Dynamic Oligopoly and Innovation: A Quantitative Analysis of Technology Spillovers and Product Market Competition

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Abstract

We develop an endogenous growth model that embeds firms' product-market rivalry and technology-spillover networks to study how product-market competition and innovation policies affect economic growth and social welfare. We extend the generalized hedonic linear demand system of Pellegrino (2024) to a dynamic setting by introducing R&D investment and technology spillover networks. Despite forward-looking R&D decisions by many firms that interact strategically over these networks, the model remains tractable. We estimate the model on U.S. publicly traded firms with patents, constructing product similarity from product descriptions and technological similarity from patent classifications. A benevolent social planner who optimally allocates R&D more than doubles the investment, raising growth by 0.29 percentage points and welfare by 4.4%. By contrast, uniform R&D subsidies increase welfare by 0.53% at most, underscoring the need for policies that account for network heterogeneity.

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1 Introduction

Product market competition and research and development (R&D) allocations are jointly determined. Firms' R&D decisions shape future productivity and product market competition, while product market competition feeds back into profits and the incentives to invest in R&D. A large body of literature studies these interactions in endogenous growth models, which feature two interfirm externalities: a business-stealing externality that depresses competitors' profits and a technology-spillover externality that raises other firms' productivity.

Both the business-stealing effect and the technology spillover effect are interfirm externalities that can be represented as networks among firms. As emphasized by Bloom et al. (2013), capturing these externalities as inter-firm relationships is essential for assessing how product-market competition affects economic growth and social welfare, and for designing optimal R&D policy. However, existing endogenous growth models abstract from these networks, either assuming monopolistic competition without strategic interaction or allowing strategic interaction among only a very few firms. The obstacle is computational because forward-looking dynamic R&D choices generate a state space that explodes with the number of strategically interacting firms.

This paper builds an endogenous growth model that explicitly includes two interfirm networks: one for product-market rivalry and one for technology spillovers. We use data on about 800 publicly listed U.S. firms that hold patents, deriving product similarity from product descriptions and technological proximity from patent classes. We show that this dynamic oligopoly remains tractable even when many firms make forward-looking R&D decisions. Using the estimated model, we analyze how product-market competition and R&D policies shape economic growth and social welfare.

We extend Pellegrino (2024)'s static product-market framework into an endogenous growth model that incorporates technology spillovers and strategic R&D decisions. Pellegrino (2024) applies the widely used hedonic demand system from the empirical IO literature to a general-equilibrium setting and provides an estimation strategy for that framework. By extending this static framework to a dynamic setting, we develop an endogenous growth model in which firms accumulate knowledge capital through their own R&D investments and through technology spillovers from other firms.

Our model is tractable because we formulate the firms' dynamic R&D problem as a linear-quadratic differential game, which avoids the curse of dimensionality. The tractability of our model is preserved even when there are granular firm-level shocks, i.e., shocks that affect the aggregate economy. The framework also delivers a computational advantage for

evaluating the expected growth rate and social welfare: once the equilibrium is solved, these statistics can be computed for any initial distribution of knowledge capital with minimal additional effort.

We begin by examining the theoretical properties of the model in an economy without shocks. In this deterministic economy, a matrix termed the "technology transition matrix," which captures the dynamics of technological spillovers and endogenous R&D strategies, plays a key role in describing the conditions for the existence of a balanced growth path, the economic growth rate, and the stationary distribution of knowledge capital. Specifically, the irreducibility of the technology transition matrix serves as a sufficient condition for the existence of a globally stable balanced growth path. Intuitively, the irreducibility of the technology transition matrix implies that all firms are technologically connected, leading to all firms growing at the same rate. On the balanced growth path, the growth rate of all firms is given by the dominant eigenvalue of the technology transition matrix, and the distribution of productivity across firms corresponds to a scalar multiple of the associated eigenvector.

We can estimate our framework using Compustat and publicly available data without relying on confidential data. Following Pellegrino (2024), we measure product market rivalry betwee each pair of U.S. publicly listed firms based on the product similarity measures of Hoberg and Phillips (2016). We construct the technological spillover network by following Bloom et al. (2013) and using patent classifications to measure technological proximity. Using these networks together with Compustat financial statement data, we also recover the distribution of firm-level knowledge capital implied by the model.

Using the estimated model, we compare the competitive equilibrium allocation with those chosen by a benevolent social planner who maximizes social welfare and by a monopolist who maximizes producer surplus. First, we hold the static product-market allocation fixed at its competitive equilibrium outcome and consider the allocations chosen by a constrained social planner and a monopolist who adjust only R&D allocations. The constrained monopolist cuts total R&D to 51.64% of the competitive equilibrium allocation, lowering the expected growth rate to 1.36% (-0.15 percentage points relative to competition) and reducing consumption-equivalent welfare by 1.1%. The constrained social planner instead raises total R&D to 241.95% of the competitive equilibrium level, increasing the growth rate to 1.80% (+0.29 percentage points) and welfare by 4.4%. These movements reflect the appropriability effect: the monopolist focuses on producer surplus and internalizes business-stealing and technology spillovers but not the consumer-surplus benefits of innovation, whereas the planner accounts for all of these effects.

Second, we allow both static and dynamic reallocation so that the monopolist and

the planner choose production and R&D at the same time. Under the monopolist, this joint choice lowers output by 3.7%, pushes total R&D down to 63.22% of the competitive allocation, slows growth by 0.11 percentage points, and reduces consumption-equivalent welfare by 4.3%. Product market power raises the monopolist's private return to innovation, so she invests more in R&D than in the constrained monopolist, yet still below the competitive equilibrium level. The social planner instead expands output by 9.3%, lifts total R&D to 348.72% of the competitive allocation, raises the growth rate by 0.32 percentage points, and increases welfare by 14.5%. By removing product-market misallocation, the planner strengthens the social return to innovation and therefore undertakes even more R&D investment than the constrained planner.

Finally, we examine the effect of uniform R&D subsidies. Welfare reaches its peak at a subsidy rate of 34%. At that point, expected growth rises to 1.80%—0.29 percentage points above the competitive equilibrium outcome—and consumption-equivalent welfare increases by 0.53%. This gain is substantially smaller than the 4.36% improvement realized under the constrained social planner's R&D allocation. These findings suggest that, to achieve sizable welfare gains from R&D policies, interventions must take into account the network heterogeneity of product-market rivalry and technology spillovers.

This study makes several contributions to the literature. First, it builds on the extensive research on innovation and product market competition (d'Aspremont and Jacquemin, 1988; Aghion et al., 2001, 2005; Acemoglu and Akcigit, 2012; Bloom et al., 2013; Peters, 2020; Akcigit and Ates, 2021, 2023; Liu et al., 2022; Cavenaile et al., 2023; De Ridder, 2024). Bloom et al. (2013) empirically examine how product market and technological proximity influence innovation. They show that higher product market proximity intensifies the business-stealing effect, whereas greater technological proximity enhances technology spillovers. Cavenaile et al. (2023) develop an oligopolistic general equilibrium Schumpeterian growth model that accommodates competition among a small number of firms within each industry. However, such models involving oligopolistic dynamic games encounter significant computational challenges, with computation time increasing exponentially with the number of firms. To overcome this limitation, we incorporate a linear-quadratic network game into the Schumpeterian framework. Unlike existing endogenous growth models, our approach allows us to directly match firm-level and firm-pair-level data with the model.

Our study is also related to research that incorporates oligopolistic competition into macroeconomic models Neary (2003); Atkeson and Burstein (2008); Azar and Vives (2021); Pellegrino (2024); Ederer and Pellegrino (2024). In particular, our oligopolistic market environment is based on the work of Pellegrino (2024) and Ederer and Pellegrino (2024). We extend their static model into a dynamic endogenous economic growth framework,

incorporating a network of R&D investment and technology spillovers.

The remainder of the paper proceeds as follows. Section 2 constructs an endogenous growth model incorporating the networks of product market rivalry and technology spillover. Section 3 describes the data and the estimation method of the model. Section 4 reports the characteristics of the baseline model and the results of various counterfactual analyses. Section 5 concludes.

2 Model

We develop an endogenous growth model where granular firms strategically interact in both product and technology spaces to examine the implications for firms' dynamic R&D investment, economic growth, and social welfare.

2.1 Environment

Our environment incorporates two networks: product market rivalry and technology spillovers. There are n oligopolistic firms, indexed by $i \in \{1, 2, \ldots, n\}$; they choose output and dynamic R&D investment while interacting strategically through these networks. Time is continuous and infinite, $t \in [0, \infty)$.

2.1.1 Demand

We introduce the Generalized Hedonic Linear (GHL) demand system, developed by Pellegrino (2024), to model product market rivalry among many oligopolistic firms. Products have two types of characteristics: common characteristics shared by all products, indexed by $k \in \{1, 2, \ldots, m\}$; and an idiosyncratic characteristic unique to each product, which shares the same index i as the product it corresponds to. The scalar $\psi_{ki} \geq 0$ denotes the quantity of the common characteristic k provided by product i. Each product i is described by an m-dimensional, non-negative column vector $\boldsymbol{\psi}_i$:

$$\boldsymbol{\psi}_i = \begin{bmatrix} \psi_{1i} & \psi_{2i} & \ldots & \psi_{mi} \end{bmatrix}^T$$

The length of each product's common-characteristics vector is normalized to 1 without loss of generality:

$$\sum_{k=1}^{m} \psi_{ki}^2 = 1 \quad \forall i \in \{1, 2, \ldots, n\}$$

We stack the vectors $\boldsymbol{\psi}_i$, which capture the common characteristics of each product, to form the $m \times n$ matrix $\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} \equiv \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \cdots & \boldsymbol{\psi}_n \end{bmatrix} \equiv \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{m1} & \psi_{m2} & \cdots & \psi_{mn} \end{bmatrix}$$

Aggregating characteristics across products yields the production function of the final-goods producer. Let $q_{i,t}$ be the number of units produced by firm i and consumed by households at time t, and define the n-dimensional vector $\boldsymbol{q}_t$:

$$\boldsymbol{q}_t \equiv \begin{bmatrix} q_{1,t} & q_{2,t} & \cdots & q_{n,t} \end{bmatrix}^T$$

Let y_{kt}^C be the total units of common characteristic k aggregated over all products:

$$y_{k,t}^C = \sum_{i=1}^{n} \psi_{k,i} q_{i,t}$$

Therefore, the matrix $\boldsymbol{\Psi}$ transforms units of goods into units of common characteristics:

$$\boldsymbol{y}_t^C = \boldsymbol{\Psi} \boldsymbol{q}_t \tag{1}$$

Let $y_{i,t}^I$ denote the units of idiosyncratic characteristic i. We assume that each unit of good i provides one unit of its corresponding idiosyncratic characteristic:

$$\boldsymbol{y}_t^I = \boldsymbol{q}_t \tag{2}$$

A representative final-goods producer aggregates the common and idiosyncratic characteristics using a linear–quadratic aggregator:

$$Y_t = \alpha \sum_{k=1}^{m} \left(b_{k,t}^C y_{k,t}^C - \frac{1}{2} \left(y_{k,t}^C \right)^2 \right) + (1 - \alpha) \sum_{i=1}^{n} \left(b_{i,t}^I y_{i,t}^I - \frac{1}{2} \left(y_{i,t}^I \right)^2 \right), \tag{3}$$

where $b_{k,t}^C > 0$ and $b_{i,t}^I > 0$ are characteristic-specific quality parameters, and $\alpha \in [0, 1]$ is the weight placed on common characteristics. Inserting Equations (1) and (2) into Equation (3), we obtain a vector form of aggregate output (in terms of the quantity vector $\boldsymbol{q}_t \in \mathbb{R}^n$):

$$Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2}\boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t \tag{4}$$

where $\boldsymbol{b}_t \in \mathbb{R}^n$ collects product-level qualities implied by common and idiosyncratic characteristics,

$$\boldsymbol{b}_t \equiv \alpha \boldsymbol{\Psi}^T \boldsymbol{b}_t^C + (1-\alpha)\boldsymbol{b}_t^I, \quad \boldsymbol{b}_t^C \in \mathbb{R}^m,\ \boldsymbol{b}_t^I \in \mathbb{R}^n,$$

$$\boldsymbol{\Sigma} \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \equiv \alpha \boldsymbol{\Psi}^T \boldsymbol{\Psi} + (1-\alpha)\,\boldsymbol{I}, \tag{5}$$

and $\boldsymbol{I}$ is the $n \times n$ identity matrix.

