{Z}) = 0_n$, $\Sigma_{22}(\mathbb{Z}) = 0_n$, $\Sigma_{23}(\mathbb{Z}) = 0_n$, $\Sigma_{24}(\mathbb{Z}) = 0_n$,

$$\Sigma_{3k}(\mathbb{Z}_l) = \sigma_{31}(\mathbb{Z}_l)\sigma_{k1}(\mathbb{Z}_l) + \sigma_{32}(\mathbb{Z}_l)\sigma_{k2}(\mathbb{Z}_l) + \sigma_{33}(\mathbb{Z}_l)\sigma_{k3}(\mathbb{Z}_l) + \sigma_{34}(\mathbb{Z}_l)\sigma_{k4}(\mathbb{Z}_l)$$

"good" and "bad" variety as follows: For the fundamental shocks to monetary policy, fiscal policy and technology, and preferences is negatively related to consumption. Hence, the term premium is larger, the more negative the covariance between the discount factor and the price of the bond, i.e., for fundamental shocks that move both consumption and bond prices in the same direction ("bad" shocks).

or $\Sigma_{31}(\mathbb{Z}) = 0_n$, $\Sigma_{32}(\mathbb{Z}) = 0_n$, $\Sigma_{33}(\mathbb{Z}) = \sigma_A^2 A^2$, $\Sigma_{34}(\mathbb{Z}) = 0_n$,

$$\Sigma_{4k}(\mathbb{Z}_l) = \sigma_{41}(\mathbb{Z}_l)\sigma_{k1}(\mathbb{Z}_l) + \sigma_{42}(\mathbb{Z}_l)\sigma_{k2}(\mathbb{Z}_l) + \sigma_{43}(\mathbb{Z}_l)\sigma_{k3}(\mathbb{Z}_l) + \sigma_{44}(\mathbb{Z}_l)\sigma_{k4}(\mathbb{Z}_l)$$

or $\Sigma_{41}(\mathbb{Z}) = 0_n$, $\Sigma_{42}(\mathbb{Z}) = 0_n$, $\Sigma_{43}(\mathbb{Z}) = 0_n$, $\Sigma_{34}(\mathbb{Z}) = \sigma_z^2 z^2$, together with the solution of the HJB equation, which yields

$$\lambda \equiv \lambda(\mathbb{Z}), \quad V \equiv V(\mathbb{Z}), \quad \pi \equiv \pi(\mathbb{Z}), \quad y \equiv y(\mathbb{Z})$$

and derivatives

$$\lambda_{\mathbb{Z}} \equiv \lambda_{\mathbb{Z}}(\mathbb{Z}), \quad V_{\mathbb{Z}} \equiv V_{\mathbb{Z}}(\mathbb{Z}), \quad \lambda_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}), \quad V_{\mathbb{Z}\mathbb{Z}} \equiv V_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z})$$

with

$$\lambda_{\mathbb{Z}} \equiv \begin{bmatrix} \lambda_i(\mathbb{Z}) \\ \lambda_v(\mathbb{Z}) \\ \lambda_A(\mathbb{Z}) \\ \lambda_z(\mathbb{Z}) \end{bmatrix}, \quad V_{\mathbb{Z}} \equiv \begin{bmatrix} V_i(\mathbb{Z}) \\ V_v(\mathbb{Z}) \\ V_A(\mathbb{Z}) \\ V_z(\mathbb{Z}) \end{bmatrix},$$

$$\lambda_{\mathbb{Z}\mathbb{Z}} \equiv \begin{bmatrix} \lambda_{ii}(\mathbb{Z}) & \lambda_{iv}(\mathbb{Z}) & \lambda_{iA}(\mathbb{Z}) & \lambda_{iz}(\mathbb{Z}) \\ \lambda_{vi}(\mathbb{Z}) & \lambda_{vv}(\mathbb{Z}) & \lambda_{vA}(\mathbb{Z}) & \lambda_{vz}(\mathbb{Z}) \\ \lambda_{Ai}(\mathbb{Z}) & \lambda_{Av}(\mathbb{Z}) & \lambda_{AA}(\mathbb{Z}) & \lambda_{Az}(\mathbb{Z}) \\ \lambda_{zi}(\mathbb{Z}) & \lambda_{zv}(\mathbb{Z}) & \lambda_{zA}(\mathbb{Z}) & \lambda_{zz}(\mathbb{Z}) \end{bmatrix}, \quad V_{\mathbb{Z}\mathbb{Z}} \equiv \begin{bmatrix} V_{ii}(\mathbb{Z}) & V_{iv}(\mathbb{Z}) & V_{iA}(\mathbb{Z}) & V_{iz}(\mathbb{Z}) \\ V_{vi}(\mathbb{Z}) & V_{vv}(\mathbb{Z}) & V_{vA}(\mathbb{Z}) & V_{vz}(\mathbb{Z}) \\ V_{Ai}(\mathbb{Z}) & V_{Av}(\mathbb{Z}) & V_{AA}(\mathbb{Z}) & V_{Az}(\mathbb{Z}) \\ V_{zi}(\mathbb{Z}) & V_{zv}(\mathbb{Z}) & V_{zA}(\mathbb{Z}) & V_{zz}(\mathbb{Z}) \end{bmatrix}$$

or

$$\tilde{\lambda}_{\mathbb{Z}} \equiv \lambda_{\mathbb{Z}}/\lambda, \quad \tilde{V}_{\mathbb{Z}} \equiv V_{\mathbb{Z}}/V, \quad \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}/\lambda, \quad \tilde{V}_{\mathbb{Z}\mathbb{Z}} \equiv V_{\mathbb{Z}\mathbb{Z}}/V$$

such that

$$\mathrm{tr}\Big(\Sigma(S_l)\lambda_{\mathbb{Z}\mathbb{Z}}\Big) = \sigma_i^2\lambda_{ii} + \sigma_A^2 A_l^2 \lambda_{AA} + \sigma_z^2 z_l^2 \lambda_{zz},$$

and $\Phi_{\mathbb{Z}_j}$ and $\Phi_{\mathbb{Z}_j\mathbb{Z}_k}$ where

$$\begin{aligned} P_N^{(N)}(S_l) &\approx \Phi_N(N_l, \mathbb{Z}_l)u = \Phi_N(S_l)u, \quad l = 1, \ldots, n, \\ P_j^{(N)}(S_l) &\approx \Phi_{\mathbb{Z}_j}(N_l, \mathbb{Z}_l)u = \Phi_{\mathbb{Z}_j}(S_l)u, \quad l = 1, \ldots, n, \quad j = 1, \ldots, d, \\ P_{jk}^{(N)}(S_l) &\approx \Phi_{\mathbb{Z}_j\mathbb{Z}_k}(N_l, \mathbb{Z}_l)u = \Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)u, \quad l = 1, \ldots, n, \quad j, k = 1, \ldots, d, \end{aligned}$$

evaluates the lth row of the basis matrix $\Phi(S_l)$, respectively its derivatives $\Phi_N(S_l)$, $\Phi_{\mathbb{Z}_j}(S_l)$,

or $\Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)$, such that at each collocation node $S_l \in \mathbb{R}^4$ (each lth row of the grid S)

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\mathbb{Z}_j}(S_l)u + \tfrac{1}{2}\sum_{j=1}^{n_d}\Sigma_{jj}(S_l)\Phi_{\mathbb{Z}_j\mathbb{Z}_j}(S_l)u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d}\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)u \\
&-\Phi_N(S_l)u - i_l\Phi(S_l)u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d}\tilde{\lambda}_{\mathbb{Z}_j}(S_l)\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_k}(S_l)u \qquad (83)
\end{aligned}$$

Hence, we may construct the collocation matrix $\mathbf{\Phi}$, whose typical lth elements are

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\mathbb{Z}_j}(S_l) + \tfrac{1}{2}\left(\Sigma_{11}(S_l)\Phi_{\mathbb{Z}_1\mathbb{Z}_1}(S_l) + \Sigma_{33}(S_l)\Phi_{\mathbb{Z}_3\mathbb{Z}_3}(S_l) + \Sigma_{44}(S_l)\Phi_{\mathbb{Z}_4\mathbb{Z}_4}(S_l)\right) \\
&-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_{\mathbb{Z}_1}(S_l)\Sigma_{11}(S_l)\Phi_{\mathbb{Z}_1}(S_l) + \tilde{\lambda}_{\mathbb{Z}_3}(S_l)\Sigma_{33}(S_l)\Phi_{\mathbb{Z}_3}(S_l) + \tilde{\lambda}_{\mathbb{Z}_4}(S_l)\Sigma_{44}(S_l)\Phi_{\mathbb{Z}_4}(S_l)
\end{aligned}$$

or

$$\begin{aligned}
&g_1(S_l)\Phi_i(S_l) + g_2(S_l)\Phi_v(S_l) + g_3(S_l)\Phi_A(S_l) + g_4(S_l)\Phi_z(S_l) \\
&+\tfrac{1}{2}\left(\sigma_i^2\Phi_{ii}(S_l) + \sigma_A^2 A_l^2\Phi_{AA}(S_l) + \sigma_z^2 z_l^2\Phi_{zz}(S_l)\right) \\
&-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_i\sigma_i^2\Phi_i(S_l) + \tilde{\lambda}_A A_l^2\sigma_A^2\Phi_A(S_l) + \tilde{\lambda}_z z_l^2\sigma_2^2\Phi_z(S_l)
\end{aligned}$$

such that $\mathbf{\Phi}u = 0$ together with the n boundary conditions requiring $\Phi(0,\mathbb{Z})u = 1$.

Note that the collocation matrix $\mathbf{\Phi}$, whose typical lth elements reads

$$\begin{aligned}
&(\theta\phi_\pi(\pi(S_l) - \pi_{ss}) - \theta(i_l - i_{ss}))\Phi_i(S_l) + (\cdot)v_l\Phi_v(S_l) - (\rho_A\log A_l - \tfrac{1}{2}\sigma_A^2)A_l\Phi_A(S_l) + \mu z_l\Phi_z(S_l) \\
&+\tfrac{1}{2}\left(\sigma_i^2\Phi_{ii}(S_l) + \sigma_A^2 A_l^2\Phi_{AA}(S_l) + \sigma_z^2 z_l^2\Phi_{zz}(S_l)\right) \\
&-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_i\sigma_i^2\Phi_i(S_l) + \tilde{\lambda}_A A_l^2\sigma_A^2\Phi_A(S_l) + \tilde{\lambda}_z z_l^2\sigma_2^2\Phi_z(S_l)
\end{aligned}$$

is used to solve for the unknown coefficients u solving the fundamental PDE (64)

$$\begin{aligned}
&g(\mathbb{Z}_t,\mathbb{X}_t;\mathbb{Y}_t)^\top P_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top P_{\mathbb{Z}\mathbb{Z}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)} i_t \\
&= -\tilde{\lambda}_{\mathbb{Z}}^\top\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top P_{\mathbb{Z}}^{(N)}
\end{aligned}$$

3.3.2. Technology shocks (stationary version)

For example, the model with technology shocks for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 3$ relevant states. We use n_0 nodal points for N, $n_1,\ldots,n_3$ for the relevant states $\hat{\mathbb{Z}} = (i,v,A)$, such that $n \equiv n_0\times n_1\times n_2\times n_3 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 1890$ ($n = 70\times 3\times 3\times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi}u = 0_n$ and the n boundary conditions $\Phi(0,\hat{\mathbb{Z}})u = 1_n$. In particular,

the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 \end{bmatrix}$$

such that

$$\Sigma_{jk}(\hat{\mathbb{Z}}_l) \equiv \sigma_{j1}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{j2}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{j3}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l),$$

denotes the lth element in the $n \times 1$ matrix Σ_{jk} for $j = 1, \ldots, 3$ and $k = 1, \ldots, 3$, i.e.,

$$\Sigma_{1k}(\hat{\mathbb{Z}}_l) = \sigma_{11}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{12}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{13}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l)$$

or $\Sigma_{11}(\hat{\mathbb{Z}}) = \sigma_i^2 \cdot 1_n$, $\Sigma_{12}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{13}(\hat{\mathbb{Z}}) = 0_n$,

$$\Sigma_{2k}(\hat{\mathbb{Z}}_l) = \sigma_{21}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{22}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{23}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l)$$

or $\Sigma_{21}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{22}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{23}(\hat{\mathbb{Z}}) = 0_n$,

$$\Sigma_{3k}(\hat{\mathbb{Z}}_l) = \sigma_{31}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{32}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{33}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l)$$

or $\Sigma_{31}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{32}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{33}(\hat{\mathbb{Z}}) = \sigma_A^2 A^2$, together with the solution of the HJB equation, which yields

$$\hat{\lambda} \equiv \hat{\lambda}(\hat{\mathbb{Z}}), \quad \hat{V} \equiv \hat{V}(\hat{\mathbb{Z}}), \quad \pi \equiv \pi(\hat{\mathbb{Z}}), \quad \hat{y} \equiv \hat{y}(\hat{\mathbb{Z}})$$

and derivatives

$$\hat{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad \hat{V}_{\hat{\mathbb{Z}}} \equiv \hat{V}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}})$$

with

$$\hat{\lambda}_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_i(\hat{\mathbb{Z}}) \\ \hat{\lambda}_v(\hat{\mathbb{Z}}) \\ \hat{\lambda}_A(\hat{\mathbb{Z}}) \end{bmatrix}, \quad \hat{V}_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{V}_i(\hat{\mathbb{Z}}) \\ \hat{V}_v(\hat{\mathbb{Z}}) \\ \hat{V}_A(\hat{\mathbb{Z}}) \end{bmatrix},$$

$$\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_{ii}(\hat{\mathbb{Z}}) & \hat{\lambda}_{iv}(\hat{\mathbb{Z}}) & \hat{\lambda}_{iA}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{vi}(\hat{\mathbb{Z}}) & \hat{\lambda}_{vv}(\hat{\mathbb{Z}}) & \hat{\lambda}_{vA}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{Ai}(\hat{\mathbb{Z}}) & \hat{\lambda}_{Av}(\hat{\mathbb{Z}}) & \hat{\lambda}_{AA}(\hat{\mathbb{Z}}) \end{bmatrix}, \quad \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{V}_{ii}(\hat{\mathbb{Z}}) & \hat{V}_{iv}(\hat{\mathbb{Z}}) & \hat{V}_{iA}(\hat{\mathbb{Z}}) \\ \hat{V}_{vi}(\hat{\mathbb{Z}}) & \hat{V}_{vv}(\hat{\mathbb{Z}}) & \hat{V}_{vA}(\hat{\mathbb{Z}}) \\ \hat{V}_{Ai}(\hat{\mathbb{Z}}) & \hat{V}_{Av}(\hat{\mathbb{Z}}) & \hat{V}_{AA}(\hat{\mathbb{Z}}) \end{bmatrix}$$

or

$$\tilde{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}} \equiv \hat{V}_{\hat{\mathbb{Z}}}/\hat{V}, \quad \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{V}$$

such that

$$\mathrm{tr}\Big(\Sigma(S_l)\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\Big) = \sigma_i^2\hat{\lambda}_{ii} + \sigma_A^2 A_l^2 \hat{\lambda}_{AA},$$

and $\Phi_{\hat{\mathbb{Z}}_j}$ and $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}$ where

$$\begin{aligned}
P_N^{(N)}(S_l) &\approx \Phi_N(N_l, \hat{\mathbb{Z}}_l)u = \Phi_N(S_l)u, \quad l = 1, \ldots, n, \\
P_j^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j}(S_l)u, \quad l = 1, \ldots, n, \quad j = 1, \ldots, d, \\
P_{jk}^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u, \quad l = 1, \ldots, n, \quad j, k = 1, \ldots, d,
\end{aligned}$$

evaluates the lth row of the basis matrix $\Phi(S_l)$, respectively its derivatives $\Phi_N(S_l)$, $\Phi_{\hat{\mathbb{Z}}_j}(S_l)$, or $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)$, such that at each collocation node $S_l \in \mathbb{R}^4$ (each lth row of the grid S)

