Reference Points as Information*

Mark Dean Benjamin Enke Thomas Graeber
Pietro Ortoleva

Preliminary – please do not circulate

February 1, 2026

Abstract

We develop a model of reference dependence based on the idea that reference points provide *comparative information* to people who are uncertain how to value objects or actions. When people are uncertain how much they value an option, they may still be able to tell whether it is better or worse than a reference point on each dimension, thus providing information. In a Bayesian framework, value uncertainty and comparative information generate regularities such as thinking in gain-loss categories; income targeting and bunching; a preference for advantages over tradeoffs; behavioral attenuation; coherent arbitrariness; contrast, assimilation and decoy effects; and simultaneous overreaction to small and underreaction to large changes. The model implies new predictions about how these effects depend on uncertainty. We then develop a theory of endogenous reference points based on which option is most informative for choice by reducing value uncertainty. This yields several novel predictions, including when the reference point is determined by the status quo, expected outcomes, similarity-based memory, or other options.

*We thank Vincent Marohl for outstanding research assistance. Dean: Department of Economics, Columbia University, mark.dean@columbia.edu. Enke: Department of Economics, Harvard University, and NBER, enke@fas.harvard.edu. Graeber: Department of Economics, University of Zurich, thomas.graeber@econ.uzh.ch. Ortoleva: Department of Economics and SPIA, Princeton University, pietro.ortoleva@princeton.edu.

1 Introduction

A central insight of behavioral economics is that valuation and choice are reference dependent. Standard models capture this by assuming that reference points directly alter the utility of outcomes—for example, that people *dislike* outcomes below the reference point, as in classic Prospect Theory (Kahneman and Tversky, 1979). In this class of models, people know how much utility they get from each option, and the reference point affects this utility.

This paper proposes an alternative view: reference dependence arises because reference points provide *comparative information* to decision makers who are unsure how to value objects or actions. We show that the simple ideas of value uncertainty and learning from reference points generate reference dependence (thinking in gain-loss categories), income targeting and bunching, and several other empirical regularities often treated as unrelated to reference dependence. This framework also yields a new theory of endogenous reference point formation, with several new predictions.

We start from the premise that people are often uncertain about the utility value of options. For instance, they may not know how to translate attributes into utility, they may be uncertain about the characteristics of a good, or the good may be too complex to evaluate. However, even when absolute utility assessments are difficult, it may still be feasible to determine whether an outcome is better than a reference point. For instance, I may be unable to assess the absolute utility of making \$90 today, but I know that it is less than the utility of the \$100 I typically make. The idea behind this paper is that such comparisons provide information that generates reference dependence.

Consider the Tversky and Kahneman (1991) improvements vs. tradeoffs effect. Suppose I must choose between two new jobs relative to my current one (the reference). Job A pays more for fewer hours (two improvements). Job B pays even more, but for more hours (tradeoff). Because I know the utilities of my current wage and hours, comparing them to Job A allows me to rule out low-utility realizations, increasing my posterior evaluation of Job A in both dimensions. For Job B, instead, the tradeoff means that my utility beliefs get truncated from above in one dimension and from below in the other, producing conflicting informational "boosts." That positive and negative dimension-by-dimension comparisons boost utility evaluations up and down is the hallmark characteristic of reference dependence, yet here it arises purely through an informational channel—the decision maker is uncertain about utility but can easily compare each option with the known reference point. In line with the evidence, this increases the probability of choosing Job A over Job B.

Deriving reference dependence from information theory offers three main advantages. First,

it highlights a tight relationship between reference dependence and value uncertainty, with associated new comparative statics predictions. Second, the model links reference dependence to a wide range of other behavioral phenomena—from contrast effects to coherent arbitrariness—within the same framework. Third, treating reference points as information allows us to endogenize them: in our model, the endogenous reference point is optimal in the sense of being the most informative comparator for subsequent decisions by reducing value uncertainty. This generates sharp predictions about when the reference point is given by expected outcomes, the status quo, similarity-based memory, or other options. Our informational view of reference dependence is not in conflict with reference-dependent preferences; both may exist, and while some predictions are identical, others are not, potentially encouraging future empirical tests.

Why comparative information? As we summarize in Appendix A, a core insight in cognitive psychology is that performance on absolute judgment tasks (evaluating magnitudes) is substantially poorer than on comparative tasks (judging which of two stimuli is brighter, longer, heavier, etc.), suggesting that cognition is optimized for detecting differences rather than encoding absolute levels (Laming, 2009, 1984; Braida and Durlach, 1972; Garner, 1953).[1] Moreover, this body of work shows that while comparisons are easier than absolute evaluations, ordinal comparisons are especially precise (Luce and Green, 1974; Parducci, 1965; Laming, 1997). This evidence has led to the view that even ostensibly absolute cardinal judgments are often constructed from underlying (partially ordinal) comparisons (Braida and Durlach, 1972; Stewart et al., 2005, 2006). Our model formalizes these key ideas: comparisons provide information when absolute assessments are imperfect, and ordinal comparisons are particularly precise.

Setup. The model rests on three core premises. First, the decision maker (DM) is uncertain about how to value option attributes. We are agnostic about whether this uncertainty reflects preference uncertainty, aggregation uncertainty, or something else. The DM holds a prior over the utility contribution of each attribute, which can be interpreted as the expected utility associated with that attribute before observing the option at hand.

Second, the DM has access to a reference point, the utility of which is fully or partially known. For example, the DM may have experienced an outcome, and thus (possibly imperfectly) remembers its utility. The DM receives both absolute and comparative information, dimension-by-dimension. First, he receives an absolute signal about the utility of the attribute. Second, a comparative ordinal signal about whether the outcome is better than the reference point in that dimension. This comparative, ordinal information is the novelty of our setup and it implies that

[1] Even though they model reference points as affecting utility, Kahneman and Tversky (1979) already note that "the perceptual apparatus is attuned to the evaluation of changes or differences rather than to the evaluation of absolute magnitudes."

our DM effectively partitions outcomes into gain-loss categories. We allow this ordinal signal to be noisy such that the DM gets the correct signal with a higher probability when the utility difference between the option and the reference point is large. This implies that, on average, our DM responds not only to ordinal comparisons but also to the underlying cardinal differences.

