Demand for Sovereign Safe Assets*

Ehsan Ebrahimy
IMF
eebrahimy@imf.org

February 27, 2026

Abstract

What do low and declining rates imply about fiscal sustainability? I show that a risky government bond behaves like a safe asset when government default significantly reduces the returns to other assets. A systemic government debt can compress the level and the changes in government bond yields, and even have an upward-sloping demand curve for intermediate levels of borrowing up to an inflection point. This happens because of a demand externality which creates complementarity in investors' demand. This demand externality can offset the traditional undersupply due to government market power and lead to an oversupply of government debt and an excessive level of aggregate risk. Whenever government debt is the safest asset, an increase in default risk can lead to a negative correlation and a widening gap between the bond yield and the CDS premium. The possibility of upward-sloping demand curve is robust to having more than one government debt. Moreover, the complementarity in demand can create complementarity in supply leading to a global fiscal cycle. This implies that 'total' elasticity, i.e., inverse of macro multiplier, can be significantly *larger* than the elasticity of demand for government debt. The existence of an upward-sloping demand curve depends on the reversed hazard rate of the distribution of future tax revenues. These results suggest more caution in interpreting low real rates because the price of a safe asset and its default risk can comove positively before government borrowing rates start rising again as debt levels surpass the inflection point.

*I thank Leonardo Martinez, Ricardo Reis, Divya Kirti, Tamon Asonuma, Damien Capelle, and Wenhao Li for helpful comments and discussions. I especially thank Ruchir Agarwal for very helpful comments. I also thank Drilona Emrullahu for excellent research assistance. This paper has also benefited from participants in the IMF Research Department seminars, and the IMF Analytical Corners. All errors are my own. The views expressed herein are those of the author and should not be attributed to the IMF, its Executive Board, or its management.

1 Introduction

Despite a significant surge in government borrowing after the Financial Crisis of 2008 and during the COVID-19 pandemic, real interest rates on government bonds have been declining in many economies (Figure 1). It has been argued that low real interest rates might imply that more government borrowing have low fiscal costs (Blanchard (2019)). Others have cautioned that even though the current rates are low, more borrowing can make government subject to bad self-fulfilling equilibria (Farhi and Maggiori (2018), Cochrane (2021)). What do these low and declining rates imply about the riskiness of government debt? Should we expect these low rates to persist indefinitely?

In this paper, I study the demand for 'systemic' government debt. I show that a risky government bond behaves like a safe asset, in the sense that it is subject to flight to safety, if government default significantly reduces the return to other existing assets ,i.e., entails systemic risk. When government bond is a safe asset, a negative shift in or an increase in riskiness of the distribution of future tax revenues *increase* the bond price and the default risk at the same time. Such changes in future economic prospects also decrease the price of other assets, and increase the risk premium. I show that when the demand for government bond is downward sloping, more systemic government debt can increase the slope of and flatten the demand curve, i.e., yield compression. Moreover, when government debt is systemic, the government might face an upward-sloping demand curve for intermediate levels of borrowing: more issuance *increases* both the bond price and the default risk. I also show the upward-sloping segment of the demand for government debt always ends in an inflection point: for high enough level of debt, further supply decreases the price. My findings implies that when government bond is systemic, changes in bond price can be a misleading indicator of the default risk. Moreover, the existence of inflection points suggests that the real rates will start rising for high enough levels of government debt.

All else equal, higher systemic risk of government debt makes an upward-sloping demand curve more likely. Factors that increase systemic risk include higher fraction of government debt to other investors' assets, higher riskiness of the payoffs to other assets, and higher investors' indebtedness.

A brief description of the model is as follows. I consider an economy with two dates and two assets: government bond and a real asset or "project" that is in fixed supply. There are three agents, a government, a continuum of investors and a representative household. The government has to raise resources by issuing bonds to pay for an exogenous amount of spending. Investors are the only agents that have resources to purchase government bonds at the initial date. One can think of investors in my model as banks or shadow

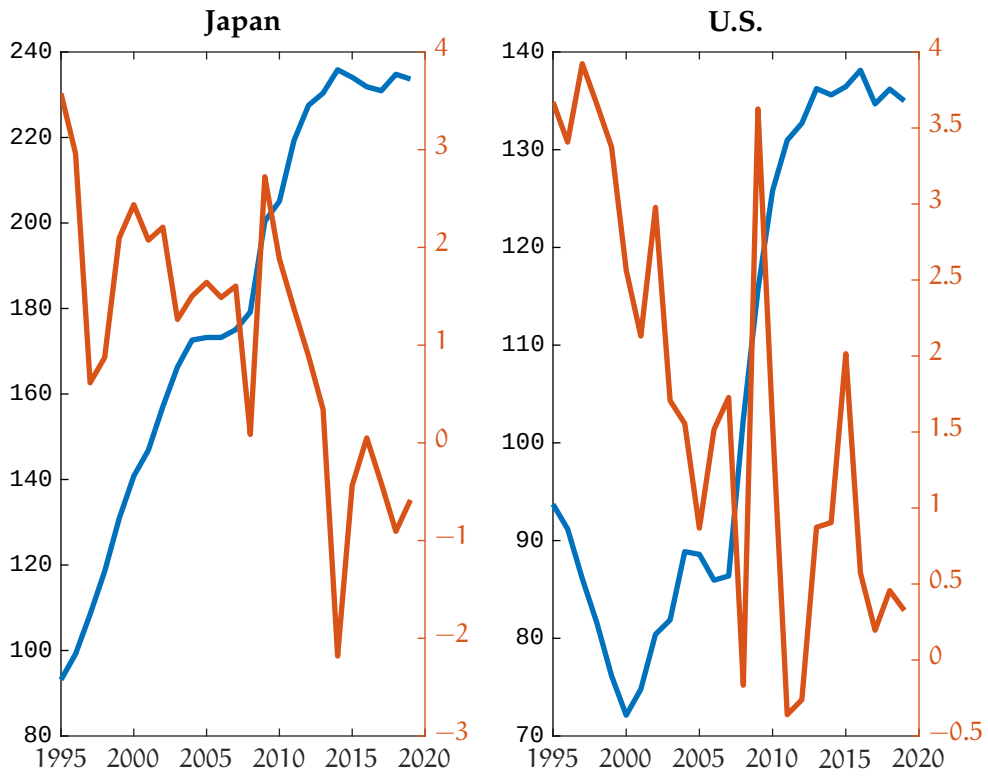


Figure 1: Government debt to GDP in blue (left axis), and long-term (10 years) real rates in red (right axis). Both are in percents. Data is from OECD and the author's calculations.

banks. Household's future income is stochastic and can be taxed, up to a limit, by the government to repay its debts to investors. The government pays its debts in full as long as there are enough tax revenues, and otherwise defaults. In the event of a default, investors are paid an exogenous fraction $0 \leqslant z < 1$ of the promised debt payments. Besides government bonds, investors hold a real asset or "project", at the beginning of the initial period. The investor needs to pay a fixed and exogenous maintenance cost for the project in the second period, in order to guarantee a high return on the project. This maintenance cost can capture any liquidity needs in a stylized way. Examples include rolling over maturing debt or financing working capital. If the maintenance cost is not paid, the project would have a low return with an exogenous probability $0 < p \leqslant 1$, which is independent across the continuum of projects. The only resources the investor can use in the second period to cover the project's maintenance cost are government bonds' payoff. Hence, when government defaults, investors holding the bonds may not be able to pay for the project maintenance costs. This implies that government default affects the return on projects. The probability p is a measure of correlation between the risks to the government bonds and the project. A higher p implies a higher likelihood of low returns

to both assets happening at the same time, and consequently more aggregate risk.

The intuition behind the yield compression and upward-sloping demand curve is as follows. Higher supply of government bonds affects the price through two channels. First, even absent default, a higher level of borrowing by the government increases the consumption of investors and reduces the expected marginal utility of an extra unit of investment in the bond. This unambiguously reduces the price of bond. I call this the *diminishing expected marginal utilities* effect. At the same time, a higher level of borrowing increases the probability of default. When project's expected payoff depends on the incident of default, this increases the demand for insurance by investors and tends to increase the price of bond. The reason is that default reduces the return to the project significantly so that government bond still remains the safest asset. I call this the *change in aggregate risk* effect. The *change in aggregate risk* effect can dampen the movements in the government bond yield. This yield compression is more significant, when government debt becomes more systemic. If the *change in aggregate risk* effect is large enough, the sum of *diminishing expected marginal utilities* and *change in aggregate risk* effects becomes positive, implying an upward-sloping demand curve.

I demonstrate that, in general, the net impact of *diminishing expected marginal utilities* and *change in aggregate risk* effects depends on the reversed hazard rate of the future tax revenues distribution. I show that when the density function of future tax revenues is convex on its left tail, more systemic government debt increases the slope of demand curve and can lead to yield compression. Moreover, when the return to other assets in the no-default state is high enough, investors face an upward-sloping demand curve for some values of bond holding. Convexity of the density function on the left tail provides a sufficient, and not necessary, condition for these results. Nonetheless, this condition holds for many distributions including normal, log normal, and Weibull distributions.

There are two necessary conditions on the parameters of the model for the government bonds to have an (partially) upward-sloping demand curve. First, given the high and low values for the project return, the default value or the recovery rate of the bond should not be too low, i.e., haircuts on government debt is not too large. The reason is that when the default value is too low, government bond becomes riskier than the project. In this case, higher risk of default lowers the (negative) risk premium on the bond and reduces the price of bond. Second, the probability p has to be higher than a threshold, which is a function of the return profile of the project. The reason is that a high value of p is needed so that default can create significant aggregate risk for the investors.

I show that, unlike the yield on government debt, CDS premium always has a positive correlation with the level of government debt and the implied default risk whenever the

volume CDS contracts is small relative to the rest of investors' portfolio. This implies that, whenever government debt is the safest asset available to the investor, an increase in default risk lead to a negative correlation and a widening gap between the bond yield and the CDS premium.

The *change in aggregate risk* effect can be thought of as the change in safety premium of government debt. When government debt becomes systemic, while remaining the safest possible asset, an increase in government debt can simultaneously increase the default risk and safety premium. This is consistent with the fact that safety premium on U.S. government debt has increased after 2000 despite a significant increase in government debt (Del Negro et al. (2017)).

My results is complementary to the emerging literature on the importance of demand composition of government debt. This paper highlights that who holds government debt can matter a great deal as emphasized in recent works (Fang et al. (2025) and Chaudhary et al. (2025)).

At the same time, these results might add nuance to the measurement of the demand for government debt and its impact on the yields in this literature. Following Koijen and Yogo (2019), a recent body of works attempt to estimate the demand for sovereign debt by different investors (Koijen and Yogo (2025), Fang et al. (2025), Eren et al. (2025), and Chaudhary et al. (2025)). There are two assumptions typically made in this literature. First, they assume an endowment economy, meaning that fiscal policy does not directly affect the returns to other investments by the marginal investor. And second, they use an iso-elastic demand specification in the form of a logit (or nested logit) model. My model suggests that the demand elasticity of government debt that serves as a safe asset might vary depending on the level of debt, and in extreme cases can be positive, i.e., yield compression or upward-sloping demand curve for intermediate level of borrowing. Moreover, and unlike more conventional environments, the 'total' elasticity incorporating the general equilibrium effects of demand for government debt might be significantly *larger* than the elasticity estimated in the literature. Therefore, the aforementioned assumptions might not be innocuous and that they might impede a full understanding of the nature of the demand for government debt.[1]

I investigate the normative implications of my model by introducing a welfare maximizing government. I show numerically that, in contrast to the traditional undersupply of safe sovereign debt due to market power, the supply of government debt can be ex-

[1]Moreover, Koijen and Yogo (2019) assumes that the demand for assets is downward sloping to prove the existence of equilibrium. While this assumption is reasonable in many settings, as the paper argues, it might not hold in certain environments such as the one considered in this paper.

cessive in my model precisely when government debt is the safest alternative available to the investors. The reason behind this oversupply is a *demand externality* which makes individual investor's demand complement with one another. The extreme version of this complementarity occurs when there is an upward-sloping demand curve: purchasing more government debt by one investor encourages others to buy more. The reason is that, at any given price, an investor's buying up more debt increases the likelihood of default at the margin for all other investors. Since government debt is still the safest asset available to investors, this increase in aggregate risk encourages all other investors to purchase more government debt as well. This complementarity can lead to excessive level of demand for debt and too much aggregate risk, when z is high enough.

In a two-country extension of my benchmark model, I examine how the existence of an upward-sloping demand curve depends on the correlation between the two countries' incomes. I numerically show that both positively and negatively correlated incomes can lead to upward-sloping demand curve. Moreover, I demonstrate that in the two-country model suggests complementarity in supply of government debt: more supply by one government increases the price of the other, thereby encouraging more supply of debt by the second government. This complementarity in supply of government debt can provide a microfoundation for the existence of the *global fiscal cycle* (Jiang and Richmond (2025)).

I show that when the investors have significant amount of liabilities on their balance sheets the upward-sloping demand curve for government bonds can result even if the returns to the real project in the default and no-default states are the same. Moreover, using numerical simulations, I show that risk premium, i.e., relative price of government bonds to the project, increases with more government borrowing whenever the recovery rate of the project is lower than that of the government bonds in the default state. In this extension, in addition to the project, investors also hold old government bonds from previous periods, and enter the initial period with previously issued liabilities. One natural interpretation of the investor is a bank with holdings of government debt, other investments in real assets such as loans, and significant amounts of deposit liabilities on its balance sheet. I assume that both the old bonds and old liabilities mature in the second period. The key assumption is that a default on newly issued bonds leads to a default on the old bonds. The extended model predicts that under these conditions, when the level of outstanding government debt and past liabilities held by the banking system are large enough, the demand curve can become upward-sloping: higher level of outstanding public debt held by banks implies lower government bond yield. This implies that a high level of old bonds holdings makes an upward-sloping demand curve for new bonds more likely.

The rest of the paper is organized as follows. Section 2 discusses the related literature. In Section 3, I present the benchmark model. I discuss the balance sheet effect in Section 6 as an extension and robustness of the results in Section 7. Finally, Section 8 concludes.

2 Related Literature

Liu (2019) and Cochrane (2020) find that higher debt to GDP ratios are associated with lower real interest rates. These findings are consistent with an upward-sloping demand curve for the U.S. debt. Liu (2019) finds that higher debt to GDP ratios are associated with higher measures of fiscal uncertainty, which can explain the the effect of higher debt on risk premia and real rates. My model is complementary to the mechanism in Liu (2019).

Krishnamurthy and Vissing-Jorgensen (2012) find that a higher U.S. debt to GDP reduces both liquidity and safety premium of Treasury debt. In contrast, Liu (2019) and Liu et al. (2020) find that an increase in the U.S. debt to GDP ratio predicts higher excess stock returns and is correlated with higher credit risk premia. My model is consistent with the latter findings.

He et al. (2019) present a model of safe asset determination in a two country setup. They find that a larger bond issuance and better relative fundamentals can make a country's bond safe. He et al. (2019) also find that a deterioration in the world-level fundamental can increase the price of the safe country's bond, while reducing the price of the other, i.e., flight to safty. In my model, government bonds can become safe asset when government default dampens the return to the other assets. Moreover, *even* a deterioration in fundamentals of the issuer of the safe asset increases the price of bonds and increases the risk premia.

In Farhi and Maggiori (2018), the supply of the safe (or reserve) asset can make the hegemon country susceptible to self-fulfilling confidence-crisis. Although there is no confidence-crisis in my model, I show the nature of a safe asset such as U.S. Treasuries can weaken or even reverse the impact of higher risks on the bond price, and risk premia.

Several recent papers, including Chernov et al. (2020) and Dittmar et al. (2019), study the U.S. sovereign default risk. Chernov et al. (2020) show that the U.S. CDS premiums, which have been elevated since the Financial Crisis of 2008, reflect the endogenous risk-adjusted probabilities of fiscal default. Dittmar et al. (2019) examine the relative pricing of nominal Treasury bonds and Treasury inflation-protected securities (TIPS) in the presence of United States default risk. They show that most of the relative mispricing after the crisis is due to default risk.

Gennaioli et al. (2014) find that domestic government debt held by banks is, on average, 11.8% of their total assets. They also document that 67% of government defaults are

accompanied by banking crises in their sample. They use these observations to argue that an important reason countries pay their debt is that sovereign default is very costly for the banking system and hence for the real economy. Unlike this model, Gennaioli et al. (2014) are not concerned with changes government yields: the borrowing rate on external debt in Gennaioli et al. (2014) is always exogenous and constant. Moreover, while this paper focuses on countries producing safe assets at a global level, Gennaioli et al. (2014) focus on emerging market and low income countries.

Shortage of safe assets has been argued to be a major cause of the decline in the real rates (Caballero et al. (2017), Del Negroa et al. (2019)).[2] My theory suggests that the significant increase in the supply of government debt and its systemic risks especially after the Financial Crisis of 2007-2008, may have exacerbated this "shortage" and contributed to even lower levels for real rates. In other words, supply of systemic safe assets by governments may have increased the demand for the same assets.

3 Model

There are two periods $t = 0, 1$. The economy consists of a government, a representative household, and a measure one of ex-ante identical investors. The model economy can capture both a closed economy and the global economy. The main difference in these two interpretations is that investors are domestic in one and foreign in the other.[3]

The household has no income at $t = 0$ but receives a stochastic endowment y_1 at $t = 1$. I assume that only a maximum fraction $0 < \tau \leqslant 1$ of y_1 can be taxed by the government, and that τy_1 is distributed according a cumulative distribution function F. Suppose that the household consumes only in $t = 1$. To avoid a large utility cost to the household, the government has to finance a deficit equal to g_0 units of $t = 0$ consumption goods. To do that, the government can sell b_1 units of bonds to the investors, at time $t = 0$, each unit of which pays off one unit of consumption goods at $t = 1$. Let q_0 denote the price of this bond in units of time $t = 0$ consumption goods. The government uses the tax receipts τy_1 to pay back its debt. Since y_1 is stochastic, for high enough b_1, the government may not be able to pay its debt obligations in full and defaults. If the government defaults on its debt, I assume that the creditors can recover zb_1, where $0 \leqslant z < 1$. I assume that the

[2]There are more than a few explanations for the secular decline in the real interest rates. I consider my model as a complementary to the explanation based on safe asset shortage. For more theories on the decline of the real rates, see Del Negroa et al. (2019), and references therein.

[3]When significant quantities of government debt is held externally, a government default can adversely affect the balance sheets of foreign holders of the debt. For example, significant fractions of U.S. and German debt are held by foreign investors. In the case of U.S. government debt, a default can be even more costly due to the especial role of dollar assets in the global trade and finance. See Gourinchas (2019).

government debt pays off at the beginning of $t = 1$. Each investor has an endowment y_0 at $t = 0$, and the following preferences:

$$U^* = c_0 + \mathbb{E}_0\left[\ln(c_1)\right].$$

c_0 and c_1 are consumption at the current and future dates. Note that there is no time discounting for simplicity. The log functional form in the second period is in line with other works that investigate the implications of intermediary risk aversion and risk bearing capacity for asset prices (He and Krishnamurthy (2012), and He and Krishnamurthy (2013)).

Each investor also holds a project, i.e., real asset, at the initial date that pays off at the end of $t = 1$. Before its completion, the investor needs to spend $\delta > 0$ units of time $t = 1$ consumption goods at the beginning of $t = 1$ in order to maintain the project. If the full maintenance cost δ is paid, the project yields a payoff of a_H. Otherwise, if δ is not paid, the project's payoff decreases to $0 < a_L < a_H$ with probability $p \in [0, 1]$, or remains a_H with probability $1 - p$. The maintenance cost in my model is similar to the liquidity shock in Holmström and Tirole (1998). The difference is that unlike Holmström and Tirole (1998), the return in the event of not covering the liquidity needs can be stochastic while the size of the liquidity shock is not. I assume that the realization of project's payoff across different projects are i.i.d across investors, and that this idiosyncratic risk can not be insured. The following assumption ensures that it is always optimal for the investor to pay the maintenance cost if she can:

Assumption 1. $p \geqslant \frac{\delta}{a_H - a_L}$.

One can think of investors as banks, shadow banks, or any other entity that might entail significant negative spillovers to the real sector of the economy under financial stress. Government defaults often lead to severe contractions in output (Mendoza and Yue (2012), Trebesch and Zabel (2017)). These contractions can be even more severe when government defaults are accompanied by banking crises (Gennaioli et al. (2014)). Gennaioli et al. (2014) find that domestic government debt held by banks is, on average, 11.8% of their total assets. They also document that 67% of government defaults are accompanied by banking crises in their sample. Government default can also be detrimental to the economy, when a significant fraction of government debt is held by domestic shadow banks. For example, 12% of U.S. government debt is held by U.S. mutual funds.[4] This suggests that it is plausible to think of investors in my model as banks or part of the shadow banking system.

Project in this model can capture real investments by banks and shadow banks. Banks

[4]See https://www.cbo.gov/publication/56309, for the composition of U.S. debt holders.

and shadow banks often need to provide liquidity to their borrowers in the corporate sector to pay off short term debt or to use as working capital. The maintenance cost in my model, is an stylized way to capture this type of liquidity provision. In this interpretation, the probability p depends on factors such as the leverage of the borrowers in the household or non-financial corporate sector. The reason is that liquidity problems are more likely to turn into solvency problems, when borrowers have high leverage.

Note that Assumption 1 implies that $a_H - \delta \geqslant a_L$. The timing is as follows. At $t = 0$, investor decides how many government bonds to buy. At $t = 1$, first, the government pays off its debt or defaults. Next, investor pays the maintenance cost of the project, using what she receives from the government. Finally, the project pays off and investor consumes whatever consumption goods are left after paying for the maintenance.

Discussion of the Assumptions The assumption that the household does not have any endowments at $t = 0$, is to simplify the analysis. Similar results can be obtained, if the household can also purchase government bonds, as long as it is the investors who price the assets at the margin.

The implicit assumption that payoff structure of the project a_L, a_H, and p are independent of the realizations of household income y_1 simplifies the analysis. In Section 7.1, I present a variation of the model in which this assumption is relaxed. I show that the main qualitative result of the model on upward-sloping demand curve remains intact in this case.

The quasi-linear preferences for investors also helps to further simplify the characterization of the demand for government borrowing. It makes the analysis tractable by eliminating the income effects but is not essential for the key results. For other preferences, as long as the income effect is not too large, the results obtained in my model should hold. In Section 7.2, I show that upward-sloping demand curve can result even with the traditional preferences, e.g., log utility in both periods. Moreover, if the marginal investor already holds government bonds before purchasing any new bonds at the initial date, the income effect of a higher bond price can increase, rather than decrease, the investor's demand.

Note that, default in my model can capture the possibility of either a fiscal default or higher than expected inflation. Moreover, I assume that government does not default strategically: it pays its debt obligations as long as its feasible to do so. But, one can write a similar model with strategic default that yields the same qualitative predictions. To do that, I can introduce a stochastic fixed cost of default. This cost reflects reputational cost or costs associated with losing access to world market. It can depend on macroeconomic conditions, which make the cost stochastic.