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l)u + \tfrac{1}{2}\sum_{j=1}^{n_d}\Sigma_{jj}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_j}(S_l)u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d}\Sigma_{jk}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u \\
&-\Phi_N(S_l)u - i_l\Phi(S_l)u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d}\tilde{\lambda}_{\hat{\mathbb{Z}}_j}(S_l)\Sigma_{jk}(S_l)\Phi_{\hat{\mathbb{Z}}_k}(S_l)u
\end{aligned} \tag{84}$$

Hence, we may construct the collocation matrix $\mathbf{\Phi}$, whose typical lth elements are

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l) + \tfrac{1}{2}\left(\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1\hat{\mathbb{Z}}_1}(S_l) + \Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3\hat{\mathbb{Z}}_3}(S_l)\right) - \Phi_N(S_l) - i_l\Phi(S_l) \\
&+\tilde{\lambda}_{\hat{\mathbb{Z}}_1}(S_l)\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1}(S_l) + \tilde{\lambda}_{\hat{\mathbb{Z}}_3}(S_l)\Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3}(S_l)
\end{aligned}$$

or

$$\begin{aligned}
&g_1(S_l)\Phi_i(S_l) + g_2(S_l)\Phi_v(S_l) + g_3(S_l)\Phi_A(S_l) + \tfrac{1}{2}\left(\sigma_i^2\Phi_{ii}(S_l) + \sigma_A^2 A_l^2\Phi_{AA}(S_l)\right) \\
&-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_i\sigma_i^2\Phi_i(S_l) + \tilde{\lambda}_A A_l^2\sigma_A^2\Phi_A(S_l)
\end{aligned}$$

used to pin down the coefficients u which solve the fundamental PDE (64)

$$\begin{aligned}
&g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top P_{\hat{\mathbb{Z}}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top P_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)} i_t \\
&= -\tilde{\lambda}_{\hat{\mathbb{Z}}}^\top\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top P_{\hat{\mathbb{Z}}}^{(N)}
\end{aligned}$$

such that $\mathbf{\Phi}u = 0$ together with the n boundary conditions requiring $\Phi(0, \hat{\mathbb{Z}})u = 1$.

Table 14 confirms our presumption that $\tilde{\lambda}_A < 0$ and $\lambda_z < 0$, such that only monetary policy shocks create unfavorable shifts in the simple NK model, $\tilde{\lambda}_i > 0$. Although positive

shocks create favorable shifts in the state A_t (and z_t), technology shocks are "bad" shocks, because they increase the term premium in the NK model, meaning they tend to cause the discount factor and bond yields to move in the same direction. To be more exact, the temporary shocks are less sensitive for the discount factor (inelastic) than permanent shocks (elastic). However, while prices do respond to temporary technology shocks (inelastic), they do *not* respond to permanent shocks (unelastic). Why? Simply because the equilibrium inflation rate is stationary and does *not* depend on z_t. Further, temporary technology shocks generate "bad" inflation (increase the inflation risk premium), because nominal prices are more responsive than real prices, $\tilde{P}^{(N)}_{A,N} > \tilde{P}^{(N)}_{A,R}$. Similarly, positive shocks to the nominal spot rate reflect an unfavorable shift in i_t, and thus monetary policy shocks are "bad" fundamental shocks. On the other hand, they generate "good" inflation (decrease the inflation risk premium) since nominal yields are less responsive (inelastic) than real yields (elastic), such that $-\tilde{P}^{(N)}_{i,N} < -\tilde{P}^{(N)}_{i,R}$.[27]

Similar to Rudebusch and Swanson (2012), the NK model generates a countercyclical term premium in response to technology shocks.[28] Clearly, the outcome depends on the solution method. While the effects on yields are comparable, a (small) time variation in the term premium arises only in the nonlinear solution. In contrast, the NK affine term structure model generates an acyclical premium that remains constant over time. To put the magnitudes into perspective, using the parameterization in Table 2 for the model with technology shocks, a 5 percent positive technology shock reduces the term premium by about 0.5 percent at the five-year horizon and by approximately 0.2 percent at the ten-year horizon. In response to monetary policy shocks, the term premium displays procyclical behavior: a 100 basis point reduction in nominal interest rates increases the five-year term premium by roughly 0.38 percent. Both nominal and real term premiums respond similarly, leaving the inflation risk premium largely unaffected.

3.3.3. Long-run risks

In the long-run risks (LRR) model, embedded in the NK model with time-varying inflation target for $\delta < \infty$, has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 5$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_5$ for the relevant states $\hat{\mathbb{Z}} = (i, v, A, x, \pi^*)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 \times n_3 \times n_4 \times n_5 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 17010$ ($n = 70 \times 3 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular,

[27] Prices of existing bonds tend to fall if interest rates rise (more sensitive for assets with longer duration), yields have a negative covariance with consumption and create positive term premium. Prices of existing bonds tend to rise with positive (temporary) technology shocks and also create positive term premium.

[28] In the flexible price limit, the sign reverses and produces counterfactual responses; that is, technology shocks would decrease the nominal term premium and generate a procyclical premium.

the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_l^*) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_l^*)/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_l^*) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \\ \rho_x(\mu - x_l) \\ \rho_\pi(\pi_{ss} - \pi_l^*) \end{bmatrix},$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 & 0 \\ 0 & 0 & 0 & \sigma_z\sigma_x & 0 \\ \sigma_{\pi,i} & 0 & \sigma_{\pi,A} & \sigma_{\pi,x} & \sigma_\pi + \sigma_{\pi,z} \end{bmatrix},$$

or

$$\Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 & \sigma_i\sigma_{\pi,i} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 & \sigma_A A_l \sigma_{\pi,A} \\ 0 & 0 & 0 & \sigma_z^2\sigma_x^2 & \sigma_z\sigma_x\sigma_{\pi,x} \\ \sigma_i\sigma_{\pi,i} & 0 & \sigma_A A_l\sigma_{\pi,A} & \sigma_z\sigma_x\sigma_{\pi,x} & \sigma_\pi^2 + \sigma_{\pi,i}^2 + \sigma_{\pi,A}^2 + \sigma_{\pi,x}^2 + \sigma_{\pi,z}^2 \end{bmatrix}.$$

Tables 15 (uncorrelated), 17 (positive correlation), and 18 (negative correlation) show that $\tilde{\lambda}_x < 0$ and $\lambda_\pi < 0$, for different parameterization, such that both shocks create favorable shifts in the states x_t and π_t^*. In addition to the standard NK model shocks, we show the elasticities for shocks to technological growth (real long run risks) and shocks to the inflation target (nominal long run risks). We allow for correlated shocks, enabling the central bank to systematically adjust its inflation target in response to other shocks. For uncorrelated shocks (diagonal covariance matrix), we find that shocks to technological growth are of the "good" variety, because they decrease the term premium, meaning they tend to cause the discount factor and yields to move in opposite direction. Furthermore, technology growth shocks generate "good" inflation (decrease the inflation risk premium), because nominal yields are more responsive than real yields, $-\tilde{P}_{x,N}^{(N)} > -\tilde{P}_{x,R}^{(N)}$. For nominal long-run risks, the results are more nuanced: While shocks to the inflation target lead to a lower term premium for nominal bonds, they actually increase the term premium for real (or inflation-protected) bonds. Both effects contribute to the result that monetary policy target rate shocks generate "good" inflation (decrease the inflation risk premium). Summarizing, our results show that with a diagonal covariance matrix both nominal and real long-run risks unambiguously decrease the inflation risk premium.

Most of the LRR results stand in contrast to the term premium from "bad" technology

and monetary policy shocks. Depending on which channel dominates, the term premium thus varies across maturities. Using Table 2 settings, the NK model produces a hump-shaped term premium—positive at shorter maturities and negative at longer ones.

Piazzesi and Schneider (2006) document a negative correlation between consumption and inflation and interpret inflation as bad news for future consumption. However, if both consumption and inflation are endogenous, the relationship becomes more nuanced: Indeed, with technology shocks, higher inflation signals lower future growth (a negative technology shock). Yet this is not necessarily the case, positive shocks to technology growth – i.e., real long-run risks, lead to higher inflation and higher future consumption (cf. Table 3).[29] Even if inflation and bond yields are endogenous, their dynamic responses to structural shocks may differ, enabling mutual predictability. To account for empirical linkages, correlation can also be introduced directly via the covariance matrix Σ.

For the case of correlated shocks (full covariance matrix), the elasticities do not change significantly (cf. Tables 15 to 18). This allows us to use back of the envelope calculations to identify conditions, under which long run risks may actually increase the term premium or generate "bad" inflation. The reason is that not only the elasticities with respect to the shocks $P^{(N)}_{\mathbb{Z}_j}$ matter, but also the size of the shocks in the covariance matrix, $\Sigma_j(\mathbb{Z})P^{(N)}_{\mathbb{Z}}$, is relevant for computing term premia (and hence the inflation risk premium).

3.3.4. Stochastic volatility

In the model with stochastic volatility for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 5$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_5$ for the relevant states $\hat{\mathbb{Z}} = (i, v, A, x, \sigma)$, such that $n \equiv n_0 \times n_1 \times \ldots \times n_5 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 17010$ ($n = 70 \times 3 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \\ \rho_x(\mu - x_l) \\ \rho_\sigma(\sigma_z - \sigma_l) \end{bmatrix},$$

[29] Following Haubrich, Pennacchi, and Ritchken (2012), inflation risk premium has been remarkably stable (cf. Figure 7). Others argue that inflation risk premia were likely positive in earlier decades, and appear to have been on average close to zero or even negative (cf. Cieslak and Pflueger, 2023).

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 & 0 \\ 0 & 0 & 0 & \sigma_l \sigma_x & 0 \\ 0 & 0 & 0 & 0 & \sigma_\sigma \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_l^2 \sigma_x^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\sigma^2 \end{bmatrix}.$$

Table 20 sheds light on the effects of long-run risks with stochastic volatility. It shows that $\tilde{\lambda}_\sigma > 0$, indicating that volatility shocks induce small but adverse shifts in uncertainty. While positive shocks lead to unfavorable shifts in σ_t, volatility shocks are "good" shocks because they actually decrease the term premium, causing the discount factor and yields to move in opposite directions. However, these shocks are less sensitive to the discount factor (inelastic) than technology growth shocks, which are approximately unitary elastic. At the same time, they do not significantly impact the yields of both the nominal and real 10-year bonds, which both decline by approximately the same amount as the SDF. Furthermore, volatility shocks generate "good" inflation by decreasing the inflation risk premium, as nominal prices respond slightly more than real prices, i.e., $\tilde{P}^{(N)}_{\sigma,N} > \tilde{P}^{(N)}_{\sigma,R}$.

Similar to the LRR model, the SV model reduces the term premium generated from "bad" technology and monetary policy shocks. Depending on which channel dominates, the term premium varies across maturities, even switches from being positive and negative. Using Table 2 settings, the NK model produces a hump-shaped term premium—positive at shorter maturities and negative at longer ones. Bansal and Shaliastovich (2013) show that bond risk premia rise with higher uncertainty about expected inflation and fall with uncertainty about expected growth. While the authors document a robust empirical link between bond premia and the volatilities of expected growth and expected inflation, we find a negative link for the uncertainty about expected growth and also for the volatility of that uncertainty (stochastic volatility) for our benchmark parameterization (cf. Table 3), but the elasticities are quantitatively negligible. As discussed above, correlated technology shocks can also generate a positive link between inflation uncertainty and bond premia – more in line with the empirical data. What matters is the implied covariance between endogenous bond yields and the discount factor.

3.3.5. Preference shocks

In the model with preference shocks for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 4$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_4$ for the relevant states $\hat{\mathbb{Z}} = (i, v, d, r)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 \times n_4 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 5670$ ($n = 70 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n

Chebychev nodes $\mathbf{\Phi}u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + \left(\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta\right)v_l \\ -(\rho_d \log d_l - \frac{1}{2}\sigma_d^2)d_l \\ \rho_r(\rho - r_l) \end{bmatrix},$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_d d_l & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_d^2 d_l^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}.$$

Table 21 presents the results for preference shocks. It shows that $\tilde{\lambda}_d > 0$ and $\tilde{\lambda}_r > 0$, indicating that positive preference shocks lead to adverse shifts in the states d_t and r_t. Both shocks are considered "bad" shocks, as they increase the term premium, causing the discount factor and yields to move in the same direction. However, note that the shocks to the subjective rate of time preference is much more sensitive to the discount factor (elastic) than for the temporary shock d_t (inelastic). At the same time, they do significantly impact the yields of both the nominal and real 10-year bonds, which both increase inelastically. Furthermore, preference shocks generate "bad" inflation by increasing the inflation risk premium, as nominal prices respond more than real prices.

Are these results quantitatively important? The answer is affirmative, as our findings support those of Gomez-Cram and Yaron (2021), who argue that macro-finance models place too much emphasis on the volatility of expected inflation news as a source of variation in nominal bond yield shocks. Instead, the authors suggest that preference shocks generate fluctuations in real rates, which appear to be the dominant driver of nominal yield shocks. Indeed, when we rely on stochastic volatility movements to explain variations in the term premium, the effects are an order of magnitude smaller than those of preference shocks, both for the SDF and for the prices of Treasury bonds (cf. Tables 19 and 21).

In the NK model, preference shocks generate counter-cyclical term and inflation risk premiums. Small adverse shocks to preferences and the discount rate, leading to a 0.5 percent output drop, raise the five-year term premium by 1.2 percent and 6.2 percent, respectively, and the inflation risk premium by 5.9 percent. For the ten-year bond, the term premium increases by 0.5 percent and 3 percent, respectively, with a 0.5 percent rise in the inflation risk premium.