Matrix $\boldsymbol{\Sigma}$ captures product-market rivalry. From Equation (4), larger σ_{ij} imply a larger quadratic penalty when q_i and q_j are both high, so σ_{ij} measures the substitutability between products i and j. Because each column $\boldsymbol{\psi}_i$ of $\boldsymbol{\Psi}$ is normalized ($\sum_k \psi_{ki}^2 = 1$), the diagonal elements satisfy $\sigma_{ii} = \alpha\, \boldsymbol{\psi}_i^T \boldsymbol{\psi}_i + (1-\alpha) = 1$ for all i.

The decomposition $\boldsymbol{\Sigma} = \alpha \boldsymbol{\Psi}^T \boldsymbol{\Psi} + (1-\alpha)\boldsymbol{I}$ shows that off-diagonal substitutabilities are governed by product similarity: each element of $\boldsymbol{\Psi}^T \boldsymbol{\Psi}$ is $\left[\boldsymbol{\Psi}^T \boldsymbol{\Psi}\right]_{ij} = \boldsymbol{\psi}_i^T \boldsymbol{\psi}_j$, the cosine similarity between the common-characteristics vectors of products i and j in $\mathbb{R}^m$. The weight α maps similarity into substitutability. As shown by Pellegrino (2024), these properties allow us to estimate substitutabilities using product-similarity data.

2.1.2 Technology

Each firm has a linear production technology in labor:

$$q_{i,t} = a_{i,t} l_{i,t} \tag{6}$$

where $a_{i,t}$ is the labor productivity of firm i at time t and $l_{i,t}$ is the production labor employed by firm i at time t.

Firms possess their own knowledge capital and use it to improve labor productivity and product quality. As shown in Equation (4), $b_{i,t}$ is the value (in final-good units) generated by the first unit of product i, so we interpret $b_{i,t}$ as product quality. Firms allocate knowledge capital between labor productivity $a_{i,t}$ and product quality $b_{i,t}$:

$$\zeta a_{i,t} + b_{i,t} = z_{i,t} \tag{7}$$

where $\zeta > 0$ governs the trade-off between improving labor productivity and product quality[1].

We model the law of motion for knowledge capital, which rises through technology spillovers from technologically similar firms and through the firm's own R&D investment. Let $\tilde{\mathbf{\Omega}} \in \mathbb{R}^{n \times n}$ denote technological proximity between firms:

$$\tilde{\mathbf{\Omega}} = \begin{bmatrix} \tilde{\omega}_{11} & \tilde{\omega}_{12} & \cdots & \tilde{\omega}_{1n} \\ \tilde{\omega}_{21} & \tilde{\omega}_{22} & \cdots & \tilde{\omega}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\omega}_{n1} & \tilde{\omega}_{n2} & \cdots & \tilde{\omega}_{nn} \end{bmatrix}$$

where each element $\tilde{\omega}_{ij} \geq 0$ denotes the technological proximity between firm i and firm j. We set the diagonal to zero by normalization: $\tilde{\omega}_{ii} = 0$ for all i. Let $\boldsymbol{x}_t \in \mathbb{R}^n$ denote the vector of R&D efforts:

$$\boldsymbol{x}_t \equiv \begin{bmatrix} x_{1,t} & x_{2,t} & \dots & x_{n,t} \end{bmatrix}^T,$$

and let $\boldsymbol{W}_t \in \mathbb{R}^n$ denote independent standard Wiener processes across firms:

$$\boldsymbol{W}_t \equiv \begin{bmatrix} W_{1,t} & W_{2,t} & \dots & W_{n,t} \end{bmatrix}^T$$

Knowledge capital follows a multi-dimensional geometric Brownian motion:

$$d\boldsymbol{z}_t = \left((\mathbf{\Omega} - \delta \boldsymbol{I})\, \boldsymbol{z}_t + \mu \boldsymbol{x}_t \right) dt + \gamma \boldsymbol{z}_t \circ d\boldsymbol{W}_t \tag{8}$$

where $\mathbf{\Omega} = \beta \tilde{\mathbf{\Omega}} \in \mathbb{R}^{n \times n}$, and β, μ, γ, and δ are positive scalars. We assume that the magnitude of technology spillovers between two firms is proportional to the technological proximity $\tilde{\mathbf{\Omega}}$, and β is the parameter that converts technological proximity into the magnitude of technology spillovers. μ governs the efficiency of R&D, γ the magnitude of shocks, and δ the depreciation rate of knowledge capital.

[1]Assuming optimal allocation of knowledge capital between labor-productivity and quality improvements is necessary for a balanced-growth-path equilibrium. Similarly, Grossman et al. (2017) and Jones and Liu (2024) obtain a balanced growth path by assuming two productivity-enhancing technologies (capital and human capital; capital productivity and automation, respectively) and allocating resources between them.

2.1.3 Resources and Household Preferences

We close the model with resource constraints and representative household preferences. Production labor is supplied inelastically:

$$L = \sum_i l_{i,t}. \tag{9}$$

The final good is used for consumption and R&D investment. Following the endogenous growth literature (e.g. Acemoglu et al. 2016), we assume that the innovation elasticity for each firm is 0.5. In other words, as the firm-level R&D outcome increases, the R&D costs imposed on the firm increase quadratically:

$$C_t + \sum_i x_{i,t}^2 = Y_t. \tag{10}$$

where C_t is consumption of the final good at time t. Finally, we assume that the representative household is risk-neutral and discounts the future at rate $\rho > 0$:

$$\mathbb{E}_0 \left[\int_0^\infty \exp\left(-\rho t\right) C_t \, dt \right]$$

2.2 Equilibrium

We focus on the Markov Perfect Equilibrium, and therefore, we can separate the problem into static and dynamic components. First, we analyze the static problem of competition in the product market, and then we address the dynamic problem of R&D investments.

2.2.1 Problem of Final Good Producer

The final good is produced competitively. In each period, the final goods producer chooses the quantity of each product that maximizes profits, given the prices.

$$\max_{\boldsymbol{q}_t} P_t Y_t - \boldsymbol{q}_t^T \boldsymbol{p}_t$$

where $\boldsymbol{p}_t$ is the vector of product prices at time t and P_t is the final good price at time t. The profit maximization problem of the final goods producer results in the following linear inverse demand function:

$$\frac{\boldsymbol{p}_t}{P_t} = \boldsymbol{b}_t - \boldsymbol{\Sigma} \boldsymbol{q}_t \tag{11}$$

2.2.2 Static Problem of Firms: Production Decision and Technology Choice

We assume that firms maximize the real value to ensure the independence of the allocation from the choice of numeraire (Azar and Vives, 2021). Thus, in the static problem, firms maximize the real profits of the firms. Each firm's real gross profit *before* subtracting R&D cost, $\pi_{i,t}$, is given by

$$\begin{aligned}\pi_{i,t} &= \frac{p_{i,t}}{P_t} q_{i,t} - \frac{w_t}{P_t} l_{i,t} \\ &= \left(b_{i,t} - \frac{1}{a_{i,t}} \frac{w_t}{P_t} - \sum_j \sigma_{ij} q_{j,t} \right) q_{i,t} \end{aligned} \tag{12}$$

In the first line on the right-hand side, the labor cost is subtracted from the real revenue. The second line is obtained by substituting the inverse demand Equation (11) and the production function Equation (6).

Each firm i simultaneously decides the product quantity $q_{i,t}$ and how to allocate its knowledge capital between improving labor productivity $a_{i,t}$ and product quality $b_{i,t}$, taking as given the output of other firms $\{q_{j,t}\}_{j \neq i}$, real wage w_t/P_t, and its own knowledge capital $z_{i,t}$ under the knowledge capital constraint Equation (7). The FOC w.r.t. $q_{i,t}$ gives

$$0 = b_{i,t} - \frac{1}{a_{i,t}} \frac{w_t}{P_t} - 2q_{i,t} - \sum_{j \neq i} \sigma_{ij} q_{j,t} \tag{13}$$

Also, from the optimality conditions for $a_{i,t}$ and $b_{i,t}$, we obtain

$$a_{i,t} = \sqrt{\frac{w_t}{\zeta P_t}} \tag{14}$$

$$b_{i,t} = z_{i,t} - \sqrt{\frac{\zeta w_t}{P_t}}. \tag{15}$$

Intuitively, firms are more inclined to improve labor productivity when labor costs w_t/P_t are higher. Inserting Equation (14) into Equation (9), we obtain

$$\sqrt{\frac{\zeta w_t}{P_t}} = \frac{\zeta}{L} \sum_i q_{i,t}. \tag{16}$$

Combining Equations (13) to (16), we obtain the expression for the quantities as a linear function of knowledge capital:

$$\boldsymbol{q}_t = \boldsymbol{N} \boldsymbol{z}_t \tag{17}$$

where $\boldsymbol{N}$ is an $n \times n$ matrix

$$\boldsymbol{N} \equiv \left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma} + \boldsymbol{I}\right)^{-1},$$

$\boldsymbol{J}$ is an $n \times n$ matrix where all elements are 1, and $\boldsymbol{I}$ is an $n \times n$ identity matrix (see Section A.1 for the derivation). The first term inside the parentheses, $2\frac{\zeta}{L}\boldsymbol{J}$, captures the labor cost, while the second term $\boldsymbol{\Sigma}$ represents the network of product market rivalry.

Given the linear relationship between the quantity $\boldsymbol{q}_t$ and knowledge capital z_t, we can express the firm's static objective in a quadratic form of the firm's knowledge capital vector z_t:

$$\pi_{i,t} = q_{i,t}^2 = \boldsymbol{z}_t^T \boldsymbol{N}_i^T \boldsymbol{N}_i \boldsymbol{z}_t$$

where $\boldsymbol{N}_i$ denotes the i-th row of $\boldsymbol{N}$.

2.2.3 Dynamic Problem of Firms: R&D Decision

Given a static production strategy profile, we consider a dynamic R&D game. In the dynamic game, given other players' strategies $\left\{x_{j,t}\right\}_{j\neq i, t\geq 0}$, firm i chooses R&D effort $\{x_{i,t}\}_{t\geq 0}$ to maximize the expected discounted present value of profits:

$$\max_{\{x_{i,t}\}_{t\geq 0}} \quad V^i(\boldsymbol{z}_0) \equiv \mathbb{E}_0\left[\int_0^\infty \exp\left(-\rho t\right)\left\{\pi_{i,t} - x_{i,t}^2\right\} dt\right]$$

while the state evolves according to Equation (8). Here, we utilize the fact that the interest rate is derived from the household's Euler equation $r_t = \rho$. The dynamic problem gives the following HJB equation:

$$\rho V^i(\boldsymbol{z}) = \max_{x_i}\left\{\boldsymbol{z}^T \boldsymbol{N}_i^T \boldsymbol{N}_i \boldsymbol{z} - x_i^2 + V_{\boldsymbol{z}}^i(\boldsymbol{z})\left[\left(\boldsymbol{\Omega} - \delta \boldsymbol{I}\right)\boldsymbol{z} + \mu \boldsymbol{x}\right] + \frac{\gamma^2}{2}\boldsymbol{z}^T V_{\boldsymbol{zz}}^i(\boldsymbol{z})\,\boldsymbol{z}\right\}. \tag{18}$$

The first term on the right-hand side of the HJB equation represents gross profit. The second term represents R&D costs. Therefore, the first two terms represent the firm's net instantaneous profit. The third term reflects the impact of changes in knowledge capital due to technology spillovers and R&D on firm i's value. The fourth term represents the expected effect of all firms' idiosyncratic shocks on firm i's value.

This dynamic game is a linear quadratic differential game, which significantly simplifies the analysis. This simplification is achieved because the following guess and verify

approach holds. Guess that the value can be expressed as a quadratic form of knowledge capital:

$$V^i(\boldsymbol{z}) = \boldsymbol{z}^T \boldsymbol{X}^i \boldsymbol{z}. \tag{19}$$

The first and second derivatives of the value function are then given by:

$$V_z^i(\boldsymbol{z}) = 2\boldsymbol{z}^T \boldsymbol{X}^i \tag{20}$$

$$V_{zz}^i(\boldsymbol{z}) = 2\boldsymbol{X}^i \tag{21}$$

In a linear quadratic differential game, firms' strategies are expressed as linear functions of the state variables. Let $\boldsymbol{X}_i^i$ denote the i-th column of $\boldsymbol{X}^i$. The first-order condition with respect to x_i gives

$$x_i = \left(\mu \boldsymbol{X}_i^i\right)^T \boldsymbol{z}. \tag{22}$$

Given firms' linear strategies, the law of motion for knowledge capital $\boldsymbol{z}_t$ can be rewritten as a geometric Brownian motion with constant coefficients. Define $\tilde{\boldsymbol{X}}$ as the matrix obtained by stacking $\boldsymbol{X}_i^i$:

$$\tilde{\boldsymbol{X}} \equiv \begin{bmatrix} \boldsymbol{X}_1^1 & \cdots & \boldsymbol{X}_n^n \end{bmatrix}^T \tag{23}$$

Then, the law of motion is rewritten as

$$d\boldsymbol{z}_t = \boldsymbol{\Phi} \boldsymbol{z}_t dt + \gamma \boldsymbol{z}_t \circ d\boldsymbol{W}_t \tag{24}$$

where the matrix $\boldsymbol{\Phi}$ that captures the drift of the stochastic process is an $n \times n$ matrix such that

$$\boldsymbol{\Phi} \equiv \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix} \equiv \boldsymbol{\Omega} - \delta \boldsymbol{I} + \mu^2 \tilde{\boldsymbol{X}}.$$

We refer to the matrix $\boldsymbol{\Phi}$ as the *technology transition matrix*, which consists of exogenous technology spillover matrix $\boldsymbol{\Omega}$, depreciation $-\delta \boldsymbol{I}$, and endogenous R&D matrix $\mu^2 \tilde{\boldsymbol{X}}$.

