Optimal Immigration and Long-Run Growth*

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Abstract

We introduce immigration into an overlapping-generations growth model. A planner chooses the rate of immigration to maximize the utility of a representative native consumer in the steady state, subject to providing each immigrant with a threshold level of utility. We derive a sufficient condition for the optimal immigration rate to be positive, identifying the components of the marginal cost and marginal benefit of immigration. The planner's optimal steady state can be implemented in a competitive economy using a fiscal package consisting of an entry fee for immigrants, a consumption tax, and government bonds.

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Immigration policy has become a major divisive issue in the United States, with passionate arguments put forth by people on all sides of questions relating to immigration. In this paper, we adopt a long-term macroeconomic perspective on the impact and desirability of immigration. We develop and analyze a simple neoclassical growth model of an economy that is an attractive destination for an unlimited supply of potential immigrants. We abstract from many of the rich set of considerations that arouse such passion in current debates and assume that all people, regardless of country of birth, have identical productivity when working in the destination country. We also assume that all consumers, regardless of country of birth, have identical utility functions that depend only on their own consumption in each period of life. We ignore other considerations that might affect utility, such as an affinity for people of similar national origin or a taste for cultural diversity associated with a population consisting of people of various national origins. Our goal is to use neoclassical growth theory, extended to include immigration, to analyze the amount of immigration that maximizes a well-defined social welfare function.

The social welfare function we adopt is the lifetime utility of a representative native consumer in the steady state. Notably, the utility of immigrants does not enter this social welfare function. Instead, the utility of immigrants is treated as a constraint–immigrants must receive a level of utility in the destination country that is greater than or equal to a threshold level of utility. We assume that children born in the destination country are treated as natives, regardless of whether their parents are natives or immigrants. (Had we adopted the alternative assumption that all descendants of immigrants are treated as immigrants, the steady state with immigration would have a vanishingly small share of natives.) Finally, to address the concern that immigrants may consume public goods that are costly to produce, we assume that public goods are rival and non-excludable so that each immigrant and each native consume the same amount of costly public goods.

In a closed economy populated only by natives with constant productivity and a constant population growth rate, steady-state consumption per person is maximized by the capital-labor ratio for which the marginal product of capital, $f'(k)$, net of the depreciation rate of capital, δ, equals the population growth rate. Phelps (1961) dubbed such a steady state the

Golden Rule because it is the steady state chosen by a planner that treats every consumer in every generation equally—implementing the Biblical dictum to treat others as you would like to be treated. In this paper, the planner maximizes the utility of the representative native-born person in the steady state. In this case, the Golden Rule is narrow in scope, applying only to natives. As mentioned above, the consumption and utility of immigrants enters the planner's decision only as a constraint. The planner respects the threshold level of immigrants' utility but does not attempt to maximize their utility. Recognizing that the planner departs from the Biblical dictum, we will use the term Golden Rule in a narrow technical sense meaning that the planner maximizes the steady-state utility of given class of consumers—in this case, natives.

Our strategy in this paper is first to analyze the planner's problem to choose the amount of capital per worker, k, and the ratio of the number of immigrant workers to native workers, m, to maximize the social welfare function. Then we turn our attention to a competitive economy and analyze a package of taxes and government bonds that induces the steady state of the competitive economy to match the steady state chosen by the planner. In Section 2, we assume that all consumers live for only one period and give birth to an exogenous number, $1+\gamma > 0$, of children at the end of their lives. The planner chooses the levels of k and m that jointly maximize steady-state consumption per native. The optimal steady state is characterized by an extension of the Golden Rule: the rate at which a unit of consumption in a given period can be transformed into consumption in the following period, $1+f'(k)-\delta$, equals the endogenous gross growth rate of population, $\Gamma(m) \equiv (1+\gamma)(1+m)$. In the special case in which $m=0$, the extended Golden Rule becomes $f'(k)-\delta=\gamma$, as in Phelps (1961) in the absence of immigration. We derive a sufficient condition for the optimal steady-state value of m to be positive: the consumption of a native in the Phelpsian Golden Rule in an economy without immigrants must exceed the sum of (1) the threshold level of consumption that a potential immigrant requires to willingly immigrate; and (2) the amount of capital needed to equip each of the immigrant's $1+\gamma$ children with capital equal to the per-worker capital-labor ratio in the Phelpsian Golden Rule.

A limitation of specifying one-period lifetimes is that consumers in a decentralized econ-

omy have no saving decision and hold no capital. As a step toward developing a competitive economy with a meaningful saving decision by individual consumers, Sections 3 and 4 extend lifetimes to two periods. Consumers work only in the first period of life, but they consume in both periods. For ease of exposition, all consumers–both native and immigrant–have identical utility functions that are a weighted average of the logarithm of their own consumption in each period. In Section 3, the planner maximizes the utility of a representative native in the steady state, subject to immigrants' threshold level of utility (rather than simply consumption) required in order to willingly immigrate. The mapping from consumption to utility depends on the endogenous rate at which consumption when young can be transformed into consumption when old. For immigrants, this endogeneity implies that the optimal allocation of consumption when young and consumption when old is endogenous, even though the level of utility is maintained equal to an exogenous constant threshold. Despite these additional complications, the extended Golden Rule derived in Section 2 continues to hold without any change: $1 + f'(k) - \delta = \Gamma(m) \equiv (1+\gamma)(1+m)$.

Since, as described above, consumers work in only the first period of life, but consume in both periods of life, the steady-state dependency ratio, which is the ratio of the number of non-working consumers to the number of working consumers, is the reciprocal of endogenous growth rate $\Gamma(m) \equiv (1+\gamma)(1+m)$. An increase in m increases $\Gamma(m)$ and thus reduces the dependency ratio. Section 3 shows that the planner's optimization problem can be viewed as incorporating a consumers' intertemporal consumption allocation problem depending on the marginal rate of transformation, $1 + f'(k) - \delta$, or equivalently, as a static optimization problem allocating a given amount of aggregate consumption to the cohort of working young consumers and the contemporaneous cohort of dependent old consumers, who are $\frac{1}{\Gamma(m)}$ times as numerous. Because the extended Golden Rule holds in the planner's optimal steady state, the intertemporal tradeoff ratio between consumption when young and consumption when old, $1 + f'(k) - \delta$, equals the intratemporal tradeoff, $\Gamma(m)$, between the consumption of contemporaneous young and old consumers.

Section 4 analyzes a version of the model with two-period lifetimes from Section 3, organized as a competitive economy. We show that if the rate of immigration in the planner's

optimal steady state is positive, the steady state in the competitive economy can be induced to match the planner's optimal steady state using three fiscal tools: (1) an entry fee charged to all young immigrants at the end of their first period of life; (2) a consumption tax levied at the same rate on all consumers regardless of national origin and regardless of age; and (3) government bonds. Because there is no risk in this model, the rate of return on government bonds equals the rate of return on capital. Hence, the extended Golden Rule implies that in the planner's optimal steady state, the rate of return on government bonds equals the economy's growth rate. Therefore, in the optimal steady state, the government earns zero seignorage on bonds. That is, bonds are not a means of paying for government expenditures in the steady state, so bonds do not enter the government's budget constraint, when the competitive economy implements the planner's optimum. The role of government bonds is to induce the competitive economy to hold the optimal size of the capital stock by setting the amount of bonds equal to the amount of saving less the optimal capital stock.

1 Related Literature

This paper is related to two strands of literature: (1) the economic effects of immigration; and (2) analyses of the Golden Rule, which is the maximization of steady-state consumption (or utility) in a dynamic economy. In this section, we focus on the subsets of these literatures most relevant to the determination of the rate of immigration that maximizes steady-state welfare of natives in the host country.

Some of the early immigration literature emphasizes the impact of illegal, as opposed to legal, immigration. Indeed, the word "illegal" appears in the title of Ethier (1986), Hazari and Sgro (2003), and Moy and Yip (2006). The earliest of these three papers discusses the impact of illegal immigration in a static economy without capital. Hazari and Sgro (2003) analyze the impact of illegal immigration in a dynamic economy with capital in which illegal immigrants, as well as natives, have infinite-horizons. Illegal immigrants receive a lower wage than equally productive natives. Hazari and Sgro (2003) finds that "if illegal migrants and domestic labour are perfect substitutes, then illegal migration necessarily lowers the long-run

per capita consumption of domestic residents."(p 142) Using the model from Hazari and Sgro (2003), Moy and Yip (2006) provides an example in which native labor and immigrant labor are perfect substitutes in production, yet, contrary to Hazari and Sgro (2003), Moy and Yip (2006) finds than an increase in immigration can lead to a higher or lower level of consumption per native in the steady state. Moy and Yip (2006) attribute the difference in findings to "an indirect intertemporal effect that is overlooked by Hazari and Sgro." (p. 2473) More recently, Correani et al. (2015), in a framework similar to that used by Hazari and Sgro (2003) and Moy and Yip (2006), finds that if native labor and immigrant labor are perfect substitutes in production, then, depending on the elasticity of output with respect to the capital stock in a Cobb-Douglas production function, an increase in immigration can either increase or decrease the steady-state level of consumption per capita.

While the studies described above focus on the impact on steady-state consumption per native, several other studies analyze the fiscal impact of immigration. Recognizing that the fiscal impact of an immigrant depends on the immigrant's age at the time of immigration, as well as the skill of the immigrant, Storesletten (2000) calculates for the United States "the smallest increase in annual immigration required to balance the budget, given that the government is free to choose the distribution of age and skills of new immigrants, while restricted to keeping the current tax and spending policies unchanged." (p. 301). In a similar vein, though applied to Sweden, where the public sector is a larger share of GDP than in the United States, Storesletten (2003) computes the present value of the taxes paid by an immigrant minus the amount of government expenditure associated with that immigrant and finds that, for young working-age immigrants, this net present value is positive, though for the average immigrant, this net present value is negative.

Ben-Gad (2004) measures the impact of immigration on welfare as the sum of the impact on labor income and the impact on capital income. In this analysis, immigration reduces the capital-labor ratio, driving up the rate of return on capital, which benefits native owners of capital. However, if immigrants consume more publicly provided goods than they pay in taxes, the overall impact of immigration on welfare may be negative.

Within the set of papers that examine the fiscal impact of immigration, some papers

focus on the role of public pensions or Social Security. Lee and Miller (2000), Naito (2014), and Schou (2006) present simulation results from models that include age-dependent public transfers. Razin and Sadka (1999, 2000), develop and apply an elegant overlapping generations model of immigration in a country with a pay-as-you-go Social Security system. Young workers are born with low skills and heterogeneous levels of innate ability that determine how much time must be diverted from employment to acquire a high level of skills. Low-ability individuals find it too costly to acquire the high level of skills and hence remain as workers with low skills and a low wage rate. High-ability individuals choose to devote some time to acquire a high level of skills and earn a high wage rate, but work for a smaller amount of time than low-skill workers. In both papers, workers contribute a fraction of their wage income when young to pay for a publicly-provided retirement benefit that is equal for all old consumers. In a small open economy, in which factor prices are exogenously given, these papers show that an increase in immigration is Pareto improving and domestic consumers would choose the immigration rate to be as high as possible. However, Razin and Sadka (2000) shows, using numerical simulations, that if factor prices are endogenously determined, as would be the case in a closed economy, or an open economy with limited factor mobility, then it is possible that migration imposes a loss on all income groups in every generation, or imposes losses on some groups while yielding gains to other groups. In this case of endogenous factor prices, the optimal rate of immigration is not addressed.

Shea and Woodfield (1996) examines optimal immigration in a model without capital. Output is a function of skilled labor and unskilled labor, and unskilled labor can be transformed into skilled labor by using some skilled labor to teach unskilled labor. Interestingly, this paper uses the term "Golden Rule education policy" to describe the education policy that maximizes steady-state consumption per worker.

The literature on the Golden Rule capital-labor ratio begins with Phelps (1961), which shows that the level of the capital-labor ratio that maximizes consumption per unit of labor in the steady state of a closed economy is characterized by a marginal product of capital, net of depreciation, equal to the growth rate of the economy. A steady state in which the capital-labor ratio is higher than the Golden Rule capital-labor ratio has lower consumption

per unit of labor than in the Golden Rule steady state. Such a steady state is said to have a dynamically inefficient overaccumulation of capital. Diamond (1965) develops an overlapping generations model with capital and demonstrates that it is possible for the competitive steady state to have a dynamically inefficient overaccumulation of capital. Cass (1972) develops a complete characterization of dynamic inefficiency for economies both outside the steady state and in the steady state. Abel et al. (1989) develops sufficient conditions to declare a stochastic economy either dynamically inefficient or dynamically efficient, though many stochastic economies cannot be declared either dynamically efficient or dynamically inefficent by these sufficient conditions.

In most of the Golden Rule literature, including the papers cited above, the growth rate of the labor force is exogenous. However, the introduction of a permanent stream of immigration increases the growth rate of the labor force. Some early papers on the Golden Rule have endogenized the growth rate of the labor force through channels other than immigration. Davis (1969) specifies the population growth rate to depend endogenously on the level of income. Samuelson (1980) endogenizes the size of the labor force by endogenizing the labor force participation rate to be an increasing function of the wage rate. In both of these papers, the capital-labor ratio that maximizes consumption per person in the steady state is no longer characterized by Phelps's condition that the net marginal product of capital equals the growth rate. However, as we emphasize in the current paper, the introduction of a permanent flow of immigration, which effectively endogenizes the growth rate of population and labor force, does not change the Phelpsian Golden Rule prescription that the capital-labor ratio that maximizes steady-state consumption per person is attained when the net marginal product of capital equals the growth rate of the labor force. To reconcile these findings, we note that the labor force in both Davis (1969) and Samuelson (1980) is a function of an endogenous variable, but the growth rate of the labor force in our model is a function of a decision variable–namely the rate of immigration–chosen by the planner.

2 One-Period Lifetimes

We begin with a simple demographic structure in which each person lives for only one period. A generation t person is born at the beginning period t and gives birth to $1+\gamma > 0$ generation $t+1$ people at the end of period t, which is the beginning of period $t+1$. The economy is closed in the sense that it does not trade goods, services, or assets with other countries. However, in each period t, M_t people born outside the country immigrate into the country and join the N_t native-born people in the employed labor force,

$$L_t = N_t + M_t. \tag{1}$$

Like natives, each immigrant gives birth to $1+\gamma$ children, and since these children are born in the country of interest, they are natives. Since all people in the labor force in period t give birth to $1+\gamma$ native-born children in generation $t+1$, we have

$$N_{t+1} = (1+\gamma)(N_t + M_t) = (1+\gamma)(1+m_t)N_t, \tag{2}$$

where $m_t \equiv \frac{M_t}{N_t}$ is the ratio of immigrants to natives in generation t. Thus, $(1+\gamma)(1+m_t)$ is the gross growth rate of natives from generation t to generation $t+1$.

