Optimal Push, Pull, and Failure Funding for Global Health

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Abstract

What mix of push and pull funding best accelerates biomedical innovation when social value exceeds private return? We study this question in the context of drug and vaccine development for diseases like malaria and tuberculosis, where commercial incentives are weak. Governments and nonprofits, including the National Institutes of Health and the Gates Foundation, typically rely primarily on push funding (e.g., grants) with occasional pull mechanisms (e.g., prizes or priority review vouchers). We propose a third tool: failure insurance, in which the funder reimburses part of a firm's testing costs only if the product fails. Using a model of adverse selection and moral hazard, we characterize the optimal mix of push, pull, and failure insurance using infinite-dimensional linear programming and duality theory. Calibrated to 29 communicable diseases, the model recommends failure insurance for tuberculosis and large pull funding for malaria. These findings challenge push-heavy practice and offer new guidance for how public and philanthropic funders can better support high-value products with limited commercial potential.

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1 Introduction

Malaria and tuberculosis each cause over half a million deaths annually (World Health Organization, 2024). Yet commercial incentives to develop new drugs for diseases concentrated in low-income countries remain weak. In addition, markets fail to fully capture the broader societal benefits of innovation and reduced disease transmission.

Governments and foundations attempt to address these market failures by funding drug development. For example, in 2022, the Gates Foundation, with assets exceeding $70 billion, invested $624 million in global health research and development (R&D), including over $100 million each for HIV/AIDS, malaria, and tuberculosis (Impact Global Health, 2023). Similarly, the Wellcome Trust, with assets exceeding $30 billion, invested $106 million in global health R&D (Impact Global Health, 2023).

However, funding new product development presents incentive challenges. Funders typically have less information than firms about a drug's likelihood of success and the firm's capabilities. Funders also cannot fully monitor effort, creating moral hazard. To secure funding, firms may exaggerate a drug's potential and then exert minimal effort, expecting failure regardless. In some cases, these incentive failures lead to outright fraud. For example, in the U.S. Agency for International Development's malaria vaccine program, researchers diverted grant funds and were indicted for theft and conspiracy. The project director later pleaded guilty to accepting kickbacks, filing false tax returns, and making false statements (Desowitz, 1991; Kremer, 2002). Some view fraud as a major concern in global health funding (García, 2019). Even in the absence of fraud, firms may shirk by assigning weaker staff, under-enrolling patients, skimping on follow-up, or neglecting clinical trial quality. Firms can plausibly attribute poor performance to external constraints, especially because trials for tropical diseases often occur in low-income settings with weak infrastructure, and it is costly for the funder to monitor behavior.

Governments and nonprofits primarily rely on push and pull mechanisms to motivate investment. Push mechanisms, such as grants, subsidize R&D costs up front. Pull mechanisms, such as prizes, advance market commitments, and priority review vouchers, reward success and can strengthen accountability by conditioning payment on observable outcomes. A practical difference is timing: pull rewards are typically paid years after testing begins, which means firms must finance development in the interim. Firms tend to discount delayed payments more heavily than large philanthropic funders.[1]

[1]For example, the discount rate for firms might be 10.5% (DiMasi, Grabowski, and Hansen, 2016), but 3% for funders (Wilkinson et al., 2016). Nevertheless, if the pull mechanism is sufficiently generous, a firm

We propose a third mechanism: failure funding. If a project fails, the funder reimburses a share of the firm's testing costs, plus interest.[2] This approach resembles a forgivable loan and provides risk sharing. Funders should not, however, combine pull and failure funding within the same contract. If the funder promises a payment in both outcomes—success and failure—then part of the payment is guaranteed. That guaranteed component is equivalent to push support, except it is delivered later. Because firms discount delayed payments more heavily than funders do, it is more cost-effective to provide any guaranteed support up front.

Failure funding has several advantages. First, it mitigates adverse selection by reducing incentives to "sandbag" where a firm understates a project's chance of success to qualify for support. Second, paying only upon failure can reduce the funder's expected outlays relative to push funding, because the funder pays nothing if the project succeeds. Finally, failure funding need not create moral hazard when reimbursement is capped and the commercial market is moderately attractive; in this case, the firm still prefers to succeed because its expected profits exceed the failure payment.

We formalize these trade-offs in a principal-agent framework. The model captures both adverse selection (private information about the ex ante probability of success) and moral hazard (unobservable effort). We assume that testing costs are known to the funder, based on the experience of funders like the Gates Foundation that support tropical disease product development, as well as data from the Drugs for Neglected Diseases initiative (2019). We formulate the funder's problem as an infinite-dimensional linear program with incentive constraints and derive analytical solutions using duality theory. The results offer policy characterizations across a range of parameters.

Our analysis reveals that diseases fall into distinct funding categories based on consumer welfare and profit potential. First, funders should combine pull and push funding for many tropical diseases, including dengue, malaria, onchocerciasis, and schistosomiasis. Second, push funding alone is optimal for diseases such as hepatitis B. Third, for diseases like trichomoniasis with moderate profit potential, funders should combine push funding with insurance.[3] Fourth, for diseases like tuberculosis, funding failure alone is optimal due to

can raise capital based on the expected value of the pull. For instance, Medicines Development for Global Health received funding from Adjuvant Capital to develop its onchocerciasis drug, backed by the anticipated proceeds from a priority review voucher (Ridley, Ganapathy, and Kettler, 2021).

[2]Failure funding could be implemented as a contingent reimbursement with a clawback provision. The funder pays if the trial fails to meet its pre-specified primary endpoints. If the firm subsequently obtains marketing authorization for the same drug and disease from a major regulator (e.g., the FDA) using evidence that relies on data from the reimbursed trial, the firm must repay the reimbursement plus interest.

[3]We refer to reimbursing firms for failed drugs as "insurance," but this should not be confused with health insurance for patients.

moderate commercial value and high public health benefit. Some drugs, such as those for leprosy, should not be funded at all if late-stage testing costs exceed $100 million, because expected welfare gains do not justify the cost. Furthermore, diseases like HIV/AIDS require no public support for late-stage development, given their strong commercial returns.[4]

Our calibrated model challenges current global health funding in four ways. First, it calls for substantially less reliance on push grants. Second, it introduces failure funding for diseases such as tuberculosis, a tool which appears absent from existing practice. Third, it recommends greater use of pull mechanisms for high-burden tropical diseases, including dengue, malaria, onchocerciasis, and schistosomiasis. Our results support large pull incentives on the order of $500 million, supplemented by modest push funding. Current pull tools, such as priority review vouchers, are worth far less (Ridley, Ganapathy, and Kettler, 2021). Malaria in particular justifies a pull commitment exceeding $1 billion. Fourth, it emphasizes disease-specific strategies. Malaria, tuberculosis, and HIV/AIDS should be treated differently despite similar burdens because their commercial prospects diverge. Prior proposals recommended identical large advance market commitments for each disease (Berndt et al., 2007), overlooking commercial differences. Funding for late-stage testing should be rebalanced away from the current HIV/AIDS-first ordering toward malaria and other high-burden diseases with little or no commercial potential.

This study makes both theoretical and practical contributions. On the theoretical side, it is the first to model and calibrate the optimal mix of push, pull, and insurance mechanisms under both adverse selection and moral hazard. It is also the first to show that rewarding failure, and not success, can be optimal when commercial value is moderate. On the practical side, our calibration exercise suggests shifting global health funding to greater emphasis on pull and insurance mechanisms, and relatively more late-stage investment in malaria than in HIV/AIDS.

While focused on global health, our framework extends to other innovation domains, such as renewable energy. More broadly, our model offers a tool for matching funding strategies to the characteristics of a given market and product.

The paper proceeds as follows. The remainder of this section reviews the literature. Section 2 introduces the model, and Section 3 presents the main theoretical results. Section 4 analyzes benchmark cases in which either adverse selection or moral hazard is absent. Section 5 calibrates the model to a set of communicable diseases and associated interven-

[4]If the commercial return for HIV/AIDS were lower, for example, for heat-stable formulations in low-income countries, then funding could still be optimal.

tions. Section 6 discusses practical implications. Supplementary tables and figures appear in Appendix A, and all proofs are provided in Appendices B–F.

1.1 Literature on Principal-Agent Models

Principals may reward failure for two reasons. First, funding failure can motivate firms to reveal poor performance, whether due to low demand (Simester and Zhang, 2010) or poor outcomes (Mayer, 2022). Second, failure funding can promote experimentation. Without it, agents may prefer safer trials (Bolton et al., 2024). Rewarding failure encourages early exploration and improves long-term outcomes (Manso, 2011). Furthermore, the principal may reward the agent for exerting effort and truthfully reporting a negative signal (Keppo and Li, 2025).

We propose a third rationale: to avoid unnecessary subsidies. Failure funding discourages "sandbagging," where firms understate success probabilities to qualify for push or pull support. This helps address adverse selection. Moral hazard is not an issue if payments are capped and the commercial market is moderately attractive, so firms still prefer to succeed.

A foundational model of innovation is Aghion and Tirole (1994), which addresses moral hazard in product development. The probability of success depends jointly on the agent's effort and the principal's investment, making it well-suited to innovation policy.

Our work relates to Rietzke and Chen (2020), where agents know their probability of success, make contractible investments, and exert non-contractible effort. We differ in five key ways: we allow rewards for failure; incorporate differing discount rates between the principal and agent; exclude payments from the firm to the funder, reflecting the context of global health; allow testing costs to exceed social benefits, making inaction optimal in some cases; and provide a full characterization of the optimal mechanism, including explicit optimality conditions and closed-form funding amounts across all model parameters.

We also relate to Lach, Neeman, and Schankerman (2021), who study optimal loans under moral hazard and adverse selection. In their model, successful agents repay loans, mitigating information rents. In contrast, our model assumes no repayment, complicating contract design.

More broadly, our work relates to innovation contract design. Crama, Reyck, and Degraeve (2008) analyze milestone-based contracts with royalties in profitable markets. We study socially valuable but unprofitable products. Bergmann and Friedl (2008) apply the LEN framework (Holmstrom and Milgrom, 1987) to model optimal compensation for a risk-averse R&D agent. Other relevant papers include Hellmann and Thiele (2011) on task

selection, Xiao and Xu (2012) on royalty renegotiation, Bhattacharya, Gaba, and Hasija (2015) on double-sided moral hazard, Schlapp, Oraiopoulos, and Mak (2015) on effort and information acquisition, and Crama, De Reyck, and Taneri (2017) on licensing contracts with control rights and buy-backs.

1.2 Literature on Drug Development

Average clinical success rates vary across diseases, and this heterogeneity is partly due to selection. Firms avoid low-probability projects when expected commercial returns are limited. For example, the Phase II success rate for infectious diseases is 23%, compared to 11% for cancer drugs (Biotechnology Innovation Organization, 2021), perhaps reflecting higher commercial returns for cancer drugs. However, these are averages across many products and indications, so the success-probability threshold that triggers investment for any particular product may differ from the averages.[5]

We assume that costs are known because we focus on sophisticated global-health funders, such as the Gates Foundation, that have access to detailed cost data from sources such as Drugs for Neglected Diseases initiative (2019). These funders are well-positioned to evaluate value and enforce contracts. In contrast, costs may be poorly observed in fast-moving crises or for novel diseases; this was the case during COVID-19 (Snyder, Hoyt, and Gouglas, 2023). Likewise, costs are often uncertain for rare diseases with limited prior experience (Xu et al., 2025).

Priority review vouchers are the main existing pull mechanism for tropical diseases. Proposed by Ridley, Grabowski, and Moe (2006) and enacted in 2007, the program awards a transferable FDA voucher for expedited review to firms that develop drugs for eligible tropical diseases. Because vouchers are tradable, small firms can benefit even if the second drug is developed by another company (Gans and Ridley, 2013). Vouchers have sold for over $100 million (Ridley, Ganapathy, and Kettler, 2021). We show that this amount is an order of magnitude smaller than the incentive needed for malaria, but appropriate for some tropical diseases.

Advance market commitments (AMCs) are another pull mechanism. AMCs promote both innovation and access by committing funders to purchase qualifying products at a pre-agreed price (Berndt et al., 2007; Kremer, Levin, and Snyder, 2022). AMCs have been used

[5]Drug development is risky, especially for firms with small portfolios. Investors mitigate this risk through diversification (e.g., megafunds; Lo and Thakor 2022) or by securitizing expected revenues. Firm behavior also matters: improved trial design and reduced patient burden can increase retention and trial success (Song, Mehrotra, and Rajapakshe, 2023).

in two cases: a pneumococcal vaccine commitment and Operation Warp Speed during the COVID-19 pandemic (Kremer, Levin, and Snyder, 2022).

Berndt et al. (2007) proposed a $3 billion AMC for HIV/AIDS, malaria, and tuberculosis based on projected commercial returns for a blockbuster drug. Our approach instead tailors incentives to disease-specific characteristics. For malaria, our recommendation aligns with theirs. For HIV/AIDS and tuberculosis, we propose different strategies reflecting differences in market potential.

While we focus on development incentives, post-approval subsidies may be needed to ensure access in low-income settings if the pull is not explicitly tied to access. Patent pools can help by licensing intellectual property to generic manufacturers (Wang, 2022). Funders can also subsidize retailer stocking or consumer purchases. Retail subsidies are preferable when demand is uncertain and products have long shelf lives (Taylor and Xiao, 2014), particularly when funder budgets are large (Ding, Freeman, and Zorc, 2025). When budgets are limited, consumer subsidies may be more effective (Ding, Freeman, and Zorc, 2025).

2 Model

A principal determines whether, when, and how much to fund an agent. The principal in this context refers to a funder such as the Gates Foundation or the Wellcome Trust.

Our model represents a drug-developing firm and its investors as a single agent. We omit inter-firm competition because neglected diseases seldom attract many firms.[6]

The firm has a candidate drug for a possible clinical trial followed by submission to a regulator, such as the FDA. We call the clinical trial and the regulatory approval process a "test," the result of which is uncertain.

Before the test, the firm privately knows the probability of success, based on earlier testing of the drug and the firm's capabilities. In order for the test to be successful, the firm needs to exert costly effort by carefully designing the study and recruiting appropriate providers and patients. The effort is not directly observable by the funder. Therefore, our setting involves both adverse selection and moral hazard. Specifically, we model the firm's effort cost as c and the probability of the test being successful given the firm exerts effort as θ, which is the firm's private information (adverse selection). The funder believes that θ

[6]Neglected diseases, by definition, have few drugs in the pipeline or marketed. Resistance in bacteria and viruses creates demand for multiple treatments, yet limited commercial returns discourage entry. We include certain non-neglected diseases, such as HIV, but the model predicts they merit no subsidy.

is drawn from a distribution with support $\Theta \equiv [\theta_L, \theta_H]$, probability density function f, and cumulative distribution function F (common prior distribution).

If the firm shirks instead, the test will definitely fail, and the corresponding cost to the firm is $\alpha\, c$ for some $\alpha \in [0, 1]$.[7] Here, shirking refers to exerting minimal effort in conducting clinical trials, such as under-investing in infrastructure, hiring inadequately trained personnel, or failing to recruit and follow up with patients appropriately. The value α measures the severity of moral hazard: if $\alpha = 0$, the firm incurs no cost from shirking, and moral hazard is the most severe. If $\alpha = 1$, on the other hand, shirking costs the same as exerting effort, and moral hazard is eliminated. One can interpret a strictly positive α as capturing the fixed cost associated with the test, including, for example, filing paperwork to get permission, opening clinical sites to make the trial accessible, and compensating investigators and participants.[8]

The funder can decide how much to fund the firm before and after the testing result is realized. The funding provided after the test can be conditioned on the test result. Because we analyze a single late-stage trial, the model includes only one signal: success or failure.

Following the revelation principle (Myerson, 1981), we study incentive-compatible direct mechanisms without loss of generality. That is, the funder publicly commits to a funding policy in the beginning based on the agent's reported value $\theta \in \Theta$, which specifies: (1) whether to induce testing $q(\theta) \in \{0, 1\}$; (2) an amount of upfront payment $\hat{m}_0(\theta)$ (*push*); and (3) amounts of subsequent payments conditional on test results. In particular, the payment after a successful test is denoted as $\hat{m}_S(\theta)$ (*pull*), and after a failed test as $\hat{m}_F(\theta)$ (*insurance*).

A successful drug brings the funder a value ω, representing consumer welfare. Therefore, when a firm with success probability θ reports truthfully and exerts effort in the test, the funder's expected payoff $u_F(\theta)$ is the potential consumer welfare $\theta\, \omega\, q(\theta)$ minus its expected payments in both periods, i.e.,

$$u_F(\theta) \equiv \left[\theta\, \omega - \hat{m}_0(\theta) - \theta\, \hat{m}_S(\theta) - (1 - \theta)\, \hat{m}_F(\theta)\right] q(\theta). \tag{1}$$

We assume that the funder does not discount future payments. In practice, the Gates Foundation's discount rate is not zero but is approximately 3% (Wilkinson et al., 2016).

[7] In Appendix K, we show that if the firm's testing cost is linear in effort, the binary effort assumption can be made without loss of generality.

[8] An equivalent way to frame the cost structure is as follows: the firm pays a fixed cost $\alpha\, c$ upon submitting for testing, and then chooses effort. Low effort incurs no additional cost and leads to certain failure, while high effort adds a cost of $(1-\alpha)\, c$. This framing clarifies that $\alpha\, c$ reflects fixed costs, while $(1-\alpha)\, c$ represents the incremental cost of exerting effort. We thank an anonymous reviewer for this suggestion.

However, assuming a zero discount rate is without loss of generality. It is reasonable to assume that the funder has a lower discount rate because it is not cash-constrained and prioritizes long-term societal benefits. Hence, we can normalize the funder's discount factor to 1 by appropriately rescaling the decision variables. Details are provided in Appendix L.

If the regulator approves the drug, the firm receives a profit of π from the market. The present value of the profit is $\delta\,\pi$, where $\delta \in [0,1]$ is the firm's discount factor, reflecting the fact that the firm discounts future payments due to cash constraints, the length of testing, and the risks of product failure.

Let $u_1(\theta, \hat{\theta})$ denote the firm's expected payoff when its true type is θ, but it reports $\hat{\theta}$ and exerts effort in the test. With probability $q(\hat{\theta})$, its expected payoff is composed of three components: (1) in period 1, the firm bears the testing cost c and receives an upfront payment of $\hat{m}_0(\hat{\theta})$; (2) the test succeeds with probability θ, and therefore the firm receives a commercial profit π along with a subsequent payment of $\hat{m}_S(\hat{\theta})$; and (3) the trial fails with the remaining probability $1-\theta$, and the firm receives $\hat{m}_F(\hat{\theta})$ as specified by the mechanism. To summarize, we have

$$u_1(\theta, \hat{\theta}) \equiv \Big[-c + \hat{m}_0(\hat{\theta}) + \delta\,\theta\,\Big(\hat{m}_S(\hat{\theta}) + \pi\Big) + \delta\,(1-\theta)\,\hat{m}_F(\hat{\theta})\Big]\; q(\hat{\theta}). \tag{2}$$

If the firm of type θ misreports as $\hat{\theta}$ and shirks, with probability $q(\hat{\theta})$, it receives an upfront payment $\hat{m}_0(\hat{\theta})$ and bears the partial testing cost $\alpha\,c$, followed by a subsequent payment $\hat{m}_F(\hat{\theta})$ because the test will fail. Its total payoff boils down to

$$u_0(\hat{\theta}) \equiv \Big[-\alpha\,c + \hat{m}_0(\hat{\theta}) + \delta\,\hat{m}_F(\hat{\theta})\Big]\; q(\hat{\theta}). \tag{3}$$

A feasible mechanism must satisfy the incentive compatibility constraint, meaning that a firm with type θ should not benefit from misreporting its type as $\hat{\theta}$ or shirking, i.e.,[9]

$$u_1(\theta,\theta) \geq \max\Big\{u_0(\hat{\theta}), u_1(\theta,\hat{\theta})\Big\}, \quad \forall(\theta,\hat{\theta}) \in \Theta^2, \tag{IC}$$

where $u_1(\theta,\theta)$ represents the type θ firm's expected payoff when it reports truthfully and exerts effort.

Furthermore, assuming the firm's outside option has a value of zero, the following indi-

[9]Consistent with standard practice in the contract theory literature, we focus on mechanisms that induce full effort from a participating agent, while recognizing that in some settings, tolerating shirking could be better.

vidual rationality constraint must hold to guarantee the firm's participation

$$u_1(\theta, \theta) \geq 0, \quad \forall \theta \in \Theta. \tag{IR}$$

Finally, we recall the constraints on q,

$$q(\theta) \in \{0, 1\}, \quad \forall \theta \in \Theta, \tag{4}$$

and impose the following limited liability constraints, restricting all monetary transfers from the funder to the firm

$$\hat{m}_0(\theta),\ \hat{m}_S(\theta),\ \hat{m}_F(\theta) \geq 0, \quad \forall \theta \in \Theta. \tag{5}$$

The funder's problem can be stated as follows:

$$\max_{\{q(\cdot), \hat{m}_0(\cdot), \hat{m}_S(\cdot), \hat{m}_F(\cdot)\} \in \hat{\Omega}} \int_{\theta_L}^{\theta_H} u_F(\theta)\, \mathrm{d}F(\theta), \tag{6}$$

where the feasible set $\hat{\Omega}$ is determined by constraints (IC), (IR), (4), and (5).

To implement the optimal solution of (6), the funder does not need the firm to report the probability of success θ, and does not dictate the firm's participation. Instead, the funder extends a take-it-or-leave-it contract, specifying payment levels for push, pull, and insurance. The firm can either accept or reject the offer. If the firm accepts, it effectively reveals that its probability of success is above the threshold for $q(\theta)$ to be 1. If the common prior distribution is uniform, we show that such an indirect mechanism is indeed optimal. Under general distributions, such a take-it-or-leave-it offer performs well. Specifically, it consistently achieves at least 99% of the optimal value under various Beta distributions (see Appendix I for details).

In order to transform the non-linear optimization problem (6) into a linear one, we introduce the following alternative decision variables,

$$m_0(\theta) \equiv q(\theta)\, \hat{m}_0(\theta), \quad m_S(\theta) \equiv q(\theta)\, \hat{m}_S(\theta), \quad \text{and} \quad m_F(\theta) \equiv q(\theta)\, \hat{m}_F(\theta), \quad \forall \theta \in \Theta. \tag{7}$$

If the firm is induced to test ($q(\theta) = 1$), the quantities m_0, m_S, and m_F are equal to $\hat{m}_0$, $\hat{m}_S$, and $\hat{m}_F$, respectively. If $q(\theta) = 0$, on the other hand, they all take the value 0. We also relax the binary constraint on q such that

$$0 \leq q(\theta) \leq 1, \quad \forall \theta \in \Theta. \tag{8}$$

Although we allow $q(\theta) \in [0,1]$ for tractability, we later show that all optimal solutions under the uniform prior distribution satisfy $q(\theta) \in \{0,1\}$.

We can rewrite both parties' payoff functions in (1), (2), and (3) as

$$u_F(\theta) = \theta\,\omega\,q(\theta) - m_0(\theta) - \theta\,m_S(\theta) - (1-\theta)\,m_F(\theta), \tag{9}$$

$$u_1(\theta,\hat{\theta}) = -c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,\theta\,\Big[m_S(\hat{\theta}) + \pi\,q(\hat{\theta})\Big] + \delta\,(1-\theta)\,m_F(\hat{\theta}),\ \text{and} \tag{10}$$

$$u_0(\hat{\theta}) = -\alpha\,c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,m_F(\hat{\theta}), \tag{11}$$

which are all linear functions of the decision variables.

We impose the following feasibility conditions on m's:

$$m_0(\theta),\ m_S(\theta),\ m_F(\theta) \geq 0, \quad \forall\theta \in \Theta. \tag{12}$$

This is a relaxation of the optimization problem (6) because we ignore the constraint that enforces $m_0(\theta) = m_S(\theta) = m_F(\theta) = 0$ if $q(\theta) = 0$. The optimal solution to this relaxed problem satisfies the ignored constraint automatically, and hence it remains optimal for the original problem.

Define the funder's expected utility as

$$U_F(q, m_0, m_S, m_F) \equiv \int_{\theta_L}^{\theta_H} u_F(\theta)\,\mathrm{d}F(\theta), \tag{13}$$

where u_F is specified in (9). The relaxed funder's problem can be formulated as follows:

$$u^* \equiv \max_{\{q(\cdot),m_0(\cdot),m_S(\cdot),m_F(\cdot)\}\in\Omega} U_F(q, m_0, m_S, m_F), \tag{14}$$

where the feasible set Ω is determined by (IC), (IR), (8), and (12).

The following result implies that under the optimal mechanism, the funder never needs to reward success and failure simultaneously.

Lemma 1. *For any mechanism $\{q, m_0, m_S, m_F\} \in \Omega$, there exists a mechanism $\{q, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F\} \in \Omega$ such that $\tilde{m}_S(\theta)\cdot\tilde{m}_F(\theta) = 0$ for any $\theta \in \Theta$, and $U_F\left(q, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F\right) \geq U_F\left(q, m_0, m_S, m_F\right)$.*

That is, any feasible mechanism that offers pull and insurance together is weakly dominated by another one that does not. Intuitively, when both rewards m_S and m_F are strictly positive, conditional on being induced to test ($q = 1$), the funder can increase m_0 by $\delta \cdot \min\{m_S, m_F\}$ while decreasing both m_S and m_F by $\min\{m_S, m_F\}$. This change does

not affect the firm's incentive while saving the funder $(1-\delta)\cdot\min\{m_S, m_F\}$ due to the firm's time discounting.

3 Optimal Mechanism

In this section, we characterize the optimal mechanism that solves the funder's problem (14). First, we show the optimal solution under the following assumption on the common prior distribution of the agent's private information in Sections 3.1 and 3.2.

Assumption 1. *The common prior of the agent's private information θ follows a uniform distribution on* $[0, 1]$.

The results obtained in this section under Assumption 1 are intuitive and easy to implement. Then, in Section 3.3, we study the problem for general distributions.

Before going into details, we provide an overview of the optimal mechanisms when $\delta \geq 1/2$. The optimal solution to the funder's problem (14) follows a simple threshold policy under Assumption 1. That is, the funder incentivizes the firm to test if and only if the success rate θ is above a threshold. Furthermore, the optimal compensations $m_0(\theta)$, $m_S(\theta)$, and $m_F(\theta)$ are constants that do not change with θ as long as it is above the threshold. Thus, the funder can implement this optimal mechanism as a simple *take-it-or-leave-it offer*: the firm will accept the offer if and only if its privately known success probability exceeds the threshold.

The main results are summarized in Figure 1. We identify seven regions based on profit and consumer welfare, holding testing cost, shirking cost, and discounting fixed. Below, we describe the optimal policy in each region.

Region (1): Do nothing. Both consumer welfare and profit are too low to justify investment.

Region (2): Pull and push funding. Consumer welfare is high, but profit is low. The funder offers some push funding, but it is capped to prevent shirking. The funder supplements the push with pull.

Region (3): Push only. Profit is higher, so pull funding is no longer needed.

Region (4): Push and insurance. Profit is even higher, so insurance replaces push.

Region (5): Insurance only. Profit is high enough that only insurance is needed. Total funding is constant, but firms test progressively riskier drugs as the commercial potential rises.

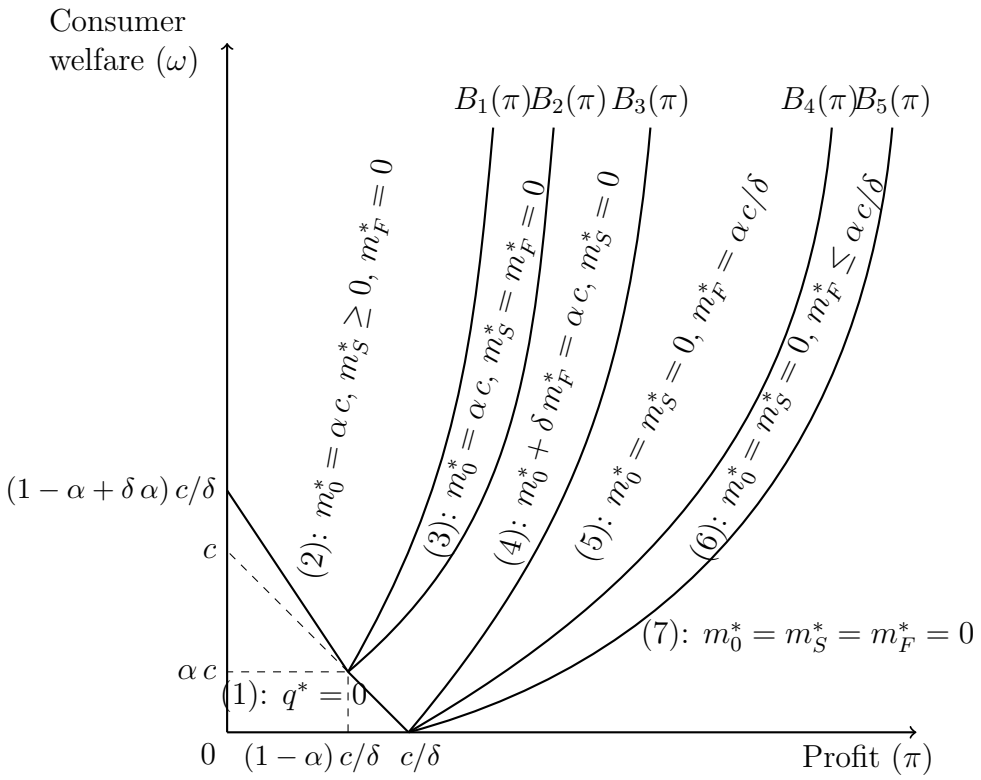


Figure 1: Optimal mechanisms under given profit π and consumer welfare ω if $\delta \geq 1/2$.

Region (6): Less insurance. As profit continues to rise, the funder reduces insurance.
Region (7): Do nothing. Profit is so high that the firm will invest without support.

Figure 2 displays the optimal mechanism for increasing π values under a fixed ω. The partition of the x-axis represents the seven cases described above. Figure 2(a) depicts the present value of the maximum payments, and (b) illustrates the testing threshold. As the profit π potential increases, the testing threshold is non-increasing for Regions (2) to (5), as well as in Region (7).

In Appendix H, we quantify the benefit of allowing the insurance funding to reward failure. Our numerical tests show that not using insurance funding can imply a 20% suboptimality (Figure 9).

Next, we describe the optimal solution structures in detail in Sections 3.1 and 3.2. In Section 3.3, we relax the assumption of a uniformly distributed θ and demonstrate the robustness of the result.

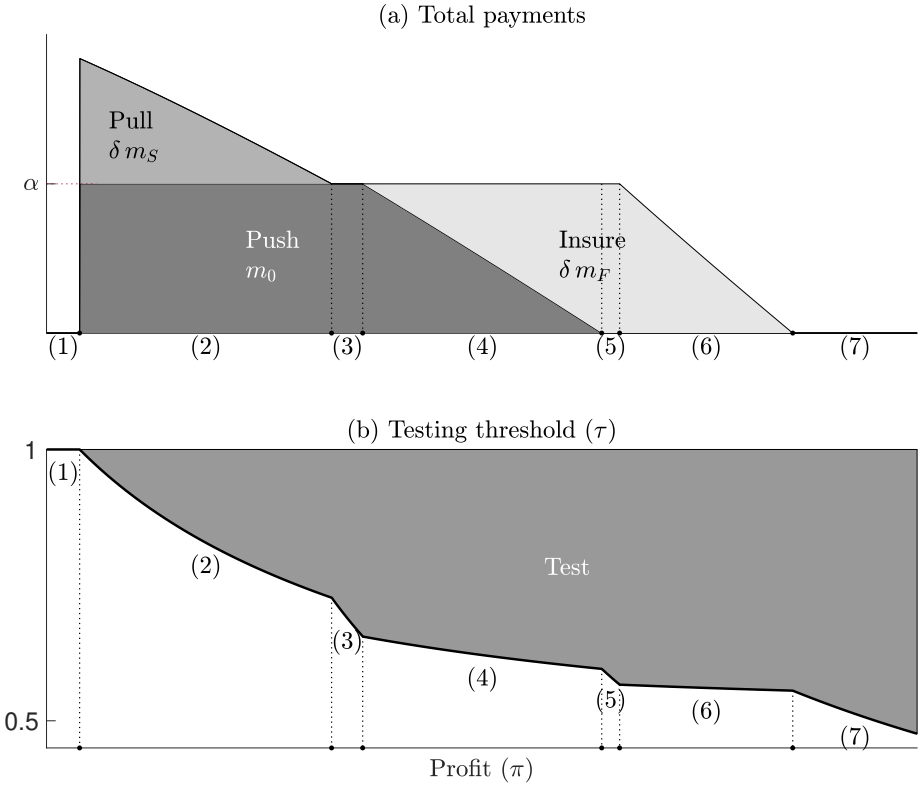


Figure 2: The total payment is the sum between the upfront payment m_0 and either payment upon success $\delta\, m_S$ or payment upon failure $\delta\, m_F$, which captures the maximum present value of payments. The testing threshold (τ) under the optimal mechanism is non-increasing as profit potential rises for these parameter values ($\alpha = 0.5$, $c = 1$, $\delta = 0.6$, and $\omega = 1.2$).

3.1 Optimal Solutions if the Firm Does Not Heavily Discount the Future

In this subsection, we focus on the case $\delta \geq 1/2$, that is, when the firm does not heavily discount future payoffs. This assumption is empirically plausible: large-scale Phase II or III testing typically takes about two years per phase (DiMasi, Grabowski, and Hansen, 2016). Assuming four years of testing and a 16% discount rate – the average for a large sample of firms (Gormsen and Huber, 2025) – yields a discount factor greater than 0.5. The complementary case ($\delta < 1/2$) is analyzed in Section 3.2.

Proposition 1 considers the case in which both the welfare ω and the profit π are relatively low. In this scenario, the optimal mechanism is not to develop the drug. This result

corresponds to Region (1) in Figure 1.[10]

Proposition 1. *If* $\omega + \delta\,\pi \leq c$ *or* $\omega + \pi \leq (1 - \alpha + \delta\,\alpha)\,c/\delta$*, the optimal solution to* (14) *is*

$$q^*(\theta) = m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0,\ \forall \theta \in [0, 1].$$

In order to show this result, we need to verify optimality in two separate cases: (1) $\omega + \delta\,\pi \leq c$; or (2) $c < \omega + \delta\,\pi \leq (1 - \alpha + \delta\,\alpha)\,c + (1 - \delta)\,\omega$. In case (1), the social surplus $\omega + \delta\,\pi$ is smaller than the testing cost, implying that it is socially inefficient to induce testing. Hence, the optimal strategy for a centralized social planner is to induce no testing and provide no funding. In case (2), even though the social surplus exceeds the testing cost, due to the firm's discounting of future income and the relatively small profit π, the compensation to induce the firm's honest effort would be too high to be worth doing. It is worth noting that if the firm does not discount future income, i.e., $\delta = 1$, case (2) is reduced to $c < \omega + \delta\,\pi \leq c$ and hence vanishes.