3.3.6. Flexible prices

The flexible-price limit of the NK model for $\delta \to \infty$ with $s_{g,t} \equiv 1$, $d_t \equiv 1$, $x_t \equiv \mu$, $\sigma_t \equiv \sigma_z$ has $n_d \equiv \dim(\mathbb{Z}) = 3$ states including wealth for the value function (cf. Proposition 5). For the fundamental pricing equation of zero-coupon bonds, we add the interest rate, take out real financial wealth, and use n_0 nodal points for N, and $n_1, ..., n_3$ for $\mathbb{Z} = (i, A, z)$, respectively, such that $n \equiv \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and a grid of $n = 1890$ ($n = 70 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(y(S_l))/(y_{ss}z_l) - 1) - \theta(i_l - i_{ss}) \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \\ \mu z_l \end{bmatrix},$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 \\ 0 & \sigma_A A_l & 0 \\ 0 & 0 & \sigma_z z_l \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 \\ 0 & \sigma_A^2 A_l^2 & 0 \\ 0 & 0 & \sigma_z^2 z_l^2 \end{bmatrix},$$

together with the solution of the HJB equation, which yields

$$\lambda \equiv \lambda(\mathbb{Z}), \quad V \equiv V(\mathbb{Z}), \quad \pi \equiv \pi(\mathbb{Z}), \quad y \equiv y(\mathbb{Z})$$

and derivatives

$$\lambda_{\mathbb{Z}} \equiv \lambda_{\mathbb{Z}}(\mathbb{Z}), \quad V_{\mathbb{Z}} \equiv V_{\mathbb{Z}}(\mathbb{Z}), \quad \lambda_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}), \quad V_{\mathbb{Z}\mathbb{Z}} \equiv V_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z})$$

with

$$\lambda_{\mathbb{Z}} \equiv \begin{bmatrix} 0 \\ \lambda_A(\mathbb{Z}) \\ \lambda_z(\mathbb{Z}) \end{bmatrix}, \quad V_{\mathbb{Z}} \equiv \begin{bmatrix} 0 \\ V_A(\mathbb{Z}) \\ V_z(\mathbb{Z}) \end{bmatrix},$$

$$\lambda_{\mathbb{Z}\mathbb{Z}} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_{AA}(\mathbb{Z}) & \lambda_{Az}(\mathbb{Z}) \\ 0 & \lambda_{zA}(\mathbb{Z}) & \lambda_{zz}(\mathbb{Z}) \end{bmatrix}, \quad V_{\mathbb{Z}\mathbb{Z}} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{AA}(\mathbb{Z}) & V_{Az}(\mathbb{Z}) \\ 0 & V_{zA}(\mathbb{Z}) & V_{zz}(\mathbb{Z}) \end{bmatrix}$$

or

$$\tilde{\lambda}_{\mathbb{Z}} \equiv \lambda_{\mathbb{Z}}/\lambda, \quad \tilde{V}_{\mathbb{Z}} \equiv V_{\mathbb{Z}}/V, \quad \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}/\lambda, \quad \tilde{V}_{\mathbb{Z}\mathbb{Z}} \equiv V_{\mathbb{Z}\mathbb{Z}}/V,$$

such that

$$\mathrm{tr}\Big(\sigma(S_l)\sigma(S_l)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\Big) = \mathrm{tr}\Big(\Sigma(S_l)\lambda_{\mathbb{Z}\mathbb{Z}}\Big) \quad = \quad \sigma_A^2 A_l^2 \lambda_{AA} + \sigma_z^2 z_l^2 \lambda_{zz},$$

and $\Phi_{\mathbb{Z}_j}$ and $\Phi_{\mathbb{Z}_j\mathbb{Z}_k}$ where

$$\begin{aligned}
P_N^{(N)}(S_l) &\approx \Phi_N(N_l,\mathbb{Z}_l)u = \Phi_N(S_l)u, \quad l=1,\dots,n,\\
P_j^{(N)}(S_l) &\approx \Phi_{\mathbb{Z}_j}(N_l,\mathbb{Z}_l)u = \Phi_{\mathbb{Z}_j}(S_l)u, \quad l=1,\dots,n, \quad j=1,\dots,d,\\
P_{jk}^{(N)}(S_l) &\approx \Phi_{\mathbb{Z}_j\mathbb{Z}_k}(N_l,\mathbb{Z}_l)u = \Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)u, \quad l=1,\dots,n, \quad j,k=1,\dots,d,
\end{aligned}$$

evaluates the lth row of the basis matrix $\Phi(S_l)$, respectively its derivatives $\Phi_N(S_l)$, $\Phi_{\mathbb{Z}_j}(S_l)$, or $\Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)$, such that at each collocation node $S_l \in \mathbb{R}^4$ (each lth row of the grid S)

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\mathbb{Z}_j}(S_l)u + \tfrac{1}{2}\sum_{j=1}^{n_d}\Sigma_{jj}(S_l)\Phi_{\mathbb{Z}_j\mathbb{Z}_j}(S_l)u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d}\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_j\mathbb{Z}_k}(S_l)u\\
&-\Phi_N(S_l)u - i_l\Phi(S_l)u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d}\tilde{\lambda}_{\mathbb{Z}_j}(S_l)\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_k}(S_l)u. \qquad (85)
\end{aligned}$$

Hence, we may construct the collocation matrix $\mathbf{\Phi}$, whose typical lth elements are

$$\begin{aligned}
&\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\mathbb{Z}_j}(S_l) - \Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_{\mathbb{Z}_2}(S_l)\Sigma_{22}(S_l)\Phi_{\mathbb{Z}_2}(S_l) + \tilde{\lambda}_{\mathbb{Z}_3}(S_l)\Sigma_{33}(S_l)\Phi_{\mathbb{Z}_3}(S_l)\\
&\qquad +\tfrac{1}{2}\left(\Sigma_{11}(S_l)\Phi_{\mathbb{Z}_1\mathbb{Z}_1}(S_l) + \Sigma_{22}(S_l)\Phi_{\mathbb{Z}_2\mathbb{Z}_2}(S_l) + \Sigma_{33}(S_l)\Phi_{\mathbb{Z}_3\mathbb{Z}_3}(S_l)\right)
\end{aligned}$$

or

$$\begin{aligned}
&g_1(S_l)\Phi_i(S_l) + g_2(S_l)\Phi_A(S_l) + g_3(S_l)\Phi_z(S_l) - \Phi_N(S_l) - i_l\Phi(S_l)\\
&+\tfrac{1}{2}\left(\sigma_i^2\Phi_{ii}(S_l) + \sigma_A^2A_l^2\Phi_{AA}(S_l) + \sigma_z^2z_l^2\Phi_{zz}(S_l)\right) + \tilde{\lambda}_AA_l^2\sigma_A^2\Phi_A(S_l) + \tilde{\lambda}_z\sigma_z^2z_l^2\Phi_z(S_l)
\end{aligned}$$

such that $\mathbf{\Phi}u = 0$ together with the n boundary conditions requiring $\Phi(0,\mathbb{Z})u = 1$.

Note that the collocation matrix $\mathbf{\Phi}$, whose typical lth elements reads

$$\begin{aligned}
&(\theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(y(S_l)/(y_{ss}z_l) - 1) - \theta(i_l - i_{ss}))\Phi_i(S_l)\\
&-(\rho_A\log A_l - \tfrac{1}{2}\sigma_A^2)A_l\Phi_A(S_l) + \mu z_l\Phi_z(S_l) - \Phi_N(S_l) - i_l\Phi(S_l)\\
&+\tfrac{1}{2}\left(\sigma_{ii}^2\Phi_{ii}(S_l) + \sigma_A^2A_l^2\Phi_{AA}(S_l) + \sigma_z^2z_l^2\Phi_{zz}(S_l)\right) + \tilde{\lambda}_AA_l^2\sigma_A^2\Phi_A(S_l) + \tilde{\lambda}_z\sigma_z^2z_l^2\Phi_z(S_l)
\end{aligned}$$

is used to solve for the unknown coefficients u in order to satisfy the PDE (64)

$$\begin{aligned}
&g_1(\mathbb{Z}_t,\mathbb{X}_t;\mathbb{Y}_t)\tilde{P}_i^{(N)} + g_2(\mathbb{Z}_t,\mathbb{X}_t;\mathbb{Y}_t)\tilde{P}_A^{(N)} + g_3(\mathbb{Z}_t,\mathbb{X}_t;\mathbb{Y}_t)\tilde{P}_z^{(N)}\\
&+\tfrac{1}{2}(\sigma_i^2\tilde{P}_{ii}^{(N)} + \sigma_A^2A_t^2\tilde{P}_{AA}^{(N)} + \sigma_z^2z_t^2\tilde{P}_{zz}^{(N)}) - \tilde{P}_N^{(N)} - i_t = -\tilde{\lambda}_A\sigma_A^2A_t^2\tilde{P}_A^{(N)} - \tilde{\lambda}_z\sigma_z^2z_t^2\tilde{P}_z^{(N)}
\end{aligned}$$

where $\tilde{P}_N^{(N)} \equiv (\partial P_t^{(N)}/\partial N)/P_t^{(N)}$ such that

$$\begin{gathered}\theta(\phi_\pi \pi(\mathbb{Z}_t) - \pi_{ss}) + \phi_y(A_t l_t)/y_{ss} - 1) - (i_l - i_{ss}))\tilde{P}_i^{(N)} \\ -(\rho_A \log A - \tfrac{1}{2}\sigma_A^2)A_t \tilde{P}_A^{(N)} + \mu z_t \tilde{P}_z^{(N)} \\ +\tfrac{1}{2}(\sigma_i^2 \tilde{P}_{ii}^{(N)} + \sigma_A^2 A_t^2 \tilde{P}_{AA}^{(N)} + \sigma_z^2 z_t^2 \tilde{P}_{zz}^{(N)}) - \tilde{P}_N^{(N)} - i_t = -\tilde{\lambda}_A \sigma_A^2 A_t^2 \tilde{P}_A^{(N)} + \gamma \sigma_z^2 z_t \tilde{P}_z^{(N)}\end{gathered}$$

and in the ATSM we have $\tilde{P}_z = \tilde{P}_{zz} = 0$.

3.3.7. Flexible prices (stationary version)

The flexible-price limit of the NK model for $\delta \to \infty$ with $s_{g,t} \equiv 1$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 3$ relevant states for the value function (cf. Proposition 5).[30] For the fundamental pricing equation, we need to add the interest rate and use n_0 nodal points for N, and $n_1, ..., n_4$ for $\hat{\mathbb{Z}} = (i, d, A, r)$, such that $n \equiv \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 5670$ ($n = 70 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ -(\rho_d \log d_l - \tfrac{1}{2}\sigma_d^2)d_l \\ -(\rho_A \log A_l - \tfrac{1}{2}\sigma_A^2)A_l \\ \rho_r(\rho - r_l) \end{bmatrix},$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & & 00 & 0 \\ 0 & \sigma_d d_l & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 \\ 0 & \sigma_d^2 d_l^2 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix},$$

together with the solution of the HJB equation, which yields[31]

$$\hat{\lambda} \equiv \hat{\lambda}(\hat{\mathbb{Z}}), \quad V \equiv V(\hat{\mathbb{Z}}), \quad \pi \equiv \pi(\hat{\mathbb{Z}}), \quad \hat{y} \equiv \hat{y}(\hat{\mathbb{Z}})$$

and derivatives

$$\hat{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad V_{\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}})$$

[30] Though fiscal policy affects the natural rate, we abstract from fiscal policy shocks $s_{g,t}$ as they play a minor role in the standard equilibrium selection of "active-monetary" and "passive-fiscal" NK models.

[31] See Online Appendix N.1 for the analytical solution for output (also inflation, the costate and the value function for the case of unit elasticity in Case **??**).

with

$$\lambda_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_i(\hat{\mathbb{Z}}) \\ \hat{\lambda}_d(\hat{\mathbb{Z}}) \\ \hat{\lambda}_A(\hat{\mathbb{Z}}) \\ \hat{\lambda}_r(\hat{\mathbb{Z}}) \end{bmatrix}, \quad V_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} V_i(\hat{\mathbb{Z}}) \\ V_d(\hat{\mathbb{Z}}) \\ V_A(\hat{\mathbb{Z}}) \\ V_r(\hat{\mathbb{Z}}) \end{bmatrix},$$

$$\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_{ii}(\hat{\mathbb{Z}}) & \hat{\lambda}_{id}(\hat{\mathbb{Z}}) & \hat{\lambda}_{iA}(\hat{\mathbb{Z}}) & \hat{\lambda}_{ir}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{di}(\hat{\mathbb{Z}}) & \hat{\lambda}_{dd}(\hat{\mathbb{Z}}) & \hat{\lambda}_{dA}(\hat{\mathbb{Z}}) & \hat{\lambda}_{dr}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{Ai}(\hat{\mathbb{Z}}) & \hat{\lambda}_{Ad}(\hat{\mathbb{Z}}) & \hat{\lambda}_{AA}(\hat{\mathbb{Z}}) & \hat{\lambda}_{Ar}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{ri}(\hat{\mathbb{Z}}) & \hat{\lambda}_{rd}(\hat{\mathbb{Z}}) & \hat{\lambda}_{rA}(\hat{\mathbb{Z}}) & \hat{\lambda}_{rr}(\hat{\mathbb{Z}}) \end{bmatrix}, \quad V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} V_{ii}(\hat{\mathbb{Z}}) & V_{id}(\hat{\mathbb{Z}}) & V_{iA}(\hat{\mathbb{Z}}) & V_{ir}(\hat{\mathbb{Z}}) \\ V_{di}(\hat{\mathbb{Z}}) & V_{dd}(\hat{\mathbb{Z}}) & V_{dA}(\hat{\mathbb{Z}}) & V_{dr}(\hat{\mathbb{Z}}) \\ V_{Ai}(\hat{\mathbb{Z}}) & V_{Ad}(\hat{\mathbb{Z}}) & V_{AA}(\hat{\mathbb{Z}}) & V_{Ar}(\hat{\mathbb{Z}}) \\ V_{ri}(\hat{\mathbb{Z}}) & V_{rd}(\hat{\mathbb{Z}}) & V_{rA}(\hat{\mathbb{Z}}) & V_{rr}(\hat{\mathbb{Z}}) \end{bmatrix}$$

or

$$\tilde{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}}/V, \quad \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/V,$$

such that

$$\mathrm{tr}\Big(\sigma(S_l)\sigma(S_l)^\top \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\Big) = \mathrm{tr}\Big(\Sigma(S_l)\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\Big) \;\; = \;\; \sigma_i^2\hat{\lambda}_{ii} + \sigma_d^2 d_l^2 \hat{\lambda}_{dd} + \sigma_A^2 A_l^2 \hat{\lambda}_{AA} + \sigma_r^2 \hat{\lambda}_{rr},$$

and $\Phi_{\hat{\mathbb{Z}}_j}$ and $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}$ where

$$\begin{aligned} P_N^{(N)}(S_l) &\approx \Phi_N(N_l, \hat{\mathbb{Z}}_l)u = \Phi_N(S_l)u, \quad l = 1, \ldots, n, \\ P_j^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j}(S_l)u, \quad l = 1, \ldots, n, \quad j = 1, \ldots, d, \\ P_{jk}^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u, \quad l = 1, \ldots, n, \quad j, k = 1, \ldots, d, \end{aligned}$$

evaluates the lth row of the basis matrix $\Phi(S_l)$, respectively its derivatives $\Phi_N(S_l)$, $\Phi_{\hat{\mathbb{Z}}_j}(S_l)$, or $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)$, such that at each collocation node $S_l \in \mathbb{R}^4$ (each lth row of the grid S)

$$\begin{aligned} &\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l)u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_j}(S_l)u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u \\ &-\Phi_N(S_l)u - i_l\Phi(S_l)u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} \tilde{\lambda}_{\mathbb{Z}_j}(S_l)\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_k}(S_l)u. \end{aligned} \tag{86}$$