Finally, there are other micro-foundations that can generate the same demand function for government bonds. I study one alternative in later sections, which uses the balance sheet effect of government bond holdings.

3.1 Equilibrium

I start with the investor's optimal consumption-saving problem. When $b_1 < \delta$, the investor cannot pay the maintenance costs, whether or not government defaults. In contrast, when $\frac{\delta}{z} \leqslant b_1$, the investor can always pay for the maintenance costs of the project. In Lemma 1, I provide the resulting demand curve in these two cases.

Next, I focus on the most interesting case, in which the investor can pay for the maintenance cost only when the government is not in default. Note that this happens only if $\delta \leqslant b_1 < \frac{\delta}{z}$. Within this range, investor solves:

$$\begin{cases} U = \max_{c_0, b_1} c_0 + \left\{(1-\pi_D)\ln(a_H - \delta + b_1) + \pi_D\left[(1-p)\ln(a_H + zb_1) + p\ln(a_L + zb_1)\right]\right\}, \\ \\ \text{s.t.} \quad c_0 + q_0 b_1 = y_0\,. \end{cases}$$

π_D is the probability of default by the government, and q_0 is the price of bond, which are taken as given by the investor. Given the quasi-linear preferences, the first order condition (FOC) with respect to c_0^* implies a Lagrange multiplier for the budget constraint equal to one. Using this fact along with the FOC with respect to b_1 give the following equation for the demand for government bonds:

$$q_0 = \frac{1-\pi_D}{a_H - \delta + b_1} + \pi_D\left[\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1}\right]. \tag{1}$$

Government defaults only if the tax receipts are less than the debt payments, or $b_1 > \tau y_1$. Therefore, the probability of default is equal to $\pi_D = F(b_1)$. Putting this back into 1, gives the following result:

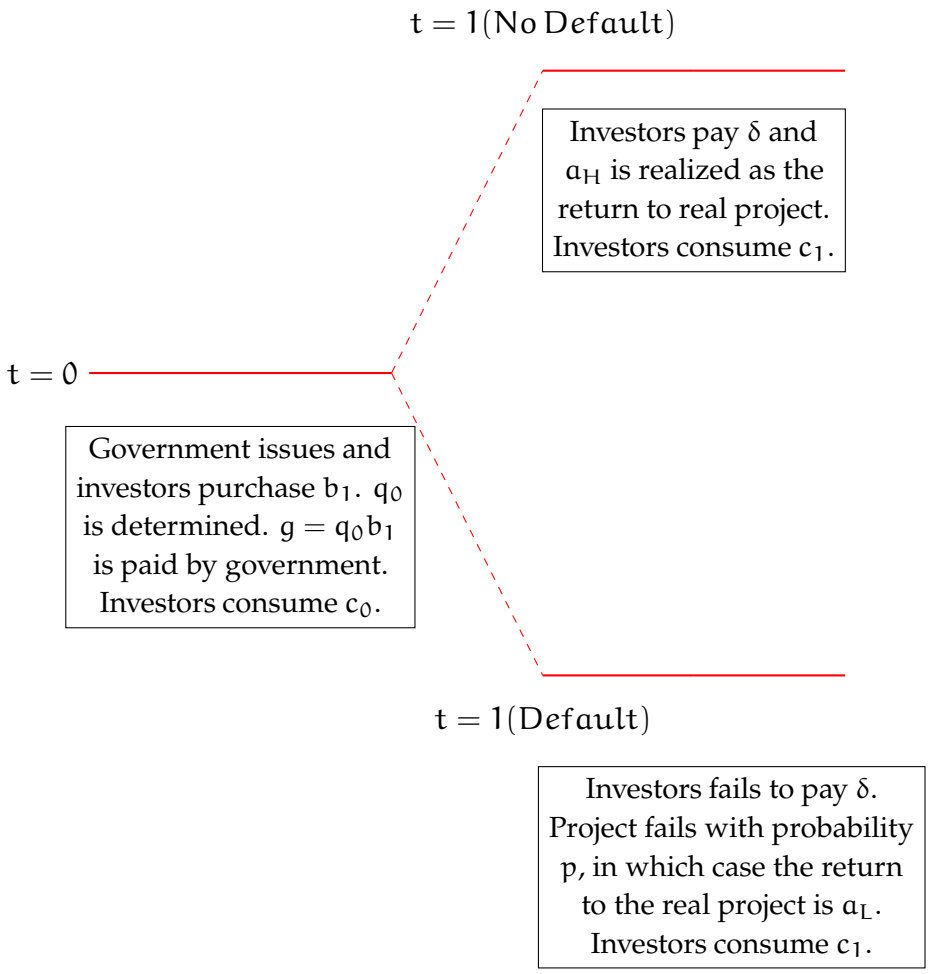


Figure 2: Model Timeline Diagram.

LEMMA 1. *The following characterizes the demand for government bonds by investors:*

$$q_0 = \begin{cases} \left[\frac{(1-p)}{a_H+b_1} + \frac{p}{a_L+b_1}\right](1-F(b_1)) + \left[\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1}\right]F(b_1) & \text{if} \quad b_1 < \delta\,, \\ \frac{1-F(b_1)}{a_H-\delta+b_1} + \left[\frac{(1-p)z}{a_H+zb_1} + \frac{pz}{a_L+zb_1}\right]F(b_1) & \text{if} \quad \delta \leqslant b_1 < \frac{\delta}{z}\,, \\ \frac{1-F(b_1)}{a_H-\delta+b_1} + \frac{zF(b_1)}{a_H-\delta+zb_1} & \text{if} \quad \frac{\delta}{z} \leqslant b_1\,. \end{cases}$$

Moreover, $\frac{\partial q_0}{\partial b_1} < 0$ *for all* $b_1 \notin [\delta, \frac{\delta}{z})$.

Whenever $b_1 \notin [\delta, \frac{\delta}{z})$, the demand for government bonds is downward sloping. Also, note that when there is no default, i.e., $z = 1$, investors always pay the maintenance costs for $b_1 \geqslant \delta$, and do not pay the maintenance costs for $b_1 < \delta$. In this case the demand

collapses to $q_0 = \frac{1}{a_H - \delta + b_1}$ and $\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1}$, for $b_1 \geqslant \delta$ and $b_1 < \delta$, respectively. In either case, the demand is strictly decreasing in b_1.

Now, consider the case of $z = 0$. In this case, the second term on the RHS of the demand schedule above is zero. Given that $F(b_1)$ is increasing in b_1, the RHS and hence q_0 is unambiguously decreasing in b_1. This implies that in order to have a upward-sloping demand curve, one needs $z > 0$. More generally, taking the derivative with respect to b_1, one obtains:

$$\frac{\partial q_0}{\partial b_1} = \underbrace{-\Big\{\frac{1 - F(b_1)}{(a_H - \delta + b_1)^2} + \Big[\frac{(1-p)z^2}{(a_H + zb_1)^2} + \frac{pz^2}{(a_L + zb_1)^2}\Big] F(b_1)\Big\}}_{A} + \tag{2}$$

$$\tag{3}$$

$$+ \underbrace{\Big[\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1} - \frac{1}{a_H - \delta + b_1}\Big] f(b_1)}_{B} . \tag{4}$$

$f(b_1)$ is the probability density function associated with $F(b_1)$. There are two parts to RHS of the equation above. The first part, denoted by A, is the result of *diminishing expected marginal utility* of consumption: holding more bonds results in more consumption, and a lower marginal utility of purchasing an extra unit of bond. This effect is present even absent any risks, and is unambiguously negative implying a lower price.

The second part of RHS in 2, denoted by B, captures the *change in aggregate risk*. B appears in the equation for the price because an increase in borrowing creates more default risk. This component takes into account the increase in risk that is due to higher probability of default, measured by $f(b_1)$. B also depends on the difference between marginal utilities of an extra unit of bond across the two states of default and no-default. When this difference is negative, purchasing one extra unit of bond creates reduces the insurance provided by the government bond, because the investor gains less utility in default relative to the no-default state. Therefore, in this case an increase in b_1 lowers the the bond price even further. This is the case, for example, when p is too low. The reason is that, given $a_H - \delta < \frac{a_H}{z}$, when p is too low, the marginal utility gained from an extra unit of bond is always lower in the state of default. Moreover, when $a_H - \delta < \frac{a_L}{z}$, a marginal increase in bond holding reduces B, regardless of the value of p.

B becomes positive when p is high and $a_H - \delta > \frac{a_L}{z}$. If the increase in the aggregate risk is large enough, the net effect of A and B can become positive. In such a case, government bonds face an upward-sloping demand curve. In other words, the increase in the riskiness of the safe asset can increase the aggregate risk and the demand for insurance, thereby

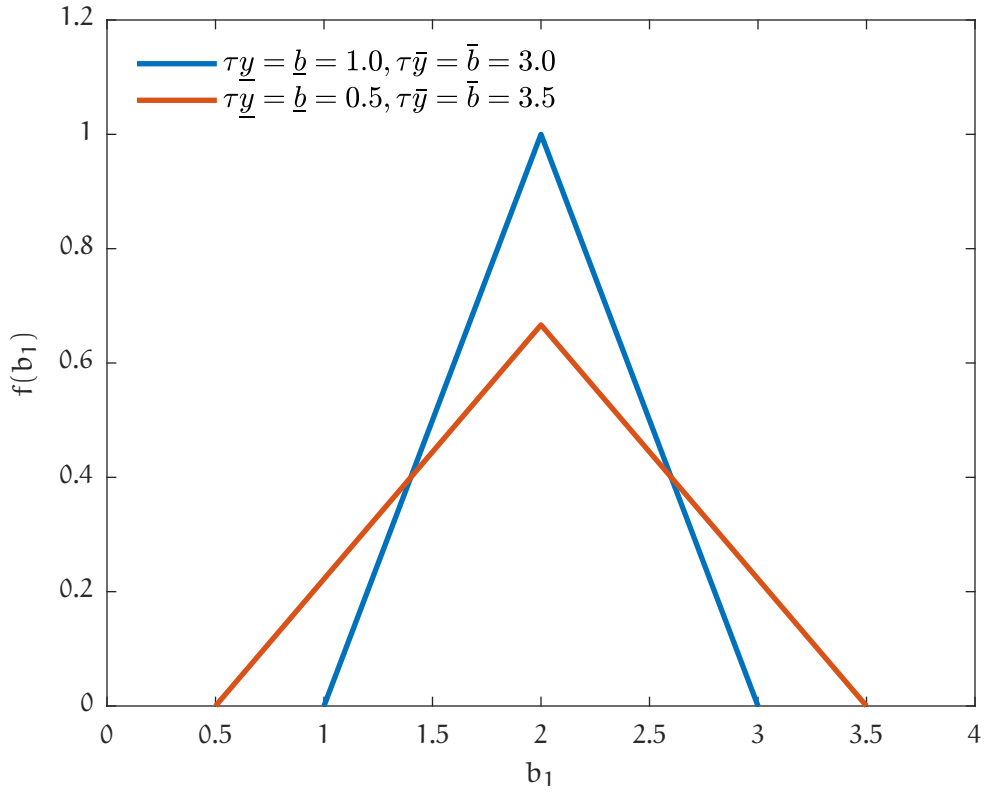


Figure 3: Triangular distributions of future tax revenues. The red density function is a mean preserving spread of the blue density function.

increasing the demand for the safe asset.

Finally, to close the model, one needs to impose the marketing clearing in the bonds market. The supply curve of government bonds dictates that $g_0 = q_0 b_1$. This defines the supply curve as a decreasing function of b_1. The intersection between the supply and demand gives the equilibrium price and quantity (Figure 4).

3.2 Relative Safety

Flight to safety is a well documented phenomenon during periods of heightened risk, such as the Financial Crisis of 2007-2008 or the COVID-19 pandemic. Flight to safety seems intuitive when government bond is default free. But, how does a worsening of economic fundamentals of the safe asset affect its demand, and the demand for other assets?

To simplify the analysis, I assume $p = 1$ for this subsection. Price of bond becomes:

$$q_0 = \frac{1 - F(b_1)}{a_H - \delta + b_1} + \frac{zF(b_1)}{a_L + zb_1}. \tag{5}$$

To study flight to safety, consider a negative shock to future tax revenues that changes

the cumulative density to $\tilde{F}(b_1)$. Suppose that $\tilde{F}(b_1)$ first order stochastically dominates $F(b_1)$: $F(b_1) < \tilde{F}(b_1)$, for all b_1. If $\frac{a_L}{z} < a_H - \delta$, this negative shift *increases* the price of bond in 20, because $\frac{1}{a_H - \delta + b_1} < \frac{z}{a_L + z b_1}$.

Although a worse prospect of future tax revenues distribution increases the default risk of the bond, such a negative shock increases the bond price. The reason is that while bond becomes riskier as a result of a negative shock to future tax revenues, it remains the safer asset. The following lemma shows a similar result when the new income distribution is a mean-preserving spread of the old one:

LEMMA 2. *Suppose that the future tax revenues distribution $f(b_1)$ changes to another symmetric triangular distribution $\tilde{f}(b_1)$. Assume that $\tilde{f}(b_1)$ has the same mean as $f(b_1)$, and the domain $[\underline{b}'', \overline{b}'']$, where $\underline{b}'' = \underline{b} - \epsilon$ and $\overline{b}'' = \overline{b} + \epsilon$. Then, the price of bond is higher under $\tilde{f}(b_1)$, for all $b_1 < \frac{\underline{b}+\overline{b}}{2}$.*

Put simply, an increase in the risk without any change in the mean of the income distribution increases the bond price.

The analysis so far assumed $\delta \leqslant b_1 < \frac{\delta}{z}$. When $b_1 \notin [\delta, \frac{\delta}{z})$, however, the above results do not hold. In particular, one can show that when b_1 is too high or too low, a first order stochastically dominant shift in $F(b_1)$ reduces q_0. This implies that government bonds behave like a safe asset, when the relative supply of bonds to the value of project in the aggregate, i.e., $\frac{b_1}{a_H - \delta}$, is within an intermediate range. Given that investors in this model can be thought of as banks or shadow banks, this also implies that government debt needs to be held in large enough quantities relative to the size of financial intermediaries' balance sheets in order to start behaving like a safe asset.

3.3 Upward-Sloping Demand Curve

The following conditions are necessary for an upward-sloping demand curve to exist:

PROPOSITION 1. *Two necessary conditions for an upward-sloping demand curve are:*

$$\begin{cases} p \geqslant \frac{a_L(1-z)}{(a_H - a_L)z}, \\ \\ \frac{a_L}{a_H - \delta} < z. \end{cases}$$

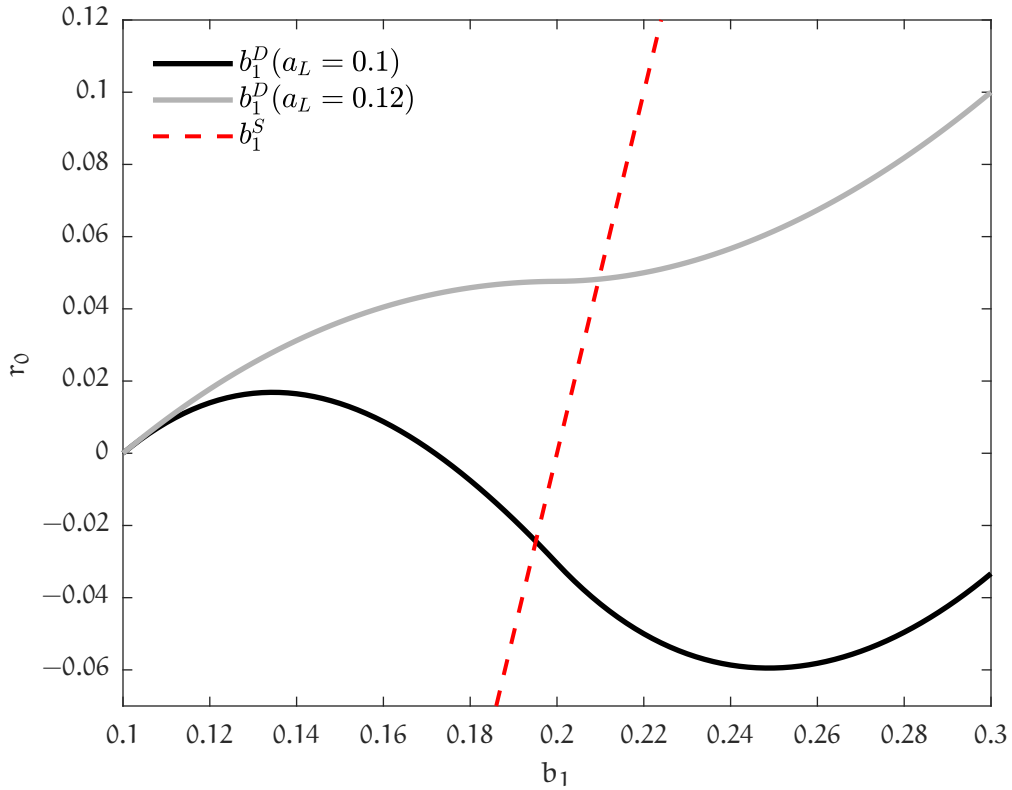


Figure 4: Supply and demand for government bonds. Demand curves are illustrated for two values of a_L, in terms of the yield $r_1 = \frac{1}{q_0} - 1$. The rest of parameter values are $a_H = 1$, $z = 0.15$, $\delta = 0.1$, $\tau\underline{y} = 0.1$, and $\tau\bar{y} = 0.3$.

Moreover, one has:

$$\frac{\partial q_0}{\partial b_1}|_{p=1} > 0 \Rightarrow \frac{\partial^2 q_0}{\partial p \partial b_1} > 0\,.$$

The value of p is a measure of correlation between the payoffs of the two available assets, namely government bond and the project. When p is high, default by the government lowers the expected payoff to the project significantly, which increases the difference in expected marginal utilities of an extra unit of bond across default and no-default. This implies higher aggregate risk and increases the demand for insurance by investors and the price of bond.

Given z, the second necessary condition in Proposition 1 states that a_L must be low enough relative to a_H or, in other words, government default should inflict a sizable damage to the rest of the economy. As mentioned before, the second necessary condition also implies that z can not be too low. The second necessary condition implies that the

liquidation value of the real project in the default state as a fraction of the return in the no-default state, i.e., the recovery rate, is lower than that of the government bond. In other words, government bond is safer than the real project. Note that default in this model can capture both a fiscal default and inflation.

The meaning of the second result in Proposition 1 is as follows. When demand for bond is upward sloping at $p = 1$, lowering p decreases the slope of the demand curve: it is less likely to have an upward-sloping demand curve for lower values of p.

It is important to note that whenever the demand curve for b_1 is upward sloping, an increase in government expenditures g_0 increases the equilibrium price of government bonds. A positive relationship between b_1 and bond price q_0 implies a positive relationship between $g_0 = q_0 b1$ and bond price: the demand curve is upward sloping no matter how one measures government debt. I state this simple but important observation in the following lemma.

LEMMA 3. *If* $\frac{\partial q_0}{\partial b_1} \geqslant 0$ *in equilibrium, then* $\frac{\partial q_0}{\partial g_0} \geqslant 0$.

To construct an example of an upward-sloping demand curve for government bonds, I assume that $f(b_1)$ is a triangular distribution (Figure 3). Later, I discuss what can be shown for somewhat more general class of distributions. Suppose that $f(b_1)$ is a symmetric triangular distribution over $[\underline{b}, \overline{b}]$, which is defined as follows:

$$f(b_1) \equiv \begin{cases} \frac{4(b_1 - \underline{b})}{(\overline{b} - \underline{b})^2} & b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}], \\ \\ \frac{4(\overline{b} - b_1)}{(\overline{b} - \underline{b})^2} & b_1 \in [\frac{\underline{b}+\overline{b}}{2}, \overline{b}]. \end{cases} \tag{6}$$

where $\underline{b} \equiv \tau \underline{y}$ and $\overline{b} \equiv \tau \overline{y}$ are the minimum and maximum values of tax collected by the government at $t = 1$. I also make the following assumptions to ensure that tax receipts are always enough to pay the default value to the investors, and that $\delta \leqslant b_1 \leqslant \frac{\delta}{z}$:

ASSUMPTION 2. $z\overline{y} < \underline{y}$, $\delta < \underline{b}$, *and* $\overline{b} < \frac{\delta}{z}$.

From now on, I also set $p = 1$ to further simplify the analysis in this and subsequent sections. The cumulative density function for $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$ is:

$$F(b_1) = 2\left(\frac{b_1 - \underline{b}}{\overline{b} - \underline{b}}\right)^2. \tag{7}$$

For all $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, 1 implies:

$$\frac{\partial q_0}{\partial b_1} \propto \frac{a_H - \frac{a_L}{z} - \delta}{\frac{a_L}{z} + b_1} \Big(\frac{b_1 - \underline{b}}{\overline{b} - \underline{b}}\Big)^2 \Big\{\frac{2}{b_1 - \underline{b}} - \Big[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\} - \frac{1}{a_H - \delta + b_1}. \quad (8)$$

The term in the braces in 8 is always positive. Moreover, for values of b_1 close to $\underline{b}$, the first term in the expression on the RHS is close to zero, which implies that the slope of the demand curve is unambiguously negative. But, for a given a value of $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, a large enough value of a_H implies a positive slope of demand curve at b_1:

Proposition 2. *For any $\tilde{b}_1 \in (\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, if $a_H - \delta - \frac{a_L}{z}$ is large enough, one has $\frac{\partial q_0}{\partial b_1} > 0$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\overline{b}}{2}]$. Moreover, for given values of a_H and a_L, there exists $\hat{b}_1 > \underline{b}$ such that $\frac{\partial q_0}{\partial b_1} < 0$, for all $b_1 \in [\underline{b}, \tilde{b}_1]$.*

For low values of borrowing, the demand curve is downward sloping. But, there is a point at which the slope of the demand curve turns positive, and stays positive at least until the mean of the distribution, $\frac{\underline{b}+\overline{b}}{2}$. The reason behind an upward-sloping demand curve is that what matters is not the absolute safety of the bond but its riskiness *relative* to other investments. In other words, government bond is a safe asset because it remains safer than the share even when default risk goes up.

A higher value of $a_H - \delta - \frac{a_L}{z}$ can reflect more riskiness of the project's payoff. Higher riskiness can be due to a higher a_H, a lower a_L, or a higher p. As discussed before, examples of project in this model include investments by banks and shadow banks. One factor that can increase the riskiness of these investments in the real world is high leverage of both borrowers and lenders. More than a decade after the Financial Crisis of 2008, banks and many shadow banks remain highly leveraged entities. The non-financial corporate debt as a fraction of GDP has been on the rise in Euro area, the U.S., and China.[5] Although borrowers and lenders are not explicitly modeled, one can think of a higher leverage of banks and shadow banks, i.e., investors, to be reflected in a higher upside a_H. A higher leverage in the household or the corporate sector, on the other hand, can be captured by a higher p: liquidity problems are more likely to become solvency problems when firms have high leverage. I study the effect of investors' indebtedness more explicitly in Section 6.