To better express the following results, we define the self-funding threshold

$$\bar{\tau} \equiv \frac{c}{\delta\,\pi}, \tag{15}$$

such that for a success rate $\theta > \bar{\tau}$, the expected profit $\delta\,\pi\,\theta$ exceeds the testing cost c. Such a firm always has an incentive to test even without support from the funder.

Next, Proposition 2 focuses on the scenario where the consumer welfare ω is relatively large compared to the firm's profit π. In this case, the optimal mechanism is to provide an upfront payment $\alpha\,c$ and a subsequent one conditional on success, which reflects the funder's intuitive desire to hold the firm accountable. This result corresponds to Region (2) in Figure 1.

Proposition 2. *For any* $(1 - \alpha)\,\bar{\tau} \leq 1$*, define function*

$$B_1(\pi) \equiv \pi\,\frac{1 - (1 - \alpha)\,(1 - \alpha - 2\,\delta\,\alpha)\,\bar{\tau}^2}{2\,(1 - \alpha)^2\,\bar{\tau}^2}. \tag{16}$$

[10] It is without loss of generality to normalize one of the following three parameters, c, π, or ω, to 1. In the Appendix, we set $c = 1$ for simplicity. In this section, we retain all of them to more explicitly express how various quantities depend on these parameters.

If $\omega + \pi \geq (1 - \alpha + \delta\,\alpha)\,c/\delta$ *and* $\omega \geq B_1(\pi)$*, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, & \forall \theta \in [0, \tau), \\ 1, & \forall \theta \in [\tau, 1], \end{cases} \qquad m_0^*(\theta) = \begin{cases} 0, & \forall \theta \in [0, \tau), \\ \alpha\, c, & \forall \theta \in [\tau, 1], \end{cases}$$

and

$$m_S^*(\theta) = \begin{cases} 0, & \forall \theta \in [0, \tau), \\ \pi\ [(1-\alpha)\,\bar{\tau}/\tau - 1]\,, & \forall \theta \in [\tau, 1], \end{cases} \qquad m_F^*(\theta) = 0, \quad \forall \theta \in [0, 1],$$

where $\tau \in [0, (1-\alpha)\,\bar{\tau}]$ *is the unique solution to*

$$2\,\delta\ (\omega + \pi)\ \tau^3 + c\ (\alpha - 2\,\delta\,\alpha - 1)\ \tau^2 - c\,(1-\alpha) = 0.$$

In the (π, ω) parametric space, the policy introduced in Proposition 2 is optimal when the pair is not too close to $(0, 0)$ and the welfare ω is above the boundary $B_1(\pi)$. Equivalently, the profit π is not excessively high compared to ω. In such scenarios, even a firm with a high success rate requires compensation to participate. The upfront transfer is set at size $\alpha\, c$ due to moral hazard: a larger upfront payment would incentivize a firm with a low success rate to misreport as a high type and shirk, resulting in a strictly positive expected payoff. The testing threshold τ is determined by optimality conditions, and the subsequent payment upon success, m_S^*, is designed to ensure that the threshold type is indifferent between participating and not. Furthermore, as shown in Proposition 2 and Figure 2(a), the total payment $m_0^* + \delta\, m_S^*$ decreases with increasing profit π.

As the profit π continues increasing, successful firms have already received enough incentives from the commercial market. In this case, the optimal policy now only involves an upfront payment of $\alpha\, c$ (push) and no after-test payment (pull). The conditions are formally established in Proposition 3, and demonstrated in Region (3) of Figure 1.

Proposition 3. *For any* $(1-\alpha)\,\bar{\tau} \leq 1$*, define*

$$B_2(\pi) \equiv B_1(\pi) - \pi\, \frac{(1-\delta)\,[1 - (1-\alpha)\,\bar{\tau}]}{(1-\alpha)^2\,\bar{\tau}^2} \leq B_1(\pi). \tag{17}$$

If $B_2(\pi) \leq \omega < B_1(\pi)$*, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, & \forall \theta \in [0, \tau), \\ 1, & \forall \theta \in [\tau, 1], \end{cases} \qquad m_0^*(\theta) = \begin{cases} 0, & \forall \theta \in [0, \tau), \\ \alpha\, c, & \forall \theta \in [\tau, 1], \end{cases} \qquad m_S^*(\theta) = m_F^*(\theta) = 0, \quad \forall \theta \in [0, 1], \tag{18}$$

where $\tau \equiv (1-\alpha)\,\bar{\tau}$.

The two boundaries, $B_1(\pi)$ and $B_2(\pi)$, intersect at $\pi = (1-\alpha)\,c/\delta$, where $\omega = \alpha\,c$. The condition $B_2(\pi) \leq \omega < B_1(\pi)$ in Proposition 3 can also be equivalently described as the profit π falls into an interval for any given $\omega \geq \alpha\,c$. It is worth mentioning that when the firm does not discount future cash flow, i.e., $\delta = 1$, we have $B_1(\pi) = B_2(\pi)$, and Region (3) in Figure (1) disappears.

As we increase the profit π further, it becomes sufficient to induce the firm's effort. However, for a firm with a relatively low success rate to be willing to test, it becomes optimal to "reward failure." We formally present the result in Proposition 4, which captures Region (4) of Figure 1.

Proposition 4. *For any* $(1-\alpha)\,\bar{\tau} \leq 1$, *define*

$$B_3(\pi) \equiv B_2(\pi) - \pi\,\frac{\alpha\,\left[3\,(2\,\delta-1) + (2+\alpha-4\,\delta\,\alpha-2\,\delta)\,\bar{\tau} + (2\,\delta\,\alpha-2\,\alpha+1)\,\bar{\tau}^2\right]}{2\,(1-\alpha)^2\,\bar{\tau}} \leq B_2(\pi).$$

If $\omega + \delta\,\pi > c$ *and* $B_3(\pi) \leq \omega < B_2(\pi)$, *the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ 1, & \forall \theta \in [\tau,1], \end{cases} \qquad m_0^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ \delta\,\pi\,\left[\alpha\,\bar{\tau} + (1-\alpha)\,\bar{\tau}/\tau - 1\right], & \forall \theta \in [\tau,1], \end{cases}$$

and

$$m_F^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ \pi\,\left[1-(1-\alpha)\,\bar{\tau}/\tau\right], & \forall \theta \in [\tau,1], \end{cases} \qquad m_S^*(\theta) = 0,\ \forall \theta \in [0,1],$$

where the threshold $\tau \in [0,1]$ *is uniquely determined by*

$$2\,\delta\,(\omega+\pi)\,\tau^3 - \left[2\,\delta\,\pi\,(1-\delta) + (1-\alpha+2\,\delta\,\alpha)\,c\right]\,\tau^2 - (1-\alpha)\,(2\,\delta-1)\,c = 0.$$

As illustrated in Figure 2(a), the present value of the total payment that the firm receives is at least $m_0^* + \delta\,m_F^* = \alpha\,c$ in Region (4). Therefore, even if the drug fails, the firm is partially compensated, which encourages its participation. In other words, rewarding failure provides insurance for the firm. The threshold τ in Proposition 4 guarantees that if the firm's success probability θ is above this threshold, the present value from success $(\delta\,\pi\,\theta)$ is sufficiently high to motivate the firm's participation and exerting effort.

Intuitively, push payments m_0 and failure-contingent payments m_F both help induce testing by agents with lower success probabilities. However, their incentive properties differ.

With m_0, the payment is fixed across types, so high-θ agents extract the same transfer as low-θ agents, which raises information rents. In contrast, the expected transfer from m_F is $(1-\theta)\, m_F$, which is decreasing in θ. This means that m_F provides relatively more support to low-θ agents while limiting the rents captured by high-θ agents. As we show here, this targeting property makes m_F strictly more efficient than m_0 in certain regions.

The expression of $m_0^*(\theta) = \delta\,\pi\,[\alpha\,\bar{\tau} + (1-\alpha)\,\bar{\tau}/\tau - 1]$ in Proposition 4 eventually becomes zero for sufficiently high profit π. If we increase π further, m_0^* will stay at zero, while the threshold τ decreases, allowing drugs with an even lower prior chance of success to be tested. In this case, the firm only receives a payment from the funder if the test fails. The detailed result is presented in Proposition 5, which is represented by Region (5) of Figure 1.

Proposition 5. *When $\bar{\tau} \leq 1$, define*

$$B_4(\pi) \equiv B_3(\pi) - \pi\,\frac{(2\,\delta - 1)\;(1-\bar{\tau})\;(1-\alpha\,\bar{\tau})^2}{2\,(1-\alpha)^2\,\bar{\tau}^2} \leq B_3(\pi).$$

If $B_4(\pi) \leq \omega < B_3(\pi)$, the following solution is optimal to (14)*:*

$$q^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ 1, & \forall\theta \in [\tau,1], \end{cases} \qquad m_F^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ \alpha\,c/\delta, & \forall\theta \in [\tau,1], \end{cases} \qquad m_0^*(\theta) = m_S^*(\theta) = 0, \quad \forall\theta \in [0,1],$$

where $\tau = (1-\alpha)\,\bar{\tau}/(1-\alpha\,\bar{\tau})$.

If the profit π grows even higher, the funder can start lowering its support level and rely more and more on the commercial market. The corresponding optimality conditions are characterized in Proposition 6, depicting Region (6) of Figure 1.

Proposition 6. *For any $\bar{\tau} \leq 1$, define*

$$B_5(\pi) \equiv B_4(\pi) - \pi\,\frac{\alpha\;(1-\bar{\tau}^2)}{2\;(1-\alpha)\;\bar{\tau}} \leq B_4(\pi).$$

If $B_5(\pi) \leq \omega < B_4(\pi)$, the following solution is optimal to (14)*: $m_0^*(\theta) = m_S^*(\theta) = 0,\ \forall\theta \in [0,1]$,*

$$q^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ 1, & \forall\theta \in [\tau,1], \end{cases} \quad m_F^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ \pi\,(\bar{\tau}-\tau)/(1-\tau), & \forall\theta \in [\tau,1], \end{cases}$$

where $\tau \equiv (1+\bar{\tau})/[2\,(\omega/\pi+1)]$.

Starting from this region, the funder no longer guarantees to cover the cost of shirking $\alpha\,c$ for the participating firm. The reason is that the profit is high enough so that the firm

does not need to use the highest insurance level ($\alpha\, c/\delta$) to lure a drug with a relatively low chance of success to be tested. In fact, the testing threshold $\tau \equiv (1+\bar{\tau})/[2\,(\omega/\pi+1)]$ is increasing in π when $\omega > \bar{\tau}\,\pi$. That is, it can be optimal to exclude more types of firms as the profit π increases. The optimality conditions drive this feature: the benefit of tightening the testing threshold, which saves money for the funder, dominates giving a drug with a relatively low ex ante efficacy level a chance to be tested.

Finally, when the profit π is high enough to incentivize testing, the funder does not need to interfere. This is shown in Region (7) of Figure 1 and summarized in Proposition 7 below.

Proposition 7. *If $\omega < B_5(\pi)$, the following solution is optimal to* (14)*:*

$$q^*(\theta) = \begin{cases} 0, & \forall \theta \in [0,\tau), \\ 1, & \forall \theta \in [\tau,1], \end{cases} \qquad m_0^*(\theta) = m_S^*(\theta) = m_F^*(\theta) = 0, \quad \forall \theta \in [0,1], \tag{19}$$

where $\tau = \bar{\tau}$.

We prove the results in this section using duality for the (infinite-dimensional) linear optimization problem (14). In particular, we first follow the standard step on the (IC) constraint from Myerson (1981) to eliminate the decision variable $m_0(\theta)$. The objective function then involves three virtual valuation functions, corresponding to coefficients of $q(\theta)$, $m_S(\theta)$, and $m_F(\theta)$, respectively. We then construct a dual feasible solution for each of the seven cases. The corresponding dual objective values are upper bounds for the optimal objective values in (14). However, these upper bounds can be achieved using the primal solutions proposed in the aforementioned seven propositions, which validates optimality. The detailed proofs are provided in Appendix C.

The following corollary further states that as the profit level π keeps increasing, the principal generally funds less promising drugs for testing, except in Region (6) of Figure 1; the same monotonicity consistently holds across all regions as the consumer welfare ω increases. Furthermore, increases in either the commercial profit π or the consumer welfare ω make both parties ex ante better off.

Corollary 1. *If $\delta \geq 1/2$, fixing other model parameters,*

(a) the optimal testing threshold τ according to Propositions 1-5 and 7 is non-increasing in π;

(b) the optimal testing threshold τ according to Propositions 1-7 is non-increasing in ω; and

(c) both the funder and the firm's expected payoffs are non-decreasing in π and in ω.

Figure 2(b) illustrates the non-increasing testing threshold τ with profit π in Regions (1)-(5) and (7). In this particular parameter setting, the threshold is decreasing in Region (6) as well.

3.2 Optimal Solutions if the Firm Heavily Discounts the Future

To address cases where the firm heavily discounts future payments, we extend the model to consider discount factors (δ) below 0.5. This situation may arise if Phase III testing is unusually long or if the firm's discount rate is substantially higher than average. When the firm places less value on future payoffs, mechanisms that rely on delayed rewards, such as pull and insurance, become less effective.

The optimality conditions are unchanged in the cases described in Propositions 1 (no testing) and 2 (push and pull). The proof for the case $\delta < 1/2$ is incorporated into these propositions.

The next proposition summarizes the optimal solution in the remaining cases, including maximal push, less push, and commercial.

Proposition 8. *Define*

$$\tilde{B}_2(\pi) \equiv B_1(\pi) - \pi \frac{-(1-2\,\delta)\,(1-\alpha)^2\,\bar{\tau}^2 - 2\,\delta\,(1-\alpha)\,\bar{\tau} + 1}{2\,(1-\alpha)^2\,\bar{\tau}^2} \tag{20}$$

and

$$\tilde{B}_5(\pi) \equiv \tilde{B}_2(\pi) - \pi \frac{\alpha\,\delta\,(\bar{\tau}+1)}{(1-\alpha)\,\bar{\tau}} \leq \tilde{B}_2(\pi). \tag{21}$$

If $\delta < 1/2$, the optimal policy is given as follows:

(a) If $\tilde{B}_2(\pi) \leq \omega < B_1(\pi)$, the optimal solution to (14) is given by (18).

(b) If $\omega + \delta\,\pi > c$ and $\tilde{B}_5(\pi) \leq \omega < \tilde{B}_2(\pi)$, the optimal solution to (14) is given by: $m_S^*(\theta) = m_F^*(\theta) = 0,\ \forall\theta \in [0,1]$,

$$q^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ 1, & \forall\theta \in [\tau,1], \end{cases} \quad \textit{and} \quad m_0^*(\theta) = \begin{cases} 0, & \forall\theta \in [0,\tau), \\ c - \delta\,\pi\,\tau, & \forall\theta \in [\tau,1], \end{cases}$$

where $\tau \equiv (1+\bar{\tau})/[\omega/(\delta\,\pi) + 2]$.

(c) If $\omega < \tilde{B}_5(\pi)$, the optimal solution to (14) is given by (19).

A key difference from the $\delta \geq 1/2$ case is that the funder never uses insurance to induce testing. The intuition is that when the firm places less value on future payoffs, it becomes too costly for the funder to rely on future payments to provide incentives. The pull mechanism still arises because the amount of push m_0 is bounded above by the moral hazard constraint. Consequently, when the commercial profit π is sufficiently low, the funder must use m_S to provide additional incentives beyond m_0.

Figure 7 in Appendix A summarizes the five possible cases when the firm's discount factor δ is below 1/2. For the cases already analyzed in the previous section, we retain the same numbering. We denote by 3' the new case, defined in Proposition 8, case (b), where the optimal policy is to grant an upfront payment m_0 smaller than $\alpha\, c$.

Analogous to Corollary 1, we summarize the comparative statics in the following corollary.

Corollary 2. *If $\delta < 1/2$, fixing other model parameters,*

(a) the optimal testing threshold τ, as characterized in Propositions 1-2 and in Proposition 8(a) and (c), is non-increasing in π;

(b) the optimal testing threshold τ according to Propositions 1-2 and 8 is non-increasing in ω; and

(c) both the funder and the firm's expected payoffs are non-decreasing in π and in ω.

Hence, when the firm values future rewards less, pull and insurance mechanisms are less effective. In such settings, the funder should rely primarily on push funding. Pull incentives may still be used when the disease burden is high and push support has already been maximized. Insurance should not be used. As in the baseline model, if the disease burden is low or the commercial market is strong, the funder should not intervene.

3.3 General Common-Prior Distributions

Now we study the model without Assumption 1. That is, we allow the common prior for drug efficacy to take general distributions. The corresponding optimal solution to (14) may be much more complex than the one illustrated in Sections 3.1 and 3.2. In particular, it may involve a multi-menu contract that is hard to implement and therefore less practically relevant. Therefore, in this section, we explore the optimal take-it-or-leave-it mechanisms. For any given testing threshold, we characterize the optimal funding levels in closed form. Our result indicates that under the optimal threshold, the funding levels demonstrate very similar characteristics as the solution in Sections 3.1 and 3.2.

Recall the definition of self-funding threshold $\bar{\tau}$ in (15). If a drug's success rate is higher than $\bar{\tau}$, the firm has sufficient incentive to test. Therefore, the funder only needs to evaluate any potential threshold in $\check{\Theta} \equiv \left[\theta_L,\ \min\left\{\bar{\tau}, \theta_H\right\}\right]$.

The optimization problem (14) with the following additional constraints computes the best take-it-or-leave-it offer for a given testing threshold $\check{\tau} \in \check{\Theta}$:

$$\begin{aligned} &q(\theta) = m_0(\theta) = m_S(\theta) = m_F(\theta) = 0,\ \forall \theta \in [\theta_L, \check{\tau}),\ \text{and} \\ &q(\theta) = 1,\ m_0(\theta) = \check{m}_0,\ m_S(\theta) = \check{m}_S,\ m_F(\theta) = \check{m}_F,\ \forall \theta \in [\check{\tau}, \theta_H], \end{aligned} \tag{22}$$

for some decision variables $\check{m}_0$, $\check{m}_S$, $\check{m}_F$. More precisely, we consider the following optimization problem,

$$u^{\mathrm{TH}}(\check{\tau}) \equiv \max_{\{\check{m}_0, \check{m}_S, \check{m}_F\} \in \bar{\Omega}(\check{\tau})} \int_{\theta_L}^{\theta_H} u_F(\theta)\, \mathrm{d}F(\theta), \tag{23}$$

where u_F follows (9), and the feasible set $\bar{\Omega}(\check{\tau})$ is determined by (IC), (IR), (8), (12), and (22).

Lemma 2. *For any $\check{\tau} \in \check{\Theta}$, we have*

$$u^{\mathrm{TH}}(\check{\tau}) = \max_{\{\check{m}_0, \check{m}_S, \check{m}_F\} \in \bar{\Omega}'(\check{\tau})} \int_{\check{\tau}}^{\theta_H} \left[\theta\, \omega - \check{m}_0 - \theta\, \check{m}_S - (1-\theta)\, \check{m}_F\right] \mathrm{d}F(\theta), \tag{24}$$

in which the feasible set $\bar{\Omega}'(\check{\tau})$ is defined by the following constraints:

$$\check{m}_S - \check{m}_F + \pi \geq 0, \tag{25}$$

$$\check{m}_0 + \delta\, \check{\tau}\, \check{m}_S + \delta\, (1 - \check{\tau})\, \check{m}_F = c - \delta\, \pi\, \check{\tau}, \tag{26}$$

$$\check{m}_0 + \delta\, \check{m}_F \leq \alpha\, c,\ \textit{and} \tag{27}$$

$$\check{m}_0,\ \check{m}_S,\ \check{m}_F \geq 0. \tag{28}$$

The following proposition identifies the optimal solution to the linear optimization problem (24) in closed form.

Proposition 9. *Define functions*

$$\mathcal{G}(\check{\tau}) \equiv \alpha + \left[(1-\alpha)/\check{\tau} - 1/\bar{\tau}\right] \quad \textit{and} \quad \mathcal{H}(\check{\tau}) \equiv \delta\, (1 - \check{\tau})\, \left[1 - F(\check{\tau})\right] - \int_{\check{\tau}}^{\theta_H} (1-\theta)\, \mathrm{d}F(\theta).$$

The following defined $(\check{m}_0^*, \check{m}_S^*, \check{m}_F^*)$ is optimal to (23) as well as (24) for a given $\check{\tau} \in \check{\Theta}$:

1. *when $\mathcal{G}(\check{\tau}) \geq \alpha$, we have $\check{m}_0^* = \alpha\, c$, $\check{m}_S^* = \pi\, [\mathcal{G}(\check{\tau}) - \alpha]\, \bar{\tau}$, and $\check{m}_F^* = 0$;*

2. *when* $0 \leq \mathcal{G}(\check{\tau}) < \alpha$, *and* $\mathcal{H}(\check{\tau}) \geq 0$, *we have* $\check{m}_0^* = \mathcal{G}(\check{\tau})\, c$, $\check{m}_S^* = 0$, *and* $\check{m}_F^* = \pi\left[\alpha - \mathcal{G}(\check{\tau})\right]\bar{\tau}$;

3. *when* $\mathcal{G}(\check{\tau}) < \alpha$ *and* $\mathcal{H}(\check{\tau}) < 0$, *we have* $\check{m}_0^* = \delta\,\pi\,(\bar{\tau} - \check{\tau})$ *and* $\check{m}_S^* = \check{m}_F^* = 0$; *and*

4. *when* $\mathcal{G}(\check{\tau}) < 0$ *and* $\mathcal{H}(\check{\tau}) \geq 0$, *we have* $\check{m}_0^* = \check{m}_S^* = 0$ *and* $\check{m}_F^* = \pi\,(\bar{\tau} - \check{\tau})/(1 - \check{\tau})$.

Note that under Assumption 1 (θ is uniformly distributed on $[0, 1]$), we have

$$\mathcal{H}(\check{\tau}) = \left(\delta - \frac{1}{2}\right)(1 - \check{\tau})^2 \geq 0,\ \forall \delta \geq \frac{1}{2}.$$

This implies that Case 3 of Proposition 9, which requires $\mathcal{H}(\check{\tau}) < 0$, does not arise in the optimal solution under Assumption 1 and $\delta \geq 1/2$. In fact, the condition $\mathcal{G}(\check{\tau}) < \alpha$ in Case 3 directly implies that the corresponding $\check{m}_0^* < \alpha\, c$. That is, the funder only uses push funding at a level strictly less than $\alpha\, c$, similar to case (b) of Proposition 8.

In order to obtain the best take-it-or-leave-it offer, we use the optimal solution described in Proposition 9 together with a line search for the optimal threshold value $\check{\tau}$. Figure 5 in Appendix A displays the optimal threshold policy when the firm's private information follows the Beta distributions as depicted in Figure 5(a). Regions (1)–(7) in panels (b)-(f) correspond to the optimal solutions defined in Propositions 1–7 (and hence regions (1)-(7) in Figure 1), respectively. Consistent with the previous section, we use (3') to represent Case 3 of Proposition 9, which involves only an upfront payment less than $\alpha\, c$.

As shown in the figure, the overall structure of the parameter regions where different optimal solutions arise largely mirrors that of the uniform distribution case. We further present Figure 6 in Appendix A, which resembles Figure 2 for the uniform distribution case. This figure illustrates that as the commercial profit π increases, policy structures (1) to (7) become optimal in sequence, consistent with the uniform distribution case. Between (3) and (4), however, it may be optimal to provide only push funding less than $\alpha\, c$, as illustrated in (3'). The optimal testing threshold τ decreases in π in panel (b).

Overall, the general structure of the optimal solution established in Sections 3.1 and 3.2 appears robust across general common prior distributions. Finally, Figure 10 of Appendix I illustrates that even though a simple take-it-or-leave-it offer may not always be optimal, it consistently performs within 99% of the optimal value.

4 Benchmark Comparisons

To illustrate the roles of adverse selection and moral hazard, we examine counterfactual scenarios in which each element is absent. In Sections 4.1 and 4.2, we characterize the optimal mechanism when either moral hazard or adverse selection is removed. In Appendix G, we examine the benchmark with neither adverse selection nor moral hazard.

4.1 No Moral Hazard

Here we consider a second-best benchmark with only adverse selection but no moral hazard, i.e., $\alpha = 1$. The optimal solutions introduced in Propositions 1–8 still hold, although some of the regions disappear. We summarize the optimal mechanisms in the following corollary.

Corollary 3. *Assuming $\alpha = 1$, it is optimal to not offer any pull funding. That is, $m_S^*(\theta) = 0$ for all $\theta \in [0,1]$. Furthermore, we summarize the optimal solution q^*, m_0^*, and m_F^* to (14) in the following cases:*

(a) If $\omega + \delta\,\pi \leq c$, which satisfies the condition of Proposition 1, it is optimal not to test the drug.

(b) Suppose $\omega + \delta\,\pi > c$ and $\delta \geq 1/2$.

(i) If $\delta\,\pi \leq c$, this corresponds to Proposition 4, with a testing threshold $\tau = 1 - \dfrac{\omega + \delta\,\pi - c}{\omega + \pi} \in (0,1)$.

(ii) If $\dfrac{\delta\,\pi}{1 + 2\,\omega/\pi} < c < \delta\,\pi$, this corresponds to Proposition 6, with a testing threshold $\tau = \dfrac{1 + \bar{\tau}}{2\,(\omega/\pi + 1)}$.

(iii) If $0 \leq c \leq \dfrac{\delta\,\pi}{1 + 2\,\omega/\pi}$, this corresponds to Proposition 7, which implies letting the commercial market take over.

(c) Suppose $\omega + \delta\,\pi > c$ and $\delta < 1/2$.

(i) If $\dfrac{\delta\,\pi}{1 + \omega/(\delta\,\pi)} \leq c$, this corresponds to Proposition 8(b), with a testing threshold $\tau = \dfrac{1 + \bar{\tau}}{\omega/(\delta\,\pi) + 2}$.

(ii) If $0 \leq c < \dfrac{\delta\,\pi}{1 + \omega/(\delta\,\pi)}$, this corresponds to Proposition 8(c).

According to Corollary 3, Regions (2), (3), and (5) in Figures 1 and 7 vanish when there is no moral hazard. In the remaining regions, Cases (b)(i), (b)(ii), and (c)(i) of Corollary 3 require funding for drug development. The first two cases involve rewarding failure ($m_F > 0$). Hence, rewarding failure is a robust strategy even if the funder is not concerned with moral hazard. This is because when effort is observable, the funder does not need to reward success, and funding for failure helps cover testing costs.

More specifically, in Case (b)(i), the reward for failure is exactly the profit π. Therefore, as long as θ is above the threshold τ, the firm's expected payoff does not depend on the test outcome. This highlights the fact that the reward for failure serves as "insurance." In fact, the present value of the total reward exactly equals the testing cost c. That is, the funder extracts all the surplus, and the firm receives no information rent.

The situation is different for Cases (b)(ii) and (c)(i). For the drug with boundary type $\theta = \tau$, the firm's expected net payoff equals zero. Any drug with θ above the threshold τ receives a positive information rent. Overall, the different cases, especially Case (b)(i), imply that the model, even without moral hazard, does not reduce to the standard screening problem studied in Myerson (1981).

Without moral hazard, pull funding is unnecessary because trial outcomes depend solely on chance rather than effort. In this case, the funder has no need to differentiate between success and failure and instead relies on a combination of push funding and insurance. As the profit potential π increases, both push funding and insurance decline. In Case (b)(i), push funding decreases with π and drops to zero when $\pi = c/\delta$. In Case (b)(ii), insurance also declines and becomes zero as π increases, transitioning into Case (b)(iii). In Case (c), the amount of push funding decreases to zero at the boundary between Cases (c)(i) and (c)(ii).

4.2 No Adverse Selection

Here, we consider a second-best benchmark with moral hazard, but without adverse selection. The success probability θ is therefore public information.

This analysis highlights the value of information. Comparing outcomes in this setting to the base case from the previous section quantifies the value to the funder of learning about whether the firm is likely to succeed with the drug.

In this setting, the value θ is fixed, and hence the decision variables q, m_0, m_S, and m_F

are scalars instead of functions. The incentive constraint (IC) is reduced to

$$-c\,q + \delta\,\theta\,\left(m_S + \pi\,q\right) + \delta\,(1-\theta)\,m_F \geq -\alpha\,c\,q + \delta\,m_F. \tag{29}$$

The corresponding funder's problem is formulated as follows:

$$\max_{\{q,m_0,m_S,m_F\}\in\Omega_{\mathrm{SB}}} \ \theta\,\omega\,q - m_0 - \theta\,m_S - (1-\theta)\,m_F, \tag{30}$$

where the feasible set Ω_{SB} is determined by (29) and the following linear inequality constraints:

$$-c\,q + m_0 + \delta\,\theta\,\left(m_S + \pi\,q\right) + \delta\,(1-\theta)\,m_F \geq 0, \tag{31}$$

$$0 \leq q \leq 1, \text{ and} \tag{32}$$

$$m_0,\ m_S,\ m_F \geq 0. \tag{33}$$

The next proposition characterizes the funder's optimal mechanism according to (30).

Proposition 10. *The following four cases summarize the optimal solution to the funder's problem without adverse selection* (30)*:*

(a) if $\theta \in [0,\ \max\{c/(\omega+\delta\,\pi), (\delta\,\alpha-\alpha+1)\,c/(\delta\,(\omega+\pi))\})$*, it is optimal to not induce testing and provide no funding, i.e.,*

$$q = m_0 = m_S = m_F = 0;$$

(b) if $\theta \in [(\delta\,\alpha-\alpha+1)\,c/(\delta\,(\omega+\pi)),\ (1-\alpha)\,\bar{\tau})$*, it is optimal to induce testing and provide a combination of push and pull funding with an expected present value of* $c-\delta\,\pi\,\theta$*, i.e.,*

$$q = 1, \quad m_0 = \alpha\,c, \quad m_S = \frac{(1-\alpha)\,c}{\delta\,\theta} - \pi, \quad m_F = 0;$$

(c) if $\theta \in [\max\{c/(\omega+\delta\,\pi), (1-\alpha)\,\bar{\tau}\},\ \bar{\tau})$*, it is optimal to induce testing and provide a push funding of amount* $c-\delta\,\pi\,\theta$*, i.e.,*

$$q = 1, \quad m_0 = c - \delta\,\pi\,\theta, \quad m_S = m_F = 0; \text{ and}$$

(d) if $\theta \in [\bar{\tau}, 1]$, the firm is willing to test without the funder, i.e.,

$$q = 1, \quad m_0 = m_S = m_F = 0.$$

Note that the condition $c/(\omega + \delta\,\pi) > (\delta\,\alpha - \alpha + 1)\,c/(\delta\,(\omega + \pi))$ in Case (a) is equivalent to $c/(\omega + \delta\,\pi) > (1-\alpha)\,\bar{\tau}$ in Case (c), which further implies that $(\delta\,\alpha - \alpha + 1)\,c/(\delta\,(\omega + \pi)) < (1-\alpha)\,\bar{\tau}$, or, the disappearance of Case (b). Therefore, the intervals for θ in the cases presented in Proposition 10 are indeed mutually exclusive and collectively exhaustive.

To summarize, when the success rate θ is public information, the funder should not offer any funding if the drug is ex ante very ineffective (Case (a)), and let the commercial market take over if the drug is sufficiently promising (Case (d)). For intermediate levels of θ, the funder should rely on either push only (Case (c)) or pull plus push (Case (b)).

It is also worth noting that in both Cases (b) and (c), the present value of the firm's expected value is $m_0 + \delta\,\theta\,(m_S + \pi) = c$, which equals the testing cost. In Case (b), the drug is less promising ex ante, which implies that the expected profit from potential success is not high enough for the firm to exert effort. Therefore, the funder needs to provide some pull ($m_S > 0$) to guarantee that the incentive constraint (29) is satisfied. Indeed, this constraint binds at the optimum. In Case (c), on the other hand, the probability of success θ is higher, and the expected profit from success is sufficiently high so that the funder no longer needs to provide additional reward for success. In this case, the constraint (29) is not even binding at optimality. In both cases, the push funding ($m_0 > 0$) is to guarantee the firm's participation.

Without adverse selection, the funder has no reason to provide insurance ($m_F = 0$). Insurance is used to counteract the firm's incentive to under-report the drug's probability of success, but in this case, the funder already knows the true probability.

5 Calibration

We calibrate the model using parameter values drawn from the literature on drug development and global health. This section describes our calibration methods and presents the resulting funding recommendations.

5.1 Calibration Methods

We specify seven key parameters: testing cost, disease burden, value of life, discount factor, success probability, relative shirking cost, and profit. These values illustrate how the model

applies in practice. While some parameters are held constant across diseases for simplicity, the framework allows for disease-specific variation.

Testing cost. We assume Phase III testing cost of either \$100 million or \$400 million (Table 1). The lower estimate is based on DNDi reports citing €60–190 million for a new chemical entity (Drugs for Neglected Diseases initiative, 2019). The upper bound reflects Phase III out-of-pocket costs ranging from \$126 million to \$628 million (Wouters, McKee, and Luyten, 2020).[11] We incorporate capital costs and failure rates separately, bringing the economic cost above \$1 billion.

Shirking. If the firm shirks, it still incurs 70% of testing costs ($\alpha = 0.7$), implying partial effort, such as enrolling patients but reducing staff quality or oversight. We test robustness to $\alpha = 0.35$ and $\alpha = 1$ (no moral hazard). We also examine policies in which the funder is uncertain about α and seeks to maximize its worst-case payoff (Appendix J).

Disease burden. We use 2021 World Health Organization (WHO) data on disability-adjusted life years (DALYs) lost by income group (World Health Organization, 2024), focusing on infectious and parasitic diseases. We exclude category-level entries (e.g., "Other STDs") and diseases with highly effective, low-cost vaccines (e.g., measles). Countries are classified as high-income (GNI per capita $\geq$ \$14,000) or otherwise. DALYs are denoted as DALY_h (high-income) and DALY_l (low- and middle-income). We also report whether the disease is classified as neglected Table 3). We assume each successful product reduces disease-specific DALYs by 10%, reflecting modest market share and effectiveness. For example, Naci et al. (2025) report that the median drug adds half a year of full health; if used by 20% of patients, this implies a 10% burden reduction. In 2015, anti-malarials reduced burden by 16% (Artemether-Lumefantrine), 12% (Artesunate-Amodiaquine), and 1% (Artesunate-Sulfadoxine-Primaquine) (Hassoun, 2015).