Third, our DM is Bayesian about his utility uncertainty. Our model thus builds on the "Bayesian brain" literature in cognitive science (Anderson, 2013; Oaksford and Chater, 2007; Gershman, 2021; Griffiths et al., 2024). However, what matters for our model is not so much that the DM is literally Bayesian, but rather that they combine imperfect utility information with comparative signals, and recognize the informational value of reference points.

Posteriors after comparative information. Focusing on Normally distributed priors and signals, we show that posterior beliefs admit a simple structure: the sum of an *absolute* utility evaluation of the outcome and a *relative*, reference-dependent component that depends on the distance between the outcome and the reference point. This reference-dependent term captures the positive or negative informational boost from the comparison.

This structure parallels classic models of reference dependence, in particular those combining absolute and relative utility (e.g. Sugden, 2003; Kőszegi and Rabin, 2006). However, there are several key distinctions. First, in our framework, reference dependence emerges purely through informational mechanisms and is directly driven by value uncertainty—in the limit, a DM with no value uncertainty exhibits no reference dependence. This predicts that reference effects will be more pronounced when a decision is difficult or unfamiliar. Second, reference dependence typically weakens when there is limited information about the utility value of the referent. This formalizes the widespread intuition that reference points must "sink in" to affect behavior. Third, classical models in the prospect theory tradition predict a contrast effect with respect to the reference point—higher reference points imply lower utility evaluations. In our model, the same is true (for informational reasons). However, our setup also predicts an assimilation effect with respect to *evaluations* of the reference point; intuitively, when the utility of the reference point is believed to be high, a positive ordinal signal means that the utility of the option is probably also high.

Contrast, attenuation and coherent arbitrariness. Relative to the benchmark of no value uncertainty, the DM's perceived utility is excessively sensitive around the reference point—where gain-loss categorization shifts—but is otherwise insufficiently sensitive to variation in outcomes. Therefore, our DM overreacts to small changes around the reference point but underreacts to large ones. Intuitively, uncertainty induces belief compression toward the prior, leading to attenuation, except around the reference point, where ordinal information produces a contrast effect.

The model, therefore, naturally accounts for the sometimes-puzzling empirical evidence on the coexistence of insensitivity and overreaction (e.g., Augenblick et al., 2025; Graeber et al., 2025).

Furthermore, the model captures the well-known evidence on coherent arbitrariness (Ariely et al., 2003). In our framework, initial valuations in the absence of a reference point are volatile, yet ordinal signals render subsequent valuations internally coherent. For instance, a DM may face high uncertainty regarding his willingness to pay (WTP) for 10 packets of pasta; yet, having estimated this value, he infers that his WTP for 11 packets must be higher. This yields stable relative behavior despite large underlying value uncertainty.

Income targeting and bunching. While our model features no built-in asymmetry between gains and losses, it nonetheless predicts a class of behaviors traditionally attributed to loss aversion, such as income targeting in labor supply decisions (e.g., Camerer et al., 1997; Abeler et al., 2011) or bunching at reference points (e.g., Pope and Schweitzer, 2011; Allen et al., 2017). Intuitively, the ordinal comparative signal relative to the reference point implies that the DM's perceived marginal utility is higher around the reference point than elsewhere. As a result, over a range of wages, our DM will work "just enough" to exceed the reference income level and avoid the negative informational "boost" from getting less than what he knows. This mechanism produces bunching around the reference point and a locally downward-sloping labor supply curve—the key stylized facts documented in the literature. This mechanism of a higher perceived marginal utility right below compared to right above the reference point is not too dissimilar from how Prospect Theory generates targeting and bunching, except that here it purely reflects beliefs rather than true utility.

Theory of endogenous reference points. Because reference points in our model only reduce value uncertainty, we propose that the endogenous reference point is the alternative that is, on average, most useful for evaluating the current choice. This theory of endogenous reference points is an as-if account: it does not require individuals to consciously choose reference points. Rather, just as we typically bring to mind memories that are broadly relevant to the situation at hand, we may also bring to mind useful reference points. For example, when assessing the value of a high-quality wine, we naturally compare it to other high-quality wines we have tried, rather than to the worst ones.

Formally, this reduces to a problem of optimal information choice. We show that, in valuation problems with quadratic losses, the informational value of a reference point admits a simple closed-form characterization: when all potential reference points are equally well-known, the most informative reference point is precisely the expected outcome. When the options near the expected outcome are not well known, there is a bias-variance trade-off: a well-known option, like the status quo, can be more informative if it is not too different from what typically happens.

In addition, options that provide clearer ordinal signals are more informative. For example, potential reference points presented in the same format as the option will have an informational advantage.

We also link our view of reference points as information to work on similarity-based memory. We consider environments in which the outcome distribution is itself unknown, such as when the DM does not know the average utility of the objects in a store. In this setting, we prove that with Normally distributed signals, the most informative reference point in the presence of a contextual cue is given by a formula that *exactly* matches the canonical similarity-based memory functional form used in psychology and economics (Kahana, 2012; Bordalo et al., 2020). This shows that the assumptions of value uncertainty and informative comparisons provide a possible microfoundation for the influential work on memory-based reference points.

Choice: Overweighting advantages and menu effects. The idea that DMs learn from comparisons applies not only to reference points but also to other choice options. We model choice assuming that the DM receives ordinal signals about the relative ranking of options within each dimension. We show that this naturally leads the DM to favor options with many advantages. For instance, an extreme case is that when value uncertainty is sufficiently high, the DM acts as if he counts advantages, with each dimension weighted by its utility relevance and its uncertainty. Although this resembles familiar heuristics, it is the Bayesian procedure in our setup.