Figure 4 is an illustration of an partially upward-sloping demand curve for government bonds. In Lemma 1, I have already shown that demand for government debt is downward sloping when $b_1 \notin [\delta, \frac{\delta}{z})$. But, Figure 4 shows that demand curve becomes downward sloping in the left and right tails of the domain of b_1, even when $b_1 \in [\delta, \frac{\delta}{z})$. I show in the next subsection that this property is rather general and does not depend on the particular distribution used in this subsection.

[5] See April 2019 issue of the Global Financial Stability Report, published by IMF.

Liu (2019) finds that an increase in U.S. government debt is associated with lower short term rate. Cochrane (2020) also finds that large amounts of debt correspond to low subsequent returns. These findings are consistent with an upward-sloping demand curve for U.S. debt.

3.4 Measurement of Default Risk

The possibility of a partially upward-sloping demand curve for government debt can make the measurement of default risk more complicated, as a decrease in the government bond yields can indicate a decreasing or an increasing risk of default depending on where we are on the demand curve. Can Credit Default Swaps (CDS) give us a better view of what is happening to the default risk?

To investigate this question suppose that investors can purchase CDS contracts each of which pays $1-z$ units of consumption goods in the event of default. Investors must pay a premium s_1 at $t=1$ if there is no default. The investor's problem becomes:

$$\begin{cases} U = \max_{c_0, b_1, \kappa_1} c_0 + \left\{(1-\pi_D)\ln(a_H - \delta + b_1 - s_1\kappa_1) + \pi_D \ln(a_L + zb_1 + (1-z)\kappa_1)\right\}, \\ \\ \text{s.t.} \quad c_0 + q_0 b_1 = y_0\,. \end{cases}$$

κ_1 is the number of CDS contracts purchased by the investor. The above representation uses the fact that CDS contracts do not require an initial transfer. The FOC with respect to κ_1 gives an equation for the CDS premium s_1:

$$s_1 = \frac{(1-z)z\pi_D(a_H - \delta + b_1 - s_1\kappa_1)}{(1-\pi_D)(\frac{a_L}{z} + b_1 + \frac{1-z}{z}\kappa_1)}\,. \tag{9}$$

The following proposition establishes that, unlike the yield on government debt, CDS premium always has a positive correlation with the level of government debt and the implied default risk whenever the volume CDS contracts is small relative to the rest of investors' portfolio:

Proposition 3. *When CDS contracts are in zero net supply, the CDS premium, s_1, defined in 9 is always an increasing function of b_1 and the likelihood of default $\pi_D = F(b_1)$, for all $b_1 \in [\underline{b}, \bar{b}]$.*

The assumption that κ_1 is small relative to b_1 is consistent with the gross and net volume of CDS contracts for countries deemed to be producers of safe assets (O'Kane (2012)). Note that while the relationship between CDS premium and default risk is monotonic, it might be highly non-linear. Proposition 3 implies that when the demand for

government debt becomes upward sloping, an increase in b_1 moves bond yields and the CDS premium in opposing directions, thereby widening the gap between the two. The following lemma shows that this phenomenon is not specific to changes in default risk stemming from changes in b_1:

LEMMA 4. *Whenever $\frac{a_L}{a_H-\delta} < z$ and CDS contracts are in zero net supply, an increase in the likelihood of default π_D, for a fixed level of b_1, increases the CDS premium s_1, lowers the yield on government debt $r_1 = \frac{1}{q_0} - 1$, and therefore widens the spread $s_1 - r_1$.*

The opposing impacts of default risk on bond yields and CDS premium in Lemma 4 and Proposition 3 is consistent with the data (Klingler and Lando (2018)). The conditions under which Lemma 4 holds is a bit more general than those underlying Proposition 3, as Lemma 4 holds as long as government debt is the safest investment available to the investors and there is no need for an upward-sloping demand curve.

3.5 General Income Distributions

In this section, I examine two questions under a more general income distribution. First, under what conditions one has $\frac{\partial q_0}{\partial b_1} > 0$? And second, is there a sense in which a more systemic government debt can change the behavior of $\frac{\partial q_0}{\partial b_1}$, even when $\frac{\partial q_0}{\partial b_1} \leqslant 0$? Using 1 at $p = 1$, I obtain:

$$\frac{\partial q_0}{\partial b_1} \propto \frac{(a_H - \delta - \frac{a_L}{z})F(b_1)}{\frac{a_L}{z} + b_1}\Big\{\frac{f(b_1)}{F(b_1)} - \Big[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\} - \frac{1}{a_H - \delta + b_1}$$

Since $\frac{a_L}{z} + b_1 < a_H - \delta + b_1$, a sufficient condition for $\frac{\partial q_0}{\partial b_1} > 0$ is $\zeta(b_1) > 1$, where:

$$\zeta(b_1) = (a_H - \delta - \frac{a_L}{z})F(b_1)\Big\{\frac{f(b_1)}{F(b_1)} - \Big[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\}$$

Consider a continuous distribution $f(b_1)$, which has a bounded support $[\underline{b}, \overline{b}]$. If $f(\underline{b}) > 0$, it is clear that $\zeta(b_1) > 0$ for b_1 close to $\underline{b}$, as $F(b_1)$ is arbitrarily close to zero. In such a case, a high enough $a_H - \delta - \frac{a_L}{z}$ implies that $\zeta(b_1) > 1$ and consequently $\frac{\partial q_0}{\partial b_1} > 0$.

Now consider the case $f(\underline{b}) = f(\overline{b}) = 0$. A necessary condition for $\frac{\partial q_0}{\partial b_1} > 0$ is that the expression in braces in the definition of $\zeta(b_1)$, is positive. The first term in the braces, which is the reversed hazard function $\frac{f(b_1)}{F(b_1)}$, is the derivative of $\ln(F(b_1))$. Hence, the value

of $\frac{f(b_1)}{F(b_1)}$ depends on the convexity of $f(b_1)$. Therefore, I can state the following result:

LEMMA 5. *Suppose that $f(b_1)$ is a continuous density function over $[\underline{b}, \overline{b}]$ that satisfies $f(\underline{b}) = f(\overline{b}) = 0$, and $f(b_1) > 0$ for $(\underline{b}, \overline{b})$. Assume that $f(b_1)$ is convex for $b_1 \in [\underline{b}, \hat{b}]$, for some $\underline{b} < \hat{b} \leqslant \overline{b}$. Then:*

$$\begin{cases} \frac{\partial q_0}{\partial b_1}|_{\hat{b}} > 0, \quad \text{for} \quad \text{large} \quad a_H - \delta - \frac{a_L}{z}, \\ \\ \frac{\partial^2 q_0}{\partial \frac{a_L}{z} \partial b_1}|_{\hat{b}} < 0. \end{cases}$$

Note that $\frac{\partial q_0}{\partial b_1} < 0$ for values of b_1 close to either $\underline{b}$ or $\overline{b}$, since $f(\underline{b}) = f(\overline{b}) = 0$. Therefore, b_1 needs to be within an intermediate range, i.e., neither too low nor too high, for the demand curve to be upward-sloping. This also implies that an upward-sloping segment for intermediate values of b_1 always comes with (at least) two inflection points. The lower inflection point is the value of b_1, where the price of government debt starts to *increase* with more supply. And the second inflection point is the value of b_1 at which, the price of government debt starts to *decrease* with more supply again.

Lemma 5 implies that as $a_H - \delta - \frac{a_L}{z}$ becomes larger, a larger neighborhood of $\hat{b}$ and, consequently, a larger fraction of the interval $b_1 \in [\underline{b}, \hat{b}]$ exhibits upward-sloping demand curve.

When $\frac{a_L}{z} < a_H - \delta$, a marginal increase in borrowing by the government increases the probability of default by $f(b_1)$, which increases the demand for insurance and the risk premium on government bonds. This is the *change in aggregate risk* effect. At the same time, such an increase in borrowing reduces the demand for government bond due to the *diminishing expected marginal utility* effect. $\zeta(b_1) > 0$ is a necessary condition for the net of these two effects to be positive.

As can be seen in the expression for $\zeta(b_1)$, one determinant of the net impact of these two effects is the reversed hazard function $\frac{f(b_1)}{F(b_1)}$. When $f(b_1)$ is convex at its left tale, an increase in b_1 increases $\frac{f(b_1)}{F(b_1)}$. This makes it more likely to have an upward-sloping demand curve, for high enough b_1. Many unimodal distributions, such as normal, log-normal and Weibull distributions, are convex on their left tale.[6] But, note that convexity is a sufficient and not a necessary condition for an upward-sloping demand curve.

Lemma 5 also provides some insight into the second question posed at the beginning of this section. A decrease in $\frac{a_L}{z}$, for example due to a lower a_L, can be thought as

[6]The left-tale of Weibull density function is convex for $k > 2$, where k is the shape parameter.

the equivalent of making the government debt more systemic. The second inequality in Lemma 5 implies that such a decrease, increases the slope of the demand curve $\frac{\partial q_0}{\partial b_1}$. Note that this is the case even if $\frac{\partial q_0}{\partial b_1} < 0$. This result implies changes in government bond yield is compressed, when the demand curve looks normal, i.e., $\frac{\partial q_0}{\partial b_1} < 0$:

Corollary 1. *Suppose that* $\frac{\partial q_0}{\partial b_1} < 0$. *Then,* $\frac{\partial^2 q_0}{\partial \frac{a_L}{z} \partial b_1} < 0$ *implies* $\frac{\partial^2 1/q_0}{\partial \frac{a_L}{z} \partial b_1} > 0$.

4 Welfare Implications

Some recent works have argued that U.S. market power in producing global safe assets has substantial welfare costs for the rest of the world. The underlying cause behind such welfare costs is thought to be the undersupplying of safe assets by a monopoly producer. For example, estimate that the U.S. supply of treasuries is below the global optimum by as much as 30%. In this section, I show that the welfare implications of government borrowing can change significantly, when government default reduces the return to other real assets as in my benchmark model.

In the benchmark model, the government only needed to raise an exogenous amount of resources, g, at the initial period. To evaluate the welfare implications, I need to introduce a welfare maximizing government to the benchmark model. The notion of the investor in the benchmark model is flexible and allows for both domestic and foreign entities that invest in government bonds. But, in this section, it is more natural to think of investors as foreign entities representing the welfare of the rest of the world. I assume that the government is maximizing the following objective function at $t = 0$:

$$\mathcal{U}^g = \ln(y_0^g + q_0 b_1) + \mathbb{E}_0\left[\ln(y_1^g - (1 - \mathbb{I}_D(1 - z))b_1)\right].$$

where $\mathbb{I}_D$ is equal to one if government is in default and zero otherwise. The question I would like to examine in this stylized setup is whether or under what conditions there is an undersupply of government debt at the level of debt that maximizes the above government objective function.

First, I use numerical simulations to illustrate the main normative results. The solid lines in Figure 5 show ex-ante welfare of the investors and the government at $t = 0$ for different values of b_1, when government default reduces the return to the real assets held by the investors so much that it renders government debt the relatively safer asset, i.e., $\frac{a_L}{z} < a_H - \delta$. The maximizing level of borrowing for the government in this scenario is

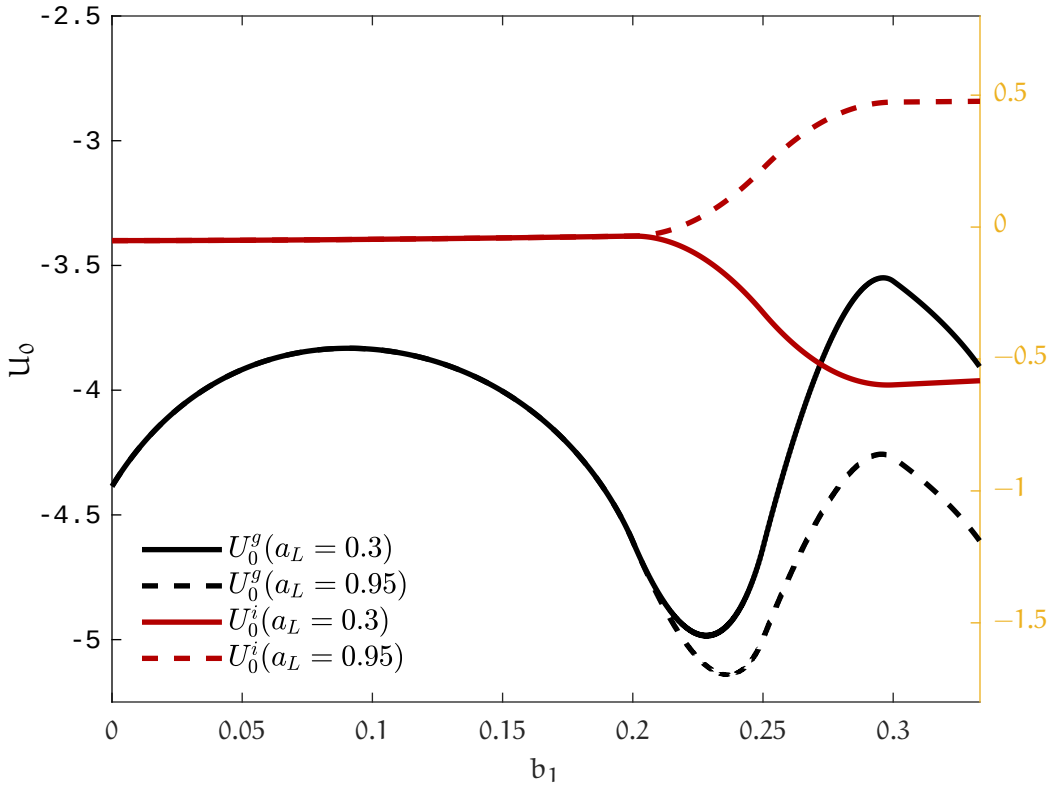


Figure 5: Welfare of the government and investors for $a_L = 0.3$ (solid lines) and $a_L = 0.95$ (dashed lines). Other parameter values are $a_H = 1.15$, $\delta = 0.2$, $z = 0.6$, $\tau\underline{y} = 0.2$, and $\tau\bar{y} = 0.3$.

$b_1^{g-max} = 0.296$. At the same time, the level of borrowing that maximizes and minimizes investors' welfare are $b_1^{i-max} = 0.201$ and $b_1^{i-min} = 0.298$ respectively. Suppose that a global planner can change b_1 by ϵ and also implement a transfer ν between the government and the investors at $t = 0$. After such a reallocation, utilities of the two types of agents will become:

$$\begin{cases} U^i = & y_0 + \nu - q_0(b_1+\epsilon)(b_1+\epsilon) + \big\{(1-\pi_D(b_1+\epsilon))\ln(a_H - \delta + (b_1+\epsilon)) + \\ & +\pi_D(b_1+\epsilon)\ln(a_L + z(b_1+\epsilon))\big\}, \\ U^g = & \ln(y_0^g - \nu + q_0(b_1+\epsilon)(b_1+\epsilon)) + \mathbb{E}_0\big[\ln(y_1^g - ((1-\mathbb{I}_D)(1-z)+z)(b_1+\epsilon))\big]. \end{cases}$$

The question is whether the planner can make both government and investors better off, and if so how. For the parameter values of Figure 5, the planner can increase government welfare, $\Delta U^g = 2.87 \times 10^{-4}$, by setting $\epsilon = -0.005$ and $\nu = 0.001$. Moreover, reducing

b_1 any further or increasing it cannot lead to Pareto improvement. This implies that the b_1^{g-max} is too high, rather than too low, from a global perspective. Nonetheless, b_1^{g-max} is very close to the global optimum, as the reduction required to achieve the globally optimum level of government debt is only 1.7%.

One can contrast the solid lines with the dashed lines in Figure 5. The dashed lines show ex-ante welfare of the investors and the government at $t = 0$ for different values of b_1, when government default does not have any impact on the return to the real assets held by the investors, i.e., $\frac{a_L}{z} = a_H - \delta$. Using tilde to indicate the new optima, the maximizing level of debt for the government in this scenario is $\tilde{b}_1^{g-max} = 0.091$. The maximizing and minimizing levels of debt for investors are on the two boundaries, i.e., $\tilde{b}_1^{i-min} = 0$ and $\tilde{b}_1^{i-max} = 0.333$ respectively. Note that unlike the case corresponding to the solid lines in Figure 5, $\tilde{b}_1^{g-max} < \tilde{b}_1^{i-max}$, which suggests that $\tilde{b}_1^{g-max}$ is likely to be too low from a global perspective. In fact, the planner can increase $\tilde{b}_1^{g-max}$ by setting $\epsilon = 0.0125$ and $\nu = 0.0011$ to achieve the global optimum. This amounts to 14% *increase* in government borrowing.

What explain the difference between the two cases in Figure 5? There are two key forces at play in both cases. The first is the well-known undersupply due to the government's market power. This first effect tends to keep the level of government borrowing too low. The second, as in the benchmark model, is the endogenous increase in demand for insurance due to systemic risk created by the possibility of default. This second effect tends to increase the demand for government debt, and consequently encouraging more supply, as it remains the safest option for the investors to hedge the increasing aggregate risk. In the case corresponding to the dotted lines in Figure 5, the second effect is absent. This leads to an undersupply of government debt. But for the parameter values corresponding to the solid lines in Figure 5, the second effect is present and is, in fact, strong enough to dominate the first, resulting in oversupply of the safe asset. The following lemma shows that the second effect dominates if and only if the demand curve is upward sloping at b_1^{g-max}:

LEMMA 6. *Suppose that the planner can change b_1, and implement a (net) transfer ϵ from investors to the government at $t = 0$. Then a Pareto improvement requires the planner to lower b_1 below (increase b_1 above) the competitive equilibrium value if and only if $\frac{dq_0}{db_1} < 0$ ($\frac{dq_0}{db_1} > 0$).*

Lemma 6 implies that it is necessary that government default creates systemic risk while government debt being the safest available asset, i.e., $\frac{a_L}{z} < a_H - \delta$, for the second effect to dominate. But it is not a sufficient condition. For example, consider parameter values corresponding to the solid lines in Figure 5 except that $z = 0.4$. In this case, one

still has $\frac{a_L}{z} < a_H - \delta$. But starting at the equilibrium that maximizes government welfare, reducing government debt cannot lead to Pareto improvement. In contrast, at $z = 0.4$, a global planner can make Pareto improvement by increasing government borrowing. In other words, z has to be high enough to make the second effect dominate.

4.1 Complementarity in Investors' Demand

Why can government debt be excessively high in equilibrium? When z is high enough and government default is systemic, investors' decisions to purchase debt, within the range of b_1 over which the demand curve is upward sloping, entail an externality that creates complementarity in investors' purchases: each investor's decision to buy more debt encourages others to buy more as well. The reason is as follows. At a given price q_0, an investor's buying up more debt increases the likelihood of default at the margin for all other investors. Since government debt is still the safest asset available to investors, this increase in aggregate risk encourages all other investors to purchase more government debt as well. This complementarity can lead to excessive level of demand for debt and too much aggregate risk, when z is high enough.

An underlying reason behind this demand externality is the assumption of atomistic investors, which implies that individual investors take price and the likelihood of default as given. This suggests that coordination among investors, or a large non-atomistic investor might eliminate equilibria with excessive government debt and aggregate risk. For example suppose that all investors decide to consolidate and are represented by one single large investor who takes into account the impact of her purchases on the likelihood of default. I assume that she still takes the price as given.[7] When her maximization has an interior solution, her demand function becomes:[8]

$$\hat{q}_0 = \frac{1-\pi_D}{a_H - \delta + b_1} + \frac{\pi_D z}{a_L + z b_1} + \underbrace{f(b_1) \ln\left(\frac{a_L + z b_1}{a_H - \delta + b_1}\right)}_{C}. \tag{10}$$

The new term C appearing in the above demand function captures the fact that the single large investor internalizes the individual investor's demand externality. To see this more precisely, one can split C into two terms as follows:

[7] In equilibrium, the price and default risk will be tightly related and jointly determined.

[8] $\hat{q}_0$ describes the demand curve only for certain values of b_1. See Appendix A for an elaborate characterization of the single large investor's demand.

$$f(b_1)\ln\left(\frac{a_L + zb_1}{a_H - \delta + b_1}\right) = \underbrace{f(b_1)\ln(z)}_{D} + \underbrace{f(b_1)\ln\left(\frac{\frac{a_L}{z} + b_1}{a_H - \delta + b_1}\right)}_{E}. \tag{11}$$

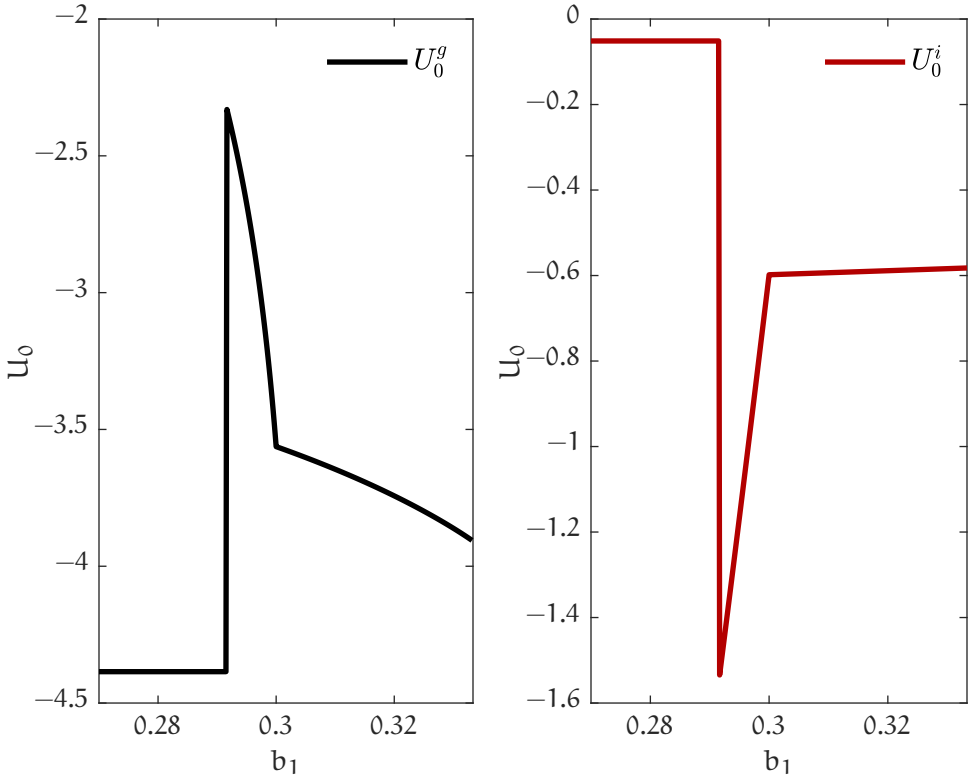


Figure 6: Welfare of the government and investors when there is a single large investor. Parameter values are $a_L = 0.3$, $a_H = 1.15$, $\delta = 0.2$, $z = 0.6$, $\tau\underline{y} = 0.2$, and $\tau\bar{y} = 0.3$.