Value of life. For low-income countries, we assume a value of \$2,000 per DALY saved ($\omega_\text{l}/\text{DALY}_\text{l}$), based on income level (Kazibwe et al., 2022). Per capita income is \$1,700 in sub-Saharan Africa and \$2,400 in South Asia (World Bank, 2025). We test robustness with \$6,000. For high-income countries, we assume \$12,000 per DALY saved ($\omega_\text{h}/\text{DALY}_\text{h}$), equal to one-quarter of per capita income (\$48,000) (World Bank, 2025), with the remainder attributed to profit or variable costs (Figure 8).

Discount factor. We assume six years of testing without revenue, followed by ten years of sales. With a 10.5% annual discount rate (DiMasi, Grabowski, and Hansen, 2016;

[11] These align with Mulcahy et al. (2025), who estimate total out-of-pocket costs of \$360 million across all phases. If Phase III represents two-thirds, that implies \$240 million—consistent with our range.

Wouters et al., 2024), this implies a discount factor of 0.55 for payments received at the time of approval, with smaller weights applied to later profits. In practice, the discount factor may be higher if the funder reduces the firm's risk or if Phase III testing lasts only four years (DiMasi, Grabowski, and Hansen, 2016). We test robustness to both higher and lower discount factors.

Success probability. We assume a uniform prior and test robustness using alternative Beta distributions.

Profit. In low-income countries, we assume zero profit ($\pi_{\text{l}}/\text{DALY}_{\text{l}} = 0$), reflecting cost-based pricing. For example, Coartem and Sirturo are both sold for under $1 per day. In high-income countries, we assume profit of $24,000 per DALY averted ($\pi_{\text{h}}/\text{DALY}_{\text{h}}$), about 62% of the United Kingdom's $39,000 willingness-to-pay threshold. The remaining 38% covers variable costs, consistent with Wouters et al. (2024), who estimate variable costs at 25–60%. This profit estimate also aligns with income-based valuation, as it represents about half of average per capita income in high-income countries ($48,000), with the rest attributed to costs and consumer surplus (Figure 8). We round up to $1 million if the present value of profit is lower.

We calculate the present value of profit per drug, π, at the beginning of the 10-year profit period as follows:

$$\pi = \sum_{t=1}^{10} \frac{\text{share} \times \text{DALY}_{\text{h}} \times (\pi_{\text{h}}/\text{DALY}_{\text{h}})}{(1+\hat{\delta})^{t-1}}. \tag{34}$$

We calculate the present value of consumer welfare by summing up ten years with no discount:

$$\omega = 10 \times \text{share} \times [\text{DALY}_{\text{l}} \times (\omega_{\text{l}}/\text{DALY}_{\text{l}}) + \text{DALY}_{\text{h}} \times (\omega_{\text{h}}/\text{DALY}_{\text{h}}))]. \tag{35}$$

Based on these parameters, we determine the optimal funding mechanism according to Propositions 1-9.

5.2 Calibration Results

We focus on three high-burden communicable diseases: tuberculosis, malaria, and HIV/AIDS. Although they impose similar global burdens, they require different policy responses. For example, tuberculosis and malaria each caused over 5 million life-years lost in low- and middle-income countries in 2021. However, their burden in high-income countries differs by two orders of magnitude (Table 3). These differences influence commercial potential and

Notation	Description	Main	Robust
c	Testing cost	\$400 million	\$100 million
α	Relative cost of shirking	70%	35%
$\omega_{\mathrm{l}}/\mathrm{DALY}_{\mathrm{l}}$	Value per DALY for low and middle-income	\$2,000	\$6,000
$\omega_{\mathrm{h}}/\mathrm{DALY}_{\mathrm{h}}$	Value per DALY for high-income countries	\$12,000	\$12,000
$\pi_{\mathrm{l}}/\mathrm{DALY}_{\mathrm{l}}$	Profit per DALY for low and middle-income countries	\$0	\$0
$\pi_{\mathrm{h}}/\mathrm{DALY}_{\mathrm{h}}$	Profit per DALY for high-income countries	\$24,000	\$24,000
share	Share of profit and DALYs per product	10%	10%
$\hat{\delta}$	Annual discount rate for firm	10.5%	10.5%
δ	Discount factor for firm $1/(1+\hat{\delta})^4$	0.55	0.67
θ	Probability of technical success	$\mathcal{U}_{[0,1]}$	Various
π	Profit	Equation (34)	Equation (34)
ω	Consumer welfare	Equation (35)	Equation (35)
	Time until profit due to clinical trial	6 years	4 years
	Time on the market	10 years	10 years

Table 1: Model parameters and their assumed values used in the main analysis and robustness check. Disease burden is measured in disability-adjusted life years (DALYs), as reported by the World Health Organization.

lead to distinct optimal funding strategies.

Tuberculosis. The optimal policy is insurance: reimbursing a share of testing costs if the product fails. This is because commercial rewards are sufficient for success, given the high burden in high-income countries. If testing costs are very high or the firm heavily discounts future revenue, the policy shifts to push. If testing costs are low, the funder should not invest.

Malaria. The optimal policy is pull funding on the order of a billion dollars supplemented by push funding that covers a share of testing costs (Table 2, Figure 3). This recommendation is robust to changes in key parameters, including a higher value of life, a lower cost share under shirking (Figure 4), a discount factor below 0.5 (Table 4), alternative priors on success probability (Tables 5 and 6), and a wide range of testing costs (Table 7). In particular, pull remains the preferred strategy when Phase III testing costs range from \$1 million to \$5 billion. For a drug that reduces malaria burden by 10% and requires \$100 million for Phase III testing, our model recommends a pull reward of roughly \$1 billion. The prize

should increase as testing costs rise (Table 2). Even at this lower bound, the optimal pull commitment for malaria far exceeds current practice. Priority review vouchers have typically sold for about $100 million for several years (Ridley, Ganapathy, and Kettler, 2021), roughly an order of magnitude below the optimal level for a malaria drug.

HIV/AIDS. The optimal policy is no funding for Phase III testing, because commercial incentives are sufficient. A product that reduces 10% of the HIV/AIDS burden in high-income countries is expected to generate $18 billion in present-value profit. This aligns with real-world data; for example, Biktarvy generated nearly $13 billion in sales in 2023.[12] If testing costs are under $3.3 billion, public funding is unnecessary. If costs exceed that threshold, the funder should offer insurance to reimburse firms in case of failure (Table 7).

Communicable diseases with moderate market potential. For acute hepatitis A, chlamydia, cysticercosis, echinococcosis, food-borne trematodes, trichomoniasis, and tuberculosis, insurance is optimal when testing costs are $100 million. For acute hepatitis A, genital herpes, trichomoniasis, and tuberculosis, insurance is optimal even at $400 million. Insurance is recommended for diseases like these with moderate markets, unless there is no adverse selection. In that case, the funder offers a mix of push and pull (Table 8).

High-burden tropical diseases with no profit potential. For dengue, lymphatic filariasis, malaria, onchocerciasis, and schistosomiasis, the optimal policy consists of pull funding above $100 million, combined with push funding to cover a portion of testing costs. This result is robust to variation in testing costs, the value of life, time-to-market, the severity of moral hazard, and assumptions about the probability of success. Across different priors, including Beta distributions and a uniform prior, policy recommendations remain consistent (Table 5). For these diseases, push funding plus priority review vouchers might be sufficient, if testing costs are sufficiently low.

Low-burden tropical diseases with no profit potential. Blinding trachoma, leprosy, trichuriasis, and trypanosomiasis also lack commercial potential but have smaller disease burdens. These diseases merit public funding only if testing costs are relatively low. If the expected welfare gains do not justify the costs, the optimal policy is not to invest (Table 2).

The optimal funding policies remain stable for most diseases as we vary four key parameters: shirking, the value of a life-year, the firm's discount rate, and the distribution of the probability of success (Figure 4). Although these parameters influence the recommended funding amounts, the core policy prescriptions are largely unchanged. Moreover,

[12]Sales data are based on the authors' calculations using IQVIA MIDAS.

	Testing cost 400				Testing cost 100			
	Policy	**Push**	**Pull**	**Ins.**	**Policy**	**Push**	**Pull**	**Ins.**
Leprosy	(1) Nothing	0	0	0	(1) Nothing	0	0	0
Gonorrhoea		0	0	0	(2) Pull+push	70	12	0
Trachoma		0	0	0		70	83	0
Trichuriasis		0	0	0		70	104	0
Trypanosomiasis		0	0	0		70	59	0
Leishmaniasis	(2) Pull+push	280	285	0		70	15	0
Rabies		280	588	0		70	163	0
Dengue		280	516	0		70	158	0
Schistosomiasis		280	565	0		70	201	0
Whooping cough		280	902	0		70	343	0
Lymphatic filariasis		280	505	0		70	195	0
Malaria		280	2083	0		70	809	0
Yellow fever		280	330	0		70	155	0
Ascariasis		280	394	0		70	180	0
Hookworm disease		280	348	0		70	165	0
Onchocerciasis		280	540	0		70	236	0
Syphilis	(3) Push	280	547	0	(3) Push	70	0	0
Chagas disease		280	107	0		70	0	0
Acute hepatitis E	(4) Push+insure	280	86	0	(4) Push+insure	60	0	18
Chlamydia		280	58	0		49	0	37
Trematodes		280	12	0	(5) Insure	0	0	127
Cysticercosis	(3) Push	280	0	0	(6) Insure less	0	0	105
Echinococcosis		280	0	0		0	0	102
Acute hepatitis A	(4) Push+insure	214	0	120		0	0	47
Trichomoniasis		218	0	113		0	0	46
Genital Herpes		148	0	240	(7) Commercial	0	0	0
Tuberculosis	(6) Insure less	0	0	243		0	0	0
Acute hepatitis C	(7) Commercial	0	0	0		0	0	0
HIV/AIDS		0	0	0		0	0	0

Table 2: Optimal policies depend on disease characteristics. Units are millions of dollars. "Ins." refers to the funder partially insuring the firm's losses. In this and subsequent tables, "trematodes" refers to food-borne trematodes. Source: authors' calculations using 2021 data from the World Health Organization on disability-adjusted life years lost.

the recommended strategies are a significant departure from current practice.

6 Conclusion

Our analysis shows how funders should optimally allocate three tools: pull funding, push funding, and insurance against failure. We focus on settings where commercial profits are

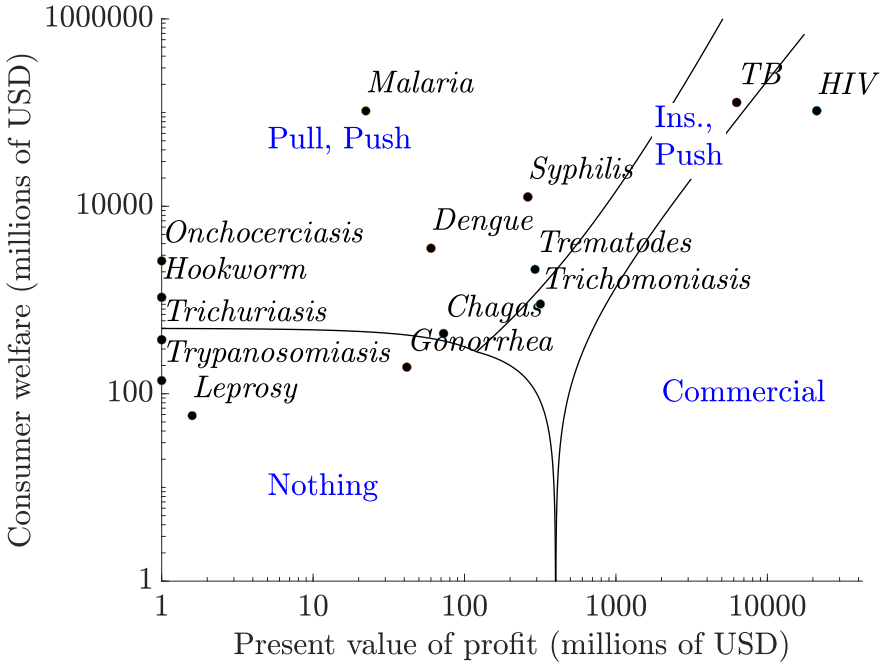


Figure 3: Optimal funding mechanisms for selected communicable diseases. The testing cost is $400 million. To reduce clutter, the figure combines Regions (2) and (3) (both involve some push) and Regions (4)–(6) (all involve insurance). Source: authors' calculations using 2021 data from the World Health Organization on disability-adjusted life years lost.

insufficient to cover testing costs, but the social value of the investment is high. To address both moral hazard and adverse selection, we solve an optimization problem in which the funder chooses how to support each firm type. We use duality theory to characterize the optimal solution and derive explicit formulas for when to use each funding tool.

Pull funding, which rewards success, directly mitigates moral hazard and should be used more extensively. Push funding is useful when firms are less patient than funders, but should represent only a small share of total support—contrary to current practice.

Failure insurance mitigates adverse selection by discouraging firms from obtaining funding by understating a product's potential. However, both insurance and push funding are potentially vulnerable to moral hazard. To address this, the funder should cover only part of the testing costs through either mechanism. Capping support ensures that the firm retains "skin in the game" and remains motivated to exert effort. In addition, insurance should be used only when a viable commercial market exists, so firms have something to gain from success. As a result, insurance may be optimal for products treating acute hepatitis A, trichomoniasis, and tuberculosis, but not malaria.

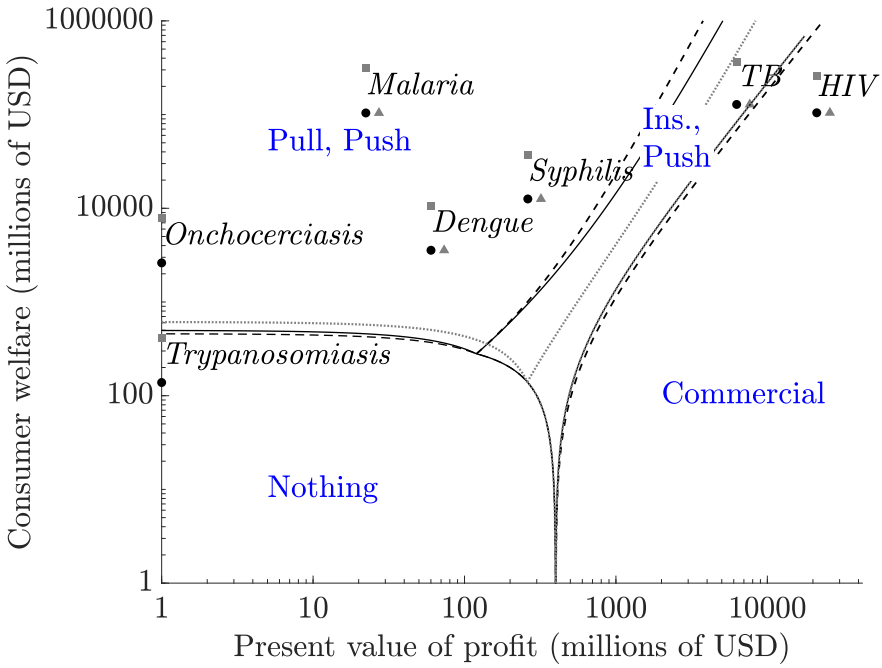


Figure 4: Optimal funding mechanisms for selected communicable diseases are robust to tripling the value of a life-year in low-income countries (gray squares), reducing the time until profit to 4 years (gray triangles and dashed lines), or halving the share of testing cost paid when shirking (dotted lines) relative to the baseline in Figure 3.

For malaria, the optimal policy is primarily pull funding, supplemented by limited push support. Recommended pull funding is on the order of a billion dollars, with larger amounts justified by low success probabilities or high testing costs. Pull funding ensures firms have a strong incentive to succeed despite limited commercial prospects. Optimal push funding is an order of magnitude smaller. An example of pull funding is the priority review voucher, but if the voucher value is only $100 million per product (Ridley, Ganapathy, and Kettler, 2021), then it is nsufficient for malaria, dengue, lymphatic filariasis, onchocerciasis, and schistosomiasis. Funders could enhance this by offering a "priority plus" reward—such as a prize or purchase guarantee, in addition to the voucher.

For tuberculosis, expected profits may be low due to limited market returns, low success probabilities, or high testing costs. In such cases, the funder should offer failure insurance or push funding. The same applies to diseases such as hepatitis A and trichomoniasis. When expected profits are high, no external funding is needed because the market can sustain development on its own.

For HIV/AIDS, commercial incentives are strong enough that no public funding for Phase

III testing is required under plausible assumptions. Only if out-of-pocket testing costs are an order of magnitude higher does failure insurance become optimal. Governments and foundations spend more on HIV/AIDS R&D than on any other infectious or parasitic disease (Impact Global Health, 2023). These funds could instead support procurement of existing HIV/AIDS treatments for low-income populations or development of products for diseases with weaker commercial potential.

We prescribe optimal policies for late-stage testing, where the firm is within six years of potentially earning revenue under reasonable assumptions about the discount rate. Even when the timeline exceeds six years or the discount rate is higher, the policy recommendations for malaria and HIV/AIDS remain essentially unchanged. For tuberculosis, however, if the testing timeline is especially long, the model recommends shifting from deferred support to immediate support through push funding.

Although our focus is on global health, the framework applies more broadly, including to sectors such as renewable energy. By combining push, pull, and insurance tools in a targeted way, funders can better align incentives, spur innovation, and improve social outcomes.

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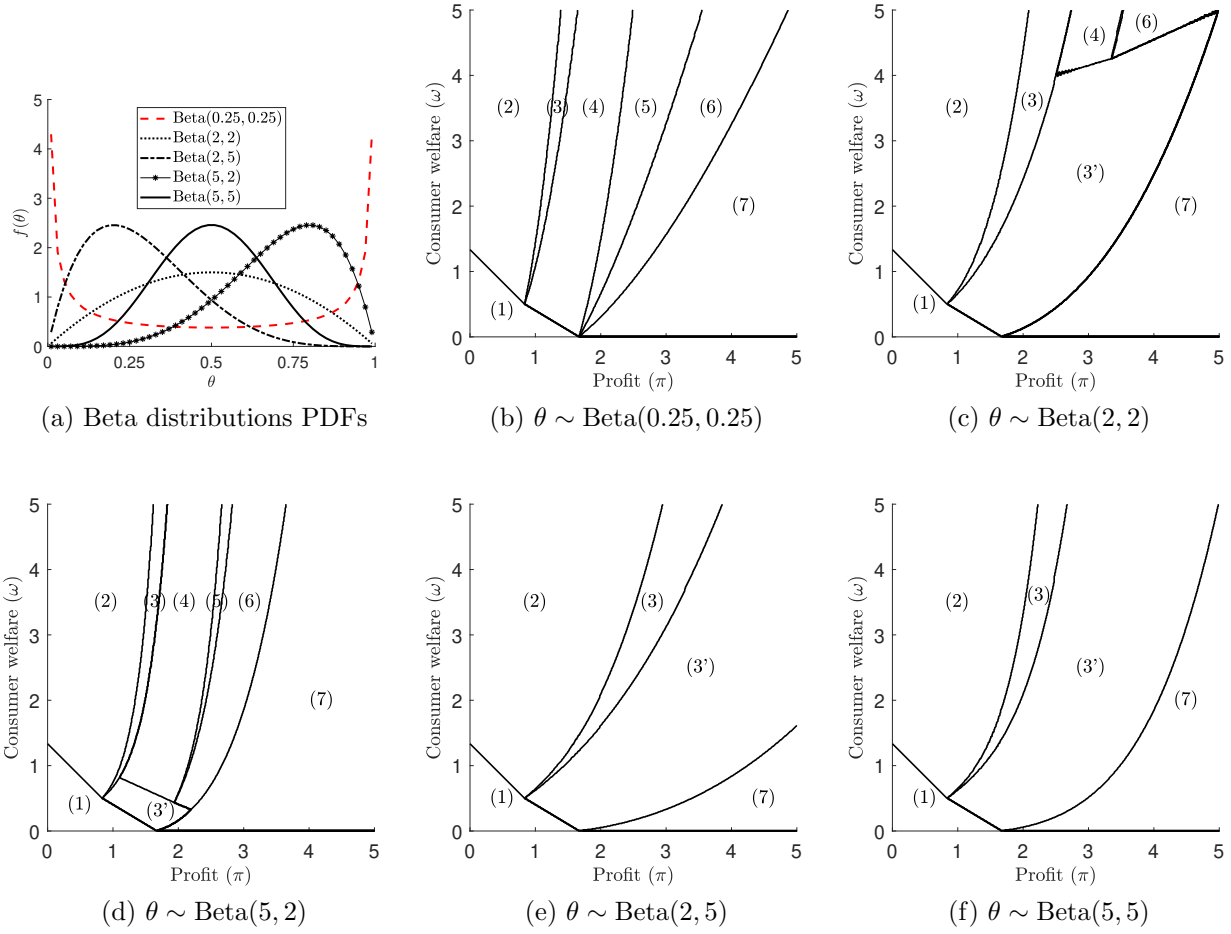
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Online Appendix

A Additional Figures and Tables



(a) Beta distributions PDFs

(b) $\theta \sim \text{Beta}(0.25, 0.25)$

(c) $\theta \sim \text{Beta}(2, 2)$

(d) $\theta \sim \text{Beta}(5, 2)$

(e) $\theta \sim \text{Beta}(2, 5)$

(f) $\theta \sim \text{Beta}(5, 5)$

Figure 5: Optimal threshold policies when θ is not uniformly distributed ($\alpha = 0.5$, $c = 1$, and $\delta = 0.6$).

Disease	Type	Neglected	Annual DALYs (000) by income group High ($DALY_h$)	Low+Middle ($DALY_l$)
Acute hepatitis A	Viral	0	74	1792
Acute hepatitis C	Viral	0	206	342
Acute hepatitis E	Viral	0	13	219
Ascariasis	Parasitic	1	0	681
Chagas disease	Parasitic	1	8	170
Chlamydia	Bacterial	0	13	149
Cysticercosis	Parasitic	1	43	1063
Dengue	Viral	1	7	1742
Echinococcosis	Parasitic	1	34	535
Gential Herpes	Viral	0	45	213
Gonorrhoea	Bacterial	0	5	68
HIV/AIDS	Viral	0	2419	37749
Hookworm disease	Parasitic	1	0	534
Leishmaniasis	Parasitic	1	13	857
Leprosy	Bacterial	1	0	28
Lymphatic filariasis	Parasitic	1	3	1320
Malaria	Parasitic	0	3	52046
Onchocerciasis	Parasitic	1	0	1307
Rabies	Viral	1	9	2588
Schistosomiasis	Parasitic	1	4	1851
Syphilis	Bacterial	0	30	6111
Trachoma	Bacterial	1	0	134
Trematodes	Parasitic	1	33	861
Trichomoniasis	Parasitic	0	36	236
Trichuriasis	Parasitic	1	0	188
Trypanosomiasis	Parasitic	1	0	69
Tuberculosis	Bacterial	0	716	59867
Whooping cough	Bacterial	0	3	5294
Yellow fever	Viral	0	0	495

Table 3: Harm from communicable diseases, measured in disability-adjusted life years (DALYs) lost. "Neglected" indicates whether the World Health Organization classifies the disease as a neglected tropical disease. Source: Authors' calculations using 2021 DALY data from the World Health Organization.

	Malaria	**Tuberculosis**	**HIV**	**Trypanosomiasis**
(1) Nothing	-	-	-	95 to 5,000
(2) Pull+push	1 to 5,000	2,475 to 5,000	-	1 to 95
(3) Push	-	390 to 2,475	-	-
(3') Less Push	-	109 to 390	1,404 to 5,000	-
(7) Commercial	-	1 to 109	1 to 1,404	-

Table 4: Critical values of testing costs (in millions of US dollars) for a case where the discount factor is less than one half. Here, the discount rate is 16.5% and the time to approval is six years. Testing costs are assumed to range from $1 million to $5 billion.

Disease	Uniform (flat)	Beta(2,2) (mod. flat)	Beta(5,5) (centered)	Beta(2,5) (left-skew)	Beta(5,2) (right-skew)	Beta(.25,.25) (U-shape)
Leprosy, Gonorrhoea, Trachoma, Trichuriasis, Trypanosomiasis	(1) Nothing					
Acute hepatitis E, Ascariasis, Chagas disease, Chlamydia, Dengue, Hookworm disease, Leishmaniasis, Lymphatic filariasis, Malaria, Onchocerciasis, Rabies, Schistosomiasis, Syphilis, Whooping cough, Yellow fever	(2) Pull+push					
Trematodes, Echinococcosis, Cysticercosis	(3) Push				(4)	
Genital Herpes, Trichomoniasis, Acute hepatitis A, Tuberculosis, Acute hepatitis C	(4) Push+insure	(3') Less push		(3)	(6)	(5) Insure
	(6) Less insure			(3')		
HIV/AIDS	(7) Commercial					

Table 5: Optimal policy identifiers for various distributions of the probability of success (θ).

	Malaria			Tuberculosis			HIV		
	Push	**Pull**	**Insure**	**Push**	**Pull**	**Insure**	**Push**	**Pull**	**Insure**
Original	280	2083	0	0	0	243	0	0	0
Testing cost (c): \$100MM	70	809	0	0	0	0	0	0	0
Life value (ω/DALY): \$6,000	280	3042	0	0	0	510	0	0	0
Cost of shirking (α): 0.35	140	3527	0	0	0	243	0	0	0
Time until profit: 4 years	280	1818	0	0	0	110	0	0	0
Mod. flat: Beta(2,2)	280	1780	0	0	0	0	0	0	0
Centered: Beta(5,5)	280	1271	0	0	0	0	0	0	0
Left skew: Beta(2,5)	280	2935	0	124	0	0	0	0	0
Right skew: Beta(5,2)	280	820	0	0	0	0	0	0	0
U-shape: Beta(.25,.25)	280	1909	0	0	0	510	0	0	0

Table 6: Funding mechanisms for malaria, tuberculosis, and HIV under varying model parameters and prior distributions.

	Malaria	**Tuberculosis**	**HIV**	**Trypanosomiasis**
(1) Nothing	-	-	-	111 to 5,000
(2) Pull+push	1 to 5,000	4,570 to 5,000	-	1 to 111
(3) Push	-	1,955 to 4,570	-	-
(4) Push+insure	-	1,500 to 1,955	-	-
(5) Insure	-	878 to 1,500	-	-
(6) Insure less	-	267 to 878	3,302 to 5,000	-
(7) Commercial	-	1 to 267	1 to 3,302	-

Table 7: Critical values of testing costs (millions of US dollars). Testing costs are assumed to range from \$1 million to \$5 billion.

	No moral hazard			No adverse selection		
	Push	**Pull**	**Insure**	**Push**	**Pull**	**Insure**
Gonorrhoea	-	-	-	-	-	-
Trachoma	-	-	-	-	-	-
Leprosy	-	-	-	-	-	-
Trichuriasis	-	-	-	-	-	-
Trypanosomiasis	-	-	-	-	-	-
Genital Herpes	3	-	722	222	-	-
Cysticercosis	21	-	691	229	-	-
Trichomoniasis	83	-	577	257	-	-
Echinococcosis	98	-	549	264	-	-
Trematodes	108	-	532	268	-	-
Syphilis	138	-	477	280	9	-
Acute hepatitis E	283	-	212	-	-	-
Chlamydia	288	-	204	-	-	-
Leishmaniasis	290	-	201	280	285	-
Rabies	321	-	145	280	341	-
Chagas disease	327	-	132	-	-	-
Dengue	340	-	109	280	376	-
Schistosomiasis	361	-	70	280	415	-
Whooping cough	374	-	48	280	438	-
Lymphatic filariasis	376	-	44	280	441	-
Malaria	378	-	41	280	445	-
Yellow fever	396	-	7	-	-	-
Ascariasis	398	-	4	280	481	-
Hookworm disease	399	-	1	-	-	-
Onchocerciasis	400	-	-	280	485	-
Tuberculosis	-	-	243	-	-	-
Acute hepatitis A	-	-	638	110	-	-
Acute hepatitis C	-	-	-	-	-	-
HIV/AIDS	-	-	-	-	-	-

Table 8: Optimal policies without moral hazard or adverse selection. Without loss of adverse selection, we assume that the probability of success θ is 0.45. Units are millions of dollars. Disease burden is measured in disability-adjusted life years (DALYs), as reported by the World Health Organization in 2021.

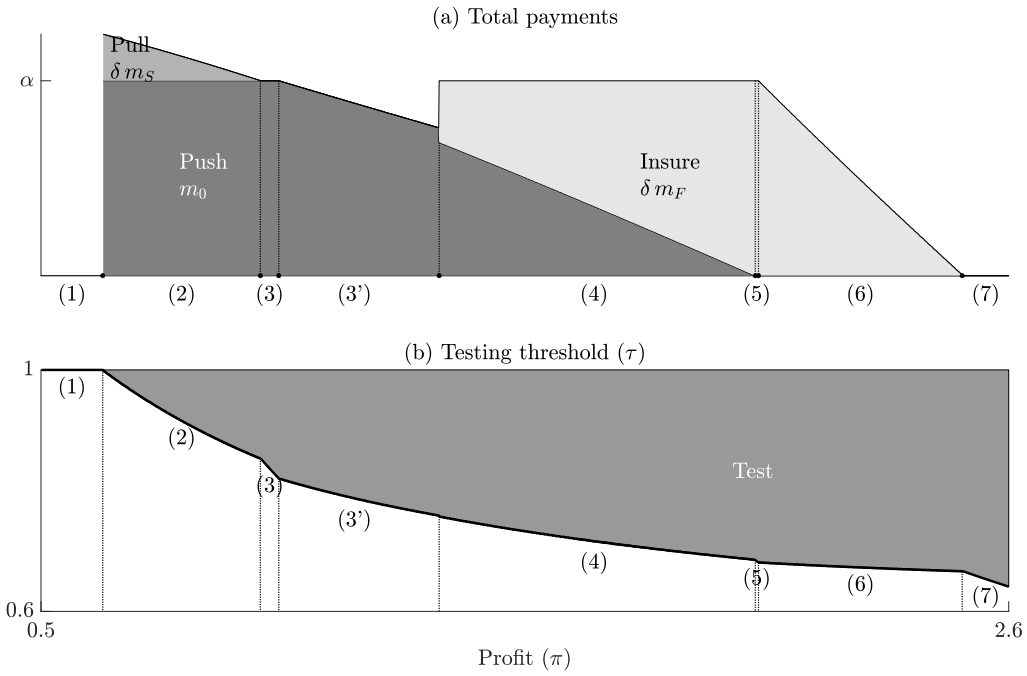


Figure 6: The total payment is the sum between the upfront payment m_0 and either payment upon success $\delta\, m_S$ or payment upon failure $\delta\, m_F$, which captures the maximum present value of payments ($\theta \sim \text{Beta}(5, 2)$, $\alpha = 0.5$, $c = 1$, $\delta = 0.6$, and $\omega = 0.7$).

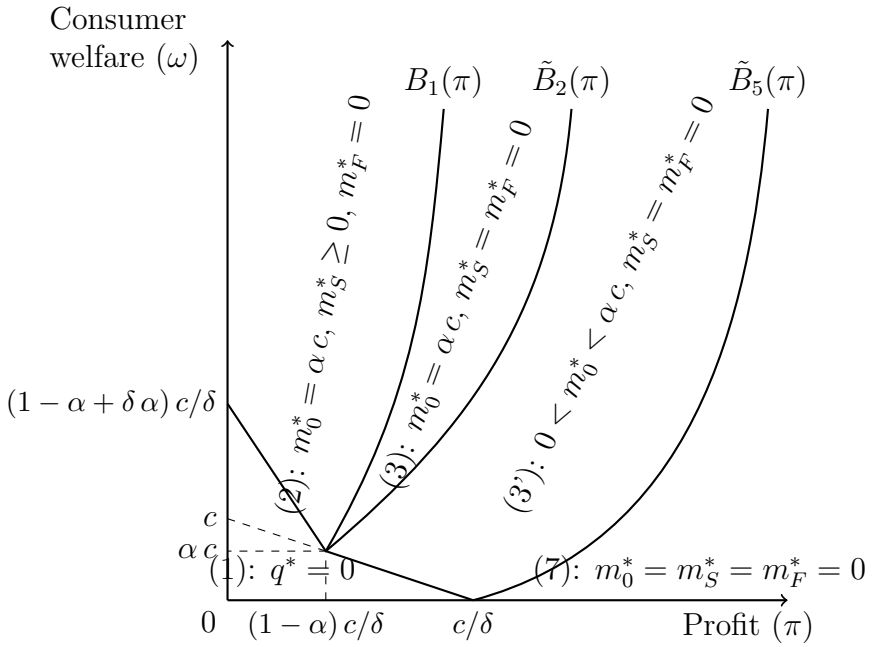


Figure 7: Optimal mechanisms under given profit π and consumer welfare ω if $\delta < 1/2$.

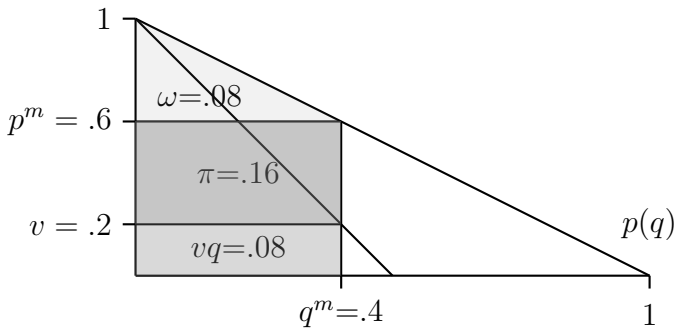


Figure 8: Assuming linear demand ($p(q)$), a monopoly price (p^m), and constant variable cost (v) set at 20% of the maximum willingness to pay, total surplus is split as follows: firm profit (π) is 50%, consumer welfare (ω) is 25%, and variable cost is the remainder.