The model also generates classic decoy effects, including asymmetric dominance. Intuitively, introducing a dominated decoy produces additional positive comparisons for the target option. This mechanism parallels existing informational models of menu effects (Natenzon, 2019). However, unlike those models, ours predicts such effects even when a good does not globally dominate the decoy, as long as there are sufficiently many dimensions of asymmetric dominance.

Caution. Finally, we extend our model to incorporate caution—uncertainty aversion with respect to one's own value uncertainty (Cerreia-Vioglio et al., 2015, 2024), akin to a form of ambiguity aversion (Gilboa and Marinacci, 2011). Under this additional assumption, our model captures the endowment effect and other forms of status quo bias. Intuitively, comparative ordinal signals disproportionately reduce value uncertainty for options closer to the reference point. The interaction between uncertainty aversion and the ordinal signal thus means that our DM values options more highly when they are (close to) the reference option.

Reference dependence as information and / or preferences? Our starting point is that when DMs are unsure about valuations, reference points contain information. We show that incorporating this informational channel produces several widely documented behavioral patterns traditionally associated with reference dependence. Of course, this does not rule out the existence

of genuine reference-dependent preferences or loss aversion. We deliberately exclude such preferences to isolate the explanatory power of information. We view empirically distinguishing the relative importance of these channels as an important topic for future research.

Related literature. Our paper contributes to the recent literature on the cognitive foundations of behavioral economics. We connect the literature on cognitive noise and uncertainty to work on reference dependence. Relative to existing models of value uncertainty or cognitive noise (e.g., Gabaix, 2014, 2019; Woodford, 2020; Khaw et al., 2021; Vieider, 2024), we model reference points as information. The closest antecedent is Villas-Boas (2024), who also models a DM who receives an ordinal signal relative to a reference point, but does not examine the applications or endogenous reference points we consider. Our idea of learning from reference points is also inspired by Natenzon (2019) and subsequent work by Shubatt and Yang (2024), who model DMs that learn from other choice options through comparisons. Augenblick et al. (2025) study a setting in which people understand the direction in which they need to update beliefs, but not the magnitude. In Woodford (2012), efficient coding yields a steeper slope of perceived utility around the prior, yet his model does not study reference points. Wu (2024) presents a non-Bayesian model of contrast and assimilation. In a similar vein, our paper relates to the extensive literature on incomplete preferences or preference imprecision.[2]

Second, our paper builds on the enormous literature on reference dependence (Barberis, 2013), including work that has endogenized the reference point as expectations (Gul, 1991; Loomes and Sugden, 1986; Kőszegi and Rabin, 2006) or similarity-based memory (Bordalo et al., 2020). Unlike these approaches, we study reference effects from the perspective of value uncertainty and information, which also yields new predictions about what the reference point is. More broadly, we relate to the large body of work that emphasizes the comparative nature of cognition, including in attention (e.g., Bordalo et al., 2012; Kőszegi and Szeidl, 2013; Bushong et al., 2021) or belief formation (e.g., Bordalo et al., 2016; Ba et al., 2025). Section 8 discusses limitations of our model and which psychological mechanisms—such as genuine loss aversion, selective attention, entirely uninformative cues, or categorizations other than gain-loss—we do not capture.

Finally, there is some empirical evidence consistent with our model's implication that the magnitude of reference effects increases with value uncertainty. Graeber ⓡ Enke (2026) show experimentally that contrast and assimilation with respect to comparison points weaken when the decision problem is less difficult. Graeber et al. (2025) show that investor responses to

[2]Among recent contributions, see Masatlioglu and Ok (2005, 2014); Sagi (2006); Butler and Loomes (2007, 2011); Ok et al. (2011); Agranov and Ortoleva (2017); Cubitt et al. (2015); Chakraborty (2021); Nishimura and Ok (2021); Cerreia-Vioglio et al. (2024) and many references therein.

earnings surprises exhibit stronger reference effects when assets are more difficult to value.

The paper proceeds as follows. Section 2 illustrates the core intuition of our model. Sections 3 and 4 present the general model and apply it to different regularities. Section 5 endogenizes the reference point. Section 6 studies binary choice, Section 7 caution and Section 8 concludes.

2 An Illustrative Example

To build intuition, we sketch a simple special case of our model here. Suppose a DM evaluates the utility of a one-dimensional option, $U(x) = u_x$. For instance, he may evaluate his willingness-to-pay for a car, where the relevant attribute is MPG. The DM is uncertain about the value of u_x. He is Bayesian and has a Normally distributed prior, $u_x \sim \mathcal{N}(\tilde{u}_x, \sigma_p^2)$. This is represented by the dashed line in the top left panel of Figure 1.

The DM receives two types of information about u_x. First, through deliberation, he gets a cardinal signal about the level of u_x, $s = u_x + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. Here, σ_ϵ^2 reflects the DM's uncertainty in assessing the utility. Second, the DM has a reference point r with utility u_r. The DM receives an ordinal signal o that tells him whether the option he evaluates is better or worse than the reference point, dimension-by-dimension: $u_x - u_r \gtrless \nu$. For now, assume that the utility of the reference point is known and that $\nu = 0$, meaning that the ordinal comparison is noiseless. As explained below, when ν is not constant, the ordinal signal is noisy in such a way that the probability of getting the correct comparison increases in the cardinal utility difference. Thus, in our full model, cardinal—rather than only ordinal—differences matter for average behavior