Hence, we may construct the collocation matrix $\mathbf{\Phi}$, whose typical lth elements are

$$\begin{aligned} &\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l) - \Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_{\hat{\mathbb{Z}}_1}(S_l)\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1}(S_l) \\ &+\tilde{\lambda}_{\hat{\mathbb{Z}}_2}(S_l)\Sigma_{22}(S_l)\Phi_{\hat{\mathbb{Z}}_2}(S_l) + \tilde{\lambda}_{\hat{\mathbb{Z}}_3}(S_l)\Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3}(S_l) + \tilde{\lambda}_{\hat{\mathbb{Z}}_4}(S_l)\Sigma_{44}(S_l)\Phi_{\hat{\mathbb{Z}}_4}(S_l) \\ &+\tfrac{1}{2}\left(\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1\hat{\mathbb{Z}}_1}(S_l) + \Sigma_{22}(S_l)\Phi_{\hat{\mathbb{Z}}_2\hat{\mathbb{Z}}_2}(S_l) + \Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3\hat{\mathbb{Z}}_3}(S_l) + \Sigma_{44}(S_l)\Phi_{\hat{\mathbb{Z}}_4\hat{\mathbb{Z}}_4}(S_l)\right) \end{aligned}$$

or

$$
\begin{aligned}
&g_1(S_l)\Phi_i(S_l) + g_2(S_l)\Phi_d(S_l) + g_3(S_l)\Phi_A(S_l) + g_4(S_l)\Phi_r(S_l) \\
&+\tfrac{1}{2}\left(\sigma_{ii}^2\Phi_{ii} + \sigma_d^2 d_l^2\Phi_{dd}(S_l) + \sigma_A^2 A_l^2\Phi_{AA}(S_l) + \sigma_{rr}^2\Phi_{rr}\right) \\
&-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_i\sigma_i^2\Phi_i(S_l) + \tilde{\lambda}_d\sigma_d^2 d_l^2\Phi_d(S_l) + \tilde{\lambda}_A A_l^2\sigma_A^2\Phi_A(S_l) + \tilde{\lambda}_r\sigma_r^2\Phi_r(S_l)
\end{aligned}
$$

such that $\mathbf{\Phi}u = 0$ together with the n boundary conditions requiring $\Phi(0, \hat{\mathbb{Z}})u = 1$.

Table 1: "Good" and "bad" shocks in the NK model

Variables (cyclical)	Shocks in the NK model						
	monet. policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	discount rate r_t	infl. target π_t^*
output	$-$	$+$	$+$	$-$	$+$	$+$	$+$
marginal costs	$-$	$-$	$+$	$-$	$+$	$+$	$+$
inflation	$-$	$-$	$+$	$-$	$+$	$+$	$+$
value function	$+$	$+$	$+$	$-$	$-$	$+$	$+$
discount factor	$+$	$-$	$-$	$+$	$+$	$+$	$-$
yields (nominal)	$+$	$-$	$+$	$-$	$+$	$+$	$+$
yields (real)	$+$	$-$	$+$	$-$	$+$	$+$	$-$
term premium (nominal)	$+$	$+$	$-$	$-$	$+$	$+$	$-$
term premium (real)	$+$	$+$	$-$	$-$	$+$	$+$	$+$
inflation risk premium	$-$	$+$	$-$	$-$	$+$	$+$	$-$

Notes: All variables are computed for the parameterization in Table 2 (no correlated shocks) assuming a diagonal matrix $\Sigma = \text{diag}(\Sigma(\mathbb{Z}_t)) = \text{diag}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top)$. We show qualitative effects on cyclical variables based on their derivatives. Results for the term premia and inflation risk premium are based on the decomposition using forward rates, rather than derivatives. Effects on yields, term premia, and inflation risk premia are computed for a zero-coupon 10Y Treasury bond.

Table 2: Sign restrictions of IRFs in the NK model (cyclicality)

Variables	Sign restrictions of IRFs in the NK model						
	monet. policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	discount rate r_t	inflation target π_t^*
term premium (nominal)	$-$	$-$	$+$	$-$	$-$	$-$	$+$
term premium (real)	$-$	$-$	$+$	$-$	$-$	$-$	$+$
inflation risk premium	$+$	$+$	$-$	$-$	$-$	$-$	$-$

Notes: All variables are computed for the parameterization in Table 2. Effects on yields, term premia, and inflation risk premia are computed for a zero-coupon 10Y Treasury bond. Results for the term premia and inflation risk premium are based on the derivatives using the comparison between the physical and risk neutral evaluation, showing the sign restrictions in IRFs or cyclical properties.

Table 3: 'Good' and 'bad' shocks in the flexible price limit

Variables (cyclical)	Shocks in the flexible price limit						
	monet. policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	discount rate r_t	infl. target π_t^*
output	0	+	0	0	0	0	0
marginal costs	0	0	0	0	0	0	0
inflation	+	+	−	+	−	−	0
value function	0	+	+	−	+	−	0
discount factor	0	−	−	+	+	+	0
yields (nominal)	+	+	−	+	−	−	−
yields (real)	0	−	+	−	+	+	0
term premium (nominal)	0	−	+	+	−	−	0
term premium (real)	0	+	−	−	+	+	0
inflation risk premium	0	−	+	+	−	−	0

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$ sd shocks of 1 percent)

Table 4: Sign restrictions of IRFs in the flexible price limit (cyclicality)

Variables	Sign restrictions of IRFs in the flexible price limit						
	monetary policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	discount rate r_t	inflation target π_t^*
term premium (nominal)	0	0	0	0	0	+	0
term premium (real)	0	0	0	0	0	−	0
inflation risk premium	0	0	0	0	0	+	0

Notes: All variables are computed for the parameterization in Table 2 using the nonlinear solution.

4. Conclusion

This paper analyzes the determinants of the term premium on nominal bonds within a micro-founded New Keynesian (NK) framework, offering a novel global solution to better understand how investors price economic risks. It is the first to provide a full analytical investigation of uncertainty in a nonlinear NK model. Our key contribution is the classification of fundamental shocks as "good" or "bad" based on their impact on the term premium, which sheds light on the term structure of interest rates. While monetary policy shocks generate "good" inflation by reducing the inflation risk premium, they actually raise the riskiness of nominal bonds, thereby increasing the term premium. In contrast, other demand shocks (related to real and nominal long-run risks) tend to lower both the inflation risk premium and the nominal term premium, whereas preference shocks, shocks to the discount rate, and supply-side technology shocks increase both, underscoring the importance of distinguishing between different shock types in bond pricing.

Furthermore, by employing a global solution method with recursive preferences and endogenous consumption and inflation dynamics, our model captures nonlinear effects that local approximations often miss. This provides deeper insights into yield curve dynamics and the associated risk premia demanded by investors. We find that temporary and persistent shocks to interest rates and inflation are required to consistently match key macroeconomic aggregates with the observed term structure. One interpretation is that monetary policy actions or other temporary shocks may trigger persistent variations in target rates, a perspective supported by empirical data (Uribe, 2012). Our model helps to resolve apparent term structure anomalies and confirms that uncertainty shocks are isomorphic to discount rate shocks (Barsky, Justiniano, and Melosi, 2014).

We believe that this paper is a starting point for several lines of research. Our analysis has important implications for both theoretical and empirical research. By establishing identifying restrictions for economic shocks in empirical models, we bridge the gap between theoretical asset pricing and macro-finance ATSM. Additionally, our framework provides a foundation for extending macro-finance term structure models to include heterogeneous agents (HANK) and idiosyncratic income shocks (cf. Kaplan, Moll, and Violante, 2018) or alternative policy scenarios such as fiscal dominance. Future research should explore the implications of incorporating heterogeneous beliefs, market segmentation, liquidity effects, and/or including a financial sector (e.g., Brunnermeier and Sannikov, 2014) to refine our understanding of the mechanisms driving risk and term premia. Though our identification of shocks provides a useful benchmark, eventually, the data should be used to jointly pin down the parameters together with the shocks. One key advantage of the continuous-time approach is that the model solution is consistent with different frequencies of macro and financial data (cf. Christensen, Posch, and van der Wel, 2016).

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A. Appendix

A.1. The Feynman-Kac formula

The Feynman-Kac formula is a useful generalization of Kolmogorov's backward equation (Øksendal, 2000, Theorems 8.2.1 and 8.1.1) and links the probability theory to the theory of partial differential equations (PDEs). Let X_t be the solution to the Itô process,

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

and let $h : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that

$$f(t,x) = \mathbb{E}\left[h(X_t)|\mathcal{F}_0\right]$$

denotes the conditional expectation of $h(X_t)$ given the information set $\mathcal{F}_0$. Then $f(t,\cdot)$ for each t satisfies the PDE (Kolmogorov's backward equation, Theorem 8.1.1)

$$f_1'(t,x) = f_2'(t,x)\mu(x) + \tfrac{1}{2}f_{22}''(x)\sigma(x)^2 \tag{A.1}$$

with boundary condition $f(0,x) = h(x)$.

Moreover, let $q : \mathbb{R} \to \mathbb{R}$ be a C^1 function, and let

$$g(t,x) = \mathbb{E}\left[\exp\left(-\int_0^t q(X_s)\,\mathrm{d}s\right)h(X_t)|\mathcal{F}_0\right]$$

Then $g(t,\cdot)$ for each t satisfies the PDE (Feynman-Kac, Theorem 8.2.1)

$$g_1'(t,x) = g_2'(t,x)\mu(x) + \tfrac{1}{2}g_{22}''(t,x)\sigma(x)^2 - q(x)g(t,x) \tag{A.2}$$

with boundary condition $g(0,x) = h(x)$.

B. Tables and Figures

B.1. Tables

Table 1: Summary of the solution algorithm

Step 1	(Initialization)	Provide an initial guess for the unknown derivatives for a given set of collocation nodes and basis functions.
Step 2	(Solution)	Solve the time-domain boundary value problem at each node (with curvature terms fixed) and compute the implied controls.
Step 3	(Update)	Update value function, costate, and refresh curvature terms from the updated derivatives.
Step 4	(Iteration)	Repeat Steps 2 and 3 until convergence.

Table 2: Parameterization

ϑ	1	Frisch labor supply elasticity
γ	2	coefficient of relative risk aversion
η	0.55	elasticity of intertemporal substitution
ρ	0.03	subjective rate of time preference, $\rho = -4 \log 0.9925$
ψ	1	preference for leisure
δ	0.65	Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$
ε	25	elasticity of substitution intermediate goods
s_g	0	share of government consumption
μ	0.02	average growth rate technology
ρ_d	0.4214	autoregressive component preference shock, $\rho_d = -4 \log 0.9$
ρ_r	0.4214	autoregressive component discount rate shock, $\rho_g = -4 \log 0.9$
ρ_A	0.4214	autoregressive component technology shock, $\rho_A = -4 \log 0.9$
ρ_g	0.4214	autoregressive component government shock, $\rho_g = -4 \log 0.9$
ρ_x	0.4214	autoregressive component technology growth shock, $\rho_g = -4 \log 0.9$
ρ_σ	0.4214	autoregressive component technology volatility shock, $\rho_g = -4 \log 0.9$
ρ_π	0.4214	autoregressive component inflation target shock, $\rho_g = -4 \log 0.9$
σ_i	0.01	variance monetary policy shock
σ_d	0.01	variance preference shock
σ_A	0.01	variance technology shock
σ_g	0	variance government shock
σ_z	0.02	variance permanent technology shock
σ_x	0.01	variance technology growth shock
σ_r	0.01	variance discount rate shock
σ_σ	0.01	variance of stochastic volatility shock
σ_π	0.01	variance inflation target shock
ϕ_π	4	inflation response Taylor rule
ϕ_y	0	output response Taylor rule
θ	0.5	interest rate smoothing Taylor rule
π_{ss}	0.02	inflation target rate
χ	1	indexation at steady-state inflation rate, $\chi = 1$ is full indexation

Table 3: Empirical and Model-Based Unconditional Moments (Technology Shocks)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0850	0.0864	0.0851	0.0851	0.0700	0.0853
$SD(i)/100$		0.0146	0.0070	0.0066	0.0066		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(r)/100$		0.0300	0.0300	0.0300	0.0300	0.0200	0.0300
$SD(r)/100$		0.0000	0.0000	0.0000	0.0000		
macro data							
$SD(c)$	1.464	0.773	0.701	0.685	0.685		
$SD(l)$	0.466	0.316	0.589	0.554	0.554		
$SD(w)$	1.592	1.046	1.110	0.991	0.991		
interest rates							
$\mathbb{E}(i)$	2.845	8.502	8.636	8.513	8.513		
$SD(i)$	2.364	1.460	0.700	0.665	0.665		
$\mathbb{E}(i-\pi)$	0.677	6.515	6.636	6.516	6.516		
$SD(i-\pi)$	2.235	0.593	0.924	0.921	0.921		
$\mathbb{E}(r^n)$	2.052	6.515	6.636	6.515	6.515		
$SD(r^n)$	0.854	0.593	0.592	0.593	0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.501	8.639	8.516	8.516		
$\mathbb{E}(y^{(5)})$	3.757	8.479	8.642	8.519	8.519		
$\mathbb{E}(y^{(10)})$	4.250	8.453	8.643	8.520	8.520		
$SD(y^{(10)})$	1.963	0.564	0.131	0.129	0.129		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.007	0.007	0.007		
$SD(y^{(10)}-i)$	1.315	0.914	0.610	0.577	0.577		
$\mathbb{E}(f^{(10)})$	5.570	8.411	8.643	8.520	8.520		
$SD(f^{(10)})$	1.900	0.139	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.997	1.997		
$SD(\pi)$	0.993	1.656	0.331	0.358	0.358		
$\mathbb{E}(BE^{(10)})$	2.136	1.931	2.000	1.996	1.996		
$SD(BE^{(10)})$	0.394	0.213	0.006	0.005	0.005		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.502	8.636	8.513	8.513		
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.997	1.997		