In period t, capital, K_t, and labor, L_t, are used to produce homogeneous output equal to $F(K_t, L_t)$, where $F(K_t, L_t)$ is strictly increasing, concave, and linearly homogeneous in K_t and L_t. Also, $F(0, L_t) = 0$, $F(K_t, 0) = 0$, $F_K(0, L_t) = \infty$, $F_L(K_t, 0) = \infty$, $F_K(\infty, L_t) = 0$, and $F_L(K_t, \infty) = 0$. Natives each consume c_t and immigrants each consume x_t so aggregate consumption in period t is $N_t c_t + M_t x_t$. Aggregate gross investment in period t is $K_{t+1} - (1-\delta)K_t$, where $\delta \in [0,1]$ is the depreciation rate of capital per period. The government buys an amount of output equal to $g > 0$ per person to provide rival, non-excludable goods, so total government purchases in period t equal $(N_t + M_t)g$. The sum of aggregate consumption, gross investment, and government purchases equals total output.

Thus, the expenditure identity for GDP is

$$N_t c_t + M_t x_t + K_{t+1} - (1-\delta) K_t + (N_t + M_t) g = F(K_t, L_t). \tag{3}$$

Divide both sides of equation (3) by N_t and use $m_t \equiv \frac{M_t}{N_t}$ to obtain

$$c_t + m_t x_t + \frac{K_{t+1}}{K_t}\frac{K_t}{L_t}\frac{L_t}{N_t} - (1-\delta)\frac{K_t}{L_t}\frac{L_t}{N_t} + (1+m_t) g = \frac{L_t}{N_t}\frac{1}{L_t} F(K_t, L_t). \tag{4}$$

Define the capital-labor ratio in period t, $k_t \equiv \frac{K_t}{L_t}$, and the intensive form of the production function $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$, where $f'(k) > 0$, $f(k) - k f'(k) > 0$, and $f''(k) < 0$ for all $k > 0$, $f(0) = 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. These assumptions imply that[1]

$$\lim_{k \to 0} k f'(k) = 0. \tag{5}$$

Use the definitions $k_t \equiv \frac{K_t}{L_t}$ and $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$ to rewrite equation (4) as

$$c_t + m_t x_t + \frac{K_{t+1}}{K_t} k_t (1+m_t) - (1-\delta) k_t (1+m_t) + (1+m_t) g = (1+m_t) f(k_t). \tag{6}$$

Henceforth, we confine attention to steady states, which are characterized by constant values of $\frac{M_t}{N_t} = m$ and $\frac{K_t}{L_t} = k$. The values of c_t and x_t are constant in a steady state and the steady-state values of these variables are indicated simply by dropping the time subscript. Equation (2) implies that in a steady state, the gross growth rate of N_t is

$$\Gamma(m) \equiv (1+\gamma)(1+m). \tag{7}$$

The constancy of $\frac{M_t}{N_t}$ implies that the gross growth rate of M_t is also $\Gamma(m)$. Therefore, the gross growth rate of the labor force, $L_t = N_t + M_t$, is $\Gamma(m)$, and the constancy of the capital-labor ratio, $\frac{K_t}{L_t}$, implies that the gross growth rate of K_t is equal to $\Gamma(m)$.

[1] Use $k f'(k) > 0$ for $k > 0$, $f(k) - k f'(k) > 0$ for $k > 0$ and $f(0) = 0$ to deduce that $0 \leq \lim_{k \to 0} k f'(k) \leq \lim_{k \to 0} f(k) = 0$, so $\lim_{k \to 0} k f'(k) = 0$.

In a steady state, the GDP expenditure identity in equation (6) can be written as

$$c = C(k, m) - mx, \tag{8}$$

where

$$C(k, m) \equiv (1 + m)\left[f(k) + (1 - \delta)k - \Gamma(m)k - g\right] \tag{9}$$

is the aggregate consumption of natives and immigrants, normalized by the number of natives.

We assume that there is an unlimited supply of potential immigrants who are willing to immigrate into the country provided that their consumption in the destination country, x, is at least as high as an exogenous threshold $\xi > 0$. One can think of ξ as the amount of consumption the potential immigrant would have in their country of origin, plus the costs of traveling to the new country, getting adjusted, and myriad other considerations. It is sufficient for our purposes that such a threshold value of x exists.

For given values of ξ and g, the planner chooses k, m, and x to maximize steady-state consumption per native, $c = C(k, m) - mx$ in equation (8). Because consumers live for only one period, there is no need to introduce a utility function in this section. Provided that utility is strictly increasing in consumption, maximizing c is equivalent to maximizing the utility derived from c; for immigrants, a threshold level of consumption ξ is equivalent to a threshold level of utility to be willing to immigrate into the country.[2]

Formally, the planner solves

$$\max_{k>0, m\geq 0, x} C(k, m) - mx \qquad \text{s.t. } x \geq \xi. \tag{10}$$

Because x enters the maximand in equation (10) negatively and linearly, the optimal value

[2]In Section 3, all consumers live for two-periods, and we introduce utility functions and a threshold level of utility for potential immigrants.

of x is ξ. Therefore, the planner's problem simplifies to

$$\max_{k>0,m\geq 0} C(k,m) - m\xi. \tag{11}$$

Partially differentiate $C(k,m) - m\xi$ with respect to k and set $\frac{\partial C(k,m)}{\partial k}$ equal to zero to obtain

$$1 + f'(k) - \delta = \Gamma(m). \tag{12}$$

In order to ensure that equation (12) has a solution for any $m \geq 0$, we assume that $\gamma+\delta > 0$.[3]

Equation (12) implies that the net marginal product of capital, $f'(k) - \delta$, equals the net growth rate of the economy, $\Gamma(m) - 1$. The equality of the net marginal product of capital and the net growth rate of the economy is the celebrated Golden Rule from Phelps (1961), extended to include endogenous growth resulting from immigration.

Definition 1 *Define $k(m)$ as the unique value of k that satisfies the extended Golden Rule condition in equation (12) for a given value of m.*

The definition of $k(m)$ and equation (12) imply that

$$\lim_{m\to\infty} k(m) = 0 \tag{13}$$

and

$$1 + f'(k(m)) - \delta = \Gamma(m), \tag{14}$$

which implies that

$$k'(m) = \frac{1+\gamma}{f''(k(m))} < 0. \tag{15}$$

[3]We require that the value of k that satisfies equation (12) has $f'(k) > 0$. Since equation (12) implies that $f'(k) = \Gamma(m) - (1-\delta) = (1+m)(1+\gamma) - (1-\delta) = m(1+\gamma) + \gamma + \delta$, we assume that $\gamma + \delta > 0$.

Use equation (14) to express normalized aggregate consumption in equation (9) when $k = k(m)$ as a function of only m

$$C(k(m), m) = (1+m)\left[f(k(m)) - k(m) f'(k(m)) - g\right]. \tag{16}$$

Lemma 1 $\frac{dC(k(m),m)}{dm} = f(k(m)) - k(m) f'(k(m)) - g - \Gamma(m) k(m)$.

Definition 2 *Define $k_G \equiv k(0)$ as the Phelpsian Golden Rule value of k in the absence of immigration. Also, define $c_G \equiv C(k_G, 0)$ as consumption per capita in the Phelpsian Golden Rule.*

Setting $m = 0$ in equation (16), the definitions of k_G and c_G imply

$$c_G = f(k_G) - k_G f'(k_G) - g. \tag{17}$$

To guarantee that $c_G > 0$ we assume

Assumption 1 $g < f(k_G) - k_G f'(k_G)$.

Having defined the function $k(m)$, the planner's problem can be written as the choice of a single scalar, m, as $\max_{m \geq 0} \Omega(m)$, where

$$\Omega(m) \equiv C(k(m), m) - m\xi. \tag{18}$$

Differentiate $\Omega(m)$ with respect to m, and use Lemma 1 to obtain

$$\Omega'(m) = f(k(m)) - k(m) f'(k(m)) - g - \Gamma(m) k(m) - \xi. \tag{19}$$

The following definition, which uses the expression for c_G in equation (17), will be helpful in characterizing a sufficient condition for the planner's optimal value of m to be positive.

Definition 3 $\bar{\xi} \equiv f(k_G) - k_G f'(k_G) - (1+\gamma) k_G - g = c_G - (1+\gamma) k_G$.

In order for $\overline{\xi}$ to be positive, it must be the case that $f(k_G) - k_G f'(k_G) - (1+\gamma) k_G > 0$, which is ensured by the following assumption.[4]

Assumption 2 *Assume that* $\alpha_G \equiv \frac{k_G f'(k_G)}{f(k_G)} < \frac{\gamma+\delta}{2(\gamma+\delta)+1-\delta} \leq \frac{1}{2}$.

As we show later in this section,[5] in some parameter configurations there is a knife-edge value of ξ for which $\max_{m \geq 0} \Omega(m)$ is attained by two values of m, one of which is zero and the other of which is positive. To avoid any ambiguity, we define the optimal steady-state value of m to be

$$m^* \equiv \min \left\{ \arg \max_{m \geq 0} \Omega(m) \right\}. \tag{20}$$

Thus, if $m^* > 0$, we can conclude that $\Omega(m^*) > \Omega(0)$, so $m = m^*$ strictly dominates $m = 0$.

Proposition 1 *If* $\xi < \overline{\xi} \equiv f(k_G) - k_G f'(k_G) - (1+\gamma) k_G - g$, *then*

1. $\Omega'(0) = \overline{\xi} - \xi > 0$
2. $\lim_{m \to \infty} \Omega'(m) = -g - \xi < 0$
3. $0 < m^* < \infty$
4. $0 < k^* = k(m^*) < k_G$

Since the proof of Proposition 1 is both simple and instructive, we include it in the text.

Proof of Proposition 1. Assume that $\xi < \overline{\xi}$. Evaluate $\Omega'(m)$ in equation (19) at $m = 0$, using $k(0) = k_G$, to obtain $\Omega'(0) = f(k_G) - f'(k_G) k_G - (1+\gamma) k_G - g - \xi = \overline{\xi} - \xi > 0$, which is Statement 1. Use $\lim_{m \to \infty} k(m) = 0$ from equation (13) along with the expression for $\Omega'(m)$ from equation (19) and the fact, from equation (5), that $\lim_{k \to 0} k f'(k) = 0$ to obtain $\lim_{m \to \infty} \Omega'(m) = -g - \xi < 0$, which is Statement 2. Statement

[4] $f(k_G) - k_G f'(k_G) - (1+\gamma) k_G > 0$ is equivalent to $f(k_G) - k_G f'(k_G) > (1+\gamma) k_G$, which can be written as $f(k_G) - \alpha_G f(k_G) > (1+\gamma) \frac{\alpha_G f(k_G)}{f'(k_G)}$. Since $f(k_G) > 0$, this condition is $1 - \alpha_G > (1+\gamma) \frac{\alpha_G}{f'(k_G)}$ $= \frac{1+\gamma}{\gamma+\delta} \alpha_G$, or equivalently, $1 > \left(\frac{1-\delta+\gamma+\delta}{\gamma+\delta} + \frac{\gamma+\delta}{\gamma+\delta} \right) \alpha_G = \frac{2(\gamma+\delta)+1-\delta}{\gamma+\delta} \alpha_G$, and finally $\alpha_G < \frac{\gamma+\delta}{2(\gamma+\delta)+1-\delta} \leq \frac{1}{2}$.

[5] See panel (c) of Figure 2.

1 implies that m^* cannot equal 0, and Statement 2 implies that m^* cannot be infinite. Therefore, m^* is positive and finite, which is Statement 3. Finally, equation (12) implies that $1 + f'(k^*) - \delta = \Gamma(m^*) > \Gamma(0) = 1 + f'(k_G) - \delta$. Therefore, since $\Gamma(m^*)$ is finite, $f'(k^*)$ is finite and hence $k^* > 0$. Also, since $f'(k^*) > f'(k_G)$ and $f''(k) < 0$, we have $k^* = k(m^*) < k_G$, which completes the proof of Statement 4. ■

Statement 1 in Proposition 1 implies that $\xi < \overline{\xi}$ is a sufficient condition for $m^* > 0$. We now interpret this sufficient condition and show its implications. Use the definition of $\overline{\xi}$ and rearrange $\xi < \overline{\xi}$ to obtain

$$\underbrace{f(k_G) - f'(k_G)\, k_G}_{\text{marginal product of labor}} > \underbrace{\xi + g}_{\text{output to immigrant}} + \underbrace{(1+\gamma)\, k_G}_{\text{new capital needed}}. \tag{21}$$

Suppose that the economy is in a Golden Rule steady state with zero immigration so that $k = k_G$. Now consider the impact of an immigrant. The left hand side of equation (21), which is the marginal product of labor, shows the additional output produced by the immigrant. The right hand side of equation (21) shows the amount of output that must be diverted from the consumption of natives in the form of: (1) ξ units of consumption for the immigrant; (2) g additional units of government purchases; and (3) the cost of providing each of the immigrant's $1 + \gamma$ children with k_G units of capital. If the immigrant's contribution to output on the left hand side of equation (21) exceeds the costs to the natives on the right hand side, then the optimal immigration rate is positive.

An alternative, but equivalent, implication of the sufficient condition $\xi < \overline{\xi}$ follows from subtracting g from both sides of equation (21) and using $c_G = f(k_G) - k_G f'(k_G) - g$ from equation (17) to obtain

$$\underbrace{c_G}_{\text{consumption of native}} > \underbrace{\xi}_{\text{consumption of immigrant}} + \underbrace{(1+\gamma)\, k_G}_{\text{new capital needed}}. \tag{22}$$

The condition in equation (22) is stronger than simply "the consumption of an immigrant, ξ, is less than the consumption of a native, c_G." Equation (22) requires that the consumption of an immigrant is less than the consumption of a native by at least the cost of providing

each of the immigrant's $1+\gamma$ children with k_G units of capital.

The condition $\xi < \overline{\xi}$ is a sufficient condition for $m^* > 0$, but it is not a necessary condition for $m^* > 0$. Indeed, Figure 2 panel (b) (discussed later in this section) illustrates a case in which $\xi > \overline{\xi}$ and yet $m^* > 0$.

Equation (19) implies that if $\Omega'(m^*) = 0$, then

$$f(k(m^*)) - k(m^*) f'(k(m^*)) = \xi + g + \Gamma(m^*) k(m^*). \tag{23}$$

We interpret equation (23) using the marginal benefit to the planner of an additional immigrant on the left hand side

$$MB(m) \equiv f(k(m)) - k(m) f'(k(m)), \tag{24}$$

which is simply the marginal product of labor, and the marginal cost to the planner of an additional immigrant on the right hand side

$$MC(m) \equiv \xi + g + \Gamma(m) k(m), \tag{25}$$

which comprises (a) $\xi + g$, the amount of output consumed by an additional immigrant in the form of private consumption (ξ) and consumption of public goods (g), and (b) the cost of creating enough additional capital to give each of the immigrant's $1+\gamma$ children, who will be natives, the amount of capital, $\frac{L_t k(m)}{N_t} = (1+m) k(m)$, that is allocated to natives. The cost of creating this additional capital for the $1+\gamma$ children is thus $(1+\gamma)(1+m) k(m) = \Gamma(m) k(m)$. Therefore, if $m^* > 0$, the planner's optimal steady state is characterized by $MB(m^*) = MC(m^*)$.

