Market Access, Competition, and Dynamic Innovation Traps with Non-Convex R&D*

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Abstract

How does market access affect innovation when R&D is non-convex? We develop an infinite-horizon model with endogenous entry, heterogeneous fixed R&D costs, and a recursive value of successful innovation. Market access raises frontier rents by expanding demand, but it also attracts entry and compresses those rents through business stealing. We first characterize the recursive Bellman-entry block and provide primitive conditions under which it is well behaved. We then show that, when the induced transition surplus is negative near the boundaries and positive at some interior state, the model features a stable corner trap, an unstable interior threshold, and at least one high-innovation steady state. We divide the comparative static with respect to market access into two parts. First, we derive an exact elasticity condition under which the current-flow effect reverses sign. Second, we establish a separate local condition under which the unstable threshold tightens. In the exact-three regime, temporary adverse shocks can have permanent effects once they push the economy below the unstable threshold. We also show, in a separate conditional subsidy proposition, that bounded R&D support can eliminate the interior crossing.

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"This process of Creative Destruction is the essential fact about capitalism."

— — Joseph A. Schumpeter, Capitalism, Socialism and Democracy, Part II, Chapter VII

1 Introduction

Why do some economies drift gradually while others appear to fall abruptly into persistent innovative stagnation? In standard models with convex R&D, weaker market access or harsher competitive conditions typically reduce innovation smoothly. Yet many slowdowns look more fragile than that. Once frontier investment falls far enough, recovery often remains weak even after the original disturbance is partly reversed.

This paper argues that non-convex R&D can change the comparative statics of market access in a sharply non-monotone way. Market access has two opposing effects on innovative activity. A larger accessible market raises the rents available to successful innovators. At the same time, better access attracts entry and compresses those rents through competition. With convex R&D, the balance between these forces mainly changes levels. With lumpy R&D, the same balance can instead determine whether firms innovate at all, whether the innovation ecosystem replenishes itself, and whether the economy remains above a critical threshold.

The model is deliberately built around that interaction. In each period a continuum of potential firms decides whether to enter. Entry is endogenous because the option value of innovation affects the zero-profit condition. After entry, firms draw heterogeneous fixed costs of R&D. A successful innovator earns a frontier rent that depends positively on market access and on the current innovation state, but negatively on the amount of entry-induced competition. Because the value of success is recursive, current innovation choices feed into future profitability through the law of motion for the innovation state. This joint determination of entry, innovation, and future rents is the main upgrade relative to a purely reduced-form threshold model.

The paper delivers three main results. First, equilibrium entry and innovative activity

are jointly pinned down by a recursive frontier-rent equation and a free-entry condition. Heterogeneous fixed costs imply that aggregate innovation varies continuously with the state even though each individual firm still faces a non-convex investment problem; see equation (4) and Proposition 5.1. Second, the model yields a trap region once the transition surplus $F_A(z) = T_A(z) - z$ is negative near the boundaries and positive at some interior state. In that case the zero-capability state is locally stable, there is an unstable interior threshold, and there is at least one high-innovation steady state. Third, market access is non-monotone only conditionally. Larger markets can tighten a local unstable branch when the induced competition effect dominates the direct market-size effect in current flow and when the local continuation component moves in the same direction. In the exact-three regime, temporary adverse shocks that push the economy into the lower basin then have permanent consequences, whereas a separate conditional subsidy result shows how sufficiently strong bounded R&D support can remove the interior threshold.

This paper sits at the intersection of work on competition and innovation, trade and dynamic growth, and threshold models of development. On the competition side, Aghion et al. (2005) and Vives (2008) study how rivalry reshapes innovation incentives. On the trade side, Rivera-Batiz and Romer (1991), Melitz and Ottaviano (2008), and Sampson (2016) connect market size and selection to innovation outcomes. On the trap side, Murphy et al. (1989), Krugman (1991), and Howitt and Mayer-Foulkes (2005) emphasize threshold dynamics and stagnation. Section 2 discusses these connections in more detail. The difference here is that the trap comes from the interaction of market access, endogenous entry, and non-convex R&D inside a recursive innovation system rather than from a generic big-push externality.

The paper is therefore not organized around sunspots or strong-form equilibrium indeterminacy. Its core claim is narrower and, for the present model, more defensible: the interaction of market access, competition, and lumpy R&D generates unstable thresholds, multiple steady states, and persistent trap dynamics. Because Assumption 4.2 delivers a unique Bellman fixed point R_A for each admissible A, the multiplicity studied here concerns steady states

of the one-dimensional transition map rather than multiple continuation values at a given state. Expectation-driven fluctuations are left to future extensions once the baseline recursive structure is fully pinned down.

Road map. Section 2 positions the paper in the literature. Section 3 then previews the main results. Section 4 presents the environment and equilibrium definition. Section 5 develops the recursive mechanism and transition map. Section 6 states the formal results on trap regions, market access, and hysteresis. Section 7 discusses policy implications, and Section 8 concludes. A detailed appendix collects the derivations, full proofs, and benchmark cases.

2 Related Literature

The paper sits at the intersection of three literatures.

The first is the literature on competition and innovation. The Schumpeter–Arrow tension already suggests that competition can either erode innovation rents or induce firms to innovate in order to escape rivalry; see Schumpeter (1942) and Arrow (1962). Aghion and Howitt (1992) places that logic in a creative-destruction growth model. Aghion et al. (2005) and Vives (2008) show how the same force can generate non-monotone or inverted-U links between competitive pressure and innovative effort. Klette and Kortum (2004), Acemoglu and Linn (2004), Acemoglu et al. (2018), and Akcigit and Kerr (2018) place these forces in dynamic environments with firm heterogeneity, reallocation, and growth. More recent work by Aghion et al. (2023), De Loecker et al. (2020), and De Ridder (2024) ties innovation incentives to rents, market power, and the equilibrium distribution of profits. The present paper does not revisit the inverted-U in a reduced-form competition index. Instead, it studies how market access changes entry and how entry changes the dynamic value of innovation when firms face lumpy R&D costs.

The second is the literature on market access, trade, and growth. Romer (1990), Rivera-

Batiz and Romer (1991), Grossman and Helpman (1991a), and Grossman and Helpman (1991b) establish the basic market-size logic: larger markets raise the return to innovation. Melitz (2003), Melitz and Ottaviano (2008), Atkeson and Burstein (2010), and Sampson (2016) show how trade exposure reshapes selection, entry, markup pressure, and innovation incentives. Buera and Oberfield (2020) and Perla et al. (2021) extend that logic to the international diffusion of ideas. On the empirical side, Bloom et al. (2016) and Coelli et al. (2022) show that trade liberalization can induce innovation and technology upgrading. The result in Arkolakis et al. (2018) is especially relevant for the present paper because it shows that pro-competitive effects of trade are theoretically elusive rather than mechanical. The broader scale-effects debate, including Jones (1995) and Young (1998), is also relevant because the present mechanism is about local threshold reversals rather than balanced-growth scale effects. This paper keeps the market-size logic of the trade-and-growth literature but adds an endogenous entry margin that can overturn it locally.

The third is the literature on multiple steady states, coordination failure, and development traps. Murphy et al. (1989), Azariadis and Drazen (1990), Cooper and John (1988), and Krugman (1991) emphasize complementarities, threshold effects, and history dependence. Galor and Zeira (1993), Acemoglu and Zilibotti (1997), and Howitt and Mayer-Foulkes (2005) show how similar nonlinearities can generate persistent development differences and stagnation clubs. The paper belongs to that tradition in spirit, but the underlying force is different. The trap here is not driven by a generic aggregate-production externality or incomplete credit markets. It arises because market access, entry, and non-convex R&D jointly determine whether the innovation ecosystem replenishes itself.

3 Result Preview

Before turning to the formal model, it is useful to preview the paper's logic.

First, the model replaces the reduced-form innovation surplus in the earlier draft with a

recursive frontier-rent object and a parameterized product-market block. Entry and innovation are jointly determined. More profitable innovation raises the option value of entry; more entry compresses frontier rents; and the resulting mass of successful innovators determines next period's innovation state.

Second, heterogeneous fixed costs convert the firm-level non-convex R&D problem into a smooth aggregate innovation schedule; formally, $m_A(z) = q\, n_A(z)\, G(qR_A(z))$ in equation (4). This generates an endogenous transition map

$$T_A(z) = (1-\delta)z + \lambda m_A(z)$$

that can display a three-crossing trap region rather than a mechanically piecewise threshold. In the benchmark economy, that shape is tied to three primitive objects: a complementarity term z^α in frontier rents, a hump-shaped density of firms at the R&D margin, and entry-mediated business stealing. The formal three-crossing result remains conditional on the interior-hump requirement in Definition 5.3 and Theorem 6.1.