The HJB equation can be transformed into an algebraic Riccati equation, enabling the dynamic oligopoly problem with a number of firms to be solved in a computationally feasible timeframe. Inserting Equations (19) to (22) into the HJB Equation (18), and rearranging, we obtain stacked algebraic Riccati equations

$$0 = \boldsymbol{Q}_i - \mu^2 \boldsymbol{X}_i^i \left(\boldsymbol{X}_i^i\right)^T + \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}^i + \boldsymbol{X}^i \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right) \tag{25}$$

Table 1: Growth Rate of Variables in Deterministic Economy

Growth Rate	Variable
0	$l_{i,t}$
g	$a_{i,t}, b_{i,t}, z_{i,t}, q_{i,t}, x_{i,t}, p_{i,t}/P_t$
$2g$	$\pi_{i,t}, C_t, Y_t, w_t/P_t$

Note: Growth rates are evaluated along the balanced-growth-path equilibrium in the deterministic economy (no shocks).

for $i = 1, 2, ..., n$. Note that the terms $\mu^2 \boldsymbol{X}_i^i \left(\boldsymbol{X}_i^i\right)^T$ and $\boldsymbol{\Phi}$ depend on the $\boldsymbol{X}^j$ of other firms $j \neq i$. Therefore, to solve for $\boldsymbol{X}^i$, we need to simultaneously solve the algebraic Riccati equations for all firms.

Here, we define the stability of the solution.

Definition 1. The solution of the stacked algebraic Riccati Equation (25) is *stable* if all the eigenvalues of $\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}$ have negative real parts.

The requirement for stability of the solution corresponds to the condition commonly imposed in growth theory, which requires the discount rate to be sufficiently large to ensure that the utility over an infinite horizon is finite.

In the following, we assume the existence of such a solution.

Assumption 1. *There exists a stabilizing solution* $\left(\boldsymbol{X}^1, \boldsymbol{X}^2, \ldots, \boldsymbol{X}^n\right)$ *of the stacked algebraic Riccati Equation* (25)*.*

2.2.4 Balanced Growth Path in a Deterministic Economy

In this subsection, we characterize the existence of a balanced growth path and its equilibrium in a deterministic economy using the technology transition matrix $\boldsymbol{\Phi}$.

We impose the following assumption on $\boldsymbol{\Phi}$, which is a sufficient condition for the existence of a balanced growth path equilibrium.

Assumption 2. $\boldsymbol{\Phi}$ *is irreducible.*

One definition of an irreducible matrix is as follows. Consider a directed graph with n vertices labeled $1, ..., n$, and an edge from vertex i to vertex j when $\phi_{ij} > 0$. Then, the matrix $\boldsymbol{\Phi}$ is irreducible if and only if its associated graph is strongly connected (i.e. every vertex is reachable from every other vertex). Intuitively, this assumes that all firms have direct or indirect technological connections with each other.

Here, we define the balanced growth path equilibrium in our economy.

Definition 2. Balanced Growth Path Equilibrium: A *balanced growth path equilibrium* is a linear Markov perfect equilibrium such that the productivity of all firms grows at the same constant rate.

In the following proposition, we present sufficient conditions for the existence of a balanced growth path and characterize the equilibrium.

Proposition 1. *Let Assumptions Assumption 1 and Assumption 2 hold. Then, there exists the largest positive eigenvalue g of $\mathbf{\Phi}$ and a strictly positive eigenvector $\boldsymbol{z}$ associated with g. Moreover, there exists a globally stable balanced growth path equilibrium such that the productivity growth rate of all firms is given by g, and the productivity distribution is given by a scalar multiple of $\boldsymbol{z}$.*

Proof. The proposition follows from the Perron-Frobenius theorem. □

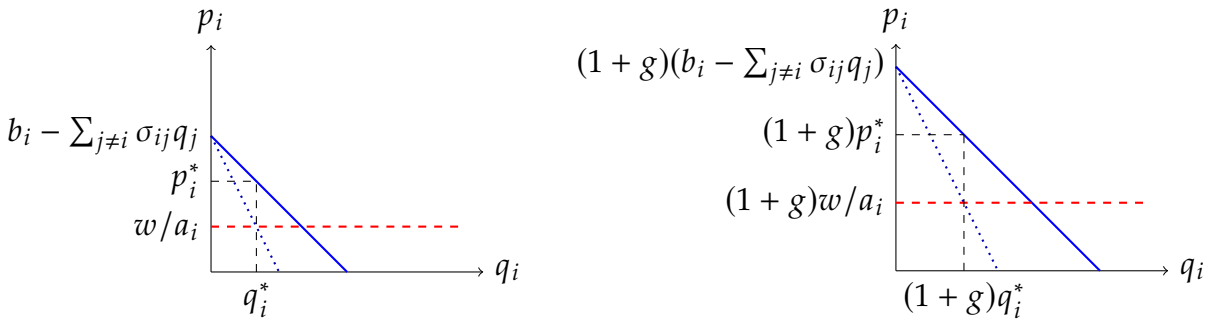
Proposition 1 implies that there exists a unique and globally stable balanced-growth-path equilibrium when the technology transition matrix $\mathbf{\Phi}$ is irreducible. It also implies that the equilibrium productivity growth rate is given by the leading eigenvalue of $\mathbf{\Phi}$, and the stationary distribution is characterized by the associated eigenvector. Recall that the technology transition matrix consists of an exogenous technology-spillover matrix and an endogenous R&D strategy-profile matrix. Therefore, this result indicates that our model is an endogenous growth model that achieves a balanced growth path. The growth rates of variables on the balanced growth path are summarized in Table 1.

Figure 1 illustrates a partial equilibrium in a deterministic economy to provide an intuitive understanding of how a balanced-growth-path equilibrium is achieved. This diagram depicts the residual demand (blue), marginal revenue (dotted line), and marginal cost (red). In equilibrium, as quality (b_i), quantity (q_i), price (p_i), and marginal cost (w/a_i) all grow at the same rate g, the partial-equilibrium diagram expands homothetically at this common rate along both axes. Consequently, ratios such as revenue to cost and consumer surplus to producer surplus remain constant along the path. This constancy allows the economy to achieve balanced growth. The figure also suggests that growth in both labor productivity a_i and quality b_i is necessary. For example, if only quality b_i grows and labor productivity a_i does not, it is impossible for $q_i = a_i l_i$ to exhibit sustained growth because labor supply is inelastic.

2.2.5 Social Welfare

Now, we demonstrate that once the equilibrium is solved, the representative household's utility can be calculated for any initial distribution with minimal additional computational cost. To characterize the household's value, we introduce the household's HJB equation.

Figure 1: Partial Equilibrium Diagram and Its Proportional Growth



Note: The left panel depicts the static equilibrium for a firm's product, showing the residual demand (blue), marginal revenue (dotted line), and marginal cost (red) curves. The right panel illustrates how the equilibrium price, quantity, marginal cost, and the intercept of the residual demand scale proportionally with the growth factor $(1+g)$.

Let $V(z_0)$ denote the expected value of the representative household in the competitive equilibrium given the initial distribution z_0, which is defined as

$$V(z_0) = \mathbb{E}_0 \left[\int_0^\infty \exp(-\rho t) \left\{ Y_t - \sum_i x_{i,t}^2 \right\} dt \middle| z_0 \right].$$

Note that the total output is given by

$$Y_t = z_t^T \boldsymbol{Q} z_t \tag{26}$$

where

$$\boldsymbol{Q} = \boldsymbol{N}^T \left(\frac{\zeta}{L} \boldsymbol{J} + \frac{1}{2} \boldsymbol{\Sigma} + \boldsymbol{I} \right) \boldsymbol{N}$$

(see Section A.2 for the derivation). The HJB equation for the representative household is given by

$$\rho V(z) = z^T \boldsymbol{Q} z - \mu^2 \tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}} + V_z(z) \boldsymbol{\Phi} z + \frac{\gamma^2}{2} z^T V_{zz}(z) z$$

The first term on the right-hand side of the HJB equation represents output. The second term represents the total R&D cost. Therefore, the first two terms represent the household's instantaneous consumption. The third and fourth terms represent the expected change in the household's value due to the stochastic process of knowledge capital. Note that the level of consumption is determined as the difference between total output Y_t and the total R&D expenditure $\sum_i x_{i,t}^2$, which result from firms' behavior under the Euler equation $r_t = \rho$. Therefore, in the representative household's HJB equation, which is solved after

the firm's problem, there are no choice variables for the representative household.

Since the household's problem is linear–quadratic, the household's value can be expressed in a quadratic form of knowledge capital z. Guess that the value is expressed as a quadratic form of knowledge capital:

$$V(z) = z^T \boldsymbol{X} z.$$

We obtain the following Lyapunov equation by substituting $V(z)$ and its derivatives into the HJB equation:

$$0 = \boldsymbol{Q} - \mu^2 \tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}} + \boldsymbol{X}\left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right) + \left(\boldsymbol{\Phi} - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X} \tag{27}$$

Therefore, $\boldsymbol{X}$ is obtained as the solution of the Lyapunov Equation (27), and the welfare in the competitive equilibrium with initial productivity vector z_0 is given by $V(z_0) = z_0^T \boldsymbol{X} z_0$. This result indicates that once the equilibrium is solved, we can easily compute the expected utility of the representative household for any given initial distribution.

2.2.6 Total Value of Firms

When competition in product markets and R&D competition is relaxed, producer surplus increases relative to total surplus. To capture the dynamic producer surplus, this section characterizes the total value of the firms.

Let V^P denote the total value of firms in the competitive equilibrium, which is defined as

$$V^P(z_0) = \mathbb{E}_0\left[\int_0^\infty \exp\left(-\rho t\right)\left(\sum_i \pi_{i,t} - \sum_i x_{i,t}^2\right) dt \,\middle|\, z_0\right]$$

Note that aggregate real profit of firms is written in vector form as

$$\begin{aligned}\sum_i \pi_{i,t} &= \boldsymbol{q}_t^T \boldsymbol{q}_t \\ &= z_t^T \boldsymbol{N}^T \boldsymbol{N} z_t.\end{aligned}$$

Comparing with the problem of calculating social welfare in Section 2.2.5, it becomes clear that solving the modified version of the Lyapunov equation (27), in which $\boldsymbol{Q}$ is replaced with $\boldsymbol{N}^T\boldsymbol{N}$. When we express the solution of the Lyapunov equation with $\boldsymbol{X}^P$, the total

value of firms is given by $V^P(\boldsymbol{z}_0) = \boldsymbol{z}_0^T \boldsymbol{X}^P \boldsymbol{z}_0$.

2.3 Counterfactual Scenarios

We examine multiple counterfactual scenarios and compare them with the allocation under the competitive equilibrium. First, we analyze the allocation chosen by a benevolent social planner who aims to maximize social welfare. Additionally, we consider the monopoly case, in which a monopolist determines production and R&D to maximize total firm value.

2.3.1 Social Planner

To analyze the socially optimal allocation of R&D and its impact on economic growth and social welfare, we formulate the social planner's problem. The social planner aims to maximize the representative household's lifetime utility, subject to technological constraints. Similar to the competitive equilibrium, the social planner's problem can be separated into a static problem and a dynamic problem.