B Proofs of Section 2

Proof. Proof of Lemma 1. For any feasible mechanism $\Xi \equiv \{q(\cdot), m_0(\cdot), m_S(\cdot), m_F(\cdot)\} \in \Omega$, we propose the following mechanism $\tilde{\Xi} \equiv \{\tilde{q}(\cdot), \tilde{m}_0(\cdot), \tilde{m}_S(\cdot), \tilde{m}_F(\cdot)\}$, where

$$\begin{aligned}
&\tilde{q}(\theta) = q(\theta), && \forall \theta \in \Theta,\\
&\tilde{m}_0(\theta) = m_0(\theta) + \delta \, \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta,\\
&\tilde{m}_S(\theta) = m_S(\theta) - \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta,\\
&\tilde{m}_F(\theta) = m_F(\theta) - \min\{m_S(\theta), m_F(\theta)\}, && \forall \theta \in \Theta.
\end{aligned}$$

With a slight abuse of notation, we use $\hat{u}_1(\theta, \hat{\theta}; \Xi)$ and $\hat{u}_0(\theta; \Xi)$ to denote the firm's expected payoff when its true type is θ and it reports $\hat{\theta}$, and when it reports θ and shirks, respectively, both under mechanism Ξ. It is straightforward to verify that the firm's payoff functions remain the same, i.e.,

$$\begin{aligned}
\hat{u}_1(\theta, \hat{\theta}; \tilde{\Xi}) = & -c\,\tilde{q}(\hat{\theta}) + \tilde{m}_0(\hat{\theta}) + \delta\,\theta\,\left[\tilde{m}_S(\hat{\theta}) + \pi\,\tilde{q}(\hat{\theta})\right] + \delta\,(1-\theta)\,\tilde{m}_F(\hat{\theta})\\
= & -c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,\theta\,\left[m_S(\hat{\theta}) + \pi\,q(\hat{\theta})\right] + \delta\,(1-\theta)\,m_F(\hat{\theta})\\
= & \,\hat{u}_1(\theta, \hat{\theta}; \Xi)
\end{aligned}$$

and

$$\hat{u}_0(\hat{\theta}; \tilde{\Xi}) = -\alpha\,c\,\tilde{q}(\hat{\theta}) + \tilde{m}_0(\hat{\theta}) + \delta\,\tilde{m}_F(\hat{\theta}) = -\alpha\,c\,q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\,m_F(\hat{\theta}) = \hat{u}_0(\hat{\theta}; \Xi).$$

By construction, we know that $\tilde{m}_0(\cdot)$, $\tilde{m}_S(\cdot)$, and $\tilde{m}_F(\cdot)$ are all nonnegative. Therefore, the proposed mechanism $\tilde{\Xi}$ is feasible.

It remains to verify that the funder's expected payoff under truth-telling is weakly higher, which follows from

$$\begin{aligned}
u_F(\theta; \tilde{\Xi}) &= \theta\,\omega\,\tilde{q}(\theta) - \tilde{m}_0(\theta) - \theta\,\tilde{m}_S(\theta) - (1-\theta)\,\tilde{m}_F(\theta)\\
&= \theta\,\omega\,q(\theta) - m_0(\theta) - \theta\,m_S(\theta) - (1-\theta)\,m_F(\theta) + (1-\delta)\,\min\{m_S(\theta), m_F(\theta)\}\\
&\geq u_F(\theta; \Xi).
\end{aligned}$$

This completes the proof. □ □

C Proofs of Section 3.1

By dividing the objective function and both sides of constraints (IC) and (IR) by c, we can, without loss of generality, normalize the model by setting $c = 1$ and replacing the parameters (c, ω, π) with $(1, \omega/c, \pi/c)$.

For notational simplicity, we set $c = 1$ for the remainder of the appendix. Accordingly, the optimality conditions and solutions presented in the main text apply with c replaced by 1. For instance, the boundary B_1 defined in equation (16) simplifies to

$$B_1(\pi) \equiv \frac{\delta^2\,\pi^3 - (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}.$$

We omit other resulting simplified expressions for brevity.

Reformulation of the funder's Problem (14)**.** We use $u(\theta)$ to denote the firm's expected payoff when it reports its type truthfully and exerts effort, i.e.,

$$u(\theta) \equiv u_1(\theta, \theta), \quad \forall \theta \in \Theta.$$

The (IC) constraint can be rewritten as two linear inequalities, i.e.,

$$u(\theta) \geq u_1(\theta, \hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2 \tag{36}$$

and

$$u(\theta) \geq u_0(\hat{\theta}), \quad \forall (\theta, \hat{\theta}) \in \Theta^2. \tag{37}$$

The following lemma is standard in mechanism design and allows us to eliminate the decision variable m_0 (See Krishna (2009), Chapter 5).

Lemma 3. *The incentive constraint* (36) *holds if and only if*

$$m_S(\cdot) - m_F(\cdot) + \pi\, q(\cdot) \text{ nondecreasing} \tag{38}$$

and

$$m_0(\theta) = u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + q(\theta) - \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] - \delta\, (1 - \theta)\, m_F(\theta). \tag{39}$$

Proof. Proof of Lemma 3 First, we prove constraint (36) implies (38) and (39). Constraint (36) can be rewritten as

$$u(\theta) \geq u(\hat{\theta}) + \delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta})\right].$$

Flipping the role of θ and $\hat{\theta}$ yields

$$u(\hat{\theta}) \geq u(\theta) + \delta\, (\hat{\theta} - \theta)\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)]\,.$$

Combining the two inequalities above yields

$$\delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta})\right] \leq u(\theta) - u(\hat{\theta}) \leq \delta\, (\theta - \hat{\theta})\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)]\,,$$

implying the monotonicity condition (38).

Invoking envelope theorem yields

$$u'(\theta) = \delta\, [m_S(\theta) - m_F(\theta) + \pi\, q(\theta)]\,,$$

which implies that

$$u(\theta) = u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y.$$

Solving for m_0 yields (39).

To prove the other direction, without loss of generality, fix $0 \leq \hat{\theta} < \theta \leq 1$, and we have

$$\begin{aligned} u(\theta) = &\ u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\ = &\ u(\theta_L) + \delta \int_{\theta_L}^{\hat{\theta}} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + \delta \int_{\hat{\theta}}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\ = &\ u(\hat{\theta}) + \delta \int_{\hat{\theta}}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \\ \geq &\ u(\hat{\theta}) + \delta\, (\theta - \hat{\theta})\, \left[m_S(\hat{\theta}) - m_F(\hat{\theta}) + \pi\, q(\hat{\theta}) \right], \end{aligned}$$

where the last inequality follows from the monotonicity condition (38).

This completes the proof. □ □

The "envelope" condition (39) allows us to eliminate m_0 and rewrite the funder's problem in terms of the "virtual valuation" functions defined by

$$\begin{aligned} w_q(\theta) &\equiv \theta\, \omega + \delta\, \pi\, \theta - 1 - \delta\, \pi\, \frac{1 - F(\theta)}{f(\theta)}, \\ w_S(\theta) &\equiv -(1 - \delta)\, \theta - \delta\, \frac{1 - F(\theta)}{f(\theta)}, \text{ and} \\ w_F(\theta) &\equiv -(1 - \delta)\, (1 - \theta) + \delta\, \frac{1 - F(\theta)}{f(\theta)}, \end{aligned}$$

The funder's problem (14) can now be reformulated as follows:

$$\max_{\{q(\cdot), m_S(\cdot), m_0(\cdot), u(\theta_L)\} \in \Omega_1} -u(\theta_L) + \int_{\theta_L}^{\theta_H} [w_q(\theta)\, q(\theta) + w_S(\theta)\, m_S(\theta) + w_F(\theta)\, m_F(\theta)]\, \mathrm{d}\theta, \tag{40}$$

where the feasible set Ω_1 is defined by (8), (38), and the following linear inequality constraints: for any $(\theta, \hat{\theta}) \in \Theta^2$,

$$\delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y$$

$$\geq \delta \int_{\theta_L}^{\hat{\theta}} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + (1 - \alpha)\, q(\hat{\theta}) - \delta\, \hat{\theta}\, \left[m_S(\hat{\theta}) + \pi\, q(\hat{\theta}) \right] + \delta\, \hat{\theta}\, m_F(\hat{\theta}), \tag{41}$$

$$u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y \geq 0, \tag{42}$$

$$u(\theta_L) + \delta \int_{\theta_L}^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\, \mathrm{d}y + q(\theta) - \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] - \delta\, (1 - \theta)\, m_F(\theta) \geq 0, \text{ and} \tag{43}$$

$$m_S(\theta),\ m_F(\theta) \geq 0. \tag{44}$$

The inequalities (41), (42), and (43) are the moral hazard constraint, the individual rationality constraint, and the feasibility constraint on m_0, respectively.

Proof. Proof of Proposition 1. Notice that our candidate primal solution is always feasible.

The remaining proof is partitioned into two parts: (1) $\omega + \delta\,\pi \leq 1$; and (2) $1 < \omega + \delta\,\pi \leq (1 - \alpha + \delta\,\alpha) + (1 - \delta)\,\omega$.

For part (1), if $\delta \geq \frac{1}{2}$, define

$$\begin{aligned}
&\chi_L \equiv 1 \\
&\eta(\theta) \equiv \frac{1}{2\,\delta}\left(\frac{2\,\delta - 1}{\theta^2} + 1\right) - 1, && \forall \theta \in (0, 1], \\
&\xi(\theta) \equiv -\left[w_S(\theta) + w_F(\theta)\right], && \forall \theta \in [0, 1], \text{ and} \\
&\beta(\theta) \equiv (1 - \alpha - \delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi \int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta), && \forall \theta \in (0, 1].
\end{aligned}$$

Because

$$w_S(\theta) + w_F(\theta) = -1 + \delta \leq 0,$$

the nonnegativity of ξ holds automatically. Since $\delta \geq \frac{1}{2}$, we know that η is decreasing, implying that for any $\theta \in [0, 1]$,

$$\eta(\theta) \geq \eta(1) = 0.$$

Differentiating β yields

$$\beta'(\theta) = -2\,\delta\,\pi\,\eta(\theta) + (1 - \alpha - \delta\,\pi\,\theta)\;\eta'(\theta) - w_q'(\theta) = -\frac{(1 - \alpha)\,(2\,\delta - 1)}{\delta\,\theta^3} - (\omega + \pi) \leq 0,$$

and hence, the nonnegativity of β boils down to

$$\beta(1) = 1 - \omega - \delta\,\pi \geq 0,$$

which is implied by the optimality condition.

We have

$$\begin{aligned}
&\int_0^1 \eta(\theta)\;\left[\delta \int_0^\theta \left[m_S(y) - m_F(y) + \pi\,q(y)\right]\mathrm{d}y + (1 - \alpha)\,q(\theta) - \delta\,\theta\;\left[m_S(\theta) + \pi\,q(\theta)\right] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\
&\quad + \chi_L\;\left[-u(0)\right] + \int_0^1 \beta(\theta)\;\left[-q(\theta)\right]\mathrm{d}\theta + \int_0^1 \xi(\theta)\;\left[-m_S(\theta)\right]\mathrm{d}\theta \\
&= -u(0) + \int_0^1 \left[w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)\right]\mathrm{d}\theta.
\end{aligned}$$

Therefore, the objective function is bounded by 0. Substituting our conjectured primal solution into the objective function also yields 0, establishing the optimality.

If $\delta < \frac{1}{2}$, define

$$\begin{aligned}
&\chi_L = 1, \\
&\beta(\theta) = -w_q(\theta), \quad \forall \theta \in [0, 1], \\
&\xi(\theta) = -w_S(\theta), \quad \forall \theta \in [0, 1], \\
&\nu(\theta) = -w_F(\theta), \quad \forall \theta \in [0, 1].
\end{aligned}$$

The nonnegativity of the multipliers follows directly from their definitions. Since

$$\begin{aligned}
&\chi_L\,[-u(0)] + \int_0^\tau \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta + \int_0^1 \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_0^1 \nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta \\
=\; &-u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta,
\end{aligned}$$

the objective function is bounded above by 0, which is the same as the objective value achieved by the candidate solution.

For part (2), we begin with the case in which $\delta \geq \frac{1}{2}$. Let

$$\begin{aligned}
\Psi(\theta) \equiv\; & 2\,\delta^2\,\pi\,(\omega+\pi)\,\theta^3 \\
& - \delta\,\left[2\,\delta\,\pi^2\,(1-\delta) + (1-\alpha)\,(3\,\omega + 4\,\pi) + 2\,\delta\,\alpha\,\pi\right]\,\theta^2 \\
& + 2\,(1-\alpha)\,\left[2\,\delta\,\alpha + 2\,\delta\,\pi\,(1-\delta) + (1-\alpha)\right]\,\theta \\
& + \delta\,(1-\alpha)\,\omega - 2\,\delta\,\alpha\,(1-\alpha) - 2\,(1-\delta)\,(1-\alpha)^2
\end{aligned}$$

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\eta(\theta) \equiv \begin{cases} \dfrac{1}{2\,\delta}\left(\dfrac{C_2}{\theta^2}+1\right) - 1, & \forall \theta \in (0,\tau), \\ \dfrac{C_1}{2\,\delta\,\pi\,(\alpha - 1 + \delta\,\pi\,\theta)^2} - \dfrac{\omega}{2\,\delta\,\pi} - 1, & \forall \theta \in [\tau,1], \end{cases} \\
&\xi(\theta) \equiv \begin{cases} -\left[w_S(\theta) + w_F(\theta)\right], & \forall \theta \in [0,\tau), \\ -\delta\,\theta\,\eta(\theta) + \delta\,\displaystyle\int_\theta^1 \eta(y)\,\mathrm{d}y - w_S(\theta), & \forall \theta \in [\tau,1], \end{cases} \\
&\beta(\theta) \equiv (1 - \alpha - \delta\,\pi\,\theta)\,\eta(\theta) + \delta\,\pi\,\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta), \qquad \forall \theta \in (0,\tau], \text{ and} \\
&\nu(\theta) \equiv \delta\,\theta\,\eta(\theta) - \delta\,\int_\theta^1 \eta(y)\,\mathrm{d}y - w_F(\theta), \qquad \forall \theta \in [\tau,1],
\end{aligned}$$

where

$$\begin{aligned}
&\tau = \sup\left\{\theta \in \Theta : \Psi(\theta) = 0\right\}, \\
&C_1 = (1 - \delta\,\pi - \alpha)\,(\omega + \delta\,\pi\,\omega - \alpha\,\omega - 2\,\delta\,\alpha\,\pi), \text{ and} \\
&C_2 = \frac{\tau\,\left[\delta\,(\omega+\pi)\,\tau^2 + \left(\alpha - 2\,\delta\,\pi - 1 + 2\,\delta^2\,\pi - 2\,\alpha\,\delta\right)\,\tau + (2 - \delta\,\omega - 2\,\alpha - 2\,\delta + 4\,\alpha\,\delta)\right]}{\alpha - 1 + \delta\,\pi\,\tau}.
\end{aligned} \tag{45}$$

The threshold τ is chosen such that η is continuous, i.e.,

$$\lim_{\theta \uparrow \tau} \eta(\theta) = \lim_{\theta \downarrow \tau} \eta(\theta).$$

First, we prove that $C_1 \geq 0$. We have

$$\delta\,\pi - 1 + \alpha \leq \delta \cdot \frac{1-\alpha}{\delta} - 1 + \alpha = 0.$$

Fix π, and the second term $(\delta\,\pi + 1 - \alpha)\,\omega - 2\,\delta\,\alpha\,\pi$ is increasing in ω. Hence, its minimum is attained at $\omega = 1 - \delta\,\pi$. We have

$$(\delta\,\pi + 1 - \alpha)\,\omega - 2\,\delta\,\alpha\,\pi \geq h(\omega) \equiv (-\omega + 2 - \alpha)\,\omega - 2\,\alpha\,(1-\omega) = -\omega^2 + (\alpha+2)\,\omega - 2\,\alpha.$$

The minimum on $\left[\alpha, \frac{1}{\delta}\,(1 - \alpha + \delta\,\alpha)\right]$ is attained at one of the two endpoints, and we have

$$h(\alpha) = 0$$

and

$$h\left(\frac{1}{\delta}\,(1-\alpha+\delta\,\alpha)\right) = \frac{(1-\alpha)\,\overbrace{(\alpha + 2\,\delta - \alpha\,\delta - 1)}^{=\alpha\,(1-\delta)+(2\,\delta-1)\geq 0}}{\delta^2} \geq 0.$$

Therefore, the constant C_1 is nonnegative, and the multiplier η is nondecreasing on $[\tau, 1]$.

Next, we verify the existence of τ on $[0,1]$. We have

$$\Psi(1) = 2\,\delta\,\underbrace{(\omega + \delta\,\pi - 1)}_{\geq 0}\,\underbrace{(\alpha + \delta\,\pi - 1)}_{\leq 0} \leq 0$$

and

$$\Psi'(\theta) = 2\,\underbrace{(\alpha + \delta\,\pi\,\theta - 1)}_{\leq \alpha + \delta\,\frac{1-\alpha}{\delta} - 1 = 0}\,\underbrace{[3\,\delta\,(\omega+\pi)\,\theta - 2\,\delta\,\pi\,(1-\delta) - \alpha\,(2\,\delta - 1) - 1]}_{\text{increasing in } \theta}$$

Because $\Psi'(0) > 0 > \Psi'(1)$, where the second inequality follows from

$$\begin{aligned} 3\,\delta\,(\omega+\pi) - 2\,\delta\,\pi\,(1-\delta) - \alpha\,(2\,\delta-1) - 1 = &\ 3\,\delta\,(\omega + \delta\,\pi) + \delta\,\pi\,(1-\delta) - \alpha\,(2\,\delta-1) - 1 \\ \geq &\ 3\,\delta + \delta\,\pi\,(1-\delta) - \alpha\,(2\,\delta-1) - 1 \\ \geq &\ 3\,\delta - \alpha\,(2\,\delta-1) - 1 \\ \geq &\ 3\,\delta - (2\,\delta - 1) - 1 \\ = &\ \delta > 0, \end{aligned}$$

function Ψ is increasing and decreasing on $\left(-\infty, \frac{2\,\delta\,\pi\,(1-\delta)+\alpha\,(2\,\delta-1)+1}{3\,\delta\,(\omega+\pi)}\right]$ and $\left[\frac{2\,\delta\,\pi\,(1-\delta)+\alpha\,(2\,\delta-1)+1}{3\,\delta\,(\omega+\pi)}, 1\right]$, respectively.

To prove the existence of τ, it suffices to show that there exists $\hat{\theta} \in (-\infty, 1]$ such that $\Psi(\hat{\theta}) \geq 0$:

1. if $\hat{\theta} \geq 0$, the existence of a zero point for the function Ψ on the interval $[0,1]$ follows directly; and
2. if $\hat{\theta} < 0$, we know that

$$\Psi\left(\frac{2\,\delta\,\pi\,(1-\delta) + \alpha\,(2\,\delta-1) + 1}{3\,\delta\,(\omega+\pi)}\right) \geq \Psi(\hat{\theta}) \geq 0,$$

and hence the desired result still holds.

Because η is increasing on $[\tau, 1]$, we have

$$\eta(\tau) \geq 0 \quad \Leftrightarrow \quad \tau \geq \frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right) \quad \Leftarrow \quad \Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right)\right) \geq 0,$$

where the last step is implied by the monotonicity of Ψ.

Therefore, to prove the existence of τ and the nonnegativity of η, it suffices to show that

$$\Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{C_1}{\omega+2\,\delta\,\pi}}\right)\right) \geq 0,$$

which can be rewritten as

$$\begin{aligned}
&(\omega+\pi)\sqrt{-\frac{(\alpha+\delta\,\pi-1)\,(\omega-\alpha\,\omega+\delta\,\pi\,\omega-2\,\delta\,\pi\,\alpha)}{\omega+2\,\delta\,\pi}}-(1-\alpha)\,(\omega+\pi)+\delta\,\pi\,(1+\pi-\delta\,\pi)\leq 0,\\
\Leftrightarrow\ &-\frac{(\alpha+\delta\,\pi-1)\,(\omega-\alpha\,\omega+\delta\,\pi\,\omega-2\,\delta\,\pi\,\alpha)}{\omega+2\,\delta\,\pi}\leq\left(1-\alpha+\frac{\delta\,\pi\,(\delta\,\pi-\pi-1)}{\omega+\pi}\right)^2,\\
\Leftrightarrow\ &g(\omega)\geq 0,
\end{aligned}$$

where

$$\begin{aligned}
g(\omega)\equiv\ &\delta^2\,\pi^2\,\omega^3+2\,\delta\,\pi^2\,\left(-2\,\alpha\,\delta+\alpha+\delta+\delta\,\pi-1\right)\,\omega^2\\
&+\delta\,\pi^2\,\left[\left(\delta^3-2\,\delta^2+2\,\delta\right)\,\pi^2+\left(2\,\alpha-2\,\alpha\,\delta-4\,\alpha\,\delta^2+2\,\delta^2-2\right)\,\pi+4\,\alpha\,\delta-3\,\delta-2\,\alpha+2\right]\,\omega\\
&+2\,\delta\,\pi^3\,(\delta\,\pi-1)\left[\delta\,(1-\delta)^2\,\pi+\alpha+2\,\delta-2\,\alpha\,\delta-\delta^2-1\right].
\end{aligned}$$

Fix $\pi \in \left[0, \frac{1-\alpha}{\delta}\right]$, and the second-order derivative of g is increasing, i.e.,

$$g''(\omega) \geq g''(1-\delta\,\pi) = 2\,\delta\,\pi^2\,\left(2\,\alpha+5\,\delta+2\,\delta\,\pi-4\,\alpha\,\delta-3\,\delta\,\pi^2-2\right) \geq 0,$$

which implies that

$$g'(\omega) \geq g'(1-\delta\,\pi) = -2\,\delta\,\pi^2\,\left(2\,\delta-1\right)\,\left(\pi-\delta\,\pi+1\right)\,\left(\alpha+\delta\,\pi-1\right) \geq 0$$

Because $g(1-\delta\,\pi)=0\geq 0$, the desired result holds automatically.

By $\eta(\tau)\geq 0$ and

$$\delta\,\left[2\,\eta(\theta)+\theta\,\eta'(\theta)\right]=1-2\,\delta\leq 0,\quad \forall\theta\in[0,\tau],$$

we know that η is decreasing on $[0,\tau]$, guaranteeing its nonnegativity.

It remains to prove the nonnegativity of β, ξ, and ν. For any $\theta\in[0,\tau]$, we have

$$\beta'(\theta)=-2\,\delta\,\pi\,\underbrace{\eta(\theta)}_{\geq 0}+\underbrace{(1-\alpha-\delta\,\pi\,\theta)}_{\geq 0}\,\underbrace{\eta'(\theta)}_{\leq 0}-\underbrace{w_q'(\theta)}_{=2\,\delta\,\pi+\omega\geq 0}\leq 0,$$

and the desired property is guaranteed by $\beta(\tau)=0$.

Similarly, for any $\theta \in [\tau, 1]$, we have

$$\xi'(\theta) = -\delta \left[2 \underbrace{\eta(\theta)}_{\geq 0} + \theta \underbrace{\eta'(\theta)}_{\geq 0} \right] - \underbrace{w_S'(\theta)}_{\geq 0} \leq 0,$$

and the desired property boils down to

$$\xi(1) = -\delta \, \eta(1) - w_S(1) \, f(1) = \frac{\delta \, (\omega + \pi) - (1 - \alpha + \delta \, \alpha)}{\alpha + \delta \, \pi - 1} \geq 0,$$

which is implied by $\delta \, (\omega + \pi) \leq 1 - \alpha + \delta \, \alpha$.

Finally, for any $\theta \in [\tau, 1]$, we have

$$\nu'(\theta) = \delta \left[2 \underbrace{\eta(\theta)}_{\geq 0} + \theta \underbrace{\eta'(\theta)}_{\geq 0} \right] - \underbrace{w_F'(\theta)}_{\leq 0} \geq 0,$$

and hence, the desired property boils down to $\nu(\tau) \geq 0$, which holds automatically.

We have

$$\begin{aligned}
&\int_0^1 \eta(\theta) \left[\delta \int_0^\theta [m_S(y) - m_F(y) + \pi \, q(y)] \, \mathrm{d}y + (1 - \alpha) \, q(\theta) - \delta \, \theta \, [m_S(\theta) + \pi \, q(\theta)] + \delta \, \theta \, m_F(\theta) \right] \mathrm{d}\theta \\
&\quad + \chi_L \, [-u(0)] + \int_0^\tau \beta(\theta) \, [-q(\theta)] \, \mathrm{d}\theta + \xi_L \, [-m_S(0)] + \int_0^1 \xi(\theta) \, [-m_S(\theta)] \, \mathrm{d}\theta + \int_\tau^1 \nu(\theta) \, [-m_F(\theta)] \, \mathrm{d}\theta \\
&= -u(0) + \int_0^1 [w_q(\theta) \, q(\theta) + w_S(\theta) \, m_S(\theta) + w_F(\theta) \, m_F(\theta)] \, \mathrm{d}\theta.
\end{aligned}$$

The nonnegativity of all the multipliers we define imply that, the objective function is bounded by 0. Substituting our conjectured primal solution into the objective function also yields 0, establishing the optimality.

We now prove part (2) in the case $\delta < \frac{1}{2}$. Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\eta(\theta) \equiv \frac{C_1}{2 \, \delta \, \pi \, (\alpha - 1 + \delta \, \pi \, \theta)^2} - \frac{\omega}{2 \, \delta \, \pi} - 1, \quad \forall \theta \in [\tau, 1], \\
&\xi(\theta) \equiv \begin{cases} \delta \int_\tau^1 \eta(y) \, \mathrm{d}y - w_S(\theta), & \forall \theta \in [0, \tau), \\ -\delta \, \theta \, \eta(\theta) + \delta \int_\theta^1 \eta(y) \, \mathrm{d}y - w_S(\theta), & \forall \theta \in [\tau, 1], \end{cases} \\
&\beta(\theta) \equiv \delta \, \pi \int_\tau^1 \eta(y) \, \mathrm{d}y - w_q(\theta), \quad \forall \theta \in (0, \tau], \text{ and} \\
&\nu(\theta) \begin{cases} -\delta \int_\tau^1 \eta(y) \, \mathrm{d}y - w_F(\theta), & \forall \theta \in [0, \tau), \\ \delta \, \theta \, \eta(\theta) - \delta \int_\theta^1 \eta(y) \, \mathrm{d}y - w_F(\theta), & \forall \theta \in [\tau, 1], \end{cases}
\end{aligned}$$

where C_1 is defined by (45), and the threshold τ is determined by

$$\Upsilon(\tau) \equiv (1-\alpha-\delta\,\pi\,\tau)^2 - \frac{(1-\alpha-\delta\,\pi)\,(\delta\,\pi\,\omega - 2\,\alpha\,\delta\,\pi + (1-\alpha)\,\omega)}{\omega + 2\,\delta\,\pi} = 0.$$

Since $\alpha + \delta\,\pi > 1$, the nonnegativity of η requires

$$1-\alpha-\delta\,\tau\,\pi \geq 0 \quad \Leftrightarrow \quad \tau \leq \frac{1-\alpha}{\delta\,\pi}.$$

Because β is decreasing and satisfies $\beta(\tau) = 0$, its nonnegativity follows.

Since w_S is increasing in θ, the nonnegativity of ξ on $[0,\tau]$ boils down to

$$w_q(\tau) - \pi\,w_S(0) \geq 0 \quad \Leftrightarrow \quad \tau \geq \frac{1}{\omega + 2\,\delta\,\pi}.$$

For any $\theta \in [\tau, 1]$, we have

$$\xi'(\theta) = -2\,\delta\,\eta(\theta) - \delta\,\theta\,\eta'(\theta) - 2\,\delta + 1$$

and

$$\xi''(\theta) = -3\,\delta\,\eta'(\theta) - \delta\,\theta\,\eta''(\theta).$$

Since $\eta'(\theta) \geq 0$ and $\eta''(\theta) \geq 0$, we know that $\xi''(\theta) \leq 0$. The concavity implies that the minimum is attained at either τ or 1. Therefore, the desired nonnegativity condition is reduced to $\xi(1) \geq 0$, which is equivalent to

$$1 - \alpha + \alpha\,\delta \geq \delta\,(\omega + \pi).$$

Since w_F is increasing in θ, the nonnegativity of ν on $[0,\tau]$ is reduced to

$$\nu(\tau) \geq 0 \quad \Leftrightarrow \quad \tau \leq \frac{1 + (1-\delta)\,\pi}{\omega + \pi}.$$

For any $\theta \in [\tau, 1]$, we have

$$\nu(\theta) = \frac{1}{\pi}\,\left[(1-\alpha)\,\eta(\theta) - w_q(\theta)\right] - w_F(\theta).$$

Taking the derivative yields

$$\nu'(\theta) = \frac{1}{\pi}\,\left[(1-\alpha)\,\eta'(\theta) - w_q'(\theta)\right] - w_F'(\theta).$$

Since

$$(1-\alpha)\,\eta'(\theta) - w_q'(\theta) = \delta\,\pi\,\left(2\,\eta(\theta) + \theta\,\eta'(\theta)\right) > 0,$$

the nonnegativity of ν on this interval is guaranteed by $\nu(\tau) \geq 0$.

It remains to verify the existence of

$$\frac{1}{\omega + 2\,\delta\,\pi} \leq \tau \leq \min\left\{\frac{1-\alpha}{\delta\,\pi}, \frac{1 + (1-\delta)\,\pi}{\omega + \pi}\right\}$$

satisfying $\Upsilon(\tau) = 0$.

We first verify that there exists a zero point between $\tau_L \equiv \frac{1}{\omega+2\,\delta\,\pi}$ and $\tau_H \equiv \frac{1+(1-\delta)\,\pi}{\omega+\pi}$. We have

$$\Upsilon(\tau_L) = \frac{\left[(1-\alpha)\,\omega+\delta\,\pi-\alpha\,\delta\,\pi\right]^2}{(\omega+2\,\delta\,\pi)(\omega+\delta\,\pi)}(\omega+\delta\,\pi-1) \geq 0$$

and

$$\Upsilon(\tau_H) = \frac{\delta}{\omega+\pi}\left[\delta\,(\omega+\pi)-(1-\alpha+\alpha\,\delta)\right] \leq 0.$$

The quadratic function Υ is minimized at $\frac{1-\alpha}{\delta\,\pi}$, and hence, we have

$$\Upsilon\left(\frac{1-\alpha}{\delta\,\pi}\right) \leq \Upsilon(\tau_H) \leq 0.$$

Since

$$\begin{aligned}
&\int_\tau^1 \eta(\theta)\,\left[\delta\int_0^\theta \left[m_S(y)-m_F(y)+\pi\,q(y)\right]\mathrm{d}y+(1-\alpha)\,c\,q(\theta)-\delta\,\theta\,\left[m_S(\theta)+\pi\,q(\theta)\right]+\delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta\\
&+\chi_L\,\left[-u(0)\right]+\int_0^\tau \beta(\theta)\,\left[-q(\theta)\right]\mathrm{d}\theta+\int_0^1 \xi(\theta)\,\left[-m_S(\theta)\right]\mathrm{d}\theta+\int_0^1 \nu(\theta)\,\left[-m_F(\theta)\right]\mathrm{d}\theta\\
=\;&-u(0)+\int_0^1\left[w_q(\theta)\,q(\theta)+w_S(\theta)\,m_S(\theta)+w_F(\theta)\,m_F(\theta)\right]\mathrm{d}\theta,
\end{aligned}$$

the objective function is bounded above by 0, establishing the optimality of our candidate solution.

This completes the proof. □ □

Proof. Proof of Proposition 2. To verify our candidate primal solution is feasible, we need to prove that

$$\frac{1-\alpha}{\delta\,\tau}-\pi \geq 0 \quad \Leftrightarrow \quad \tau \leq \frac{1-\alpha}{\delta\,\pi}.$$

Define

$$h(\theta) \equiv 2\,\delta\,(\omega+\pi)\,\theta^3+(\alpha-2\,\delta\,\alpha-1)\,\theta^2+\alpha-1$$

Taking the derivative yields

$$h'(\theta) = 6\,\delta\,(\omega+\pi)\,\theta^2+2\,(\alpha-2\,\delta\,\alpha-1)\,\theta = 2\,\theta\,\underbrace{\left[3\,\delta\,(\omega+\pi)\,\theta+(\alpha-2\,\delta\,\alpha-1)\right]}_{\text{increasing in }\theta}.$$

We have $\lim_{\theta\downarrow 0} h'(\theta) < 0$ and

$$h'(1) = 2\,\left[3\,\delta\,(\omega+\pi)+(\alpha-2\,\delta\,\alpha-1)\right] \geq 2\,\left[3\,(1-\alpha+\delta\,\alpha)+(\alpha-2\,\delta\,\alpha-1)\right] > 0.$$

Therefore, the derivative h' is first negative and then positive, and the function h is first decreasing and then increasing on $[0,1]$. Because

$$h(0) = -(1-\alpha) \leq 0$$

and

$$h\left(\frac{1-\alpha}{\delta\,\pi}\right) = \frac{1}{\delta^2\,\pi^3}\cdot\left[-\delta^2\,\pi^3+(1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi+2\,\omega\,(1-\alpha)^2\right] \geq 0,$$

where the second inequality follows from the optimality condition $\omega \geq B_1(\pi)$, the unique existence of $\tau \in \left[0, \frac{1-\alpha}{\delta\,\pi}\right]$ is guaranteed.

Define

$$\chi_L \equiv 1,$$

$$\eta(\theta) \equiv \begin{cases} \eta_1(\theta) = \dfrac{C_3}{\theta^2} + \dfrac{1}{2\,\delta} - 1, & \forall \theta \in (0, \hat{\tau}), \\ \eta_2(\theta) = \dfrac{C_2}{(1-\alpha-\delta\,\pi\,\theta)^2} - \dfrac{\omega}{2\,\delta\,\pi} - 1, & \forall \theta \in [\hat{\tau}, \tau), \\ \eta_3(\theta) = \dfrac{1}{2\,\delta}\left(\dfrac{1}{\theta^2} + 1\right) - 1, & \forall \theta \in [\tau, 1], \end{cases}$$

$$\xi(\theta) \equiv \begin{cases} -\left[w_S(\theta) + w_F(\theta)\right], & \forall \theta \in [0, \hat{\tau}], \\ -\delta\,\theta\,\eta_2(\theta) + \delta\left[\displaystyle\int_\theta^\tau \eta_2(y)\,\mathrm{d}y + \int_\tau^1 \eta_3(y)\,\mathrm{d}y\right] - w_S(\theta), & \forall \theta \in [\hat{\tau}, \tau], \end{cases}$$

$$\nu(\theta) \equiv \begin{cases} \delta\,\theta\,\eta_2(\theta) - \delta\left[\displaystyle\int_\theta^\tau \eta_2(y)\,\mathrm{d}y + \int_\tau^1 \eta_3(y)\,\mathrm{d}y\right] - w_F(\theta), & \forall \theta \in [\hat{\tau}, \tau], \\ -\left[w_S(\theta) + w_F(\theta)\right], & \forall \theta \in [\tau, 1], \end{cases}$$

$$\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta_1(\theta) + \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta), \qquad \forall \theta \in (0, \hat{\tau}], \text{ and}$$

$$\sigma(\theta) \equiv -(1-\alpha-\delta\,\pi\,\theta)\;\eta_3(\theta) - \delta\,\pi\int_\theta^1 \eta_3(y)\,\mathrm{d}y + w_q(\theta), \qquad \forall \theta \in [\tau, 1],$$

where

$$\hat{\tau} = \sup\left\{\theta \in [0,\tau] : (1-\alpha)\;\left[\frac{C_2}{(1-\alpha-\delta\,\theta\,\pi)^2} - \frac{\omega}{2\delta\pi} - 1\right] \leq (\omega+\pi)\,\theta - (1-\delta)\,\pi - 1\right\},$$

$$C_2 = \frac{1}{2\,\delta}\left(1 + \frac{1}{\tau^2} + \frac{\omega}{\pi}\right)(1-\alpha-\delta\,\tau\,\pi)^2, \text{ and}$$

$$C_3 = \hat{\tau}^2\left[\frac{C_2}{(1-\alpha-\delta\,\hat{\tau}\,\pi)^2} - \frac{\omega}{2\,\delta\,\pi} - \frac{1}{2\,\delta}\right].$$

The thresholds $\hat{\tau}$ and τ are chosen such that $\beta(\hat{\tau}) = \sigma(\tau) = 0$, implying the continuity of η at $\hat{\tau}$ and τ, i.e.,

$$\lim_{\theta\uparrow\hat{\tau}} \eta_1(\theta) = \lim_{\theta\downarrow\hat{\tau}} \eta_2(\theta) \quad \text{and} \quad \lim_{\theta\uparrow\tau} \eta_2(\theta) = \lim_{\theta\downarrow\tau} \eta_3(\theta).$$

Let $G(\theta)$ for $\theta \in [0,\tau]$ be defined as

$$G(\theta) = (1-\alpha)\left(\frac{C_2}{(1-\alpha-\delta\,\theta\,\pi)^2} - \frac{\omega}{2\,\delta\,\pi} - 1\right) - \left[(\omega+\pi)\,\theta - (1-\delta)\,\pi - 1\right].$$

If $\hat{\tau}$ exists in $[0,\tau]$, it is defined by $G(\hat{\tau}) = 0$.