The first term D captures the direct effect of default internalized by the single large investor. Since $z < 1$, default implies less resources for the inventors in the bad state, and the single investor's objective function adds a penalty to internalize the direct demand effect on the default and to lower b_1. The second term E captures the investors' demand externality internalized by the single large investor. Note that $E < 0$ if and only if $\frac{a_L}{z} < a_H - \delta$, implying that when government default lowers the return to other assets, there is an additional penalty for demanding a high amount of debt b_1. When $E < 0$, the demand curve for the single large investor always lies below that of the case of atomistic investors. In other words, $\hat{q}_0(b_1) < q_0(b_1)$, where $\hat{q}_0(b_1)$ is the price of government debt at which the single large investor is willing to purchase b_1.

Figure 7 shows ex-ante welfare of the government and the single large investor for different levels of b_1. The maximum of U_0^g and the minimum of U_0^i happen at the

same value of $\hat{b}_1 = 0.291$. Note that this value is less than the maximizing level of government debt in the case of atomistic investors, i.e., $\hat{b}_1 < b_1^{g-max} = 0.296$. Due to non-monotonicity of $\hat{q}_0$ with respect to b_1, the single large investor never chooses any value of $b_1 > 0$ below $\hat{b}_1$: $b_1 = 0$ gives a strictly higher utility to the single large investor than any value $b_1 \in (0, \hat{b}_1)$.[9] Moreover, a global planner cannot make any Pareto improvement by reducing or increasing $\hat{b}_1$, implying that the equilibrium is efficient.

5 Access to Other Safe Assets

In my benchmark model, I have assumed that investors do not have access to any other private or public safe assets. But, in the real world, investors may have access to more than one type of safe asset. If the second bond is risk free and that its fixed supply is equal to b^{rf*}, the demand curve for b_1 is given by Lemma 1, and its slope is given by 2, with $a_L^* = a_L + b^{rf*}$ and $a_H^* = a_H + b^{rf*}$. In order for the first government bond to remain safer than the project, one needs to have:

$$a_H^* - \delta > \frac{a_L^*}{z} \iff b^{rf*} < \frac{(a_H - \delta - \frac{a_L}{z})z}{1-z}.$$

Therefore, as long as b^{rf*} is not too large, the results in the previous sections hold. This condition is based on the assumption of fixed supply, but it can also be a useful approximation when the supply of the second government bond is inelastic enough.

But, why should access to other government bonds be limited? In the case of banks, there are various reasons why they might prefer or even be encouraged to hold own government debt. These include especial treatment of own government bonds by capital and liquidity regulations, central banks' liquidity operations, risk taking, and financial repression (Dell'Ariccia et al. (2018)).

For the rest of this section, I consider a more realistic extension of the benchmark model in which investors have access to two different government bonds, both of which are subject to default.

5.1 A Two-Country Model

Consider the benchmark model extended to include two countries issuing government debt. I use the same notation as in the benchmark model with hat for the second country, e.g., b_1 and $\hat{b}_1$ represent bonds issued by the two countries. For example, one can think

[9] See Appendix A for details.

of the two bonds as U.S. treasuries and the basket of sovereign bonds produced by the rest of the world. As in the benchmark model, there is a continuum of investors who have access to a real project with payoff a_H if the investor can pay for the costs of maintaining the project δ, and $a_L < a_H$ otherwise. Investors can invest in the two government bonds in order to cover δ and make sure the high return to their real project at $t = 1$.

I start by analyzing the case in which default on any of the two government bonds makes the investor unable to cover δ. In such equilibria, the investors solve the following maximization problem:

$$\begin{cases} U^i = \max_{c_0, b_1, \hat{b}_1} c_0 + \left\{ \pi_{D,\hat{ND}} \ln(a_L + z b_1 + \hat{b}_1) + \pi_{ND,\hat{D}} \ln(a_L + b_1 + \hat{z}\hat{b}_1) + \right. \\ \left. + \pi_{ND,\hat{ND}} \ln(a_H - \delta + b_1 + \hat{b}_1) + \pi_{D,\hat{D}} \ln(a_L + z b_1 + \hat{z}\hat{b}_1) \right\}, \\ \text{s.t.} \quad c_0 + q_0 b_1 + \hat{q}_0 \hat{b}_1 = y_0 . \end{cases}$$

D and $\hat{D}$ (ND and $\hat{ND}$) indicate default (no default) by the firs or second government at $t = 1$. The terms $\pi_{i,j}$ indicate the probability that the two government bonds are in state i and j, for $i \in \{D, ND\}$ and $j \in \{\hat{D}, \hat{ND}\}$, respectively.

The FOCs with respect to b_1 and $\hat{b}_1$ characterize the two demand curves:

$$\begin{cases} q_0 = \frac{z\pi_{D,\hat{ND}}}{a_L + z b_1 + \hat{b}_1} + \frac{\pi_{ND,\hat{D}}}{a_L + b_1 + \hat{z}\hat{b}_1} + \frac{\pi_{ND,\hat{ND}}}{a_H + b_1 + \hat{b}_1} + \frac{z\pi_{D,\hat{D}}}{a_L + z b_1 + \hat{z}\hat{b}_1}, \\ \hat{q}_0 = \frac{\pi_{D,\hat{ND}}}{a_L + z b_1 + \hat{b}_1} + \frac{\hat{z}\pi_{ND,\hat{D}}}{a_L + b_1 + \hat{z}\hat{b}_1} + \frac{\pi_{ND,\hat{ND}}}{a_H + b_1 + \hat{b}_1} + \frac{\hat{z}\pi_{D,\hat{D}}}{a_L + z b_1 + \hat{z}\hat{b}_1}. \end{cases} \tag{12}$$

To do numerical simulation, I set $\tau = \hat{\tau} = 1$ and assume that $(y_1, \hat{y}_1) \sim G$, where the cumulative distribution function (cdf) G is constructed using a Gaussian copula as follows:

$$G(y_1, \hat{y}_1) = \Phi_2(\phi_1^{-1}(F(y_1)), \phi_1^{-1}(F(\hat{y}_1))) . \tag{13}$$

F is assumed to be the cdf of a symmetric triangular distribution over $[\underline{y}, \bar{y}]$, as defined in the benchmark model with one government. ϕ_1^{-1} is the inverse of standard normal cdf, and Φ_2 is the cdf of the two dimensional normal distribution with the following mean and covariance matrix:

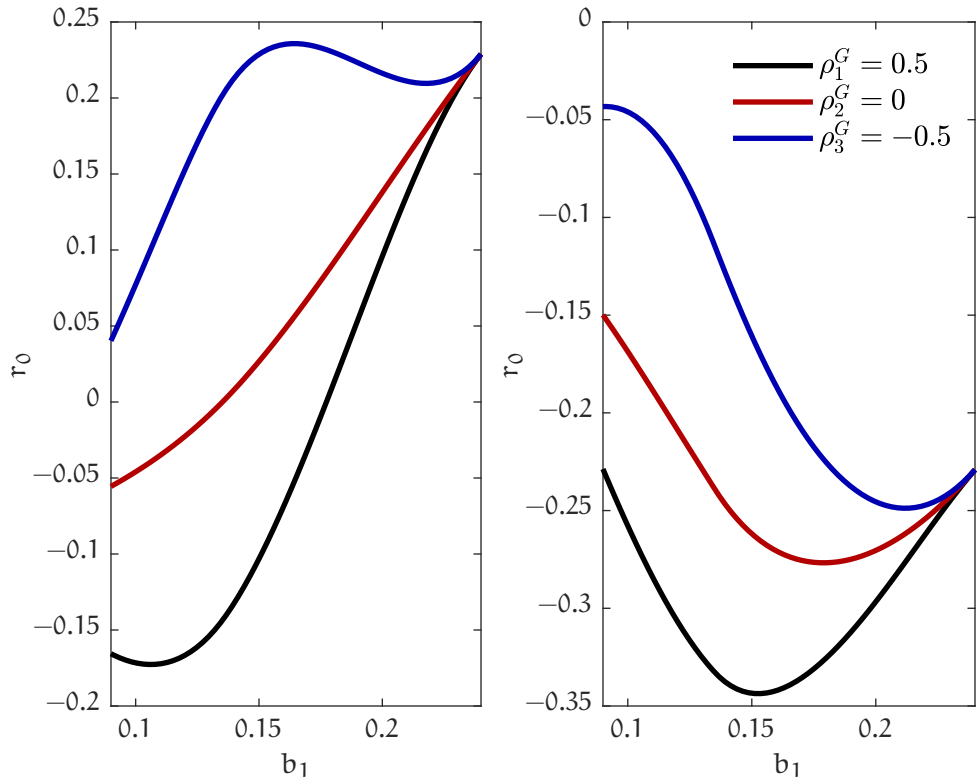


Figure 7: Demand for government debt for different values of correlation coefficient ρ^G, between y_1 and $\hat{y}_1$.

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

ρ is the correlation coefficient between the two variables in the two-dimensional normal distribution, i.e., $\phi_1^{-1}(F(y_1))$ and $\phi_1^{-1}(F(\hat{y}_1)$, which can range from -1 to 1. In all numerical simulations, ρ^G, the correlation coefficient of y_1 and $\hat{y}_1$, stays very close to ρ and has the same domain as ρ.

Note that G is constructed in such a way that the marginal distribution of both y_1 and $\hat{y}_1$ is F. In other words, when the second government debt is low enough and debt is paid back with certainty, the model looks similar to the benchmark model with an adjustment to the payoff of the real project, i.e., a_L and a_H.

5.2 Upward-Sloping Demand Curve in the Two-Country Case

Figure 7 is an illustration of the different behavior of demand for one government debt, b_1, given $\hat{b}_1 = 0.1$ and different values of correlation between the two countries income,

y_1 and $\hat{y}_1$. The two panels in the figure correspond to two different values of $\hat{b}_1$.

The black curve corresponds to a correlation coefficient of $\rho^G = 0.5$. When countries' incomes are significantly positively correlated, the default events become also significantly correlated for close values of b_1 and $\hat{b}_1$. This implies that, at least to a first order approximation, the resulting demand curve should look similar to the demand curve in the case of one single bond subject to a similar default risk profile. The demand curve in this case looks like a truncated version of the demand curve in the benchmark model, because the issuance by the second government $\hat{b}_1$ shifts the demand curve for b_1 to the right: the upward-sloping part of the demand curve starts at the beginning rather than the middle. Moreover the higher $\hat{b}_1$ implies a longer upward-sloping segment of the demand curve. Note that in this case investing in the second bond cannot provide a better hedge against aggregate risk.

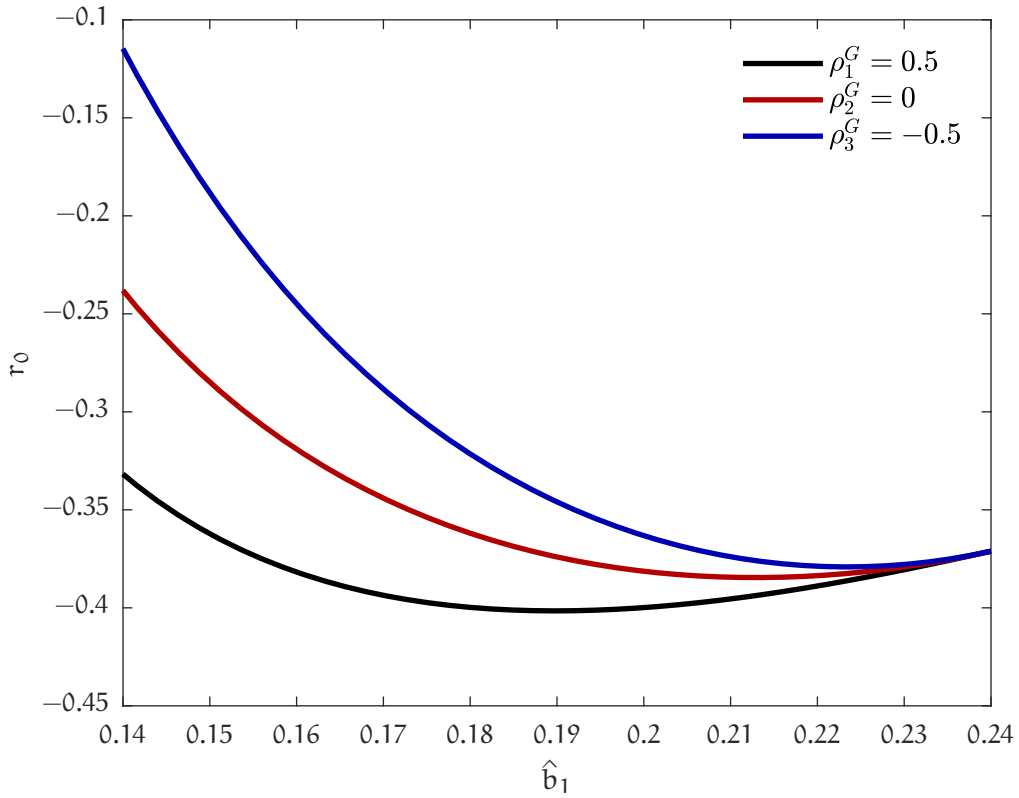


Figure 8: Demand for government debt for different values of correlation coefficient ρ^G, between y_1 and $\hat{y}_1$.

The blue curve in Figure 7 shows the demand schedule when countries' incomes are negatively correlated. When $\hat{b}_1 = 0.1$ (left panel), the demand curve resembles the benchmark model: yield is increasing in b_1 except in the middle segment resulting in two

extremums. In this case, and in contrast to the case of positive correlation, when b_1 is low and relatively close to $\hat{b}_1 = 0.1$, $\hat{b}_1$ and b_1 provide some insurance against each other's default because they are of comparable size and the incomes are negatively correlated. Therefore, when b_1 is low, an increasing in b_1 does not increase aggregate risk and demand for insurance as in the case of positive correlation, leading to a downward-sloping demand curve.

But, when $\hat{b}_1 = 0.14$ is significantly larger than b_1, b_1 can not provide enough insurance against $\hat{b}_1$ default. This is because, although the there is a negative correlation between the incidence of default by the two governments, loss given default is much larger for one government relative to the other, i.e., $(1-z)\hat{b}_1 > (1-z)b_1$. Since the government issuing b_1 is already exposed to significant aggregate risk due to insufficient insurance provided by the negatively correlated incomes, a marginal increase in b_1, when b_1 is low, can significantly increase exposure to aggregate risk at the margin. This implies that the demand curve is upward-sloping from the start within the same domain.

Finally, the red curve in Figure 7 shows the demand schedule when there is no correlation between countries incomes. In this case, the demand curve is downward-sloping for low levels of $\hat{b}_1$ but become upward-sloping at the beginning when $\hat{b}_1$ is higher.

To get further insight into the different behaviour of the demand curve under positive and negative correlations, one can write the bond price as follows:

$$q_0 = \pi_{\hat{ND}}\left\{\frac{z\pi_{D|\hat{ND}}}{a_L + zb_1 + \hat{b}_1} + \frac{\pi_{ND|\hat{ND}}}{a_H + b_1 + \hat{b}_1}\right\} + \pi_{\hat{D}}\left\{\frac{\pi_{ND|\hat{D}}}{a_L + b_1 + \hat{z}\hat{b}_1} + \frac{z\pi_{D|\hat{D}}}{a_L + zb_1 + \hat{z}\hat{b}_1}\right\}.$$

It suffices to explain the terms in the first braces above. $\pi_{D|\hat{ND}}$ and $\pi_{ND|\hat{ND}}$ are conditional probabilities of b_1 being and not being in default, if $\hat{b}_1$ is not in default. Moreover, $\pi_{\hat{ND}}$ is the unconditional probability of $\hat{b}_1$ not being in default.

Note that both $\pi_{\hat{ND}}$ and $\pi_{\hat{D}}$ are unconditional probabilities and therefore independent of the value of b_1. The derivative of q_0 with respect to b_1 include four terms. I collect these four terms in two different expressions, η_1 and η_2:

$$\frac{\partial q_0}{\partial b_1} = \eta_1 + \eta_2 .$$

The first expression, η_1, include the following two terms, each stemming from one of

the expressions in braces, are:

$$\begin{cases} \eta_1 = \pi_{\widehat{ND}} \left\{ \frac{z}{a_L + z b_1 + \widehat{b}_1} \frac{\partial \pi_{D|\widehat{ND}}}{\partial b_1} + \frac{1}{a_H + b_1 + \widehat{b}_1} \frac{\partial \pi_{ND|\widehat{ND}}}{\partial b_1} \right\} + \\ \\ \pi_{\widehat{D}} \left\{ \frac{z}{a_L + z b_1 + \widehat{z} \widehat{b}_1} \frac{\partial \pi_{D|\widehat{D}}}{\partial b_1} + \frac{1}{a_L + b_1 + \widehat{z} \widehat{b}_1} \frac{\partial \pi_{ND|\widehat{D}}}{\partial b_1} \right\}. \end{cases} \tag{14}$$

The two terms in 14 reflect the *change in aggregate risk*. Moreover, while the first term in 14 is positive as long as $\frac{a_L + \widehat{b}_1}{a_H + \widehat{b}_1} < z$, the second term is always negative. Therefore, the demand curve can become upward-sloping only if the size of the first (positive) term is large enough. Taking the derivatives of conditional cumulative distribution functions in 14, one can rewrite the 14 as:

$$\begin{cases} \pi_{\widehat{ND}} \left\{ \frac{z}{a_L + z b_1 + \widehat{b}_1} - \frac{1}{a_H + b_1 + \widehat{b}_1} \right\} g(b_1 \,|\, \widehat{b}_1 \leqslant \hat{y}_1) + \\ \\ + \pi_{\widehat{D}} \left\{ \frac{z}{a_L + z b_1 + \widehat{z} \widehat{b}_1} - \frac{1}{a_L + b_1 + \widehat{z} \widehat{b}_1} \right\} g(b_1 \,|\, \widehat{b}_1 \geqslant \hat{y}_1). \end{cases} \tag{15}$$

I have used in the above the fact that:

$$\begin{cases} \eta_1 = \frac{\partial \pi_{D|\widehat{ND}}}{\partial b_1} = -\frac{\partial \pi_{ND|\widehat{ND}}}{\partial b_1} = g(b_1 \,|\, \widehat{b}_1 \leqslant \hat{y}_1), \\ \\ \frac{\partial \pi_{D|\widehat{D}}}{\partial b_1} = -\frac{\partial \pi_{ND|\widehat{D}}}{\partial b_1} = g(b_1 \,|\, \widehat{b}_1 \geqslant \hat{y}_1). \end{cases} \tag{16}$$

where $g(b_1 \,|\, \widehat{b}_1 \geqslant \hat{y}_1)$ and $g(b_1 \,|\, \widehat{b}_1 \leqslant \hat{y}_1)$ are the conditional probability distribution of y_1 at $y_1 = b_1$ given that $\widehat{b}_1$ is or is not in default. When $\frac{a_L + \widehat{b}_1}{a_H + \widehat{b}_1} < z$, the relative size of the two positive and negative terms in 15 depend partly on $\pi_{\widehat{ND}} g(b_1 \,|\, \widehat{b}_1 \leqslant \hat{y}_1)$ and $\pi_{\widehat{D}} g(b_1 \,|\, \widehat{b}_1 \geqslant \hat{y}_1)$. The second expression, η_2, including the last two terms in $\frac{\partial q_0}{\partial b_1}$ is the derivative of marginal utilities in different future states holding the conditional probabilities of default constant. As in the benchmark model, these other terms are reflections of the *diminishing expected marginal utility*, and therefore are all negative:

$$\begin{cases} \eta_2 = -\pi_{\widehat{ND}} \left\{ \frac{z^2 \pi_{D|\widehat{ND}}}{(a_L + z b_1 + \widehat{b}_1)^2} + \frac{\pi_{ND|\widehat{ND}}}{(a_H + b_1 + \widehat{b}_1)^2} \right\} - \\ \\ -\pi_{\widehat{D}} \left\{ \frac{\pi_{ND|\widehat{D}}}{(a_L + b_1 + \widehat{z} \widehat{b}_1)^2} + \frac{z^2 \pi_{D|\widehat{D}}}{(a_L + z b_1 + \widehat{z} \widehat{b}_1)^2} \right\}. \end{cases} \tag{17}$$

Consider the case of $\widehat{b}_1 = 0.1$. In this case, one has $\pi_{\widehat{ND}} = 0.78$ and $\pi_{\widehat{D}} = 0.22$.

Moreover, Figure 9 illustrates the probability density functions for the default and no default in $\hat{b}_1$. It is clear from Figure 9 that $g(b_1 \,|\, \hat{b}_1 \leqslant \hat{y}_1)$ is lower and $g(b_1 \,|\, \hat{b}_1 \geqslant \hat{y}_1)$ is higher when $\rho^G = 0.5$. This implies that η_1 is larger under the negatively correlated incomes, i.e., $\rho^G = -0.5$. This is intuitive because, around $b_1 = 0.1$, the change and level of probability of default in b_1 should be higher under as the negative correlation between countries incomes, when $\hat{b}_1$ is not in default, and consequently the *change in aggregate risk* effect is more significant under a negative correlation. But, why is q_0 increasing around $b_1 = 0.1$ when the correlation is positive? The reason is that η_2, reflecting the *diminishing expected marginal utility*, is higher under $\rho^G = 0.5$ by a large enough margin that dominates the *change in aggregate risk*.