Let $c^* \equiv \Omega(m^*)$ be the optimal amount of consumption per native in the planner's optimal steady state.

Proposition 2 *If $m^* > 0$, then the optimal steady-state value of consumption per native is $c^* = \xi + (1+m^*) \Gamma(m^*) k(m^*)$.*

Proposition 2 immediately implies that if $m^* > 0$, then in the planner's optimal steady

state, the consumption of each native, c^*, exceeds the consumption of each immigrant.

2.1 Cobb-Douglas Production Function

For the remainder of Section 2, we assume that the production function is Cobb-Douglas, specifically, $f(k) = Ak^{\alpha}$, with $A > 0$ and $0 < \alpha < 1$. With this specification of the production function, the marginal benefit of m from equation (24) is

$$MB(m) = (1-\alpha) f(k(m)) \tag{26}$$

and the marginal cost of m from equation (25) is

$$MC(m) = \xi + g + \alpha f(k(m)) + (1-\delta) k(m). \tag{27}$$

Remark 1 *In an economy with a capital-intensive Cobb-Douglas production function, i.e., with $\alpha \geq \frac{1}{2}$ (which violates Assumption 2) the planner's optimal steady state will have zero immigration.*[6]

Remark 1 follows from equations (26) and (27), which imply $MB(m) - MC(m) = (1-2\alpha) f(k(m)) - [(1-\delta) k(m) + \xi + g]$. Thus, if $\alpha \geq \frac{1}{2}$, we have $MB(m) - MC(m) < 0$ for all positive m, and therefore, $m^* = 0$. Except for Remark 1, we maintain Assumption 2, $\alpha_G < \frac{1}{2}$, throughout the paper.

Lemma 2 *If $f(k) = Ak^{\alpha}$ and $\alpha < \frac{1}{2}$, then $\Omega''(m) \lesseqgtr 0$ as $m \gtreqless \Theta \equiv 2\frac{1-\alpha}{1-2\alpha}\frac{1-\delta}{1+\gamma} - 1$.*

If $m^* > 0$, we have $\Omega(m^*) > \Omega(0)$, $\Omega'(m^*) = 0$, and $\Omega''(m^*) < 0$. Lemma 2 implies that $\Omega(m)$ attains a local maximum at no more than one positive value of m. These features of $\Omega(m)$, together with $\frac{d\Omega'(m^*)}{d\xi} = -1$, which follows from equation (19), immediately imply the following the following proposition.

[6] Correani et al. (2015) find that "the long-run effect of migration on per-capita domestic consumption is positive when production is 'sufficiently' reactive to capital." (p. 442). That finding is derived in a competitive economy where the flow of immigrants is determined by the wage differential between immigrants' home and destination country, and immigrants are assumed not to save; in contrast, the model in the current paper focusses on a planners' optimal allocation, including the planner's optimal choice of the rate of immigration, and assumes that immigrants and natives have identical intertemporal preferences.

Proposition 3 *If* $m^* > 0$*, then* $\frac{dm^*}{d\xi} = \frac{1}{\Omega''(m^*)} < 0$.

Lemma 2 implies that if $\Theta < 0$, then $\Omega(m)$ is strictly concave for all $m \geq 0$. Therefore, Statement 1 of Proposition 1 implies the following remark.

Remark 2 *If* $\Theta < 0$*, then* $\xi < \overline{\xi}$ *is necessary as well as sufficient for* $m^* > 0$.

To explore the source of the potential non-concavity of the planner's objective function, $\Omega(m)$, that is, the potential for $\Omega''(m) > 0$, it is helpful to rewrite $\Omega'(m)$ in equation (19) as $\Omega'(m) = MB(m) - MC(m)$ so $\Omega''(m) = MB'(m) - MC'(m)$. The slopes of $MB(m)$ and $MC(m)$ in equations (26) and (27), respectively, are $MB'(m) = (1-\alpha) f'(k(m)) k'(m) < 0$ and $MC'(m) = [\alpha f'(k(m)) + 1 - \delta] k'(m) < 0$ since $k'(m) < 0$. Thus, $MB(m)$ and $MC(m)$ both slope downward. If the marginal cost curve slopes downward less (more) steeply than the marginal benefit curve at a given value of m, then the objective function is concave (convex) at that value of m.

Figure 1 illustrates two distinct cases: $\Theta < 0$ in panel (a) and $\Theta > 0$ in panel (b).

Case I: $\Theta < 0$. Lemma 2 implies that $MB'(m) - MC'(m) = \Omega''(m) < 0$ for all $m \geq 0$. Since the $MB(m)$ curve slopes downward more steeply than the $MC(m)$ curve for all $m \geq 0$, the two curves intersect at most once for $m \geq 0$. If the vertical intercept of the $MB(m)$ curve is higher than the vertical intercept of the $MC(m)$, as in Panel (a), then $\Omega'(0) > 0$, so the optimal value of m is located at the intersection, where $MB(m)$ crosses $MC(m)$ from above.

Case II: $\Theta > 0$. Lemma 2 implies that $MB'(m) - MC'(m) = \Omega''(m) \gtreqless 0$ as $m \lesseqgtr \Theta$. Thus, as shown in Panel (b) of Figure 1, for $m < \Theta$, the $MC(m)$ curve slopes downward more steeply than the $MB(m)$ curve, so the objective function $\Omega(m)$ is convex; for $m > \Theta$, the $MC(m)$ curve slopes downward less steeply than the $MB(m)$ curve, so the objective function $\Omega(m)$ is concave. As Panel (b) is drawn, the vertical intercept of $MC(m)$ is higher than the vertical intercept of $MB(m)$ and the two curves intersect twice. If the vertical intercept of $MC(m)$ were lower than the vertical intercept of $MB(m)$, the two curves would intersect only once, and the $MB(m)$ curve would intersect the $MC(m)$ curve from above at that intersection.

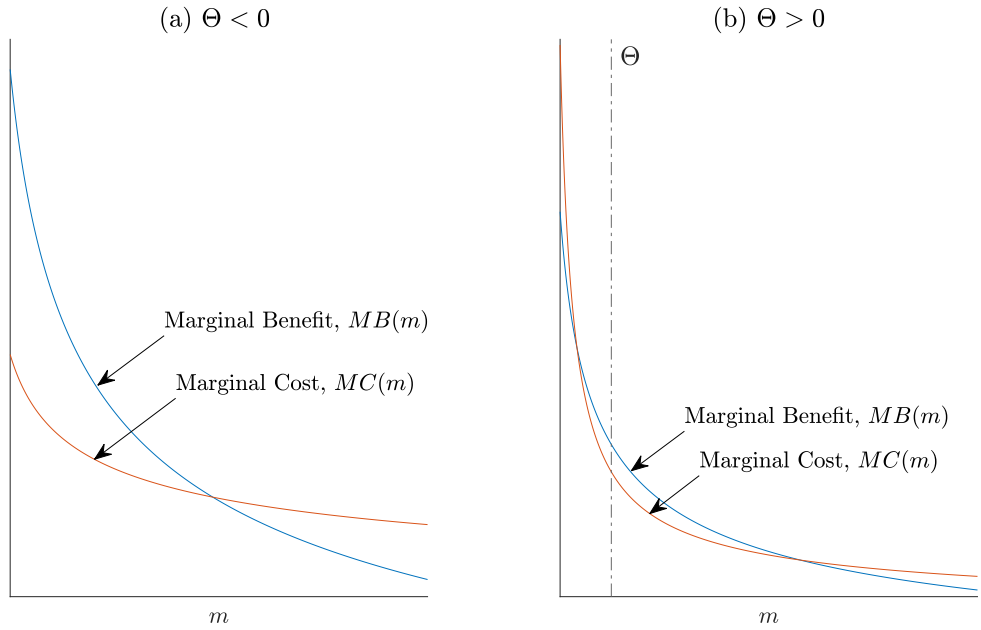


Figure 1: Marginal benefit, $MB(m)$, and marginal cost, $MC(m)$, curves. In Panel (a), $\Theta < 0$ so $MB\,(m)$ slopes downward more steeply than $MC\,(m)$ for all $m \geq 0$; at the unique intersection of these two curves, $m = m^*$. In Panel (b), $\Theta > 0$ so $MB\,(m)$ slopes downward less steeply than $MC\,(m)$ for all $0 \leq m < \Theta$, and $MB\,(m)$ slopes downward more steeply than $MC\,(m)$ for $m > \Theta$; m^* equals the larger of the two values of m at which these two curves intersect.

To summarize Lemma 2, the objective function $\Omega\,(m)$ can have a convex segment with $m \geq 0$ only if $\Theta > 0$, which requires that the depreciation rate, δ, is less than one. Thus, if $\delta = 1$, the objective function $\Omega\,(m)$ is concave for all $m \geq 0$.

Lemma 3 *If* $f\,(k) = Ak^{\alpha}$ *with* $A > 0$ *and* $0 < \alpha < \frac{1}{2}$, *and* $\Theta > 0$, *then there exists a unique finite* $\widehat{\xi} > \overline{\xi}$ *such that* $\max_{m>0} \Omega\,(m) = \Omega\,(0)$ *if* $\xi = \widehat{\xi}$.

Figure 2 illustrates a numerical example using a Cobb-Douglas production function, $f\,(k) = Ak^{\alpha}$, where $A = 100$ and $\alpha = 0.3$. In addition, $\gamma = 0.1$, $g = 40$, and $\delta = 0.6$ in all four panels of this figure. The values of these parameters are chosen simply to illustrate salient features of $\Omega\,(m)$–they are not to be considered a calibration. These parameter values imply $\overline{\xi} = 74.409$, $k_G = 214.517$, $f\,(k_G) = 500.540$, $c_G = 310.378$, and $\Theta = 0.273$. The value of Θ, along with Lemma 2, implies that $\Omega\,(m)$ is strictly convex for non-negative

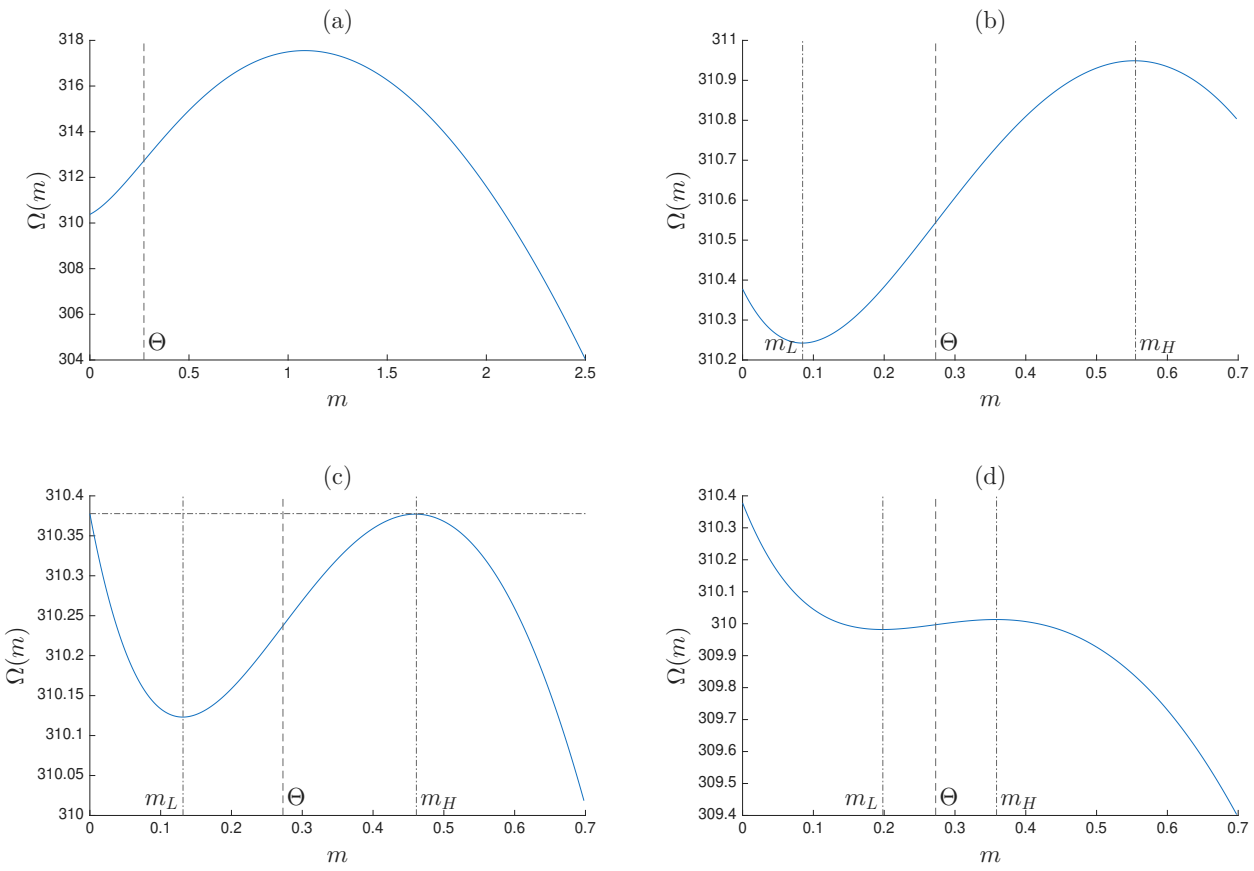


Figure 2: All four panels this figure show the value of $\Omega(m)$ on the vertical axis as function of m on the horizontal axis, for $f(k) = Ak^{\alpha}$, where $A = 100$ and $\alpha = 0.3$, and $\gamma = 0.1$, $g = 40$, and $\delta = 0.6$. These parameter values imply $\overline{\xi} = 74.409$ and $\Theta = 0.273$. The values of ξ in panels (a), (b), (c), and (d) are 70, 79, 79.118, and 80, respectively.

values of $m < 0.273$, and is strictly concave for $m > 0.273$. The only parameter that has different values in the four panels is the value of immigrants' consumption threshold, ξ, which increases from panels (a) to (b) to (c) to (d).