Third, market access does not simply shift the innovation schedule upward. It raises gross rents, but it also attracts entry and compresses those rents. The unstable threshold therefore moves non-monotonically only under an explicit two-step condition: current-flow rents reverse when $\varepsilon_A^n(z^M) > 1/\tau$, and the threshold itself tightens only if the full transition map shifts downward at the unstable branch.

Fourth, the paper is organized around trap dynamics and hysteresis. A temporary deterioration in market access can permanently move the economy to the low-innovation basin if it pushes the state below the unstable threshold. Conversely, sufficiently strong reforms or support-preserving R&D subsidies, meaning interventions that keep the effective innovation cutoff inside the original support of fixed costs, can remove the interior crossing in the conditional policy experiment studied in Section 7.

4 Environment

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Market access is summarized by a scalar $A > 0$, which can be interpreted as inverse trade costs or effective external demand. The aggregate innovation state is $z_t \in [0, 1]$. A higher z_t means that frontier implementation capabilities, specialized suppliers, and complementary know-how are more abundant.

Notation. The central equilibrium objects are: market access A, innovation state z, entry mass $n_A(z)$, frontier rent $R_A(z)$, successful innovation flow $m_A(z)$, and transition map $T_A(z)$.

4.1 A microfounded benchmark

At the start of period t, a continuum of potential firms decides whether to enter after observing (A, z_t). Entering requires paying a sunk cost $\kappa > 0$. A unit mass of consumers has CES demand over active varieties, and the reduced-form expenditure term A scales total spending in the accessible market. If n firms enter, the symmetric operating profit of a non-innovating producer is

$$\pi^0(A, n) = \mu A n^{-1},$$

where $\mu > 0$ is the operating-profit share implied by the product-market block.

After entering, each firm draws a fixed R&D cost $f \in [0, \bar{f}]$. A firm that attempts frontier innovation succeeds with probability $q \in (0, 1)$. Conditional on success, the firm becomes a temporary frontier leader in its product line. The current flow rent of that frontier position is

$$\pi^F(A, z, n) = \chi A z^{\alpha} n^{-\tau}, \tag{1}$$

where $\chi > 0$, $\alpha > 1$, and $\tau \in (0, 1]$. The exponent $\alpha > 1$ is a reduced-form complementarity parameter. One heuristic interpretation is that successful implementation requires a bundle of complementary inputs whose joint availability rises more than linearly with z. I do not require α to be an integer, so the task language is mnemonic rather than literal. The trap

results should therefore be read as conditional on this reduced-form complementarity, not as a full microfoundation of implementation frictions. The parameter τ captures business stealing. When entry rises, frontier rents are compressed at elasticity τ.

Let $R_A(z)$ denote the continuation value of holding the frontier at the beginning of a period when the aggregate state is z. The recursive frontier-rent equation is

$$R_A(z) = \chi A z^{\alpha} n_A(z)^{-\tau} + \beta R_A(T_A(z)). \tag{2}$$

The fixed-cost distribution is specialized to a rescaled Beta$(2,2)$ benchmark with a cubic CDF,

$$G(f) = 3\left(\frac{f}{\bar{f}}\right)^2 - 2\left(\frac{f}{\bar{f}}\right)^3, \qquad g(f) = \frac{6f(\bar{f}-f)}{\bar{f}^3}, \qquad f \in [0, \bar{f}]. \tag{3}$$

This choice is useful because the density is zero at both boundaries. Aggregate innovative activity is therefore naturally flat when frontier rents are either extremely weak or already high enough to cover almost every cost draw. Nothing essential depends on Beta$(2,2)$ per se beyond tractability and the zero-boundary density property shared by many interior-peaked distributions. The benchmark is chosen because it yields closed-form option values and keeps the fixed-cost margin transparent.

A firm with cost draw f attempts innovation if and only if

$$qR_A(z) \geq f.$$

Hence the mass of successful innovators at state z is

$$m_A(z) = q\, n_A(z)\, G(qR_A(z)). \tag{4}$$

The aggregate innovation state evolves according to

$$T_A(z) = (1-\delta)z + \lambda m_A(z), \tag{5}$$

where $\delta \in (0,1)$ is the decay rate of frontier capability and $\lambda > 0$ converts successful innovation into next period's state.

The ex ante value of entry includes the option value of drawing a low fixed cost. Under (3),

$$\Omega(x) = \int_0^x (x - f)\, \mathrm{d}G(f).$$

For $x \in [0, \bar{f}]$ this becomes

$$\Omega(x) = \frac{x^3}{\bar{f}^2} - \frac{x^4}{2\bar{f}^3}. \tag{6}$$

Then the free-entry condition is

$$A\mu n_A(z)^{-1} + \Omega(qR_A(z)) = \kappa. \tag{7}$$

Equations (2)–(7) jointly determine entry, innovation, and state dynamics.

For any $L > 0$, let

$$\mathcal{X}_L := \left\{ R : [0,1] \to \left[0, \frac{\bar{f}}{q}\right] : R \text{ is increasing and } L\text{-Lipschitz} \right\}.$$

Assumption 4.1 (Benchmark primitives)**.** The following conditions hold.

1. $\mu, \chi, \kappa, \bar{f}, \lambda > 0$, $\alpha > 1$, $\tau \in (0,1]$, and $q, \beta, \delta \in (0,1)$.

2. $\kappa > \Omega(\bar{f}) = \bar{f}/2$, so free entry remains well defined for every innovation rent in $[0, \bar{f}/q]$.

3. $$\frac{q\chi A n_{\min}(A)^{-\tau}}{1-\beta} \le \bar{f}, \qquad n_{\min}(A) = \frac{\mu A}{\kappa},$$

   so equilibrium frontier rents remain inside the support of G.

4. $$\lambda q n_{\max}(A) \le \delta, \qquad n_{\max}(A) = \frac{\mu A}{\kappa - \bar{f}/2},$$

   so the transition map remains in $[0,1]$.

Items 2 and 3 work jointly. Item 2 guarantees that the free-entry denominator stays positive once item 3 keeps the equilibrium rent inside the support of the fixed-cost distribution.

Assumption 4.2 (Recursive Regularity)**.** For every fixed A in the admissible range, there exists $L_A < \infty$ such that the Bellman-entry operator

$$(\mathcal{B}_A R)(z) = \chi A z^{\alpha} n(R, z; A)^{-\tau} + \beta R((1-\delta)z + \lambda q\, n(R, z; A) G(qR(z)))$$

maps $\mathcal{X}_{L_A}$ into itself and is a contraction in the sup norm, where

$$n(R, z; A) = \frac{\mu A}{\kappa - \Omega(qR(z))}.$$

Appendix A.3 gives primitive sufficient conditions for Assumption 4.2. Together with Assumption 4.1, this turns the support and contraction requirements into inequalities on primitives rather than on equilibrium objects.

Call a market-access level A *admissible* if Assumptions 4.1 and 4.2 hold at that value. Unless stated otherwise, the equilibrium-existence and dynamic results below are stated for admissible A.

Proposition 4.3 (Primitive sufficient conditions for a nonempty admissible region)**.** *Suppose there exist $A_0 > 0$ and $L_0 > 0$ such that the inequalities in Lemma A.3 hold strictly at (A_0, L_0). Then there exists an open interval $\mathcal{A}$ containing A_0 such that every $A \in \mathcal{A}$ is admissible. In particular, on $\mathcal{A}$ the Bellman-entry operator maps $\mathcal{X}_{L_0}$ into itself and is a contraction.*

Proof. The bounds in Lemma A.3 are continuous in A. Strict feasibility at (A_0, L_0) therefore survives on a neighborhood of A_0, which yields the claim. □

Appendix A.3 records the same continuity argument in full.