Static Problem First, we characterize the static allocation in the social planner's problem. The social planner maximizes the final good production given the vector of knowledge capital $\boldsymbol{z}_t$ in each period:

$$\max_{\boldsymbol{a}_t, \boldsymbol{b}_t, \boldsymbol{l}_t, \boldsymbol{q}_t} \; Y_t = \boldsymbol{q}_t^T \boldsymbol{b}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

subject to

$$\begin{aligned} q_{i,t} &= a_{i,t} l_{i,t} \quad i = \{1, 2, ..., n\} \\ z_{i,t} &= \zeta a_{i,t} + b_{i,t} \quad i = \{1, 2, ..., n\} \\ L &= \sum_i l_{i,t} \end{aligned} \tag{28}$$

The first constraint represents the production technology of each firm, the second governs the allocation of each firm's knowledge capital between labor productivity and product quality, and the third ensures the labor market clears. By solving the planner's static problem, we obtain a solution of quantities as a linear function of knowledge capital:

$$\boldsymbol{q}_t^* = \boldsymbol{N}^* \boldsymbol{z}_t$$

and a quadratic form for output of knowledge capital

$$Y_t^* = \frac{1}{2} \boldsymbol{z}_t^T \boldsymbol{N}^* \boldsymbol{z}_t$$

where

$$\boldsymbol{N}^* = \left(2\frac{\zeta}{L} \boldsymbol{J} + \boldsymbol{\Sigma} \right)^{-1}$$

(see Section A.3).

Dynamic Problem Next, we characterize the dynamic allocation in the social planner's problem. The social planner's dynamic problem is formulated as follows:

$$V^*(\boldsymbol{z}_0) \equiv \max_{\{\boldsymbol{x}_t\}_{t \geq 0}} \mathbb{E}_0 \left[\int_0^\infty \exp\left(-\rho t\right) \left\{ \frac{1}{2} \boldsymbol{z}_t^T \boldsymbol{N}^* \boldsymbol{z}_t - \boldsymbol{x}_t^T \boldsymbol{x}_t \right\} dt \right],$$

subject to the law of motion of knowledge capital Equation (24). The HJB equation of the social planner's value V^* derived from the dynamic problem is given by

$$\rho V^*(\boldsymbol{z}) = \max_{\boldsymbol{x} \geq 0} \left\{ \frac{1}{2} \boldsymbol{z}^T \boldsymbol{N}^* \boldsymbol{z} - \boldsymbol{x}^T \boldsymbol{x} + V_z^*(\boldsymbol{z})^T \left[(\boldsymbol{\Omega} - \delta \boldsymbol{I}) \boldsymbol{z} + \mu \boldsymbol{x} \right] + \frac{\gamma^2}{2} \boldsymbol{z}^T V_{zz}^*(\boldsymbol{z}) \boldsymbol{z} \right\} \tag{29}$$

The first term on the right-hand side of the HJB equation represents output. The second term represents the total R&D cost. Therefore, the first two terms represent the household's instantaneous consumption. The third and fourth terms represent the expected change in the household's value due to the stochastic process of knowledge capital.

Since the social planner's problem is also linear-quadratic, household utility can be expressed in a quadratic form of knowledge capital $\boldsymbol{z}$. Guess that the value is expressed as a quadratic form of knowledge capital:

$$V^*(\boldsymbol{z}) = \boldsymbol{z}^T \boldsymbol{X}^* \boldsymbol{z} \tag{30}$$

The socially optimal allocation of R&D is a linear function of knowledge capital $\boldsymbol{z}$, as in the competitive equilibrium:

$$\boldsymbol{x} = \mu \boldsymbol{X}^* \boldsymbol{z} \tag{31}$$

Therefore, again, the law of motion for knowledge capital $\boldsymbol{z}_t$ is rewritten as a geometric Brownian motion with diagonal multiplicative noise:

$$d\boldsymbol{z}_t = \boldsymbol{\Phi}^* \boldsymbol{z}_t dt + \gamma \boldsymbol{z}_t \circ d\boldsymbol{W}_t$$

where the optimal technology transition $\mathbf{\Phi}^*$ is given by

$$\mathbf{\Phi}^* \equiv \mathbf{\Omega} + \mu^2 \boldsymbol{X}^* - \delta \boldsymbol{I}$$

Inserting Equation (30) and its first and second derivatives with respect to z and Equation (31) into Equation (29), we obtain a single algebraic Riccati equation:

$$0 = \frac{1}{2}\boldsymbol{N}^* - \mu^2 \left(\boldsymbol{X}^*\right)^2 + \boldsymbol{X}^* \left(\mathbf{\Phi}^* - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right) + \left(\mathbf{\Phi}^* - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)\boldsymbol{X}^* \tag{32}$$

Therefore, $\boldsymbol{X}^*$ is obtained as the solution of the single algebraic Riccati equation (32). Remember that, in the competitive equilibrium, a Riccati equation exists for each firm, requiring simultaneous solutions of these equations. In contrast, for the social planner problem, we only need to solve a single Riccati equation, making the computation much simpler.

The social planner's problem yields a quadratic form for social welfare under the optimal allocation: $V^*(z_0) = z_0^T \boldsymbol{X}^* z_0$. This result allows us to calculate the expected utility of the representative household for any initial distribution z_0 with minimal additional computation once $\boldsymbol{X}^*$ is obtained.

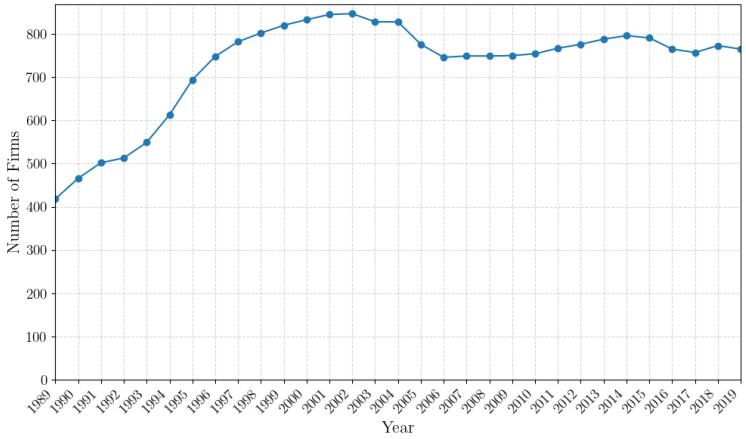
2.3.2 Monopoly

As a second counterfactual scenario, we consider the monopoly case, where competition in both the product market and R&D is fully concentrated. In the monopoly case, the monopolist determines all production and R&D investments to maximize the total firm value. Since we only need to consider the dynamic problem of a single agent (the monopolist), the equilibrium can be determined by solving a single Riccati equation. For details on equilibrium characteristics and the solution, see Section A.4.

2.3.3 Different Competitive Environment for Production and R&D

We also consider counterfactual scenarios where the competitive environment for production differs from that of R&D. For instance, we can analyze a scenario in which firms competitively determine production quantities, and given this static competitive equilibrium, a benevolent social planner then selects the optimal allocation of R&D. This problem can be solved as follows: starting from an initial productivity distribution z_0, the social

Figure 2: Number of Firms



Note: This figure shows the trend in the number of sample firms. The potential set of firms are U.S. publicly traded firms included in DISCERN 2, a dataset linking patents and public firms. Among these, we select firms with positive profit (revenue minus the cost of goods sold) and holding at least ten patents during the year or within two years before or after the year (a total of five years).

planner maximizes the value

$$V^{**}(z_0) \equiv \max_{\{x_t\}_{t\geq 0}} \mathbb{E}_0 \left[\int_0^\infty \exp(-\rho t) \left\{ z_t^T Q z_t - x_t^T x_t \right\} dt \right],$$

subject to

$$dz_t = \left((\Omega - \delta I) z_t + \mu x_t \right) dt + \gamma z_t \circ dW_t.$$

Note that, unlike in Section 2.3.1, the social planner's gross instantaneous return is replaced by $z_t^T Q z_t$ in place of $\frac{1}{2} z_t^T N^* z_t$. Therefore, we obtain the constrained social planner's allocation by solving the modified version of the Riccati equation (32), in which $N^*/2$ is replaced with Q. Denoting the solution of the modified Riccati equation by X^{**}, the social welfare under the constrained optimal allocation is given by $V^{**}(z_0) = z_0^T X^{**} z_0$.

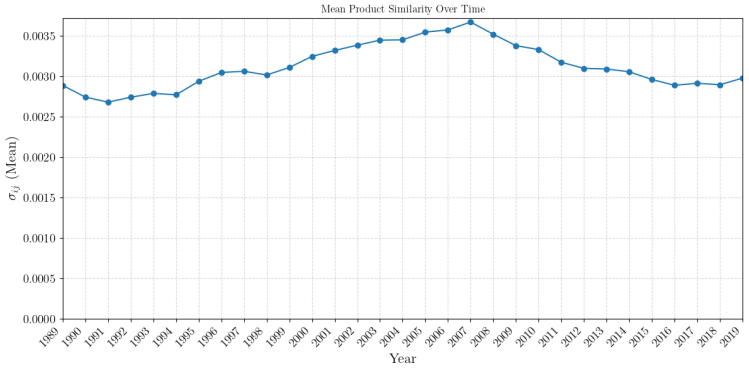
3 Data and Estimation

We map model parameters, including two networks—product proximity and technology proximity—to the observed data. In this section, we explain the data used and the method for estimating the model. The period of data used for our analysis is from 1989 to 2019. Figure 2 plots the number of sample firms by year, while Table 2 summarizes the estimated parameters and their sources.

Table 2: Parameters

Notation	Description	Value	Source
$\mathbf{\Psi}^T\mathbf{\Psi}$	Product market proximity		Hoberg and Phillips (2016)
$\mathbf{\Omega}$	Technological proximity		USPTO patent classification
α	Product proximity → Substitutability	.12	Pellegrino (2024)
β	Technology proximity → Spillover	0.024	Estimate the law of motion
γ	St.d. of idiosyncratic shocks	.00	Deterministic economy in the baseline
ζ/L	Labor augmentation efficiency	.0040	Compustat, Cost of goods sold
ρ	Discount rate	.10	
μ	R&D coefficient	0.054	1.6% economic growth rate
δ	Depreciation rate	0.015	2.6% economic growth rate

Figure 3: Mean σ_{ij}



Note: This figure plots the cross-firm mean of the product proximity σ_{ij} by year, 1989–2019. Following Pellegrino (2024), we construct σ_{ij} using the Hoberg and Phillips (2016) text-based product similarity—cosine similarity of the 10-K Business Description. The mean is the simple average over all $i \neq j$ pairs in the sample each year.

3.1 Product Market Proximity and Substitutability

Following Pellegrino (2024), we measure product market proximity, denoted as $\mathbf{\Psi}^T\mathbf{\Psi}$, using the dataset developed by Hoberg and Phillips (2016). This measure is based on the cosine similarity of the words found in the Business Description sections of 10-K filings. Hoberg and Phillips (2016) construct a vocabulary comprising 61,146 words that firms use to describe the features and characteristics of their products. For each firm i, they generate a vector of word frequencies, where each element of the vector represents the number of times a specific word appears in the firm's product description. Their analysis demonstrates that cosine similarity effectively identifies industry groupings and predicts competitive relationships between firms, outperforming other industry classification methods.

We adopt the value of α, which governs the degree of horizontal differentiation between good, from Pellegrino (2024). Pellegrino (2024) estimates α to align with the inverse cross-price elasticity of demand observed by Nevo (2001), as this parameter links the cosine similarity between products to the magnitude of cross-price elasticity. Pellegrino (2024)

shows that the estimated price elasticities of demand derived from his demand system are well aligned with those estimated in the industrial organization literature. Specifically, Pellegrino (2024) compares them to the estimated price elasticities of demand in the markets for automobiles Berry et al. (1995), ready-to-eat cereal Nevo (2001), and computers Goeree (2008). Table 2 in Pellegrino (2024) shows that the magnitude of the own-price elasticities of demand for each firm and the cross-price elasticities between firms, derived from their estimated framework, are similar to the values estimated in the literature. Figure 3 plots the average product proximity across all pairs of firms from 1989 to 2019. The figure indicates no clear trend in product proximity over this period.

3.2 Technological Proximity

For technological proximity, we use patent data and follow the methodologies of Bloom et al. (2013) and Lucking et al. (2019) to construct measures of technological proximity between firms. Specifically, we utilize the DISCERN 2 dataset (Arora et al., 2024), which links data on U.S. publicly listed firms from the Compustat database to their patents. Following Bloom et al. (2013), we calculate technological proximity between firms using the Jaffe measure based on the Cooperative Patent Classification.