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 4: Empirical and Model-Based Unconditional Moments (Technology Shocks)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0850	0.0852	0.0851	0.0851	0.0700	0.0853
$SD(i)/100$		0.0146	0.0070	0.0066	0.0066		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(r)/100$		0.0300	0.0300	0.0300	0.0300	0.0200	0.0300
$SD(r)/100$		0.0000	0.0000	0.0000	0.0000		
macro data							
$SD(c)$	1.464	0.773	0.701	0.685	0.685		
$SD(l)$	0.466	0.316	0.589	0.554	0.554		
$SD(w)$	1.592	1.046	1.110	0.991	0.991		
interest rates							
$\mathbb{E}(i)$	2.845	8.502	8.524	8.513	8.513		
$SD(i)$	2.364	1.460	0.700	0.665	0.665		
$\mathbb{E}(i-\pi)$	0.677	6.515	6.524	6.516	6.516		
$SD(i-\pi)$	2.235	0.593	0.924	0.921	0.921		
$\mathbb{E}(r^n)$	2.052	6.515	6.524	6.515	6.515		
$SD(r^n)$	0.854	0.593	0.592	0.593	0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.501	8.527	8.516	8.516		
$\mathbb{E}(y^{(5)})$	3.757	8.479	8.530	8.519	8.519		
$\mathbb{E}(y^{(10)})$	4.250	8.453	8.530	8.520	8.520		
$SD(y^{(10)})$	1.963	0.564	0.131	0.129	0.129		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.007	0.007	0.007		
$SD(y^{(10)}-i)$	1.315	0.914	0.610	0.577	0.577		
$\mathbb{E}(f^{(10)})$	5.570	8.411	8.531	8.520	8.520		
$SD(f^{(10)})$	1.900	0.139	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.997	1.997		
$SD(\pi)$	0.993	1.656	0.331	0.358	0.358		
$\mathbb{E}(BE^{(10)})$	2.136	1.931	2.000	1.996	1.996		
$SD(BE^{(10)})$	0.394	0.213	0.006	0.005	0.005		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.502	8.524	8.513	8.513		
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.997	1.997		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = \sigma_A = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 5: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0852	0.0864	0.0850	0.0850	0.0700	0.0853
$SD(i)/100$		0.0160	0.0099	0.0102	0.0101		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.784	0.810	0.799		
$SD(l)$	0.466	0.317	0.686	0.751	0.726		
$SD(w)$	1.592	1.045	1.461	1.623	1.557		
interest rates							
$\mathbb{E}(i)$	2.845	8.523	8.636	8.504	8.502		
$SD(i)$	2.364	1.604	0.990	1.024	1.008		
$\mathbb{E}(i-\pi)$	0.677	6.511	6.636	6.512	6.510		
$SD(i-\pi)$	2.235	0.595	1.291	1.302	1.300		
$\mathbb{E}(r^n)$	2.052	6.511		6.513	6.511		
$SD(r^n)$	0.854	0.595		0.591	0.595		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.522	8.638	8.506	8.504		
$\mathbb{E}(y^{(5)})$	3.757	8.499	8.629	8.497	8.495		
$\mathbb{E}(y^{(10)})$	4.250	8.467	8.621	8.488	8.487		
$SD(y^{(10)})$	1.963	0.698	0.263	0.263	0.263		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	−0.015	−0.016	−0.015		
$SD(y^{(10)}-i)$	1.315	0.973	0.787	0.822	0.805		
$\mathbb{E}(f^{(10)})$	5.570	8.412	8.611	8.478	8.476		
$SD(f^{(10)})$	1.900	0.202	0.019	0.019	0.019		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.002	−0.002	−0.002		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.013	2.000	1.992	1.992		
$SD(\pi)$	0.993	1.786	1.420	1.419	1.422		
$\mathbb{E}(BE^{(10)})$	2.136	1.949	1.973	1.965	1.965		
$SD(BE^{(10)})$	0.394	0.265	0.082	0.082	0.082		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.025	−0.025	−0.025		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.523	7.000	8.504	8.502		
$\mathbb{E}(\pi^{(1)})$	2.385	2.013	2.000	1.992	1.992		

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = \sigma_i = \sigma_x = \sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0$)

Table 6: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0852	0.0852	0.0850	0.0850	0.0700	0.0853
$SD(i)/100$		0.0160	0.0099	0.0102	0.0101		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.784	0.810	0.799		
$SD(l)$	0.466	0.317	0.686	0.751	0.726		
$SD(w)$	1.592	1.045	1.461	1.623	1.557		
interest rates							
$\mathbb{E}(i)$	2.845	8.523	8.524	8.504	8.502		
$SD(i)$	2.364	1.604	0.990	1.024	1.008		
$\mathbb{E}(i-\pi)$	0.677	6.511	6.524	6.512	6.510		
$SD(i-\pi)$	2.235	0.595	1.291	1.302	1.300		
$\mathbb{E}(r^n)$	2.052	6.511		6.513	6.511		
$SD(r^n)$	0.854	0.595		0.591	0.595		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.522	8.525	8.506	8.504		
$\mathbb{E}(y^{(5)})$	3.757	8.499	8.516	8.497	8.495		
$\mathbb{E}(y^{(10)})$	4.250	8.467	8.508	8.488	8.487		
$SD(y^{(10)})$	1.963	0.698	0.263	0.263	0.263		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	−0.015	−0.016	−0.015		
$SD(y^{(10)}-i)$	1.315	0.973	0.787	0.822	0.805		
$\mathbb{E}(f^{(10)})$	5.570	8.412	8.498	8.478	8.476		
$SD(f^{(10)})$	1.900	0.202	0.019	0.019	0.019		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.002	−0.002	−0.002		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.013	2.000	1.992	1.992		
$SD(\pi)$	0.993	1.786	1.421	1.419	1.422		
$\mathbb{E}(BE^{(10)})$	2.136	1.949	1.972	1.965	1.965		
$SD(BE^{(10)})$	0.394	0.265	0.082	0.082	0.082		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.025	−0.025	−0.025		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.523	8.524	8.504	8.502		
$\mathbb{E}(\pi^{(1)})$	2.385	2.013	2.000	1.992	1.992		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = \sigma_i = \sigma_x = \sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0$)

Table 7: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0852	0.0852	0.0844	0.0843	0.0700	0.0849
$SD(i)/100$		0.0160	0.0168	0.0182	0.0181		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0244	0.0244	0.0244		
macro data							
$SD(c)$	1.464	0.773	1.101	1.218	1.218		
$SD(l)$	0.466	0.317	0.648	1.008	1.003		
$SD(w)$	1.592	1.045	2.198	2.839	2.834		
interest rates							
$\mathbb{E}(i)$	2.845	8.523	8.523	8.438	8.431		
$SD(i)$	2.364	1.604	1.684	1.816	1.814		
$\mathbb{E}(i-\pi)$	0.677	6.511	6.523	6.451	6.446		
$SD(i-\pi)$	2.235	0.595	1.835	1.891	1.889		
$\mathbb{E}(r^n)$	2.052	6.511			6.511		
$SD(r^n)$	0.854	0.595			0.595		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.522	8.517	8.431	8.424		
$\mathbb{E}(y^{(5)})$	3.757	8.499	8.463	8.375	8.367		
$\mathbb{E}(y^{(10)})$	4.250	8.467	8.425	8.337	8.329		
$SD(y^{(10)})$	1.963	0.698	0.489	0.489	0.489		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	−0.099	−0.101	−0.102		
$SD(y^{(10)}-i)$	1.315	0.973	1.236	1.383	1.380		
$\mathbb{E}(f^{(10)})$	5.570	8.412	8.376	8.289	8.281		
$SD(f^{(10)})$	1.900	0.202	0.034	0.033	0.033		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.051	−0.049	−0.050		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.001	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.013	2.000	1.987	1.986		
$SD(\pi)$	0.993	1.786	2.925	2.914	2.915		
$\mathbb{E}(BE^{(10)})$	2.136	1.949	1.870	1.861	1.858		
$SD(BE^{(10)})$	0.394	0.265	0.180	0.179	0.179		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.109	−0.107	−0.108		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.523	8.523	8.438	8.431		
$\mathbb{E}(\pi^{(1)})$	2.385	2.013	2.000	1.987	1.986		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = \sigma_i = \sigma_x = \sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0.01$, $\eta = 0.55$)

Table 8: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0852	0.0852	0.0850	0.0849	0.0700	0.0864
$SD(i)/100$		0.0160	0.0175	0.0190	0.0189		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0244	0.0244	0.0244		
macro data							
$SD(c)$	1.464	0.773	1.001	1.143	1.143		
$SD(l)$	0.466	0.317	1.228	1.485	1.485		
$SD(w)$	1.592	1.045	2.581	3.216	3.214		
interest rates							
$\mathbb{E}(i)$	2.845	8.523	8.524	8.500	8.486		
$SD(i)$	2.364	1.604	1.747	1.897	1.895		
$\mathbb{E}(i-\pi)$	0.677	6.511	6.524	6.536	6.525		
$SD(i-\pi)$	2.235	0.595	2.544	2.573	2.573		
$\mathbb{E}(r^n)$	2.052	6.511			4.952		
$SD(r^n)$	0.854	0.595			0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.522	8.523	8.499	8.485		
$\mathbb{E}(y^{(5)})$	3.757	8.499	8.465	8.432	8.423		
$\mathbb{E}(y^{(10)})$	4.250	8.467	8.418	8.383	8.377		
$SD(y^{(10)})$	1.963	0.698	0.562	0.564	0.564		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	−0.105	−0.118	−0.109		
$SD(y^{(10)}-i)$	1.315	0.973	1.300	1.463	1.461		
$\mathbb{E}(f^{(10)})$	5.570	8.412	8.359	8.319	8.317		
$SD(f^{(10)})$	1.900	0.202	0.041	0.040	0.040		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.056	−0.064	−0.056		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.013	2.000	1.964	1.961		
$SD(\pi)$	0.993	1.786	3.283	3.257	3.257		
$\mathbb{E}(BE^{(10)})$	2.136	1.949	1.858	1.807	1.820		
$SD(BE^{(10)})$	0.394	0.265	0.185	0.185	0.185		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.131	−0.147	−0.131		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.523	8.524	8.500	8.486		
$\mathbb{E}(\pi^{(1)})$	2.385	2.013	2.000	1.964	1.961		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = \sigma_i = \sigma_x = \sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = -0.01$, $\eta = 0.55$)

Table 9: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	ATSM $\eta = 0.55$	ATSM $\eta = 1.5$	Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0852	0.0852	0.0627	0.0851	0.0700	0.0853
$SD(i)/100$		0.0160	0.0070	0.0060	0.0064		
$\mathbb{E}(A)$		1.0001	1.0000	1.0000	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0000	0.0000	0.0000		
macro data							
$SD(c)$	1.464	0.773	0.701	1.338	0.674		
$SD(l)$	0.466	0.317	0.589	0.783	0.525		
$SD(w)$	1.592	1.045	1.110	1.431	0.904		
interest rates							
$\mathbb{E}(i)$	2.845	8.523	8.524	6.267	8.513		
$SD(i)$	2.364	1.604	0.700	0.600	0.641		
$\mathbb{E}(i-\pi)$	0.677	6.511	6.524	4.267	6.516		
$SD(i-\pi)$	2.235	0.595	0.924	0.895	0.919		
$\mathbb{E}(r^n)$	2.052	6.511					
$SD(r^n)$	0.854	0.595					
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.522	8.527	6.269	8.516		
$\mathbb{E}(y^{(5)})$	3.757	8.499	8.530	6.271	8.520		
$\mathbb{E}(y^{(10)})$	4.250	8.467	8.530	6.272	8.520		
$SD(y^{(10)})$	1.963	0.698	0.131	0.086	0.128		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	0.007	0.005	0.007		
$SD(y^{(10)}-i)$	1.315	0.973	0.610	0.543	0.553		
$\mathbb{E}(f^{(10)})$	5.570	8.412	8.531	6.272	8.521		
$SD(f^{(10)})$	1.900	0.202	0.009	0.006	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.008	0.011		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.013	2.000	2.000	1.997		
$SD(\pi)$	0.993	1.786	0.331	0.348	0.373		
$\mathbb{E}(BE^{(10)})$	2.136	1.949	2.000	2.000	1.996		
$SD(BE^{(10)})$	0.394	0.265	0.006	0.005	0.005		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.523	8.524	6.267	8.513		
$\mathbb{E}(\pi^{(1)})$	2.385	2.013	2.000	2.000	1.997		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = \sigma_i = \sigma_x = 0.01$, $\sigma_\pi = 0$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0$)

Table 10: Empirical and Model-Based Unconditional Moments (Stochastic Volatility)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	ATSM	Model Two-Step	Nonlinear	Steady-states determ.	stoch.
state variables							
$\mathbb{E}(i)/100$		0.0855	0.0864	0.0848	0.0848	0.0700	0.0850
$SD(i)/100$		0.0146	0.0070	0.0066	0.0066		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\sigma)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\sigma)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.701	0.684	0.684		
$SD(l)$	0.466	0.317	0.589	0.549	0.549		
$SD(w)$	1.592	1.046	1.110	0.980	0.979		
interest rates							
$\mathbb{E}(i)$	2.845	8.553	8.636	8.476	8.476		
$SD(i)$	2.364	1.462	0.700	0.665	0.665		
$\mathbb{E}(i-\pi)$	0.677	6.482	6.636	6.483	6.483		
$SD(i-\pi)$	2.235	0.607	0.924	0.924	0.924		
$\mathbb{E}(r^n)$	2.052	6.482	6.636		6.482		
$SD(r^n)$	0.854	0.607	0.593		0.607		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.551	8.639	8.479	8.479		
$\mathbb{E}(y^{(5)})$	3.757	8.530	8.642	8.482	8.482		
$\mathbb{E}(y^{(10)})$	4.250	8.504	8.643	8.482	8.482		
$SD(y^{(10)})$	1.963	0.566	0.131	0.131	0.131		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.007	0.007	0.007		
$SD(y^{(10)}-i)$	1.315	0.915	0.610	0.575	0.575		
$\mathbb{E}(f^{(10)})$	5.570	8.461	8.643	8.483	8.483		
$SD(f^{(10)})$	1.900	0.140	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.071	2.000	1.993	1.993		
$SD(\pi)$	0.993	1.667	0.331	0.364	0.364		
$\mathbb{E}(BE^{(10)})$	2.136	2.015	2.000	1.992	1.992		
$SD(BE^{(10)})$	0.394	0.215	0.006	0.005	0.005		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.553	8.636	8.476	8.476		
$\mathbb{E}(\pi^{(1)})$	2.385	2.071	2.000	1.993	1.993		