In Panel (a), $\xi = 70 < \overline{\xi}$, so Statement 1 of Proposition 1 implies that $\Omega'(0) > 0$ and Lemma 2 implies that $\arg\max_{m \geq 0} \Omega(m)$ is positive and unique. In Panel (b), $\xi = 79 > \overline{\xi}$, so Statement 1 of Proposition 1 implies that $\Omega'(0) < 0$. There are two values of $m > 0$ for which $\Omega'(m) = 0$, the smaller of which, m_L, locally minimizes $\Omega(m)$, and the larger of which, m_H, locally maximizes $\Omega(m)$. Since $\Omega(m_H) > \Omega(0)$, we have $m^* = m_H > 0$. In Panel (c), $\xi = 79.118 > \overline{\xi}$, and as shown in this panel, $\Omega(m_H) = \Omega(0)$. Therefore, $\{\arg\max_{m \geq 0} \Omega(m)\} = \{0, m_H\}$, that is, $\arg\max_{m \geq 0} \Omega(m)$ is not unique. The

	$\alpha < \frac{1}{2}$		$\alpha \geq \frac{1}{2}$
	$\Theta \leq 0$ $\Omega(m)$ concave	$\Theta > 0$ $\Omega(m)$ concave iff $m \geq \Theta$	
$\xi < \overline{\xi}$	$m^* > 0$	$m^* > \Theta$	$m^* = 0$
$\overline{\xi} \leq \xi < \widehat{\xi}$	$m^* = 0$	$m^* \geq \Theta$	
$\xi \geq \widehat{\xi}$	$m^* = 0$	$m^* = 0$	

Table 1: This table shows the value of m^* for various configuration of α, Θ, and ξ.

definition of m^* in equation (20) resolves this multiplicity by choosing the smaller value, so $m^* = 0$. As discussed earlier, this resolution criterion implies that whenever m^* defined in equation (20) is positive, $\Omega(m^*)$ strictly dominates $\Omega(0)$, so the optimal rate of immigration is unambiguously positive. Finally, in panel (d), $\xi = 80 > \overline{\xi}$. In this case, the consumption threshold of immigrants is high enough that $\Omega(m_H) < \Theta(0)$, so $m^* = 0$.

Table 1 summarizes the value of m^* for various parameter configurations.

Remark 3 *If $\Theta > 0$, then if m^* turns out to be positive, it cannot be less than Θ. Thus, the function $m^*(\xi)$, which is the optimal value of m for given value of ξ, holding A, α, δ, and γ fixed, has a discontinuity at $\xi = \widehat{\xi}$.*

For the example in Figure 2, the discontinuity occurs at $\widehat{\xi} = 79.118$; $m^*(\xi) > \Theta > 0$ for $\xi < 79.118$ and $m^*(\xi) = 0$ for $\xi \geq 79.118$.

3 Two-Period Lifetimes

In this section, we extend the lifetimes of individual consumers from one period to two periods, so that in a competitive economy, individual consumers have a meaningful consumption/saving decision and hold capital. Extending lifetimes to two periods introduces some additional considerations. Consumers who live for two periods obtain utility from consumption in both periods. The planner maximizes the utility of a representative native in the steady state to each immigrant attaining an an exogenous threshold level of utility. This constraint on immigrants' utility is complicated by the fact that lifetime utility from consumption in their new country depends on the rate at which consumption in one period

can be transformed into consumption in the following period, which is an endogenous variable. Another important consideration is that in the two-period version of the model, only young consumers work, but all consumers consume, so the dependency ratio is the ratio of old consumers to young consumers. Not only is the dependency ratio positive, it is endogenous. Despite these additional considerations, much of the analytic apparatus developed for the case of one-period lifetimes continues to be useful when we extend lifetimes to two periods. In fact, the one-period lifetime version of the model is a special case of the two-period version we introduce in this section when the intertemporal discount factor, β, equals zero.

We begin by extending the national income accounting framework to reflect the assumption that all consumers live for two periods. As in the case of one-period lifetimes, each consumer, whether a native or an immigrant, gives birth to $1+\gamma$ children at the end of the first period of their life. The generations overlap so that in each period, there are young and old natives as well as young and old immigrants. All young consumers work and all old consumers are retired, so total employed labor in period t is $L_t \equiv N_t + M_t$, where, as in the version with one-period lifetimes, N_t natives are born at the beginning of period t and M_t foreign-born young consumers immigrate into the country at the beginning of period t. In period t, each of the N_t young natives consumes c_t^y and each of the M_t young immigrants consumes x_t^y. Also in period t, the N_{t-1} old natives each consume c_t^o and the M_{t-1} old immigrants each consume x_t^o, so aggregate consumption in period t is $N_t c_t^y + N_{t-1} c_t^o + M_t x_t^y + M_{t-1} x_t^o$. As in Section 2, aggregate gross investment in period t is $K_{t+1} - (1-\delta) K_t$. For simplicity, we scale government purchases of goods to the number of young consumers, so that, as in Section 2, government purchases of goods equal $(N_t + M_t) g$. Therefore, the GDP expenditure identity is

$$N_t c_t^y + N_{t-1} c_t^o + M_t x_t^y + M_{t-1} x_t^o + K_{t+1} - (1-\delta) K_t + (N_t + M_t) g = F(K_t, L_t). \quad (28)$$

Divide both sides of equation (28) by N_t to obtain

$$c_t^y + \frac{N_{t-1}}{N_t} c_t^o + \frac{M_t}{N_t} x_t^y + \frac{M_{t-1}}{N_{t-1}} \frac{N_{t-1}}{N_t} x_t^o + \frac{K_{t+1}}{K_t} \frac{K_t}{L_t} \frac{L_t}{N_t} - (1-\delta) \frac{K_t}{L_t} \frac{L_t}{N_t} + \left(1 + \frac{M_t}{N_t}\right) g = \frac{L_t}{N_t} \frac{1}{L_t} F(K_t, L_t).$$

(29)

Recall that $m_t \equiv \frac{M_t}{N_t}$, $k_t \equiv \frac{K_t}{L_t}$, and $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$ to rewrite equation (29) as

$$\begin{array}{r} c_t^y + \frac{N_{t-1}}{N_t} c_t^o + m_t x_t^y + m_{t-1} \frac{N_{t-1}}{N_t} x_t^o \\ + \frac{K_{t+1}}{K_t} k_t (1 + m_t) - (1 - \delta) k_t (1 + m_t) + (1 + m_t) g \end{array} = (1 + m_t) f(k_t). \tag{30}$$

In a steady state, $k_t = k$, $m_t = m$, $c_t^y = c^y$, $c_t^o = c^o$, $x_t^y = x^y$ and $x_t^o = x^o$ are all constant. As in Section 2, the gross growth rates of K_t, L_t, M_t, and N_t are all equal to $\Gamma(m) \equiv (1 + m)(1 + \gamma)$ in a steady state. Therefore, in a steady state, equation (30) can be written, after isolating the consumption of natives on the left hand side, as

$$c^y + \frac{1}{\Gamma(m)} c^o = C(k, m) - m \left[x^y + \frac{1}{\Gamma(m)} x^o \right], \tag{31}$$

where, from equation (9), $C(k, m) \equiv (1 + m)[f(k) + (1 - \delta) k - \Gamma(m) k - g]$ is the aggregate consumption of natives and immigrants normalized by the number of young natives.

The planner chooses c^y, c^o, x^y, x^o, k, and m to maximize the utility of a representative native in the steady state, $u(c^y, c^o)$, subject to the constraint in equation (31) and the constraint that the utility of a representative immigrant in the steady state, $u(x^y, x^o)$, is at least as high as a threshold value, $\overline{u}$,

$$u(x^y, x^o) \geq \overline{u}, \tag{32}$$

which is the analog of the constraint $x \geq \xi$ in Section 2. As in Section 2, this constraint will hold as an equality. To obtain the optimal values of x^y and x^o, we need to introduce a specific utility function. For simplicity, we assume that the utility function is logarithmic and write the binding version of the constraint in equation (32) as

$$(1 - \beta) \ln x^y + \beta \ln x^o = \overline{u}, \tag{33}$$

where $0 \leq \beta < 1$.

For a given value of m, the planner chooses x^y and x^o to minimize the amount of consumption devoted to immigrants in each period in the steady state subject to the constraint in equation (33). This minimized value of consumption devoted to immigrants is $mX(m)$, where

$$X(m) \equiv \min_{x^y, x^o} x^y + \frac{1}{\Gamma(m)} x^o \quad \text{s.t.} \quad (1-\beta)\ln x^y + \beta \ln x^o = \overline{u}. \tag{34}$$

We show in equations (A.8) and (A.9) in Appendix A that

$$X(m) = \xi (1+m)^{-\beta} \leq \xi, \tag{35}$$

where

$$\xi \equiv \exp(\overline{u})(1-\beta)^{-(1-\beta)} \beta^{-\beta} (1+\gamma)^{-\beta} > 0. \tag{36}$$

The definition of ξ in equation (36) is the analog of ξ in the case of one-period lifetimes in Section 2.

Now substitute $X(m)$ for $x^y + \frac{1}{\Gamma(m)} x^o$ in equation (31). We assume that natives have the same logarithmic utility function that immigrants have, so the planner's problem can be written as[7]

$$\max_{c^y, c^o} (1-\beta)\ln c^y + \beta \ln c^o \quad \text{s.t.} \quad c^y + \frac{1}{\Gamma(m)} c^o = C(k,m) - mX(m). \tag{37}$$

With the logarithmic utility function in equation (37), the optimal values of consumption by natives are[8]

$$c^y = (1-\beta)[C(k,m) - mX(m)] \tag{38}$$

[7]The optimization problem in equation (37) is meaningful only if $C(k,m) - mX(m) > 0$. The definition of $V(m)$ in equation (44) implies that the planner will never choose (k,m) for which $C(k,m) - mX(m) \leq 0$.

[8]See equations (A.4) and (A.5) in Appendix A, with c^y replacing z^y, c^o replacing z^o, $\Gamma(m)$ replacing $1+r$, and $C(k,m) - mX(m)$ replacing ϕ.

and

$$c^o = \Gamma(m)\,\beta\,[C(k,m) - mX(m)]\,. \tag{39}$$

Evaluating the logarithmic utility function of natives at the optimal values of c^y and c^o in equations (38) and (39), respectively, yields

$$\begin{aligned} u(c^y, c^o) &= \ln[C(k,m) - mX(m)] + \beta \ln \Gamma(m) \\ &\quad + (1-\beta)\ln(1-\beta) + \beta \ln \beta. \end{aligned} \tag{40}$$

Partially differentiate the right hand side of equation (40) with respect to k and set the partial derivative equal to zero to obtain $\frac{\partial C(k,m)}{\partial k} = 0$, just as in the case with one-period lifetimes in Section 2. Therefore, the expression for the extended Golden Rule in equation (12) in Section 2 continues to hold with two-period lifetimes. Thus, the function relating the extended Golden Rule capital-labor ratio to m, $k(m)$, is unchanged from Section 2, though, of course, the optimal values of m and k may differ from their values in that section.

Definition 4 *In a steady state with immigration ratio m and $k = k(m)$, define $\mathcal{C}(m) \equiv c^y + \frac{1}{\Gamma(m)} c^o$.*

$\mathcal{C}(m)$ has two alternative interpretations: it is the total consumption of natives in period t in the extended Golden Rule steady state, $N_t c_t^y + N_{t-1} c_t^o$, divided by N_t; it is also the present value of a native's consumption over the two-period lifetime, $c^y + \frac{1}{1+r} c^o$, in the extended Golden Rule steady state, in which the shadow gross discount rate, $1+r$, equals $\Gamma(m)$.

The definition of $\mathcal{C}(m)$ implies that $\mathcal{C}(0) = C(k(0), 0)$ so equation (16) and the definition $k_G \equiv k(0)$ imply that

$$\mathcal{C}(0) = f(k_G) - k_G f'(k_G) - g = c_G, \tag{41}$$

where the second equality follows from equation (17).

Use the definition of $\mathcal{C}(m)$ to rewrite the constraint in equation (37) as

$$\mathcal{C}(m) + mX(m) = C(k(m), m). \tag{42}$$

Divide both sides of equation (42) by $1+m$ and use the expression for $C(k(m), m)$ from equation (16) to obtain

$$\frac{1}{1+m}\mathcal{C}(m) + \frac{m}{1+m}X(m) = f(k(m)) - k(m) f'(k(m)) - g. \tag{43}$$

Equation (43) shows that for $m > 0$, $f(k(m)) - k(m) f'(k(m)) - g$ lies between $\mathcal{C}(m)$ and $X(m)$.

The planner's problem can be formulated as the choice of a single scalar, m, by substituting $k(m)$ for k in equation (40), ignoring additive constants, and using equation (42) to obtain $\max_{m \geq 0} V(m)$, where

$$V(m) \equiv \begin{cases} \ln \mathcal{C}(m) + \beta \ln \Gamma(m), & \text{if } C(k(m), m) - mX(m) > 0 \\ -\infty, & \text{if } C(k(m), m) - mX(m) \leq 0 \end{cases}. \tag{44}$$

The second line in equation (44) guarantees that the planner will never choose a value of m for which, $C(k(m), m) - mX(m) \leq 0$.

Lemma 4 *There exists a finite $\widehat{m} > 0$ such that $V(m) = -\infty$ for all $m \geq \widehat{m}$.*

Lemma 4 is useful in proving the following proposition.

Proposition 4 $0 \leq \arg\max_{m \geq 0} V(m) < \infty$.

We have not shown that $\arg\max_{m \geq 0} V(m)$ is unique. Hence, as in Section 2, to avoid any ambiguity, we define the optimal steady-state value of m to be

$$m^{**} \equiv \min \left\{ \arg\max_{m \geq 0} V(m) \right\}. \tag{45}$$

Therefore, if $m^{**} > 0$, we know that $V(m^{**}) > V(0)$ and the optimal rate of immigration in the steady state is unambiguously positive. The two asterisks in m^{**} indicate that we are considering two-period lifetimes, in contrast to m^*, which refers to one-period lifetimes.

Lemma 5 *For any non-negative m for which $\mathcal{C}(m) > 0$, $V'(m) = \frac{1}{1+m}\frac{1}{\mathcal{C}(m)}H(m)$, where $H(m) \equiv \mathcal{C}(m) - X(m) - (1+m)\Gamma(m)k(m) + \beta[\mathcal{C}(m) + mX(m)]$.*

The following lemma uses Lemma 5 to obtain two alternative, but equivalent, expressions for $V'(0)$, which plays a key role in a sufficient condition for m^{**} to be positive.

Lemma 6 *Define $\overline{\overline{\xi}} \equiv (1+\beta)\overline{\xi} + \beta(1+\gamma)k_G \geq \overline{\xi} > 0$. Then*

1. $V'(0) = \frac{1}{\mathcal{C}(0)}\left(\overline{\overline{\xi}} - \xi\right)$
2. $V'(0) = \frac{1}{\mathcal{C}(0)}[\beta\mathcal{C}(0) - (1+\gamma)k_G + \mathcal{C}(0) - X(0)]$.

The expression for $V'(0)$ in Statement 1 is presented in terms of parameters directly related to the threshold level of utility for potential immigrants. The expression for $V'(0)$ in Statement 2 is presented in terms of consumption and capital in the Phelpsian Golden Rule, which is the planner's optimal steady state in the absence of immigration.

Proposition 4, which states that m^{**} is finite and Statement 1 of Lemma 6, which implies that $\xi < \overline{\overline{\xi}}$ is a sufficient condition for $V'(0) > 0$, together immediately imply the following proposition.

Proposition 5 *If $\xi < \overline{\overline{\xi}}$, then*

1. $0 < m^{**} < \infty$
2. $0 < k^{**} < k_G$.

The next step is to characterize the values of $\mathcal{C}(m)$ and $X(m)$ in the planner's optimal steady state when $m^{**} > 0$. When $m^{**} > 0$, the first-order condition $V'(m^{**}) = 0$ and Lemma 5 imply that $H(m^{**}) = 0$. The following proposition follows immediately from $H(m^{**}) = 0$.