Denote the vector of the share of patents of firm i in any given technology class by T_i. When we use the Jaffe measure (Jaffe, 1986), the technological proximity $\tilde{\omega}_{ij}$ is measured by

$$\tilde{\omega}_{ij} = \frac{T_i T_j'}{\left(T_i T_i'\right)^{1/2} \left(T_j T_j'\right)^{1/2}}$$

We measure T_i using both subclass-level and group-level classifications.

Similar to Liu and Ma (2024), we assume that the sum of the weights for technology spillover is set to the same size for all firms from the perspective of the recipients of technology:

$$\tilde{\omega}_{ij} = \frac{\hat{\omega}_{ij}}{\sum_{k \neq i} \hat{\omega}_{ik}}, \qquad j \neq i.$$

3.3 Knowledge Capital Distribution

Next, we identify firms' profits $\{\pi_{i,t}\}$, quantity $\boldsymbol{q}_t$, the ratio of technology transformation rate to labor supply ζ/L, and knowledge capital $\boldsymbol{z}_t$ step-by-step. The identified knowledge capital $\boldsymbol{z}_t$ is used for estimating its law of motion and as an initial distribution for numerical analysis.

Table 3: Estimates of the law of motion for knowledge capital

	(1)	(2)	(3)
Dependent variable:	$\Delta \log z_{i,t}$	$\Delta \log z_{i,t}$	$\Delta \log z_{i,t}$
$\sum_{j \neq i} \tilde{\omega}_{ij,t} \frac{z_{j,t}}{z_{i,t}}$	0.026^{**}	0.024^{**}	0.073^{*}
	(0.010)	(0.010)	(0.041)
$\log z_{i,t}$	-0.132^{***}	-0.162^{***}	-0.088^{**}
	(0.013)	(0.013)	(0.038)
$\frac{x_{i,t}}{z_{i,t}}$		0.514^{***}	
		(0.064)	
Year fixed effects	✓	✓	✓
Firm fixed effects	✓	✓	✓
IV			✓
IV first-stage F-statistic			49.198
No. of observations	14,442	14,442	14,442
R^2	0.713	0.716	–
Within R^2	0.072	0.080	–

Note: The dependent variable is $\Delta \log z_{i,t} \equiv \log z_{i,t+1} - \log z_{i,t}$. The spillover regressor is the technological-proximity-weighted knowledge capital, $\sum_{j \neq i} \tilde{\omega}_{ij,t} \, z_{j,t}/z_{i,t}$, where $\tilde{\omega}_{ij,t}$ denotes technological proximity. Controls include firm and year fixed effects, $\log z_{i,t}$, and R&D intensity $x_{i,t}/z_{i,t}$ when indicated. Column (3) reports IV estimates using the instrument $\sum_{j \neq i} \tilde{\omega}_{ij,t} \, z_{j,t}^{\text{TAX}}$, where $z_{j,t}^{\text{TAX}}$ is constructed by cumulating predicted R&D over the prior five years based on tax-induced variation in the R&D user cost. Standard errors are two-way clustered by year and 4-digit NAICS industry. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

The firm-level profit $\pi_{i,t}$ is measured as "Revenue -- Total" (revt) minus the production costs represented by "Costs of Goods Sold" (cogs) in the Compustat database. Nominal variables such as revenue and the cost of goods sold are deflated using the GDP deflator. In our model, a firm's profit is given by the square of its production quantity, $\pi_{i,t} = q_{i,t}^2$. Using this relationship, we derive the vector of production quantities. Next, we identify ζ/L using the following relationship in our model:

$$\frac{\zeta}{L} \boldsymbol{q}_t^T \boldsymbol{J} \boldsymbol{q}_t = \text{total production cost at time } t.$$

We calculate the total production cost as the sum of the production costs for all firms in our sample. Note that in our model, ζ and L cannot be separately identified. Finally, we identify the knowledge capital of firms $\boldsymbol{z}_t$ using Equation (17).

3.4 Law of Motion of Knowledge Capital

Using each firm's identified knowledge capital, we estimate the law of motion in our model to recover the technological spillover parameter β. Specifically, we estimate the following

Table 4: First stage: predicting R&D using tax credits

	R&D expenditure (1)
State tax credit component of R&D user cost	-1.163^{***} (0.294)
Federal tax credit component of R&D user cost	-34.298^{***} (3.649)
Firm fixed effects	✓
Year fixed effects	✓
No. of observations	16,197

Note: Outcome is firm-level R&D expenditure. Regressors are components of the R&D user cost based on federal and state tax credits, following Lucking et al. (2019), with firm and year fixed effects. This specification corresponds to the first stage used to construct predicted knowledge capital, $z_{j,t}^{\text{TAX}}$. Standard errors are two-way clustered by year and 4-digit NAICS industry. * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.

regression equation:

$$\log z_{i,t+1} - \log z_{i,t} = \beta \sum_{j \neq i} \tilde{\omega}_{ij,t} \frac{z_{j,t}}{z_{i,t}} + \text{Controls}_{i,t} + \epsilon_{i,t} \tag{33}$$

where $z_{i,t}$ denotes firm i's knowledge capital at time t, and $\tilde{\omega}_{ij,t}$ measures the technological proximity between firms i and j at time t. $\text{Controls}_{i,t}$ includes year and firm fixed effects, log knowledge capital $\left(\log z_{i,t}\right)$, and R&D intensity $(x_{i,t}/z_{i,t})$.

Table 3 reports estimates of Equation (33). In column (1), the coefficient on technological-proximity-weighted knowledge capital, $\sum_{j \neq i} \tilde{\omega}_{ij,t} z_{j,t}/z_{i,t}$, is positive and statistically significant, indicating that knowledge capital from technologically proximate firms contributes to a firm's own knowledge growth. Column (2) adds firms' R&D investment, $x_{i,t}$, as a control; the spillover coefficient remains positive and significant.

Column (3) addresses endogeneity using an instrumental-variables strategy that exploits tax-induced variation in the effective user cost of R&D, following Wilson (2009), Bloom et al. (2013), and Lucking et al. (2019). We first regress a firm's R&D expenditure on the user cost of R&D, constructed using tax credits in Lucking et al. (2019), to obtain predicted R&D. We then cumulate the prior five years of predicted R&D to construct predicted knowledge capital $z_{j,t}^{\text{TAX}}$ and form the instrument $\sum_{j \neq i} \tilde{\omega}_{ij,t} z_{j,t}^{\text{TAX}}$. The IV estimate of the spillover coefficient remains positive and statistically significant.

3.5 Discount Rate and R&D Coefficient

The remaining parameters that need to be estimated are the discount rate ρ and R&D coefficient μ. We set the discount rate ρ to 10%. While this value is high compared to the observed real return on safe assets (often less than 5%), it is low compared to the private return on R&D estimated in the literature. For example, Bloom et al. (2013) find the private return on R&D to be more than 20%.

We estimate the R&D coefficient μ so that the economic growth rate generated by the model matches the observed economic growth rate. Specifically, we set μ such that the economic growth rate generated by the model matches the average growth rate of GDP per capita in the United States between 1989 and 2019, which is 1.5%.

3.6 Validation

Next, we evaluate the goodness of fit of the estimated model for both target and non-target moments. Figure 4 illustrates firm-level profits, sales, and R&D expenditures, comparing the model's output with actual data. The model, as previously stated, is calibrated to replicate the firm-level profits observed in the data. As panel (a) shows, the model-generated firm-level profits almost perfectly match the data. Panel (b) compares firm-level sales, while panel (c) does the same for R&D expenditures. The correlations between the model and the data are 0.96 for the log of sales and 0.72 for the log of R&D expenditures.

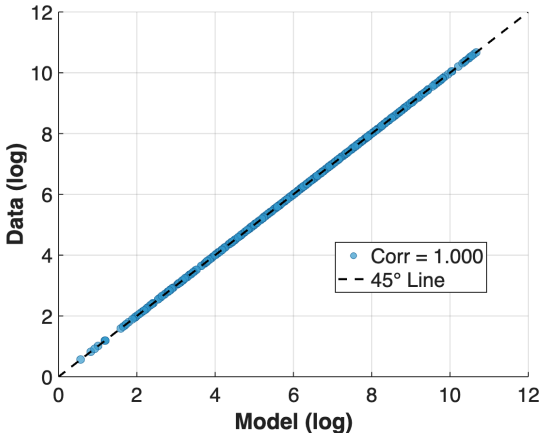
Figure 5 compares the growth rates of knowledge capital in the model and in the data. The data-based growth rate is computed from changes between 2010 and 2017. To construct the 2010 benchmark in the data, we use the 2010 network and the initial level of knowledge capital. The correlation between the model-implied and data-based growth rates of knowledge capital is 0.55.

4 Numerical Analysis

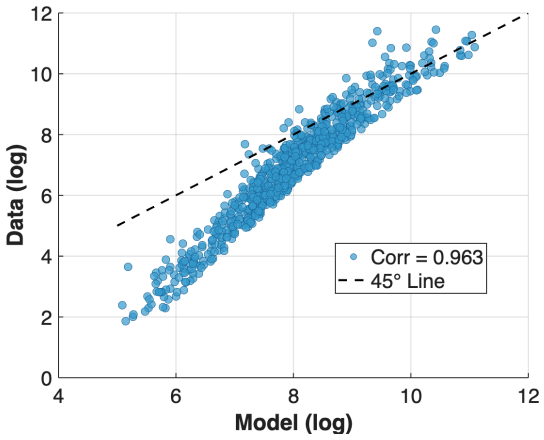
In this section, we first explain the numerical algorithm and its advantages, and then present the results of the numerical exercise.

Figure 4: Firm-Level Profits, Sales, and R&D Expenditures: Model vs. Data

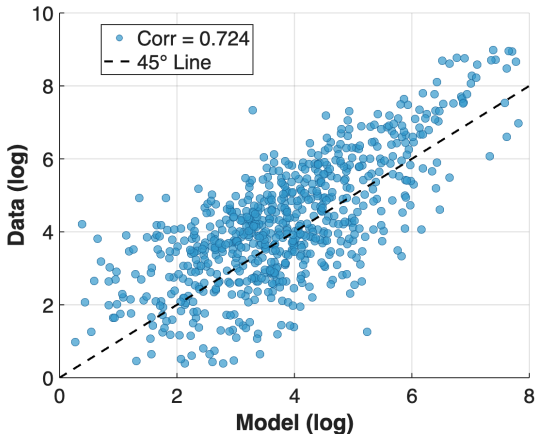
(a) log (Profits)



(b) log (Sales)

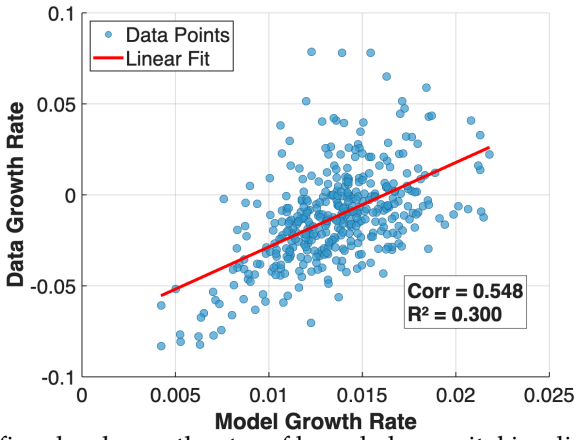


(c) log (R&D Expenditures)



Note: These panels plot firm-level profits, sales, and R&D expenditures from the model and from Compustat. Profits in the data are measured as "Revenue – Total" (revt) minus "Cost of Goods Sold" (cogs) in Compustat. The model is calibrated so that firm-level profits match those in the data.

Figure 5: Growth Rate Comparison: Model vs. Data



Note: This figure compares firm-level growth rates of knowledge capital implied by the model and computed from the data. The data-based growth rate is measured between 2010 and 2017 and is constructed using the 2010 network and the initial level of knowledge capital.

4.1 Algorithms and Computation Speed

To solve the competitive equilibrium, we solve the system of the Riccati equations backwards. Specifically, given $\left[\begin{array}{ccc} \boldsymbol{X}_\tau^1 & \cdots & \boldsymbol{X}_\tau^n \end{array}\right]$, update $\left[\begin{array}{ccc} \boldsymbol{X}_{\tau-\Delta}^1 & \cdots & \boldsymbol{X}_{\tau-\Delta}^n \end{array}\right]$ by

$$-\frac{\boldsymbol{X}_\tau^i - \boldsymbol{X}_{\tau-\Delta}^i}{\Delta} = \boldsymbol{Q}^i - \mu^2 \boldsymbol{X}_{i,\tau}^i \left(\boldsymbol{X}_{i,\tau}^i\right)^T + \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}_\tau^i + \boldsymbol{X}_\tau^i \left(\boldsymbol{\Phi}_\tau - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)$$

When the number of firms is n, each firm has an $n \times n$ matrix of undetermined coefficients $\boldsymbol{X}^i$. As shown in Figure 2, the number of sample firms ranges between 418 (in 2011) and 847 (in 2002). For $n = 700$, the total number of undetermined coefficients is $n^3 = 700^3 \simeq 343$ million. Despite this scale, an efficient vectorized algorithm allows us to solve the problem in a few minutes on a laptop.