From the cardinal signal s, the DM obtains a posterior that follows the standard Normal-Normal shrinkage formula, with mean $\mathbb{E}[u_x|s] := \tilde{u}_x^s = \lambda s + (1-\lambda)\tilde{u}_x$. This posterior is illustrated in grey in the left panel of Figure 1. What happens with an ordinal signal? The belief gets truncated—giving us the density depicted in red in the top left panel of Figure 1. Naturally, this "boosts" the posterior mean, which is given by

$$\mathbb{E}[u_x|s,o] = \underbrace{\tilde{u}_x^s}_{\text{Ref.-ind. component}} + \underbrace{\sigma_x}_{\text{Weight of ref.-dep. comp.}} \underbrace{\psi(\tilde{u}_x^s - u_r, o)}_{\text{Gain-loss component}} \tag{1}$$

While the exact formula for the gain-loss component $\psi(\cdot)$ involves the inverse Mills ratio, it

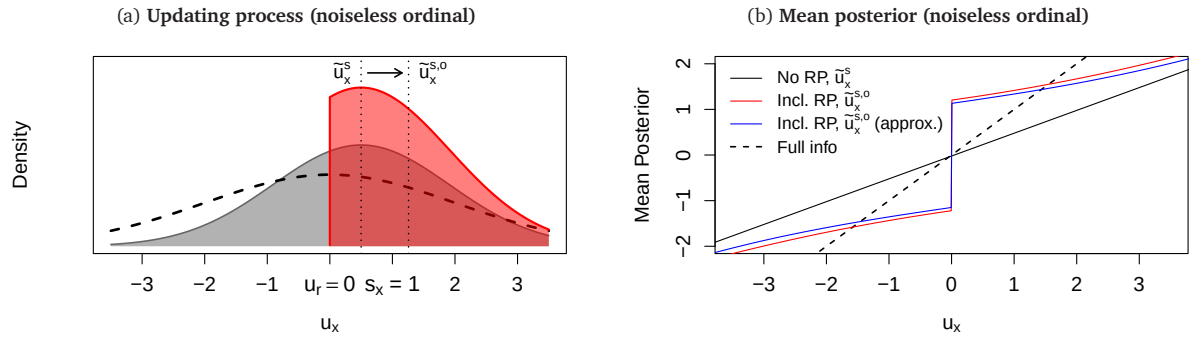


Figure 1: Illustration of the updating mechanism. The left panel shows a 'snapshot' of the posterior distributions at $u_x - u_r = 1$. In this snapshot, the prior (dashed black) is updated by a cardinal signal $s_x = 1$ to form the posterior $\tilde{u}_x^s$ (grey distribution, mean at left dotted line). The ordinal signal ($u_x > u_r$) truncates the posterior distribution, resulting in the red distribution. The arrow at the top illustrates the resulting 'boost' to the mean, which is shifted to $\tilde{u}_x^{s,o}$. The right panel plots the mean posterior as a function of u_x, where we fix $u_r = \tilde{u}_x = 0$. The figure compares the mean posterior with *only* the cardinal signal (black line, $\tilde{u}_x^s$), the posterior including the ordinal signal (red line, $\tilde{u}_x^{s,o}$), a piece-wise linear approximation (blue line), and the full-information benchmark.

admits a very good piece-wise linear approximation:

$$\psi(\tilde{u}_x^s - u_r, o) \approx \begin{cases} \max\left\{\sqrt{\frac{2}{\pi}} - \frac{2}{\pi\sigma_x}(\tilde{u}_x^s - u_r)\ ; 0\right\} & \text{if } u_x > u_r \\ -\max\left\{\sqrt{\frac{2}{\pi}} + \frac{2}{\pi\sigma_x}(\tilde{u}_x^s - u_r)\ ; 0\right\} & \text{if } u_x < u_r. \end{cases} \tag{2}$$

The top right panel of Figure 1 plots the average posterior mean as the true value of u_x varies. Two key forces are evident in this figure: *attenuation* away from the reference point, and a *comparative boost* (contrast effect) around the reference point. These two effects reflect that the posterior mean in Eq. (1) is the sum of two components. First, an *absolute* (reference-independent) term that reflects the Bayesian shrinkage combining the cardinal signal with the prior. This generates the standard Bayesian attenuation of perceived utility to variation in the option's objective value. Second, a *relative* or *comparative*, reference-dependent component. Intuitively, a signal that the utility of the option is above the utility of the reference point boosts the mean belief because it rules out low utility values.

The structure in eq. (1) is reminiscent of theories of reference-dependent preferences that model utility as a convex combination of absolute and gain-loss utility. However, here the weight of the gain-loss component endogenously reflects the DM's value uncertainty—in our model, there is no reference dependence when the DM perfectly knows u_x.

The model sketched here assumes that the utility of the reference point is perfectly known and that the ordinal signal is always correct. When we relax these assumptions below, the

mean posterior becomes smoother around the reference point, but the patterns of contrast and attenuation remain.

Which reference point? We propose that the endogenous reference point is the option that is *most informative* for a choice by reducing value uncertainty. This is an as-if formulation—clearly, individuals do not consciously choose their reference points. But just as people are known to retrieve memories that are roughly useful for the situation at hand, they may also retrieve reference points that are, on average, most useful for assessing the options they face.

As a simple illustrative example, suppose the DM knows that they will want to assess the utility u_x of an alternative and would like to minimize the Mean Square Prediction Error. The value of u_x is drawn from a Normal distribution of known mean $\tilde{u}_x$ and variance σ_x^2. Two reference points are available. First, the status quo r_{SQ}, which has a known utility value but is biased—its value differs from the expected outcome $\tilde{u}_x$ by some amount b. Second, an alternative reference point r_{A}, which is given by the expected outcome $\tilde{u}_x$, but its utility is not perfectly known, with variance σ_{A}^2. Assume for now that the DM expects no further information and no ordinal noise (both assumptions are relaxed below). Which reference point provides more information? We show that this has a simple characterization. The status quo is the most informative reference point if and only if its bias satisfies

$$b \leq b^* \approx \frac{3}{2}\sigma_x \sqrt{\log\left(1 + \frac{5}{4} \cdot \frac{\sigma_{\text{A}}^2}{\sigma_x^2}\right)}.$$

The status quo is the more informative reference point whenever its utility is not too far from the expected outcome, and when the uncertainty about the expected outcome is sufficiently large. The intuition is simple. Reference points that are too low or too high will almost always yield the same ordinal comparison, which would be uninformative. We show that, if all reference points are equally well-known, the most informative reference point is exactly at the mean of the distribution of u_x (the expected outcome). On the other hand, uncertainty about the reference point's utility diminishes its informational value. These two forces create a bias-variance tradeoff, captured by the formula above.