Definition 4.4 (Markov equilibrium)**.** For a given market-access level A, a *Markov equilibrium* is a triple (R_A, n_A, T_A) such that:

1. R_A satisfies (2);

2. n_A satisfies (7);

3. T_A is generated by (4) and (5).

5 Dynamic Innovation Traps

Proposition 5.1 (Benchmark equilibrium block)**.** *Suppose Assumptions 4.1 and 4.2 hold. Then for every admissible A there exists a unique bounded increasing Markov equilibrium (R_A, n_A, T_A). Moreover:*

1. *the entry rule is explicit:*
$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))};$$

2. *R_A is Lipschitz, while $n_A(z)$ and $m_A(z) = q\, n_A(z)\, G(qR_A(z))$ are continuous and increasing in z;*

3. *at every point where R_A is differentiable, hence almost everywhere,*
$$n'_A(z) = \frac{q\, n_A(z)\, G(qR_A(z))\, R'_A(z)}{\kappa - \Omega(qR_A(z))};$$

4. *at every point where R_A is differentiable, hence almost everywhere,*
$$m'_A(z) = q^2 n_A(z) R'_A(z) \left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))} \right].$$

Proof. The free-entry condition delivers the explicit entry rule. Because $\Omega'(\cdot) = G(\cdot) \geq 0$, higher frontier rents increase the option value of entry and therefore raise $n_A(z)$. Assumption 4.2 then implies that the Bellman operator has a unique fixed point R_A in $\mathcal{X}_{L_A}$, so R_A is increasing and Lipschitz. The associated n_A and m_A are increasing as compositions of

increasing functions. Differentiating the explicit formulas at points where R_A is differentiable yields the derivative identities. □

Appendix A derives the closed-form expressions and Appendix B.1 gives the full proof.

Proposition 5.1 moves the paper away from a black-box transition map. Entry, innovation, and next period's state are linked by explicit primitives: a CES-style operating-profit block, a complementarity parameter α, a business-stealing elasticity τ, and a non-convex cost distribution with closed-form option values.

Proposition 5.2 (Primitive slope decomposition)**.** *Suppose Assumptions 4.1 and 4.2 hold, and let (R_A, n_A, T_A) be the equilibrium from Proposition 5.1. If R_A is differentiable at z, then*

$$T'_A(z) = 1 - \delta + \lambda q^2 n_A(z) R'_A(z) \Xi_A(z), \tag{8}$$

where

$$\Xi_A(z) = g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))}. \tag{9}$$

If z^* is a steady state at which R_A is differentiable and $1 - \beta T'_A(z^*) > 0$, then $R'_A(z^*)$ satisfies the implicit slope equation

$$R'_A(z^*) = \frac{\chi A n_A(z^*)^{-\tau} z^{*\,\alpha-1} \left[\alpha - \tau z^* \frac{n'_A(z^*)}{n_A(z^*)}\right]}{1 - \beta T'_A(z^*)}. \tag{10}$$

Proof. Equation (8) follows by differentiating

$$T_A(z) = (1-\delta)z + \lambda q\, n_A(z) G(qR_A(z))$$

and substituting the formulas for $n'_A(z)$ and $m'_A(z)$ from Proposition 5.1. At a steady state $z^* = T_A(z^*)$, differentiating (2) gives

$$R'_A(z^*) = \chi A \frac{\mathrm{d}}{\mathrm{d}z}\left(z^\alpha n_A(z)^{-\tau}\right)\Big|_{z=z^*} + \beta R'_A(z^*) T'_A(z^*),$$

which yields (10). The equation is implicit because both $n'_A(z^*)/n_A(z^*)$ and $T'_A(z^*)$ depend on $R'_A(z^*)$. □

Appendix A develops the derivative formulas step by step, and Appendix B.2 gives the full proof.

Proposition 5.2 is the paper's central upgrade. The low-state and high-state slopes are no longer imposed directly on $T_A(\cdot)$. They are traced to concrete primitives. The term $z^{\alpha-1}$ captures implementation complementarity, $\Xi_A(z)$ captures how many firms are marginal at the current innovation rent, and the factor $n'_A(z)/n_A(z)$ captures entry-mediated business stealing. The regularity condition $1 - \beta T'_A(z^*) > 0$ is separate from local stability. Because $\beta < 1$, it can hold at both stable and unstable steady states provided $T'_A(z^*) < 1/\beta$.

Three primitive forces now govern the shape of the transition map. First, because $\alpha > 1$, the flow-rent gradient is weak near $z = 0$. Second, the density $g(\cdot)$ is hump-shaped, so intermediate rents bring many firms to the margin of paying the fixed cost. Third, when rents are already high, the density $g(\cdot)$ falls again and business stealing dampens the continuation value of an additional state increase. This decomposition delivers a sharper benchmark than a directly imposed transition-map shape.

Definition 5.3 (Three-crossing trap region)**.** For a given market-access level A, let

$$F_A(z) = T_A(z) - z.$$

The economy is in a *three-crossing trap region* if $F_A(0) = 0$, $F'_A(0) = -\delta < 0$, there exists $z^+ \in (0, 1)$ such that $F_A(z^+) > 0$, and $F_A(1) < 0$.

6 Formal Results

Theorem 6.1 (Conditional characterization of a trap region)**.** *Suppose Assumptions 4.1 and 4.2 hold. Let $F_A(z) = T_A(z) - z$, and assume there exists $z^+ \in (0, 1)$ such that*

$F_A(z^+) > 0$ and $T_A(1) < 1$. Then the economy is in a three-crossing trap region in the sense of Definition 5.3. In particular:

1. *$z = 0$ is a steady state, and it is locally stable because $T'_A(0) = 1 - \delta < 1$;*
2. *there exists at least one interior steady state $z^M(A) \in (0, z^+)$;*
3. *there exists at least one high steady state $z^H(A) \in (z^+, 1)$.*

Any interior steady state with $T'_A(z^M(A)) > 1$ is unstable. Any high steady state with $|T'_A(z^H(A))| < 1$ is locally stable. These are local slope tests only. If F_A has a unique zero in $(0, z^+)$ and a unique zero in $(z^+, 1)$, then these are the only positive steady states.

Proof. At $z = 0$, (4) implies $m_A(0) = 0$, so $T_A(0) = 0$. Proposition 5.2 gives $T'_A(0) = 1 - \delta < 1$ because $G(0) = g(0) = 0$. Hence $z = 0$ is locally stable and $F_A(z) < 0$ on a right neighborhood of the origin. Because $F_A(z^+) > 0$, continuity implies that F_A has at least one zero in $(0, z^+)$. Because $F_A(1) = T_A(1) - 1 < 0$, continuity implies that F_A has at least one zero in $(z^+, 1)$. The local stability and instability claims are the standard one-dimensional slope test evaluated at the relevant crossings. □

Appendix B.3 provides the full argument and spells out the local and global parts separately.

Theorem 6.1 is deliberately a conditional existence result rather than a full exact-three characterization. It shows what the recursive benchmark can establish cleanly once the interior-hump condition is verified: a stable corner trap, at least one interior threshold, and at least one high steady state. The low-state flat region comes from implementation complementarities and the zero density of marginal innovators at the origin; the high-state flattening comes from saturation and business stealing.

Corollary 6.2 (Basin characterization under exact three steady states)**.** *Suppose the hypotheses of Theorem 6.1 hold and that the only steady states are 0, $z^M(A)$, and $z^H(A)$ with*

$0 < z^M(A) < z^H(A) < 1$. *If* $T'_A(z^M(A)) > 1$ *and* $|T'_A(z^H(A))| < 1$, *then every trajectory with* $z_0 < z^M(A)$ *converges to* 0, *while every trajectory with* $z_0 > z^M(A)$ *converges to* $z^H(A)$.

Proof. Proposition 5.1 implies that m_A is increasing, so T_A is increasing. If $z_0 < z^M(A)$, then exact-three steady states imply $T_A(z_0) < z_0$ and $T_A(z_0) \geq 0$. The trajectory is therefore monotone decreasing and bounded below by zero, so it converges to the only fixed point in $[0, z^M(A)]$, namely zero. If $z_0 \in (z^M(A), z^H(A))$, then $z_0 < T_A(z_0) < z^H(A)$, so the trajectory is monotone increasing and bounded above by $z^H(A)$. If $z_0 > z^H(A)$, then $z^H(A) < T_A(z_0) < z_0$, so the trajectory is monotone decreasing and bounded below by $z^H(A)$. In both upper-region cases, the limit must be the unique fixed point $z^H(A)$. □

Appendix B.4 records the same argument in monotone-iteration form.

From this point on, references in the text to "the" unstable threshold or "the" high steady state should be read as shorthand for the exact-three regime of Corollary 6.2. Theorem 6.1 itself remains an existence result with potentially richer positive-state geometry.