Within the framework of existing endogenous economic growth models, solving the dynamic oligopoly problem involving all publicly traded firms with patents in the United States within a realistic timeframe is challenging. When considering a dynamic oligopoly with n firms, the state variables typically include the productivity of each firm, leading to a computational complexity of $O\left(2^n\right)$. For instance, Cavenaile et al. (2023) introduce a dynamic oligopoly into a Schumpeterian growth model, similar to ours. However, due to the curse of dimensionality, they analyze oligopolies with a maximum of four firms. In contrast, our model has a computational complexity of order $O\left(n^4\right)$, allowing us to analyze markets with more than 700 firms. Furthermore, while Cavenaile et al. (2023)'s model uses only six discrete grids for productivity, our model allows for a continuous state space.

Table 5: Computational Complexity and Scale

Model	Computational complexity	Number of firms	Productivity state space
Cavenaile et al. (2023)	$O(2^n)$	4	Six grids
Our model	$O(n^4)$	418–847	Continuous

4.2 Social and Private Marginal Returns to R&D

We compare private and social marginal returns to R&D. We measure the private marginal return to R&D as the increase in the weighted value of a firm's owners induced by its R&D investment:[2]

$$\text{Private marginal return to R\&D for firm } i \;=\; \mu \frac{d}{dz_i} V_z^i(\boldsymbol{z}) \;=\; 2\mu \left(\boldsymbol{X}_i^i\right)^\top \boldsymbol{z}$$

In vector form, this is:

$$\text{Private marginal return to R\&D} \;=\; 2\mu \widetilde{\boldsymbol{X}}\, \boldsymbol{z}.$$

Similarly, the social marginal return to R&D is the increase in social welfare from an additional unit of firm i's R&D:

$$\text{Social marginal return to R\&D} \;=\; \mu \frac{d}{d\boldsymbol{z}} V_z(\boldsymbol{z}) \;=\; 2\mu\, \boldsymbol{X}\, \boldsymbol{z}.$$

Figure 6 plots private and social marginal returns to R&D for each firm. The model-implied correlation between private and social marginal returns is high (0.991), but social returns to R&D tend to exceed private returns, and this tendency is more pronounced among firms with high R&D returns.

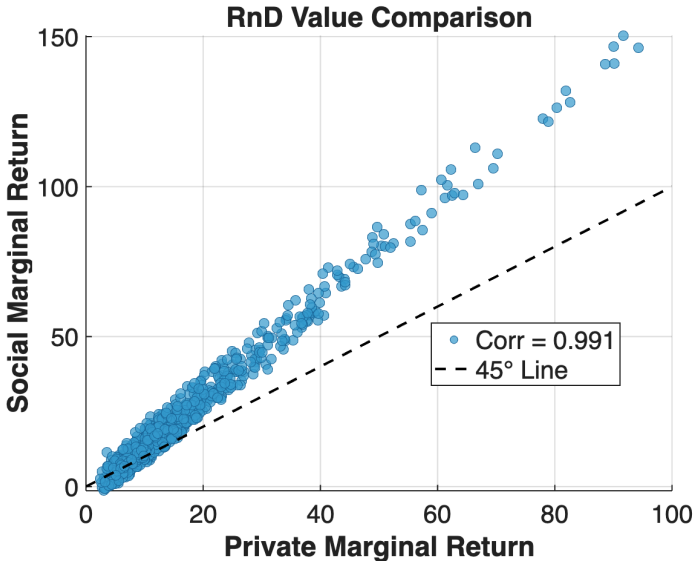
4.3 Growth Decomposition

In this section, we derive a formula that decomposes the expected economic growth rate into four terms: (i) technology spillovers, (ii) obsolescence (depreciation of knowledge capital), (iii) R&D, and (iv) a term related to shocks.

Using Equations (24) and (26), we apply Itô's lemma to obtain the stochastic process for

[2] In our model with quadratic R&D costs, the private marginal return to R&D, $2\mu \left(\boldsymbol{X}_i^i\right)^\top \boldsymbol{z}$, equals twice the investment level, $2x_i$.

Figure 6: Private and Social Marginal Returns on R&D



Note: Each point represents a firm. The horizontal axis reports the private marginal return on R&D, $2\mu\,\widetilde{\boldsymbol{X}}\boldsymbol{z}$; the vertical axis reports the social marginal return on R&D, $2\mu\,\boldsymbol{X}\boldsymbol{z}$. Returns are computed using the estimated networks and firm-level knowledge capital in 2017.

Table 6: Growth Decomposition

Description	Growth rate in 2017
Technology Spillovers	4.59%
R&D	0.54%
Obsolescence	−2.90
Total	1.51%

Note: Contributions to expected economic growth are decomposed using Equations (34) to (36). Calculations use the estimated ownership, technology spillover, and product-market rivalry networks, together with knowledge capital in 2017.

log output:

$$d\log Y_t = \left[\frac{\boldsymbol{z}_t^\top\left(\boldsymbol{Q}\boldsymbol{\Phi}+\boldsymbol{\Phi}^\top\boldsymbol{Q}\right)\boldsymbol{z}_t}{\boldsymbol{z}_t^\top\boldsymbol{Q}\boldsymbol{z}_t}+\gamma^2\left\{\frac{\sum_i z_{i,t}^2 Q_{ii}}{\boldsymbol{z}_t^\top\boldsymbol{Q}\boldsymbol{z}_t}-2\frac{\boldsymbol{z}_t^\top\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t^2\right)\boldsymbol{Q}\boldsymbol{z}_t}{\left(\boldsymbol{z}_t^\top\boldsymbol{Q}\boldsymbol{z}_t\right)^2}\right\}\right]dt + 2\gamma\frac{\boldsymbol{z}_t^\top\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t\right)}{\boldsymbol{z}_t^\top\boldsymbol{Q}\boldsymbol{z}_t}d\boldsymbol{W}_t$$

(see Section A.7 for the derivation). Because the technology transition matrix $\boldsymbol{\Phi}$ consists of the exogenous technology spillover matrix $\boldsymbol{\Omega}$, the endogenous R&D term $\mu^2\widetilde{\boldsymbol{X}}$, and

Table 7: Scenario Comparison in 2017

Scenario	Output Y	Total R&D	Growth Rate g (%)	Social Welfare V	Firm Value Share (%)
CC	100.00	100.00	1.51	100.00	26.54
CM	100.00	51.64	1.36	98.89	27.39
CS	100.00	241.95	1.80	104.36	24.09
MM	96.33	63.22	1.40	95.67	43.09
SS	109.29	348.72	1.83	114.54	-5.55

Note: CC denotes the static and dynamic competitive equilibrium, MM the static and dynamic monopoly, SS the static and dynamic social planner, CS the static competitive equilibrium with a dynamic social planner, and CM the static competitive equilibrium with a dynamic monopolist. Total R&D is normalized to the competitive equilibrium, and the firm value share equals $100 \times V^P/V$. All statistics refer to 2017, with output and social welfare normalized to the competitive equilibrium.

obsolescence $-\delta \boldsymbol{I}$, we can further decompose the first drift term:

$$\boldsymbol{Q\Phi} + \boldsymbol{\Phi}^\top \boldsymbol{Q} = (\boldsymbol{Q\Omega} + \boldsymbol{\Omega Q}) + \mu^2 \left(\boldsymbol{Q}\widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}}^\top \boldsymbol{Q} \right) - 2\delta \boldsymbol{Q}$$

Given this observation, we define the contributions of technology spillovers, obsolescence, R&D, and the Itô term to the expected economic growth rate as:

$$\frac{d \log Y_t|_{\text{spillover}}}{dt} = \frac{\boldsymbol{z}_t^\top (\boldsymbol{Q\Omega} + \boldsymbol{\Omega Q}) \boldsymbol{z}_t}{\boldsymbol{z}_t^\top \boldsymbol{Q} \boldsymbol{z}_t} \tag{34}$$

$$\frac{d \log Y_t|_{\text{R\&D}}}{dt} = \mu^2 \frac{\boldsymbol{z}_t^\top \left(\boldsymbol{Q}\widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{X}}^\top \boldsymbol{Q} \right) \boldsymbol{z}_t}{\boldsymbol{z}_t^\top \boldsymbol{Q} \boldsymbol{z}_t} \tag{35}$$

$$\frac{d \log Y_t|_{\text{Obsolescence}}}{dt} = -2\delta \tag{36}$$

$$\frac{d \log Y_t|_{\text{Itô}}}{dt} = \gamma^2 \left\{ \frac{\sum_i z_{i,t}^2 Q_{ii}}{\boldsymbol{z}_t^\top \boldsymbol{Q} \boldsymbol{z}_t} - 2 \frac{\boldsymbol{z}_t^\top \boldsymbol{Q} \text{diag}\left(\boldsymbol{z}_t^2\right) \boldsymbol{Q} \boldsymbol{z}_t}{\left(\boldsymbol{z}_t^\top \boldsymbol{Q} \boldsymbol{z}_t\right)^2} \right\} x \tag{37}$$

In Table 6, we report the results of the growth decomposition.

4.4 Counterfactual Analysis Holding Product Market Competition Fixed

Table 7 compiles five counterfactual allocations that retain the estimated 2017 product-market rivalry, technology-spillover networks, and knowledge-capital distribution. Output, total R&D expenditure, and social welfare are normalized so that the competitive equilibrium (CC) equals 100, and the firm value share reports total firm value divided by aggregate welfare.

We first examine the allocations chosen by the constrained monopolist (CM) and the social planner (CS), holding fixed the product-market allocation from the competitive equilibrium. With the static product-market allocation kept at its competitive level and the same initial distribution of knowledge capital, production quantities are unchanged. Consequently, contemporaneous output remains $Y = 100$ across these cases (CC, CM, CS).

While both constrained monopolists and constrained social planners internalize business stealing externalities and technology spillover externalities, but their R&D allocation differs because monopolists aim to maximize producer surplus while social planners aim to maximize social welfare. The monopolist (CM) cuts investment to 51.64—roughly half of the competitive equilibrium value—because internalizing the business-stealing externality dominates the gain from internalizing technology spillovers. Firms in the competitive equilibrium allocation focus on their own producer surplus, so the economy underprovides innovation—the appropriability effect. The social planner (CS) therefore raises total R&D to 241.95, more than doubling the competitive equilibrium allocation by valuing the payoff from innovation for both producers and consumers. With z_t and the static matrix $\boldsymbol{Q}$ unchanged in Equations (34) and (37), the growth column simply tracks these R&D choices: growth falls to 1.36% under the monopolist, 0.15 percentage points below the competitive equilibrium, and rises to 1.80% under the social planner, 0.29 points above the competitive equilibrium.

Because R&D is already undersupplied in the competitive equilibrium allocation, the constrained monopolist's lower R&D spending pushes welfare down, leading to a 1.11% consumption-equivalent welfare loss relative to the competitive equilibrium outcome. The planner, by contrast, values the appropriability effect and internalizes both the business-stealing and technology-spillover externalities, thereby lifting welfare 4.36% above competition. Prioritizing producer surplus leaves the monopolist with a firm value share of 27.39%—0.85 percentage points above competition—whereas the planner steers surplus toward consumers and reduces the share to 24.09%, 3.30 points below the competitive equilibrium.

4.5 Counterfactual Analysis with Static and Dynamic Reallocation

We now allow both the monopolist and the social planner to rearrange production labor at the same time that they choose dynamic R&D policies. The monopoly (MM) and planner (SS) scenarios that combine these choices are reported in the final two rows of Table 7.

Giving monopoly control over labor reduces output by 3.7% relative to the competitive equilibrium allocation. The monopolist concentrates workers in the markets with high

markups, which deepens the misallocation that already existed under competition. By contrast, the planner keeps shifting workers until marginal products line up across firms, so output rises by 9.3%.