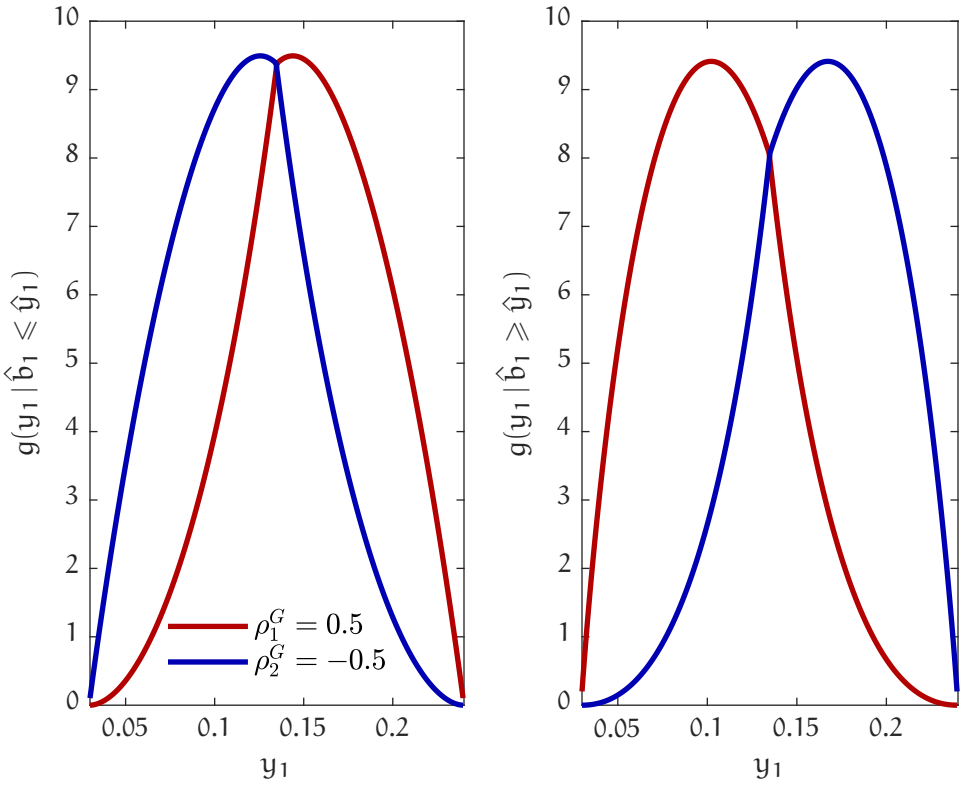


Figure 9: The probability density function of y_1 when $\hat{b}_1 = 0.1$, conditional on the government issuing $\hat{b}_1$ is (right panel) and is not (left panel) in default, for different values of ρ^G.

5.3 Complementarity in the Supply of Government Debts

An important feature of the demand curve derived in 12 is that an increase in $\hat{b}_1$ (b_1) can increase q_0 ($\hat{q}_0$). In other words, one can have:

$$\begin{cases} \frac{\partial q_0}{\partial \hat{b}_1} > 0\,, \\ \\ \frac{\partial \hat{q}_0}{\partial b_1} > 0\,. \end{cases} \tag{18}$$

Figure 8 shows the above to be true for the same parameter values used in Figure 7. All three yield curves, corresponding to different correlations between countries' incomes, are downward-sloping at the beginning. This implies that when one country increases b_1, there is more incentive for the other country to issue more $\hat{b}_1$, simply because $\hat{b}_1$ can be sold at a higher price than before.[10] This can lead to complementarity in the supply of government debt by the two countries: both countries increase or reduce issuance simultaneously.

Jiang and Richmond (2025) provide evidence for the existence of what they call the *global fiscal cycle*. They document that more than 50% of the variation in debt/GDP of the 8 developed countries which are considered potential reserve asset issuers can be explained by a single factor.[11] This might be the case, for instance, due to correlated business cycles, which can be the result of significant bilateral trade and financial linkages between the issuing countries. In such a case, default risk can become correlated since large increases in government debts become synchronized, raising the probability of default for all issuing countries. Moreover, correlated shocks, as a factor underlying synchronized fiscal expansions, can itself make the fiscal risk correlated across the issuing countries. For instance, one can see this synchronization during the Financial Crisis of 2007-2008 and the ongoing COVID-19 recession.

5.4 Demand Elasticities

A recent body of works emphasize that there is significant heterogeneity in demand elasticities across different institutional investors and that the composition of demand for government debt matters for how yields respond to a change in the supply (Fang et al. (2025) and Chaudhary et al. (2025)). My results are complementary to these works as this paper highlights that, even within a given investor type, the demand elasticities might change as the balance sheet exposure to sovereign default risk can evolve over time.

An emerging literature attempts to estimate the demand elasticities for sovereign debt using the framework introduced by Koijen and Yogo (2019) (Koijen and Yogo (2025), Fang

[10] Though, the governments in the two-country model are not maximizing welfare functions as in Section 4, it is straightforward to introduce utility maximizing governments.

[11] They include Canada, France, Germany, Japan, Netherlands, Switzerland, United Kingdom, and United States.

et al. (2025), Eren et al. (2025), and Chaudhary et al. (2025)). There are two assumptions underlying these demand estimations. First, they assume an endowment economy, meaning that fiscal policy does not affect the returns to other assets held by the marginal investor. And second, they use a within-asset-class iso-elastic demand specification in the form of a logit (and nested logit between different asset classes) model.

Below, I elaborate on the connections between my results and each of the the aforementioned assumptions. I argue that these assumptions might be impeding a more comprehensive understanding of the nature of the demand for government debt. My key results hold precisely because government default affects the returns to other assets held by the marginal investors, thereby exposing them to more aggregate risk while making other alternative investments even less appealing. These results, encapsulated in Proposition 2, Lemma 5, and the numerical simulations in the two-country extension of the model imply that the demand elasticity of government debt that serves as a safe asset might vary significantly with the level of debt. Moreover, the demand elasticities can be negative for a certain range of government debt levels.

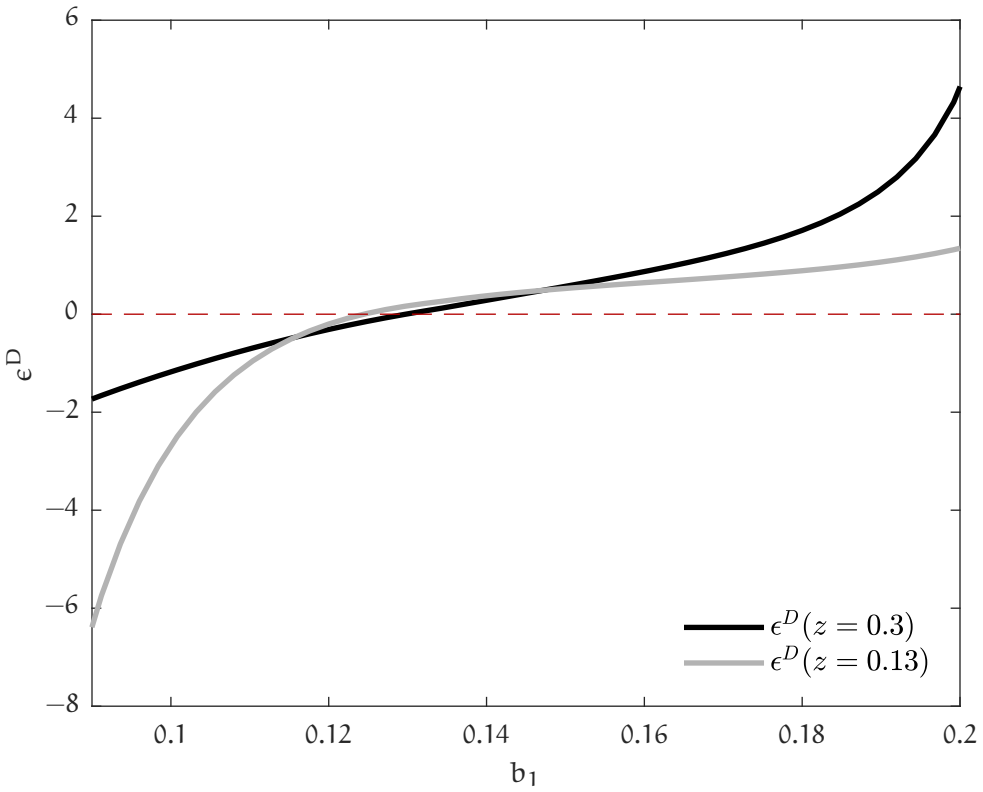


Figure 10: Partial elasticities for different values of b_1 and two different values of z. Other parameter values are $a_H = 0.9$, $\delta = 0.12$, $\tau = 1$, $\underline{y} = 0.03$, $\bar{y} = 0.24$, $\hat{b}_1 = 0.14$, and $\rho^G = 0.5$.

Koijen and Yogo (2025) find a small elasticity of 3.2 for long-term debt by averaging across countries and over time. Their sample covers asset holdings during 2003-2020, a period that has seen a significant surge in government borrowing across the globe. If demand elasticities can vary significantly with the level of debt, as this paper suggests, such variation, with the possibility that the demand elasticity might even be negative, can make an average estimate less useful in projecting policy impacts.

Figure 10 illustrates the own demand elasticities for a range of b_1 values and two different levels of z.[12] The black curve corresponds to the highest value of z, e.g., an advanced economy such as U.S. producing a globally traded safe debt. In this case, the demand curve is upward sloping, and demand elasticities are negative at the beginning. The gray curve corresponds to the case in which government debt is the safest asset available to the investors, but the demand curve is downward sloping at all values of b_1, i.e., an advance economy whose debt is the safest alternative but not globally traded. Nonetheless, the elasticities are negative for lower values of b_1, before turning positive. Rather surprisingly, the negative elasticities are larger, in absolute value, for the gray curve compared to the black curve. But, this order is reversed when one considers the equilibrium response as discussed further below.

The range of elasticities in Figure 10 are comparable to Koijen and Yogo (2025) in absolute value terms. However, there are two important differences: the demand elasticities in Figure 10 are decreasing in b_1 and they are negative for low values of b_1. What are the sources of these differences? Figure **??** shows the relationship between relative demand in fraction of shares spent on the two countries' debt and their relative price. In Koijen and Yogo (2025), this curve is assumed to have a constant slope in the entire range of the relative prices. Moreover, given the assumption of constant slope, Koijen and Yogo (2025) find the slope to be less than one in all their estimations. In contrast, Figure 10 suggests that there is significant variation in the slope of the relative demand curve expressed in fraction of shares. In fact, the slope of the relative shares is higher than one both at the beginning and at the end of the relative price range (See Appendix D). The variation in the slope of relative demand curve becomes more extreme for lower values of z: the slope can become negative for a large fraction of relative price values in the middle. An important question is whether imposing a logit structure on demand can mask this type of variations in the slope of demand curve, and consequently in actual demand elasticities.

It is important to note that, while the elasticities can be negative and the demand curve might be upward sloping for certain values of debt, the relative price of the two countries' debt, $\frac{q_0}{\hat{q}_0}$, is a decreasing function of the relative supply, $\frac{b_1}{\hat{b}_1}$. Figure 12 is an illustration of

[12] See Appendix C for more details on the calculation of demand elasticities in Figure 10.

this fact. This is consistent with the stylized facts presented in Koijen and Yogo (2025). In other words, the relative demand in fraction of quantities is always downward sloping, even when the absolute demand curves is upward sloping for certain values of b_1.

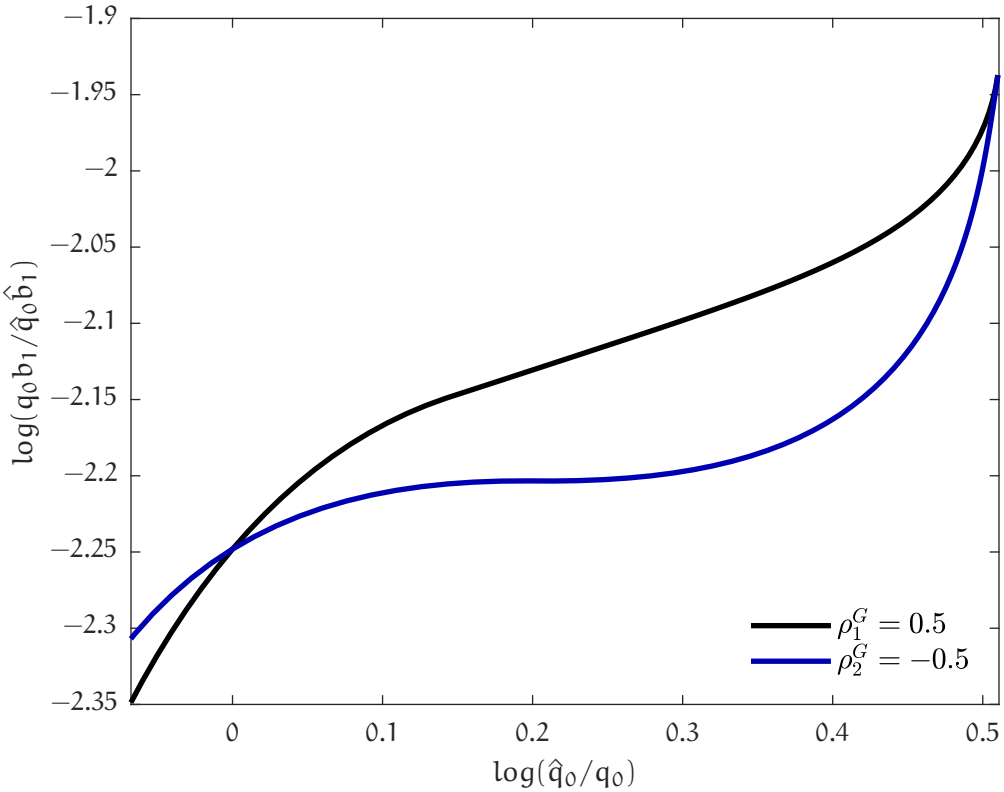


Figure 11: Relative shares of the two countries debt, $\log(\frac{q_0 b_1}{\hat{q}_0\hat{b}_1})$, as a function of their relative prices, $\log(\frac{\hat{q}_0}{q_0})$, for different values of correlation coefficient ρ^G, between y_1 and $\hat{y}_1$. Other parameter values are $\alpha_H = 0.9$, $z = \hat{z} = 0.3$, $\delta = 0.12$, $\tau = 1$, $\underline{y} = 0.03$, and $\bar{y} = 0.24$.

5.4.1 Total Elasticities and Multipliers

Figure 13 shows the price impact or multipliers, which are more important than the (partial) demand elasticities discussed above in many policy issues. It implies a much more dramatic variation in the *total* elasticity, i.e., the equilibrium response of quantities to change in the price due to an exogenous supply shift, for different values of z.[13] I choose to illustrate the multipliers because the potential infinities created by the upward-sloping demand can make the elasticity curves difficult to understand. The kinks in the curves

[13] See Chaudhary et al. (2025) and Appendix C for further details.

are the results of the triangular distribution, and would be smoothed out with a more smooth distribution of future income tax. The three curves can be thought of as total demand elasticities corresponding to different issuer countries. For the highest value of z, i.e., the black curve, the total demand elasticity is negative at the beginning before turning positive around $b_1 = 0.15$. The Great Financial Crisis might have been such an inflection point. Note that the elasticity can be arbitrarily large in absolute value close to $b_1 = 0.15$. Chaudhary et al. (2025) find that demand elasticities for treasuries have become substantially smaller after the Great Financial Crisis.[14]

At the other end, the green curve with $z = 0.06$ illustrates the price impact when government debt is riskier than the real assets held by the marginal investors, e.g., an emerging market bond. In this case, total elasticities are always positive and less than 3.1 in absolute value, i.e. $\epsilon^D > -3.1$.

For the green curve, the range of total elasticity values over the entire domain is limited. This suggests that averaging the elasticities over time, as supply increases, might not lead to dramatic errors in projections. But, that is not true for the black curve because the range of possible total elasticities are unbounded and it includes negative values for a significant fraction of b_1 values. The gray curve lies somewhere in between the green and the black curves. Given how the range of total elasticities depend on z in Figure 13, an average across different countries with different levels of government debt's safety can mask a great deal of heterogeneity in elasticities.

Koijen and Yogo (2025) note that demand elasticity might depend on the investing country. The analysis in this section adds a different dimension. It suggests that one might need an understanding of the elasticity curve, rather than a point estimate, to have a more accurate projection of the effect on the yield in the medium and long term.

6 Extension with Old Debt and Liabilities

In this section, I show that when the investors have significant amount of liabilities on their balance sheets the upward-sloping demand curve for government bonds can result even if the returns to the real project in the default and no-default states are the same. To this end, I introduce an extension of the benchmark model with two key features, namely the existence of *old* government debt, and outstanding liabilities from the past.

[14] They use the notion of *total elasticity* which includes the equilibrium effects of a change in treasury price through its impact on other asset prices. In my stylized model, there are no other traded assets beside government bonds and therefore the notion of elasticity has a different meaning.

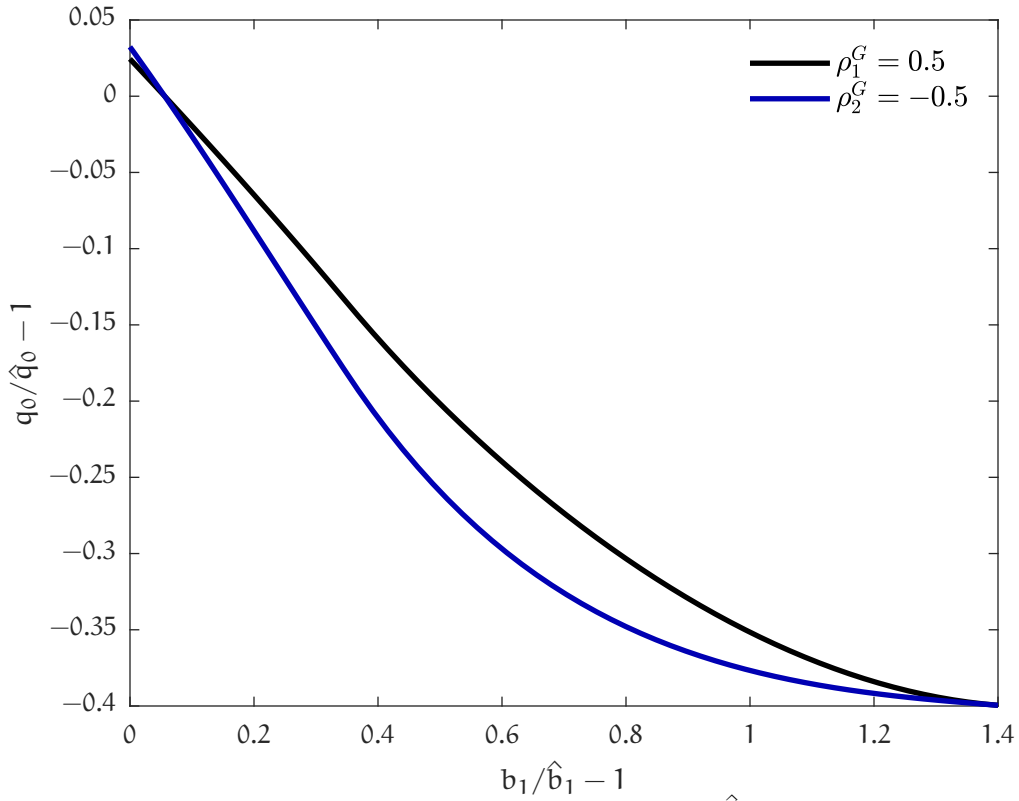


Figure 12: Relative prices of the two countries debt, $\frac{\hat{q}_0}{q_0} - 1$, as a function of their relative supply, $\frac{b_1}{\hat{b}_1} - 1$, for different values of correlation coefficient ρ^G, between y_1 and $\hat{y}_1$. Other parameter values are $a_H = 0.9$, $z = \hat{z} = 0.3$, $\delta = 0.12$, $\tau = 1$, $\underline{y} = 0.03$, and $\bar{y} = 0.24$.

One natural interpretation of the investor is a bank with holdings of government debt, other investments in real assets such as loans, and significant amounts of deposit liabilities on its balance sheet.

6.1 Balance-Sheet Effects

More specifically, consider the model from the previous sections with the same agents. The investors come into period $t = 0$ with an outstanding amount of government bonds $\tilde{b}_1$, some investment in real assets, and an outstanding level of liabilities to the household d_1, all maturing at $t = 1$. Assume that the real investment pays $a_H > 0$ in the no-default state, and $a_L \leqslant a_H$ in the case of government default. I implicitly assume the microfoundation used in Section 3 and, just to slightly simplify the notation, a_H in this section is the equivalent of $a_H - \delta$ in the benchmark model, i.e., the value of a_H is net of the maintenance costs δ. One can think of $\tilde{b}_1$ as long-term bonds purchased in a previous period, which is proportional to what Asonuma et al. (2015) label home bias. d_1 captures

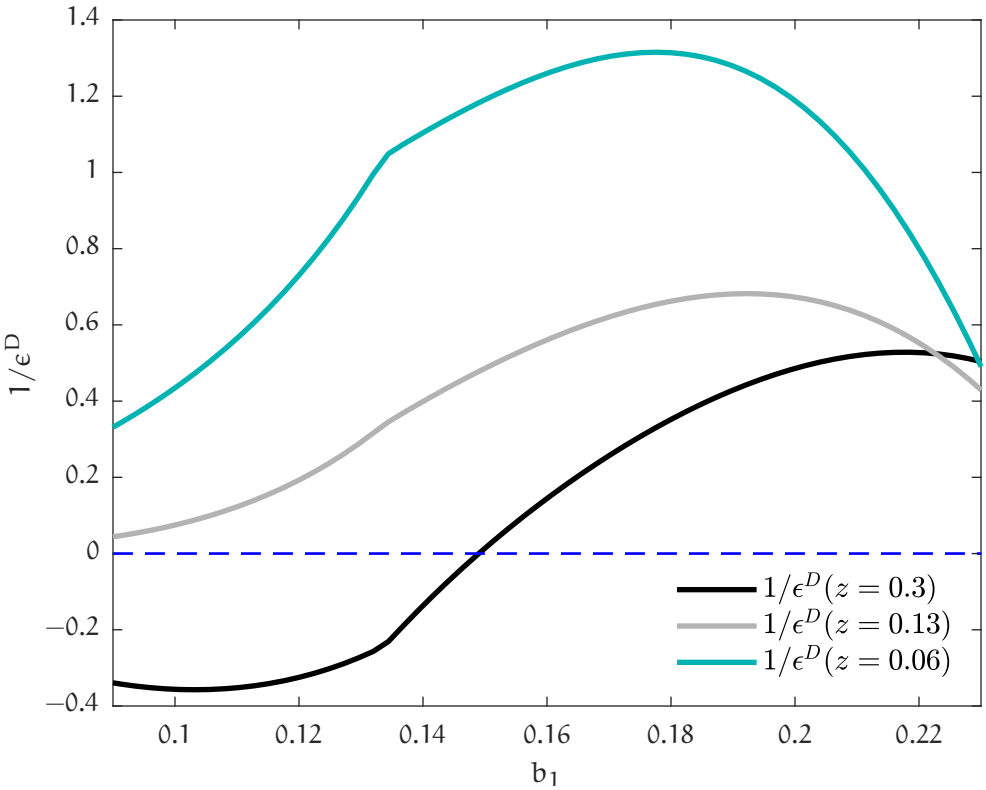


Figure 13: Inverse of total demand elasticity $1/\epsilon^D$ for different values of z. Other parameter values are $a_L = 0.03$, $a_H = 0.9$, $\hat{z} = 0.3$, $\delta = 0.12$, $\tau = 1$, $\underline{y} = 0.03$, $\bar{y} = 0.24$, $\hat{b}_1 = 0.14$, and $\rho^G = 0.5$.

different types of funding used by banks and shadow banks, including deposits, repo, and commercial papers.