This distinction also clarifies what kind of multiplicity the model delivers. For a fixed admissible A, Assumption 4.2 yields a unique continuation-value fixed point R_A, and the deterministic map T_A then generates a unique path from any initial state z_0. The multiplicity studied here is therefore multiplicity of steady states and basins, not path indeterminacy from a given initial condition.

Theorem 6.3 (Market access and competition)**.** *Define* $H(A, z) = T_A(z) - z$. *Suppose the hypotheses of Theorem 6.1 hold on an interval of admissible market-access values, and suppose moreover that for each* z *the equilibrium objects* $R_A(z)$, $n_A(z)$, *and* $T_A(z)$ *are differentiable in* A *on that interval. Let* $z^M(A)$ *denote a local unstable branch satisfying* $H(A, z^M(A)) = 0$ *and* $H_z(A, z^M(A)) = T'_A(z^M(A)) - 1 > 0$ *on that interval. Define the entry elasticity and the frontier-flow-profit elasticity by*

$$\varepsilon^n_A(z) = \frac{\partial \log n_A(z)}{\partial \log A}, \qquad \varepsilon^\pi_A(z) = \frac{\partial \log \pi^F(A, z, n_A(z))}{\partial \log A}. \tag{11}$$

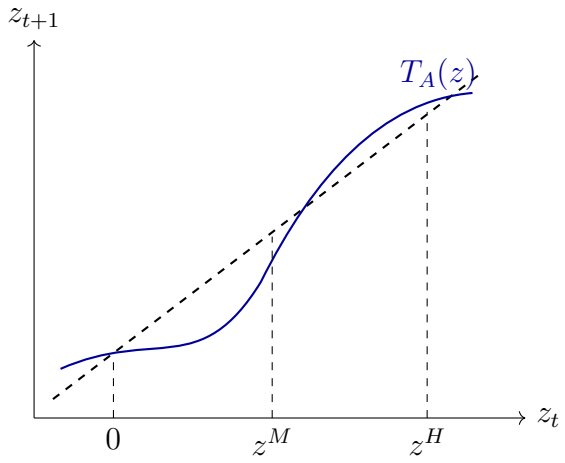


Figure 1: A schematic exact-three transition map. The baseline model has a stable corner steady state at zero, an unstable interior threshold, and a stable high steady state.

Then

$$\varepsilon_A^{\pi}(z) = 1 - \tau \varepsilon_A^{n}(z). \tag{12}$$

In particular, the current-flow frontier rent falls with market access at state z if and only if

$$\varepsilon_A^{n}(z) > \frac{1}{\tau}.$$

If

$$H_A(A, z^M(A)) < 0,$$

then the unstable threshold is locally increasing in market access:

$$\frac{\mathrm{d}z^M(A)}{\mathrm{d}A} > 0.$$

Hence a marginal improvement in market access can shrink the basin of attraction of the high-innovation regime.

Proof. Because $\pi^F(A, z, n) = \chi A z^{\alpha} n^{-\tau}$, log differentiation gives

$$\varepsilon_A^{\pi}(z) = 1 - \tau \frac{\partial \log n_A(z)}{\partial \log A} = 1 - \tau \varepsilon_A^{n}(z).$$

Hence the current-flow part of frontier rent falls with market access exactly when entry responds more than $1/\tau$ in elasticity terms. If, in addition, the full transition map shifts downward locally at the unstable branch so that $H_A(A, z^M(A)) < 0$, then $H_z(A, z^M(A)) > 0$ lets the implicit-function theorem define a local branch $z^M(A)$ and implies that the unstable cutoff rises with market access. □

Appendix B.5 restates the argument using the implicit-function formula for the unstable branch.

Theorem 6.3 separates two objects that were previously mixed together. The elasticity condition pins down exactly when the current-flow rent turns downward with market access. Threshold tightening is stronger: it additionally requires the full transition map to shift downward at the unstable branch, which is what the sign restriction on $H_A(A, z^M(A))$ records. The present paper does not fully reduce that sign restriction to primitives, and it takes local A-regularity of the equilibrium objects as given, so the threshold-tightening statement should be read as a local conditional comparative static rather than as a global characterization.

Corollary 6.4 (No endogenous entry, no competition reversal)**.** *If entry is exogenous so that $n_A(z) \equiv \bar{n}(z)$ does not respond to market access, then $\varepsilon_A^n(z) = 0$ and*

$$\varepsilon_A^\pi(z) = 1.$$

The current-flow frontier rent is therefore increasing in market access at every state, and the threshold-tightening mechanism in Theorem 6.3 disappears.

Proof. Immediate from (12). □

Appendix C.1 embeds this corollary in the benchmark comparison with exogenous entry.

Theorem 6.5 (Hysteresis and trap dynamics)**.** *Fix a baseline market-access level A satisfying Corollary 6.2, and suppose the economy starts from $z_0 > z^M(A)$. Let $\tilde{A} < A$ be a temporary*

shock to market access, and let $T_{\tilde{A}}$ denote the associated transition map. Suppose there exists $\eta > 0$ such that

$$T_{\tilde{A}}(z) \leq z - \eta \qquad \text{for every } z \in [z^M(A), z_0].$$

Define

$$m^\star(z_0; A, \tilde{A}) = \min\left\{m \in \mathbb{N} : T_{\tilde{A}}^m(z_0) < z^M(A)\right\}.$$

Then $m^\star(z_0; A, \tilde{A}) \leq \lceil (z_0 - z^M(A))/\eta \rceil < \infty$. If the shock lasts for at least $m^\star(z_0; A, \tilde{A})$ periods and market access is then restored to A, the economy converges to the corner trap at $z = 0$.

Proof. Under the temporary shock, the state evolves according to $T_{\tilde{A}}$. The uniform drift condition implies

$$z_{t+1} \leq z_t - \eta \qquad \text{whenever } z_t \in [z^M(A), z_0],$$

so the crossing time is finite and bounded by the displayed ceiling formula. By construction, after $m^\star$ periods the economy lies below the baseline unstable threshold. Once market access returns to A, Corollary 6.2 implies that the post-shock trajectory converges to zero. □

Appendix B.7 gives the full basin-crossing proof.

Theorem 6.5 formalizes the idea of a persistent innovation slowdown without hiding the crossing condition. Temporary disruptions matter permanently when they push the economy across the unstable threshold and into the lower basin. The drift condition is a propagation condition on the shocked transition map, not a primitive characterization of shock size.

7 Policy Implications

The model is informative about threshold movements in the transition map, not about welfare rankings.

Small access reforms need not enlarge the high-innovation basin. In the trap region, a marginal opening reform raises gross frontier rents but can also increase entry enough to raise

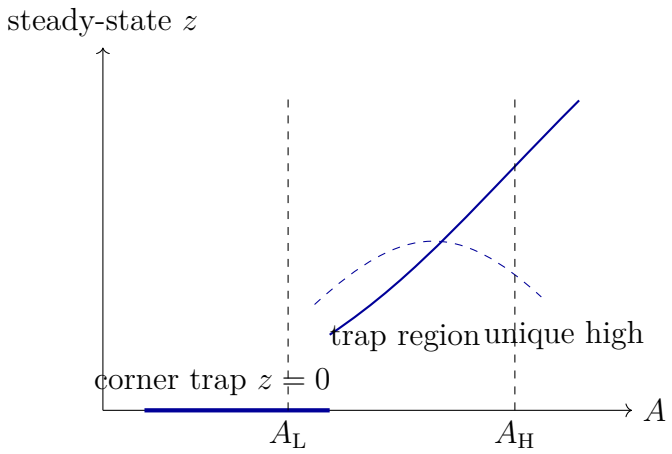


Figure 2: A schematic bifurcation diagram. For intermediate market access the economy exhibits a corner trap, a high stable branch, and an unstable branch in between.

the unstable threshold. The relevant positive question is therefore not whether market access raises expected profits on average, but whether it shifts the economy into the high-innovation basin.

R&D support instead acts directly on the fixed-cost margin. If a subsidy $s \geq 0$ lowers the effective fixed cost from f to $f - s$, then the innovation rule becomes

$$f \leq qR_A(z) + s.$$

The successful innovation flow becomes

$$m_{A,s}(z) = q\, n_{A,s}(z)\, G(qR_{A,s}(z) + s),$$

which creates a direct upward pressure on the innovation flow. Whether the full recursive transition map shifts upward is a separate equilibrium question, so the proposition below states that monotonicity as an explicit condition rather than deriving it from primitives here.