Moreover, we show below that once the decision maker does not know the distribution of outcomes but has access to a contextual cue, a well-known formula from research on similarity-based memory turns out to characterize the informationally optimal reference point.

3 Formal Framework and Reference Points as Information

3.1 Framework: Objects of Uncertain Values

A decision maker (DM) faces one or more alternatives defined by n dimensions. Each option x returns a total utility of

$$U(x) = \sum_{i=1}^{n} u_{x,i},$$

where $u_{x,i}$ denotes the utility of x from dimension i, for $i = 1, \dots n$. For example, the item x in question may be a car, and the various $u_{x,i}$ may be the utility from speed, seat comfort, *etc*.

The DM knows this structure, but is uncertain about each $u_{x,i}$.[3] This uncertainty can have different sources. First, the DM may be uncertain about the actual characteristics of the car, such as its true speed. Second, the DM may be uncertain about the utility from a given characteristic: they may know that the car's maximum speed is 280 km/h, but may be unsure of the utility this provides. Third, the DM may struggle to aggregate across dimensions. We remain agnostic about the source of the uncertainty.

The individual is Bayesian and holds priors about each unknown variable. Throughout the paper, we denote by $\tilde{z}$ the mean of the belief about the random variable z, and $\tilde{z}^A$ the posterior mean after information A. For tractability, we assume that the prior about each $u_{x,i}$ is distributed according to $\mathcal{N}(\tilde{u}_{x,i}, \sigma^2_{p,i})$ and that values are independent across dimensions of the same good ($u_{x,i}$ and $u_{x,j}$ are independent for any i, j with $i \neq j$) and across goods ($u_{x,i}$ and $u_{y,i}$ are independent for all i and all x, y with $x \neq y$).

In addition to the knowledge implicit in the prior, the DM also receives *cardinal* information about the utility in each dimension, which we interpret as the outcome of a deliberation process. For each $i = 1, \dots, N$, the DM receives a signal $s_{x,i} = u_{x,i} + \varepsilon_{x,i}$, where $\varepsilon_{x,i} \sim \mathcal{N}(0, \sigma^2_{\varepsilon,i})$.[4] When no confusion arises, we will drop the subscript x. Thus far, this is a standard choice model with value uncertainty and cardinal signals.

Before proceeding, two remarks are in order. First, we focus on Normal priors and signals for simplicity. As will be clear in the discussion that follows, while the exact functional forms of our results rely on Normality—mostly, because truncated Normal distributions are well-understood and well-behaved—the core intuition generalizes. Second, we assume additively separable utility across dimensions, again for simplicity; our results naturally extend, suitably adapted.

[3] For evidence on such uncertainty see, among many others, Butler and Loomes (2007); Cubitt et al. (2015); Agranov and Ortoleva (2017); Enke and Graeber (2023); Enke et al. (2024); Halevy et al. (2023).

[4] Each $\varepsilon_{x,i}$ is assumed to be independent of all other variables in the model.

3.2 Reference Points as Information

We study situations in which the DM evaluates option x while having (at least mental) access to another object, r, with utility $u_{r,i}$ in the same n dimensions. Here, we treat r as the reference point; later, we show how the same framework can be used to model situations in which r is something else, such as another option in the choice set. We maintain reference points as exogenous for now and endogenize them in Section 5.

We consider both situations in which the utility of r in each dimension is fully known, and when it is not. We assume that the DM believes $u_{r,i} \sim \mathcal{N}(\tilde{u}_{r,i}, \sigma^2_{r,i})$ for all i, where $\sigma^2_{r,i} = 0$ captures the case of no uncertainty. Uncertainty about the utility from the reference point may arise from limited experience with the reference point or from memory imperfections. Prior to receiving any further information, r and x are treated as independent.

Our main innovation is to assume that the DM receives *ordinal* signals, in each dimension, about whether it is the reference point r or the item at hand x that has higher utility. We allow these signals to be noisy, and assume that the DM obtains signal $o_{x,i}$ with values

$$o_{x,i} = \begin{cases} + & \text{if } u_{x,i} - u_{r,i} \geq \nu_{x,i} \\ - & \text{if } u_{x,i} - u_{r,i} < \nu_{x,i} \end{cases} \tag{3}$$

where $\nu_{x,i} \sim \mathcal{N}(0, \sigma^2_{o,i})$ for $i = 1, \ldots, n$.[5] (We will drop the subscript x when no confusion arises.) The parameter $\sigma^2_{o,i}$ captures the noise in the ordinal signal. Importantly, as we discuss in detail below, the noisiness of the ordinal signal operates in such a way that the probability of receiving the correct ordinal signal increases in the distance between the true utility of the option and the reference point. Not only does this correspond to the Probit functional form widespread in economics, but it also aligns with standard principles in psychology and economics regarding just noticeable differences.

As $\sigma^2_{o,i} \to 0$, the ordinal signal becomes perfect, regardless of the distance between option and referent. In general, we should expect the difficulty of ordinal comparisons to vary across goods or dimensions: there may be little ordinal noise in comparing car MPGs, but more in qualitative dimensions, such as seat comfort.