Proposition 6 *If* $m^{**} > 0$*, then*

$$\mathcal{C}(m^{**}) = X(m^{**}) + (1 + m^{**})\,\Gamma(m^{**})\,k(m^{**}) - \beta\left[\mathcal{C}(m^{**}) + m^{**}X(m^{**})\right].$$

Proposition 6 is a generalization of Proposition 2 to two-period lifetimes. Specifically, $\mathcal{C}(m^{**})$, $X(m^{**})$, and $(1 + m^{**})\,\Gamma(m^{**})\,k(m^{**})$ are the two-period analogs of c^*, ξ, and $(1 + m^*)\,\Gamma(m^*)\,k(m^*)$ in one-period lifetimes.

The following corollary to Proposition 6 is used to interpret m^{**} as the value of m that equates the marginal benefit and marginal cost to a representative native of an additional immigrant.

Corollary 1 *If* $m^{**} > 0$*, then* $f(k(m^{**})) - k(m^{**})\,f'(k(m^{**})) = X(m^{**}) + g + \Gamma(m^{**})\,k(m^{**}) - \frac{\beta}{1+m^{**}}\left[\mathcal{C}(m^{**}) + m^{**}X(m^{**})\right]$.

To interpret the term $-\frac{\beta}{1+m^{**}}\left[\mathcal{C}(m^{**}) + m^{**}X(m^{**})\right]$ in Corollary 1, first consider how the planner optimally allocates a given amount of consumption C to c^y and c^o. This optimization problem is

$$U(C, \Gamma(m)) \equiv \max_{c^y, c^o} (1-\beta)\ln c^y + \beta \ln c^o \quad \text{s.t. } c^y + \frac{1}{\Gamma(m)} c^o = C. \tag{46}$$

Ignoring additive constants, equations (A.1) and (A.6) in Appendix A (replacing ϕ by C and replacing $1 + r$ by $\Gamma(m)$) imply

$$U(C, \Gamma(m)) = \ln C + \beta \ln \Gamma(m). \tag{47}$$

The planner's allocation problem in equation (46) has two alternative interpretations. One interpretation is that c^y is the consumption of a young native in period t and c^o is the consumption of that consumer in period $t+1$ when old. In this case, $\Gamma(m)$ is $1 + f'(k(m)) - \delta$, which is the marginal rate of transformation between consumption goods in two adjacent periods in the steady state. Another interpretation is that equation (46) is a static problem at time t in the steady state, in which young natives in the aggregate consume $N_t c^y$ and old

natives in the aggregate consume $N_{t-1}c^o$, so the constraint is $N_t c^y + N_{t-1}c^o = N_t c^y + \frac{N_t}{\Gamma(m)}c^o = N_t C$. In this interpretation, $\Gamma(m)$ is the growth rate of the population, and $\frac{1}{\Gamma(m)}$ is the dependency ratio, defined as the ratio of the number of non-workers, who are old, to the number of workers, who are young. An increase in m increases $\Gamma(m)$ and thus reduces the dependency ratio, $\frac{1}{\Gamma(m)}$, thereby increasing $U(C, \Gamma(m))$.

The partial derivatives of the indirect utility function in equation (47) are $U_C = \frac{1}{C}$ and $U_\Gamma = \frac{\beta}{\Gamma}$, so the impact on $U(C, \Gamma(m))$ of an increase in m, measured in terms of consumption goods, is

$$\frac{U_\Gamma \Gamma'(m)}{U_C} = \frac{\beta C}{\Gamma}\Gamma'(m) = \frac{\beta C}{1+m}. \tag{48}$$

For each immigrant, the planner solves the optimization problem in equation (34), $X(m) \equiv \min_{x^y, x^o} x^y + \frac{1}{\Gamma(m)}x^o$ s.t. $(1-\beta)\ln x^y + \beta \ln x^o = \overline{u}$. Again, there are two interpretations of this problem. One interpretation is that x^y is the consumption of a young immigrant in period t, and x^o is the consumption of that immigrant in period $t+1$ when old. In this case, as for natives, $\Gamma(m)$ is $1 + f'(k(m)) - \delta$, which is the marginal rate of transformation between consumption goods in two adjacent periods in the steady state. Alternatively, the optimization problem in equation (34) can be viewed as a static problem at time t in the steady state, in which young immigrants in the aggregate consume $M_t x^y$ and old immigrants in the aggregate consume $M_{t-1}x^o$. In either interpretation, $X(m) = \xi(1+m)^{-\beta}$, as shown in equation (35). For an immigrant, if m increases, the planner maintains a level of utility equal to the threshold value $\overline{u}$ by reducing $X(m) = \xi(1+m)^{-\beta}$, by $-X'(m) = \frac{\beta}{1+m}X(m)$; reducing $X(m)$ for each of the M_t immigrants by $\frac{\beta}{1+m}X(m)$ enables the planner to increase the value of C in equation (47) for each of the N_t natives by $\frac{1}{N_t}M_t\frac{\beta}{1+m}X(m) = \frac{\beta}{1+m}mX(m)$. Adding this marginal benefit accruing to each of the natives to the marginal benefit in equation (48), yields a combined marginal benefit to each native of $\frac{\beta}{1+m}[C + mX(m)]$. Therefore, Corollary 1 can be written as $MB(m^{**}) = \widehat{MC}(m^{**})$, where the marginal benefit function, $MB(m)$, is identical to the marginal benefit function in equation (24) defined for one-period lifetimes, and the marginal cost function, $\widehat{MC}(m)$

$\equiv X(m) + g + \Gamma(m)\, k(m) - \frac{\beta}{1+m}[C + mX(m)]$, which, using the marginal cost function $MC(m)$ in equation (25), equals $MC(m)$ minus $\frac{\beta}{1+m}[C + mX(m)]$.

The next step is to determine whether $\mathcal{C}(m^{**})$ is larger than or smaller than $X(m^{**})$. The following proposition provides a sufficient condition for $\mathcal{C}(m^{**}) > X(m^{**})$.

Proposition 7 *If* $\xi < \min\left\{\overline{\overline{\xi}}, c_G\right\}$, *then* $\mathcal{C}(m^{**}) > X(m^{**})$.

To explore the possibility that $\mathcal{C}(m^{**}) \leq X(m^{**})$ in the planner's optimal steady state, we introduce the following definition.

Definition 5 *Define* $\Upsilon \equiv \Gamma(m^{**})\, k(m^{**}) - \beta[f(k(m^{**})) - k(m^{**})\, f'(k(m^{**})) - g]$.

We use Definition 5 to obtain the following corollary to Proposition 6.

Corollary 2 *If* $m^{**} > 0$, *then* $\mathcal{C}(m^{**}) - X(m^{**}) = (1 + m^{**})\,\Upsilon$.

3.1 Cobb-Douglas Production Function

To simplify the analysis of the relative sizes of $\mathcal{C}(m^{**})$ and $X(m^{**})$, we assume throughout this subsection that the production function is Cobb-Douglas.

Proposition 8 *Assume that* $m^{**} > 0$ *and that the production function is* $f(k) = Ak^{\alpha}$, *with* $A > 0$ *and* $0 < \alpha < 1$. *Then*

1. $\Upsilon = [\alpha - \beta(1-\alpha)]\, f(k(m^{**})) + (1-\delta)\, k(m^{**}) + \beta g$
2. *If* $\beta(1-\alpha) \leq \alpha$, *then* $\Upsilon > 0$ *and* $\mathcal{C}(m^{**}) > X(m^{**})$
3. *If* $\mathcal{C}(m^{**}) \leq X(m^{**})$, *then* $\beta(1-\alpha) > \alpha$.

Proposition 8 implies that a necessary condition for $\mathcal{C}(m^{**}) \leq X(m^{**})$ is $\beta(1-\alpha) > \alpha$. To interpret this restriction on α and β, consider the steady state of a laissez-faire economy without government purchases, taxes, or government bonds, with $\delta = 1$, and without immigration. In such an economy, aggregate investment, which equals aggregate saving, is

$\beta(1-\alpha)f(k)$, and aggregate capital income is $\alpha f(k)$, so the condition $\beta(1-\alpha)-\alpha>0$ implies that aggregate investment is greater than aggregate capital income in every period, which is a sufficient condition for a dynamically inefficient overaccumulation of capital in Abel et al. (1989).[9]

In the case of a Cobb-Douglas production function and a depreciation rate, δ, equal to one, Corollary 3 in Appendix B provides a function $\xi(\alpha,\beta,\gamma,A,g)$ such the first-order condition in Proposition 6 is satisfied by $\mathcal{C}(m)=X(m)$ when $\xi=\xi(\alpha,\beta,\gamma,A,g)$. Corollary 3 does not address concavity of $V(m)$ when the first-order condition is satisfied nor does it address the possibility that the first-order condition is satisfied by multiple values of m, or that the planner's optimal value of m is a corner solution with $V(0)>V(m)$ for all $m>0$. Thus, to demonstrate the possibility that $\mathcal{C}(m^{**})=X(m^{**})$ in the planner's optimal steady state, Figure 3 illustrates a numerical example for various values of g on the horizontal axis and ξ on the vertical axis, for given values of the parameters A, α, β, and γ. In the shaded area in the upper right of the diagram, the values of g and/or ξ are sufficiently high that $m^*=0$; for the unshaded area, $m^*>0$. The downward-sloping curves represent level sets of $\frac{\mathcal{C}(m^{**})}{X(m^{**})}$, with the constant value of $\frac{\mathcal{C}(m^{**})}{X(m^{**})}$ shown on each level set. Of particular interest is uppermost labelled level set, along which $\frac{\mathcal{C}(m^{**})}{X(m^{**})}=1$. This level set traces out the function $\xi(\alpha,\beta,\gamma,A,g)$ for different values of g. For the region below and to the left of this level set, $\mathcal{C}(m^{**})>X(m^{**})$. However, for the unshaded sliver of the diagram above and to the right of this level set, $\xi>\xi(\alpha,\beta,\gamma,A,g)$ and $\mathcal{C}(m^{**})<X(m^{**})$.

[9] In periods t and $t+1$ in the steady state of a laissez-faire economy without immigration, $\beta(1-\alpha)f(k)L_t = K_{t+1} = kL_{t+1}$, so $\beta(1-\alpha)f(k)=(1+\gamma)k$. Since $f(k)=\frac{1}{\alpha}f'(k)k$, we have $\beta(1-\alpha)\frac{1}{\alpha}f'(k)=1+\gamma$, which implies $f'(k)=\frac{\alpha}{\beta(1-\alpha)}(1+\gamma)=\frac{\alpha}{\beta(1-\alpha)}(\delta+\gamma)$, since $\delta=1$ by assumption. Therefore, the assumption $\beta(1-\alpha)-\alpha>0$ implies $\frac{\alpha}{\beta(1-\alpha)}<1$ and hence $f'(k)<\delta+\gamma$, which is the condition for a dynamically inefficient steady state.

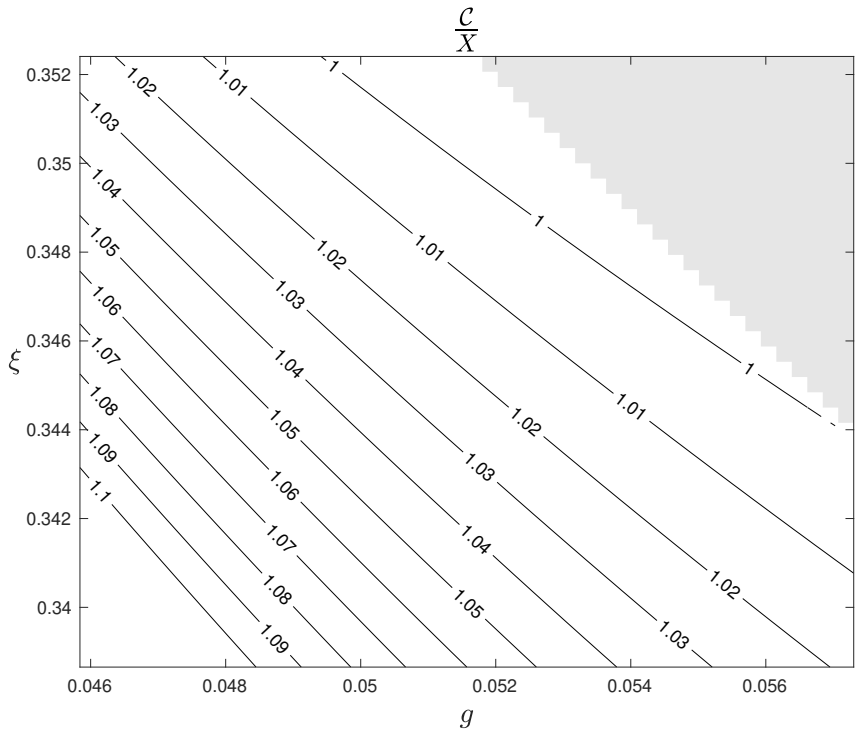


Figure 3: Optimal immigration, m^{**}, is zero in the shaded region and is positive in the unshaded region. The downward-sloping curves are level sets of $\frac{\mathcal{C}(m^{**})}{X(m^{**})}$. Along the uppermost labelled level set, $\frac{\mathcal{C}(m^{**})}{X(m^{**})} = 1$; $\mathcal{C}\left(m^{**}\right) > X\left(m^{**}\right)$ below and to the left of this level set and $\mathcal{C}\left(m^{**}\right) < X\left(m^{**}\right)$ in the unshaded region above and to the right of this level set.

4 Implementation of the Planner's Optimal Steady State in a Competitive Economy

Up to this point, we have focused on the planner's problem. Now we consider competitive economies in which consumers live for two periods. Our goal is to derive a fiscal package of taxes and government bonds that leads the steady state of a competitive economy with two-period lifetimes to have $k = k^{**}$ and $m = m^{**}$, which are the planner's optimal values of k and m, respectively, derived in Section 3.

We consider a fiscal package that pays for government purchases, g, and implements the planner's optimal steady-state values of c^y, c^o, x^y, and x^o, together with $m = m^{**}$, $k = k^{**} = k\left(m^{**}\right)$, and $w = w^{**} = f\left(k\left(m^{**}\right)\right) - k\left(m^{**}\right) f'\left(k\left(m^{**}\right)\right)$, which is the marginal product of labor. Specifically, we consider a three-part package consisting of (1) an entry

fee, τ_M, paid by a young immigrant at the end of the period in which they immigrate (after earning wages in the destination country); (2) a consumption tax at rate $\tau_C > -1$, levied on the consumption of all consumers–natives and immigrants–in both periods of life; and (3) one-period government bonds that pay an interest rate $r = f'(k(m^{**})) - \delta = \Gamma(m^{**}) - 1$.