Call a subsidy *support-preserving* if it keeps the effective innovation cutoff inside the

original support of fixed costs:

$$qR_{A,s}(z) + s \in [0, \bar{f}] \qquad \text{for every } z \in [0,1].$$

This restriction keeps the subsidized economy inside the same admissible state space as the baseline model.

Proposition 7.1 (Conditional threshold removal under support-preserving subsidies)**.** *Fix a market-access level A in the exact-three regime of Corollary 6.2. Suppose there exists $\bar{s} > 0$ such that for every $s \in [0, \bar{s}]$ the subsidized equilibrium is well defined, satisfies $qR_{A,s}(z) + s \in [0, \bar{f}]$ for all $z \in [0,1]$, and induces a transition map $T_{A,s}$ that is continuous in (z, s) and weakly increasing in s. If, for some $\bar{s}$, the map $T_{A,\bar{s}}$ has no interior fixed point in $(0,1)$, then there exists $s^\star(A) \in [0, \bar{s}]$ such that every $s \geq s^\star(A)$ eliminates the interior threshold. If, in addition, $T_{A,s}$ has a unique positive fixed point for all $s \geq s^\star(A)$, every initial condition $z_0 > 0$ converges to that high-innovation steady state.*

Proof. Because G is strictly increasing, a larger subsidy raises the static set of firms that find innovation privately worthwhile at a given continuation value. The support-preservation assumption ensures that the subsidized recursive block remains inside the same admissible state space. Continuity in (z, s) and monotonicity in s imply that crossings can disappear only through a finite threshold level. Once the interior threshold is gone, only the remaining positive fixed point governs dynamics away from zero. □

Appendix B.8 provides the complete continuity argument.

This proposition is a conditional statement about how subsidies reshape the transition map. The paper does not provide a welfare ranking of subsidies, access reforms, or policy bundles.

8 Conclusion

This paper develops a parsimonious theory of innovation traps under non-convex R&D. Market access raises the scale of innovation rents but also attracts entry, and entry compresses those rents. Once innovation requires crossing a fixed cost, the interaction of these two forces generates thresholds, multiple steady states, and history dependence. In the baseline theorem, the lower regime is anchored by a stable corner state at zero and the upper regime by at least one high-innovation steady state.

The paper's main contribution is to show that this mechanism can be organized as a recursive equilibrium with endogenous entry and a continuous aggregate innovation schedule. The resulting transition map can display a three-crossing trap region with a corner trap, an unstable interior threshold, and at least one high steady state. In the exact-three regime, temporary shocks have permanent consequences, and small reforms can be ineffective unless they move the economy across the threshold.

The framework is intentionally stylized. It is not yet a quantitative calibration, and it does not deliver welfare rankings. A natural next step is therefore to discipline the key parameters with data, especially the entry elasticity and the location of the unstable threshold, and to enrich the product-market environment enough to derive sharper normative results. Another is to study whether the same recursive structure can support genuinely expectation-driven fluctuations once local indeterminacy is introduced endogenously rather than by assumption. For the present paper, however, the central message is already clear: market access, competition, and non-convex R&D together can generate persistent innovation traps.

The appendix develops the underlying derivations, supplies full proofs, and records benchmark cases that clarify where the mechanism does and does not operate.

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A Recursive Block and Core Derivations

A.1 Closed-form Option Value

This appendix begins from the rescaled Beta$(2,2)$ benchmark

$$G(f) = 3\left(\frac{f}{\bar{f}}\right)^2 - 2\left(\frac{f}{\bar{f}}\right)^3, \qquad g(f) = \frac{6f(\bar{f}-f)}{\bar{f}^3}, \qquad f \in [0,\bar{f}].$$

The corresponding option value is

$$\Omega(x) = \int_0^x (x-f)\,\mathrm{d}G(f), \qquad x \in [0,\bar{f}].$$

The rescaled Beta$(2,2)$ benchmark is chosen for tractability and for its zero boundary density. The boundary-flatness argument does not rely on uniqueness of that distributional choice.

Lemma A.1 (Closed-form option value)**.** *For every $x \in [0,\bar{f}]$,*

$$\Omega(x) = \frac{x^3}{\bar{f}^2} - \frac{x^4}{2\bar{f}^3}.$$

Moreover,

$$\Omega'(x) = G(x), \qquad \Omega(\bar{f}) = \frac{\bar{f}}{2}.$$

Proof. Using the density g, write

$$\Omega(x) = \int_0^x (x-f)g(f)\,\mathrm{d}f = x\int_0^x g(f)\,\mathrm{d}f - \int_0^x fg(f)\,\mathrm{d}f.$$

Because $\int_0^x g(f)\,\mathrm{d}f = G(x)$, the first term is

$$xG(x) = 3\frac{x^3}{\bar{f}^2} - 2\frac{x^4}{\bar{f}^3}.$$

For the second term,

$$\int_0^x f g(f)\,\mathrm{d}f = \int_0^x \left(6\frac{f^2}{\bar{f}^2} - 6\frac{f^3}{\bar{f}^3}\right)\mathrm{d}f = 2\frac{x^3}{\bar{f}^2} - \frac{3}{2}\frac{x^4}{\bar{f}^3}.$$

Subtracting yields

$$\Omega(x) = \left(3\frac{x^3}{\bar{f}^2} - 2\frac{x^4}{\bar{f}^3}\right) - \left(2\frac{x^3}{\bar{f}^2} - \frac{3}{2}\frac{x^4}{\bar{f}^3}\right) = \frac{x^3}{\bar{f}^2} - \frac{x^4}{2\bar{f}^3}.$$

Differentiating gives

$$\Omega'(x) = 3\frac{x^2}{\bar{f}^2} - 2\frac{x^3}{\bar{f}^3} = G(x),$$

and evaluating at $x = \bar{f}$ yields $\Omega(\bar{f}) = \bar{f}/2$. □

A.2 Entry Rule and Derivative Identities

The free-entry condition is

$$A\mu n_A(z)^{-1} + \Omega(qR_A(z)) = \kappa.$$

Because the left-hand side is strictly decreasing in $n_A(z)$, the entry rule can be solved explicitly.

Lemma A.2 (Entry and innovation derivatives)**.** *Suppose $qR_A(z) \in [0, \bar{f}]$ and R_A is differentiable at z. Then*

$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))}.$$

In addition,

$$n'_A(z) = \frac{q\, n_A(z)\, G(qR_A(z))\, R'_A(z)}{\kappa - \Omega(qR_A(z))}$$

and

$$m'_A(z) = q^2 n_A(z) R'_A(z)\left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))}\right].$$

Proof. Rearranging free entry immediately gives

$$n_A(z) = \frac{\mu A}{\kappa - \Omega(qR_A(z))}.$$

Differentiate this expression with respect to z:

$$n'_A(z) = \mu A \frac{\Omega'(qR_A(z))qR'_A(z)}{[\kappa - \Omega(qR_A(z))]^2}.$$

Using Lemma A.1, $\Omega'(x) = G(x)$, so

$$n'_A(z) = \frac{\mu A q G(qR_A(z))R'_A(z)}{[\kappa - \Omega(qR_A(z))]^2} = \frac{q\, n_A(z)\, G(qR_A(z))\, R'_A(z)}{\kappa - \Omega(qR_A(z))}.$$

Now differentiate

$$m_A(z) = q\, n_A(z)\, G(qR_A(z)) :$$

$$m'_A(z) = qn'_A(z)G(qR_A(z)) + q^2 n_A(z) g(qR_A(z))R'_A(z).$$

Substituting the formula for $n'_A(z)$ yields

$$m'_A(z) = q^2 n_A(z)R'_A(z)\left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))}\right].$$

□

A.3 A Primitive Contraction Check

Assumption 4.2 packages the recursive fixed-point argument into a single regularity condition. This subsection gives one convenient sufficient condition. For a fixed market-access level A, define the class

$$\mathcal{X}_L = \left\{ R : [0,1] \to \left[0, \frac{\bar{f}}{q}\right] : R \text{ is increasing and } L\text{-Lipschitz} \right\}.$$

Also define the bounds

$$n_{\min}(A) = \frac{\mu A}{\kappa}, \qquad n_{\max}(A) = \frac{\mu A}{\kappa - \bar{f}/2},$$

$$g_{\max} = \max_{f \in [0,\bar{f}]} g(f) = \frac{3}{2\bar{f}}, \qquad N_R(A) = \frac{\mu A q}{(\kappa - \bar{f}/2)^2},$$

$$M_R(A) = \lambda q \left[N_R(A) + q n_{\max}(A) g_{\max} \right], \qquad F_R(A) = \chi A \tau n_{\min}(A)^{-\tau-1} N_R(A).$$

Lemma A.3 (A sufficient condition for Assumption 4.2)**.** *Fix A and $L > 0$. Suppose:*

1. $\chi A n_{\min}(A)^{-\tau} \le (1-\beta)\bar{f}/q$;

2. $$\chi A \alpha n_{\min}(A)^{-\tau} + F_R(A)L + \beta L\Big[(1-\delta) + M_R(A)L\Big] \le L;$$

3. $$F_R(A) + \beta + \beta L M_R(A) < 1.$$

Then the Bellman-entry operator maps $\mathcal{X}_L$ into itself and is a contraction under the sup norm.