To sum up, our model encompasses three types of uncertainty. Suppose you're estimating the value of a new TV, which has only one relevant dimension (resolution). You have a prior about the utility of owning a TV and get a cardinal signal by looking at the new TV, both of which

[5] One may also want to consider the event in which $u_{x,i} - u_{r,i} = \nu_i$ as separate and associated to a signal "=." Since this is a probability-zero event (given our absolutely continuous distributions), to avoid extra notation and since nothing changes, for simplicity, we leave this possibility aside and associate this event with the signal "+."

have uncertainty ($\sigma^2_{p,i}$ and $\sigma^2_{\epsilon,i}$). You additionally have access to the utility from a comparison TV—the one you currently own—and that you think you like pretty well, but you're not entirely sure how well ($\sigma^2_{r,i}$). Moreover, you think that the target TV has higher resolution than your current TV, but again, you're not fully certain ($\sigma^2_{o,i}$).

Cardinal comparative information. In our model, the comparative information is ordinal: the DM can tell whether $u_{x,i} \gtrless u_{r,i}$ but not by how much. While this resonates with a large literature in psychology emphasizing that ordinal reasoning is especially simple and precise (see Appendix A), there are clearly also contexts in which people have a sense for *how much* better or worse an option is compared to a reference quantity. We now discuss to what extent our model already captures this intuition, and how it can easily be extended to capture it in full.

Our DM *does* engage in cardinal reasoning, in two ways. First, the absolute signal about $u_{x,i}$ is cardinal in nature. Second, and more importantly for the present discussion, the comparative ordinal signal is specified such that the cardinal difference between the option and the reference point matters. The reason is that the DM receives the correct ordinal signal with a probability that increases in the utility difference between x and r, see eq. (3). As a result, average posterior beliefs directionally vary with the cardinal utility difference $(u_{x,i} - u_{r,i})$ in the same way as in a model with a cardinal comparative signal as specified below. This aspect of our formulation is deliberate. While we could instead have specified the probability of receiving the correct ordinal signal as independent of the cardinal utility difference, our formulation implicitly allows for cardinal differences to affect the comparative information (without giving up the importance of ordinal information) in a simple and tractable way.

However, should readers wish to incorporate a direct cardinal comparative signal into our setup, then this is also straightforward to do. Concretely, suppose that in addition to the ordinal signal, the DM also receives a cardinal comparative signal, $d_i = u_{x,i} - u_{r,i} + \zeta_i$, where $\zeta_i \sim \mathcal{N}(0, \sigma^2_{d,i})$. In fact it is possible to rewrite our model in a form that includes such a signal, albeit with some restrictions on the relevant covariance matrix. It is also straightforward to extend our results to incorporate an unrestricted version of this signal in a simple and tractable way. Both results are included in the Appendix.

3.3 Perceived Utility with Reference Points

Given the additively separable structure of $U(x)$ and independence across dimensions, we can focus on each dimension separately, as the multi-dimensional case is simply the sum across dimensions; therefore, we suppress the subscript indicating the dimension (we write u_x for $u_{x,1}$).

A standard result in Bayesian updating gives us that the posterior after the cardinal signal is

$N(\tilde{u}_x^s, \sigma_x^2)$, with $\tilde{u}_x^s = \lambda s + (1-\lambda)\tilde{u}_x$, where $\lambda := \frac{\sigma_p^2}{\sigma_p^2+\sigma_\epsilon^2}$, and $\sigma_x^2 := \frac{\sigma_\epsilon^2\sigma_p^2}{\sigma_\epsilon^2+\sigma_p^2}$. How does the ordinal signal affect it? The following proposition characterizes its effects on posterior means.[6]

Proposition 1. *The following is true:*

$$\mathbb{E}[u_x|o,s] = \tilde{u}_x^s + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\psi\left(\frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}, o\right) \tag{4}$$

$$\textit{where}\quad \psi(t,o) := \begin{cases} \frac{\phi(t)}{\Phi(t)} & \textit{if } o = + \\ -\frac{\phi(-t)}{\Phi(-t)} & \textit{if } o = -. \end{cases} \tag{5}$$

To illustrate, consider the case of a "+" ordinal signal. Then,

$$\mathbb{E}[u_x|+,s] = \underbrace{\tilde{u}_x^s}_{\text{Ref.-indep. component}} + \underbrace{\frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}}_{\text{Weight on ref.-dep.}} \underbrace{\frac{\phi\left(\frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\right)}{\Phi\left(\frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\right)}}_{\text{Ref.-dependent component}} \tag{6}$$

The posterior mean after the cardinal and ordinal signals is the sum of two components. First, an *absolute* (reference-independent) term that reflects the DM's mean belief about the utility of the option combining the cardinal signal with the prior through a Bayesian shrinkage, giving rise to attenuation.[7] Second, a *relative* or *comparative*, reference-dependent component: a "boost" in beliefs obtained from the "+" ordinal signal, where the weight of the boost depends on the DM's uncertainty. This formula is the generalization of the one used in our example in the previous section, Eq. (1), allowing for uncertainty about u_r and noise in the ordinal signal, and giving the correct expression of the boost rather than an approximation.

Through the comparative boost, the mean belief is repelled (or contrasted away) from the reference point. Formally, this boost is given by the so-called inverse Mill's ratio that has attracted much attention in the econometrics literature on correcting for sample selection bias (Heckman, 1979). Note how the boosts have the expected signs: it is immediate to show that $\psi(t,+) > 0$ and $\psi(t,-) < 0$ for all t.

Proposition 1 describes beliefs *conditional* on signal realizations. While this makes the mechanics clear, signal realizations may not be observed by the researcher. Perhaps a more empiri-

[6]Proposition 7 in Appendix B.1 characterizes the effects on the variance. The posterior variance is, on average, increasing in the distance from the reference point. Intuitively, being close to the reference point means being close to a point whose utility one knows, and the further one moves away, the larger the posterior variance becomes.

[7]We understand attenuation as $\partial\mathbb{E}_{s,o}\left[\mathbb{E}[u_x|s,o]\right]/\partial u_x < 1$.