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 11: Empirical and Model-Based Unconditional Moments (Stochastic Volatility)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0855	0.0852	0.0848	0.0848	0.0700	0.0850
$SD(i)/100$		0.0146	0.0070	0.0066	0.0066		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(x)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(x)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\sigma)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\sigma)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.702	0.684	0.684		
$SD(l)$	0.466	0.317	0.590	0.549	0.549		
$SD(w)$	1.592	1.046	1.115	0.980	0.979		
interest rates							
$\mathbb{E}(i)$	2.845	8.553	8.524	8.476	8.476		
$SD(i)$	2.364	1.462	0.704	0.665	0.665		
$\mathbb{E}(i-\pi)$	0.677	6.482	6.524	6.483	6.483		
$SD(i-\pi)$	2.235	0.607	0.927	0.924	0.924		
$\mathbb{E}(r^n)$	2.052	6.482	6.524		6.482		
$SD(r^n)$	0.854	0.607	0.606		0.607		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.551	8.527	8.479	8.479		
$\mathbb{E}(y^{(5)})$	3.757	8.530	8.529	8.482	8.482		
$\mathbb{E}(y^{(10)})$	4.250	8.504	8.530	8.482	8.482		
$SD(y^{(10)})$	1.963	0.566	0.133	0.131	0.131		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.006	0.007	0.007		
$SD(y^{(10)}-i)$	1.315	0.915	0.612	0.575	0.575		
$\mathbb{E}(f^{(10)})$	5.570	8.461	8.530	8.483	8.483		
$SD(f^{(10)})$	1.900	0.140	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011		
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.071	2.000	1.993	1.993		
$SD(\pi)$	0.993	1.667	0.333	0.364	0.364		
$\mathbb{E}(BE^{(10)})$	2.136	2.015	1.999	1.992	1.992		
$SD(BE^{(10)})$	0.394	0.215	0.006	0.005	0.005		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.553	8.524	8.476	8.476		
$\mathbb{E}(\pi^{(1)})$	2.385	2.071	2.000	1.993	1.993		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 12: Empirical and Model-Based Unconditional Moments (Preference Shocks)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.1038	0.0852	0.0834	0.0834	0.0852	0.0900
$SD(i)/100$		0.0271	0.0234	0.0279	0.0278		
$\mathbb{E}(d)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(d)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(r)/100$		0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
$SD(r)/100$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.000	1.184	1.642	1.643		
$SD(l)$	0.466	0.000	1.184	1.954	1.954		
$SD(w)$	1.592	0.000	3.204	4.976	4.977		
interest rates							
$\mathbb{E}(i)$	2.845	10.383	8.524	8.342	8.341		
$SD(i)$	2.364	2.710	2.344	2.786	2.784		
$\mathbb{E}(i-\pi)$	0.677	6.363	6.524	6.507	6.507		
$SD(i-\pi)$	2.235	3.740	2.106	2.347	2.346		
$\mathbb{E}(r^n)$	2.052	6.363	6.524	6.363	6.363		
$SD(r^n)$	0.854	3.740	3.588	3.739	3.740		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	10.372	8.534	8.351	8.350		
$\mathbb{E}(y^{(5)})$	3.757	10.222	8.543	8.350	8.347		
$\mathbb{E}(y^{(10)})$	4.250	10.005	8.528	8.331	8.327		
$SD(y^{(10)})$	1.963	1.496	0.740	0.783	0.779		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.378	0.004	−0.011	−0.014		
$SD(y^{(10)}-i)$	1.315	1.472	1.703	2.128	2.129		
$\mathbb{E}(f^{(10)})$	5.570	9.629	8.504	8.304	8.299		
$SD(f^{(10)})$	1.900	0.510	0.052	0.049	0.048		
$\mathbb{E}(TP^{(10)})$	0.836	−0.252	0.097	0.109	0.106		
$SD(TP^{(10)})$	0.749	0.015	0.000	0.003	0.004		
inflation							
$\mathbb{E}(\pi)$	2.164	4.021	2.000	1.835	1.834		
$SD(\pi)$	0.993	5.658	1.090	1.063	1.063		
$\mathbb{E}(BE^{(10)})$	2.136	3.409	1.985	1.819	1.817		
$SD(BE^{(10)})$	0.394	0.773	0.023	0.025	0.024		
$\mathbb{E}(RP^{(10)})$	0.412	−0.699	0.026	0.030	0.029		
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	10.384	8.524	8.342	8.341		
$\mathbb{E}(\pi^{(1)})$	2.385	4.021	2.000	1.835	1.834		

Notes: First-order approximation (ATSM) includes constant variances. All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_r = 0.01$, $\sigma_z = \mu = 0.02$, $\eta = 0.55$)

Table 13: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Technology Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.142	1.142
$\tilde{\lambda}_{ii}$	0.000	0.000	−0.922	−0.922
$\tilde{\lambda}_A$	−1.303	−0.720	−0.658	−0.658
$\tilde{\lambda}_{AA}$	3.004	0.000	0.194	0.194
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	6.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.656	−0.689	−0.858	−0.858
$\tilde{P}_{ii}^{(10)}$	14.773	0.470	3.631	3.631
$\tilde{P}_A^{(10)}$	−1.054	0.923	0.834	0.834
$\tilde{P}_{AA}^{(10)}$	2.229	0.874	1.126	1.125
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.016	−1.143	−1.143
$\tilde{P}_{ii}^{(10)}$	0.000	1.020	3.444	3.444
$\tilde{P}_A^{(10)}$	1.271	0.688	0.620	0.620
$\tilde{P}_{AA}^{(10)}$	0.345	0.489	0.724	0.724
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 (sd shocks of 1 percent, $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 14: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Technology Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.142	1.142
$\tilde{\lambda}_{ii}$	0.000	0.000	−0.922	−0.922
$\tilde{\lambda}_A$	−1.303	−0.719	−0.658	−0.658
$\tilde{\lambda}_{AA}$	3.004	0.000	0.194	0.194
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	6.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.656	−0.689	−0.858	−0.858
$\tilde{P}_{ii}^{(10)}$	14.773	0.470	3.631	3.631
$\tilde{P}_A^{(10)}$	−1.054	0.923	0.834	0.834
$\tilde{P}_{AA}^{(10)}$	2.229	0.874	1.126	1.125
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.016	−1.143	−1.143
$\tilde{P}_{ii}^{(10)}$	0.000	1.020	3.444	3.444
$\tilde{P}_A^{(10)}$	1.271	0.688	0.620	0.620
$\tilde{P}_{AA}^{(10)}$	0.345	0.489	0.724	0.724
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: First-order approximation (ATSM) includes constant variances. All variables are computed for the parameterization in Table 2 (sd shocks of 1 percent, $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 15: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.015	1.015
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.004	−1.003
$\tilde{\lambda}_A$	−1.301	−0.720	−0.721	−0.721
$\tilde{\lambda}_{AA}$	1.359	0.000	0.469	0.460
$\tilde{\lambda}_x$	−0.398	−2.340	−2.336	−2.336
$\tilde{\lambda}_{xx}$	−0.019	0.000	−3.329	−3.151
$\tilde{\lambda}_\pi$	0.000	−1.073	−1.071	−1.072
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.615	−1.113
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	6.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.665	−0.689	−0.682	−0.684
$\tilde{P}_{ii}^{(10)}$	14.527	0.483	3.639	3.539
$\tilde{P}_A^{(10)}$	−1.056	0.923	0.918	0.920
$\tilde{P}_{AA}^{(10)}$	0.763	0.853	1.034	1.098
$\tilde{P}_x^{(10)}$	3.520	−3.086	−3.071	−3.080
$\tilde{P}_{xx}^{(10)}$	13.843	9.566	22.635	22.508
$\tilde{P}_\pi^{(10)}$	1.941	−1.696	−1.686	−1.692
$\tilde{P}_{\pi\pi}^{(10)}$	3.376	2.909	6.900	6.726
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.009	−1.016	−1.011	−1.012
$\tilde{P}_{ii}^{(10)}$	0.210	1.061	3.484	3.413
$\tilde{P}_A^{(10)}$	1.276	0.688	0.683	0.685
$\tilde{P}_{AA}^{(10)}$	2.051	0.483	0.638	0.697
$\tilde{P}_x^{(10)}$	−4.246	−2.299	−2.285	−2.291
$\tilde{P}_{xx}^{(10)}$	17.001	5.423	15.398	15.335
$\tilde{P}_\pi^{(10)}$	−0.005	1.073	1.079	1.076
$\tilde{P}_{\pi\pi}^{(10)}$	0.709	1.207	4.309	4.150
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$ standard deviation of 1 percent). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection ($\sigma_z = 0.02$, $\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = 0$).

Table 16: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.015	1.015
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.004	−1.003
$\tilde{\lambda}_A$	−1.301	−0.719	−0.721	−0.721
$\tilde{\lambda}_{AA}$	1.359	0.000	0.469	0.460
$\tilde{\lambda}_x$	−0.398	−2.340	−2.336	−2.336
$\tilde{\lambda}_{xx}$	−0.019	0.000	−3.329	−3.151
$\tilde{\lambda}_\pi$	0.000	−1.073	−1.071	−1.072
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.615	−1.113
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	6.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.665	−0.689	−0.682	−0.684
$\tilde{P}_{ii}^{(10)}$	14.527	0.483	3.639	3.539
$\tilde{P}_A^{(10)}$	−1.056	0.923	0.918	0.920
$\tilde{P}_{AA}^{(10)}$	0.763	0.852	1.034	1.098
$\tilde{P}_x^{(10)}$	3.520	−3.087	−3.071	−3.080
$\tilde{P}_{xx}^{(10)}$	13.843	9.492	22.635	22.508
$\tilde{P}_\pi^{(10)}$	1.941	−1.696	−1.686	−1.692
$\tilde{P}_{\pi\pi}^{(10)}$	3.376	2.914	6.900	6.726
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.009	−1.016	−1.011	−1.012
$\tilde{P}_{ii}^{(10)}$	0.210	1.061	3.484	3.413
$\tilde{P}_A^{(10)}$	1.276	0.688	0.683	0.685
$\tilde{P}_{AA}^{(10)}$	2.051	0.482	0.638	0.697
$\tilde{P}_x^{(10)}$	−4.246	−2.299	−2.285	−2.291
$\tilde{P}_{xx}^{(10)}$	17.001	5.364	15.398	15.335
$\tilde{P}_\pi^{(10)}$	−0.005	1.073	1.079	1.076
$\tilde{P}_{\pi\pi}^{(10)}$	0.709	1.210	4.309	4.150
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: First-order approximation (ATSM) includes constant variances. All variables are computed for the parameterization in Table 2 ($\eta = 1.5$ standard deviation of 1 percent). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection ($\sigma_z = 0.02$, $\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = 0$).

Table 17: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.014	1.013
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.003	−1.002
$\tilde{\lambda}_A$	−1.301	−0.719	−0.721	−0.721
$\tilde{\lambda}_{AA}$	1.359	0.000	0.473	0.465
$\tilde{\lambda}_x$	−0.398	−2.340	−2.334	−2.333
$\tilde{\lambda}_{xx}$	−0.019	0.000	−3.157	−3.154
$\tilde{\lambda}_\pi$	0.000	−1.073	−1.069	−1.069
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.515	−1.515
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	0.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.665	−0.689	−0.682	−0.682
$\tilde{P}_{ii}^{(10)}$	14.527	0.482	3.526	3.517
$\tilde{P}_A^{(10)}$	−1.056	0.923	0.919	0.922
$\tilde{P}_{AA}^{(10)}$	0.763	0.858	0.778	1.027
$\tilde{P}_x^{(10)}$	3.520	−3.087	−3.074	−3.083
$\tilde{P}_{xx}^{(10)}$	13.843	9.484	23.662	21.830
$\tilde{P}_\pi^{(10)}$	1.941	−1.694	−1.686	−1.688
$\tilde{P}_{\pi\pi}^{(10)}$	3.376	2.878	6.927	6.802
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.009	−1.016	−1.009	−1.011
$\tilde{P}_{ii}^{(10)}$	0.210	1.060	3.430	3.393
$\tilde{P}_A^{(10)}$	1.276	0.687	0.683	0.685
$\tilde{P}_{AA}^{(10)}$	2.051	0.487	0.275	0.647
$\tilde{P}_x^{(10)}$	−4.246	−2.299	−2.284	−2.290
$\tilde{P}_{xx}^{(10)}$	17.001	5.358	14.563	14.800
$\tilde{P}_\pi^{(10)}$	−0.005	1.073	1.078	1.066
$\tilde{P}_{\pi\pi}^{(10)}$	0.709	1.188	4.450	4.233
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, sd of 1 percent, and correlations $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = 0.01$). In the flexible price scenario, we set $\phi_\pi = 0.5$.

Table 18: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.021	1.018
$\tilde{\lambda}_{ii}$	0.000	0.000	−0.499	−1.008
$\tilde{\lambda}_A$	−1.301	−0.719	−0.720	−0.720
$\tilde{\lambda}_{AA}$	1.359	0.000	0.465	0.465
$\tilde{\lambda}_x$	−0.398	−2.340	−2.344	−2.343
$\tilde{\lambda}_{xx}$	−0.019	0.000	−3.177	−3.182
$\tilde{\lambda}_\pi$	0.000	−1.073	−1.075	−1.075
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.524	−1.538
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	0.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.665	−0.689	−0.696	−0.687
$\tilde{P}_{ii}^{(10)}$	14.527	0.482	3.128	3.551
$\tilde{P}_A^{(10)}$	−1.056	0.923	0.910	0.916
$\tilde{P}_{AA}^{(10)}$	0.763	0.858	0.842	1.052
$\tilde{P}_x^{(10)}$	3.520	−3.087	−3.093	−3.074
$\tilde{P}_{xx}^{(10)}$	13.843	9.436	23.502	21.931
$\tilde{P}_\pi^{(10)}$	1.941	−1.694	−1.688	−1.683
$\tilde{P}_{\pi\pi}^{(10)}$	3.376	2.879	7.310	6.845
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.009	−1.016	−1.014	−1.015
$\tilde{P}_{ii}^{(10)}$	0.210	1.060	3.639	3.426
$\tilde{P}_A^{(10)}$	1.276	0.687	0.674	0.681
$\tilde{P}_{AA}^{(10)}$	2.051	0.487	1.142	0.660
$\tilde{P}_x^{(10)}$	−4.246	−2.299	−2.283	−2.283
$\tilde{P}_{xx}^{(10)}$	17.001	5.315	18.326	14.854
$\tilde{P}_\pi^{(10)}$	−0.005	1.073	1.076	1.071
$\tilde{P}_{\pi\pi}^{(10)}$	0.709	1.189	3.501	4.279
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, sd of 1 percent, and correlations $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = -0.01$). In the flexible price scenario, we set $\phi_\pi = 0.5$.