Proof. Take $R \in \mathcal{X}_L$. Because $R(z) \in [0, \bar{f}/q]$, Lemma A.1 implies

$$0 \le \Omega(qR(z)) \le \Omega(\bar{f}) = \frac{\bar{f}}{2}.$$

Hence

$$n_{\min}(A) \le n(R, z; A) \le n_{\max}(A)$$

for every z. The current flow term therefore satisfies

$$0 \le \chi A z^{\alpha} n(R, z; A)^{-\tau} \le \chi A n_{\min}(A)^{-\tau}.$$

Condition (i) ensures that

$$0 \leq (\mathcal{B}_A R)(z) \leq \frac{\bar{f}}{q},$$

so the operator preserves the value range.

To control the R-sensitivity of the operator, note that

$$\left|\frac{\partial n(R, z; A)}{\partial R}\right| = \frac{\mu A q G(qR(z))}{[\kappa - \Omega(qR(z))]^2} \leq N_R(A).$$

Hence the flow term has scalar derivative bounded by

$$\left|\frac{\partial}{\partial R}\left[\chi A z^\alpha n(R, z; A)^{-\tau}\right]\right| \leq F_R(A).$$

Similarly, Lemma A.2 implies that the transition map has scalar derivative bounded by

$$\left|\frac{\partial T(R, z; A)}{\partial R}\right| \leq M_R(A).$$

To see the bound explicitly, write

$$\frac{\partial T(R, z; A)}{\partial R} = \lambda q \left[\frac{\partial n(R, z; A)}{\partial R} G(qR(z)) + q\, n(R, z; A) g(qR(z))\right].$$

The first term is bounded by $\lambda q N_R(A)$ because $G \leq 1$, and the second is bounded by $\lambda q^2 n_{\max}(A) g_{\max}$. Summing the two bounds gives $M_R(A) = \lambda q[N_R(A) + q n_{\max}(A) g_{\max}]$. Therefore, for any $R_1, R_2 \in \mathcal{X}_L$,

$$\|\mathcal{B}_A R_1 - \mathcal{B}_A R_2\|_\infty \leq \left[F_R(A) + \beta + \beta L M_R(A)\right] \|R_1 - R_2\|_\infty.$$

Condition (iii) makes this coefficient strictly smaller than one, so $\mathcal{B}_A$ is a contraction.

It remains to check that $\mathcal{B}_A$ preserves the Lipschitz bound. The direct z-derivative of the current flow term is bounded by $\chi A \alpha n_{\min}(A)^{-\tau}$, while the indirect z-dependence through R

contributes at most $F_R(A)L$. The continuation term contributes at most $\beta L[(1-\delta)+M_R(A)L]$. Condition (ii) ensures that the total slope is bounded by L, so $\mathcal{B}_A$ maps $\mathcal{X}_L$ into itself. □

Corollary A.4 (A nonempty admissible neighborhood)**.** *Suppose there exist $A_0 > 0$ and $L_0 > 0$ such that the three inequalities in Lemma A.3 hold strictly at (A_0, L_0). Then there exists $\eta > 0$ such that every $A \in (A_0 - \eta, A_0 + \eta) \cap \mathbb{R}_+$ satisfies Assumption 4.2 with the same Lipschitz class $\mathcal{X}_{L_0}$.*

Proof. The objects $n_{\min}(A)$, $n_{\max}(A)$, $N_R(A)$, $M_R(A)$, and $F_R(A)$ are continuous in A on $\{A > 0 : \kappa > \bar{f}/2\}$. Because the inequalities in Lemma A.3 hold strictly at (A_0, L_0), they continue to hold on a sufficiently small neighborhood of A_0. The lemma then applies pointwise throughout that neighborhood. □

B Full Proofs of Main Results

B.1 Proof of Proposition 5.1

Full proof of Proposition 5.1. Fix $A > 0$. Step 1 establishes the static entry block. For any bounded increasing candidate value function R, the left-hand side of the free-entry condition,

$$\Phi(n; R, z, A) = A\mu n^{-1} + \Omega(qR(z)),$$

is continuous and strictly decreasing in n. Moreover,

$$\Phi(n; R, z, A) \to +\infty \text{ as } n \downarrow 0, \qquad \Phi(n; R, z, A) \to \Omega(qR(z)) < \kappa \text{ as } n \uparrow \infty.$$

Therefore Φ crosses κ exactly once, so there is a unique $n(R, z; A)$ solving $\Phi = \kappa$.

Step 2 establishes monotonicity in the continuation value. Since

$$\frac{\partial \Phi}{\partial n} = -A\mu n^{-2} < 0, \qquad \frac{\partial \Phi}{\partial R} = qG(qR(z)) \geq 0,$$

the implicit-function theorem implies $\partial n(R, z; A)/\partial R \geq 0$. Therefore $R_1(z) \geq R_2(z)$ implies $n(R_1, z; A) \geq n(R_2, z; A)$.

Step 3 uses Assumption 4.2. The Bellman-entry operator maps $\mathcal{X}_{L_A}$ into itself and is a contraction. Banach's fixed-point theorem therefore gives a unique bounded increasing fixed point $R_A \in \mathcal{X}_{L_A}$. In particular, R_A is Lipschitz. The associated entry rule $n_A(z) = n(R_A, z; A)$ is continuous and increasing in z, because R_A is.

Step 4 turns to innovation. Since G is increasing and R_A is increasing, the mass of successful innovators

$$m_A(z) = q\, n_A(z)\, G(qR_A(z))$$

is also continuous and increasing in z. Finally, the derivative identities follow from Lemma A.2. This proves all parts of the proposition. □

B.2 Proof of Proposition 5.2

Full proof of Proposition 5.2. Start from the transition law

$$T_A(z) = (1-\delta)z + \lambda q\, n_A(z) G(qR_A(z)).$$

Differentiating with respect to z gives

$$T'_A(z) = 1 - \delta + \lambda m'_A(z).$$

Lemma A.2 implies

$$m'_A(z) = q^2 n_A(z) R'_A(z) \left[g(qR_A(z)) + \frac{G(qR_A(z))^2}{\kappa - \Omega(qR_A(z))} \right].$$

Substituting this formula into the transition derivative yields (8) and the definition of $\Xi_A(z)$ in (9).

For the steady-state derivative formula, let z^* satisfy $z^* = T_A(z^*)$. Differentiate the frontier-rent recursion:

$$R_A(z) = \chi A z^{\alpha} n_A(z)^{-\tau} + \beta R_A(T_A(z)).$$

At z^*,

$$R'_A(z^*) = \chi A \frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\alpha} n_A(z)^{-\tau}\right)\Big|_{z=z^*} + \beta R'_A(z^*) T'_A(z^*).$$

The derivative of the current-flow term is

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\alpha} n_A(z)^{-\tau}\right) = z^{\alpha-1} n_A(z)^{-\tau}\left[\alpha - \tau z \frac{n'_A(z)}{n_A(z)}\right].$$

Rearranging the recursion derivative then gives

$$R'_A(z^*) = \frac{\chi A n_A(z^*)^{-\tau} z^{*\alpha-1}\left[\alpha - \tau z^* n'_A(z^*)/n_A(z^*)\right]}{1 - \beta T'_A(z^*)}.$$

This should be read as an implicit slope equation because $n'_A(z^*)/n_A(z^*)$ depends on $R'_A(z^*)$ through Proposition 5.1, and $T'_A(z^*)$ depends on $R'_A(z^*)$ through (8). The denominator is positive by hypothesis. This proves the proposition. □

B.3 Proof of Theorem 6.1

Full proof of Theorem 6.1. Define

$$F_A(z) = T_A(z) - z.$$

At $z = 0$, the innovation flow is zero because $G(0) = 0$, so $T_A(0) = 0$ and hence $F_A(0) = 0$. Proposition 5.2 implies

$$F'_A(0) = T'_A(0) - 1 = -\delta < 0.$$

Therefore there exists $\underline{z} \in (0, z^+)$ such that

$$F_A(\underline{z}) < 0.$$

By assumption, $F_A(z^+) > 0$. Continuity of F_A then implies that there is at least one zero in $(\underline{z}, z^+)$. Denote any such zero by $z^M(A)$.