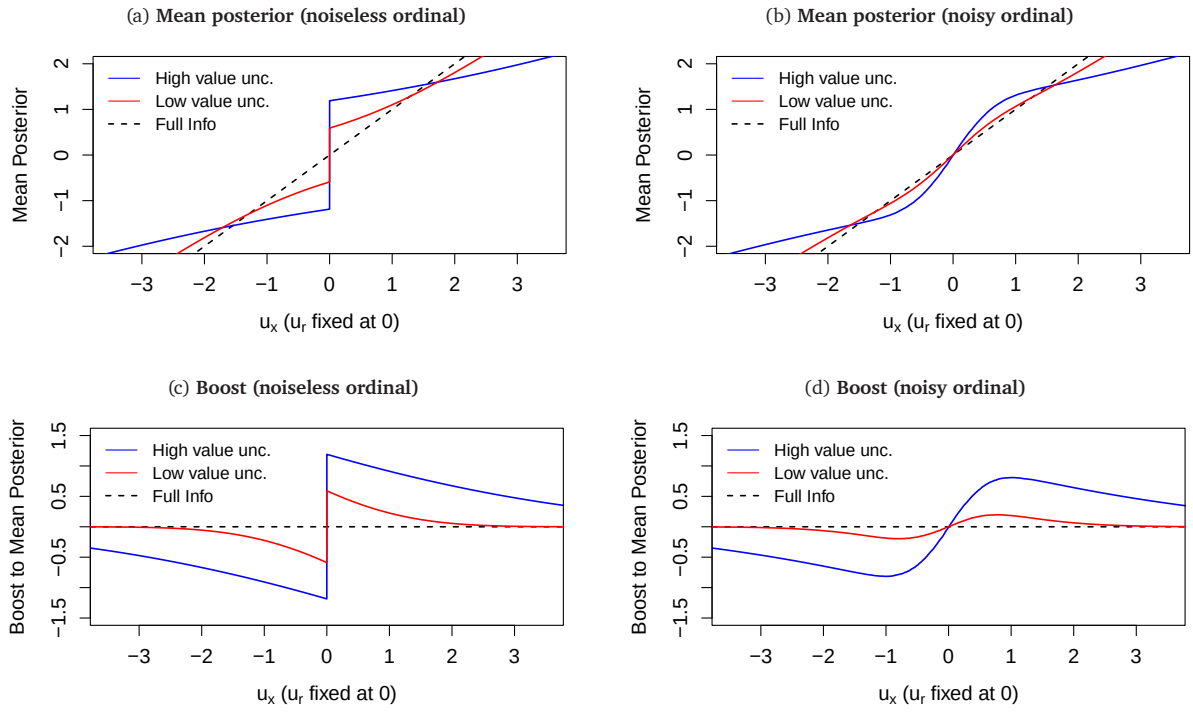


Figure 2: Illustration of the mean posterior (top panels) and its reference-dependent boost component (bottom panels) as a function of value uncertainty σ_x, separately for noiseless and noisy ordinal signals. We set $\tilde{u}_x = u_r = \tilde{u}_r = 0$ and then vary u_x (and thus s_x). All panels compare a 'low value uncertainty' DM ($\sigma_x = 0.75$, red line) with a 'high value uncertainty' DM ($\sigma_x = 2$, blue dashed line), where we set $\sigma_p = 2$. The left panels (a, c) illustrate the case with a *noisy reference, noiseless ordinal* signal ($\sigma_r = 0.25, \sigma_o = 0$). The right panels (b, d) illustrate the case with a *noisy reference, noisy ordinal* signal ($\sigma_r = 0.25, \sigma_o = 0.5$). The top row (a, b) plots the mean posterior, $\tilde{u}_x^{s,o}$, while the bottom row (c, d) plots the reference-dependent 'boost', $\psi(\cdot)$. Simulation conducted over 10,000 draws.

cally relevant metric is average beliefs across possible signal realizations. A full characterization of average beliefs is given in Proposition 8 in Appendix B. Figure 2 illustrates the average mean posterior and average reference-dependent component for different levels of σ_x. The figure fixes the reference point and thus implicitly shows comparative statics with respect to u_x.

Mean beliefs are given by the sum of the reference-dependent boost and the reference-independent beliefs based on the cardinal signal. Does this combination lead to over- or under-estimation? The answer is both, depending on how close u_x is to u_r. Suppose $u_x > u_r$ (the patterns are symmetric for the opposite case). As long as the ordinal noise is not too big and priors are aligned, the average posterior belief will be strictly above the true u_x when u_x is close to u_r, as the boost dominates. As the difference between them increases, the mean belief will eventually go below the true value—the boost becomes weaker and attenuation dominates.