I assume that a government default affects the payments of both the old and new bonds alike. Suppose that the default value of the outstanding bonds is enough to pay the outstanding liabilities, or $d_1 < z\tilde{b}_1$. As before, the future tax receipts are distributed according to a cumulative density F, which is symmetric and triangular with the domain $[\underline{b}, \overline{b}]$.

As before, the FOC and the probability of default as a function of $b_1 + \tilde{b}_1$ yield the bond price in equilibrium:

$$q_0(\tilde{b}_1, d_1; b_1) = \frac{1 - F(\tilde{b}_1 + b_1)}{(a_H + \tilde{b}_1 - d_1) + b_1} + \frac{zF(\tilde{b}_1 + b_1)}{(a_L + z\tilde{b}_1 - d_1) + zb_1}. \tag{19}$$

Note that if $d_1 = 0$, 19 is the same as 1. The following corollary highlights how d_1 can make an upward-sloping demand curve more likely:

Corollary 2. *A necessary condition for the existence of an (partially) upward-sloping demand*

curve is to have:

$$\frac{a_L - d_1}{a_H - d_1} < z\,.$$

This is an slightly modified version of the condition in Proposition 1. Given values $a_L < a_H$, a high enough d_1 can ensure that the condition in Corollary 2 hold. Importantly, this implies that the demand can become upward-sloping even when the real asset is safer than the government deb, i.e., $\frac{a_L}{a_H} > z$, if investors' liabilities are large. One can think of the the denominators in 19 as investor's future equity, i.e., $e_H = (a_H + \tilde{b}_1 - d_1) + b_1$. Given a fixed level of e_H, a higher d_1 corresponds to a higher leverage.

$\tilde{b}_1$ and b_1, enter the price equation as a sum, which facilitates the following comparative statics:

Proposition 4. *Suppose* $a_L = a_H = a$, *and* $d_1 - a \leqslant \frac{z\underline{b}}{2}$. *And, let* $\omega_0 = q_0(\tilde{b}_1 + b_1)$. *Then,* $\frac{\partial q_0}{\partial d_1} > 0$ *and if* d_1 *satisfies:*

$$\frac{1-z}{z}(d_1 - a)\left\{\frac{1}{\overline{b} - \underline{b}} - \frac{1}{\overline{b} + (1-2z)\underline{b}}\right\} \geqslant 1\,.$$

for large enough $\tilde{b}_1 + b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$, *one also has* $\frac{\partial q_0}{\partial \omega_0} > 0$.

In other words, large enough values of d_1 and $\tilde{b}_1 + b_1$ imply an upward-sloping demand curve. One case, in which the condition in Proposition 4 is satisfied, is when $\underline{b}$ is large enough. The intuition for $\frac{\partial q_0}{\partial \omega_0} > 0$ is similar to the previous sections. The effects of an increase in $b_1 + \tilde{b}_1$ are twofold. On the one hand, such an increase yields more consumption in both future states. This effect depresses the price because of a lower expected marginal utility of an extra unit of bond. On the other hand, an increase in $b_1 + \tilde{b}_1$ creates more risk of default, and aggregate risk. This latter effect, similar to the previous case, creates more demand for insurance. Whenever the second effect is dominant, an increase in bond holding increases the price of new bonds. Finally, note that $\frac{\partial q_0}{\partial \tilde{b}_1 + b_1} > 0$ implies $\frac{\partial q_0}{\partial \omega_0} > 0$, since:

$$\frac{\partial q_0}{\partial \omega_0} = \frac{\frac{\partial q_0}{\partial \tilde{b}_1 + b_1}}{\frac{\partial q_0}{\partial \tilde{b}_1 + b_1}(\tilde{b}_1 + b_1) + q_0}\,.$$

Importantly, an upward-sloping demand curve can result even when the real project

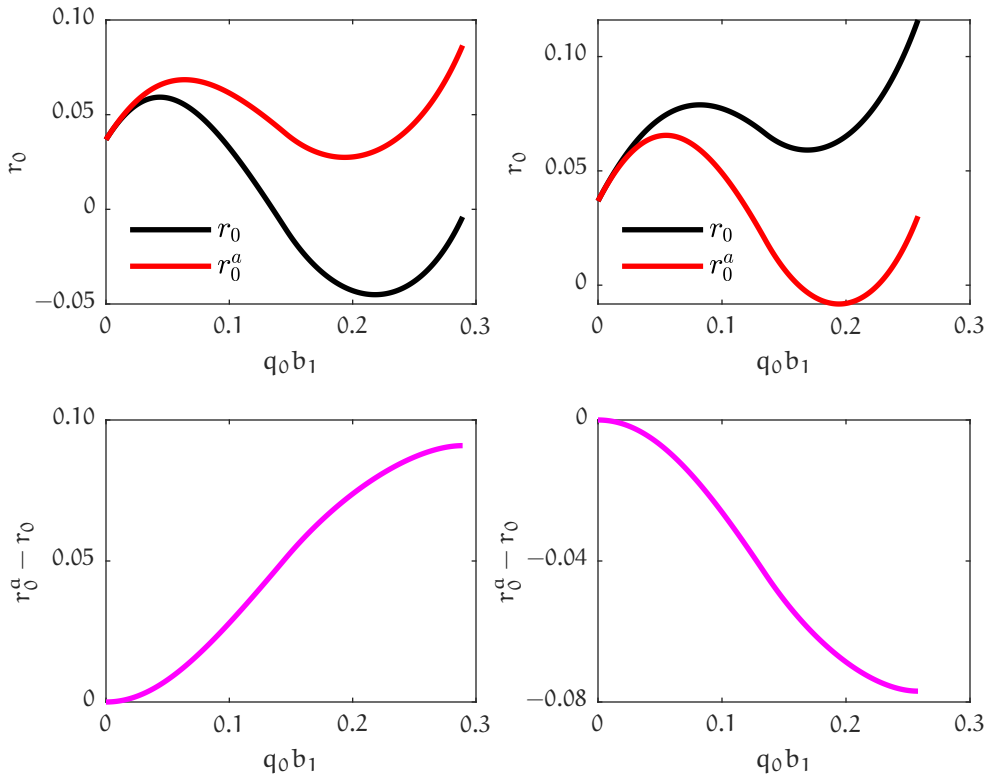


Figure 14: Yield on government bond $r_1 = \frac{1}{q_0} - 1$, yield on real asset $r_1^a = \frac{1}{q_0^a} - 1$, and the spread $r_1^a - r_1$ are illustrated as a function of $q_0 b_1$. $\frac{a_L}{a_H} = 0.55$ for the left panels and $\frac{a_L}{a_H} = 0.65$ for the right panels. The rest of parameter values are $a_H = 0.72$, $z = 0.6$, $\tau \underline{y} = 0.72$, and $\tau \bar{y} = 1.01$.

is truly safe, i.e., $a_L = a_H = a$. The reason is as follows. Unlike the benchmark model, the recovery rate of the past investments is $\frac{z\tilde{b}_1 - (d_1 - a)}{\tilde{b}_1 - (d_1 - a))}$. When $d_1 > a$, the recovery rate of past investment is strictly lower than that of the new government bonds, i.e., $\frac{z\tilde{b}_1 - (d_1 - a)}{\tilde{b}_1 - (d_1 - a))} < z$. This implies that returns to new government bonds is safer than that of the past investment portfolio.

The fact that $b_1 + \tilde{b}_1$ should be high enough in Proposition 4, implies that a high enough level of old government debt can make an upward-sloping demand curve more likely. In such a case, and as before, a higher level of b_1 leads to lower real interest rate (higher q_0) and higher aggregate default risk.

Note that the case $a_H = a_L = a$ is an extreme case, in which the investor has access to a truly safe asset. But in general, a similar result is expected to hold when $d_1 - a_L$ is large enough. Moreover, similar to the benchmark model, in order to have an upward-sloping demand curve z must be above a certain threshold.

6.2 Risk Premium

How does an increase in borrowing affect the risk premium, i.e., price of bond relative to the real asset? Suppose that investors can buy and sell shares in their projects to other investors at $t = 0$. Each share of any given project is a promise to deliver one unit of consumption in the no-default state and $\frac{a_L}{a_H}$ units of consumption if government defaults. Note that there are a total of a_H of shares for any project. Since all projects are the same, investors can buy a share of the projects pool. Hence, an investor solves:

$$\begin{cases} U = \max_{c_0,\theta_1,b_1} c_0 + \left\{(1-\pi_D)\ln(\theta_1 - d_1 + b_1 + \tilde{b}_1) + \pi_D \ln(\frac{a_L}{a_H}\theta_1 - d_1 + z(b_1+\tilde{b}_1))\right\}, \\ \\ \text{s.t.} \quad c_0 + q_0^a\theta_1 + q_0 b_1 = y_0 + q_0^a a_H\,. \end{cases}$$

q_0^a denotes the market price of the real asset on the investor's balance sheet. Using the FOCs of the above problem, one obtains:

$$q_0^a(\tilde{b}_1, d_1; b_1) = \frac{1 - F(\tilde{b}_1 + b_1)}{(a_H + \tilde{b}_1 - d_1) + b_1} + \frac{\frac{a_L}{a_H}F(\tilde{b}_1 + b_1)}{(a_L + z\tilde{b}_1 - d_1) + zb_1}\,. \tag{20}$$

In general, the risk premium can increase or decrease with respect to an increase in b_1. But, when $d_1 = 0$ one can be more specific under upward-sloping demand curve:

Lemma 7. *Suppose that $d_1 = 0$. Then, $\frac{q_0}{q_0^a}$ is increasing in b_1, if $\frac{\partial q_0}{\partial b_1} \geqslant 0$.*

Without any outstanding liabilities, an increase in default risk unambiguously increases the risk premium when the demand for bond is upward sloping. This is intuitive because, facing with a higher level of default risk, investors value safety even more. This causes a shift in their demand for bonds relative to the real asset, which increases their relative price, $\frac{q_0}{q_0^a}$, in equilibrium.

But, Lemma 7 does not hold when $d_1 > 0$. Figure 14 illustrates an example. The two panels on the left correspond to the value $\frac{a_L}{a_H} = 0.55$. The fact that $\frac{a_L}{a_H} < z$ implies that the return to real asset is riskier than the return to government bond. Hence, due to the same mechanism explained in the case of $d_1 = 0$, the yield on the real asset (inverse of its price) falls more slowly than that of the government bond. In other words, the spread in this case increases with more government borrowing. In contrast, the two panels on the right side of Figure 14 correspond to the value of $\frac{a_L}{a_H} = 0.65$, which is larger than z. In this case, the spread is decreasing in government borrowing as a higher default risk increases the demand for the real asset more than the government bond. But, despite the fact that the real asset is safer, the demand curve for government bond is upward sloping.

Krishnamurthy and Vissing-Jorgensen (2012) find that the spread between AAA-rated corporate bonds and U.S. treasury is negatively correlated with the debt to GDP ratio. They show that both the safety and liquidly premium of U.S. Treasuries contribute to the movements in this spread. However, Liu (2019) finds that an increase in U.S. government debt predicts higher risk premia in equity and bond markets. Given that $\frac{a_L}{a_H} < z$ seems to be more plausible in the case of U.S., the behavior of risk premium in this model is consistent with Liu (2019).

7 Robustness

In this section I show that the key results of the benchmark model are robust to two of the assumptions I have made to simplify the analysis. The first assumption in the benchmark model is that households' income does not depend on the real asset's payoffs. And the second assumption used to simply the benchmark model is that investors' preferences are quasilinear. I relax both of these assumptions in the following subsections.

7.1 Endogenous Taxable Income

In the stylized model discussed in Section 3, households' income that was subject to taxation is exogenous to realizations of the real asset's payoffs, i.e., a_L and a_H. But, when at least some of the investors are domestic this simplification might not hold. In this section, I provide a variation of the model where the realization of households' income is affected by the return to the real asset held by the investors, e.g., banks.

Consider a similar model as in Section 3, with one important difference in the structure of the payoffs to the real asset. Suppose that the real asset held by the investor, e.g., bank, is a loan with a face value of a_H to the household who runs a project, e.g., firm. The payoff to this project is a random variable $y \in [y_L, y_H]$ which can be taxed at a maximum rate of τ. I assume τy is distributed with a symmetric triangular density f over $[\tau y_L, \tau y_H]$. When the after-tax payoff to the project exceeds a_H, the investor is paid a_H and the rest is consumed by the household.

If the after-tax return to the project is less than a_H, all of project's return is accrued to the investor. In this case, the investor can consume the after-tax return to the project if there is no default on government bonds. But, I assume that if government defaults on its debt the investor payment is reduced to a constant amount a_L, where $a_L < (1-\tau)y_L$. This can happen, for instance, because government default puts the investor under financial distress and forces the investor to liquidate the loan at a large discount. While the precise

mechanism can be microfounded in an straightforward way, I abstract from these details to keep the analysis simple. In sum, the investor's payoff structure is as follows:

$$\begin{cases} a_H + b_1 & \text{if } \frac{\tau a_H}{1-\tau} \leqslant \tau y \\ y & \text{if } b_1 \leqslant \tau y < \frac{\tau a_H}{1-\tau} \\ a_L + z b_1 & \text{if } \tau y < b_1 \end{cases}$$

Given the above payoff structure, I can show the following:

LEMMA 8. *The demand curve for government bonds is upward sloping, i.e.,* $\frac{\partial q_0}{\partial b_1} > 0$*, for high enough values of* b_1*, if:*

$$\begin{cases} \frac{a_L}{a_H} \leqslant z\,, \\ \frac{a_L}{z} < \frac{(1-2\tau)(y_L+y_H)}{2} \\ \frac{\tau}{1-\tau} < \frac{y_L}{y_H} \end{cases}$$

Note that the first condition is the same as what we have in Section 3.

7.2 Conventional Preferences

The quasilinear preferences assumed in the benchmark model imply that there is no income effect in the demand for government bonds. In this subsection I show numerically that the key result of the benchmark model is robust to using log utility for both time periods. Suppose that all components of the benchmark model are intact except that investors' preferences are given by:

$$U^* = \ln(c_0) + \beta \mathbb{E}_0\big[\ln(c_1)\big]\,.$$

where $\beta \in (0,1)$ is the discount factor. Households' maximization problem for $p = 1$ becomes:

$$\begin{cases} U = \max_{c_0, b_1} \ln(c_0) + \beta\big\{(1-\pi_D)\ln(a_H - \delta + b_1) + \pi_D \ln(a_L + z b_1)\big\}, \\ \\ \text{s.t.} \quad c_0 + q_0 b_1 = y_0 \,. \end{cases}$$

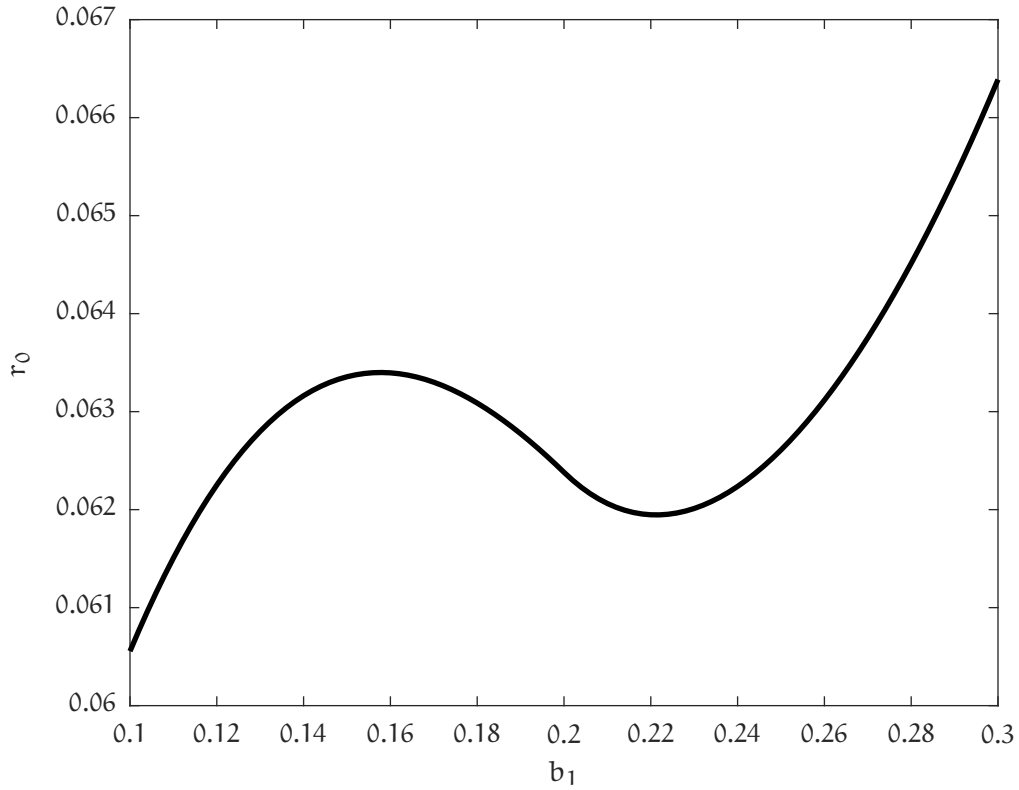


Figure 15: Demand curve is illustrated in terms of the yield $r_1 = \frac{1}{q_0} - 1$ for the case of log utility at $t = 0$. The rest of parameter values are $a_H = 1$, $a_L = 0.1$, $z = 0.16$, $y_0 = 20$, $\beta = 0.9$, $\tau\underline{y} = 1.2$, $\tau\bar{y} = 1.68$, and $\delta = 0.1$.

Using the FOCs, investors' maximization problem yields the following demand curve for government bonds:

$$q_0 = \frac{y_0}{b_1 + \frac{1}{\beta}\big\{\frac{1-F(b_1)}{a_H - \delta + b_1} + \frac{zF(b_1)}{a_L + z b_1}\big\}^{-1}} \,.$$

Unlike the benchmark model, higher income of the investors in the initial period, y_0, increases the price of bond. Taking the inverse of the above, the interest rate on government bonds consists of two terms:

$$\frac{1}{q_0} = 1 + r_1 = \frac{1}{y_0}\Big\{b_1 + \frac{1}{\beta}\Big\{\frac{1-F(b_1)}{a_H - \delta + b_1} + \frac{zF(b_1)}{a_L + z b_1}\Big\}^{-1}\Big\} \,.$$

Absent the first term in the above equation, b_1, the equation above is the same as the one determining the interest rate in the benchmark model, only scaled by a constant $\frac{1}{\beta y_0}$. Note that the first term always pushes up the interest rate on bonds as there is more borrowing by the government. This suggests that it might be more difficult to get an upward-sloping demand curve in this case. But, Figure 15 shows that the effect of second term can dominate, leading to an upward-sloping demand curve for the case of log utilities in both periods.

8 Conclusion

In this paper, I study the demand for risky government bonds as safe assets. I show that when government default significantly affects the return to other available assets, government bonds become the safe asset. An exogenous deterioration in the future prospects, in the form of a negative shift or a mean preserving spread of the future tax revenues distribution, increases the price of government bonds.

I show that government bonds can face an upward-sloping demand curve, when they are safe assets. This happens when the increase in default risks due to higher levels of issuance, significantly increases the demand for insurance. I show that, whether this is the case, depends on the return portfolio of the other assets and the reversed hazard rate of the future tax revenues distribution.

While the model is stylized, one can extend it to a fully dynamic infinite horizon setup. Such extension can be useful in estimating the contribution of the mechanisms discussed in this paper to the evolution of the real rates and risk premia. The model can have welfare implications for the conduct of fiscal policy, when the economy is at the zero lower bound. When government debt is systemic and nominal frictions make the real rate sticky, a fiscal expansion that lowers the equilibrium real interest rate due to the upward-sloping demand curve for government bonds, can depress the aggregate demand and welfare. These and other potentially interesting extensions of the model can be subjects for future research.

References

Asonuma, T., S. Bakhache, and H. Hesse (2015, February). Is banks' home bias good or bad for public debt sustainability? (IMF Working Paper WP/15/44. International Monetary Fund.).

Blanchard, O. (2019, April). Public debt and low interest rates. *American Economic Review 109*(4), 1197—-1229.

Caballero, R. J., E. Farhi, and P.-O. Gourinchas (2017). The safe assets shortage conundrum. *Journal of Economic Perspectives 31*(3), 29–46.

Chaudhary, M., J. Zhiyu Fu, and H. Zhou (2025, July). Anatomy of the treasury market: Who moves yields? *Working Paper*.

Chernov, M., L. Schmid, and A. Schneider (2020, October). A macrofinance view of u.s. sovereign cds premiums. *Journal of Finance 75*(5), 2809–2844.

Cochrane, J. (2021, February). r<g. (Working Paper. Hoover Institution, Stanford University.).

Cochrane, J. H. (2020, February). The value of government debt. (NBER Working Paper.).

Del Negro, M., D. Giannone, M. P. Giannoni, and A. Tambalotti (2017, Spring). Safety, liquidity, and the natural rate of interest. *Brookings Papers on Economic Activity*, 235–316.

Del Negroa, M., D. Giannone, M. P. Giannoni, and A. Tambalotti (2019, May). Global trends in interest rates. *Journal of International Economics 118*, 248–262.

Dell'Ariccia, G., C. Ferreira, N. Jenkinson, L. Laeven, A. Martin, C. Minoiu, and A. Popov (2018). Managing the sovereign-bank nexus. (Departmental Paper No. 18/16, International Monetary Fund, European Central Bank.).

Dittmar, R., A. Hsu, G. Roussellet, and P. Simasek (2019, October). Default risk and the pricing of u.s. sovereign bonds. (Working Paper. University of Michigan, Georgia Institute of Technology, McGill University, Georgia Institute of Technology.).

Eren, E., A. Schrimpf, and F. D. Xia (2025, March). The demand for government debt. *Working Paper*.

Fang, X., B. Hardy, and K. K. Lewis (2025, August). Who holds sovereign debt and why it matters. *Review of Financial Studies 38*(8), 2326–2361.

Farhi, E. and M. Maggiori (2018, February). A model of the international monetary system. *Quarterly Journal of Economics 133*(1), 295—-355.

Gennaioli, N., A. Martin, and S. Rossi (2014, April). Sovereign default, domestic banks, and financial institutions. *Journal of Finance 69*(2), 819–866.

Gourinchas, P.-O. (2019, May). The dollar hegemon? evidence and implications for policy makers. (Working Paper. University of Berkeley.).

He, Z. and A. Krishnamurthy (2012, April). A model of capital and crises. *Review of Economic Studies 79*(2), 735–777.

He, Z. and A. Krishnamurthy (2013, April). Intermediary asset pricing. *American Economic Review 103*(2), 732–770.

He, Z., A. Krishnamurthy, and K. Milbradt (2019, April). A model of safe asset determination. *American Economic Review 109*(4), 1230–1262.

Holmström, B. and J. Tirole (1998, February). Private and public supply of liquidity. *Journal of Political Economy 106*(1), 1–40.

Jiang, Z. and R. J. Richmond (2025, January). Reserve asset competition and the global fiscal cycle. *Working Paper*.