The assumption $T_A(1) < 1$ is equivalent to $F_A(1) < 0$. Since $F_A(z^+) > 0$, continuity implies that there is at least one further zero in $(z^+, 1)$. Denote any such zero by $z^H(A)$.

The local stability claims are standard. Any steady state satisfying $|T'_A(z^*)| < 1$ is locally stable, while any steady state satisfying $T'_A(z^*) > 1$ is unstable. This proves the theorem. □

B.4 Proof of Corollary 6.2

Full proof of Corollary 6.2. Because Proposition 5.1 implies that m_A is increasing, the transition map

$$T_A(z) = (1 - \delta)z + \lambda m_A(z)$$

is increasing on $[0, 1]$.

Fix an initial condition $z_0 < z^M(A)$. Since the only steady states are 0, $z^M(A)$, and $z^H(A)$, continuity implies

$$T_A(z) - z < 0 \qquad \text{for every } z \in (0, z^M(A)).$$

Hence $0 \leq T_A(z_0) < z_0$. Iterating gives a monotone decreasing sequence bounded below by zero, so z_t converges to some fixed point in $[0, z^M(A)]$. The only one is zero.

Now take $z_0 \in (z^M(A), z^H(A))$. Exact-three steady states imply

$$T_A(z) - z > 0 \qquad \text{for every } z \in (z^M(A), z^H(A)).$$

Monotonicity of T_A and the fact that $T_A(z^H(A)) = z^H(A)$ imply

$$z_0 < T_A(z_0) < z^H(A).$$

Thus the trajectory is monotone increasing and bounded above by $z^H(A)$, so it converges to the unique fixed point in that interval, namely $z^H(A)$.

Finally, if $z_0 > z^H(A)$, continuity implies $T_A(z) - z < 0$ on $(z^H(A), 1]$. Monotonicity of T_A and $T_A(z^H(A)) = z^H(A)$ therefore imply

$$z^H(A) < T_A(z_0) < z_0.$$

The trajectory is monotone decreasing and bounded below by $z^H(A)$, so it converges to $z^H(A)$. □

B.5 Proof of Theorem 6.3

Full proof of Theorem 6.3. Define the equilibrium fixed-point equation

$$H(A, z) = T_A(z) - z.$$

At the unstable branch, $H(A, z^M(A)) = 0$. The derivative of H with respect to z is

$$H_z(A, z^M(A)) = T'_A(z^M(A)) - 1 > 0$$

because the middle branch is unstable.

Next consider the effect of market access. The current frontier-flow profit is

$$\pi^F(A, z, n_A(z)) = \chi A z^\alpha n_A(z)^{-\tau}.$$

Its elasticity with respect to market access is

$$\varepsilon_A^{\pi}(z) = 1 - \tau \varepsilon_A^{n}(z).$$

If $\varepsilon_A^n(z^M(A)) > 1/\tau$, then the current-flow part of frontier rent falls with market access at the unstable branch. The theorem additionally assumes that the full transition map shifts downward there, so

$$H_A(A, z^M(A)) < 0.$$

The implicit-function theorem therefore gives

$$\frac{\mathrm{d}z^M(A)}{\mathrm{d}A} = -\frac{H_A(A, z^M(A))}{H_z(A, z^M(A))} > 0.$$

Hence a marginal increase in market access raises the unstable threshold and shrinks the basin of attraction of the high-innovation steady state. □

B.6 Proof of Corollary 6.4

Full proof of Corollary 6.4. If entry is exogenous, then $n_A(z)$ does not move with market access. Therefore

$$\varepsilon_A^{n}(z) = \frac{\partial \log n_A(z)}{\partial \log A} = 0.$$

Substituting into (12) gives

$$\varepsilon_A^{\pi}(z) = 1.$$

The current-flow frontier rent therefore rises one-for-one with market access. The threshold-tightening channel in Theorem 6.3 requires the frontier rent to fall locally with market access. That possibility is eliminated when entry is exogenous. □

B.7 Proof of Theorem 6.5

Full proof of Theorem 6.5. Let $z^M(A)$ denote the unstable threshold under the baseline market-access level A, and assume $z_0 > z^M(A)$. Under the temporary shock $\tilde{A} < A$, the state evolves according to

$$z_{t+1} = T_{\tilde{A}}(z_t).$$

The hypothesis of the theorem gives the uniform drift bound

$$z_{t+1} \leq z_t - \eta \qquad \text{whenever } z_t \in [z^M(A), z_0].$$

Therefore, as long as the trajectory has not crossed below $z^M(A)$, it moves downward by at least η each period. By induction,

$$z_t \leq z_0 - t\eta$$

for all such dates. Hence the crossing time is finite and satisfies

$$m^\star(z_0; A, \tilde{A}) \leq \left\lceil \frac{z_0 - z^M(A)}{\eta} \right\rceil.$$

By definition,

$$m^\star(z_0; A, \tilde{A}) = \min\left\{ m \in \mathbb{N} : T_{\tilde{A}}^m(z_0) < z^M(A) \right\}$$

is the first date at which the shocked trajectory crosses below the baseline unstable threshold.

Suppose the shock lasts for at least $m^\star$ periods. Then at the restoration date,

$$z_{m^\star} < z^M(A).$$

Once market access returns to A, subsequent dynamics are governed by the baseline map T_A. Corollary 6.2 implies that any state below $z^M(A)$ converges to the corner trap. Therefore

$$z_t \to 0.$$

This is precisely the hysteresis claim: a temporary shock has a permanent effect because it moves the state across the unstable threshold. □

B.8 Proof of Proposition 7.1

Full proof of Proposition 7.1. Let $T_{A,s}$ denote the transition map under subsidy $s \geq 0$. Because the subsidy lowers effective fixed costs from f to $f - s$, the innovation condition becomes

$$f \leq qR_{A,s}(z) + s.$$

The proposition assumes that the subsidized recursive equilibrium remains well defined on $[0, \bar{s}]$, that the support restriction continues to hold, and that the family $\{T_{A,s}\}_{s\in[0,\bar{s}]}$ is continuous in (z, s) and weakly increasing in s. Define

$$F_{A,s}(z) = T_{A,s}(z) - z$$

and let

$$\mathcal{S} = \{s \in [0, \bar{s}] : \exists z \in (0, 1) \text{ such that } F_{A,s}(z) = 0\}.$$

Because $F_{A,s}$ is continuous in (z, s) and $[0, \bar{s}] \times [0, 1]$ is compact, $\mathcal{S}$ is closed. By hypothesis, $0 \in \mathcal{S}$ and $\bar{s} \notin \mathcal{S}$, so the critical level

$$s^\star(A) = \sup \mathcal{S}$$

is well defined and satisfies $s^\star(A) < \bar{s}$.

Now take any $s > s^\star(A)$. If $T_{A,s}$ still had an interior fixed point, then $s \in \mathcal{S}$, contradicting the definition of $s^\star(A)$. Hence every $s > s^\star(A)$ eliminates the interior threshold. Because $\mathcal{S}$ is closed, the same is true at $s = s^\star(A)$. If, in addition, the remaining subsidized map has a unique positive fixed point, monotone dynamics imply convergence to that fixed point from any initial condition above zero. □

C Benchmarks That Discipline the Mechanism

C.1 Benchmark 1: Exogenous Entry

Proposition C.1 (Exogenous entry eliminates the competition reversal)**.** *If $n_A(z)$ is fixed exogenously, then the current-flow elasticity in Theorem 6.3 collapses to one, so market access always raises current innovation rents.*

Proof. With exogenous entry, $\varepsilon_A^n(z) = 0$ at every state. Equation (12) then implies $\varepsilon_A^\pi(z) = 1$. □

This is the cleanest way to see which part of the mechanism is really about business stealing. The non-monotone access result is therefore not a generic implication of larger markets; it is generated by entry-mediated rent compression.