We have

$$G(\tau) = \pi\,(1-\delta) \geq 0,$$

and the function G is convex, i.e.,

$$G''(\theta) = \frac{6\,(\delta\pi)^2\,(1-\alpha)\,C_2}{(1-\alpha-\delta\,\theta\,\pi)^4} \geq 0.$$

Because $C_2 \geq 0$ and $\hat{\tau} \leq \tau \leq \frac{1-\alpha}{\delta\,\pi}$, we know that η_2 is increasing on $[\hat{\tau}, \tau]$. The desired nonnegativity condition boils down to $\eta_2(\hat{\tau}) \geq 0$.

We first show that if $\delta \geq \frac{1}{2}$, then there exists $\hat{\tau} \in [0, \tau]$ such that $\eta_2(\hat{\tau}) \geq 0$. By the continuity of η, we have

$$\begin{aligned}
&\lim_{\theta \uparrow \tau}\ (1-\alpha-\delta\,\theta\,\pi)\ \eta_2(\theta) + \delta\,\pi \int_{\tau}^{1} \eta_3(y)\,\mathrm{d}y = w_q(\tau),\ \text{and} \\
&\lim_{\theta \uparrow \hat{\tau}}\ \delta\,\theta\,\eta(\theta) - \delta \int_{\hat{\tau}}^{1} \eta(y)\,\mathrm{d}y = w_F(\hat{\tau}),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&(1-\alpha-\delta\,\tau\,\pi)\ \left[\frac{C_2}{(1-\alpha-\delta\,\pi\,\tau)^2} - \frac{\omega}{2\,\delta\,\pi} - 1\right] + \delta\,\pi \int_{\tau}^{1} \left[\frac{1}{2\,\delta}\ \left(\frac{1}{y^2}+1\right) - 1\right] \mathrm{d}y = (\omega + 2\,\delta\,\pi)\,\tau - \delta\,\pi - 1, \\
&\delta\,\hat{\tau}\ \left(\frac{C_3}{\hat{\tau}^2} + \frac{1}{2\,\delta} - 1\right) - \delta\ \left[\int_{\hat{\tau}}^{\tau} \left[\frac{C_2}{(1-\alpha-\delta\,\pi\,y)^2} - \frac{\omega}{2\,\delta\,\pi} - 1\right] \mathrm{d}y + \int_{\tau}^{1} \left[\frac{1}{2\,\delta}\ \left(\frac{1}{y^2}+1\right) - 1\right] \mathrm{d}y\right] = (2\,\delta - 1)\,(1-\hat{\tau}).
\end{aligned}$$

Solving the system of equations above yields alternative expressions for the constants C_2 and C_3, which are mathematically equivalent to their previous definitions. Since

$$\eta_1(\theta) - \eta_2(\theta) = \frac{\Psi(\theta)}{2\,\delta\,\hat{\tau}\,\tau\,(1-\alpha-\delta\,\pi\,\hat{\tau})^2},$$

the threshold $\hat{\tau}$ satisfies $\Psi(\hat{\tau}) = 0$.

Differentiating Ψ yields

$$\Psi'(\theta) = 2\,\tau\ \underbrace{(\alpha - 1 + \delta\,\pi\,\theta)}_{\leq 0}\ \underbrace{[3\,\delta\,(\omega+\pi)\,\theta - [2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]]}_{\text{increasing in } \theta},\quad \forall \theta \in [0,\tau].$$

Because $\Psi'(0) > 0$, the function Ψ is either increasing or first increasing and then decreasing on $(0, \tau]$. Since

$$\begin{aligned}
&\Psi(\tau) \\
=\ &(\alpha - 1 + \delta\,\pi\,\tau)\ \left[2\,\delta\,(\omega+\pi)\,\tau^3 + \left(\alpha - 2\,\delta\,\pi - 1 + 2\,\delta^2\,\pi - 2\,\alpha\,\delta\right)\,\tau^2 + 2\,(1-\delta)\,(1-\alpha)\,\tau + \alpha - 1\right] \\
=\ &-2\,(1-\delta)\,\tau\ [\delta\,\pi\,\tau - (1-\alpha)]^2 \\
\leq\ &0,
\end{aligned}$$

the first case violates the existence of $\hat{\tau} \in [0, \tau]$. Therefore, the function Ψ is first increasing and then decreasing on this interval.

The desired condition $\eta_2(\hat{\tau}) \geq 0$ can be rewritten as

$$\hat{\tau} \geq \frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{2\,\delta\,\pi\,C_2}{\omega+2\,\delta\,\pi}}\right) \quad \Leftarrow \quad \Psi\left(\frac{1}{\delta\,\pi}\left(1-\alpha-\sqrt{\frac{2\,\delta\,\pi\,C_2}{\omega+2\,\delta\,\pi}}\right)\right) \geq 0,$$

where the last step is implied by the monotonicity of Ψ and can be rewritten as

$$-(\omega+\pi)^2\,\frac{(\alpha+\delta\,\pi\,\tau-1)\,\left(\omega\,\tau-\delta\,\pi^2+\delta\,\pi^2\,\tau^2-\alpha\,\omega\,\tau+\delta\,\pi\,\omega\,\tau^2-2\,\delta\,\pi\,\alpha\,\tau\right)}{\omega+2\,\delta\,\pi} \leq [(1-\alpha)\,(\omega+\pi)-\delta\,\pi\,(1+\pi-\delta\,\pi)]$$

or equivalently,

$$\begin{array}{l}(\omega+2\,\delta\,\pi)\,\left[(1-\alpha)\,(\omega+\pi)-\delta\,\pi\,(1+\pi-\delta\,\pi)\right]^2\,\tau \\ +(\omega+\pi)^2\,\left(\alpha+\delta\,\pi\,\tau-1\right)\,\left(\omega\,\tau-\delta\,\pi^2+\delta\,\pi^2\,\tau^2-\alpha\,\omega\,\tau+\delta\,\pi\,\omega\,\tau^2-2\,\delta\,\pi\,\alpha\,\tau\right) \geq 0.\end{array}$$

We can use the definition of τ to eliminate the constant term, i.e.,

$$1-\alpha = 2\,\delta\,\left(\omega+\pi\right)\,\tau^3+\left(\alpha-2\,\delta\,\alpha-1\right)\,\tau^2,$$

and the desired result boils down to $g(\omega) \geq 0$, where

$$\begin{array}{rl} g(\omega) \equiv & 3\,\delta\,(\omega+\pi)^3\,\tau\,(\omega)^2-2\,(\omega+\pi)^2\,(2\,\delta\,\alpha-\alpha+1)\,\tau\,(\omega) \\ & +(\omega+\pi)\,\left[-\delta\,\pi\,(\omega+\pi)+2\,(1-\alpha)\,\left[1-2\,\delta-(1-\delta)\,(\omega+2\,\delta\,\pi)\right]\right]+\delta\,(\omega+2\,\delta\,\pi)\,(1+\pi-\delta\,\pi)^2.\end{array}$$

Applying the implicit function theorem to the definition of τ yields

$$\tau'(\omega) = -\frac{\delta\,\tau(\omega)^2}{3\,\delta\,(\omega+\pi)\,\tau(\omega)+(\alpha-2\,\delta\,\alpha-1)}.$$

Because

$$2\,\delta\,(\omega+\pi)\,\tau^3+(\alpha-2\,\delta\,\alpha-1)\,\tau^2 = 1-\alpha \geq 0,$$

we know that

$$3\,\delta\,(\omega+\pi)\,\tau+(\alpha-2\,\delta\,\alpha-1) > 2\,\delta\,(\omega+\pi)\,\tau+(\alpha-2\,\delta\,\alpha-1) \geq 0,$$

and hence, the threshold τ is decreasing in ω.

Next, we will prove the nonnegativity of g in three steps.

Lemma 4. *The function*

$$l(\omega) \equiv 3\,\delta\,(\omega+\pi)^2\,\tau\,(\omega)^2-2\,(\omega+\pi)\,(2\,\delta\,\alpha-\alpha+1)\,\tau\,(\omega)-\delta\,\pi\,(\omega+\pi)+2\,(1-\alpha)\,\left[1-2\,\delta-(1-\delta)\,(\omega+2\,\delta\,\pi)\right]$$

is increasing.

Proof. Proof of Lemma 4. Taking the derivative yields

$$\begin{aligned} l'(\omega) = & \ 3\,\delta\,\left[2\,(\omega+\pi)\,\tau(\omega)^2 + 2\,(\omega+\pi)^2\,\tau(\omega)\,\tau'(\omega)\right] - 2\,(2\,\delta\,\alpha - \alpha + 1)\,\left[\tau(\omega) + (\omega+\pi)\,\tau'(\omega)\right] \\ & -\delta\,\pi - 2\,(1-\alpha)\,(1-\delta) \\ = & \ 4\,\delta\,(\omega+\pi)\,\tau\,(\omega)^2 + 2\,\left(\alpha - 2\,\alpha\,\delta - 1\right)\,\tau\,(\omega) - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \\ = & \ \frac{2\,(1-\alpha)}{\tau(\omega)} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi, \end{aligned}$$

where the last equality follows from the definition of τ.

When $\pi \geq \frac{1-\alpha}{\delta}$, the threshold τ is bounded by $\frac{1-\alpha}{\delta\,\pi}$, and hence, we have

$$\frac{2\,(1-\alpha)}{\tau} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \geq \delta\,\pi - 2\,(1-\delta)\,(1-\alpha) \geq (1-\alpha)\,(2\,\delta - 1) \geq 0.$$

When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$, the threshold τ is bounded by 1, and hence, we have

$$\frac{2\,(1-\alpha)}{\tau} - 2\,(1-\delta)\,(1-\alpha) - \delta\,\pi \geq 2\,\delta\,(1-\alpha) - \delta\,\pi \geq (1-\alpha)\,(2\,\delta - 1) \geq 0.$$

Therefore, function l is nondecreasing. □ □

Lemma 5. *Function g is nondecreasing.*

Proof. Proof of Lemma 5. Function g can be rewritten in terms of l, i.e.,

$$g(\omega) = (\omega+\pi)\,l(\omega) + \delta\,(\omega + 2\,\delta\,\pi)\,(1 + \pi - \delta\,\pi)^2.$$

Taking the derivative yields

$$g'(\omega) = l(\omega) + (\omega+\pi)\,l'(\omega) + \delta\,(1 + \pi - \delta\,\pi)^2.$$

Because τ is decreasing in ω, we know that l' is also increasing. Combining with the monotonicity of l, we can conclude that g' is increasing.

1. When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$, we have

$$\begin{aligned} & g'(\omega) \\ \geq & \ g'\left(\frac{1-\alpha+\delta\,\alpha}{\delta} - \pi\right) \\ = & \ \frac{\delta^2\,(1-\delta)^2\,\pi^2 + 2\,\delta^2\,\left(-2\,\delta\,\alpha + 2\,\alpha + \delta - 2\right)\,\pi + \left(-5\,\alpha^2 + 8\,\alpha - 3\right)\,\delta^2 + \left(6\,\alpha^2 - 12\,\alpha + 6\right)\,\delta - \alpha^2 + 2\,\alpha - 1}{\delta} \\ \geq & \ \left.\frac{\delta^2\,(1-\delta)^2\,\pi^2 + 2\,\delta^2\,\left(-2\,\delta\,\alpha + 2\,\alpha + \delta - 2\right)\,\pi + \left(-5\,\alpha^2 + 8\,\alpha - 3\right)\,\delta^2 + \left(6\,\alpha^2 - 12\,\alpha + 6\right)\,\delta - \alpha^2 + 2\,\alpha - 1}{\delta}\right|_{\pi=\frac{1-\alpha}{\delta}} \\ = & \ 0, \end{aligned}$$

where the second inequality follows from the numerator decreasing in π for any $\pi \in \left[0, \frac{1-\alpha}{\delta}\right]$.

2. When $\pi > \frac{1-\alpha}{\delta}$, we have

$$g'(\omega) \geq g'\left(\frac{\delta^2\,\pi^3 - (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}\right) \equiv \frac{k(\pi)}{4\,\delta\,(1-\alpha)^2},$$

where

$$\begin{aligned} k(\pi) \equiv\ & 3\,\delta^4\,\pi^4 + \left(8\,\alpha\,\delta^3 - 8\,\alpha\,\delta^4 - 8\,\delta^3 + 8\,\delta^4\right)\,\pi^3 \\ & + \left(4\,\alpha^2\,\delta^4 - 12\,\alpha^2\,\delta^3 + 6\,\alpha^2\,\delta^2 - 8\,\alpha\,\delta^4 + 20\,\alpha\,\delta^3 - 12\,\alpha\,\delta^2 + 4\,\delta^4 - 8\,\delta^3 + 6\,\delta^2\right)\,\pi^2 \\ & + \left(8\,\alpha^2\,\delta^3 - 8\,\alpha^2\,\delta^2 - 16\,\alpha\,\delta^3 + 16\,\alpha\,\delta^2 + 8\,\delta^3 - 8\,\delta^2\right)\,\pi \\ & + 4\,\left(-\alpha^4 + 6\,\alpha^3 - 12\,\alpha^2 + 10\,\alpha - 3\right)\,\delta^2 + 4\,\left(\alpha^4 - 5\,\alpha^3 + 9\,\alpha^2 - 7\,\alpha + 2\right)\,\delta \\ & - \alpha^4 + 4\,\alpha^3 - 6\,\alpha^2 + 4\,\alpha - 1. \end{aligned}$$

Because $k^{(3)}(\pi) \geq 0$, we have

$$k''(\pi) \geq k''\left(\frac{1-\alpha}{\delta}\right) = 8\,\delta^3\,(\alpha-1)\,(3\,\alpha - \delta + \alpha\,\delta - 4) \geq 0,$$

which further implies

$$k'(\pi) \geq k'\left(\frac{1-\alpha}{\delta}\right) = 8\,\delta^3\,(1-\alpha)^2\,(2-\alpha) \geq 0.$$

Therefore, we have

$$k(\pi) \geq k\left(\frac{1-\alpha}{\delta}\right) = 0,$$

which implies the monotonicity of g. □

□

Lemma 6. *Function g is nonnegative.*

Proof. Proof of Lemma 6. When $0 \leq \pi \leq \frac{1-\alpha}{\delta}$ and ω attains its minimal feasible value, we have $\tau = 1$, and hence,

$$\begin{aligned} g(\omega) &\geq g\left(\frac{1-\alpha+\delta\,\alpha}{\delta} - \pi\right) \\ &= \frac{(1-\delta)\,(\alpha+\delta\,\pi-1)^2\,\left[\delta\,(1-\delta)\,(2\,\delta-1)\,\pi + \alpha + 3\,\delta - 4\,\alpha\,\delta + 3\,\alpha\,\delta^2 - 1\right]}{\delta^2} \\ &\geq \frac{(1-\delta)\,(\alpha+\delta\,\pi-1)^2\,\left(\alpha + 3\,\delta - 4\,\alpha\,\delta + 3\,\alpha\,\delta^2 - 1\right)}{\delta^2} \\ &\geq 0. \end{aligned}$$

When $\pi > \frac{1-\alpha}{\delta}$, we have

$$g(\omega) \geq g\left(\frac{\delta^2\,\pi^3 - (1-\alpha)\,(1-\alpha-2\,\delta\,\alpha)\,\pi}{2\,(1-\alpha)^2}\right)$$

$$\equiv \frac{\pi\,\left(\delta^2\,\pi^2 - 2\,\delta^2\,\pi\,\alpha + 2\,\delta\,\pi\,\alpha + 2\,\delta^2\,\pi - 2\,\delta\,\pi - 2\,\alpha^2\,\delta + \alpha^2 + 4\,\alpha\,\delta - 2\,\alpha - 2\,\delta + 1\right)^2\,k(\pi)}{8\,\delta\,(1-\alpha)^4},$$

where

$$\begin{aligned} k(\pi) &= \delta^2\,\pi^2 + 2\,\alpha^2\,\delta - \alpha^2 - 6\,\alpha\,\delta + 2\,\alpha + 4\,\delta - 1 \\ &\geq (1-\alpha)^2 + 2\,\alpha^2\,\delta - \alpha^2 - 6\,\alpha\,\delta + 2\,\alpha + 4\,\delta - 1 \\ &= 2\,\delta\,(1-\alpha)\,(2-\alpha) \\ &\geq 0. \end{aligned}$$

□ □

We have shown that if $\delta \geq \frac{1}{2}$, then $C_3 \geq \eta_2(\hat{\tau}) \geq 0$, and hence η_1 is decreasing on $(0, \hat{\tau}]$, which completes the proof of the nonnegativity of η in this case.

If $\delta < \frac{1}{2}$ and $C_3 \geq 0$, which requires the existence of $\hat{\tau} \in [0, \tau]$, then since $\eta_2(\hat{\tau}) > C_3 \geq 0$, the required nonnegativity of η still holds. However, if $C_3 < 0$ or $\hat{\tau} = -\infty$, the proposed multipliers either violate the nonnegativity condition for sufficiently small θ or are not well-defined. We will return to this case later in the proof.

We now proceed under the assumption that $C_3 \geq 0$. Differentiating $\beta(\cdot)$ yields

$$\beta'(\theta) = -2\,\delta\,\pi\,\eta_1(\theta) + (1 - \alpha - \delta\,\pi\,\theta)\;\eta_1'(\theta) - w_q'(\theta) < 0.$$

Hence, the multiplier β is nonincreasing in θ, and the nonnegativity condition is implied by $\beta(\hat{\tau}) = 0$.

For any $\theta \in [\hat{\tau}, \tau]$, we have

$$\xi'(\theta) = -\delta\;\left[2\,\eta_2(\theta) + \theta\,\eta_2'(\theta)\right] - (2\,\delta - 1)$$

and

$$\xi''(\theta) = -\delta\;\left[3\,\eta_2'(\theta) + \theta\,\eta_2''(\theta)\right] \leq 0.$$

Thus, ξ is concave, and its minimum on this interval is attained at one of the endpoints. Consequently, the nonnegativity condition reduces to

$$\xi(\hat{\tau}) = 1 - \delta \geq 0 \quad \text{and} \quad \xi(\tau) = 0 \geq 0,$$

which both hold automatically.

Regarding ν on $[\hat{\tau}, \tau]$, we can verify that it is convex, with

$$\nu(\hat{\tau}) = 0 \quad \text{and} \quad \nu(\tau) = 1 - \delta.$$

Therefore, the desired nonnegativity property is equivalent to

$$\lim_{\theta \downarrow \hat{\tau}} \nu'(\theta) = \delta\;\left[2\,\eta_2(\hat{\tau}) + \hat{\tau}\,\eta_2'(\theta)\big|_{\theta=\hat{\tau}}\right] + 2\,\delta - 1 \geq 0,$$

which clearly holds when $\delta \geq \frac{1}{2}$. In the other case, the inequality above can be rewritten as

$$2\,\delta\;\left[\frac{C_2}{(1 - \alpha - \delta\,\pi\,\hat{\tau})^2} - \frac{\omega}{2\,\delta\,\pi} - 1\right] + \frac{2\,\delta^2\,\pi\,\hat{\tau}\,C_2}{(1 - \alpha - \delta\,\pi\,\hat{\tau})^3} + 2\,\delta - 1 \geq 0.$$

The inequality above follows from

$$\begin{aligned}
&2\,\delta\,\left[\frac{C_2}{(1-\alpha-\delta\,\pi\,\hat{\tau})^2}-\frac{\omega}{2\,\delta\,\pi}-1\right]+\frac{2\,\delta^2\,\pi\,\hat{\tau}\,C_2}{(1-\alpha-\delta\,\pi\,\hat{\tau})^3}+2\,\delta-1\\
&=\frac{2\,\delta\,C_2\,(1-\alpha)}{(1-\alpha-\delta\,\pi\,\hat{\tau})^3}-\frac{\omega}{\pi}-1\\
&\geq\frac{2\,\delta\,(1-\alpha)}{1-\alpha-\delta\,\pi\,\hat{\tau}}\,\left(\frac{\omega}{2\,\delta\,\pi}+\frac{1}{2\,\delta}\right)-\frac{\omega}{\pi}-1\\
&=\left(\frac{1-\alpha}{1-\alpha-\delta\,\pi\,\hat{\tau}}-1\right)\,\left(\frac{\omega}{\pi}+1\right)\\
&\geq 0,
\end{aligned}$$

where the first inequality follows from $C_3 > 0$.

The nonnegativity of σ follows from its monotonicity on $[\tau, 1]$ and the boundary condition $\sigma(\tau) = 0$.

We have

$$\begin{aligned}
&\chi_L\,[-u(0)]+\int_0^{\hat{\tau}}\beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta+\int_{\tau}^{1}\sigma(\theta)\,q(\theta)\,\mathrm{d}\theta+\int_0^{\tau}\xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta+\int_{\hat{\tau}}^{1}\nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta\\
&+\int_0^1\eta(\theta)\,\left[\delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi\,q(y)]\,\mathrm{d}y+(1-\alpha)\,q(\theta)-\delta\,\theta\,[m_S(\theta)+\pi\,q(\theta)]+\delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta\\
&=-u(0)+\int_0^1[w_q(\theta)\,q(\theta)+w_S(\theta)\,m_S(\theta)+w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta.
\end{aligned}$$

Hence, the objective function is bounded by

$$\int_{\tau}^{1}\sigma(\theta)\,\mathrm{d}\theta=\int_{\tau}^{1}\left[\frac{2\,\delta\,(\omega+\pi)\,\theta^3+(\alpha-2\,\delta\,\alpha-1)\,\theta^2+\alpha-1}{2\,\delta\,\theta^2}\right]\mathrm{d}\theta.$$

Substituting our conjectured primal solution into the objective function yields

$$\int_{\tau}^{1}(\theta\,\omega-m_0^*-\theta\,m_S^*)\,\mathrm{d}\theta,$$

achieving the upper bound above and, thereby, establishing the optimality. The desired result follows from

$$\begin{aligned}
&\int_{\tau}^{1}\left[\frac{2\,\delta\,(\omega+\pi)\,\theta^3+(\alpha-2\,\delta\,\alpha-1)\,\theta^2+\alpha-1}{2\,\delta\,\theta^2}\right]\mathrm{d}\theta-\int_{\tau}^{1}(\theta\,\omega-m_0^*-\theta\,m_S^*)\,\mathrm{d}\theta\\
&=\int_{\tau}^{1}\left(\pi\,\theta+\frac{\alpha-1}{2\,\delta}+\frac{\alpha-1}{2\,\delta\,\theta^2}\right)\mathrm{d}\theta-\int_{\tau}^{1}(-\theta\,m_S^*)\,\mathrm{d}\theta\\
&=\frac{\pi}{2}\,(1-\tau^2)+\frac{\alpha-1}{2\,\delta}\,(1-\tau)+\frac{1-\alpha}{2\,\delta}\,\left(1-\frac{1}{\tau}\right)+\frac{1}{2}\,(1-\tau^2)\,\left(\frac{1-\alpha}{\delta\,\tau}-\pi\right)\\
&=0.
\end{aligned}$$

Next, we focus on the case in which $\delta < \frac{1}{2}$ and $C_3 < 0$. We first consider the scenario in which

$\eta_2(\hat{\tau}) \geq 0$. Define

$$\hat{\eta}(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \tilde{\tau}], \\ \eta_1(\theta), & \text{if } \theta \in (\tilde{\tau}, \hat{\tau}], \\ \eta_2(\theta), & \text{if } \theta \in (\hat{\tau}, \tau], \\ \eta_3(\theta), & \text{if } \theta \in (\tau, 1], \end{cases}$$

$$\hat{\xi}(\theta) = \begin{cases} \delta \int_{\tilde{\tau}}^1 \hat{\eta}(y)\,\mathrm{d}y - w_S(\theta), & \text{if } \theta \in [0, \tilde{\tau}], \\ -\delta\,\theta\,\hat{\eta}(\theta) + \delta \int_{\theta}^1 \hat{\eta}(y)\,\mathrm{d}y - w_S(\theta), & \text{if } \theta \in (\tilde{\tau}, \tau], \\ 0, & \text{if } \theta \in (\tau, 1], \end{cases}$$

$$\hat{\nu}(\theta) = \begin{cases} -\delta \int_{\tilde{\tau}}^1 \hat{\eta}(y)\,\mathrm{d}y - w_F(\theta), & \text{if } \theta \in [0, \tilde{\tau}], \\ 0, & \text{if } \theta \in (\tilde{\tau}, \hat{\tau}], \\ \delta\,\theta\,\hat{\eta}(\theta) - \delta \int_{\theta}^1 \hat{\eta}(y)\,\mathrm{d}y - w_F(\theta), & \text{if } \theta \in (\hat{\tau}, 1], \end{cases}$$

$$\hat{\beta}(\theta) = \begin{cases} \delta\,\pi \int_{\tilde{\tau}}^1 \hat{\eta}(y)\,\mathrm{d}y - w_q(\theta), & \text{if } \theta \in [0, \tilde{\tau}], \\ (1 - \alpha - \delta\,\theta\,\pi)\,\hat{\eta}(\theta) + \delta\,\pi \int_{\theta}^1 \hat{\eta}(y)\,\mathrm{d}y - w_q(\theta), & \text{if } \theta \in (\tilde{\tau}, \hat{\tau}], \\ 0, & \text{if } \theta \in (\hat{\tau}, 1], \end{cases}$$

where

$$\tilde{\tau} \equiv \inf\left\{\theta \in [0,1] : \frac{C_3}{\theta^2} + \frac{1}{2\delta} - 1 \geq 0\right\}.$$

By definition, when $\theta \in [\hat{\tau}, 1]$, the multipliers have the same expressions as in the previous case, and hence the required nonnegativity properties on this interval hold automatically.

Since $C_3 < 0$, we know that η_1 is increasing on $(0, \hat{\tau}]$. Therefore, the nonnegativity of $\hat{\eta}$ follows directly from its definition.

When $\theta \in [0, \tilde{\tau}]$, the multipliers $\hat{\xi}$, $\hat{\nu}$, and $\hat{\beta}$ are increasing, decreasing, and decreasing, respectively. Therefore, the desired nonnegativity conditions boil down to

$$\begin{aligned} &\hat{\xi}(0) = -(2\,\delta - 1)\,(1 - \tilde{\tau}) + \delta \geq 0, \\ &\hat{\nu}(\tilde{\tau}) = 0 \geq 0, \\ &\hat{\beta}(\tilde{\tau}) = \delta\,\pi + 1 - \pi\,(2\,\delta - 1)\,(1 - \tilde{\tau}) > 0. \end{aligned}$$

When $\theta \in [\tilde{\tau}, \hat{\tau}]$, we have

$$\hat{\xi}(\theta) = 1 - \delta > 0.$$

It remains to verify the nonnegativity of $\hat{\beta}$ on this interval. We have

$$\hat{\beta}(\theta) = (1 - \alpha)\,\hat{\eta}(\theta) - \pi\,w_F(\theta) - w_q(\theta).$$

Taking the derivative yields

$$\hat{\beta}'(\theta) = (1 - \alpha)\,\hat{\eta}'(\theta) - \omega - \pi$$

and

$$\hat{\beta}''(\theta) = (1 - \alpha)\,\hat{\eta}''(\theta) < 0.$$

Therefore, the minimum of $\hat{\beta}$ on this interval is attained at either $\tilde{\tau}$ or $\hat{\tau}$. Since $\hat{\beta}(\tilde{\tau}) > 0 = \hat{\beta}(\hat{\tau})$,

we conclude that β is always nonnegative.

Since

$$\begin{aligned}
&\chi_L\ [-u(0)] + \int_0^{\hat{\tau}} \hat{\beta}(\theta)\ [-q(\theta)]\,\mathrm{d}\theta \\
&+ \int_{\tilde{\tau}}^1 \hat{\eta}(\theta)\ \left[\delta \int_0^{\theta} [m_S(y) - m_F(y) + \pi\, q(y)]\,\mathrm{d}y + (1-\alpha)\, c\, q(\theta) - \delta\,\theta\ [m_S(\theta) + \pi\, q(\theta)] + \delta\,\theta\, m_F(\theta)\right]\mathrm{d}\theta \\
&+ \int_{\tau}^1 \sigma(\theta)\, q(\theta)\,\mathrm{d}\theta + \int_0^{\tau} \hat{\xi}(\theta)\ [-m_S(\theta)]\,\mathrm{d}\theta + \int_0^{\tilde{\tau}} \hat{\nu}(\theta)\ [-m_F(\theta)]\,\mathrm{d}\theta + \int_{\hat{\tau}}^1 \hat{\nu}(\theta)\ [-m_F(\theta)]\,\mathrm{d}\theta \\
=\ & -u(0) + \int_0^1 [w_q(\theta)\, q(\theta) + w_S(\theta)\, m_S(\theta) + w_F(\theta)\, m_F(\theta)]\,\mathrm{d}\theta,
\end{aligned}$$

the objective function is bounded above by

$$\int_{\tau}^1 \sigma(y)\,\mathrm{d}y.$$

Since the definition of τ and the expression for σ remain the same, optimality follows.

Finally, it remains to prove the case in which $\eta_2(\hat{\tau}) < 0$ or $\hat{\tau} = -\infty$. Define

$$\tilde{\eta}(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \tau^\dagger], \\ \eta_2(\theta), & \text{if } \theta \in (\tau^\dagger, \tau], \\ \eta_3(\theta), & \text{if } \theta \in (\tau, 1], \end{cases}$$

$$\tilde{\xi}(\theta) = \begin{cases} \delta \int_{\tau^\dagger}^1 \tilde{\eta}(y)\,\mathrm{d}y - w_S(\theta), & \text{if } \theta \in [0, \tau^\dagger], \\ -\delta\,\theta\,\tilde{\eta}(\theta) + \delta \int_{\theta}^1 \tilde{\eta}(y)\,\mathrm{d}y - w_S(\theta), & \text{if } \theta \in (\tau^\dagger, \tau], \\ 0, & \text{if } \theta \in (\tau, 1], \end{cases}$$

$$\tilde{\nu}(\theta) = \begin{cases} -\delta \int_{\tau^\dagger}^1 \tilde{\eta}(y)\,\mathrm{d}y - w_F(\theta), & \text{if } \theta \in [0, \tau^\dagger], \\ \delta\,\theta\,\tilde{\eta}(\theta) - \delta \int_{\theta}^1 \tilde{\eta}(y)\,\mathrm{d}y - w_F(\theta), & \text{if } \theta \in (\tau^\dagger, 1], \end{cases}$$

$$\tilde{\beta}(\theta) = \begin{cases} \delta\,\pi \int_{\tau^\dagger}^1 \tilde{\eta}(y)\,\mathrm{d}y - w_q(\theta), & \text{if } \theta \in [0, \tau^\dagger], \\ 0, & \text{if } \theta \in (\tau^\dagger, 1], \end{cases}$$

where

$$\tau^\dagger \equiv \inf\{\theta \in [0,1] : \eta_2(\theta) \geq 0\}.$$

By definition, when $\theta \in [\tau, 1]$, the multipliers have the same expressions as in the previous case, and hence the required nonnegativity properties on this interval hold automatically.

Since $C_2 > 0$, we know that η_2 is increasing on $(0, \tau]$. Therefore, the nonnegativity of $\tilde{\eta}$ follows directly from its definition.

By definition, when $\theta \in [\tau^\dagger, \tau]$, we have

$$(1 - \alpha - \delta\,\theta\,\pi)\ \eta(\theta) + \delta\,\pi \int_{\theta}^1 \eta(y)\,\mathrm{d}y = (\omega + 2\,\delta\,\pi)\,\theta - \delta\,\pi - 1.$$

Suppose $\tau^\dagger = 0$. When θ is sufficiently close to 0, the left-hand and right-hand sides are positive

and negative, respectively, which yields a contradiction. Therefore, the threshold $\tau^\dagger$ is determined by $\eta_2(\tau^\dagger) = 0$.