Klingler, S. and D. Lando (2018, May). Safe haven cds premiums. *Review of Financial Studies 31*(5), 1856–1895.

Koijen, R. S. and M. Yogo (2019, August). A demand system approach to asset pricing. *Journal of Political Economy 127*(4).

Koijen, R. S. and M. Yogo (2025, July). Exchange rates and asset prices in a global demand system. *Working Paper*.

Krishnamurthy, A. and A. Vissing-Jorgensen (2012, April). The aggregate demand for treasury debt. *Journal of Political Economy 120*(2), 233–267.

Liu, Y. (2019, May). Government debt and risk premia. (Working Paper. University of Hong Kong.).

Liu, Y., L. Schmid, and A. Yaron (2020, January). The risks of safe assets. (Working Paper. University of Hong Kong, University of Southern California, University of Pennsylvania.).

Mendoza, E. G. and V. Z. Yue (2012, May). A general equilibrium model of sovereign default and business cycles. *Quarterly Journal of Economics 127*(2), 889–946.

O'Kane, D. (2012, January). The link between eurozone sovereign debt and cds prices. *Working Paper*.

Trebesch, C. and M. Zabel (2017, February). The output costs of hard and soft sovereign default. *European Economic Review 92*, 416–432.

A Single Large Investor's Demand

B Construction of $G(y_1, \hat{y}_1)$

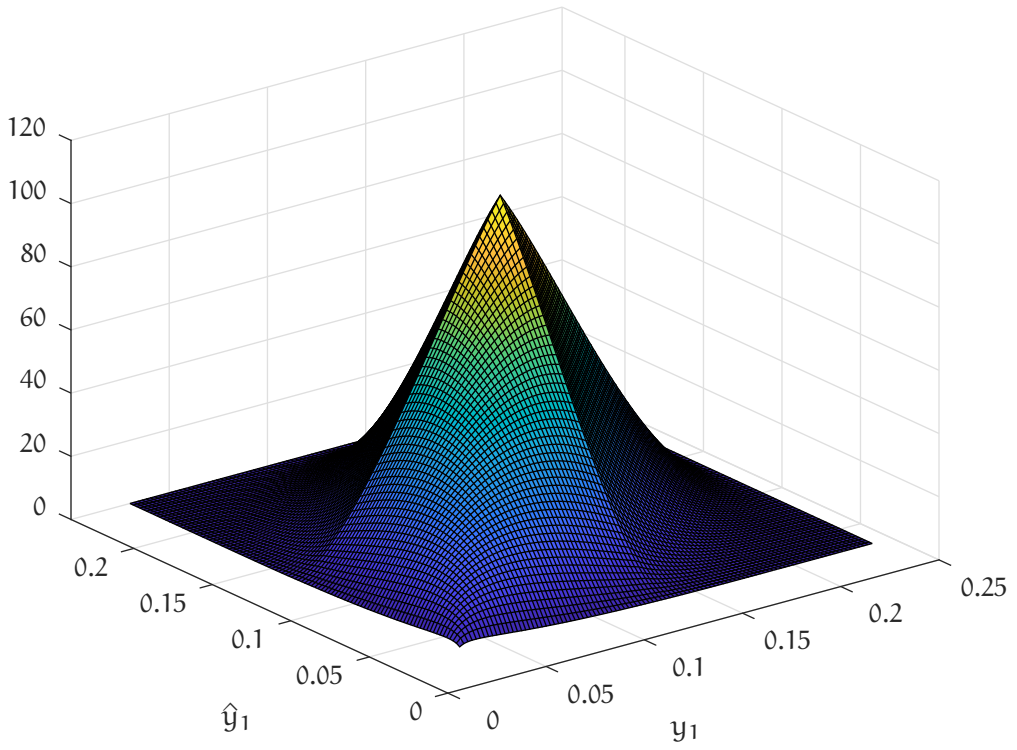


Figure 16: The probability density function $g(y_1, \hat{y}_1)$ when $\rho^G = -0.5$.

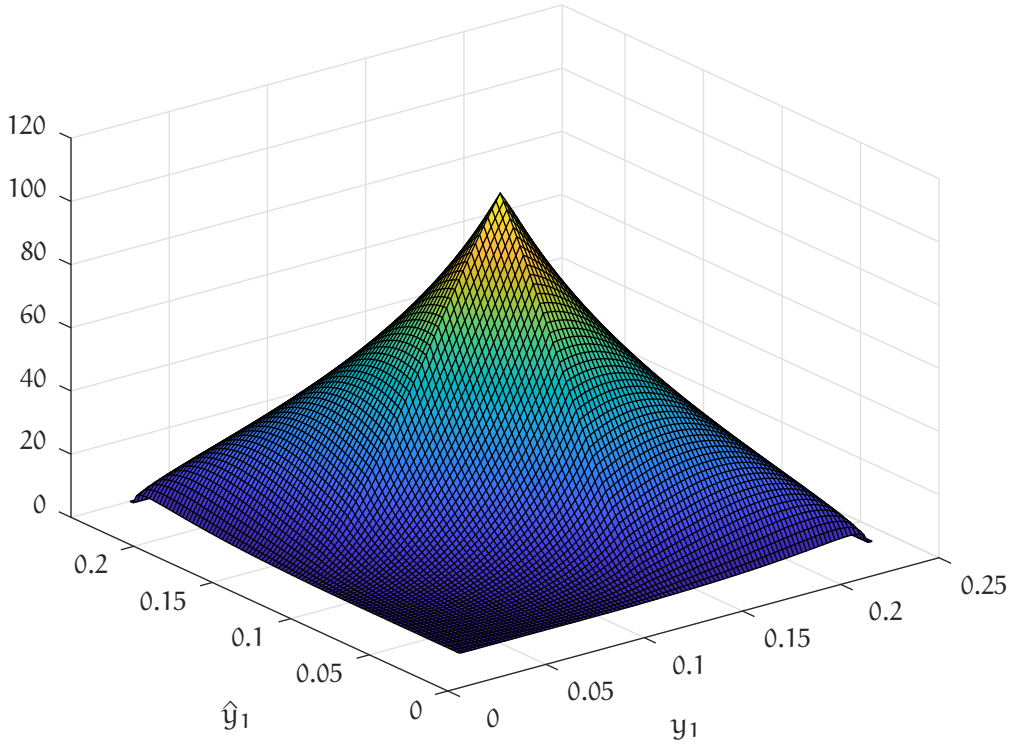


Figure 17: The probability density function $g(y_1, \hat{y}_1)$ when $\rho^G = -0.5$.

C Partial Elasticities

To clarify the distinction between the *partial* and *total* elasticities, I rewrite the demand curves of the two-country extension as follows:

$$\begin{cases} q_0 = \eta(b_1, \hat{b}_1)\,, \\ \\ \hat{q}_0 = \hat{\eta}(b_1, \hat{b}_1)\,. \end{cases} \tag{21}$$

Taking derivatives, one obtains:

$$\begin{cases} dq_0 = \eta_1 db_1 + \eta_2 d\hat{b}_1\,, \\ \\ d\hat{q}_0 = \hat{\eta}_1 db_1 + \hat{\eta}_2 d\hat{b}_1\,. \end{cases} \tag{22}$$

total demand elasticity with respect to b_1 is obtained by changing b_1 for a given (supply of) $\hat{b}_1$. The *partial* elasticity, on the other hand, can be calculated by changing b_1 while

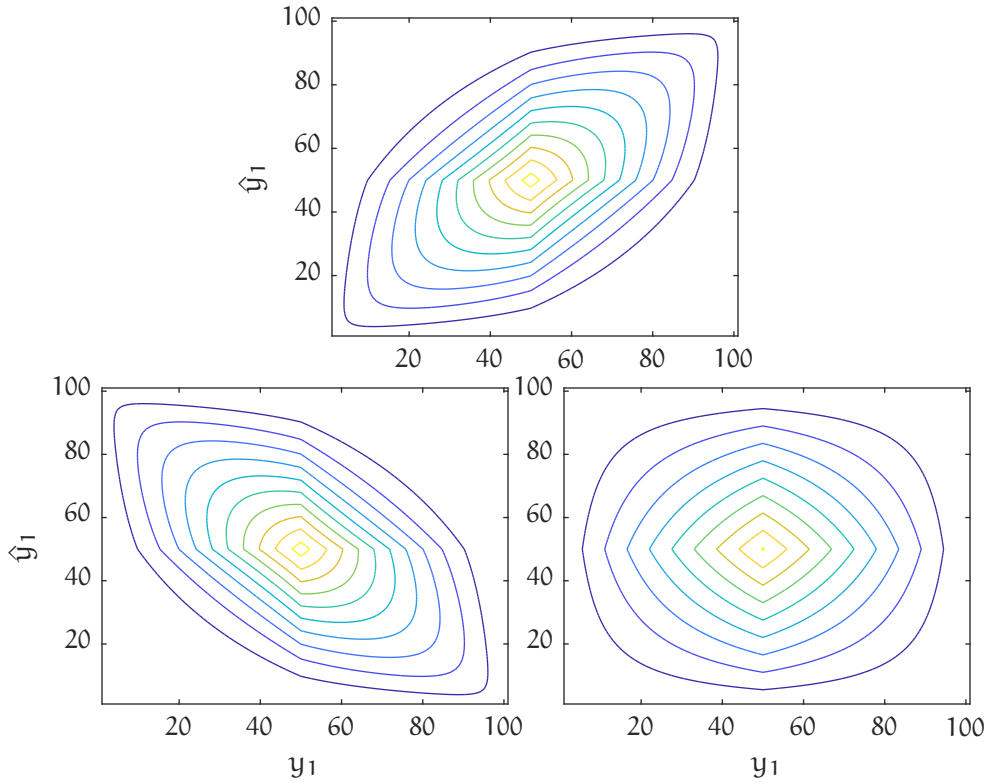


Figure 18: Contour plots of the probability density function $g(y_1, \hat{y}_1)$ for $\rho^G = 0.5$ (top), $\rho^G = -0.5$ (left), and $\rho^G = 0$ (right).

holding $\hat{q}_0$ constant. Hence, one can get:

$$\begin{cases} \epsilon^D_{Total} = -\frac{\partial b_1}{\partial q_0}|_{\hat{b}_1} = \eta_1^{-1}, \\ \\ \epsilon^D_{Partial} = -\frac{\partial b_1}{\partial q_0}|_{\hat{q}_0} = \left[\eta_1 - \frac{\hat{\eta}_1}{\hat{\eta}_2}\eta_2\right]^{-1}. \end{cases} \tag{23}$$

Figure 10 shows the value of the *partial* elasticities at each equilibrium point given the supply of b_1 and $\hat{b}_1$.

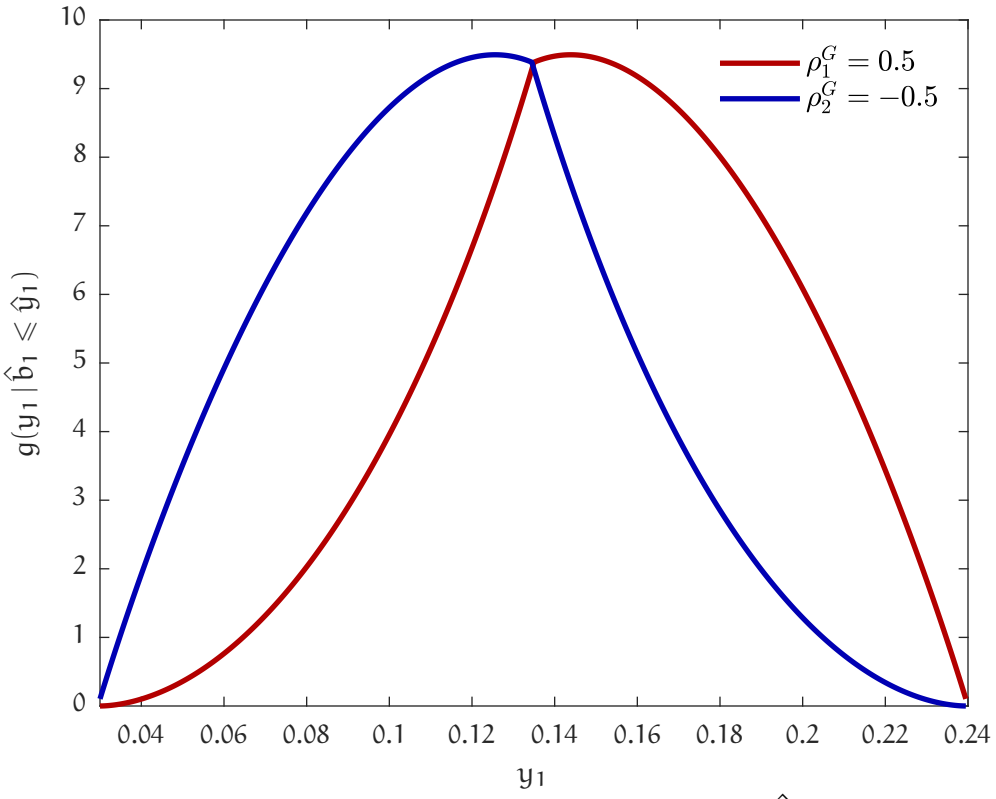


Figure 19: The probability density function of y_1 when $\hat{b}_1 = 0.14$, conditional on the government issuing $\hat{b}_1$ is not in default.

D Slope of Relative Demand Curve

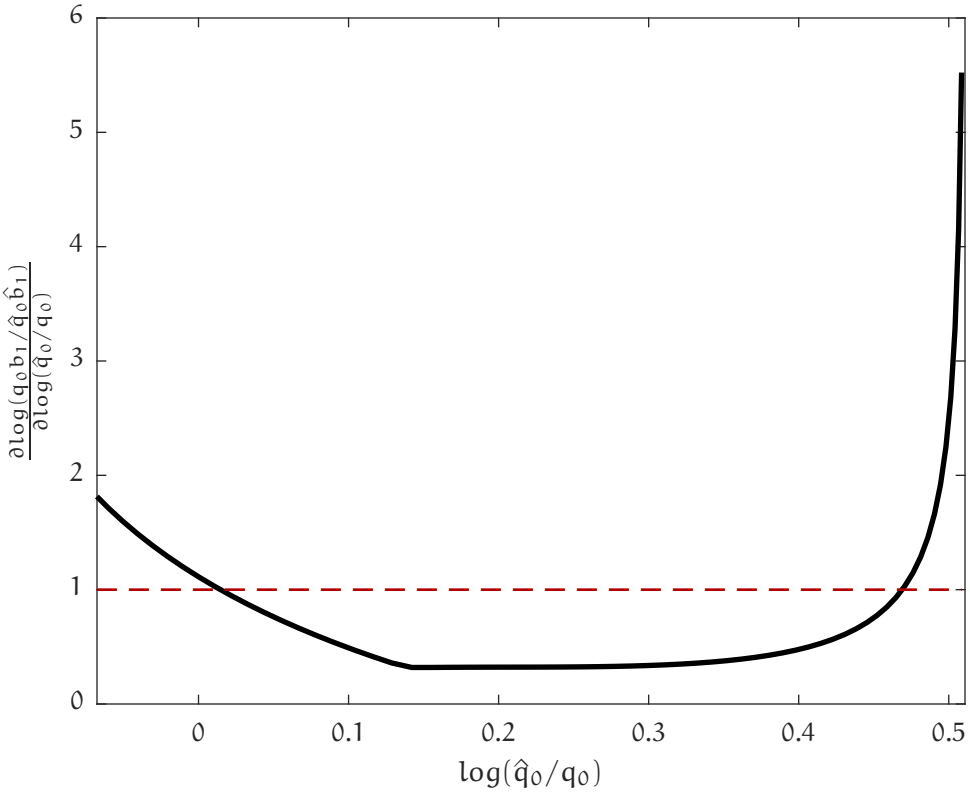


Figure 20: The slope of the relative demand curve shown in Figure 11.

E Proofs

Proof of Lemma 1. I derived the demand curve for the case $\delta \leqslant b_1 < \frac{\delta}{z}$. I assume for the moment that investors pay the maintenance costs if they can. I derive the demand and show it is downward sloping only for the case $b_1 \geqslant \frac{\delta}{z}$, as the third case $b_1 < \delta$ is very similar. When $b_1 \geqslant \frac{\delta}{z}$, investors pay the maintenance costs in all future states. Therefore, the payoffs in the default and no-default states are $a_H - \delta + z b_1$ and $a_H - \delta + b_1$, respectively. Thus, the demand function immediately follows from the FOCs of the investors. For $b_1 \geqslant \frac{\delta}{z}$, taking the derivative with respect to b_1, I obtain:

$$\frac{\partial q_0}{\partial b_1} = -\Big\{\Big[\frac{1 - F(b_1)}{(a_H - \delta + b_1)^2} + \frac{z F(b_1)}{(a_H - \delta + z b_1)^2}\Big] + \Big[\frac{1}{a_H - \delta + b_1} - \frac{z}{a_H - \delta + z b_1}\Big] f(b_1)\Big\} < 0\,.$$

Note that both terms inside the braces are strictly positive. Next, I need to show that given Assumption 1, investors choose to pay the maintenance costs if they can. Investors pay the maintenance costs, under no default, if:

$$\ln(a_H - \delta + b_1) \geqslant p \ln(a_L + b_1) + (1 - p) \ln(a_H + b_1) \iff \frac{a_H - \delta + b_1}{a_H + b_1} \geqslant \Big(\frac{a_L + b_1}{a_H + b_1}\Big)^p\,.$$

Define the function $\gamma(b_1)$ as the difference between the LHS and the RHS of the first inequality above:

$$\gamma(b_1) \equiv \ln(a_H - \delta + b_1) - (p \ln(a_L + b_1) + (1-p)\ln(a_H + b_1)) \Rightarrow$$

$$\Rightarrow \frac{\partial \gamma}{\partial b_1} = \frac{1}{a_H + b_1}\left(\frac{\delta}{a_H - \delta + b_1} - \frac{p(a_H - a_L)}{a_L + b_1}\right).$$

Under Assumption 1, the term inside the parentheses is strictly negative, hence, $\frac{\partial \gamma}{\partial b_1} < 0$. This implies that if $\gamma(b_1) > 0$, then $\gamma(\tilde{b}_1) > 0$ for all $\tilde{b}_1 < b_1$, and specifically $\gamma(zb_1) > 0$. Therefore, it suffices to find a lower bound for p that works under the no default state. Using the second inequality above, I obtain:

$$\frac{a_H - \delta + b_1}{a_H + b_1} \geqslant \left(\frac{a_L + b_1}{a_H + b_1}\right)^p \iff p \geqslant \frac{\ln(a_H + b_1) - \ln(a_H - \delta + b_1)}{\ln(a_H + b_1) - \ln(a_L + b_1)} \equiv h(b_1).$$

Taking the derivative, one has:

$$\frac{\partial h}{\partial b_1} = \frac{1}{(\ln(a_H + b_1) - \ln(a_L + b_1))^2}\Big\{\left(\frac{1}{a_H + b_1} - \frac{1}{a_H - \delta + b_1}\right)(\ln(a_H + b_1) -$$

$$- \ln(a_L + b_1)) - \left(\frac{1}{a_H + b_1} - \frac{1}{a_L + b_1}\right)(\ln(a_H + b_1) - \ln(a_H - \delta + b_1))\Big\} > 0.$$

Since $h(b_1)$ is an increasing function, if $h(b_1)$ has a limit as $b_1 \to \infty$, that limit can provide a sufficient condition on p. Both the numerator and the denominator of $h(b_1)$ go to zero as $b_1 \to \infty$. The limit, using the L'Hopital's rule is given by:

$$\lim_{b_1 \to \infty} h(b_1) = \lim_{b_1 \to \infty} \frac{\frac{1}{a_H + b_1} - \frac{1}{a_H - \delta + b_1}}{\frac{1}{a_H + b_1} - \frac{1}{a_L + b_1}} = \frac{\delta}{a_H - a_L} \lim_{b_1 \to \infty} \frac{a_L + b_1}{a_H - \delta + b_1} = \frac{\delta}{a_H - a_L}.$$

Therefore, if $p \geqslant \frac{\delta}{a_H - a_L}$, investors pay for the maintenance costs if they can. ∎

Proof of Lemma 2. Note that:

$$\frac{1}{a_H - \delta + b_1} < \frac{1}{\frac{a_L}{z} + b_1}.$$

Therefore, it suffices to show that $\tilde{F}(b_1) > F(b_1)$ for all $b_1 \in [\underline{b}, \frac{\underline{b} + \overline{b}}{2})$. To see this, note that:

$$\tilde{F}(b_1) > F(b_1) \iff \frac{b_1 - \underline{b} + \epsilon}{\overline{b} - \underline{b} + 2\epsilon} > \frac{b_1 - \underline{b}}{\overline{b} - \underline{b}} \iff \overline{b} - \underline{b} > 2(b_1 - \underline{b}) \iff b_1 < \frac{\underline{b} + \overline{b}}{2}.$$

■

Proof of Proposition 1. I can rewrite the expression in 2 as follows:

$$\frac{\partial q_0}{\partial b_1} = -\frac{1 - F(b_1)}{(a_H - \delta + b_1)^2} - \Big[\frac{(1-p)z^2}{(a_H + zb_1)^2} + \frac{pz^2}{(a_L + zb_1)^2}\Big]F(b_1) +$$

$$+ \Big[\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1} - \frac{1}{a_H - \delta + b_1}\Big]f(b_1).$$

It is clear from the above that the first two terms in the first line above are negative. If $a_H - \delta \leqslant \frac{a_L}{z}$, the third term in the second line would be non-positive regardless of the value of p, implying that $\frac{\partial q_0}{\partial b_1} < 0$. Therefore $a_H - \delta > \frac{a_L}{z}$ is a necessary condition for an upward-sloping demand curve. To have an upward-sloping demand curve, the last term in the expression for $\frac{\partial q_0}{\partial b_1}$ above needs to be strictly positive. This is the case iff:

$$\frac{(1-p)z}{a_H + zb_1} + \frac{pz}{a_L + zb_1} - \frac{1}{a_H - \delta + b_1} > 0 \iff \Big(\frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{a_H - \delta + b_1}\Big) -$$

$$- \Big(\frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{\frac{a_H}{z} + b_1}\Big)(1-p) > 0,$$

$$\iff 1 - p < \frac{z(a_H - \delta - \frac{a_L}{z})}{a_H - a_L} \frac{\frac{a_H}{z} + b_1}{a_H - \delta + b_1}.$$