Table 19: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Stochastic Volatility)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.016	1.016
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.050	−1.003
$\tilde{\lambda}_A$	−1.303	−0.720	−0.719	−0.719
$\tilde{\lambda}_{AA}$	1.680	0.000	0.478	0.460
$\tilde{\lambda}_x$	−0.397	−2.345	−2.339	−2.339
$\tilde{\lambda}_{xx}$	0.152	0.000	−3.113	−3.183
$\tilde{\lambda}_\sigma$	0.016	0.000	0.136	0.136
$\tilde{\lambda}_{\sigma\sigma}$	0.000	0.000	4.962	4.952
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	0.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.657	−0.689	−0.686	−0.686
$\tilde{P}_{ii}^{(10)}$	14.713	0.483	3.547	3.547
$\tilde{P}_A^{(10)}$	−1.054	0.922	0.919	0.919
$\tilde{P}_{AA}^{(10)}$	1.061	0.854	1.094	1.093
$\tilde{P}_x^{(10)}$	3.518	−3.099	−3.074	−3.074
$\tilde{P}_{xx}^{(10)}$	13.063	9.639	22.429	22.415
$\tilde{P}_\sigma^{(10)}$	−0.218	−0.000	0.191	0.191
$\tilde{P}_{\sigma\sigma}^{(10)}$	−6.206	−0.000	2.935	2.922
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.016	−1.014	−1.014
$\tilde{P}_{ii}^{(10)}$	−0.001	1.061	3.415	3.415
$\tilde{P}_A^{(10)}$	1.272	0.687	0.684	0.684
$\tilde{P}_{AA}^{(10)}$	1.671	0.484	0.692	0.691
$\tilde{P}_x^{(10)}$	−4.245	−2.309	−2.287	−2.287
$\tilde{P}_{xx}^{(10)}$	18.012	5.463	15.318	15.307
$\tilde{P}_\sigma^{(10)}$	0.263	−0.000	0.142	0.142
$\tilde{P}_{\sigma\sigma}^{(10)}$	7.180	−0.000	2.191	2.182
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, $\sigma_x = 0.01$, $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 20: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Stochastic Volatility)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.016	1.016
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.050	−1.003
$\tilde{\lambda}_A$	−1.303	−0.720	−0.719	−0.719
$\tilde{\lambda}_{AA}$	1.680	0.000	0.478	0.460
$\tilde{\lambda}_x$	−0.397	−2.345	−2.339	−2.339
$\tilde{\lambda}_{xx}$	0.152	0.000	−3.113	−3.183
$\tilde{\lambda}_\sigma$	0.016	0.137	0.136	0.136
$\tilde{\lambda}_{\sigma\sigma}$	0.000	0.000	4.962	4.952
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	0.000	6.000	6.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.657	−0.689	−0.686	−0.686
$\tilde{P}_{ii}^{(10)}$	14.713	0.483	3.547	3.547
$\tilde{P}_A^{(10)}$	−1.054	0.922	0.919	0.919
$\tilde{P}_{AA}^{(10)}$	1.061	0.854	1.094	1.093
$\tilde{P}_x^{(10)}$	3.518	−3.099	−3.074	−3.074
$\tilde{P}_{xx}^{(10)}$	13.063	9.641	22.429	22.415
$\tilde{P}_\sigma^{(10)}$	−0.218	0.192	0.191	0.191
$\tilde{P}_{\sigma\sigma}^{(10)}$	−6.206	0.037	2.935	2.922
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.016	−1.014	−1.014
$\tilde{P}_{ii}^{(10)}$	−0.001	1.061	3.415	3.415
$\tilde{P}_A^{(10)}$	1.272	0.687	0.684	0.684
$\tilde{P}_{AA}^{(10)}$	1.671	0.484	0.692	0.691
$\tilde{P}_x^{(10)}$	−4.245	−2.309	−2.287	−2.287
$\tilde{P}_{xx}^{(10)}$	18.012	5.464	15.318	15.307
$\tilde{P}_\sigma^{(10)}$	0.263	0.143	0.142	0.142
$\tilde{P}_{\sigma\sigma}^{(10)}$	7.180	0.021	2.191	2.182
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: First-order approximation (ATSM) includes constant variances. All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, $\sigma_x = 0.01$, $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 21: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Preference Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.017	1.066	1.066
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.029	−1.041
$\tilde{\lambda}_d$	1.022	0.570	0.552	0.552
$\tilde{\lambda}_{dd}$	0.032	0.000	−0.708	−0.702
$\tilde{\lambda}_z$	−2.000	−2.000	−2.000	−2.000
$\tilde{\lambda}_{zz}$	6.000	0.000	6.000	6.000
$\tilde{\lambda}_r$	5.718	2.270	1.738	1.738
$\tilde{\lambda}_{rr}$	−171.008	0.000	−97.007	−97.178
Nominal				
$\tilde{P}_i^{(10)}$	−3.672	−0.689	−0.750	−0.746
$\tilde{P}_{ii}^{(10)}$	11.691	0.477	3.519	3.531
$\tilde{P}_d^{(10)}$	0.804	−0.715	−0.681	−0.680
$\tilde{P}_{dd}^{(10)}$	−0.188	0.513	1.909	1.777
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	0.348	−0.954	−0.865	−0.865
$\tilde{P}_{rr}^{(10)}$	−14.513	0.914	51.837	51.871
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.016	−1.063	−1.060
$\tilde{P}_{ii}^{(10)}$	−0.000	1.042	3.379	3.388
$\tilde{P}_d^{(10)}$	−0.965	−0.533	−0.506	−0.505
$\tilde{P}_{dd}^{(10)}$	1.888	0.289	1.344	1.245
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	−2.813	−0.018	0.028	0.028
$\tilde{P}_{rr}^{(10)}$	118.906	0.007	9.015	9.006

Notes: All variables are computed for the parameterization in Table 2 (standard deviation of 1 percent), $\sigma_z = \mu = 0.02$. In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

B.2. Figures

Figure 1: US federal funds rate, 10-year treasury rate and inflation rate

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), at the monthly frequency, and the Output gap (HP Filter) at the quarterly frequency. All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

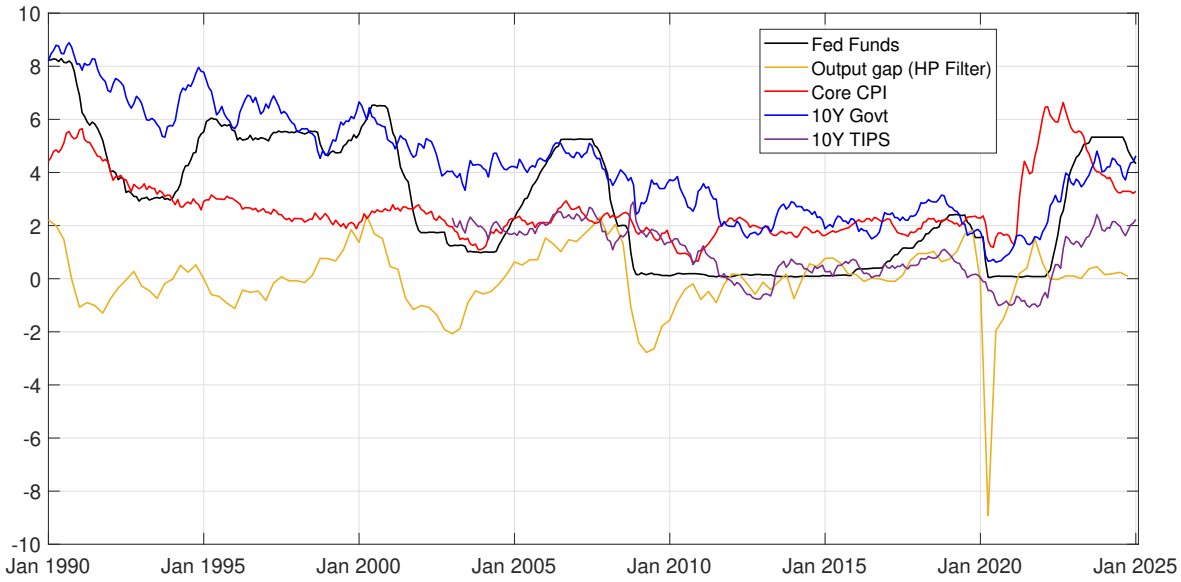


Figure 2: US federal funds rate, output gap, cyclical components

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), and different estimates of the Output gap based on potential output from the Congressional Budget Office (CBO), the Hodrick-Prescott (HP) filter, and the Beveridge-Nelson (BN) trend-cycle decomposition, and the same filter with dynamic mean adjustment (DMA). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from 1990Q1, through 2024Q4.

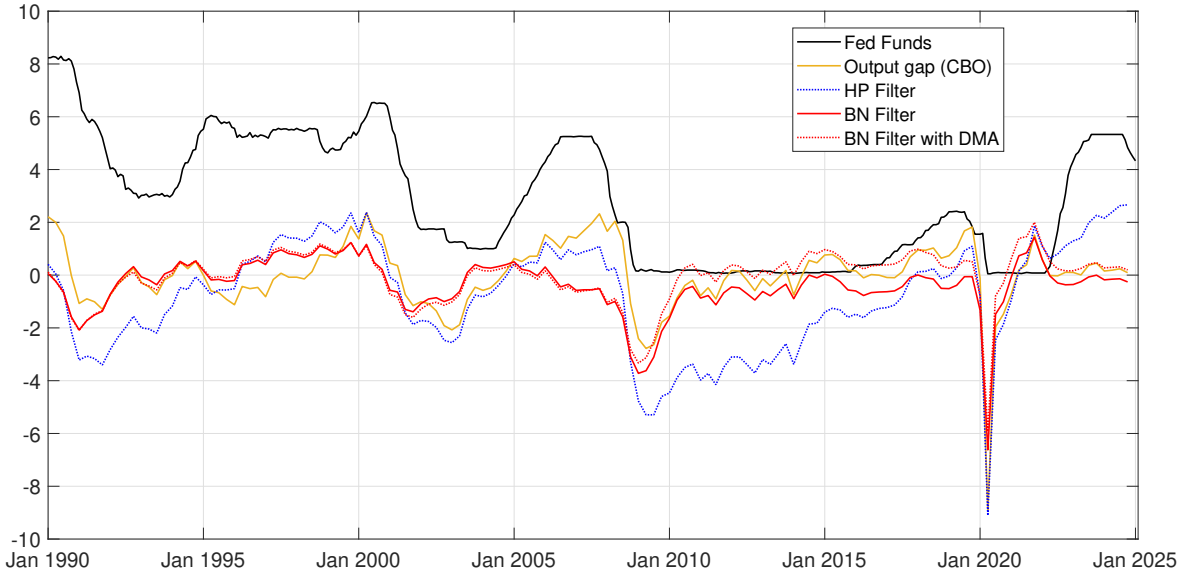


Figure 3: Macro data

In this figure we show time series plots of Real Personal Consumption Expenditures (Real Consumption), Real Compensation of Employees (Real Wages), Average Weekly Hours of Production and Nonsupervisory Employees (Hours). All series are HP filtered cyclical components, obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

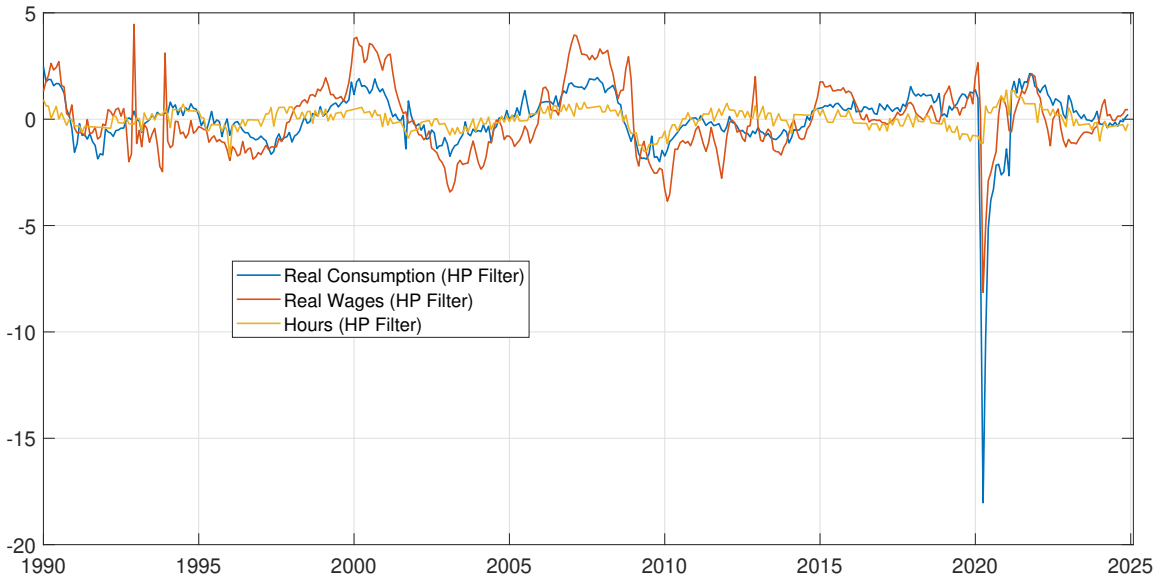


Figure 4: Interest Rates

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the Risk-free Rate (Real Rate), and the Estimate of the Natural Rate (Natural Rate) from Laubach and Williams (2003). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED) and the Federal Reserve Bank of New York. The sample runs from January 1990 to January 2025.