When $\theta \in [0, \tau^\dagger]$, the multipliers $\tilde{\xi}$, $\tilde{\nu}$, and $\tilde{\beta}$ are increasing, decreasing, and decreasing, respectively. Therefore, the desired nonnegativity conditions boil down to

$$\begin{aligned}
&\tilde{\xi}(0) = \frac{1}{\pi}\,\Big[(\omega + 2\,\delta\,\pi)\,\tau^\dagger - 1\Big],\\
&\tilde{\nu}(\tau^\dagger) = -\frac{1}{\pi}\,\Big[(\omega + 2\,\delta\,\pi)\,\tau^\dagger - \delta\,\pi - 1\Big] - (2\,\delta - 1)\,(1 - \tau^\dagger),\\
&\tilde{\beta}(\tau^\dagger) = 0 \geq 0.
\end{aligned}$$

We have

$$\tilde{\nu}(\tau^\dagger) \geq 0 \quad \Leftrightarrow \quad \tau^\dagger \leq \frac{(1-\delta)\,\pi + 1}{\omega + \pi}.$$

By definitions of $\hat{\tau}$ and $\tau^\dagger$, we have

$$(1-\alpha)\,\underbrace{\left[\frac{C_2}{(1-\alpha-\delta\,\tau^\dagger\,\pi)^2} - \frac{\omega}{2\delta\pi} - 1\right]}_{=0} \geq (\omega + \pi)\,\tau^\dagger - (1-\delta)\,\pi - 1,$$

which implies that

$$\tau^\dagger \leq \frac{(1-\delta)\,\pi + 1}{\omega + \pi}.$$

By definition, for any $\theta \in [0, \tau^\dagger]$, we have

$$\pi\,\tilde{\xi}(\theta) - \tilde{\beta}(\theta) = -\pi\,w_S(\theta) + w_q(\theta),$$

which can be rewritten as

$$\pi\,\tilde{\xi}(\theta) = \delta\,\pi\int_{\tau^\dagger}^{1}\tilde{\eta}(y)\,\mathrm{d}y - \pi\,w_S(\theta).$$

Hence, we have

$$\pi\,\tilde{\xi}(0) = \delta\,\pi\int_{\tau^\dagger}^{1}\tilde{\eta}(y)\,\mathrm{d}y - \pi\,\underbrace{w_S(0)}_{=-\delta} > 0,$$

establishing the nonnegativity of $\tilde{\xi}$ in this interval.

When $\theta \in [\tau^\dagger, \tau]$, we have

$$\tilde{\xi}'(\theta) = -2\,\delta\,\tilde{\eta}(\theta) - \delta\,\theta\,\tilde{\eta}'(\theta) - (2\,\delta - 1)$$

and

$$\tilde{\xi}''(\theta) = -3\,\delta\,\tilde{\eta}'(\theta) - \delta\,\theta\,\tilde{\eta}''(\theta).$$

Since $\tilde{\eta}'(\theta)$ and $\tilde{\eta}''(\theta)$ are both positive on this interval, we know that ξ is concave on the same interval, implying that the minimum is attained at one of the endpoints, $\tau^\dagger$ or τ. The nonnegativity properties follow from

$$\tilde{\xi}(\tau^\dagger) \geq \tilde{\xi}(0) \geq 0 \quad \text{and} \quad \tilde{\xi}(\tau) = 0 \geq 0.$$

Similarly, $\tilde{\nu}$ is convex in this interval. If the minimum is attained at one of the endpoints, the nonnegativity property holds automatically. If $\tilde{\nu}$ is first increasing and then decreasing,

The derivative at $\tau^\dagger$ is given by

$$\lim_{\theta \downarrow \tau^\dagger} \tilde{\nu}'(\theta) = \lim_{\theta \downarrow \tau^\dagger} \delta\, \theta\, \tilde{\eta}'(\theta) + 2\,\delta - 1 = \frac{2\, C_2\, \delta\, \pi}{(1-\alpha-\delta\, \tau^\dagger\, \pi)^3} + 2\,\delta - 1 = \frac{\omega + 2\,\delta\,\pi}{1-\alpha-\delta\,\tau^\dagger\,\pi} + 2\,\delta - 1.$$

If suffices to prove that

$$\lim_{\theta \downarrow \tau^\dagger} \tilde{\nu}'(\theta) \geq 0 \quad \Leftrightarrow \quad \frac{\omega + 2\,\delta\,\pi}{1-\alpha-\delta\,\tau^\dagger\,\pi} + 2\,\delta - 1 \geq 0 \quad \Leftrightarrow \quad \tau^\dagger \geq \frac{1}{\delta\,\pi}\left(1-\alpha-\frac{\omega+2\,\delta\,\pi}{1-2\,\delta}\right).$$

The last inequality holds for sure because $\omega + \delta\,\pi \geq 1$, implying that the right-hand side of the inequality is negative.

The optimality follows by the same argument as in the previous cases, since the definitions of both τ and σ remain unchanged.

This completes the proof. □ □

Proof. Proof of Proposition 3. Primal feasibility follows directly from the condition $(1-\alpha)\,\bar{\tau} \leq 1$.

Define

$$\begin{aligned}
&\chi_L \equiv 1,\\
&\xi(\theta) \equiv -\left[w_S(\theta) + w_F(\theta)\right], \quad \forall \theta \in [0,\tau) \cup [\hat{\tau},1],\\
&\nu(\theta) \equiv -\left[w_S(\theta) + w_F(\theta)\right], \quad \forall \theta \in [\tau,\hat{\tau}),\\
&\eta(\theta) \equiv \begin{cases} \dfrac{1}{2\,\delta}\left(\dfrac{C_3}{\theta^2}+1\right) - 1, & \forall \theta \in (0,\tau),\\ \dfrac{1}{2\,\delta}\left(\dfrac{C_2}{\theta^2}+1\right) - 1, & \forall \theta \in [\tau,\hat{\tau}),\\ \dfrac{1}{2\,\delta}\left(\dfrac{C_1}{\theta^2}+1\right) - 1, & \forall \theta \in [\hat{\tau},1], \end{cases}\\
&\sigma(\theta) \equiv w_q(\theta) - (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) - \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y, \quad \forall \theta \in [\tau,1],\\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi\int_\theta^1 \eta(y)\,\mathrm{d}y - w_q(\theta), \quad \forall \theta \in (0,\tau),
\end{aligned}$$

where

$$\begin{aligned}
&C_1 = 2\,\delta - 1,\\
&C_2 = 2\,\hat{\tau}\,(1-\delta) + 2\,\delta - 1 \geq 0,\\
&C_3 = 2\,(1-\delta)\,(\hat{\tau}-\tau) + 2\,\delta - 1 \geq 0,\\
&\hat{\tau} = \frac{1}{2\,\delta^2\,\pi^3\,(1-\delta)}\left[(1-\alpha)^2\,(2\,\omega+\pi) - 2\,\alpha\,\delta\,\pi\,(1-\alpha) + \delta^2\,\pi^3\,(1-2\,\delta)\right].
\end{aligned}$$

The dual feasibility conditions $\tau \leq \hat{\tau} \leq 1$ can be reformulated as

$$\frac{1}{2\,\delta^2\,\pi^3\,(1-\delta)}\left[(1-\alpha)^2\,(2\,\omega+\pi) - 2\,\alpha\,\delta\,\pi\,(1-\alpha) + \delta^2\,\pi^3\,(1-2\,\delta)\right] \geq \frac{1-\alpha}{\delta\,\pi}$$

and

$$\frac{1}{2\,\delta^2\,\pi^3\,(1-\delta)}\,\left[(1-\alpha)^2\,(2\,\omega+\pi)-2\,\delta\,\alpha\,\pi\,(1-\alpha)+\delta^2\,\pi^3\,(1-2\,\delta)\right]\leq 1.$$

Both conditions coincide with the two optimality conditions given in Proposition 3.

To prove the nonnegativity of η, we have:

- Because $C_1, C_2, C_3 \geq 0$, all the three pieces of η are decreasing.
- Because $\eta(1) = 0$, the nonnegativity of η on $[\hat{\tau}, 1]$ holds automatically.
- The nonnegativity condition for η on $[\tau, \hat{\tau})$ boils down to

$$\lim_{\theta\uparrow\hat{\tau}}\eta(\theta)=\frac{1}{2\,\delta}\,\left(\frac{2\,\hat{\tau}\,(1-\delta)+2\,\delta-1}{\hat{\tau}^2}+1\right)-1\geq 0\quad\Leftrightarrow\quad(1-2\,\delta)\,\hat{\tau}^2+2\,(1-\delta)\,\hat{\tau}+(2\,\delta-1)\geq 0,$$

or equivalently,

$$\tau\leq\hat{\tau}\leq\underbrace{\frac{1-\delta+\sqrt{5\,\delta^2-6\,\delta+2}}{2\,\delta-1}}_{\geq 1\text{ for any }\delta\in\left[\frac{1}{2},1\right]},$$

which is implied by $\tau \leq \hat{\tau} \leq 1$.

- The nonnegativity condition for η on $(0, \tau)$ boils down to

$$\lim_{\theta\uparrow\tau}\eta(\theta)=\frac{1}{2\,\delta}\,\left(\frac{2\,(1-\delta)\,(\hat{\tau}-\tau)+2\,\delta-1}{\tau^2}+1\right)-1\geq 0\quad\Leftrightarrow\quad\hat{\tau}\geq\tau+\underbrace{\frac{2\,\delta-1}{2\,(1-\delta)}\,(\tau^2-1)}_{\leq 0},$$

which is implied by $\tau \leq \hat{\tau}$.

After proving the nonnegativity of η's, we show that β and σ are also nonnegative. Taking the derivative yields

$$\forall\theta\in[0,\tau),\ \beta'(\theta)=-2\,\delta\,\pi\,\underbrace{\eta(\theta)}_{\geq 0}+\underbrace{(1-\alpha-\delta\,\pi\,\theta)}_{\geq 0,\ \forall\theta\in[0,\tau]}\,\underbrace{\eta'(\theta)}_{\leq 0}-\underbrace{w_q'(\theta)}_{=\omega+2\,\delta\,\pi\geq 0}\leq 0,$$

$$\begin{aligned}\forall\theta\in[\tau,\hat{\tau}),\quad\sigma'(\theta)&=2\,\delta\,\pi\,\eta(\theta)-(1-\alpha-\delta\,\pi\,\theta)\,\eta'(\theta)+w_q'(\theta)\\&=\frac{(1-\alpha)\,[2\,\hat{\tau}\,(1-\delta)+2\,\delta-1]+\delta\,\theta^3\,(\omega+\pi)}{\delta\,\theta^3}\\&\geq 0,\end{aligned}$$

and

$$\begin{aligned}\forall\theta\in[\hat{\tau},1],\quad\sigma'(\theta)&=2\,\delta\,\pi\,\eta(\theta)-(1-\alpha-\delta\,\pi\,\theta)\,\eta'(\theta)+w_q'(\theta)\\&=\frac{(1-\alpha)\,(2\,\delta-1)+\delta\,\theta^3\,(\omega+\pi)}{\delta\,\theta^3}\\&\geq 0.\end{aligned}$$

Therefore, the nonnegativity conditions boil down to

$$\beta(\tau)\geq 0,\quad\sigma(\tau)\geq 0,\quad\text{and}\quad\sigma(\hat{\tau})\geq 0.$$

By definition of $\hat{\tau}$, we have $\beta(\tau) = \sigma(\tau) = 0$. The remaining condition $\sigma(\hat{\tau}) \geq 0$ can be rewritten as

$$2\,\delta\,(\omega+\pi)\,\hat{\tau}^3 - [2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\;\hat{\tau}^2 + (1-\alpha)\,(1-2\,\delta) \geq 0.$$

To prove it, define

$$h(\theta) \equiv 2\,\delta\,(\omega+\pi)\,\theta^3 - [2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\;\theta^2 + (1-\alpha)\,(1-2\,\delta).$$

Taking the derivative yields

$$h'(\theta) = 6\,\delta\,(\omega+\pi)\,\theta^2 - 2\;[2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha]\;\theta.$$

We have

$$h'(\theta)\begin{cases} \leq 0, & \text{if } \theta \leq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}, \\ \geq 0, & \text{if } \theta \geq \dfrac{2\,\delta\,\pi\,(1-\delta) + (1-\alpha) + 2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}. \end{cases}$$

Therefore, function h is either first decreasing and then increasing or increasing on $[0,1]$. Because

$$h(0) = (1-\alpha)\,(1-2\,\delta) \leq 0 \quad \text{and} \quad h\,(\tau) \geq 0,$$

where the latter one follows from the optimality condition in the proposition, and $\hat{\tau} \geq \tau$, the desired property $h(\hat{\tau}) \geq 0$ holds for sure.

We have

$$\begin{aligned}
&\chi_L\;[-u(0)] + \int_0^{\tau} \beta(\theta)\;[-q(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{1} \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta \\
&+ \int_0^1 \eta(\theta)\;\left[\delta \int_0^{\theta} [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\;[m_S(\theta) + \pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\
&+ \int_0^{\tau} \xi(\theta)\;[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\hat{\tau}}^1 \xi(\theta)\;[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{\hat{\tau}} \nu(\theta)\;[-m_F(\theta)]\,\mathrm{d}\theta \\
=& -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta.
\end{aligned}$$

Hence, the objective function is bounded by

$$\int_{\tau}^1 \sigma(\theta)\,\mathrm{d}\theta = -\frac{\omega\,(1-\alpha)^2}{2\,\delta^2\,\pi^2} + \frac{\alpha\;(1-\alpha)}{\delta\,\pi} + \frac{\omega}{2} - \alpha.$$

Substituting our conjectured primal solution into the objective function yields $\int_{\tau}^1 (\theta\,\omega - \alpha)\,\mathrm{d}\theta$, attaining the upper bound above and, thereby, establishing the optimality.

This completes the proof. □ □

Proof. Proof of Proposition 4. We first verify that $B_3(\pi) \leq B_2(\pi)$ for any $\pi \geq \frac{1-\alpha}{\delta}$. When $\pi \geq \frac{1}{\delta}$, the desired condition is equivalent to

$$\phi(\pi) \equiv 3\,\delta^2\;(2\,\delta - 1)\;\pi^2 + \delta\;(2 + \alpha - 4\,\delta\,\alpha - 2\,\delta)\;\pi + 2\,\delta\,\alpha - 2\,\alpha + 1 \geq 0, \quad \forall \pi \in \left[\frac{1}{\delta}, \infty\right),$$

which follows from

$$-\frac{(2+\alpha-4\,\delta\,\alpha-2\,\delta)}{6\,\delta\,(2\,\delta-1)}-\frac{1}{\delta}=\frac{\overbrace{-\alpha-10\,\delta+4\,\delta\,\alpha+4}^{\leq -10\,\delta+4}}{6\,\delta\,(2\,\delta-1)}\leq 0$$

and

$$\phi\left(\frac{1}{\delta}\right)=-\frac{\alpha\,\overbrace{(\alpha-4\,\delta+2\,\delta\,\alpha)}^{\leq 2\,\alpha-2}}{2\,\delta\,(1-\alpha)^2}\geq 0.$$

When $\frac{1-\alpha}{\delta}\leq\pi\leq\frac{1}{\delta}$, we have $B_3(\pi)\leq 1-\delta\,\pi\leq B_2(\pi)$, completing the proof of $B_3(\pi)\leq B_2(\pi)$.

Under the given conditions, the primal feasibility condition boils down to

$$\frac{1-\alpha}{\delta\,\pi}\leq\tau\leq\mathbb{1}_{\delta\,\pi-\alpha>0}\cdot\frac{1-\alpha}{\delta\,\pi-\alpha}+\mathbb{1}_{\delta\,\pi-\alpha\leq 0}$$

We begin by proving the unique existence of the threshold τ. Define

$$h(\theta)\equiv 2\,\delta\,(\omega+\pi)\,\theta^3-\left[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha\right]\,\theta^2-(1-\alpha)\,(2\,\delta-1).$$

We have

$$h(0)=-\,(2\,\delta-1)\,\,(1-\alpha)\leq 0$$

and

$$h(1)=2\,\delta\,\,(\omega-1+\delta\,\pi)>0,$$

and hence, the existence of $\tau\in[0,1]$ is guaranteed.

Differentiating h yields

$$h'(\theta)=6\,\delta\,(\omega+\pi)\,\theta^2-2\,\left[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha\right]\,\theta.$$

We have

$$h'(\theta)\begin{cases}\leq 0, & \text{if } \theta\leq\dfrac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)},\\ \geq 0, & \text{if } \theta\geq\dfrac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}.\end{cases}$$

Therefore, function h is either first decreasing and then increasing or increasing on $[0,1]$. Because $h(0)<0<h(1)$, the threshold τ is uniquely determined.

By the monotonicity of h, to prove $\tau\geq\frac{1-\alpha}{\delta\,\pi}$, it suffices to show that

$$h\left(\frac{1-\alpha}{\delta\,\pi}\right)\leq 0,$$

which is equivalent to $\omega\leq B_2(\pi)$.

When $\pi\leq\frac{\alpha}{\delta}$, the upper bound in the primal feasibility condition for τ simplifies to $\tau\leq 1$, which holds automatically.

When $\pi>\frac{\alpha}{\delta}$, the upper bound in the primal feasibility condition for τ boils down to $\tau\leq\frac{1-\alpha}{\delta\,\pi-\alpha}$,

which follows from

$$h\left(\frac{1-\alpha}{\delta\,\pi-\alpha}\right)\geq 0 \quad \Leftrightarrow \quad \omega\geq B_3(\pi).$$

Define

$$\begin{aligned}
&\chi_L\equiv 1,\\
&\xi(\theta)\equiv -\left[w_S(\theta)+w_F(\theta)\right], && \forall\theta\in[0,1],\\
&\eta(\theta)\equiv \frac{1}{2\,\delta}\left(\frac{2\,\delta-1}{\theta^2}+1\right)-1, && \forall\theta\in(0,1],\\
&\beta(\theta)\equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta)+\delta\,\pi\int_\theta^1\eta(y)\,\mathrm{d}y-w_q(\theta), && \forall\theta\in(0,\tau],\ \text{and}\\
&\sigma(\theta)\equiv -(1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta)-\delta\,\pi\int_\theta^1\eta(y)\,\mathrm{d}y+w_q(\theta), && \forall\theta\in[\tau,1].
\end{aligned}$$

Because

$$w_S(\theta)+w_F(\theta)=-1+\delta\leq 0$$

and $\eta(\theta)\geq\eta(1)=0$, the nonnegativity of ξ and η holds automatically. Differentiating β and σ yields that these two multipliers are decreasing and increasing on $[0,\tau]$ and $[\tau,1]$, respectively. Because $\beta(\tau)=\sigma(\tau)=0$, they are guaranteed to be nonnegative.

We have

$$\begin{aligned}
&\int_0^1\eta(\theta)\;\left[\delta\int_0^\theta\left[m_S(y)-m_F(y)+\pi\,q(y)\right]\mathrm{d}y+(1-\alpha)\,q(\theta)-\delta\,\theta\;\left[m_S(\theta)+\pi\,q(\theta)\right]+\delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta\\
&+\int_0^\tau\beta(\theta)\;\left[-q(\theta)\right]\mathrm{d}\theta+\int_\tau^1\sigma(\theta)\,q(\theta)\,\mathrm{d}\theta+\int_0^1\xi(\theta)\;\left[-m_S(\theta)\right]\mathrm{d}\theta\\
=&-u(0)+\int_0^1\left[w_q(\theta)\,q(\theta)+w_S(\theta)\,m_S(\theta)+w_F(\theta)\,m_F(\theta)\right]\mathrm{d}\theta,
\end{aligned}$$

and hence, the objective function is bounded by

$$\int_\tau^1\sigma(\theta)\,\mathrm{d}\theta=\frac{(1-\tau)\;\left(1-\alpha-2\,\delta-\tau-\delta\,\pi\,\tau+2\,\alpha\,\delta+\delta\,\omega\,\tau+\alpha\,\tau+\delta\,\pi\,\tau^2+2\,\delta^2\,\pi\,\tau+\delta\,\omega\,\tau^2-2\,\alpha\,\delta\,\tau\right)}{2\,\delta\,\tau}$$

Substituting our conjectured primal solution into the objective function yields

$$\int_\tau^1\left[\theta\,\omega-m_0^*-(1-\theta)\,m_F^*\right]\mathrm{d}\theta,$$

achieving the upper bound above and, thereby, establishing the optimality. This completes the proof. □ □

Proof. Proof of Proposition 5. The primal feasibility condition $\tau\in[0,1]$ is equivalent to $\pi\geq\frac{1}{\delta}$, which is implied by $B_4(\pi)<B_3(\pi)$, as follows from their definitions.

Define

$$\chi_L\equiv\frac{1}{2\,\delta},$$

$$\eta(\theta) \equiv \frac{1}{2\,\delta}\left(\frac{2\,\delta-1}{\theta^2}+1\right)-1, \quad \forall\theta\in[\hat{\tau},1],$$

$$\gamma(\theta) \equiv -\frac{1}{2\,\delta}\left[\frac{C_2}{(1-\theta)^2}+1\right]+1, \quad \forall\theta\in[0,\hat{\tau}),$$

$$\xi(\theta) \equiv \begin{cases} \delta\,\theta\,\gamma(\theta)+\delta\int_{\hat{\tau}}^{1}\eta(y)\,\mathrm{d}y-\delta\int_{\theta}^{\hat{\tau}}\gamma(y)\,\mathrm{d}y-w_S(\theta), & \forall\theta\in[0,\hat{\tau}), \\ -\left[w_S(\theta)+w_F(\theta)\right], & \forall\theta\in[\hat{\tau},1], \end{cases}$$

$$\beta(\theta) \equiv -(1-\delta\,\pi\,\theta)\,\gamma(\theta)+\delta\,\pi\int_{\hat{\tau}}^{1}\eta(y)\,\mathrm{d}y-\delta\,\pi\int_{\theta}^{\hat{\tau}}\gamma(y)\,\mathrm{d}y-w_q(\theta), \quad \forall\theta\in[0,\tau), \text{ and}$$

$$\sigma(\theta) \equiv \begin{cases} (1-\delta\,\pi\,\theta)\,\gamma(\theta)-\delta\,\pi\int_{\hat{\tau}}^{1}\eta(y)\,\mathrm{d}y+\delta\,\pi\int_{\theta}^{\hat{\tau}}\gamma(y)\,\mathrm{d}y+w_q(\theta), & \forall\theta\in[\tau,\hat{\tau}), \\ -(1-\alpha-\delta\,\pi\,\theta)\,\eta(\theta)-\delta\,\pi\int_{\theta}^{1}\eta(y)\,\mathrm{d}y+w_q(\theta), & \forall\theta\in[\hat{\tau},1], \end{cases}$$

where

$$C_2=\frac{(1-2\,\delta)\,(1-\hat{\tau})^2}{\hat{\tau}}\leq 0$$

and the second threshold $\hat{\tau}$ is determined by

$$\begin{array}{ll} & (1-2\,\delta)\,(1-\delta\,\pi)\,\hat{\tau}^2 \\ + & \left[-2\,\delta\,(\omega+\pi)\,\tau^3+(4\,\delta\,\omega+5\,\delta\,\pi+1)\,\tau^2-2\,(\delta\,\omega+2\,\delta\,\pi+1)\,\tau+(-4\,\delta^2\,\pi+3\,\delta\,\pi+4\,\delta-1)\right]\,\hat{\tau} \\ + & (1-2\,\delta)\,(1-\delta\,\pi)=0. \end{array}$$

First, we prove the unique existence of $\hat{\tau}$ on $[\tau,1]$. Define

$$\begin{array}{rcl} h(\theta) & \equiv & (1-2\,\delta)\,(1-\delta\,\pi)\,(\theta^2+1) \\ & & +\left[-2\,\delta\,(\omega+\pi)\,\tau^3+(4\,\delta\,\omega+5\,\delta\,\pi+1)\,\tau^2-2\,(\delta\,\omega+2\,\delta\,\pi+1)\,\tau+(-4\,\delta^2\,\pi+3\,\delta\,\pi+4\,\delta-1)\right]\,\theta, \end{array}$$

and we have

$$h(1)=-\frac{(\delta\,\pi-1)^2\,\left[-\delta^2\,\pi^2+\delta\,(1-\alpha)\,\pi+\alpha+2\,\delta\,\omega\,(1-\alpha)\right]}{(\delta\,\pi-\alpha)^3}\leq 0.$$

To prove the existence of $\hat{\tau}$, it suffices to show that

$$\max_{\theta\in[\tau,1]}h(\theta)\geq 0.$$

Because $(1-2\,\delta)\,(1-\delta\,\pi)>0$, the maximum is attained at either τ or 1. Hence, the desired result boils down to $h(\tau)\geq 0$, or equivalently, $\omega\leq B_3(\pi)$. Because h is quadratic, the uniqueness of $\hat{\tau}$ holds automatically.

Differentiating β yields

$$\beta'(\theta)=2\,\delta\,\pi\,\gamma(\theta)-(1-\delta\,\pi\,\theta)\,\gamma'(\theta)-w_q'(\theta)=\frac{C_2\,(1-\delta\,\pi)}{\delta\,(1-\theta)^3}-(\omega+\pi),$$

which is increasing in θ. We have

$$\begin{aligned}
& \beta'(\theta) \le 0, \quad \forall \theta \in [0, \tau], \\
\Leftrightarrow \quad & C_2 \left(1 - \delta \, \pi\right) \le \delta \left(1 - \theta\right)^3 (\omega + \pi), \quad \forall \theta \in [0, \tau], \\
\Leftrightarrow \quad & \frac{\left(1 - 2\,\delta\right) \left(1 - \hat{\tau}\right)^2}{\hat{\tau}} \left(1 - \delta \, \pi\right) \le \delta \left(1 - \tau\right)^3 (\omega + \pi) \\
\Leftrightarrow \quad & \left(1 - 2\,\delta\right) \left(1 - \delta \, \pi\right) \left(1 - \hat{\tau}\right)^2 - \delta \, \hat{\tau} \left(1 - \tau\right)^3 (\omega + \pi) \le 0 \\
\Leftrightarrow \quad & -\hat{\tau} \left(1 - \tau\right)^2 \left[1 + 2\,\delta \, \pi + \delta \, \omega - 3\,\delta \, \tau \left(\omega + \pi\right)\right] \le 0 \\
\Leftrightarrow \quad & 1 + 2\,\delta \, \pi + \delta \, \omega - 3\,\delta \, \tau \left(\omega + \pi\right) \ge 0,
\end{aligned}$$

where the second to last step follows from the definition of $\hat{\tau}$. The last inequality can be rewritten as, for any (π, ω) in the feasible region, we have

$$2\,\delta^2 \, \pi^2 + \delta \left(-2 + \alpha\right) \pi - \alpha + \delta \left(\delta \, \pi + 2\,\alpha - 3\right) \omega \ge 0.$$

When $\pi \le \frac{3 - 2\,\alpha}{\delta}$, fix π, and the expression above is decreasing in ω. Therefore, the minimum is attained at the upper bound of ω. If the desired result holds for any $\pi \in \left[\frac{1}{\delta}, \frac{3 - 2\,\alpha}{\delta}\right]$, it holds automatically for any $\pi \in \left(\frac{3 - 2\,\alpha}{\delta}, \infty\right)$, because the expression we are interested in is increasing in π on $\left[\frac{1}{\delta}, \infty\right)$, which follows from

$$-\frac{\delta \left(-2 + \alpha + \delta \, \omega\right)}{4\,\delta^2} = \frac{2 - \alpha - \delta \, \omega}{4\,\delta} \le 1 \le \frac{1}{\delta}.$$

It suffices to show that

$$2\,\delta^2 \, \pi^2 + \delta \left(-2 + \alpha\right) \pi - \alpha + \delta \left(\delta \, \pi + 2\,\alpha - 3\right) B_3(\pi) = \frac{\overbrace{\left(1 - \delta \, \pi\right)}^{\le 0} \, g(\pi)}{2 \left(1 - \alpha\right)^2} \ge 0,$$

where

$$\begin{aligned}
g(\pi) \equiv \quad & \delta^3 \, \left(1 - 2\,\delta\right) \, \pi^3 + \delta^2 \, \left(6\,\delta - 4 + \alpha\right) \, \pi^2 + \delta \, \left(6\,\alpha^2 \, \delta - 4\,\alpha^2 - 12\,\alpha \, \delta + 8\,\alpha - 1\right) \, \pi \\
& + \alpha - 4\,\alpha^2 + 2\,\alpha^3 + 6\,\alpha^2 \, \delta - 4\,\alpha^3 \, \delta.
\end{aligned}$$

Taking the derivative yields

$$\begin{aligned}
& g'(\pi) = \left(3\,\delta^3 - 6\,\delta^4\right) \, \pi^2 + \left(12\,\delta^3 - 8\,\delta^2 + 2\,\alpha \, \delta^2\right) \, \pi + 6\,\alpha^2 \, \delta^2 - 4\,\alpha^2 \, \delta - 12\,\alpha \, \delta^2 + 8\,\alpha \, \delta - \delta, \\
& g''(\pi) = \left(6\,\delta^3 - 12\,\delta^4\right) \, \pi + 12\,\delta^3 - 8\,\delta^2 + 2\,\alpha \, \delta^2, \\
& g^{(3)}(\pi) = -6\,\delta^3 \left(2\,\delta - 1\right) < 0,
\end{aligned}$$

and hence,

$$g''(\pi) \le g'' \left(\frac{1}{\delta}\right) = -2\,\delta^2 \, \left(1 - \alpha\right) \le 0.$$

This further implies

$$g'(\pi) \le g' \left(\frac{1}{\delta}\right) = 2\,\delta \, \left(1 - \alpha\right) \underbrace{\left(2\,\alpha + 3\,\delta - 3\,\alpha\,\delta - 3\right)}_{\le 3 - 3\,\alpha + 2\,\alpha - 3 = -\alpha \le 0} \le 0$$

and

$$g(\pi) \leq g\left(\frac{1}{\delta}\right) = 2\,(1-\alpha)^2\,\underbrace{(\alpha + 2\,\delta - 2\,\alpha\,\delta - 2)}_{\leq 2-2\,\alpha+\alpha-2=-\alpha} \leq 0,$$

and hence the desired result holds.

Because β and σ are decreasing and increasing on $[0,\tau]$ and $[\tau,\hat{\tau})$, respectively, and $\beta(\tau) = \sigma(\tau) = 0$, the corresponding nonnegativity conditions hold automatically.

For any $\theta \in [\hat{\tau}, 1]$, we have

$$\sigma'(\theta) = 2\,\delta\,\pi\,\eta(\theta) - (1-\alpha-\delta\,\pi\,\theta)\;\eta'(\theta) + w_q'(\theta) = \frac{(1-\alpha)\,(2\,\delta-1)}{\delta\,\theta^3} + \omega + \pi \geq 0,$$

and hence, the desired nonnegativity condition boils down to $\sigma(\hat{\tau}) \geq 0$, which follows from

$$\begin{aligned}\sigma(\hat{\tau}) - \lim_{\theta\uparrow\hat{\tau}}\sigma(\theta) = &\; -(1-\alpha-\delta\,\pi\,\hat{\tau})\;\eta(\hat{\tau}) - (1-\delta\,\pi\,\hat{\tau})\,\gamma(\hat{\tau}) \\ = &\; -\frac{(2\,\delta-1)\;(\hat{\tau}+1)\;(1-\alpha-\delta\,\pi\,\hat{\tau}+\alpha\,\hat{\tau})}{2\,\delta\,\hat{\tau}^2} \\ \geq &\; 0.\end{aligned}$$

The last inequality follows from

$$1-\alpha-\delta\,\pi\,\hat{\tau}+\alpha\,\hat{\tau} \leq (1-\alpha) - (\delta\,\pi-\alpha)\,\tau = 0.$$

It remains to verify the nonnegativity of ξ on $[0,\hat{\tau})$. Taking the derivative yields

$$\xi'(\theta) = \delta\;\left[2\,\gamma(\theta) + \theta\,\gamma'(\theta)\right] - w_S'(\theta) = \frac{(2\,\delta-1)\;(1-\hat{\tau})^2}{\hat{\tau}\,(1-\theta)^3} \geq 0,$$

and the desired property is implied by

$$\xi(0) = \frac{1}{\hat{\tau}}\;\left(\delta - \frac{1}{2}\right)\;(1-\hat{\tau})^2 + \frac{1}{2} \geq \frac{1}{2}.$$

We have

$$\begin{aligned}&\chi_L\;[-u(0)] + \int_0^{\tau}\beta(\theta)\;[-q(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{1}\sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^1\xi(\theta)\;[-m_S(\theta)]\,\mathrm{d}\theta \\ &+ \int_{\hat{\tau}}^1\eta(\theta)\;\left[\delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\;[m_S(\theta)+\pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\ &- \int_0^{\hat{\tau}}\gamma(\theta)\left[u(0) + \delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi q(y)]\,\mathrm{d}y + q(\theta) - \delta\theta\,[m_S(\theta)+\pi q(\theta)] - \delta(1-\theta)m_F(\theta)\right]\mathrm{d}\theta \\ =&\; -u(0) + \int_0^1\left[w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)\right]\mathrm{d}\theta,\end{aligned}$$

and hence, the objective function is bounded by

$$\int_{\tau}^{\hat{\tau}}\sigma(\theta)\,\mathrm{d}\theta + \int_{\hat{\tau}}^1\sigma(\theta)\,\mathrm{d}\theta = \frac{(1-\tau)\;[\delta\,\omega\,(1+\tau) - \alpha\,(1-\tau)]}{2\,\delta}.$$

Substituting our conjectured primal solution into the objective function yields

$$\int_{\tau}^{1} \left[\theta\,\omega - (1-\theta)\cdot\frac{\alpha}{\delta}\right] \mathrm{d}\theta,$$

which attains the upper bound above. This completes the proof. □ □

Proof. Proof of Proposition 6. The primal feasibility condition boils down to

$$0 \leq \tau \leq 1 \quad \Leftrightarrow \quad \delta\,\pi - 2\,\delta\,(\omega+\pi) + 1 \leq 0$$

and

$$0 \leq \frac{1-\delta\,\pi\,\tau}{\delta\,(1-\tau)} \leq \frac{\alpha}{\delta} \quad \Leftrightarrow \quad \begin{cases} 1-\delta\,\pi\,\tau \geq 0, \\ 1-\delta\,\pi\,\tau - \alpha\,(1-\tau) \leq 0, \end{cases}$$

where the first and second conditions are for the testing threshold τ and the reward for failure, respectively. The feasibility condition for τ follows from

$$\delta\,\pi - 2\,\delta\,(\omega+\pi) + 1 = 1 - \delta\,\pi - 2\,\delta\,\omega \leq 1 - \delta\,\frac{1}{\delta} = 0.$$

The feasibility condition for m_F follows from

$$1-\delta\,\pi\,\tau = \frac{2\,(\omega+\pi) - \pi\,(\delta\,\pi+1)}{2\,(\omega+\pi)} \geq \frac{2\,[B_5(\pi)+\pi] - \pi\,(\delta\,\pi+1)}{2\,(\omega+\pi)} = 0,$$

and

$$1-\delta\,\pi\,\tau - \alpha\,(1-\tau) = \frac{-\delta^2\,\pi^2 + \delta\,(1-\alpha)\,\pi + \alpha + 2\,\delta\,\omega\,(1-\alpha)}{2\,\delta\,(\omega+\pi)} \leq 0,$$

where the last inequality follows from $\omega < B_4(\pi)$ directly.