Changes in the product market feed directly into R&D. When the monopolist chooses both production and innovation (MM), investment climbs from 51.64 in CM to 63.22 because higher markups raise the private return to new ideas, yet the level still sits roughly 37% below the competitive equilibrium allocation. Once the planner coordinates production and research, eliminating the static distortion expands output and lifts the social value of innovation, so total R&D spending jumps to 348.72—about three and a half times the competitive equilibrium allocation.[3]

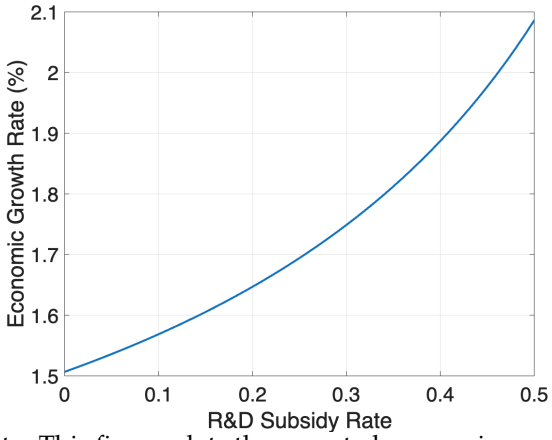
These R&D choices show up one-for-one in growth. When the monopolist selects both production and innovation, the economy grows at 1.40%—0.11 percentage points below the competitive equilibrium allocation, but 0.40 points above the CM outcome with fixed production. Allowing the planner to pick both margins raises the growth rate to 1.83%, 0.32 points above competition and 0.04 points higher than the CS scenario. Stronger planner-led R&D accelerates knowledge accumulation, whereas the monopolist's restraint keeps growth just shy of the competitive equilibrium allocation.

Letting both margins adjust magnifies the welfare stakes. When the monopolist chooses production and R&D jointly, welfare falls 4.3% below the competitive equilibrium allocation—a much steeper loss than the 1.1% decline observed when only R&D is distorted. The extra damage comes from worsening product-market misallocation, which erodes static efficiency on top of the innovation shortfall. In contrast, the planner raises welfare by 14.5% relative to competition when allowed to reallocate labor and R&D, far exceeding the 4.4% gain secured by adjusting R&D alone. Eliminating static misallocation delivers an immediate boost to output and simultaneously amplifies the social return to each unit of research.

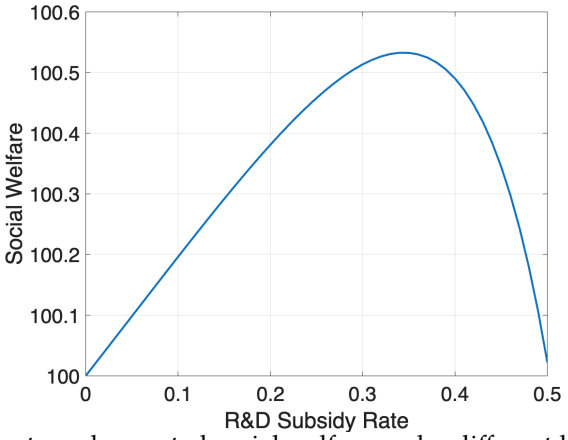
Firm value shares swing even more when both margins move. A monopolist that sets production and R&D captures 43.09% of aggregate welfare, up sharply from the 26.54% competitive share. In contrast, the planner drives the share to -5.55 because firms relinquish all static profits yet still finance positive R&D outlays, pushing nearly all surplus to consumers.

Figure 7: Economic Growth Rate and Social Welfare under R&D Subsidy

(a) Expected Growth Rate



(b) Expected Social Welfare



Note: This figure plots the expected economic growth rate and expected social welfare under different levels of R&D subsidy. The networks of technology spillovers and product market rivalry, as well as the distribution of knowledge capital, are estimated using data from 2017.

4.6 Uniform R&D Subsidy

Finally, we study a uniform subsidy to R&D. In our setting, a uniform subsidy leaves the static allocation unchanged, yet welfare can still improve because equilibrium R&D falls short of the constrained planner's allocation. Welfare reaches its maximum at a subsidy of 34% (Figure 7). At that rate, the expected growth rate increases to 1.80%—0.29 percentage points above competition (Figure 7a)—and consumption-equivalent welfare improves by 0.53% (Figure 7b). The gain is nonetheless far smaller than the 4.36% increase delivered by the constrained social planner under competitive product markets (CS). Achieving large welfare gains therefore calls for policies that leverage heterogeneity in product-market rivalry and technology spillovers rather than relying on uniform subsidies alone.

5 Conclusion

We develop an endogenous growth model that embeds firms' product-market rivalry and technology-spillover networks to study how product-market competition and innovation policies affect economic growth and social welfare. We extend the generalized hedonic linear demand system of Pellegrino (2024) to a dynamic setting by introducing R&D investment and technology spillover networks. Despite forward-looking R&D decisions by

[3]This size accords with Bloom et al. (2013), who estimate that the socially optimal level of R&D is roughly three times the observed level.

many firms that interact strategically over these networks, the model remains tractable. We estimate the model on U.S. publicly traded firms with patents, constructing product similarity from product descriptions and technological similarity from patent classifications. A benevolent social planner who optimally allocates R&D more than doubles the investment, raising growth by +0.29 percentage points and welfare by 4.4%. By contrast, uniform R&D subsidies increase welfare by 0.53% at most, underscoring the need for policies that account for network heterogeneity.

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Appendix

Appendix A Theoretical Appendix

A.1 Derivation of Static Equilibrium

Inserting the optimality condition for technology choice Equation (14) and Equation (15) into the optimality condition for quantity choice Equation (13), we obtain

$$z_{i,t} = 2\sqrt{\zeta \frac{w_t}{P_t}} + q_{i,t} + \sum_j \sigma_{ij} q_{j,t}$$

Inserting the expression for real wage Equation (16), we obtain

$$z_{i,t} = q_{i,t} + \sum_j \left(2\frac{\zeta}{L} + \sigma_{ij}\right) q_{j,t}.$$

In vector form,

$$\boldsymbol{z}_t = \left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma} + \boldsymbol{I}\right) \boldsymbol{q}_t.$$

Therefore, quantities are linear in knowledge diffusion:

$$\boldsymbol{q}_t = \boldsymbol{N}\boldsymbol{z}_t$$

where

$$\boldsymbol{N} \equiv \left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma} + \boldsymbol{I}\right)^{-1}.$$

A.2 Derivation of Output

Let $\mathbf{1}$ denote an $n \times 1$ vector with all elements equal to 1. Rewrite Equation (15) in vector form:

$$\boldsymbol{b}_t = \boldsymbol{z}_t - \sqrt{\zeta \frac{w_t}{P_t}}\mathbf{1}$$

Inserting Equation (16), we obtain

$$\begin{aligned}\boldsymbol{b}_t &= \boldsymbol{z}_t - \frac{\zeta}{L}\left(\sum_i q_{i,t}\right)\mathbf{1} \\ &= \boldsymbol{z}_t - \frac{\zeta}{L}\boldsymbol{J}\boldsymbol{q}_t.\end{aligned}$$

From Equation (17), we obtain

$$\boldsymbol{b}_t = \left(\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma} + \boldsymbol{I}\right)\boldsymbol{q}_t$$

Therefore, from Equation (4), the total output is given by

$$\begin{aligned}Y_t &= \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma} + \boldsymbol{I}\right)\boldsymbol{q}_t - \frac{1}{2}\boldsymbol{q}_t^T\boldsymbol{\Sigma}\boldsymbol{q}_t \\ &= \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\boldsymbol{J} + \frac{1}{2}\boldsymbol{\Sigma} + \boldsymbol{I}\right)\boldsymbol{q}_t \\ &= \boldsymbol{z}_t^T\boldsymbol{N}^T\left(\frac{\zeta}{L}\boldsymbol{J} + \frac{1}{2}\boldsymbol{\Sigma} + \boldsymbol{I}\right)\boldsymbol{N}\boldsymbol{z}_t \\ &= \boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{z}_t\end{aligned}$$

where

$$\boldsymbol{Q} \equiv \boldsymbol{N}^T\left(\frac{\zeta}{L}\boldsymbol{J} + \frac{1}{2}\boldsymbol{\Sigma} + \boldsymbol{I}\right)\boldsymbol{N}$$

A.3 Derivation of the Solution of Social Planner's Static Problem

Define the Lagrangian as

$$\mathcal{L} = \boldsymbol{q}_t^T\left(\boldsymbol{z}_t - \zeta\boldsymbol{a}_t\right) - \frac{1}{2}\boldsymbol{q}_t^T\boldsymbol{\Sigma}\boldsymbol{q}_t - \lambda_t\left\{\sum_i \frac{q_{i,t}}{a_{i,t}} - L\right\}$$

The FOC w.r.t. $a_{i,t}$ gives

$$a_{i,t} = \sqrt{\frac{\lambda_t}{\zeta}} \tag{38}$$

and therefore,

$$b_{i,t} = z_{i,t} - \sqrt{\zeta\lambda_t} \tag{39}$$

Inserting Equation (38) into Equation (28), we obtain

$$\sqrt{\zeta\lambda_t} = \frac{\zeta}{L}\sum_i q_{i,t} \tag{40}$$

Note that

$$\sqrt{\zeta\lambda_t}\mathbf{1} = \frac{\zeta}{L}\left(\sum_i q_{i,t}\right)\mathbf{1} = \frac{\zeta}{L}\boldsymbol{J}\boldsymbol{q}_t. \tag{41}$$

The FOC w.r.t. $\boldsymbol{q}_t$ gives

$$\boldsymbol{z}_t - \zeta\boldsymbol{a}_t - \boldsymbol{\Sigma}\boldsymbol{q}_t = \lambda_t \begin{bmatrix} 1/a_{1,t} \\ 1/a_{2,t} \\ \vdots \\ 1/a_{n,t} \end{bmatrix} \tag{42}$$

Substituting Equation (38), we obtain

$$\boldsymbol{z}_t = 2\sqrt{\zeta\lambda_t}\mathbf{1} + \boldsymbol{\Sigma}\boldsymbol{q}_t$$

Using Equation (41), we obtain

$$\boldsymbol{z}_t = \left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma}\right)\boldsymbol{q}_t$$

Therefore,

$$\boldsymbol{q}_t = \boldsymbol{N}^*\boldsymbol{z}_t \tag{43}$$

where

$$\boldsymbol{N}^* = \left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma}\right)^{-1}$$

To obtain the quadratic form of output, substitute Equation (15) into Equation (4):

$$Y_t = \boldsymbol{q}_t^T\left(\boldsymbol{z}_t - \sqrt{\zeta\lambda_t}\mathbf{1}\right) - \frac{1}{2}\boldsymbol{q}_t^T\boldsymbol{\Sigma}\boldsymbol{q}_t$$

Using Equation (41) together with Equation (43),

$$\begin{aligned} Y_t &= \boldsymbol{q}_t^T\left(\left(2\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma}\right)\boldsymbol{q}_t - \frac{\zeta}{L}\boldsymbol{J}\boldsymbol{q}_t\right) - \frac{1}{2}\boldsymbol{q}_t^T\boldsymbol{\Sigma}\boldsymbol{q}_t \\ &= \boldsymbol{q}_t^T\left(\frac{\zeta}{L}\boldsymbol{J} + \frac{1}{2}\boldsymbol{\Sigma}\right)\boldsymbol{q}_t \end{aligned}$$

$$= \frac{1}{2}\boldsymbol{q}_t^T(\boldsymbol{N}^*)^{-1}\boldsymbol{q}_t$$
$$= \frac{1}{2}\boldsymbol{z}_t^T\boldsymbol{N}^*\boldsymbol{z}_t$$

A.4 Monopoly

As a second counterfactual scenario, we consider the monopoly case, analyzing the allocation of R&D, its impact on economic growth, and social welfare in an environment where competition in both the product market and R&D is extremely concentrated. In the monopoly case, the monopolist determines all production and R&D investments to maximize the total firm value. Because the dynamic problem involves only a single agent (the monopolist), the equilibrium can be determined by solving a single Riccati equation.

Static Problem

We first characterize the monopolist's static allocation. As in the baseline static Cournot-Nash equilibrium, the monopolist takes the real wage w_t/P_t as given in the static problem. The monopolist chooses $\boldsymbol{a}_t$, $\boldsymbol{b}_t$, and $\boldsymbol{q}_t$ to maximize the producer surplus Π_t, which is defined as

$$\begin{aligned}\Pi_t &\equiv \frac{\boldsymbol{q}_t^T\boldsymbol{p}_t}{P_t} - \frac{w_t}{P_t}\sum_i l_{i,t} \\ &= \boldsymbol{q}_t^T\boldsymbol{b}_t - \boldsymbol{q}_t^T\boldsymbol{\Sigma}\boldsymbol{q}_t - \frac{w_t}{P_t}\sum_i \frac{q_{i,t}}{a_{i,t}}\end{aligned}$$

subject to the technology choice constraint $z_{i,t} = \zeta a_{i,t} + b_{i,t}$ for each firm. Solving the monopolist's static problem yields quantities as a linear function of knowledge capital:

$$\boldsymbol{q}_t^M = \boldsymbol{N}^M\boldsymbol{z}_t$$

where

$$\boldsymbol{N}^M \equiv \frac{1}{2}\left(\frac{\zeta}{L}\boldsymbol{J} + \boldsymbol{\Sigma}\right)^{-1}.$$

and the resulting profit is a quadratic function of knowledge capital:

$$\Pi_t^M = \boldsymbol{z}_t^T\boldsymbol{P}^M\boldsymbol{z}_t$$

where

$$\boldsymbol{P}^M = \boldsymbol{N}^M \boldsymbol{\Sigma} \boldsymbol{N}^M$$

(See Section A.5).