C.2 Benchmark 2: No Fixed-Cost Heterogeneity

Proposition C.2 (No fixed-cost heterogeneity flattens the middle margin)**.** *Suppose $f = 0$ almost surely so every entrant innovates. If*

$$\sup_{z\in[0,1]} \lambda q\, n_A'(z) < \delta,$$

then $T_A'(z) < 1$ for every z, so the transition map has at most one steady state.

Proof. Under degenerate fixed costs,

$$G(qR_A(z)) = 1, \qquad \Omega(qR_A(z)) = qR_A(z),$$

so

$$m_A(z) = q\, n_A(z), \qquad T_A'(z) = 1 - \delta + \lambda q\, n_A'(z).$$

The displayed bound implies $T'_A(z) < 1$ for every z. Hence the graph of T_A can cross the 45-degree line at most once. □

Now remove the hump-shaped margin created by heterogeneous fixed costs. Let $f = 0$ almost surely, so every entrant innovates and

$$G(qR_A(z)) = 1, \qquad \Omega(qR_A(z)) = qR_A(z).$$

Then

$$m_A(z) = q\, n_A(z), \qquad T'_A(z) = 1 - \delta + \lambda q\, n'_A(z).$$

The density term $g(qR_A(z))$ disappears from the slope decomposition. The middle amplification channel is therefore much weaker. A sufficient condition for the transition map to be everywhere flatter than the 45-degree line is

$$\sup_{z\in[0,1]} \lambda q\, n'_A(z) < \delta.$$

Under that bound, $T'_A(z) < 1$ for every z, so the three-crossing trap region disappears and the economy has at most one steady state. The point of the benchmark is not that multiple steady states are impossible without cost heterogeneity, but that the heterogeneous fixed-cost margin is what naturally creates the interior hump in the benchmark economy.

C.3 Benchmark 3: Low Complementarity

Proposition C.3 (Lower complementarity weakens low-state damping)**.** *For every fixed $z \in (0,1)$, the direct state derivative of current frontier rents is proportional to $z^{\alpha-1}$. Lowering α toward one therefore weakly raises that derivative pointwise and eliminates its vanishing-at-the-origin property in the limit $\alpha \downarrow 1$.*

Proof. The current-flow frontier profit is $\pi^F(A, z, n) = \chi A z^\alpha n^{-\tau}$. Holding the entry term fixed, its direct derivative with respect to z is proportional to $z^{\alpha-1}$. Since $z^{\alpha-1}$ rises pointwise

on $(0, 1)$ as α falls, the low-state flatness is weakened. □

The complementarity parameter α controls how fast frontier rents rise with the innovation state at low values of z. In the benchmark economy, the direct state derivative of current frontier rents is proportional to

$$z^{\alpha-1}.$$

When $\alpha > 1$, this term is small near the origin. As α falls toward one, the low-state flat region shrinks. The economy then leaves the neighborhood in which innovation is weak more quickly. In this sense, lower complementarity does not automatically eliminate multistability, but it weakens the low-end damping force that helps generate a trap.

C.4 Benchmark 4: No Business-Stealing Dominance

Proposition C.4 (No business-stealing dominance restores the market-size logic)**.** *If* $\tau\varepsilon_A^n(z) \leq 1$ *and* $H_A(A, z) \geq 0$ *throughout the relevant state range, then current-flow reversal and threshold tightening both disappear.*

Proof. The condition $\tau\varepsilon_A^n(z) \leq 1$ implies $\varepsilon_A^\pi(z) \geq 0$ by equation (12). The additional condition $H_A(A, z) \geq 0$ rules out a downward shift of the full transition map. Hence neither component of the access-reversal mechanism survives. □

The non-monotone market-access result requires entry to respond strongly enough that

$$\tau\varepsilon_A^n(z) > 1$$

at the unstable branch. If instead

$$\tau\varepsilon_A^n(z) \leq 1$$

throughout the relevant state range, then the current-flow frontier rent is weakly increasing in market access everywhere. If the continuation term is also weakly increasing in market

access, then

$$H_A(A, z) \geq 0$$

for every relevant (A, z), and the threshold-tightening channel disappears. In that benchmark, larger markets revert to the more standard market-size logic.

D Primitive Interpretation and Parameter Region

D.1 Economic Meaning of the Benchmark Parameters

The parameter α measures how strongly frontier activity depends on complementary capabilities. A larger α means that implementation requires more local inputs, specialized skills, or frontier-adjacent tasks. The parameter τ measures business stealing: it is the elasticity with which frontier rents are compressed by entry. The distribution G governs the mass of firms that are close to the fixed-cost margin. Finally, κ controls the level of product-market entry. When κ is low, entry reacts strongly to any increase in the value of innovation; when it is high, the competition channel is muted.

D.2 Keeping Frontier Rents Inside the Support of G

Because $z \leq 1$ and $n_A(z) \geq n_{\min}(A) = \mu A/\kappa$, the frontier-rent recursion implies the bound

$$R_A(z) \leq \bar{R}(A) := \frac{\chi A n_{\min}(A)^{-\tau}}{1-\beta}.$$

Hence a simple sufficient condition for the benchmark support restriction

$$qR_A(z) \in [0, \bar{f}]$$

to hold at every state is

$$q\bar{R}(A) \leq \bar{f}.$$

That is,

$$\frac{q\chi A n_{\min}(A)^{-\tau}}{1-\beta} \leq \bar{f}.$$

This condition is conservative, but it is convenient because it depends only on primitives and the lower bound on equilibrium entry.

D.3 Keeping the Transition Map in $[0, 1]$

Because $G(\cdot) \leq 1$ and $n_A(z) \leq n_{\max}(A) = \mu A/(\kappa - \bar{f}/2)$,

$$T_A(z) \leq (1-\delta)z + \lambda q n_{\max}(A) \leq (1-\delta) + \lambda q n_{\max}(A).$$

Therefore a sufficient condition for $T_A([0,1]) \subseteq [0,1]$ is

$$\lambda q n_{\max}(A) \leq \delta.$$

This is exactly the bounded-state condition used in Assumption 4.1. It says that the maximal flow of successful innovation cannot overwhelm depreciation by more than one full unit of the state variable.

D.4 Why the Map Is Low-Flat, Middle-Steep, and High-Flat

The slope decomposition

$$T_A'(z) = 1 - \delta + \lambda q^2 n_A(z) R_A'(z) \Xi_A(z)$$

isolates the three forces that matter.

At low states, the complementarity term makes frontier rents rise slowly with z, and the mass of marginal innovators is small because both $G(qR_A(z))$ and $g(qR_A(z))$ are close to zero. The map therefore starts close to the depreciation line.

At intermediate states, the density of marginal innovators is largest. In the rescaled Beta$(2,2)$ benchmark, $g(\cdot)$ is hump-shaped, peaking when innovation rents lie near the middle of the support. Small changes in frontier rents then move many firms across the R&D threshold, so the transition map becomes steep.

At high states, the density of marginal firms falls again because most firms are already innovating. Entry keeps rising, but that very response compresses frontier rents through the business-stealing elasticity τ. The result is a flatter high-state region.

D.5 Elasticity Interpretation of Theorem 6.3

The explicit entry rule gives

$$\log n_A(z) = \log \mu + \log A - \log\Big[\kappa - \Omega(qR_A(z))\Big].$$

Differentiating with respect to $\log A$ yields

$$\varepsilon_A^n(z) = 1 + \frac{AqG(qR_A(z))\partial_A R_A(z)}{\kappa - \Omega(qR_A(z))}.$$

The first term is the direct market-size effect. The second term is the indirect amplification that comes from the endogenous value of innovation. The elasticity condition in Theorem 6.3,

$$\varepsilon_A^n(z) > \frac{1}{\tau},$$

therefore has a clean interpretation. If business stealing is weak, so that τ is small, entry must react very strongly before the current-flow frontier rent turns downward with market access. If business stealing is strong, so that τ is close to one, a more moderate entry response is enough to generate the competition reversal.

The decomposition also shows why the current paper stops short of a full primitive characterization of threshold tightening. The sign of $\partial_A R_A(z)$ depends on how market access

changes both current rents and continuation values, and the present paper does not sign that object globally. Accordingly, the elasticity condition should be read as the exact condition for current-flow reversal, while the additional requirement $H_A(A, z^M(A)) < 0$ remains a separate local condition for the full transition map.