Define

$$\chi_L \equiv \frac{1}{2\,\delta},$$

$$\gamma(\theta) \equiv 1 - \frac{1}{2\,\delta} \geq 0, \qquad \forall\theta \in [0,1],$$

$$\beta(\theta) \equiv -(1-\delta\,\pi\,\theta)\,\gamma(\theta) - \delta\,\pi\int_{\theta}^{1}\gamma(y)\,\mathrm{d}y - w_q(\theta) = \frac{1}{2\,\delta}\,(\delta\,\pi+1) - \theta\,(\omega+\pi), \quad \forall\theta \in [0,\tau),$$

$$\sigma(\theta) \equiv (1-\delta\,\pi\,\theta)\,\gamma(\theta) + \delta\,\pi\int_{\theta}^{1}\gamma(y)\,\mathrm{d}y + w_q(\theta) = -\frac{1}{2\,\delta}\,(\delta\,\pi+1) + \theta\,(\omega+\pi), \quad \forall\theta \in [\tau,1], \text{ and}$$

$$\xi(\theta) \equiv \delta\,\theta\,\gamma(\theta) - \delta\int_{\theta}^{1}\gamma(y)\,\mathrm{d}y - w_S(\theta) = \frac{1}{2}, \qquad \forall\theta \in [0,1].$$

Because β and σ are decreasing and increasing on $[0,\tau)$ and $[\tau,1]$, respectively, and

$$\lim_{\theta\uparrow\tau}\beta(\theta) = \sigma(\tau) = 0,$$

they are always nonnegative.

We have

$$\begin{aligned}
&\chi_L\ [-u(0)] + \int_0^\tau \beta(\theta)\ [-q(\theta)]\,\mathrm{d}\theta + \int_\tau^1 \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^1 \xi(\theta)\ [-m_S(\theta)]\,\mathrm{d}\theta \\
&- \int_0^1 \gamma(\theta)\ \left[u(0) + \delta \int_0^\theta [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + q(\theta) - \delta\,\theta\ [m_S(\theta) + \pi\,q(\theta)] - \delta\,(1-\theta)\,m_F(\theta)\right]\mathrm{d}\theta \\
=& -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta,
\end{aligned}$$

and hence, the objective function is bounded by

$$\int_\tau^1 \sigma(\theta)\,\mathrm{d}\theta = \frac{(1-\tau)\ [\delta\,\tau\,(\omega+\pi) + \delta\,\omega - 1]}{2\,\delta}$$

Substituting our conjectured primal solution into the objective function yields

$$\int_\tau^1 \left[\theta\,\omega - (1-\theta)\cdot\frac{1}{1-\tau}\cdot\left(\frac{1}{\delta} - \tau\,\pi\right)\right]\mathrm{d}\theta,$$

achieving the upper bound above and, thereby, establishing the optimality. This completes the proof. □ □

Proof. Proof of Proposition 7. We prove the desired result in two steps: first, we show that the optimal mechanism is to provide no funding when $\omega = \frac{\pi}{2}\,(\delta\,\pi - 1)$; and then, we prove that if our candidate solution is optimal when the consumer welfare is ω_0, fixing all the remaining parameters, it remains optimal for any $0 \leq \omega_1 < \omega_0$. Combining the results from both steps yields the desired conclusion.

When $\omega = \frac{\pi}{2}\,(\delta\,\pi - 1)$, by Proposition 6, the testing threshold is given by

$$\tau = \frac{\delta\,\pi + 1}{2\,\delta\,(\omega + \pi)} = \frac{1}{\delta\,\pi},$$

and the optimal mechanism boils down to

$$q(\theta) = \begin{cases} 0, & \text{if } \theta \in [0,\tau), \\ 1, & \text{if } \theta \in [\tau,1], \end{cases} \qquad m_0(\theta) = m_S(\theta) = m_F(\theta) = 0, \quad \forall \theta \in \Theta.$$

This completes the proof of the first argument.

Lemma 7 states that if the optimal mechanism is to provide no funding, lowering the consumer welfare ω and keeping all the remaining parameters will not alter the optimal mechanism.

Lemma 7. *If the optimal mechanism is to provide no funding when the consumer welfare is ω_0, fixing all the remaining parameters, for any $\omega \leq \omega_0$, the optimal mechanism remains providing no funding.*

Proof. Proof of Lemma 7. Let $\tau \equiv \frac{1}{\delta\,\pi}$. Suppose there exists $\omega_1 < \omega_0$ such that the mechanism providing no funding is strictly dominated by

$$\hat{\Omega} \equiv \{\hat{q}(\theta), \hat{m}_0(\theta), \hat{m}_S(\theta), \hat{m}_F(\theta)\}.$$

Because the policy providing no funding is strictly dominated, we have

$$\int_0^1 \left[\theta\,\omega_1\,\hat{q}(\theta) - \hat{m}_0(\theta) - \theta\,\hat{m}_S(\theta) - (1-\theta)\,\hat{m}_F(\theta)\right]\mathrm{d}\theta > \omega_1 \int_\tau^1 \theta\,\mathrm{d}\theta,$$

or equivalently,

$$\omega_1 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] > \int_0^1 \left[\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)\right]\mathrm{d}\theta. \tag{46}$$

Because the feasible set of the funder's problem (14) is independent of ω, when the profit is ω_0, the mechanism $\hat{\Omega}$ remains feasible.

Since the mechanism providing no funding is optimal when the consumer welfare is ω_0, we have

$$\int_0^1 \left[\theta\,\omega_0\,\hat{q}(\theta) - \hat{m}_0(\theta) - \theta\,\hat{m}_S(\theta) - (1-\theta)\,\hat{m}_F(\theta)\right]\mathrm{d}\theta \le \omega_0 \int_\tau^1 \theta\,\mathrm{d}\theta,$$

or equivalently,

$$\omega_0 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] \le \int_0^1 \left[\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)\right]\mathrm{d}\theta \tag{47}$$

Combining (46) and (47) yields

$$\omega_0 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right] < \omega_1 \left[\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta\right]$$

Because $\omega_0 > \omega_1 \ge 0$, we have

$$\int_0^1 \theta\,\hat{q}(\theta)\,\mathrm{d}\theta - \int_\tau^1 \theta\,\mathrm{d}\theta < 0,$$

which implies

$$\int_0^1 \left[\hat{m}_0(\theta) + \theta\,\hat{m}_S(\theta) + (1-\theta)\,\hat{m}_F(\theta)\right]\mathrm{d}\theta < 0,$$

contradicting the nonnegativity of m's. This completes the proof. □ □

Combining the results above completes the proof. □ □

Proof. Proof of Corollary 1. In Region (2), the testing threshold τ is determined by the following polynomial equation:

$$2\,\delta\,(\omega+\pi)\,\tau^3 + (\alpha - 2\,\delta\,\alpha - 1)\,\tau^2 + \alpha - 1 = 0.$$

As either ω or π increases, it is clear that τ must decrease in order to maintain equality, implying a lower testing threshold.

In Region (3), it is straightforward to see that the threshold $\tau = \frac{1-\alpha}{\delta\,\pi}$ is decreasing in π and constant in ω.

In Region (4), by the implicit function theorem, we have

$$\tau'(\pi) = -\frac{2\,\delta\,\tau^3}{6\,\delta\,(\omega+\pi)\,\tau^2 - 2\,[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\tau} = -\frac{\delta\,\tau^2}{3\,\delta\,(\omega+\pi)\,\tau - [2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]},$$

and the desired result boils down to

$$3\,\delta\,(\omega+\pi)\,\tau - [2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha] \geq 0 \quad \Leftrightarrow \quad \tau \geq \frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)},$$

which follows from

$$\begin{aligned}
&2\,\delta\,(\omega+\pi)\,\tau^3 - [2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\tau^2 - (1-\alpha)\,(2\,\delta-1)\,\big|_{\tau=\frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}} \\
&= \frac{1}{9\,\delta^2\,(\omega+\pi)^2}\left[-\frac{1}{3}\,[2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]^3 - 9\,\delta^2\,(\omega+\pi)^2\,(1-\alpha)\,(2\,\delta-1)\right] \\
&< 0.
\end{aligned}$$

In the proof of Proposition 4, we have shown that function

$$2\,\delta\,(\omega+\pi)\,\theta^3 - [2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha]\,\theta^2 - (1-\alpha)\,(2\,\delta-1)$$

is either increasing or first decreasing and then increasing in θ, with the values at the two endpoints (0 and 1) being negative and positive, respectively. Therefore, the inequality above implies that $\tau \geq \frac{2\,\delta\,\pi\,(1-\delta)+(1-\alpha)+2\,\delta\,\alpha}{3\,\delta\,(\omega+\pi)}$ and thus the desired monotonicity. The monotonicity of τ in ω follows from the fact that ω appears only in the coefficient of τ^3 in the polynomial equation.

In Regions (5) and (7), the thresholds $\frac{1-\alpha}{\delta\,\pi-\alpha}$ and $\frac{1}{\delta\,\pi}$ are both decreasing in π and constant in ω.

In Region (6), the threshold $\frac{\delta\,\pi+1}{2\,\delta\,(\omega+\pi)}$ is decreasing in ω.

Since the optimal threshold τ is non-increasing in π in Propositions 1–5 and 7, it is straightforward to verify that

$$[m_S^*(\theta) - m_F^*(\theta) + \pi\,q^*(\theta)] \cdot \mathbb{1}_{\theta>\tau},$$

where the functions m_S^*, m_F^*, and q^* may also depend on π, is non-decreasing in π. Consequently, for any $\theta > \tau$,

$$u_1(\theta,\theta) = \int_\tau^\theta [m_S^*(y) - m_F^*(y) + \pi\,q^*(y)]\,\mathrm{d}y$$

is non-decreasing in π as well. Thus, the firm's expected payoff is always non-decreasing, regardless of its type θ, implying that its ex ante expected payoff is non-decreasing in π.

For the case introduced in Proposition 6, the firm's ex ante expected payoff is given by

$$\mathbb{E}[u_1(\theta)] = \int_\tau^1 [-c + \delta\,\theta\,\pi + (1-\theta)\,m_F^*]\,\mathrm{d}\theta = \frac{(\delta\,\pi - c)(2\,\delta\,\omega + \delta\,\pi - c)}{4\,\delta\,(\omega+\pi)}.$$

Taking the derivative with respect to π yields

$$\frac{\mathrm{d}}{\mathrm{d}\pi}\,\mathbb{E}[u_1(\theta)] = \frac{1}{4\,\delta}\left[\delta^2 + \frac{\delta^2\,\omega^2 - c^2}{(\omega+\pi)^2}\right] \geq 0,$$

where the inequality follows from

$$\delta^2 (\omega + \pi)^2 - c^2 \geq \delta^2 \pi^2 - c^2 \geq 0.$$

For the funder, as the firm's commercial profit π increases, it can maintain the same testing threshold while reducing the compensation level, thereby obtaining a weakly higher expected payoff. Since the optimal mechanism achieves at least this much, it follows that the funder's expected payoff U_F is non-decreasing in π.

Since the social value ω appears only in the objective function, increasing ω does not affect the feasible set. Moreover, because its coefficient is positive, the funder's expected payoff U_F must be non-decreasing in ω.

As ω increases, the optimal testing threshold τ is non-increasing. Furthermore, in Regions (2), (4), and (6), for any $\theta > \tau$, the functions $m_S^*(\theta)$, $m_0^*(\theta) + \delta (1-\theta) m_F^*(\theta)$, and $(1-\theta) m_F^*(\theta)$ are all increasing in ω. This, in turn, implies that the firm's expected payoff is non-decreasing for any type θ, and therefore so is the ex-ante expected payoff.

This completes the proof. □ □

D Proofs of Section 3.2

Proof. Proof of Proposition 8.

Regarding case (a), primal feasibility follows directly from its definition.

We establish optimality by considering three separate cases. In the first, in addition to the optimality conditions stated in the proposition, we impose the following condition:

$$\omega \geq \frac{\delta \pi \, (\pi - \delta \pi + \alpha)}{1 - \alpha} - \frac{\pi}{2}.$$

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\eta(\theta) \equiv \begin{cases} \dfrac{1}{2\,\delta} \left(\dfrac{C_2}{\theta^2} + 1 \right) - 1, & \forall \theta \in (0, \tau), \\ \dfrac{1}{2\,\delta} \left(\dfrac{C_1}{\theta^2} + 1 \right) - 1, & \forall \theta \in [\tau, \hat{\tau}], \end{cases} \\
&\xi(\theta) \equiv \begin{cases} -\delta\,\theta\,\eta(\theta) + \delta \displaystyle\int_\theta^{\hat{\tau}} \eta(y)\,\mathrm{d}y - w_S(\theta) = 1 - \delta, & \forall \theta \in [0, \tau], \\ -w_S(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases} \\
&\nu(\theta) \equiv \begin{cases} \delta\,\theta\,\eta(\theta) - \delta \displaystyle\int_\theta^{\hat{\tau}} \eta(y)\,\mathrm{d}y - w_F(\theta) = 1 - \delta, & \forall \theta \in [\tau, \hat{\tau}), \\ -w_F(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases} \\
&\sigma(\theta) \equiv \begin{cases} w_q(\theta) - (1 - \alpha - \delta\,\pi\,\theta)\ \eta(\theta) - \delta\,\pi \displaystyle\int_\theta^{\hat{\tau}} \eta(y)\,\mathrm{d}y, & \forall \theta \in [\tau, \hat{\tau}), \\ w_q(\theta), & \forall \theta \in [\hat{\tau}, 1], \text{ and} \end{cases} \\
&\beta(\theta) \equiv (1 - \alpha - \delta\,\pi\,\theta)\ \eta(\theta) + \delta\,\pi \int_\theta^{\hat{\tau}} \eta(y)\,\mathrm{d}y - w_q(\theta), \quad \forall \theta \in (0, \tau],
\end{aligned}$$

where

$$C_1 = \hat{\tau}^2 \left(\frac{1}{2\delta} - 1 + \frac{1}{\hat{\tau}} \right) > 0, \tag{48}$$

$$C_2 = \frac{1-\alpha}{2\delta^3\pi^3} \left[2\delta^2\pi^2 - 2\delta\pi^2 - 2\alpha\delta\pi + (1-\alpha)\pi + 2\omega(1-\alpha) \right], \tag{49}$$

and the threshold $\hat{\tau} \in [\tau, 1]$ is the unique solution to

$$\frac{(\delta\pi)^2(1-2\delta)}{2\delta(1-\alpha)}\hat{\tau}^2 + \frac{(\delta\pi)^2}{1-\alpha}\hat{\tau} + (1-\alpha)\left(\frac{1-2\delta}{2\delta}\right) = \frac{(\omega+\pi)(1-\alpha)}{\delta\pi} - 1. \tag{50}$$

We first verify the existence of $\hat{\tau} \in [\tau, 1]$ under the given conditions. The axis of symmetry for the quadratic function determining $\hat{\tau}$ is $-\frac{\delta}{1-2\,\delta} < 0$. Hence, the left-hand side is increasing in $\hat{\tau}$ when $\hat{\tau} \in [\tau, 1]$. The existence of $\hat{\tau}$ is thus reduced to

$$\frac{(\delta\pi)^2(1-2\delta)}{2\delta(1-\alpha)}\,\tau^2 + \frac{(\delta\pi)^2}{1-\alpha}\,\tau + (1-\alpha)\left(\frac{1-2\delta}{2\delta}\right) \le \frac{(\omega+\pi)(1-\alpha)}{\delta\pi} - 1 \le \frac{(\delta\pi)^2(1-2\delta)}{2\delta(1-\alpha)} + \frac{(\delta\pi)^2}{1-\alpha} + (1-\alpha)\left(\frac{1-2\delta}{2\delta}\right),$$

which can be simplified to

$$\tilde{B}_2(\pi) = \frac{\delta\pi\,(\delta\pi + 2\alpha - 1)}{1-\alpha} \le \omega \le \frac{-\pi(1-\alpha)^2 + 2\alpha\delta\pi(1-\alpha) + \delta^2\pi^3}{2(1-\alpha)^2} = B_1(\pi).$$

Because $C_1 > 0$, the function η is decreasing on $[\tau, \hat{\tau}]$, and hence, the nonnegativity condition is reduced to $\eta(1) \ge 0$, which clearly holds since $\delta < \frac{1}{2}$.

The additional condition guarantees that $C_2 \ge 0$. Since

$$2\,\eta(\theta) + \eta'(\theta) = \frac{1}{\delta} - 2 > 0, \quad \forall \theta \in (0, \tau),$$

and η is non-increasing in this interval, the nonnegativity is established.

Since η and w_q are decreasing and increasing on $(0, \tau)$, the minimum of β is attained at τ, the value of which is 0. The nonnegativity of σ when $\theta \in [\tau, \hat{\tau}]$ is guaranteed by $\sigma(\tau) = 0$ and its monotonicity. When $\theta \in [\hat{\tau}, 1]$, the nonnegativity of σ boils down to $\sigma(\hat{\tau}) \ge 0$, which follows from

$$w_q(\hat{\tau}) \ge w_q(\tau) = \delta\,\pi \int_{\tau}^{\hat{\tau}} \eta(y)\,\mathrm{d}y > 0.$$

The nonnegativity of ξ and ν follows directly from their definitions.

Since

$$\begin{aligned}
&\chi_L\,[-u(0)] + \int_0^{\tau} \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta \\
&+ \int_0^{\hat{\tau}} \eta(\theta) \left[\delta \int_0^{\theta} [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,q(\theta) - \delta\,\theta\,[m_S(\theta) + \pi\,q(\theta)] + \delta\,\theta\,m_F(\theta) \right] \mathrm{d}\theta \\
&+ \int_{\tau}^{1} \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^{\tau} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\hat{\tau}}^{1} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\tau}^{1} \nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta
\end{aligned}$$

$$= -u(0) + \int_0^1 [w_q(\theta)\, q(\theta) + w_S(\theta)\, m_S(\theta) + w_F(\theta)\, m_F(\theta)]\, \mathrm{d}\theta,$$

the objective function is bounded above by

$$\int_\tau^1 \sigma(y)\, \mathrm{d}y,$$

which is the same as the objective value achieved by the candidate solution

$$\int_\tau^1 (\theta\, \omega - \alpha)\, \mathrm{d}\theta.$$

The above result follows from

$$\begin{aligned}
&\int_\tau^1 \sigma(y)\, \mathrm{d}y - \int_\tau^1 (\theta\, \omega - \alpha)\, \mathrm{d}\theta \\
&= \int_\tau^{\hat{\tau}} \left[-(1-\alpha) \left(\frac{C_1}{\theta^2} + \frac{1}{2\,\delta} - 1 \right) + (\omega + \pi)\, \theta - 1 \right] \mathrm{d}\theta + \int_{\hat{\tau}}^1 [(\omega + 2\,\delta\,\pi)\, \theta - \delta\,\pi - 1]\, \mathrm{d}\theta - \int_\tau^1 (\theta\, \omega - \alpha)\, \mathrm{d}\theta \\
&= \int_\tau^{\hat{\tau}} \left[-(1-\alpha) \left[\frac{\hat{\tau}^2}{\theta^2} \left(\frac{1}{2\delta} - 1 + \frac{1}{\hat{\tau}} \right) + \frac{1}{2\,\delta} - 1 \right] + \pi\, \theta - 1 \right] \mathrm{d}\theta + \int_{\hat{\tau}}^1 (2\,\delta\,\pi\,\theta - \delta\,\pi - 1)\, \mathrm{d}\theta + \alpha\,(1-\tau) \\
&= 0.
\end{aligned}$$

Next, we consider the case subject to the following additional conditions:

$$\frac{\delta\pi(\pi - \delta\pi + 1)}{1-\alpha} - \pi \leq \omega < \frac{\delta\pi(\pi - \delta\pi + \alpha)}{1-\alpha} - \frac{\pi}{2}.$$

Define

$$\begin{aligned}
&\chi_L \equiv 1, \\
&\eta(\theta) \equiv \begin{cases} \dfrac{1}{2\,\delta} \left(\dfrac{C_2}{\theta^2} + 1 \right) - 1, & \forall \theta \in (\tilde{\tau}, \tau), \\ \dfrac{1}{2\,\delta} \left(\dfrac{C_1}{\theta^2} + 1 \right) - 1, & \forall \theta \in [\tau, \hat{\tau}], \end{cases} \\
&\xi(\theta) \equiv \begin{cases} \delta \displaystyle\int_{\tilde{\tau}}^{\hat{\tau}} \eta(y)\, \mathrm{d}y - w_S(\theta), & \forall \theta \in [0, \tilde{\tau}), \\ -\delta\,\theta\,\eta(\theta) + \delta \displaystyle\int_\theta^{\hat{\tau}} \eta(y)\, \mathrm{d}y - w_S(\theta) = 1 - \delta, & \forall \theta \in [\tilde{\tau}, \tau], \\ -w_S(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases} \\
&\nu(\theta) \equiv \begin{cases} -\delta \displaystyle\int_{\tilde{\tau}}^{\hat{\tau}} \eta(y)\, \mathrm{d}y - w_F(\theta), & \forall \theta \in [0, \hat{\tau}), \\ \delta\,\theta\,\eta(\theta) - \delta \displaystyle\int_\theta^{\hat{\tau}} \eta(y)\, \mathrm{d}y - w_F(\theta) = 1 - \delta, & \forall \theta \in [\tau, \hat{\tau}), \\ -w_F(\theta), & \forall \theta \in [\hat{\tau}, 1], \end{cases}
\end{aligned}$$

$$\sigma(\theta) \equiv \begin{cases} w_q(\theta) - (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) - \delta\,\pi\int_\theta^{\hat\tau}\eta(y)\,\mathrm{d}y, & \forall\theta\in[\tau,\hat\tau), \\ w_q(\theta), & \forall\theta\in[\hat\tau,1], \text{ and} \end{cases}$$

$$\beta(\theta) \equiv \begin{cases} \delta\,\pi\int_{\tilde\tau}^{\hat\tau}\eta(y)\,\mathrm{d}y - w_q(\theta), & \forall\theta\in[0,\tilde\tau), \\ (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi\int_\theta^{\hat\tau}\eta(y)\,\mathrm{d}y - w_q(\theta), & \forall\theta\in[\tilde\tau,\tau], \end{cases}$$

where C_1 and C_2 are defined by (48) and (49), and the thresholds $(\tilde\tau,\hat\tau) \in (0,\tau]\times[\tau,1]$ are determined by (50) and

$$-(1-\alpha)\left[\frac{1}{\tau^2}\left(\hat\tau + \frac{1-2\delta}{2\delta}\hat\tau^2\right) + \frac{1}{2\delta} - 1\right] + (\omega+\pi)\tau - 1 = 0. \tag{51}$$

In the previous case, we have shown that the existence of $\hat\tau$ is implied by $\tilde B_2(\pi) \le \omega < B_1(\pi)$. The existence of $\tilde\tau$ requires that

$$\hat\tau + \frac{1-2\delta}{2\delta}\hat\tau^2 - \frac{\tau(1-\delta)}{\delta} < 0$$

and

$$\frac{1}{\tau^2}\left[\hat\tau + \frac{1-2\delta}{2\delta}\hat\tau^2 - \frac{\tau(1-\delta)}{\delta}\right] + \frac{1}{2\delta} - 1 \ge 0.$$

After substituting $\hat\tau + \frac{1-2\delta}{2\delta}\hat\tau^2$ with $\tau^2\left[\frac{(\omega+\pi)\tau-1}{1-\alpha} - \frac{1}{2\delta} + 1\right]$, the inequalities above can be simplified to the additional condition

$$\frac{\delta\pi(\pi-\delta\pi+1)}{1-\alpha} - \pi \le \omega < \frac{\delta\pi(\pi-\delta\pi+\alpha)}{1-\alpha} - \frac{\pi}{2}.$$

Since $C_1 > 0$, the nonnegativity of η on the interval $[\tau,\hat\tau]$ is implied by $\eta(\hat\tau) > 0$. The inequality conditions above imply that $C_2 < 0$, and the desired nonnegativity condition on $[\tilde\tau,\tau)$ is implied by $\eta(\tilde\tau) = 0$.

When $\theta \in [0,\tilde\tau)$, the functions β, ξ, and ν are decreasing, increasing, and decreasing, respectively. The desired nonnegativity conditions are given by

$$\begin{aligned} &\beta(\tilde\tau) = (1-\delta)\pi - (\omega+\pi)\tilde\tau + 1 \ge 0, \\ &\xi(0) = \delta\int_{\tilde\tau}^{\hat\tau}\eta(y)\,\mathrm{d}y + \delta > \delta > 0, \\ &\nu(\tilde\tau) = 0. \end{aligned}$$

The first condition is equivalent to

$$\tilde\tau \le \frac{(1-\delta)\,\pi + 1}{\omega+\pi},$$

which can be rewritten as

$$\left(\frac{\omega+\pi}{(1-\delta)\,\pi+1}\right)^2\left[\tau^2\left[\frac{(\omega+\pi)\tau-1}{1-\alpha} - \frac{1}{2\delta} + 1\right] - \frac{\tau(1-\delta)}{\delta}\right] + \frac{1}{2\delta} - 1 \ge 0.$$

Let

$$L(x) \equiv \left(\frac{x}{(1-\delta)\,\pi+1}\right)^2 \left[\tau^2\left[\frac{\tau x-1}{1-\alpha}-\frac{1}{2\delta}+1\right]-\frac{\tau(1-\delta)}{\delta}\right]+\frac{1}{2\delta}-1,$$

where the coefficients of the cubic and quadratic terms are positive and negative, respectively. We have

$$L\left(\frac{\delta\pi(\pi-\delta\pi+1)}{1-\alpha}\right)=0.$$

Since the sign of $L'(x)$ depends on an increasing function of x, and we have

$$L'\left(\frac{\delta\pi(\pi-\delta\pi+1)}{1-\alpha}\right)=\frac{\delta\pi(1-\delta)-2\alpha\delta+\alpha+3\delta-1}{\delta^2\pi\,(\pi-\delta\pi+1)}.$$

The derivative above is positive, which follows from

$$\underbrace{\delta\pi}_{\geq 1-\alpha}(1-\delta)-2\alpha\delta+\alpha+3\delta-1\geq\delta(2-\alpha)>0.$$

Therefore, we know that the function L is non-decreasing on the interval for ω, implying the desired inequality.

When $\theta\in[\tilde{\tau},\tau)$, the nonegativity of ξ follows from its definition directly. Since

$$\beta'(\theta)=(1-\alpha)\eta'(\theta)-\omega-\pi$$

and

$$\beta''(\theta)=(1-\alpha)\eta''(\theta)<0,$$

the concavity implies that its minimum on this interval is attained at either $\tilde{\tau}$ or τ. Because $\beta(\tilde{\tau})\geq 0$ and $\beta(\tau)=0$, the nonnegativity is thus established.

When $\theta\in[\tau,\hat{\tau})$, the nonnegativity of σ follows from its monotonicity together with the fact that $\sigma(\tau)=0$.

When $\theta\in[\hat{\tau},1]$, the nonnegativity of ξ and ν follows from their definitions. Since

$$\hat{\tau}\geq\frac{\delta\,\pi+1}{\omega+2\,\delta\,\pi}\quad\Leftarrow\quad\tau=\frac{1-\alpha}{\delta\,\pi}\geq\frac{\delta\,\pi+1}{\omega+2\,\delta\,\pi}\quad\Leftrightarrow\quad\omega\geq\frac{\delta\pi\left(\delta\pi+2\alpha-1\right)}{1-\alpha},$$

the nonnegativity of σ is also established.

Since

$$\begin{aligned}
&\chi_L\,[-u(0)]+\int_0^{\tau}\beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta\\
&+\int_{\tilde{\tau}}^{\hat{\tau}}\eta(\theta)\left[\delta\int_0^{\theta}[m_S(y)-m_F(y)+\pi\,q(y)]\,\mathrm{d}y+(1-\alpha)\,q(\theta)-\delta\,\theta\,[m_S(\theta)+\pi\,q(\theta)]+\delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta\\
&+\int_{\tau}^{1}\sigma(\theta)\,q(\theta)\,\mathrm{d}\theta+\int_0^{\tau}\xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta+\int_{\hat{\tau}}^{1}\xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta+\int_0^{\tilde{\tau}}\nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta+\int_{\tau}^{1}\nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\\
=\;&-u(0)+\int_0^1[w_q(\theta)\,q(\theta)+w_S(\theta)\,m_S(\theta)+w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta,
\end{aligned}$$

the objective function is bounded above by

$$\int_\tau^1 \sigma(y)\,\mathrm{d}y.$$

Because the expression for σ coincides with that of the previous case, the optimality condition is established.

Finally, we focus on the case subject to the following additional conditions:

$$\omega < \frac{\delta\,\pi\,(\pi - \delta\,\pi + 1)}{1-\alpha} - \pi.$$

Define

$$\begin{aligned}
&\chi_L \equiv 1,\\
&\eta(\theta) \equiv \frac{1}{2\,\delta}\left(\frac{C_1}{\theta^2}+1\right) - 1, \quad \forall\theta\in[\tau,\hat\tau],\\
&\xi(\theta) \equiv \begin{cases} \delta\displaystyle\int_\tau^{\hat\tau}\eta(y)\,\mathrm{d}y - w_S(\theta), & \forall\theta\in[0,\tau],\\ -w_S(\theta), & \forall\theta\in[\hat\tau,1],\end{cases}\\
&\nu(\theta) \equiv \begin{cases} \delta\,\theta\,\eta(\theta) - \delta\displaystyle\int_\theta^{\hat\tau}\eta(y)\,\mathrm{d}y - w_F(\theta), & \forall\theta\in[0,\tau),\\ -\delta\displaystyle\int_\tau^{\hat\tau}\eta(y)\,\mathrm{d}y - w_F(\theta) = 1-\delta, & \forall\theta\in[\tau,\hat\tau),\\ -w_F(\theta), & \forall\theta\in[\hat\tau,1],\end{cases}\\
&\sigma(\theta) \equiv \begin{cases} w_q(\theta) - (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) - \delta\,\pi\displaystyle\int_\theta^{\hat\tau}\eta(y)\,\mathrm{d}y, & \forall\theta\in[\tau,\hat\tau),\\ w_q(\theta), & \forall\theta\in[\hat\tau,1],\ \text{and}\end{cases}\\
&\beta(\theta) \equiv (1-\alpha-\delta\,\pi\,\theta)\;\eta(\theta) + \delta\,\pi\int_\theta^{\hat\tau}\eta(y)\,\mathrm{d}y - w_q(\theta), \quad \forall\theta\in(0,\tau],
\end{aligned}$$

where C_1 is defined by (48), and the threshold $\hat\tau\in[\tau,1]$ is the unique solution to (50).

The unique existence of $\hat\tau\in[\tau,1]$, together with the nonnegativity of η, σ, and β, follows from the previous case.

When $\theta\in[0,\tau)$, the functions ξ and ν are increasing and decreasing, respectively. Therefore, the desired nonnegativity conditions are given by

$$\xi(0) = \delta\int_\tau^{\hat\tau}\eta(y)\,\mathrm{d}y + \delta > \delta > 0$$

and

$$\lim_{\theta\uparrow\tau}\nu(\theta) = -\delta\int_\tau^{\hat\tau}\eta(y)\,\mathrm{d}y - (2\,\delta-1)\,(1-\tau) = 1-\delta-\underbrace{\delta\,\tau\,\eta(\tau)}_{=\frac{1}{\pi}\,[(\omega+\pi)\,\tau-1]} \geq 0.$$

The latter inequality is equivalent to

$$\omega \leq \frac{\delta\,\pi\,(\pi - \delta\,\pi + 1)}{1-\alpha} - \pi.$$

The nonnegativity on $[\hat{\tau}, 1]$ follows from their definitions directly.

Since

$$\begin{aligned}
&\chi_L\,[-u(0)] + \int_0^{\tau} \beta(\theta)\,[-q(\theta)]\,\mathrm{d}\theta \\
&+ \int_{\tau}^{\hat{\tau}} \eta(\theta)\,\left[\delta \int_0^{\theta} [m_S(y) - m_F(y) + \pi\,q(y)]\,\mathrm{d}y + (1-\alpha)\,c\,q(\theta) - \delta\,\theta\,[m_S(\theta) + \pi\,q(\theta)] + \delta\,\theta\,m_F(\theta)\right]\mathrm{d}\theta \\
&+ \int_{\tau}^{1} \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^{\tau} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_{\hat{\tau}}^{1} \xi(\theta)\,[-m_S(\theta)]\,\mathrm{d}\theta + \int_0^{1} \nu(\theta)\,[-m_F(\theta)]\,\mathrm{d}\theta \\
=\ & -u(0) + \int_0^{1} [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta,
\end{aligned}$$

the objective function is bounded above by

$$\int_{\tau}^{1} \sigma(y)\,\mathrm{d}y.$$

Because the expression for σ coincides with that of the previous case, the optimality condition is established.

Under the given conditions, we have $\tau \leq 1$ and

$$0 \leq 1 - \delta\,\pi\,\tau < \alpha.$$

It is straightforward to verify the primal feasibility.

Next, we prove case (b), in which it is optimal to grant less push funding to the firm.

Under the given conditions, we have $\tau \leq 1$ and

$$0 \leq 1 - \delta\,\pi\,\tau < \alpha.$$

It is straightforward to verify the primal feasibility.

Define

$$\begin{aligned}
&\chi_L = 1, \\
&\beta(\theta) = -w_q(\theta), \quad \forall\theta \in [0,\tau], \\
&\sigma(\theta) = w_q(\theta), \quad \forall\theta \in [\tau,1], \\
&\xi(\theta) = -w_S(\theta), \quad \forall\theta \in [0,1], \\
&\nu(\theta) = -w_F(\theta), \quad \forall\theta \in [0,1].
\end{aligned}$$

The nonnegativity of β and σ follows from the definition of τ. Since ξ and ν are increasing and decreasing in θ, respectively, the nonegativity conditions are given by

$$\xi(0) = \delta > 0,$$

$$\nu(1) = 0.$$

Since

$$\chi_L\ [-u(0)] + \int_0^\tau \beta(\theta)\ [-q(\theta)]\,\mathrm{d}\theta + \int_\tau^1 \sigma(\theta)\,q(\theta)\,\mathrm{d}\theta + \int_0^1 \xi(\theta)\ [-m_S(\theta)]\,\mathrm{d}\theta + \int_0^1 \nu(\theta)\ [-m_F(\theta)]\,\mathrm{d}\theta$$
$$= -u(0) + \int_0^1 [w_q(\theta)\,q(\theta) + w_S(\theta)\,m_S(\theta) + w_F(\theta)\,m_F(\theta)]\,\mathrm{d}\theta,$$

the objective function is bounded above by

$$\int_\tau^1 \sigma(y)\,\mathrm{d}y,$$

which is the same as the objective value achieved by the candidate solution

$$\int_\tau^1 (\theta\,\omega - \delta\,\pi\,\tau)\,\mathrm{d}\theta.$$

The above result follows from

$$\int_\tau^1 \sigma(\theta)\,\mathrm{d}\theta - \int_\tau^1 (\theta\,\omega - \delta\,\pi\,\tau)\,\mathrm{d}\theta = \int_\tau^1 [(\omega + 2\,\delta\,\pi)\,\theta - \delta\,\pi - 1]\,\mathrm{d}\theta - \int_\tau^1 (\theta\,\omega + \delta\,\tau\,\pi - 1)\,\mathrm{d}\theta = 0.$$

The proof of case (c) proceeds analogously to that of Proposition 7; hence, the details are omitted.