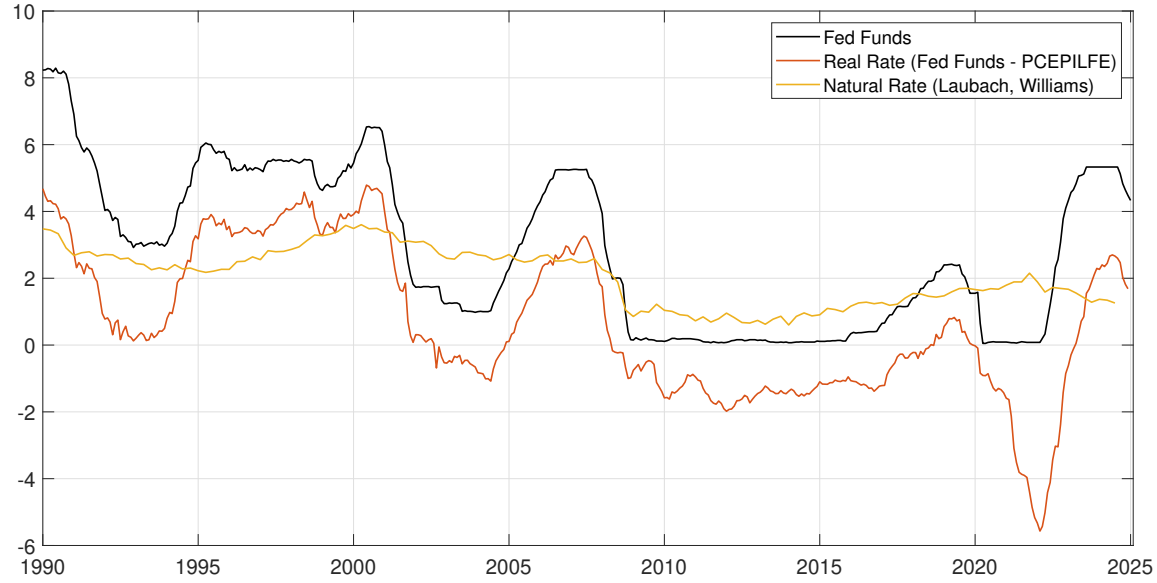


Figure 5: Term Structure of Interest Rates

In this figure we show time series plots of the Market Yield on U.S. Treasury Securities at X-Year Constant Maturity (1Y Govt, 5Y Govt, 10Y Govt), and the Spread of the 10 Year Yield over the Fed Funds Rate (Slope). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

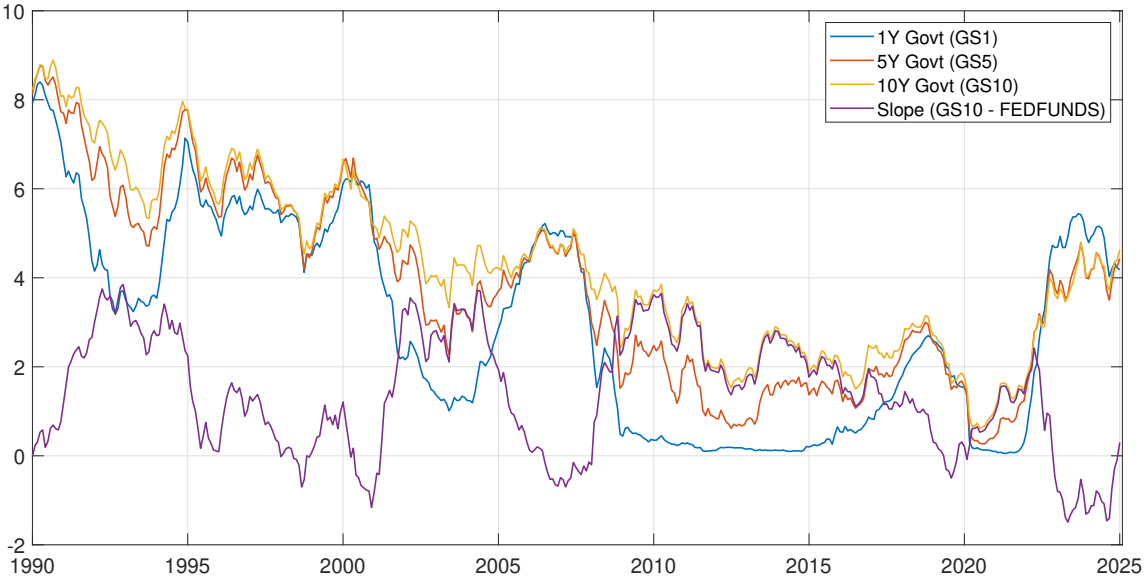


Figure 6: Forward Rates, Term Premium, and Inflation Risk Premium

In this figure we show time series plots of Fitted Instantaneous Forward Rate 10 Years Hence (Forward Rate 10Y), the Term Premium on an 10 Year Zero-Coupon Bond (Term Premium 10Y), the Instantaneous Forward Term Premium 10 Years Hence (Forward Term Premium 10Y), and the Fitted Inflation Risk Premium 10 Years (Inflation Risk Premium 10Y). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), and from the Cleveland Fed's Center for Inflation Research. The sample runs from January 1990 to January 2025.

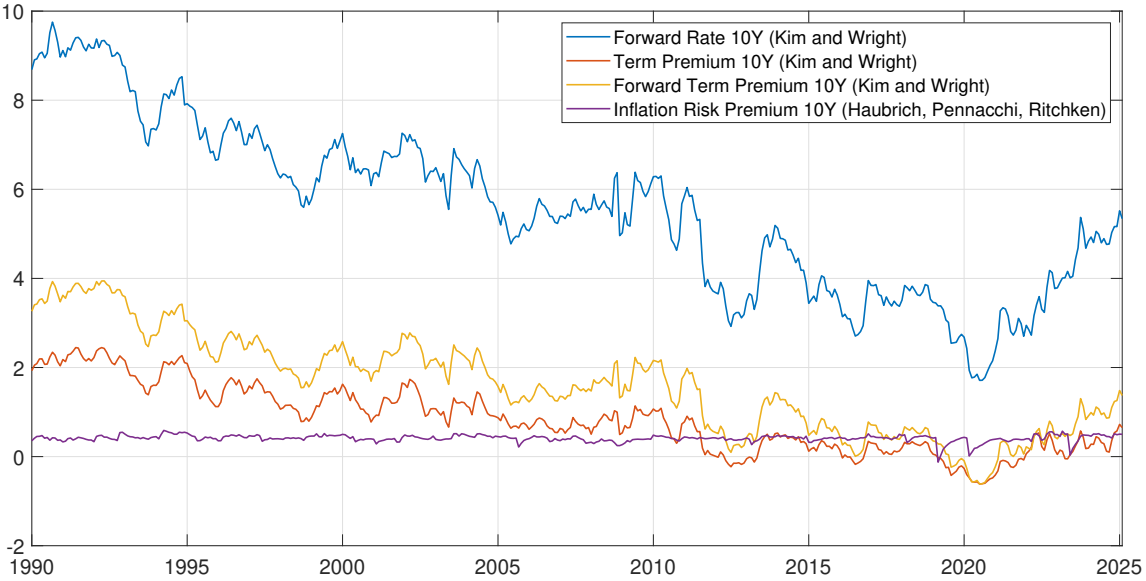


Figure 7: Inflation

In this figure we show time series plots of Percentage Changes from Year ago of the Consumer Price Index for All Urban Consumers: All Items (CPI Headline), Consumer Price Index for All Urban Consumers: All Items Less Food and Energy (CPI Core), Personal Consumption Expenditures: Chain-type Price Index (PCE Headline), and the Fed's primary measure of Personal Consumption Expenditures Excluding Food and Energy (PCE Core). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

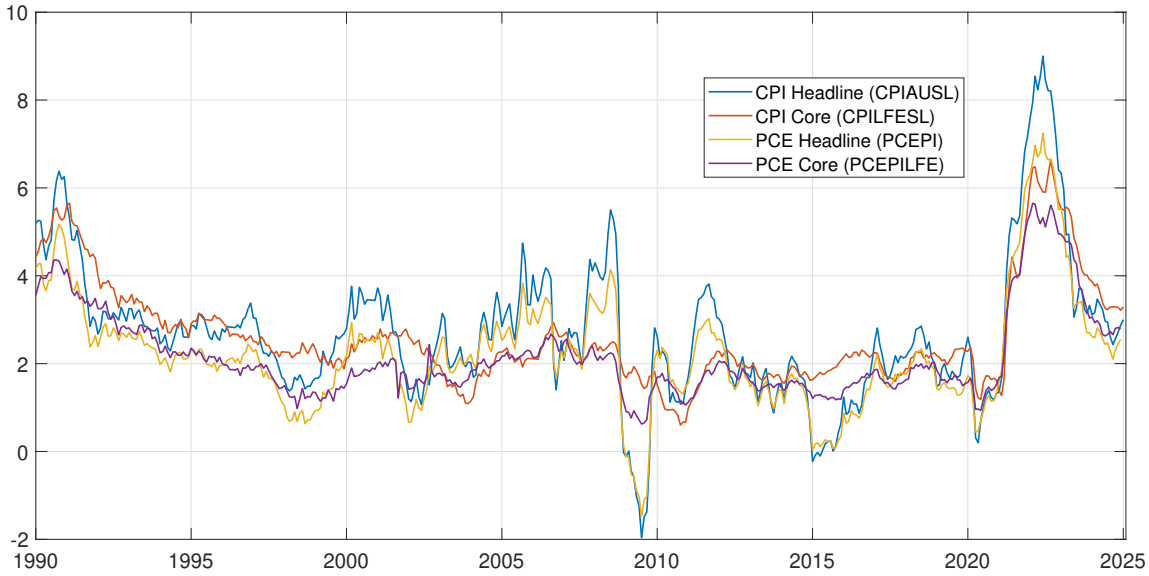


Figure 8: Forecasts

In this figure we show time series plots of Percentage Changes from Year ago of the Consumer Price Index for All Urban Consumers: All Items Less Food and Energy (Inflation), Fitted inflation expectations 1 Year (Inflation Expectation 1Y), the Fitted Instantaneous Forward Rate minus Instantaneous Forward Term Premium 1 year Hence (Expected Short Rate 1Y) based on Kim and Wright (2005), and the Inflation Compensation of the Zero-coupon Continuously Compounded 10 Years (Breakeven Inflation 10Y) from Gürkaynak, Sack, and Wright (2010). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

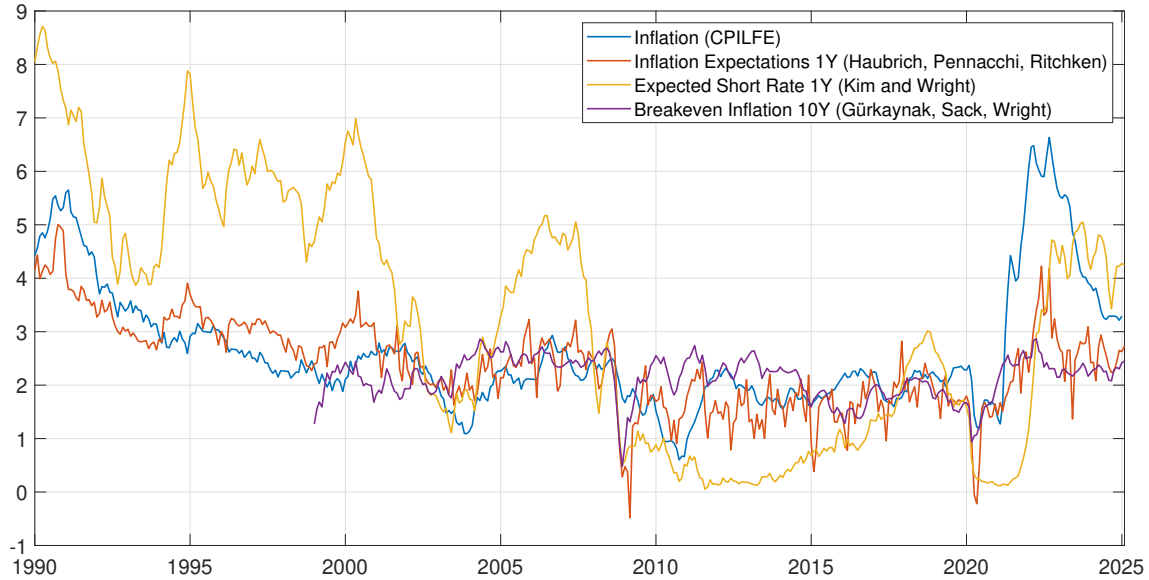


Figure 9: Yield Curve, Term Premium, Risk Premium (at deterministic steady states)

In this figure we show (from top left to bottom right) the term structure of interest rates of zero-coupon treasury bonds (yield curve), the TIPS yield curve, inflation expectations, the term premium, the TIPS premium, and the inflation risk premium. All series are obtained for the NK model (technology shocks), with state variables initialized at the deterministic steady state for the nonlinear model (downward sloping), and the (certainty-equivalent) ATSM (counterfactually upward sloping).

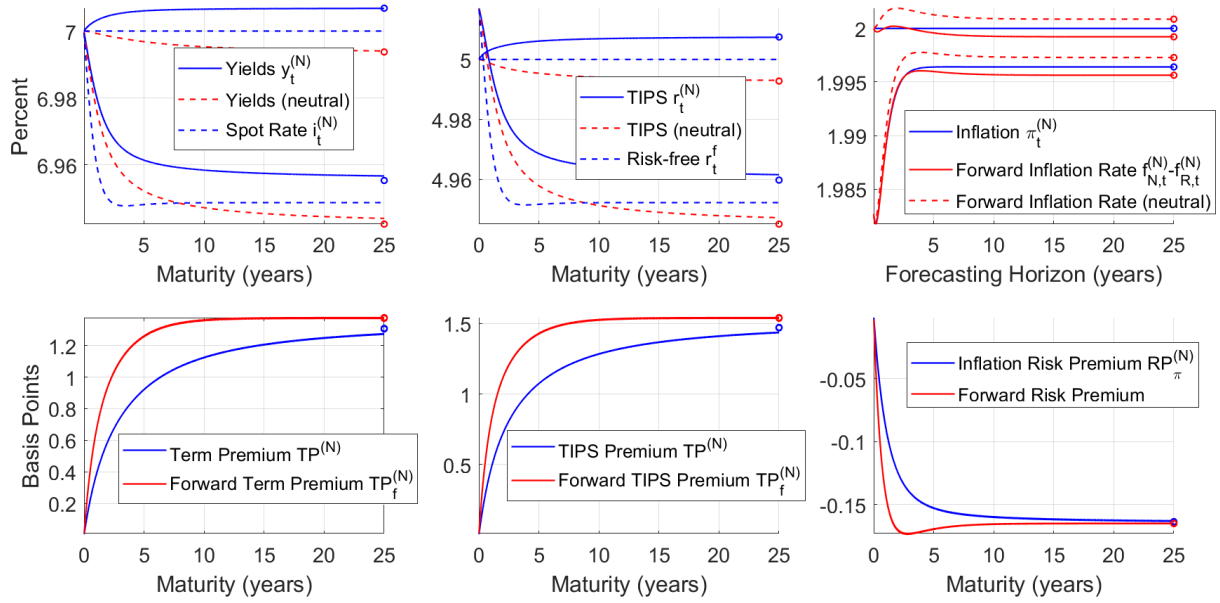


Figure 10: Yield Curve, Term Premium, Risk Premium (at mean values)

In this figure we show (from top left to bottom right) the term structure of interest rates of zero-coupon treasury bonds (yield curve), the TIPS yield curve, inflation expectations, the term premium, the TIPS premium, and the inflation risk premium. All series are obtained for the NK model (technology shocks), with state variables initialized at the unconditional mean values for the nonlinear model (hump shaped), and the (certainty-equivalent) ATSM (upward sloping).