Neither of the taxes creates a distortion. The entry tax is a tax on the act of immigrating into the destination country, but it does not distort the decision to immigrate because the combination of tax rates is chosen to allow an immigrant to purchase goods with present value $X(m^{**})$, which is sufficient to allow potential immigrants to achieve their threshold level of utility in the destination country. The consumption tax does not introduce intertemporal distortions because the tax rate is the same during both periods of a consumer's life; and since all consumers face the same consumption tax rate, there is no distortion between natives and immigrants. The role of government bonds is simply to adjust the portion of consumers' saving allocated to capital, so that the steady state of the competitive economy has the same value of k as in the planner's optimal steady state. Since the interest rate, r, equals the net growth rate of the economy, $\Gamma(m^{**}) - 1$, bonds have no impact on the government's budget constraint in the steady state. That is, with $r = \Gamma(m^{**}) - 1$ in the planner's optimal steady state, bond seignorage equals zero.

The budget constraint of a young immigrant in the planner's optimal steady state is

$$(1 + \tau_C) x^y + \frac{1}{1 + r} (1 + \tau_C) x^o = w^{**} - \tau_M. \tag{49}$$

In the planner's optimal steady state, $X(m^{**}) = x^y + \frac{1}{\Gamma(m^{**})} x^o$ and $1 + r = \Gamma(m^{**})$, so the budget constraint of an immigrant in equation (49) can be written as

$$(1 + \tau_C) X(m^{**}) = w^{**} - \tau_M. \tag{50}$$

The optimal consumption expenditure (including the consumption tax) of a young immigrant is $(1 + \tau_C) x^y = (1 - \beta)(w^{**} - \tau_M)$, so the saving of a young immigrant, after paying the

entry fee, τ_M, and spending $(1+\tau_C)\,x^y$ on consumption is

$$s_M = (w^{**} - \tau_M) - (1+\tau_C)\,x^y = \beta\,(w^{**} - \tau_M)\,. \tag{51}$$

The budget constraint of a young native in the planner's optimal steady state is

$$(1+\tau_C)\,c^y + \frac{1}{1+r}\,(1+\tau_C)\,c^o = w^{**}. \tag{52}$$

Since $1+r = \Gamma\,(m^{**})$ in the planner's optimal steady state, the budget constraint of a native in the planner's optimal steady state can be written, using $\mathcal{C}\,(m) \equiv c^y + \frac{1}{\Gamma(m)}c^o$ from Definition 4, as

$$(1+\tau_C)\,\mathcal{C}\,(m^{**}) = w^{**}. \tag{53}$$

The optimal consumption expenditure of a young native in the planner's optimal steady state is $(1+\tau_C)\,c^y = (1-\beta)\,w^{**}$, so the saving of this native is

$$s_N = w^{**} - (1+\tau_C)\,c^y = \beta w^{**}. \tag{54}$$

In period t in the steady state, the aggregate saving of the N_t young natives, who each save $s_N = \beta w^{**}$, and the M_t young immigrants, who each save $s_M = \beta\,(w^{**} - \tau_M)$, is

$$S_t = \beta\,[N_t w^{**} + M_t\,(w^{**} - \tau_M)]\,. \tag{55}$$

In the aggregate, young consumers use their saving at the end of period t to purchase capital, K_{t+1}, and one-period government bonds, B_{t+1}. Therefore,

$$K_{t+1} + B_{t+1} = S_t, \tag{56}$$

so equation (55) implies that

$$K_{t+1} + B_{t+1} = \beta N_t \left[w^{**} + m^{**} \left(w^{**} - \tau_M \right) \right]. \tag{57}$$

Dividing both sides of equation (57) by N_t implies[10] that in the planner's optimal steady state,

$$\left(1 + m^{**}\right) \Gamma \left(m^{**}\right) k^{**} + \left(1 + m^{**}\right) \Gamma \left(m^{**}\right) b = \beta \left[w^{**} + m^{**} \left(w^{**} - \tau_M \right) \right], \tag{58}$$

where $b_{t+1} \equiv \frac{B_{t+1}}{L_{t+1}} = b$ in the steady state.

In period t in the steady state, the government's budget constraint is

$$M_t \tau_M + \tau_C \left(N_t c^y + M_t x^y + N_{t-1} c^o + M_{t-1} x^o \right) = \left(N_t + M_t \right) g, \tag{59}$$

where the left hand side consists of entry tax revenue plus consumption tax revenue, and the right hand side is government purchases of output. Dividing both sides of the government budget constraint in equation (59) by N_t implies that in the planner's optimal steady state,

$$m^{**} \tau_M + \tau_C \left[\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) \right] = \left(1 + m^{**} \right) g. \tag{60}$$

Equations (50), (53), (58), and (60) form a four-equation linear system in τ_M, τ_C, and b. However equation (60) is a linear combination of equations (50) and (53),[11] so we ignore equation (60). The other three equations form a linear system that we write in matrix form

[10] Note that $\frac{1}{N_t} = \frac{L_t}{N_t} \frac{L_{t+1}}{L_t} \frac{1}{L_{t+1}} = \left(1 + m\right) \Gamma \left(m\right) \frac{1}{L_{t+1}}$.

[11] Add m^{**} times equation (50) to equation (53) and rearrange to obtain $m^{**} \tau_M = \left(1 + m^{**}\right) w^{**} - \left(1 + \tau_C\right) \left[\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) \right]$. Next rewrite this equation using equations (43) and $w^{**} = f \left(k \left(m^{**} \right) \right) - k \left(m^{**} \right) f' \left(k \left(m^{**} \right) \right)$, which imply $\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) = \left(1 + m^{**} \right) \left(w^{**} - g \right)$, to rewrite the equation in the previous sentence to obtain $m^{**} \tau_M = \left(1 + m^{**}\right) g - \tau_C \left[\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) \right]$; adding $\tau_C \left[\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) \right]$ to both sides of this equation yields $m^{**} \tau_M + \tau_C \left[\mathcal{C} \left(m^{**} \right) + m^{**} X \left(m^{**} \right) \right] = \left(1 + m^{**}\right) g$, which is equation (60).

(with the equations ordered as (53), (50), and (58)) as

$$\Lambda \times \begin{bmatrix} 1+\tau_C \\ \tau_M \\ b \end{bmatrix} = \begin{bmatrix} w^{**} \\ w^{**} \\ \beta\left(1+m^{**}\right)w^{**} - \left(1+m^{**}\right)\Gamma\left(m^{**}\right)k^{**} \end{bmatrix}, \tag{61}$$

where

$$\Lambda \equiv \begin{bmatrix} \mathcal{C}\left(m^{**}\right) & 0 & 0 \\ X\left(m^{**}\right) & 1 & 0 \\ 0 & m^{**}\beta & \left(1+m^{**}\right)\Gamma\left(m^{**}\right) \end{bmatrix}. \tag{62}$$

Since $\det\Lambda = \left(1+m^{**}\right)\Gamma\left(m^{**}\right)\mathcal{C}\left(m^{**}\right) > 0$, equations (53), (50), and (58) are independent and there is a unique solution for the fiscal parameters τ_C, τ_M, and b. Also, since Λ is triangular, the solution to the matrix equation in (61) is particularly easy to obtain as

$$\begin{bmatrix} 1+\tau_C \\ \tau_M \\ b \end{bmatrix} = \begin{bmatrix} \frac{w^{**}}{\mathcal{C}(m^{**})} \\ \left(1-\frac{X(m^{**})}{\mathcal{C}(m^{**})}\right)w^{**} \\ \beta\left[1+m^{**}\frac{X(m^{**})}{\mathcal{C}(m^{**})}\right]\frac{w^{**}}{(1+m^{**})\Gamma(m^{**})} - k^{**} \end{bmatrix}. \tag{63}$$

Equation (63) implies that the optimal value of the entry fee, τ_M, is positive if and only if $X\left(m^{**}\right) < \mathcal{C}\left(m^{**}\right)$, so Proposition 7, which states that if $\xi < \min\left\{\overline{\overline{\xi}}, \mathcal{C}\left(0\right)\right\}$, then $X\left(m^{**}\right) < \mathcal{C}\left(m^{**}\right)$, immediately proves the following proposition.

Proposition 9 *If $\xi < \min\left\{\overline{\overline{\xi}}, c_G\right\}$, then the optimal value of τ_M is positive.*

Proposition 9 provides a sufficient condition for a positive entry fee as part of a fiscal package that implements the planner's optimal steady state. This entry fee is a channel by which resources are extracted from immigrants to increase the welfare of natives. However, it is possible to have $X\left(m^{**}\right) > \mathcal{C}\left(m^{**}\right)$ in the planner's optimal steady state, in which case, the optimal value of the entry fee, τ_M, would be negative. In this situation, if $m^{**} > 0$, the benefit that natives obtain from immigration operates through the increase in the rate

of return on capital, which increases the level of utility associated with any given level of consumption; it must be the case that the benefit of the increased rate of return offsets any reduction in native's consumption. Equivalently, as discussed in Section 3, an increase in m reduces the dependency ratio, $\frac{1}{\Gamma(m)}$. The reduction in the dependency ratio means that the output produced by a given amount of workers is spread among fewer consumers, thereby allowing an increase in utility. Since the utility of each immigrant is fixed exogenously at $\overline{u}$, the increase in utility accrues only to natives.

To interpret τ_C, observe that the government budget constraint in equation (60) implies

$$\tau_C = \frac{(1 + m^{**})\, g - m^{**}\tau_M}{\mathcal{C}\,(m^{**}) + m^{**}X\,(m^{**})}, \tag{64}$$

where the denominator, $\mathcal{C}\,(m^{**}) + m^{**}X\,(m^{**})$, is positive. If the government purchases an amount of output, $(1 + m^{**})\, g$, greater than the revenue collected from the entry fee paid by immigrants, $m^{**}\tau_M$, then the tax rate on consumption, τ_C, is positive. However, if government purchases are smaller than the revenue collected from entry fees, the tax rate on consumption is negative.

The value of b in equation (63) is constructed to satisfy equation (58), which is equivalent to equation (56) along a balanced growth path. If $S_t > K_{t+1}$, then $B_{t+1} > 0$ and bonds essentially "mop up" any saving in excess of the optimal capital stock. Alternatively, if $S_t < K_{t+1}$, then $B_{t+1} < 0$ and the government lends enough funds to consumers to enable them to purchase the optimal capital stock.]

5 Conclusion

We have developed and analyzed an overlapping-generations growth model extended to allow for the possibility of immigration into a country. Except for immigration, the country is a closed economy that does not trade goods, services, or assets with other countries. Our goal is to answer two important questions related to immigration. First, is the optimal rate of immigration positive in the steady state? To address this question, we need to specify a social welfare function. Motivated by Phelps's seminal work on the Golden Rule of capital

accumulation, we specify the social welfare function to be the utility of a representative consumer in the long run–in particular, the utility of a representative native-born consumer in the long run. We consider the optimization problem solved by a planner who allocates output to consumption of natives, consumption of immigrants, and capital to be carried into the following period. We derive sufficient conditions for the optimal amount of immigration to be positive. Second, if the optimal rate of immigration of positive, what fiscal package will lead the steady state of a competitive economy to be the same as the planner's optimal steady state? We show that the planner's optimal steady state with positive immigration can be implemented by a combination of (1) an entry fee imposed on immigrants, (2) a consumption tax levied at the same rate (which could be positive, negative, or zero) on all consumers, regardless of age or country of birth, and (3) government bonds.

To distill the first question to its essential elements, in Section 2 we consider a non-overlapping generations model, in which each person lives for only one period and gives birth to a fixed number, $1+\gamma$, of children at the end of that period. Because individual lifetimes are only one period in length, there is no need to specify a utility function for consumers. Provided that utility is strictly increasing in consumption, the maximum level of utility per native in the steady state is attained by the maximum level of consumption per native in the steady state. Thus, the social welfare function is the level of consumption of a representative native in the steady state, as in Phelps's Golden Rule. The steady state that maximizes this social welfare function satisfies a simple extension of the Golden Rule: the rate at which consumption in the current period can be transformed into consumption in the following period, $1+f'(k)-\delta$, equals the growth rate of the economy, $\Gamma(m) \equiv (1+m)(1+\gamma)$. The extended Golden Rule implies that $k(m)$, defined implicitly by $1+f'(k(m))-\delta = \Gamma(m)$ is the optimal steady-state level of k when the immigration rate is m. When $m=0$, the optimal steady-state value of k is $k(0)=k_G$, which is the original Golden Rule value of k in Phelps (1961) where $f'(k_G)-\delta=\gamma$.

We assume that there is an unlimited supply of potential immigrants, provided that they receive a level of consumption in the destination country that is at least as large as an exogenous threshold, ξ. Since the social welfare function does not value the consumption

of immigrants, the planner minimizes the consumption of a representative immigrant, and thus sets that consumption equal to the threshold level, ξ. We prove that a sufficient condition for the rate of immigration to be positive in the optimal steady state is that, c_G, the consumption of a representative native in a steady state with $k = k_G$, exceeds ξ plus the cost of devoting $(1+\gamma)\,k_G$ units of output to provide capital to an immigrant's $1+\gamma$ children. When the optimal rate of immigration, m, is positive, the first-order condition for optimal m equates the consumption of a representative native to ξ plus the cost of allocating $\Gamma(m)\,k(m)$ units of output to provide capital to an immigrant's $\Gamma(m) \equiv (1+m)(1+\gamma)$ children.

In the model with one-period lifetimes, consumers do not hold capital from one period to the next. To introduce a capital investment decision for individual consumers, in Section 3 we modify the model to have overlapping generations of consumers who each live for two periods, as in the classic Diamond (1965) model. We must introduce a utility function, which, for simplicity, we assume to be logarithmic in the consumption of each period of life. Instead of the social welfare function being the level of *consumption* of a representative native in the steady state, we assume that the social welfare function is the level of *utility* of a representative native in the steady state. Similarly, there is a threshold level of *utility* rather *consumption*, that makes potential immigrants willing to immigrate. Despite these changes, the extended Golden Rule, $1 + f'(k) - \delta = (1+m)(1+\gamma)$, is unchanged from the one-period lifetime version of the model in Section 2, though specific values of k and m in the planner's optimal steady state can differ from those in the one-period lifetime version of the model. We derive sufficient conditions for the rate of immigration in the planner's optimal steady state to be positive; these conditions are similar in spirit, though more complicated in detail, than the sufficient condition in Section 2.

The complicating details with two-period lifetimes arise from an additional term in the first-order condition for optimal m, when the optimal value of m is positive. This additional term, which is added to the marginal cost of m, is $-\frac{\beta}{1+m^{**}}\left[\mathcal{C}(m^{**}) + m^{**}X(m^{**})\right]$. As discussed more fully in Section 3, this additional term has two complementary interpretations. Briefly, one interpretation is that an increase in m increases $1 + f'(k(m)) - \delta$

and increases the utility that can be obtained with any given present value of consumption over a two-period lifetime. The alternative interpretation is based on the fact that $\frac{1}{\Gamma(m)}$ is the dependency ratio, that is, the ratio of the number of nonworking old consumers to the number of working young consumers. An increase in m increases $\Gamma(m)$ and thus reduces the dependency ratio. Under either interpretation, $-\frac{\beta}{1+m^{**}}\left[\mathcal{C}\left(m^{**}\right)+m^{**}X\left(m^{**}\right)\right]$ reduces the marginal cost of m, and if β is sufficiently large, this term could exceed the cost of providing $\Gamma(m)\,k(m)$ units of capital to an immigrant's children; if that situation arises, then, unlike in Section 2 with one-period lifetimes, it is possible that in the planner's optimal steady state, each immigrant has a higher present value of consumption than does each native.