Dynamic Problem

Next, we characterize the monopolist's dynamic problem. The monopolist's dynamic problem is formulated as follows:

$$V^*(\boldsymbol{z}_0) \equiv \max_{\{\boldsymbol{x}_t\}_{t\geq 0}} \mathbb{E}_0 \left[\int_0^\infty \exp(-\rho t) \left\{ \boldsymbol{z}_t^T \boldsymbol{P}^M \boldsymbol{z}_t - \boldsymbol{x}_t^T \boldsymbol{x}_t \right\} dt \right],$$

subject to

$$d\boldsymbol{z}_t = \left((\boldsymbol{\Omega} - \delta \boldsymbol{I}) \boldsymbol{z}_t + \mu \boldsymbol{x}_t \right) dt + \gamma \boldsymbol{z}_t d\boldsymbol{W}_t$$

We obtain the monopoly allocation by solving a modified version of the Riccati equation Equation (32), in which $\boldsymbol{Q}^*$ is replaced with $\boldsymbol{P}^M$. Let $\tilde{\boldsymbol{X}}^M$ denote that solution; then the R&D allocation is given by

$$\boldsymbol{x}_t = \mu \tilde{\boldsymbol{X}}^M \boldsymbol{z}_t$$

Welfare

In contrast to the social planner's problem, the monopolist maximizes producer surplus, which is distinct from household utility. Thus, computing household utility requires an additional step. The total output is given by

$$Y_t^M = \boldsymbol{z}_t^T \boldsymbol{Q}^M \boldsymbol{z}_t$$

where

$$\boldsymbol{Q}^M \equiv \frac{1}{2} \boldsymbol{N}^M \left(2\frac{\zeta}{L} \boldsymbol{J} + 3\boldsymbol{\Sigma} \right) \boldsymbol{N}^M$$

(See Section A.6). Let V^M denote the value of the representative household under the monopoly allocation, defined as

$$V^M(\boldsymbol{z}_0) = \mathbb{E}_0 \left[\int_0^\infty \exp(-\rho t) \left(\boldsymbol{z}_t^T \boldsymbol{Q}^M \boldsymbol{z}_t - \boldsymbol{x}_t^T \boldsymbol{x}_t \right) dt \middle| \boldsymbol{z}_0 \right]$$

subject to the law of motion:

$$d\boldsymbol{z}_t = \boldsymbol{\Phi}^M \boldsymbol{z}_t dt + \gamma \boldsymbol{z}_t d\boldsymbol{W}_t$$

where the technology transition under monopoly allocation $\boldsymbol{\Phi}^M$ is given by

$$\boldsymbol{\Phi}^M \equiv \boldsymbol{\Omega} - \delta \boldsymbol{I} + \mu^2 \tilde{\boldsymbol{X}}^M.$$

Guess $V^M(\boldsymbol{z}) = \boldsymbol{z}^T \boldsymbol{X}^M \boldsymbol{z}$. Following the same steps as the derivation of welfare in competitive equilibrium, we obtain a single Riccati equation:

$$0 = \boldsymbol{Q}^M - \mu^2 \left(\tilde{\boldsymbol{X}}^M\right)^T \tilde{\boldsymbol{X}}^M + \boldsymbol{X}^M \left(\boldsymbol{\Phi}^M - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right) + \left(\boldsymbol{\Phi}^M - \frac{1}{2}\left(\rho - \gamma^2\right)\boldsymbol{I}\right)^T \boldsymbol{X}^M \tag{44}$$

Therefore, $\boldsymbol{X}^M$ is obtained as the solution of the Lyapunov equation Equation (44), and welfare in the monopoly problem is given by $V^M(\boldsymbol{z}_0) = \boldsymbol{z}_0^T \boldsymbol{X}^M \boldsymbol{z}_0$.

A.5 Derivation of the Static Solution of Monopolist's Static Problem

The FOC w.r.t. $q_{i,t}$ gives

$$0 = b_{i,t} - \frac{1}{a_{i,t}} \frac{w_t}{P_t} - 2 \sum_j \sigma_{ij} q_{j,t} \tag{45}$$

Therefore, the real profit of each firm is given by:

$$\begin{aligned}
\pi_{i,t} &= \left(b_{i,t} - \frac{1}{a_{i,t}} \frac{w_t}{P_t}\right) q_{i,t} - \sum_j \sigma_{ij} q_{i,t} q_{j,t} \\
&= 2 \sum_j \sigma_{ij} q_{j,t} q_{i,t} - \sum_j \sigma_{ij} q_{i,t} q_{j,t} \\
&= \sum_j \sigma_{ij} q_{i,t} q_{j,t}
\end{aligned} \tag{46}$$

The FOC w.r.t. $a_{i,t}$ gives

$$a_{i,t} = \sqrt{\frac{1}{\zeta} \frac{w_t}{P_t}}, \tag{47}$$

and therefore,

$$b_{i,t} = z_{i,t} - \sqrt{\zeta \frac{w_t}{P_t}} \tag{48}$$

Inserting Equation (47) into the labor market clearing condition Equation (9), we obtain

$$\sqrt{\zeta \frac{w_t}{P_t}} = \frac{\zeta}{L} \sum_i q_{i,t} \tag{49}$$

Inserting the optimality condition for technology choice Equation (47) and Equation (48) into the optimality condition for quantity choice Equation (45), we obtain

$$z_{i,t} = 2\sqrt{\frac{\zeta w_t}{P_t}} + 2 \sum_j \sigma_{ij} q_{j,t}.$$

Inserting the expression for real wage Equation (49), we obtain

$$z_{i,t} = \sum_j \left(2\frac{\zeta}{L} + 2\sigma_{ij} \right) q_{j,t}.$$

In vector form,

$$\boldsymbol{z}_t = 2 \left(\frac{\zeta}{L} \boldsymbol{J} + \boldsymbol{\Sigma} \right) \boldsymbol{q}_t. \tag{50}$$

Therefore, the quantities are expressed as linear functions of knowledge diffusion:

$$\boldsymbol{q}_t^M = \boldsymbol{N}^M \boldsymbol{z}_t \tag{51}$$

where

$$\boldsymbol{N}^M \equiv \frac{1}{2} \left(\frac{\zeta}{L} \boldsymbol{J} + \boldsymbol{\Sigma} \right)^{-1}.$$

From Equation (46), the total profit is given by

$$\Pi_t = \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t$$

Inserting Equation (51) , we obtain

$$\Pi_t = z_t^T \boldsymbol{N}^M \boldsymbol{\Sigma} \boldsymbol{N}^M z_t$$

A.6 Derivation of Output in Monopolist's Problem

Similar to competitive equilibrium, we have

$$\boldsymbol{b}_t = z_t - \frac{\zeta}{L} \boldsymbol{J} \boldsymbol{q}_t.$$

From Equation (50), we obtain

$$\boldsymbol{b}_t = \left(\frac{\zeta}{L} \boldsymbol{J} + 2\boldsymbol{\Sigma} \right) \boldsymbol{q}_t$$

Therefore, from Equation (4), the total output is given by

$$\begin{aligned} Y_t &= \boldsymbol{q}_t^T \left(\frac{\zeta}{L} \boldsymbol{J} + 2\boldsymbol{\Sigma} \right) \boldsymbol{q}_t - \frac{1}{2} \boldsymbol{q}_t^T \boldsymbol{\Sigma} \boldsymbol{q}_t \\ &= \frac{1}{2} \boldsymbol{q}_t^T \left(2\frac{\zeta}{L} \boldsymbol{J} + 3\boldsymbol{\Sigma} \right) \boldsymbol{q}_t \\ &= z_t^T \boldsymbol{Q}^M z_t \end{aligned}$$

where

$$\boldsymbol{Q}^M = \frac{1}{2} \boldsymbol{N}^M \left(2\frac{\zeta}{L} \boldsymbol{J} + 3\boldsymbol{\Sigma} \right) \boldsymbol{N}^M$$

A.7 Derivation of Stochastic Process of Output

Let $dz_t = \boldsymbol{\mu}_t dt + \boldsymbol{\Gamma}_t d\boldsymbol{W}_t$. Ito's lemma states that

$$df\left(z_t\right) = \left\{ \left(f_z\left(z_t\right) \right)^T \boldsymbol{\mu}_t + \frac{1}{2} \mathrm{Tr}\left[\boldsymbol{\Gamma}_t^T f_{zz}\left(z_t\right) \boldsymbol{\Gamma}_t \right] \right\} dt + \left(f_z\left(z_t\right) \right)^T \boldsymbol{\Gamma}_t d\boldsymbol{W}_t$$

We apply this lemma to $dz_t = \boldsymbol{\Phi} z_t dt + \gamma \mathrm{diag}\left(z_t\right) d\boldsymbol{W}_t$ and $f\left(z_t\right) = z_t^T \boldsymbol{Q} z_t = Y_t$. Note that

$$\begin{aligned} \boldsymbol{\mu}_t &= \boldsymbol{\Phi} z_t \\ \boldsymbol{\Gamma}_t &= \gamma \mathrm{diag}\left(z_t\right) \\ f_z\left(z_t\right) &= 2\boldsymbol{Q} z_t \end{aligned}$$

$$\mathbf{\Gamma}_t^T f_{zz}\left(z_t\right)\mathbf{\Gamma}_t = 2\boldsymbol{Q}$$

Therefore,

$$\begin{aligned} dY_t &= \left\{2\boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{\Phi}\boldsymbol{z}_t + \frac{1}{2}\mathrm{Tr}\left[\gamma^2\mathrm{diag}\left(\boldsymbol{z}_t\right)\left(2\boldsymbol{Q}\right)\mathrm{diag}\left(\boldsymbol{z}_t\right)\right]\right\}dt + 2\gamma\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t\right)d\boldsymbol{W}_t \\ &= \left\{\boldsymbol{z}_t^T\left(\boldsymbol{Q}\boldsymbol{\Phi} + \boldsymbol{\Phi}^T\boldsymbol{Q}\right)\boldsymbol{z}_t + \gamma^2\sum_i z_i^2\boldsymbol{Q}_{ii}\right\}dt + 2\gamma\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t\right)d\boldsymbol{W}_t \end{aligned}$$

Next, applying Ito's lemma to $\log Y_t = g\left(Y_t\right)$,

$$\begin{aligned} d\log Y_t &= \left\{\frac{1}{Y_t}\left(\boldsymbol{z}_t^T\left(\boldsymbol{Q}\boldsymbol{\Phi} + \boldsymbol{\Phi}^T\boldsymbol{Q}\right)\boldsymbol{z}_t + \gamma^2\sum_i z_{i,t}^2\boldsymbol{Q}_{ii}\right) + \frac{1}{2}\mathrm{Tr}\left[\left(-\frac{1}{Y_t^2}\right)4\gamma^2\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t^2\right)\boldsymbol{Q}\boldsymbol{z}_t\right]\right\}dt \\ &\quad + \frac{1}{Y_t}2\gamma\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t\right)d\boldsymbol{W}_t \\ &= \left[\frac{\boldsymbol{z}_t^T\left(\boldsymbol{Q}\boldsymbol{\Phi} + \boldsymbol{\Phi}^T\boldsymbol{Q}\right)\boldsymbol{z}_t}{\boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{z}_t} + \gamma^2\left\{\frac{\sum_i z_{i,t}^2\boldsymbol{Q}_{ii}}{\boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{z}_t} - 2\frac{\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t^2\right)\boldsymbol{Q}\boldsymbol{z}_t}{\left(\boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{z}_t\right)^2}\right\}\right]dt + 2\gamma\frac{\boldsymbol{z}_t^T\boldsymbol{Q}\mathrm{diag}\left(\boldsymbol{z}_t\right)}{\boldsymbol{z}_t^T\boldsymbol{Q}\boldsymbol{z}_t}d\boldsymbol{W}_t \end{aligned}$$