But, $\frac{\frac{a_H}{z} + b_1}{a_H - \delta + b_1} < \frac{\frac{a_H}{z}}{a_H - \delta}$, as the numerator is larger than the denominator. Therefore, one has:

$$\frac{\partial q_0}{\partial b_1} > 0 \Rightarrow 1 - p < \frac{z(a_H - \delta - \frac{a_L}{z})}{a_H - a_L} \frac{\frac{a_H}{z}}{a_H - \delta} = \frac{a_H}{a_H - a_L} \frac{a_H - \delta - \frac{a_L}{z}}{a_H - \delta} < \frac{a_H - \frac{a_L}{z}}{a_H - a_L} \Rightarrow,$$

$$\Rightarrow p > \frac{a_L(1-z)}{(a_H - a_L)z}.$$

To see the last result, note that we can collect all terms multiplying $1 - p$, and rewrite the expression for $\frac{\partial q_0}{\partial b_1}$ as follows:

$$\frac{\partial q_0}{\partial b_1} = -\Big(\frac{1-F(b_1)}{(a_H-\delta+b_1)^2}+\frac{f(b_1)}{a_H-\delta+b_1}\Big)+\frac{f(b_1)}{\frac{a_L}{z}+b_1}-\frac{F(b_1)}{(\frac{a_L}{z}+b_1)^2}+$$

$$+\Big\{\Big[\frac{1}{(\frac{a_L}{z}+b_1)^2}-\frac{1}{(\frac{a_H}{z}+b_1)^2}\Big]F(b_1)-\Big[\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{\frac{a_H}{z}+b_1}\Big]f(b_1)\Big\}(1-p)\,.$$

But, one has:

$$\frac{\partial q_0}{\partial b_1}|_{p=1} = -\frac{1}{(a_H-\delta+b_1)^2}+\Big[f(b_1)-\Big(\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}\Big)F(b_1)\Big]\Big(\frac{1}{\frac{a_L}{z}+b_1}-\frac{1}{a_H+b_1}\Big)$$

If $\frac{\partial q_0}{\partial b_1}|_{p=1} > 0$, it must be that $f(b_1)-\big(\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}\big)F(b_1) > 0$. This implies that the term that is multiplied by $1-p$ in the expression for $\frac{\partial q_0}{\partial b_1}$ must be strictly negative as $\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{\frac{a_H}{z}+b_1} < \frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H+b_1}$. This implies:

$$\frac{\partial q_0}{\partial b_1}|_{p=1} > 0 \Rightarrow \frac{\partial^2 q_0}{\partial p \partial b_1} > 0\,.$$

■

Proof of Lemma 3. One has:

$$\frac{\partial q_0}{\partial g_0} = \frac{\frac{\partial q_0}{\partial b_1}}{\frac{\partial g_0}{\partial b_1}} = \frac{\frac{\partial q_0}{\partial b_1}}{q_0+\frac{\partial q_0}{\partial b_1}b_1}\,.$$

It is clear from the above that if $\frac{\partial q_0}{\partial b_1} \geqslant 0$, one has $\frac{\partial q_0}{\partial g_0} \geqslant 0$. ■

Proof of Proposition 2. For the triangular distribution, the expression for $\frac{\partial q_0}{\partial b_1}$ in 2, for $p=1$ and $b_1 \in [\underline{b}, \frac{\underline{b}+\overline{b}}{2}]$ becomes:

$$\frac{\partial q_0}{\partial b_1} = \frac{1}{a_H-\delta+b_1}\Big\{\frac{a_H-\frac{a_L}{z}-\delta}{\frac{a_L}{z}+b_1}\Big(\frac{b_1-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big\{\frac{2}{b_1-\underline{b}}-\Big[\frac{1}{\frac{a_L}{z}+b_1}+\frac{1}{a_H-\delta+b_1}\Big]\Big\}-\frac{1}{a_H-\delta+b_1}\Big\}\,.$$

Note that the term in the first pair of inner braces is strictly positive. Suppose that $a_H-\frac{a_L}{z}-\delta$ is large enough, such that:

$$\lambda(\tilde{b}_1) \equiv (a_H-\frac{a_L}{z}-\delta)\Big(\frac{\tilde{b}_1-\underline{b}}{\overline{b}-\underline{b}}\Big)^2\Big\{\frac{2}{\tilde{b}_1-\underline{b}}-\Big[\frac{1}{\frac{a_L}{z}+\tilde{b}_1}+\frac{1}{a_H-\delta+\tilde{b}_1}\Big]\Big\} \geqslant 1\,.$$

The derivative of $\lambda(\tilde{b}_1)$ is:

$$\frac{\partial \lambda}{\partial \tilde{b}_1} \propto \left(1 - \frac{\tilde{b}_1 - \underline{b}}{\frac{a_L}{z} + \tilde{b}_1}\right)^2 + \left(1 - \frac{\tilde{b}_1 - \underline{b}}{a_H - \delta + \tilde{b}_1}\right)^2 > 0\,.$$

Since $\frac{1}{\frac{a_L}{z}+b_1} > \frac{1}{a_H-\delta+b_1}$ and that $\lambda(b_1) \geqslant 1$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\bar{b}}{2}]$, one has $\frac{\partial q_0}{\partial b_1} > 0$ for all $b_1 \in [\tilde{b}_1, \frac{\underline{b}+\bar{b}}{2}]$. For the last part of the proof, it suffices to note that $\lambda(b_1) \to 0$, as $b_1 \to \underline{b}$. ■

Proof of Proposition 3. At zero net supply, i.e., $\kappa_1 = 0$, the CDS premium becomes:

$$s_1 = \frac{(1-z)z\pi_D(a_H - \delta + b_1)}{(1-\pi_D)(\frac{a_L}{z} + b_1)}\,.$$

Taking derivative gives:

$$\frac{\partial s_1}{\partial b_1} = (1-z)z\left\{\frac{\frac{\partial \pi_D}{\partial b_1}(a_H - \delta + b_1)}{(1-\pi_D)^2(\frac{a_L}{z} + b_1)} - \frac{\pi_D(a_H - \delta + b_1)}{(1-\pi_D)(\frac{a_L}{z} + b_1)^2}\right\}.$$

Rearranging terms gives:

$$\frac{\partial s_1}{\partial b_1} = \frac{(1-z)z\pi_D}{(1-\pi_D)(a_H - \delta + b_1)(\frac{a_L}{z} + b_1)}\left\{\frac{\frac{\partial \pi_D}{\partial b_1}}{\pi_D(1-\pi_D)} - \left[\frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{a_H - \delta + b_1}\right]\right\}.$$

If $\frac{a_L}{a_H-\delta} \geqslant z$, the expression in the brackets is negative which, along with the fact that $\frac{\partial \pi_D}{\partial b_1} = f(b_1) > 0$, implies that $\frac{\partial s_1}{\partial b_1} > 0$.

Now, consider the case $\frac{a_L}{a_H-\delta} < z$. When $b_1 \in [\underline{b}, \frac{\underline{b}+\bar{b}}{2}]$, one has:

$$\frac{\frac{\partial \pi_D}{\partial b_1}}{\pi_D(1-\pi_D)} = \frac{1}{1-\pi_D}\frac{2}{b_1 - \underline{b}} > \frac{2}{b_1 - \underline{b}} > \frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1} > \frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{a_H - \delta + b_1}\,.$$

This implies that $\frac{\partial s_1}{\partial b_1} > 0$. Alternatively, when $b_1 \in [\frac{\underline{b}+\bar{b}}{2}, \bar{b}]$, one has:

$$\frac{\frac{\partial \pi_D}{\partial b_1}}{\pi_D(1-\pi_D)} = \frac{1}{\pi_D}\frac{2}{\bar{b} - b_1} > \frac{2}{\bar{b} - b_1} > \frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1} > \frac{1}{\frac{a_L}{z} + b_1} - \frac{1}{a_H - \delta + b_1}\,.$$

The above inequalities hold because $\bar{b} - b_1 < b_1 < \frac{a_L}{z} + b_1 < a_H - \delta + b_1$. The above implies that $\frac{\partial s_1}{\partial b_1} > 0$ which concludes the proof. ■

Proof of Lemma 4. When CDS contracts are in zero net supply, the premium is given by:

$$s_1 = \frac{(1-z)z\pi_D(a_H - \delta + b_1)}{(1-\pi_D)(\frac{a_L}{z} + b_1)}.$$

It is clear from the above that an increase in π_D, for a fixed level of b_1, unambiguously increases s_1. For government debt one has:

$$q_0 = \frac{1}{a_H - \delta + b_1} + \pi_D\Big\{\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\Big\}.$$

It is clear from the above that when $\frac{a_L}{a_H - \delta} < z$, an increase in π_D for a fixed level of b_1 increases q_0, and therefore, decreases r_1. ∎

Proof of Lemma 5. Since $f(b_1)$ is convex for $b_1 \in (\underline{b}, \hat{b})$, one has:

$$f(b_1) \leqslant \lambda f(\underline{b}) + (1-\lambda)f(\hat{b}).$$

Where, $b_1 = \lambda\underline{b} + (1-\lambda)\hat{b}$. This implies that the area under $f(b_1)$ between $\underline{b}$ and $\hat{b}$, which is equal to $F(\hat{b})$, is not larger than the area of the triangle with the base $\hat{b} - \underline{b}$ and height of $f(\hat{b})$. Therefore, one has:

$$\frac{f(\hat{b})}{F(\hat{b})} \geqslant \frac{f(\hat{b})}{\frac{1}{2}(\hat{b} - \underline{b})f(\hat{b})} = \frac{2}{\hat{b} - \underline{b}}.$$

Hence, $f(\hat{b}) - \big(\frac{1}{\frac{a_L}{z} + \hat{b}} + \frac{1}{a_H + \hat{b}}\big)F(\hat{b}) > 0$. This implies that if $a_H - \frac{a_L}{z} - \delta$ is large enough, it must be that $\frac{\partial q_0}{\partial b_1}|_{\hat{b}} > 0$.

For the second inequality, define $\eta \equiv \frac{a_L}{z}$. Using the expression for $\frac{\partial q_0}{\partial b_1}$ and taking the derivative with respect to η, I obtain:

$$\frac{\partial^2 q_0}{\partial\eta\partial b_1} \propto \frac{-(a_H - \delta + b_1)}{(\eta + b_1)^2}\Big\{\frac{f(b_1)}{F(b_1)} - \Big[\frac{1}{\eta + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\} + \frac{a_H - \delta - \eta}{\eta + b_1}\frac{1}{(\eta + b_1)^2} \Rightarrow$$

$$\frac{\partial^2 q_0}{\partial\eta\partial b_1} \propto \frac{a_H - \delta - \eta}{\eta + b_1} - (a_H - \delta + b_1)\Big\{\frac{f(b_1)}{F(b_1)} - \Big[\frac{1}{\eta + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\} \Rightarrow$$

$$\frac{\partial^2 q_0}{\partial\eta\partial b_1} < 0 \iff \frac{f(b_1)}{F(b_1)} > \frac{1}{\eta + b_1} + \frac{1}{a_H - \delta + b_1} + \frac{a_H - \delta - \eta}{(a_H - \delta + b_1)(\eta + b_1)} = \frac{2}{\eta + b_1}.$$

But, I established above that $\frac{f(\hat{b})}{F(\hat{b})} \geqslant \frac{2}{\hat{b}-\underline{b}}$. Given that $\frac{2}{\hat{b}-\underline{b}} > \frac{2}{\eta+\hat{b}}$, the proof is complete. ■

Proof of Lemma 6. Changes in the welfare of government and investors are given by:

$$\begin{cases} dU^g = \frac{dU^g}{d\epsilon} d\epsilon + \frac{dU^g}{db_1} db_1 \,, \\ \\ dU^i = \frac{dU^i}{d\epsilon} d\epsilon + \frac{dU^i}{db_1} db_1 \,. \end{cases}$$

But, one has $\frac{dU^i}{d\epsilon} = -1$, $\frac{dU^g}{d\epsilon} > 0$, and:

$$\begin{cases} \frac{dU^g}{db_1}|_{CE} = 0 \,, \\ \\ \frac{dU^i}{db_1}|_{CE} = \frac{\partial U^i}{\partial b_1}|_{CE} + \frac{\partial U^i}{\partial q_0} \frac{dq_0}{db_1}|_{CE} = -b_1 \frac{dq_0}{db_1}|_{CE} \,. \end{cases}$$

In the above, the first equality follows from the fact that government is maximizing its welfare at the competitive equilibrium incorporating the impact of its debt supply on q_0. Moreover, one must have $\frac{\partial U^i}{\partial b_1}|_{CE} = 0$ in equilibrium given investors are maximizing their welfare taking q_0 as given, and that $\frac{\partial U^i}{\partial q_0} = -b_1$. These imply the second equality. The above equalities imply:

$$\begin{cases} dU^g|_{CE} = \frac{dU^g}{d\epsilon} d\epsilon \,, \\ \\ dU^i|_{CE} = -(d\epsilon + b_1 \frac{dq_0}{db_1}|_{CE} db_1) \,. \end{cases}$$

Hence, for any Pareto improvement one must have $d\epsilon > 0$. The reallocation $\{d\epsilon, db_1\}$ is a Pareto improvement if and only if:

$$d\epsilon + b_1 \frac{dq_0}{db_1}|_{CE} db_1 \leqslant 0 \iff b_1 \frac{dq_0}{db_1}|_{CE} db_1 \leqslant -d\epsilon \,.$$

It is clear that for the above condition to hold one must have $db_1 < 0$ if and only if $\frac{dq_0}{db_1}|_{CE} > 0$. ■

Proof of Lemma 7. Using 20, I obtain:

$$\frac{q_0^a}{q_0} = a_H - \delta - (a_H - \frac{a_L}{z} - \delta)\frac{\frac{zF(b_1)}{a_L+zb_1}}{\frac{zF(b_1)}{a_L+zb_1} + \frac{1-F(b_1)}{a_H-\delta+b_1}}.$$

Define:

$$\zeta_1(b_1) \equiv \frac{\frac{zF(b_1)}{a_L+zb_1}}{\frac{zF(b_1)}{a_L+zb_1} + \frac{1-F(b_1)}{a_H-\delta+b_1}}.$$

Then, one has:

$$\frac{1}{\zeta_1(b_1)} = 1 + \frac{(1-F(b_1))(\frac{a_L}{z}+b_1)}{F(b_1)(a_H-\delta+b_1)}.$$

If I define $\zeta_2(b_1) \equiv \frac{(1-F(b_1))(\frac{a_L}{z}+b_1)}{F(b_1)(a_H-\delta+b_1)}$, then:

$$\frac{\partial \zeta_2}{\partial b_1} \propto (a_H - \frac{a_L}{z} - \delta)(1-F(b_1))F(b_1) - (a_H-\delta+b_1)(\frac{a_L}{z}+b_1)f(b_1) =$$

$$-(a_H-\delta+b_1)(\frac{a_L}{z}+b_1)\Big\{f(b_1) - \big[\frac{1}{\frac{a_L}{z}+b_1} - \frac{1}{a_H-\delta+b_1}\big](1-F(b_1))F(b_1)\Big\}.$$

From the proof of Proposition 1, one knows that if $\frac{\partial q_0}{\partial b_1}|_{p=1} > 0$, it must be that $f(b_1) - \big(\frac{1}{\frac{a_L}{z}+b_1} + \frac{1}{a_H+b_1}\big)F(b_1) > 0$. This immediately implies that, if $\frac{\partial q_0}{\partial b_1}|_{p=1} > 0$ then $\frac{\partial \zeta_2}{\partial b_1} < 0$. But, if $\zeta_2(b_1)$ is decreasing in b_1, $\zeta_1(b_1)$ must be increasing, and $\frac{q_0^a}{q_0}$ ($\frac{q_0}{q_0^a}$) must be decreasing (increasing) in b_1. ■

Proof of Corollary 1. Let $\eta \equiv \frac{a_L}{z}$. Given that $\frac{\partial 1/q_0}{\partial b_1} = -\frac{1}{q_0^2}\frac{\partial q_0}{\partial b_1}$, I obtain:

$$\frac{\partial^2 1/q_0}{\partial \eta \partial b_1} = \frac{2}{q_0^3}\frac{\partial q_0}{\partial \eta}\frac{\partial q_0}{\partial b_1} - \frac{1}{q_0^2}\frac{\partial^2 q_0}{\partial \eta \partial b_1}.$$

It is clear that $\frac{\partial q_0}{\partial \eta} < 0$. Hence, if $\frac{\partial q_0}{\partial b_1} < 0$ and $\frac{\partial^2 q_0}{\partial \eta \partial b_1} < 0$, one must have $\frac{\partial^2 1/q_0}{\partial \eta \partial b_1} > 0$. ■

Proof of Corollary ??. Using 1 implies:

$$q_0 = \frac{1}{a_H - \delta + b_1}.$$

Moreover, 2, when $p = 1$, implies:

$$\frac{\partial q_0}{\partial b_1} = \frac{1}{a_H - \delta + b_1}\Big\{\frac{a_H - \frac{a_L}{z} - \delta}{\frac{a_L}{z} + b_1}F(b_1)\Big\{\frac{f(b_1)}{F(b_1)} - \Big[\frac{1}{\frac{a_L}{z} + b_1} + \frac{1}{a_H - \delta + b_1}\Big]\Big\} - \frac{1}{a_H - \delta + b_1}\Big\}.$$

Assuming $\frac{a_L}{z} = a_H - \delta$ implies $a_H - \frac{a_L}{z} - \delta = 0$, which gives:

$$\frac{\partial q_0}{\partial b_1} = -\frac{1}{(a_H - \delta + b_1)^2} \to \frac{\partial q_0}{\partial b_1}\frac{b_1}{q_0} = -\frac{b_1}{a_H - \delta + b_1}.$$

Therefore, the demand elasticity is:

$$\epsilon^D = -\frac{a_H - \delta + b_1}{b_1}.$$

It is clear from the above that $|\epsilon^D|$ is decreasing in b_1. ■

Proof of Proposition 4. The first part of the proposition is evident from the equation for q_0. Let $\hat{d}_1 \equiv d_1 - a$. The price of new bonds is:

$$q_0 = \frac{1 - F(\tilde{b}_1 + b_1)}{\tilde{b}_1 - \hat{d}_1 + b_1} + \frac{zF(\tilde{b}_1 + b_1)}{z\tilde{b}_1 - \hat{d}_1 + zb_1}.$$

Taking the derivative, I obtain:

$$\frac{\partial q_0}{\partial \tilde{b}_1 + b_1} = \frac{1}{\tilde{b}_1 + b_1 - \hat{d}_1}\times$$

$$\times\Big\{\frac{2\frac{1-z}{z}\hat{d}_1}{\tilde{b}_1 + b_1 - \frac{\hat{d}_1}{z}}\Big(\frac{\tilde{b}_1 + b_1 - \underline{b}}{\overline{b} - \underline{b}}\Big)^2\Big\{\frac{2}{\tilde{b}_1 + b_1 - \underline{b}} - \Big[\frac{1}{\tilde{b}_1 + b_1 - \frac{\hat{d}_1}{z}} + \frac{1}{\tilde{b}_1 + b_1 - \hat{d}_1}\Big]\Big\} - \frac{1}{\tilde{b}_1 + b_1 - \hat{d}_1}\Big\}.$$

Define:

$$\hat{\lambda}(\tilde{b}_1 + b_1) \equiv 2\frac{1-z}{z}\hat{d}_1\Big(\frac{\tilde{b}_1 + b_1 - \underline{b}}{\overline{b} - \underline{b}}\Big)^2\Big\{\frac{2}{\tilde{b}_1 + b_1 - \underline{b}} - \Big[\frac{1}{\tilde{b}_1 + b_1 - \frac{\hat{d}_1}{z}} + \frac{1}{\tilde{b}_1 + b_1 - \hat{d}_1}\Big]\Big\}.$$

It suffices to show that $\hat{\lambda}(\frac{\underline{b}+\overline{b}}{2}) \geqslant 1$. To see this, note that:

$$\hat{\lambda}(\frac{\underline{b}+\overline{b}}{2}) = \frac{1-z}{z}\hat{d}_1\Big\{\frac{2}{\overline{b} - \underline{b}} - \Big[\frac{1}{\underline{b} + \overline{b} - \frac{2\hat{d}_1}{z}} + \frac{1}{\underline{b} + \overline{b} - 2\hat{d}_1}\Big]\Big\} > \frac{1-z}{z}\hat{d}_1\Big\{\frac{1}{\overline{b} - \underline{b}} - \frac{1}{\underline{b} + (1-2z)\overline{b}}\Big\} \geqslant 1.$$

where I have used the fact that $\hat{d}_1 \leqslant \frac{z\tilde{b}}{2}$. This implies that $\frac{\partial q_0}{\partial b_1} > 0$ for high enough values of $\tilde{b}_1 + b_1$. Then, the second part of the proposition follows given that:

$$\frac{\partial q_0}{\partial \omega_0} = \frac{\frac{\partial q_0}{\partial \tilde{b}_1 + b_1}}{\frac{\partial q_0}{\partial \tilde{b}_1 + b_1}(\tilde{b}_1 + b_1) + q_0}.$$

■

Proof of Lemma 8. Similar to Section 3, using similar FOCs with respect to c_0 and b_1 give the bond price:

$$q_0 = \int_{\tau y_L}^{b_1} \frac{z}{a_L + z b_1} dF(x) + \int_{b_1}^{\frac{\tau a_H}{1-\tau}} \frac{\tau}{x} dF(x) + \int_{\frac{\tau a_H}{1-\tau}}^{\tau y_H} \frac{1}{a_H + b_1} dF(x).$$

Taking the derivative with respect to b_1 in the above yields:

$$\frac{\partial q_0}{\partial b_1} = \frac{f(b_1)}{\frac{a_L}{z} + b_1} - \frac{\tau f(b_1)}{b_1} - \frac{F(b_1)}{(\frac{a_L}{z} + b_1)^2} - \frac{1 - F(\frac{\tau a_H}{1-\tau})}{(a_H + b_1)^2}.$$

Using the fact that $\frac{a_L}{a_H} < z$, I obtain:

$$\frac{\partial q_0}{\partial b_1} > \frac{(1-\tau)b_1 - \frac{\tau a_L}{z}}{(\frac{a_L}{z} + b_1)b_1} f(b_1) - \frac{1}{(\frac{a_L}{z} + b_1)^2}.$$

Hence, it is sufficient to have:

$$((1-\tau)b_1 - \frac{\tau a_L}{z})(\frac{a_L}{z} + b_1) f(b_1) > b_1.$$

Given the triangular density f, and evaluating the above at the mean of the distribution $b_1 = \frac{\tau(y_L + y_H)}{2}$, the sufficient condition becomes:

$$(\frac{\tau(y_L + y_H)}{2} - \frac{\tau a_L}{(1-\tau)z})(\frac{a_L}{z} + \frac{\tau(y_L + y_H)}{2}) > \frac{\tau^2(y_H^2 - y_L^2)}{4(1-\tau)}.$$

Given the assumption that $\frac{a_L}{z} < \frac{(1-2\tau)(y_L+y_H)}{2}$, it is sufficient to have $\frac{\tau^2(y_H+y_L)^2}{4} > \frac{\tau^2(y_H^2-y_L^2)}{4(1-\tau)}$. But, this last condition follows from $\frac{\tau}{1-\tau} < \frac{y_L}{y_H}$. ■