Having characterized the planner's optimal steady state, we turn, in Section 4, to the implementation question. Specifically, we take as given the planner's optimal steady-state allocations of consumption to young natives, young immigrants, old natives, and old immigrants, as well as the optimal immigration rate, m^{**}, and the optimal capital-labor ratio, k^{**}. We then derive a fiscal package consisting of an entry fee for immigrants, a uniform consumption tax on all consumers, and government bonds, that lead a competitive economy to attain these optimal allocations. The consumption tax collects enough revenue to pay for any excess of government purchases over revenue from the entry fee. Because the extended Golden Rule implies that rates of return on capital and on government bonds equal the population growth rate in the optimal steady state, seignorage from bonds is zero in that steady state. The role of government bonds, which could be in either positive or negative supply, is to induce the steady-state amount of capital held in consumers' portfolios to match the capital-labor ratio dictated by the extended Golden Rule. A positive amount of government bonds absorbs saving in excess of the Golden Rule capital stock, thereby avoiding a dynamically inefficient overaccumulation of capital.

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Appendix

A Two-Period Consumption Problem with Log Utility

Consider a person who lives for two periods and has logarithmic utility

$$u\left(z^y, z^o\right) = (1-\beta)\ln z^y + \beta \ln z^o \tag{A.1}$$

where z^y is consumption when young and z^o is consumption when old. The person chooses z^y and z^o to

$$\max_{z^y, z^o} (1-\beta)\ln z^y + \beta \ln z^o \quad \text{s.t.} \quad z^y + \frac{1}{1+r} z^o = \phi, \tag{A.2}$$

which can be written as

$$\max_{z^y} (1-\beta)\ln z^y + \beta \ln\left[(1+r)\left(\phi - z^y\right)\right]. \tag{A.3}$$

The first-order condition associated with the maximization in equation (A.3) is $\frac{1-\beta}{z^y} = \frac{\beta}{\phi - z^y}$, which implies

$$z^y = (1-\beta)\,\phi \tag{A.4}$$

and hence

$$z^o = (1+r)\,\beta\phi. \tag{A.5}$$

Evaluating the utility function $u\left(z^y, z^o\right)$ from equation (A.1) at the optimal values of z^y and z^o in equations (A.4) and (A.5), respectively, yields

$$u\left(z^y, z^o\right) = \ln\phi + \beta\ln(1+r) + (1-\beta)\ln(1-\beta) + \beta\ln\beta. \tag{A.6}$$

A.1 Planner's Optimal Steady State

In the planner's optimal steady state, equation (12) implies that $1+r = 1+f'(k)-\delta = \Gamma(m) = (1+m)(1+\gamma)$. The value of ϕ for which the maximized value of $u(z^y, z^o)$ equals $\overline{u}$ is

$$\phi = \exp(\overline{u})(1-\beta)^{-(1-\beta)}\beta^{-\beta}(1+\gamma)^{-\beta}(1+m)^{-\beta}. \tag{A.7}$$

Since $\phi = z^y + \frac{1}{1+r}z^o = z^y + \frac{1}{\Gamma(m)}z^o$ in the maximization problem in equation (A.2), it is the value of $X(m)$ in the minimization problem in equation (34), which is the dual to the maximization problem (with z^y replacing x^y and z^o replacing x^o) in equation (A.2). Therefore,

$$X(m) = \xi(1+m)^{-\beta}, \tag{A.8}$$

where

$$\xi \equiv \exp(\overline{u})(1-\beta)^{-(1-\beta)}\beta^{-\beta}(1+\gamma)^{-\beta} > 0. \tag{A.9}$$

B Selected Proofs, Lemmas, and Corollary

Proof of Lemma 1. Differentiate $C(k(m), m)$ in equation (16) with respect m to obtain $\frac{dC(k(m),m)}{dm} = f(k(m)) - k(m)f'(k(m)) - g - (1+m)k(m)f''(k(m))k'(m)$. Equation (15) implies that $f''(k(m))k'(m) = 1+\gamma$, so $\frac{dC(k(m),m)}{dm} = f(k(m)) - k(m)f'(k(m)) - g - (1+m)k(m)(1+\gamma) = f(k(m)) - k(m)f'(k(m)) - g - \Gamma(m)k(m)$. ■

Proof of Proposition 2. If $m^* > 0$, then the optimal value of m is an interior solution and hence $\Omega'(m^*) = 0$, so equation (23), which is implied by $\Omega'(m^*) = 0$, holds. Evaluate equation (16) at $m = m^*$ and substitute the result into equation (18) to obtain the GDP expenditure identity in the planner's optimal steady state, $c^* + m^*\xi = (1+m^*)[f(k(m^*)) - k(m^*)f'(k(m^*)) - g]$. Finally, substitute the expression for $f(k(m^*))$ from equation (23) into the GDP expenditure identity to obtain $c^* = \xi + (1+m^*)\Gamma(m^*)k(m^*)$.

■

Proof of Lemma 2. Differentiate the expression for $\Omega'(m)$ in equation (19) with respect to m, and use $f''(k(m))k'(m) = 1+\gamma$ from equation (15) to obtain $\Omega''(m) = -\frac{1+\gamma}{f''(k(m))}[2k(m)f''(k(m)) + \Gamma(m)]$. Therefore, since $f''(k(m)) < 0$, we have $\Omega''(m) \lesseqgtr 0$ as $h(m) \equiv 2k(m)f''(k(m)) + \Gamma(m) \lesseqgtr 0$. The production function $f(k) = Ak^{\alpha}$ implies $f'(k) = \alpha A k^{\alpha-1}$ and $f''(k) = (\alpha-1)\alpha A k^{\alpha-2}$, so $kf''(k) = (\alpha-1)f'(k)$. Therefore, $h(m) = 2(\alpha-1)f'(k(m)) + \Gamma(m) = 2(\alpha-1)[1+f'(k)-\delta-(1-\delta)] + \Gamma(m) = 2(\alpha-1)[\Gamma(m)-(1-\delta)]+\Gamma(m) = (2\alpha-1)\Gamma(m) - 2(\alpha-1)(1-\delta) = -(1-2\alpha)(1+\gamma)(1+m) + 2(1-\alpha)(1-\delta) = (1-2\alpha)(1+\gamma)\left[2\frac{1-\alpha}{1-2\alpha}\frac{1-\delta}{1+\gamma} - 1 - m\right]$. Now use the definition $\Theta \equiv 2\frac{1-\alpha}{1-2\alpha}\frac{1-\delta}{1+\gamma} - 1$ to obtain $h(m) = (1-2\alpha)(1+\gamma)(\Theta - m)$. Therefore, since Assumption 2 implies that $\alpha = \alpha_G < \frac{1}{2}$, we have $(1-2\alpha)(1+\gamma) > 0$, which implies $h(m) \lesseqgtr 0$ as $m \gtreqless \Theta$ and completes the proof. ■

Proof. of Lemma 3 To highlight the dependence of $\Omega(m)$ on ξ as well as m, write $\Omega(m,\xi) \equiv (1+m)[f(k(m)) - k(m)f'(k(m)) - g] - m\xi$. Lemma 2 implies that if $\Theta > 0$, then $\arg\max_{m\geq 0}\Omega(m,\xi) \in \{0\} \cup \{m : m \geq \Theta\}$. If $\Omega_m(0,\xi) \geq 0$, then $\max_{m\geq 0}\Omega(m,\xi) > \Omega(0,\xi)$, so $\xi \neq \widehat{\xi}$. Statement 1 of Proposition 1 implies $\Omega_m(0,\xi) = \overline{\xi} - \xi$, so $\widehat{\xi}$ cannot be less than or equal to $\overline{\xi}$.

Lemma 2 implies that if $\Theta > 0$, then for a given value of ξ, the function $\Omega(m,\xi)$ for $m > 0$ has a unique local maximum with respect to m; let $\overline{m}(\xi)$ denote the value of m at this local maximum and note that $\overline{m}(\xi) \geq \Theta$. Now consider a constant denoted $\Delta > 0$ and observe that $\Omega\left(m, \frac{\Delta}{\Theta}\right) = \Omega(m,0) - m\frac{\Delta}{\Theta}$. If $m \geq \Theta$, then $m\frac{\Delta}{\Theta} \geq \Delta$ and hence $\Omega\left(m, \frac{\Delta}{\Theta}\right) \leq \Omega(m,0) - \Delta$, which implies that $\max_{m\geq\Theta}\Omega\left(m, \frac{\Delta}{\Theta}\right) = \Omega\left(\overline{m}\left(\frac{\Delta}{\Theta}\right), \frac{\Delta}{\Theta}\right) \leq \Omega\left(\overline{m}\left(\frac{\Delta}{\Theta}\right), 0\right) - \Delta \leq \max_{m\geq\Theta}\Omega(m,0) - \Delta$. Now let $\Delta = \max_{m\geq\Theta}\Omega(m,0) - \Omega(0,0)$, which implies $\max_{m\geq\Theta}\Omega\left(m, \frac{\Delta}{\Theta}\right) \leq \Omega(0,0) = \Omega\left(0, \frac{\Delta}{\Theta}\right)$, where the equality at the end of this expression follows from the fact that $\Omega(0,\xi)$ is independent of ξ. Since $\Omega(m,\xi)$ is decreasing in ξ for $m \geq \Theta$ and $\Omega(0,\xi)$ is invariant to ξ, it follows that if $\xi > \frac{\Delta}{\Theta}$, then $\max_{m\geq\Theta}\Omega(m,\xi) < \Omega(0,\xi)$, so $\widehat{\xi}$ cannot be greater than $\frac{\Delta}{\Theta}$. Hence, $\widehat{\xi} > \overline{\xi}$ is finite. ■

Proof of Lemma 4. Observe that equation (12) implies $1 + f'(k(m)) - \delta = \Gamma(m) = (1+\gamma)(1+m)$, which implies that (1) $\lim_{m\to\infty} f'(k(m)) = \infty$ and hence $\lim_{m\to\infty} k(m) = 0$

and (2) $\lim_{m\to\infty}\Gamma(m)k(m)=\lim_{m\to\infty}[1+f'(k(m))-\delta]k(m)=\lim_{m\to\infty}f'(k(m))k(m)$ $=0$, where the final equality follows from equation (5). From equation (16), $C(k(m),m)$ $=(1+m)[f(k(m))-k(m)f'(k(m))-g]$, so $C(k(m),m)-mX(m)=(1+m)A(m)$, where $A(m)\equiv f(k(m))-k(m)f'(k(m))-g-\frac{m}{1+m}X(m)$. Observe that $\lim_{m\to\infty}A(m)$ $=\lim_{m\to\infty}\left[f(k(m))-k(m)f'(k(m))-g-\frac{m}{1+m}X(m)\right]=$
$\lim_{m\to\infty}\left[-g-\frac{m}{1+m}X(m)\right]\leq -g<0$. Therefore, there exists a finite $\widehat{m}$ such that $A(m)\leq 0$ and hence $C(k(m),m)-mX(m)\leq 0$ for all $m\geq\widehat{m}$. Thus, equation (42) implies that $\mathcal{C}(m)\leq 0$ for all $m\geq\widehat{m}$, and the definition of $V(m)$ in equation (44) implies that $V(m)=-\infty$ for all $m\geq\widehat{m}$. ■

Proof of Proposition 4. Define $\widehat{V}(m)\equiv\max\{V(m),V(0)-\varepsilon\}$ for small $\varepsilon>0$. Lemma 4 implies that if $m\geq\widehat{m}$, where $\widehat{m}$ is finite, then $V(m)=-\infty$, and hence $\widehat{V}(m)=V(0)-\varepsilon$ $<V(0)=\widehat{V}(0)$. Therefore, if $\widehat{V}(m)$ attains a maximum, it must be attained for $0\leq m\leq\widehat{m}$. $\widehat{V}(m)$ is continuous on the closed and bounded domain $\{m:0\leq m\leq\widehat{m}\}$, so it attains a maximum on this set. That is, $\arg\max_{m\geq 0}\widehat{V}(m)\in\{m:0\leq m\leq\widehat{m}\}$. Observe that $\max_{0\leq m\leq\widehat{m}}\widehat{V}(m)=\max_{0\leq m\leq\widehat{m}}V(m)$, so $\arg\max_{0\leq m\leq\widehat{m}}V(m)=\arg\max_{0\leq m\leq\widehat{m}}\widehat{V}(m)\in$ $\{m:0\leq m\leq\widehat{m}\}$. ■

Proof of Lemma 5. Since $\mathcal{C}(m)>0$, we have

$$V(m)=\ln\mathcal{C}(m)+\beta\ln\Gamma(m)=\ln[C(k(m),m)-mX(m)]+\beta\ln\Gamma(m).$$

Therefore, $V'(m)=\frac{1}{\mathcal{C}(m)}\left\{\frac{dC(k(m),m)}{dm}-X(m)-mX'(m)\right\}+\frac{\beta}{1+m}$. Differentiate $X(m)=\xi(1+m)^{-\beta}$ from equation (35) with respect to m to obtain $X'(m)=-\beta\frac{X(m)}{1+m}$. Use this expression for $X'(m)$ and rearrange the resulting equation to obtain $V'(m)=$
$\frac{1}{\mathcal{C}(m)}\left\{\frac{dC(k(m),m)}{dm}-X(m)+\beta\frac{m}{1+m}X(m)+\frac{\beta}{1+m}(C(k(m),m)-mX(m))\right\}$, which can be simplified to $V'(m)=\frac{1}{\mathcal{C}(m)}\times\left\{\frac{dC(k(m),m)}{dm}+\beta\frac{C(k(m),m)}{1+m}-X(m)\right\}$. Use $\frac{dC(k(m),m)}{dm}=\frac{C(k(m),m)}{1+m}-$ $\Gamma(m)k(m)$, which is implied by Lemma 1, to obtain $V'(m)=$
$\frac{1}{\mathcal{C}(m)}\times\left\{\frac{C(k(m),m)}{1+m}-\Gamma(m)k(m)+\beta\frac{C(k(m),m)}{1+m}-X(m)\right\}=$
$\frac{1}{\mathcal{C}(m)}\times\left\{(1+\beta)\frac{C(k(m),m)}{1+m}-\Gamma(m)k(m)-X(m)\right\}=$
$\frac{1}{\mathcal{C}(m)}\frac{1}{1+m}\times\{(1+\beta)C(k(m),m)-(1+m)\Gamma(m)k(m)-(1+m)X(m)\}$. Now use $\mathcal{C}(m)+$

$mX(m) = C(k(m), m)$ from equation (42) to obtain
$V'(m) = \frac{1}{\mathcal{C}(m)}\frac{1}{1+m} \times \{\mathcal{C}(m) + mX(m) - (1+m)\Gamma(m)k(m) - (1+m)X(m) + \beta[\mathcal{C}(m) + mX(m)]\}$
$= \frac{1}{\mathcal{C}(m)}\frac{1}{1+m} \times \{\mathcal{C}(m) - X(m) - (1+m)\Gamma(m)k(m) + \beta[\mathcal{C}(m) + mX(m)]\}$, so $V'(m) = \frac{1}{\mathcal{C}(m)}\frac{1}{1+m} \times H(m)$, where $H(m) \equiv \mathcal{C}(m) - X(m) - (1+m)\Gamma(m)k(m) + \beta[\mathcal{C}(m) + mX(m)]$.