This completes the proof. □ □

Proof. Proof of Corollary 2.

The monotonicity of the testing threshold τ with respect to π and ω has been established in Regions (1)-(3) and (7) in Corollary 1. In Region (3'), it is straightforward to verify that $\tau = \frac{\delta\,\pi + c}{\omega + 2\,\delta\,\pi}$ is decreasing in ω.

Parts (c) and (d) follow from a similar argument as in the proof of Corollary 1. In Region (3'), the firm's ex ante expected payoff is given by

$$\mathbb{E}[u_1(\theta)] = \int_\tau^1 (-c + m_0^* + \delta\,\theta\,\pi)\,\mathrm{d}\theta = \frac{\delta\,\pi}{2}\left(\frac{\omega + \delta\,\pi - c}{\omega + 2\,\delta\,\pi}\right)^2.$$

Taking the derivative yields

$$\frac{\mathrm{d}\mathbb{E}[u_1(\theta)]}{\mathrm{d}\pi} = \frac{\delta\,(\omega + \delta\,\pi - c)\ [\omega\,(\omega + \delta\,\pi - c) + 2\,\delta\,\pi\,(\delta\,\pi + c)]}{2\,(\omega + 2\,\delta\,\pi)^3} \geq 0,$$

establishing the monotonicity of the firm's ex ante expected payoff in π in the new region.

This completes the proof. □ □

E Proofs of Section 3.3

Proof. Proof of Lemma 2. It is straightforward to verify that the funder's expected payoff is reduced

to

$$\int_{\check{\tau}}^{\theta_H} [\theta\,\omega - \check{m}_0 - \theta\,\check{m}_S - (1-\theta)\,\check{m}_F]\,\mathrm{d}F(\theta)$$

under the threshold mechanism (22).

Invoking Lemma 3, we know that the (IC) constraint is equivalent to (37), (38), and (39). By substituting the decision variables with the threshold mechanism defined in (22), the incentive constraints are further reduced as follows:

$$\check{m}_S - \check{m}_F + \pi \geq 0, \text{ and} \tag{25}$$
$$\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F = c - \delta\,\pi\,\check{\tau}, \tag{26}$$
$$\check{m}_0 + \delta\,\check{m}_F \leq \alpha\,c, \tag{27}$$
$$-c + \check{m}_0 + \delta\,\theta\,(\check{m}_S + \pi) + \delta\,(1-\theta)\,\check{m}_F \geq -\alpha\,c + \check{m}_0 + \delta\,\check{m}_F, \quad \forall\theta \in [\theta_L, \theta_H], \tag{52}$$

where (25) and (26) come from (38) and (39), respectively. Constraints (27) and (52) are derived from the moral hazard constraint (37), ensuring that firms of types below and above $\check{\tau}$ have no incentive to (misreport and) shirk. Constraint (52) is implied by (25) and (27).

The participation constraint (IR) and the feasibility condition for q hold automatically by construction. The nonnegativity condition (12) boils down to (28).

This completes the proof. □ □

Proof. Proof of Proposition 9. It is straightforward to verify the feasibility of our candidate solution. Given $\check{\tau}$, the term $\omega \int_{\check{\tau}}^{\theta_H} \theta\,\mathrm{d}F(\theta)$ is constant with respect to the decision variables. For notational simplicity, we omit this constant from the objective function. The simplified "objective function" is thus

$$\int_{\check{\tau}}^{\theta_H} [-\check{m}_0 - \theta\,\check{m}_S - (1-\theta)\,\check{m}_F]\,\mathrm{d}F(\theta).$$

In case 1, define

$$\phi = -\frac{1}{\delta\,\check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta\,\mathrm{d}F(\theta), \quad \eta = -\left[1 - F(\check{\tau})\right] + \frac{1}{\delta\,\check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta\,\mathrm{d}F(\theta), \text{ and } \nu = (1-\delta)\,\left[1 - F(\check{\tau})\right] \geq 0.$$

The nonnegativity of η follows from its monotonicity in $\check{\tau}$ and

$$\left[-\left[1 - F(\check{\tau})\right] + \frac{1}{\delta\,\check{\tau}} \int_{\check{\tau}}^{\theta_H} \theta\,\mathrm{d}F(\theta)\right]_{\check{\tau}=\theta_H} = 0.$$

We have

$$\phi\,[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F] + \eta\,(\check{m}_0 + \delta\,\check{m}_F) - \nu\,\check{m}_F = (\phi+\eta)\,\check{m}_0 + \delta\,\check{\tau}\,\phi\,\check{m}_S + [\delta\,(1-\check{\tau})\,\phi + \delta\,\eta - \nu]\,\check{m}_F,$$

and hence, the objective function is bounded by $(1 - \delta\,\pi\,\check{\tau})\,\phi + \alpha\,\eta$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 2, define

$$\phi = \frac{1}{\delta\,\check{\tau}} \left[-\delta\,\left[1 - F(\check{\tau})\right] + \int_{\check{\tau}}^{\theta_H} (1-\theta)\,\mathrm{d}F(\theta)\right],$$

$$\eta = \frac{1-\check{\tau}}{\check{\tau}}\,[1-F(\check{\tau})] - \frac{1}{\delta\,\check{\tau}}\int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta),\ \text{and}$$
$$\xi = (1-\delta)\,[1-F(\check{\tau})] \geq 0,$$

where the nonnegativity of η follows from $\mathcal{H}(\check{\tau}) \geq 0$. We have

$$\phi\,[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F] + \eta\,(\check{m}_0 + \delta\,\check{m}_F) - \xi\,\check{m}_S = (\phi+\eta)\,\check{m}_0 + (\delta\,\check{\tau}\,\phi - \xi)\,\check{m}_S + [\delta\,(1-\check{\tau})\,\phi + \delta\,\eta]\,\check{m}_F,$$

and hence, the objective function is bounded by $(1-\delta\,\pi\,\check{\tau})\,\phi + \alpha\,\eta$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 3, define

$$\phi = -\,[1-F(\check{\tau})]\,,$$
$$\xi = -\delta\,\check{\tau}\,[1-F(\check{\tau})] + \int_{\check{\tau}}^{\theta_H}\theta\,\mathrm{d}F(\theta) \geq 0,\ \text{and}$$
$$\nu = -\delta\,(1-\check{\tau})\,[1-F(\check{\tau})] + \int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta),$$

where the nonnegativity of ν follows from $\mathcal{H}(\check{\tau}) \leq 0$. The nonnegativity of ξ follows from that of η in case 1. We have

$$\phi\,[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F] - \xi\,\check{m}_S - \nu\,\check{m}_F = \phi\,\check{m}_0 + (\delta\,\check{\tau}\,\phi - \xi)\,\check{m}_S + [\delta\,(1-\check{\tau})\,\phi - \nu]\,\check{m}_F,$$

and hence, the objective function is bounded by $(1-\delta\,\pi\,\check{\tau})\,\phi$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

In case 4, define

$$\phi = -\frac{1}{\delta\,(1-\check{\tau})}\int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta),$$
$$\gamma = [1-F(\check{\tau})] - \frac{1}{\delta\,(1-\check{\tau})}\int_{\check{\tau}}^{\theta_H}(1-\theta)\,\mathrm{d}F(\theta),\ \text{and}$$
$$\xi = \frac{1}{1-\check{\tau}}\left[\int_{\check{\tau}}^{\theta_H}\theta\,\mathrm{d}F(\theta) - \check{\tau}\,[1-F(\check{\tau})]\right],$$

where the nonnegativity of γ and ξ follows from $\mathcal{H}(\check{\tau}) \geq 0$ and its monotonicity, respectively. We have

$$\phi\,[\check{m}_0 + \delta\,\check{\tau}\,\check{m}_S + \delta\,(1-\check{\tau})\,\check{m}_F] - \gamma\,\check{m}_0 - \xi\,\check{m}_S = (\phi-\gamma)\,\check{m}_0 + (\delta\,\check{\tau}\,\phi - \xi)\,\check{m}_S + \delta\,(1-\check{\tau})\,\phi\,\check{m}_F,$$

and hence, the objective function is bounded by $(1-\delta\,\pi\,\check{\tau})\,\phi$. Substituting our candidate policy into the objective function attains this upper bound, establishing its optimality.

This completes the proof. □ □

F Proofs of Section 4

Proof. Proof of Proposition 10. We prove the desired result in the following steps.

When $\theta \geq \frac{1}{\delta\,\pi}$, ignore constraints (29) and (31), and the optimal solution to the relaxed problem is given by

$$q = 1, \quad m_0 = m_S = m_F = 0,$$

satisfying the two ignored constraints automatically.

Next, we prove the following two lemmas to simplify the problem.

Lemma 8. *Without loss of generality, we can focus on mechanisms satisfying* $m_F = 0$.

Proof. Proof. Given any mechanism $(\hat{q}, \hat{m}_0, \hat{m}_S, \hat{m}_F)$ with $\hat{m}_F > 0$, we propose the following mechanism $(\tilde{q}, \tilde{m}_0, \tilde{m}_S, \tilde{m}_F)$, where

$$\begin{aligned}
&\tilde{q} = \hat{q},\\
&\tilde{m}_0 = \hat{m}_0 + \delta\,(1-\theta)\,\hat{m}_F,\\
&\tilde{m}_S = \hat{m}_S,\\
&\tilde{m}_F = 0.
\end{aligned}$$

The new mechanism is feasible and generates a (weakly) higher expected payoff for the principal. This completes the proof. □ □

Lemma 9. *When* $\theta < \frac{1}{\delta\,\pi}$, *the agent's expected payoff is always* 0.

Proof. Proof. Suppose the agent's expected payoff is strictly positive when $\theta < \frac{1}{\delta\,\pi}$. The principal can always lower the compensation levels m's to make (IR) hold with equality:

1. first, if $m_0 > 0$, the principal can always lower m_0 without affecting the moral hazard constraint; and
2. if the agent's expected payoff is still strictly positive after setting $m_0 = 0$, it is always feasible to decrease m_S to $\tilde{m}_S$ to make (IR) hold with equality. The moral hazard constraint is automatically satisfied, because

$$-(1-\alpha)\,q + \delta\,\theta\,(\tilde{m}_S + \pi\,q) \geq -q + \delta\,\theta\,(\tilde{m}_S + \pi\,q) = 0.$$

This completes the proof. □

When $\theta < \frac{1}{\delta\,\pi}$, we can use constraint (IR) to eliminate m_0, i.e.,

$$m_0 = q - \delta\,\theta\,(m_S + \pi\,q).$$

The funder's problem can be reformulated as follows:

$$\begin{aligned}
\max_{q, m_S}\ & (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,\\
\text{subject to}\ & -\left[\delta\,\theta\,\pi - (1-\alpha)\right]\,q - \delta\,\theta\,m_S \leq 0, && (\eta)\\
& -q \leq 0, && (\beta)\\
& q \leq 1, && (\sigma)\\
& (\delta\,\theta\,\pi - 1)\,q + \delta\,\theta\,m_S \leq 0, && (\gamma)
\end{aligned}$$

$$-m_S \leq 0, \tag{ξ}$$

where the Greek letters at the end of each inequality represent the shadow prices associated with this constraint.

When $\max\left\{\frac{1}{\omega+\delta\,\pi}, \frac{1-\alpha}{\delta\,\pi}\right\} \leq \theta \leq \frac{1}{\delta\,\pi}$, define

$$\sigma \equiv \theta\,\omega + \delta\,\theta\,\pi - 1, \quad \xi \equiv \theta\,(1-\delta).$$

It is straightforward to verify that both σ and ξ are nonnegative. We have

$$\sigma \cdot q - \xi \cdot m_S = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by

$$\sigma = \theta\,\omega + \delta\,\theta\,\pi - 1.$$

Plugging our conjectured solution into the objective function yields the upper bound, establishing the optimality.

When $\frac{\delta\,\alpha-\alpha+1}{\delta\,(\omega+\pi)} \leq \theta \leq \frac{1-\alpha}{\delta\,\pi}$,define

$$\eta \equiv \frac{1-\delta}{\delta}, \quad \sigma \equiv \frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right].$$

It is straightforward to verify that both η and σ are nonnegative. We have

$$\eta \cdot \left[-\left[\delta\,\theta\,\pi - (1-\alpha)\right]\,q - \delta\,\theta\,m_S\right] + \sigma \cdot q = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by

$$\sigma = \frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right].$$

Plugging our conjectured solution into the objective function yields

$$(\theta\,\omega + \delta\,\theta\,\pi - 1) - \theta\,(1-\delta)\left(\frac{1-\alpha}{\delta\,\theta} - \pi\right) = \frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi) - (\delta\,\alpha - \alpha + 1)\right],$$

attaining the upper bound.

When $\theta \leq \frac{1}{\omega+\delta\,\pi}$, define

$$\beta \equiv -\theta\,\omega - \delta\,\theta\,\pi + 1, \quad \xi \equiv \theta\,(1-\delta).$$

It is straightforward to verify that both β and ξ are nonnegative. We have

$$-\beta \cdot q - \xi \cdot m_S = (\theta\,\omega + \delta\,\theta\,\pi - 1)\,q - \theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by 0. Plugging our conjectured solution into the objective function yields the upper bound.

When $\theta \leq \frac{\delta\,\alpha-\alpha+1}{\delta\,(\omega+\pi)}$, define

$$\eta \equiv \frac{1-\delta}{\delta}, \quad \beta \equiv -\frac{1}{\delta}\left[\delta\,\theta\,(\omega+\pi)-(\delta\,\alpha-\alpha+1)\right].$$

It is straightforward to verify that both η and β are nonnegative. We have

$$\eta \cdot \left[-\left[\delta\,\theta\,\pi-(1-\alpha)\right]\,q-\delta\,\theta\,m_S\right]-\beta\cdot q=\left(\theta\,\omega+\delta\,\theta\,\pi-1\right)q-\theta\,(1-\delta)\,m_S,$$

and hence, the objective function is bounded by 0. Plugging our conjectured solution into the objective function yields the upper bound.

This completes the proof. □ □

G No Moral Hazard and No Adverse Selection

For completeness, we present the first-best optimal solution with neither adverse selection nor moral hazard. The funder's problem boils down to

$$\max_{\{q,m_0,m_S,m_F\}\in\Omega_{\mathrm{FB}}} \theta\,\omega\,q-m_0-\theta\,m_S-(1-\theta)\,m_F, \tag{53}$$

where the feasible set Ω_{FB} is determined by (31), (32), and (33). We present the optimal funding mechanism in the following proposition.

Proposition 11. *The following three cases summarize the optimal solution to the funder's problem without asymmetric information* (53)*:*

(a) if $\theta \in [0, c/(\omega+\delta\,\pi))$, it is optimal to not induce testing, i.e.,

$$q=m_0=m_S=m_F=0;$$

(b) if $\theta \in [c/(\omega+\delta\,\pi), \bar{\tau}]$, it is optimal to induce testing with a push amount of $c-\delta\,\theta\,\pi$, i.e.,

$$q=1, \quad m_0=c-\delta\,\theta\,\pi, \quad m_S=m_F=0;$$

(c) if $\theta \in [\bar{\tau}, 1]$, the firm is willing to test without the funder, i.e.,

$$q=1, \quad m_0=m_S=m_F=0.$$

Proof. Proof of Proposition 11. If $\theta \geq \frac{1}{\delta\,\pi}$, the firm is capable of self-funding, and the optimal mechanism is to grant no external funding.

If $\theta < \frac{1}{\delta\,\pi}$, the individual rationality constraint (31) must bind; otherwise, the funder can strictly improve its own payoff by reducing the funding amount. We can solve (31) as an equality for m_0, i.e.,

$$m_0=q-\delta\,\theta\,(m_S+\pi\,q)-\delta\,(1-\theta)\,m_F,$$

and substitute it back into the objective function, yielding

$$\left(\theta\,\omega+\delta\,\theta\,\pi-1\right)q-(1-\delta)\,\theta\,m_S-(1-\delta)\,(1-\theta)\,m_F.$$

If $\theta\,\omega + \delta\,\theta\,\pi - 1 \leq 0$, it is optimal to set $q = 0$, i.e., not to fund the test.

If $\theta\,\omega + \delta\,\theta\,\pi - 1 > 0$, ignoring the nonnegativity constraint on m_0, it is optimal to set $q = 1$, $m_S = m_F = 0$, and the resulting $m_0 = 1 - \delta\,\theta\,\pi > 0$, satisfying the previously ignored constraint.

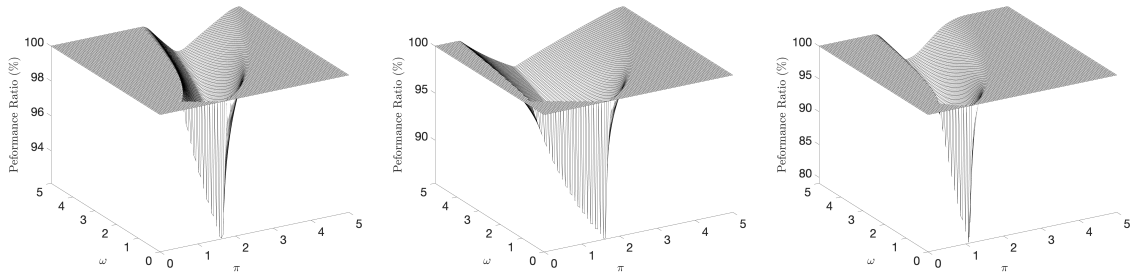
This completes the proof. □ □

When there are no incentive issues, i.e., both the probability of success and the agent's effort are publicly observable, the result is intuitive. The funder induces testing if and only if the expected societal value is higher than the cost, that is, $\theta\,(\omega + \delta\,\pi) \geq c$, corresponding to Cases (b) and (c). In this case, if the firm's expected payoff is insufficient to cover the testing cost (Case (b)), the funder pays the shortfall using push funding to ensure the firm's participation.

H No Reward for Failure

In this section, we quantify the value of allowing the mechanism to reward failure under Assumption 1. That is, consider a benchmark case in which the funder is not allowed to reward failure, i.e., add the constraint $m_F(\theta) = 0$ for all $\theta \in [0, 1]$ to the optimization problem (14). Denote $\hat{u}$ to be the corresponding objective value, in place of u^* in (14). Consider the performance ratio without rewarding failure, i.e.,[13]

$$\text{Performance Ratio} \equiv \frac{\hat{u}}{u^*}.$$



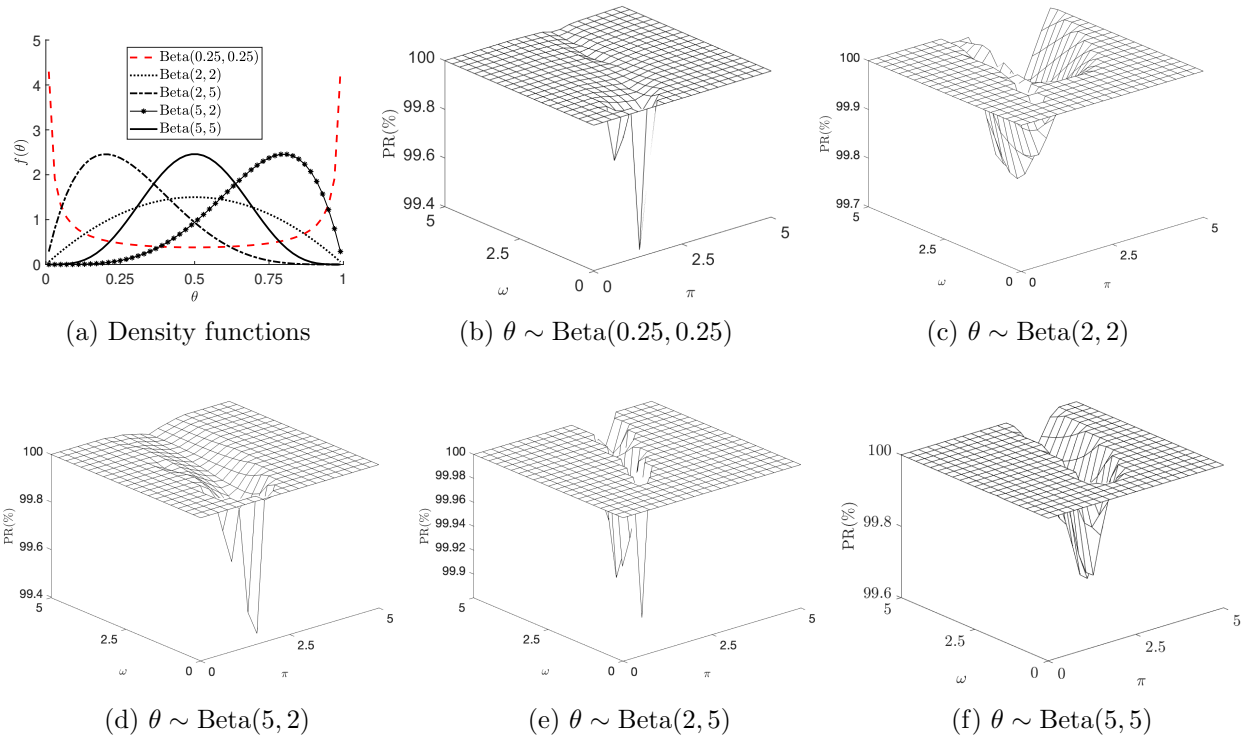
(a) $\alpha = 0.5$ and $\delta = 0.6$ (b) $\alpha = 0.9$ and $\delta = 0.6$ (c) $\alpha = 0.5$ and $\delta = 0.9$

Figure 9: Rerformance ratios (in percentage) when rewarding failure is prohibited under different values of α and δ ($c = 1$ and $\theta \sim \mathcal{U}_{[0,1]}$).

Figures 9(a) and (b) compare the performance ratios for different α values. It appears that the parameter region where the performance ratios are strictly below 100% expands as the value of α increases, and the minimal ratio can be as low as 85% in Figure 9(b). This highlights the importance of considering the possibility of rewarding failure in policy design.

Figures 9(a) and (c) compare the performance ratios under various discount factors δ. As δ increases from 0.6 to 0.9, the minimal performance ratio decreases from 92% in Figure 9(a) to 79% in (c). The intuition is that as the firm discounts the future less (i.e., as δ increases), future payments become more effective, making the reward for failure a more effective tool.

[13]When $u^* = \hat{u} = 0$, for completeness of the figures, we set the performance ratio to 100%.



(a) Density functions (b) $\theta \sim \text{Beta}(0.25, 0.25)$ (c) $\theta \sim \text{Beta}(2, 2)$

(d) $\theta \sim \text{Beta}(5, 2)$ (e) $\theta \sim \text{Beta}(2, 5)$ (f) $\theta \sim \text{Beta}(5, 5)$

Figure 10: Suboptimality of the simple threshold policy when θ is not uniformly distributed.

I Suboptimality of the Take-It-or-Leave-It Mechanism

Figure 10 compares its performance with that of the optimal mechanism numerically. That is, we plot the performance ratio

$$\text{PR} \equiv \frac{\max_{\check{\tau} \in [0,1]} u^{\text{TH}}(\check{\tau})}{u^*}$$

under various values of π and ω. We consider the five Beta distributions in Figure 5. Panels (b) and (c) illustrate that the best take-it-or-leave-it offer achieves at least 99% of the maximal objective value in all cases considered.

J Uncertainty about the Cost of Shirking

Suppose the funder does not have an accurate estimate of the firm's cost of shirking, α, and only knows that it lies within an interval $\alpha \in [\underline{\alpha}, \bar{\alpha}]$. The funder then solves a robust optimization problem that maximizes its expected payoff under the worst-case realization of α:

$$u^{MAXMIN} \equiv \max_{\{q(\cdot), \hat{m}_0(\cdot), \hat{m}_S(\cdot), \hat{m}_F(\cdot)\} \in \hat{\Omega}} \min_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \int_{\theta_L}^{\theta_H} u_F(\theta) \, \mathrm{d}F(\theta), \tag{54}$$

where the feasible set $\hat{\Omega}$ is identical to that defined in Section 2 and is determined by constraints (IC), (IR), (4), and (5).

The following proposition states that the funder's robust optimal strategy is to implement the optimal funding mechanism corresponding to $\alpha = \underline{\alpha}$.

Proposition 12. *The funder's robust optimal mechanism that solves the maximin problem in* (54) *is the same as the optimal mechanism derived under* $\alpha = \underline{\alpha}$.

Proof. Proof of Proposition 12. By the max-min inequality, we have

$$u^{MAXMIN} \leq u^{MINMAX} \equiv \min_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \max_{\{q(\cdot), \hat{m}_0(\cdot), \hat{m}_S(\cdot), \hat{m}_F(\cdot)\} \in \hat{\Omega}} \int_{\theta_L}^{\theta_H} u_F(\theta) \, \mathrm{d}F(\theta).$$

Since the feasible set $\hat{\Omega}$ expands as α increases, nature (the adversary) minimizes the funder's payoff by choosing $\alpha = \underline{\alpha}$. The funder's best response is therefore to implement the optimal funding mechanism corresponding to this worst-case value. The corresponding objective function value serves as an upper bound for u^{MAXMIN}.

Suppose the funder adopts the optimal funding mechanism corresponding to $\alpha = \underline{\alpha}$. For any realization of $\alpha \in [\underline{\alpha}, \bar{\alpha}]$, this mechanism remains feasible: it satisfies all incentive and participation constraints and continues to induce all firms with success rates above the testing threshold τ to participate. Since the funder's objective function does not directly depend on α, the expected payoff remains unchanged across all α in the admissible range. Therefore, this strategy achieves the value of u^{MINMAX}, establishing its optimality in the robust problem.

This completes the proof. □ □

Let $\tilde{u}(\alpha, \hat{\alpha})$ denote the funder's expected payoff when the actual relative cost of shirking is α, while the funder implements the optimal policy derived under $\hat{\alpha}$. The performance ratio is then defined as

$$\text{Performance Ratio} = \frac{\tilde{u}(\alpha, \underline{\alpha})}{\tilde{u}(\alpha, \alpha)} \times 100\%,$$

where the denominator represents the funder's expected payoff when the true value of α is known.

Figure 11 presents the performance ratio under a scenario where the funder applies the mechanism optimized for $\underline{\alpha} = 0.35$, whereas the true cost of shirking is considerably larger ($\alpha = 0.7$). In the lower-left corner, we have $\tilde{u}(\alpha, \underline{\alpha}) = \tilde{u}(\alpha, \alpha) = 0$, so the performance ratio is undefined and thus omitted from the figure. The line indicates the boundary B_5, which separates Regions (6) and (7); to the right of this curve, the commercial market takes over. As shown in the figure, the performance ratio is high for the six representative diseases: it exceeds 80% for dengue, is above 90% for malaria and syphilis, and reaches almost 100% for TB and exactly 100% for HIV. For Trypanosomiasis, the optimal policy is not to induce testing under either value of α.

K Continuous Effort Level

In this section, we show that the assumption of binary effort can be made without loss of generality when the testing cost is linear in the firm's effort level.

Given its private information θ, the firm may exert an intermediate level of effort that yields a lower probability of success $\tilde{\theta} \leq \theta$. We assume that, conditional on the firm's type θ and its chosen

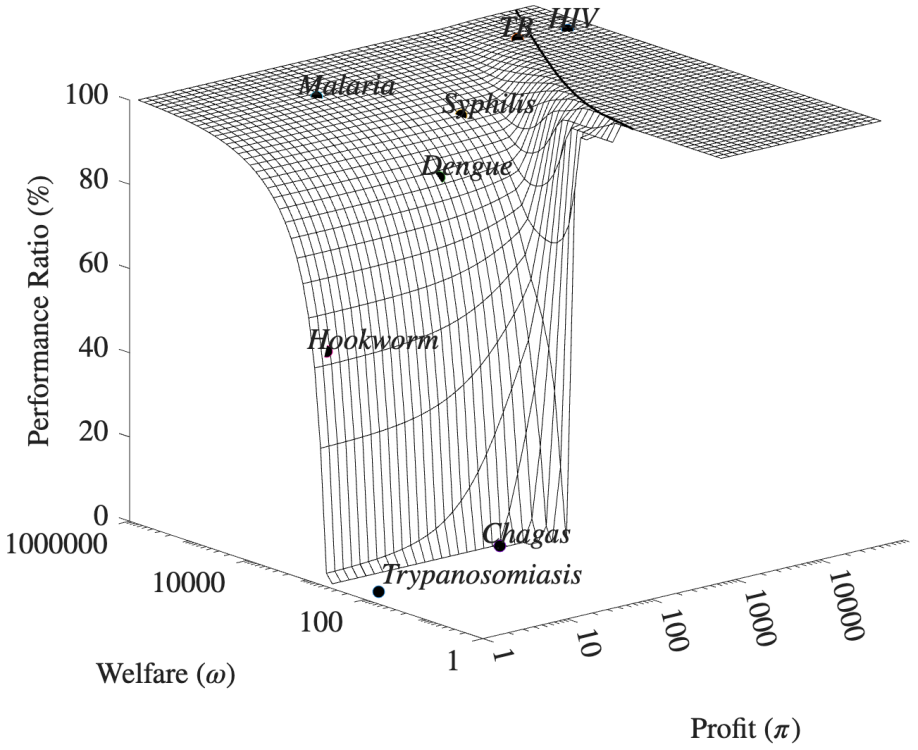


Figure 11: Performance ratio when the funder implements the optimal mechanism for $\underline{\alpha} = 0.35$, while the true value is $\alpha = 0.7$, with $c = 1$ and $\delta = 0.55$.

probability of success $\tilde{\theta}$, the associated cost $\tilde{c}$ is a convex combination of the cost of exerting no effort (αc) and the cost of exerting full effort (c). Formally,

$$\tilde{c}(\theta, \tilde{\theta}) = \frac{\tilde{\theta}}{\theta}\, c + \left(1 - \frac{\tilde{\theta}}{\theta}\right) \alpha\, c.$$

We have $\tilde{c}(\theta, 0) = \alpha\, c$ and $\tilde{c}(\theta, \theta) = c$.

Let $\tilde{u}(\theta, \hat{\theta}, \tilde{\theta})$ denote the firm's expected payoff when its true type is θ, but it reports $\hat{\theta}$ and exerts effort such that the success rate is $\tilde{\theta} \leq \theta$ in the test. We have

$$\tilde{u}(\theta, \hat{\theta}, \tilde{\theta}) \equiv \left[-\tilde{c}(\theta, \tilde{\theta}) + \hat{m}_0(\hat{\theta}) + \delta\, \tilde{\theta}\, \left(\hat{m}_S(\hat{\theta}) + \pi\right) + \delta\, (1 - \tilde{\theta})\, \hat{m}_F(\hat{\theta})\right]\, q(\hat{\theta}). \tag{55}$$

We focus on mechanisms that induce the firm's full effort. A feasible mechanism must satisfy the incentive compatibility constraint, meaning that a firm with type θ should not benefit from misreporting its type as $\hat{\theta}$ or shirking (i.e., $\tilde{\theta} < \theta$), i.e.,

$$\tilde{u}(\theta, \theta, \theta) \geq \tilde{u}(\theta, \hat{\theta}, \tilde{\theta}), \quad \forall (\theta, \hat{\theta}, \tilde{\theta}) \in \Theta^3 \ \text{ with } \ \tilde{\theta} \leq \theta, \tag{56}$$

where $u_1(\theta, \theta, \theta)$ represents the type θ firm's expected payoff when it reports truthfully and exerts effort.

The following lemma establishes that, without loss of generality, it suffices to restrict the right-hand side of (56) to (2) and (3).

Lemma 10. *The incentive compatibility constraint* (56) *holds if and only if condition* (IC) *is satisfied.*

Proof. Proof of Lemma 10 The proof of the necessary part ((56) $\implies$ (36) and (37)) is straightforward. Setting $\tilde{\theta} = \theta$ and 0 on the right-hand side of (56) yields (36) and (37), respectively.

It remains to prove the sufficient part ((36) and (37) $\implies$ (56)). Multiplying constraint (36) by $\frac{\tilde{\theta}}{\theta}$ and constraint (37) by $1 - \frac{\tilde{\theta}}{\theta}$, and then summing the two, yields

$$\begin{aligned}
&- c\, q(\theta) + m_0(\theta) + \delta\, \theta\, [m_S(\theta) + \pi\, q(\theta)] + \delta\, (1 - \theta)\, m_F(\theta) \\
&\geq \frac{\tilde{\theta}}{\theta}\, \left[-c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\, \theta\, \left[m_S(\hat{\theta}) + \pi\, q(\hat{\theta})\right] + \delta\, (1 - \theta)\, m_F(\hat{\theta})\right] + \left(1 - \frac{\tilde{\theta}}{\theta}\right)\, \left[-\alpha\, c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta\, m_F(\hat{\theta})\right] \\
&= \tilde{u}(\theta, \hat{\theta}, \tilde{\theta}),
\end{aligned}$$

which completes the proof. □ □

L Funder's Discount Rate

Let δ_0 and δ_1 denote the funder's and the firm's discount rates. Given the committed mechanism

$$\mathcal{M} \equiv \{q(\theta), m_0(\theta), \tilde{m}_S(\theta), \tilde{m}_F(\theta)\},$$

the payoff functions of both parties can be written as follows:

$$\begin{aligned}
u_F(\theta) &= -\, m_0(\theta) + \delta_0\, \theta\, [\tilde{\omega}\, q(\theta) - \tilde{m}_S(\theta)] - \delta_0\, (1-\theta)\, \tilde{m}_F(\theta), \\
u_1(\theta, \hat{\theta}) &= -\, c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta_1\, \theta\, \left[\tilde{m}_S(\hat{\theta}) + \tilde{\pi}\, q(\hat{\theta})\right] + \delta_1\, (1-\theta)\, \tilde{m}_F(\hat{\theta}), \\
u_0(\hat{\theta}) &= -\, \alpha\, c\, q(\hat{\theta}) + m_0(\hat{\theta}) + \delta_1\, \tilde{m}_F(\hat{\theta}).
\end{aligned}$$

Let

$$\delta \equiv \frac{\delta_1}{\delta_0}, \quad \omega \equiv \delta_0\, \tilde{\omega}, \quad \pi \equiv \delta_0\, \tilde{\pi}, \quad m_S(\theta) \equiv \delta_0\, \tilde{m}_S(\theta), \quad m_F(\theta) \equiv \delta_0\, \tilde{m}_F(\theta).$$

The payoff functions can be expressed as (9), (10), and (11), in Section 2. Therefore, without loss of generality, we normalize the funder's discount factor to 1.