■

Proof of Lemma 6. To evaluate $V'(m)$ at $m = 0$, note that $k(0) = k_G$, $\mathcal{C}(0) = c_G$, $\Gamma(0) = 1 + \gamma$, and $X(0) = \xi$ to evaluate $H(m) \equiv \mathcal{C}(m) - X(m) - (1+m)\Gamma(m)k(m) + \beta[\mathcal{C}(m) + mX(m)]$ from Lemma 5 at $m = 0$ to obtain $H(0) = (1+\beta)c_G - (1+\gamma)k_G - \xi$. Now use $\overline{\xi} = c_G - (1+\gamma)k_G$ from Definition 3 to obtain $H(0) = (1+\beta)\overline{\xi} + \beta(1+\gamma)k_G - \xi$ $= \overline{\overline{\xi}} - \xi$. Finally, Lemma 5 implies $V'(0) = \frac{1}{\mathcal{C}(0)}H(0) = \frac{1}{\mathcal{C}(0)}\left(\overline{\overline{\xi}} - \xi\right)$, which proves Statement 1.

Now rewrite $\overline{\xi} = c_G - (1+\gamma)k_G$ from above as $\overline{\xi} = \mathcal{C}(0) - (1+\gamma)k_G$. Therefore, $\overline{\overline{\xi}} \equiv (1+\beta)\overline{\xi} + \beta(1+\gamma)k_G = (1+\beta)[\mathcal{C}(0) - (1+\gamma)k_G] + \beta(1+\gamma)k_G = (1+\beta)\mathcal{C}(0) - (1+\gamma)k_G$. Finally, substitute $\overline{\overline{\xi}} = (1+\beta)\mathcal{C}(0) - (1+\gamma)k_G$ and $X(0) = \xi$ into Statement 1 to obtain $V'(0) = \frac{1}{\mathcal{C}(0)}\left(\overline{\overline{\xi}} - \xi\right) = \frac{1}{\mathcal{C}(0)}[\beta\mathcal{C}(0) - (1+\gamma)k_G + \mathcal{C}(0) - X(0)]$, which is Statement 2. ■

Proof of Proposition 5. Assume that $\xi < \overline{\overline{\xi}}$. From Statement 1 of Lemma 6, $V'(0) = \frac{1}{c_G}\left(\overline{\overline{\xi}} - \xi\right) > 0$, so $m^{**} > 0$. Also, the definition of m^{**} in equation (45) and Lemma 4, which defines and characterizes $\widehat{m}$, imply that $m^{**} \leq \widehat{m} < \infty$, which completes the proof of Statement 1. Statement 2 follows from $k^{**} \equiv k(m^{**})$ and (1) the facts that $k(m)$ is strictly decreasing, $k(0) = k_G$, and $m^{**} > 0$, so $k(m^{**}) < k_G$; and (2) $1 + f'(k(m^{**})) - \delta = (1+m)(1+\gamma)$ is finite since m^{**} is finite, so $k(m^{**}) > 0$. ■

Proof of Proposition 6. Since $m^{**} > 0$, we have $V'(m^{**}) = 0$ and Lemma 5 implies that $H(m^{**}) = \mathcal{C}(m^{**}) - X(m^{**}) - (1+m^{**})\Gamma(m^{**})k(m^{**}) + \beta[\mathcal{C}(m^{**}) + m^{**}X(m^{**})] = 0$, which proves the proposition.

■

Proof of Corollary 1. Divide both sides of the equation in Proposition 6 by $1 + m^{**}$ to obtain $\frac{1}{1+m^{**}}\mathcal{C}(m^{**}) = \frac{1}{1+m^{**}}X(m^{**}) + \Gamma(m^{**})k(m^{**}) - \frac{\beta}{1+m^{**}}[\mathcal{C}(m^{**}) + m^{**}X(m^{**})]$. Then use equation (43) to substitute $\frac{1}{1+m^{**}}\mathcal{C}(m^{**}) = -\frac{m^{**}}{1+m^{**}}X(m^{**}) + f(k(m^{**})) - k(m^{**})f'(k(m^{**})) -$

g for $\frac{1}{1+m^{**}}\mathcal{C}(m^{**})$ to obtain $f(k(m^{**}))-k(m^{**})f'(k(m^{**}))=X(m^{**})+g+\Gamma(m^{**})k(m^{**})-\frac{\beta}{1+m^{**}}[\mathcal{C}(m^{**})+m^{**}X(m^{**})]$. ■

Proof of Proposition 7. The assumption $\xi<\min\left\{\overline{\overline{\xi}},c_G\right\}$ implies that $\xi<\overline{\overline{\xi}}$, which together with Statement 1 of Lemma 6 implies that $V'(0)>0$, and hence $m^{**}>0$. Define $U_M(m)\equiv\ln X(m)+\beta\ln(1+f'(k(m))-\delta)+(1-\beta)\ln(1-\beta)+\beta\ln\beta$ and $U_N(m)\equiv\ln\mathcal{C}(m)+\beta\ln(1+f'(k(m))-\delta)+(1-\beta)\ln(1-\beta)+\beta\ln\beta$ as the values of utility of immigrants and natives, respectively, in the steady state with immigration ratio m and $k=k(m)$. Observe that $U_N(m)-U_M(m)=\ln\mathcal{C}(m)-\ln X(m)$, which implies $U_N(0)-U_M(0)=\ln\mathcal{C}(0)-\ln X(0)=\ln c_G-\ln\xi>0$, where the inequality follows from the assumption $\xi<\min\left\{\overline{\overline{\xi}},c_G\right\}$. Since $m^{**}\equiv\min\{\arg\max_{m\geq0}V(m)\}$ and $m^{**}>0$, we have $U_N(m^{**})>U_N(0)\geq U_M(0)=\overline{u}=U_M(m^{**})$. Therefore, $\ln\mathcal{C}(m^{**})-\ln X(m^{**})=U_N(m^{**})-U_M(m^{**})>0$, so $\mathcal{C}(m^{**})>X(m^{**})$, which proves the proposition. ■

Proof of Corollary 2. Proposition 6 implies that if $m^{**}>0$, then $\mathcal{C}(m^{**})-X(m^{**})=(1+m^{**})\Gamma(m^{**})k(m^{**})-\beta[\mathcal{C}(m^{**})+m^{**}X(m^{**})]$. Equation (43) implies $\mathcal{C}(m^{**})+m^{**}X(m^{**})=(1+m^{**})[f(k(m^{**}))-k(m^{**})f'(k(m^{**}))-g]$, so $\mathcal{C}(m^{**})-X(m^{**})=(1+m^{**})\{\Gamma(m^{**})k(m^{**})-\beta[f(k(m^{**}))-k(m^{**})f'(k(m^{**}))-g]\}=(1+m^{**})\Upsilon$, which completes the proof of the corollary. ■

Proof of Proposition 8. Assume that $m^{**}>0$. Substitute $\Gamma(m^{**})=1+f'(k(m^{**}))-\delta$ from equation (14) into the definition $\Upsilon\equiv\Gamma(m^{**})k(m^{**})-\beta[f(k(m^{**}))-k(m^{**})f'(k(m^{**}))-g]$ to obtain $\Upsilon=f'(k(m^{**}))k(m^{**})+(1-\delta)k(m^{**})-\beta[f(k(m^{**}))-k(m^{**})f'(k(m^{**}))]+\beta g$. With the Cobb-Douglas production function $f(k)=Ak^{\alpha}$, we have $k(m^{**})f'(k(m^{**}))=\alpha f(k(m^{**}))$, so that $\Upsilon=[\alpha-\beta(1-\alpha)]f(k(m^{**}))+(1-\delta)k(m^{**})+\beta g$, which proves Statement 1. Now assume that $\beta(1-\alpha)\leq\alpha$, so that the three components of Υ, namely, $[\alpha-\beta(1-\alpha)]f(k(m^{**}))$, $(1-\delta)k(m^{**})$, and βg are each non-negative. If $\beta=0$, the first component, $[\alpha-\beta(1-\alpha)]f(k(m^{**}))$, is positive so $\Upsilon>0$; if $\beta>0$, then the third component, βg, is positive so $\Upsilon>0$. Hence, for all non-negative β such that $\beta(1-\alpha)\leq\alpha$, we have $\Upsilon>0$, which along with Corollary 2 implies that $\mathcal{C}(m^{**})>X(m^{**})$, which completes the proof of Statement 2. Statement 3 is simply the contrapositive of Statement 2. ■

Corollary 3 *Assume that the production function is* $f(k)=Ak^{\alpha}$, $\beta(1-\alpha)-\alpha>0$, $\delta=1$,

$g > 0$, $A > \left(\frac{1+\gamma}{\alpha}\right)^{\alpha}\left[\frac{\beta(1-\alpha)-\alpha}{\beta}\frac{1}{g}\right]^{\alpha-1}$, *and* $\xi = \xi(\alpha,\beta,\gamma,A,g) \equiv \frac{\alpha g}{\beta(1-\alpha)-\alpha}\left(\frac{\alpha}{1+\gamma}A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}}\right)^{\beta}$. *Then the first-order condition in Proposition 6 is satisfied by* $m = \widetilde{m}$, *where*

1. $\widetilde{m} = \frac{\alpha}{1+\gamma}A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}} - 1 > 0$
2. $k(\widetilde{m}) = A^{-\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{1}{\alpha}}$
3. $\mathcal{C}(\widetilde{m}) = X(\widetilde{m}) = \frac{\alpha}{\beta(1-\alpha)-\alpha}g$.

Proof of Corollary 3. The assumption $A > \left(\frac{1+\gamma}{\alpha}\right)^{\alpha}\left[\frac{\beta(1-\alpha)-\alpha}{\beta}\frac{1}{g}\right]^{\alpha-1}$ implies that $\widetilde{m} = \frac{\alpha}{1+\gamma}A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}} - 1 > \frac{\alpha}{1+\gamma}\left(\left(\frac{1+\gamma}{\alpha}\right)^{\alpha}\left[\frac{\beta(1-\alpha)-\alpha}{\beta}\frac{1}{g}\right]^{\alpha-1}\right)^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}} - 1 = \left[\frac{\beta(1-\alpha)-\alpha}{\beta}\frac{1}{g}\right]^{\frac{\alpha-1}{\alpha}} \times \left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}} - 1 = 0$, so $\widetilde{m} > 0$, as stated in Statement 1. Since $\delta = 1$, $f'(k(\widetilde{m})) = \Gamma(\widetilde{m}) = (1+\widetilde{m})(1+\gamma) = \alpha A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}}$. To verify $f'(k(\widetilde{m})) = \alpha A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}}$, use the value of $k(\widetilde{m})$ from Statement 2 in $f'(k(\widetilde{m})) = \alpha A(k(\widetilde{m}))^{\alpha-1}$ to obtain $f'(k(\widetilde{m})) = \alpha A\left(A^{-\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{1}{\alpha}}\right)^{\alpha-1} = \alpha A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}}$, which is the verification. Now substitute $k(\widetilde{m}) = A^{-\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{1}{\alpha}}$ into the production function to obtain $f(k(\widetilde{m})) = A(k(\widetilde{m}))^{\alpha} = \frac{\beta g}{\beta(1-\alpha)-\alpha}$. Since $\Gamma(\widetilde{m}) = f'(k(\widetilde{m}))$, we have $k(\widetilde{m})\Gamma(\widetilde{m}) = k(\widetilde{m})f'(k(\widetilde{m})) = \alpha f(k(\widetilde{m})) = \alpha\frac{\beta g}{\beta(1-\alpha)-\alpha}$.

Equation (43) implies that $\frac{1}{1+\widetilde{m}}\mathcal{C}(\widetilde{m}) + \frac{\widetilde{m}}{1+\widetilde{m}}X(\widetilde{m}) = f(k(\widetilde{m})) - k(\widetilde{m})f'(k(\widetilde{m})) - g = (1-\alpha)f(k(\widetilde{m})) - g = \frac{\alpha}{\beta(1-\alpha)-\alpha}g$. Also, $\Gamma(\widetilde{m}) = f'(k(\widetilde{m}))$ so $k(\widetilde{m})\Gamma(\widetilde{m}) = \alpha f(k(\widetilde{m})) = \alpha\frac{\beta g}{\beta(1-\alpha)-\alpha} = \beta\left[\frac{1}{1+\widetilde{m}}\mathcal{C}(\widetilde{m}) + \frac{\widetilde{m}}{1+\widetilde{m}}X(\widetilde{m})\right]$. Therefore, $(1+\widetilde{m})k(\widetilde{m})\Gamma(\widetilde{m}) = \beta[\mathcal{C}(\widetilde{m}) + \widetilde{m}X(\widetilde{m})]$, so Proposition 6 implies $\mathcal{C}(\widetilde{m}) = X(\widetilde{m})$, which the first part of Statement 3 of Corollary 3.

Set $\mathcal{C}(\widetilde{m}) = X(\widetilde{m})$ in $(1+\widetilde{m})k(\widetilde{m})\Gamma(\widetilde{m}) = \beta[\mathcal{C}(\widetilde{m}) + \widetilde{m}X(\widetilde{m})]$ to obtain $\Gamma(\widetilde{m})k(\widetilde{m}) = \beta\mathcal{C}(\widetilde{m})$. Equation (14) implies that if $\delta = 1$, then $\Gamma(\widetilde{m}) = f'(k(\widetilde{m}))$ so $\Gamma(\widetilde{m})k(\widetilde{m}) = f'(k(\widetilde{m}))k(\widetilde{m}) = \alpha f(k(\widetilde{m})) = \frac{\alpha\beta g}{\beta(1-\alpha)-\alpha}$, which equals $\beta\mathcal{C}(\widetilde{m})$, where $\mathcal{C}(\widetilde{m})$ is the expression in Statement 3.

Finally, $\xi = X(\widetilde{m})(1+\widetilde{m})^{\beta} = \frac{\alpha g}{\beta(1-\alpha)-\alpha}\left(\frac{\alpha}{1+\gamma}A^{\frac{1}{\alpha}}\left[\frac{\beta g}{\beta(1-\alpha)-\alpha}\right]^{\frac{\alpha-1}{\alpha}}\right)^{\beta}$, as assumed in the statement of Corollary 3. ■

Proof of Proposition 9. Since Propositions 7 and 9 share the same assumption, we have that $\mathcal{C}(m^{**}) > X(m^{**})$. Therefore, the solution for τ_M in equation (63) implies that $\tau_M = \left(1 - \frac{X(m^{**})}{\mathcal{C}(m^{**})}\right) w^{**} > 0$. ■