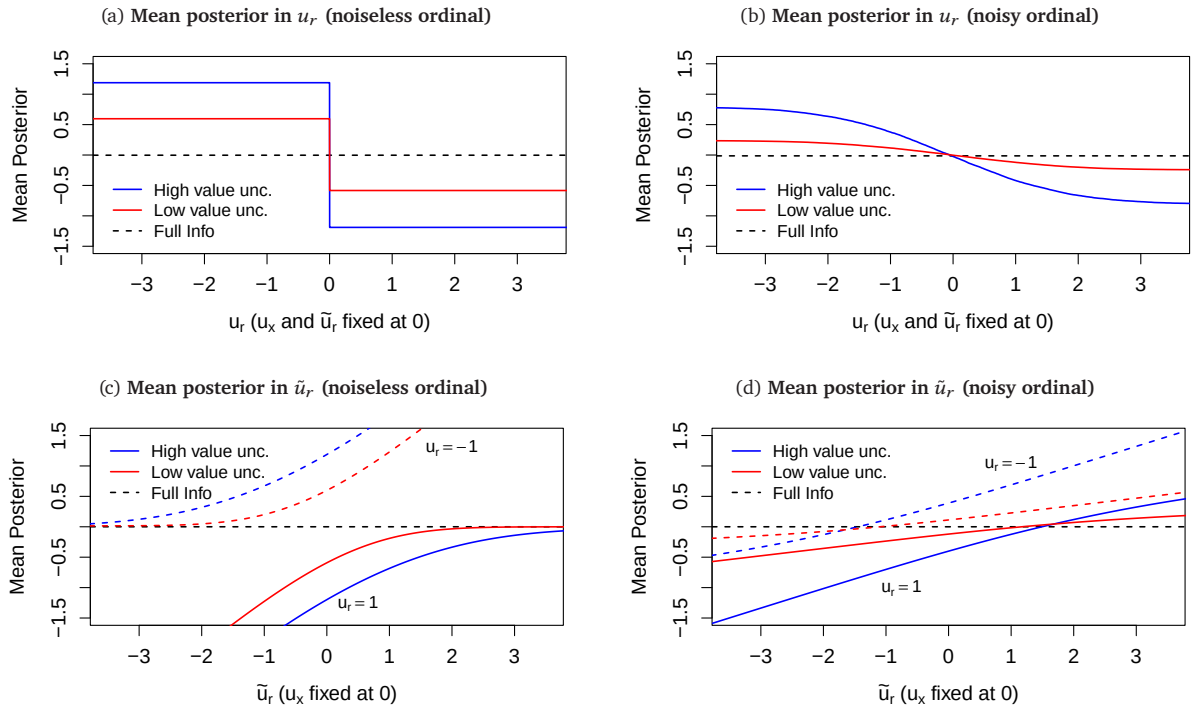


Figure 3: Illustration of comparative statics with respect to reference quantities. The top panels show the mean posterior as a function of the reference point u_r, holding constant $u_x = \tilde{u}_r = 0$. The bottom panels show the mean posterior as a function of the reference evaluation $\tilde{u}_r$, holding $u_x = 0$ and fixing $u_r \in \{-1, 1\}$ (solid lines: $u_r = 1$; dashed lines: $u_r = -1$). The left panels (a,c) illustrate the case with a *noisy reference, noiseless ordinal* signal ($\sigma_r = 0.25$, $\sigma_o = 0$). The right panels (b,d) illustrate the case with a *noisy reference, noisy ordinal* signal ($\sigma_r = 0.25$, $\sigma_o = 1.5$). All panels compare $\tilde{u}_x^{s,o}$ for a 'low value uncertainty' DM ($\sigma_x = 0.75$, red lines) with a 'high value uncertainty' DM ($\sigma_x = 2$, blue lines), where we set $\sigma_p = 2$ and fix prior $\tilde{u}_x = 0$. Simulation conducted over 10,000 draws.

3.4 Comparative Statics

We now study how the mean posterior (which we also refer to as 'evaluation') varies as a function of (i) the option, u_x, (ii) the reference point, u_r, and (iii) the different types of uncertainty in the model, $\sigma_x^2, \sigma_r^2, \sigma_o^2$. We first discuss the results at an intuitive level and then provide a formal statement of the impact of these variables on mean beliefs. Appendix B.4 additionally provides comparative static results for the reference dependent boost in eq. (4), both conditional on and averaging across signal realization.

Comparative statics with respect to the option. An increase in the value of the option can impact beliefs through two channels: by changing the distribution of the cardinal signal received about u_x, and by changing the probability of receiving a positive ordinal signal.

Conditional on receiving a positive ordinal signal, Proposition 1 tells us that higher beliefs $\tilde{u}_x^s$ will lead to a *lower* boost. This follows from the fact that $\psi(t, +)$ is a decreasing function, and is also intuitive: the impact of learning that u_x is above u_r should be smaller if you already think

$u_x - u_r$ is large. Thus, an increase in u_x will, on average, increase $\tilde{u}_x^s$ and decrease the impact of a positive boost. By a symmetric argument the same change will increase the magnitude of a negative boost.

Absent ordinal noise, a change in the value of u_x only changes ordinal relationships as it moves from right below to right above u_r. This is the case illustrated in panel (c) of figure 2. Below u_r an increase in u_x acts to increase the magnitude of the negative boost. At u_r there is a discontinuous jump as the ordinal signal switches from negative to positive. Above u_r, further increases act to decrease the magnitude of the positive boost.

In the case of ordinal noise, the effect of an increase in u_x will be smoother: the probability of receiving a positive signal will increase continuously, rather than jump discontinuously. This is the case shown in panel (d) of Figure 2. Generically the maximal average boost will now occur strictly above u_r: close to u_r the DM is just as likely to receive a positive as a negative ordinal signal, meaning the average boost will be close to zero. This can be seen in Appendix E Figure 11, which shows a heatmap of the full distribution of the mean posterior for a given value of $u_x - u_r$.

To obtain the effect of changes in u_x on mean beliefs, the effect of the boost must be combined with the effect on the reference independent portion of beliefs. By standard attenuation arguments, these will respond sluggishly due to the noise in the cardinal signal. The overall effect can be seen in the top panels of Figure 2: beliefs are attenuated away from the reference point and excessively sensitive close to it, where the implicit change in gain-loss categorizations through the ordinal signal produces a contrast effect. Absent ordinal noise such over-sensitivity only happens for changes that move u_x from below to above the reference point. With ordinal noise other changes close to the reference can also lead to over-sensitivity.

Comparative statics with respect to the reference point. The mean posterior in eq. (4) has novel comparative statics implications regarding the role of the reference point. As with the option itself, two distinct—but interrelated—reference quantities enter the mean posterior: the beliefs about the reference point — $\tilde{u}_r$ — which affects the size of the boost, and the true utility of the reference point — u_r — which affects the probability of which ordinal signal will be received.

Fixing the ordinal signal, an increase in $\tilde{u}_r$ *increases* the evaluation of the reference point, through the size of the boost. Intuitively, one's belief about the utility of an option is higher when you are told that it is better than an object that has a utility of one million than when you are told it is better than one that has a utility of zero. This produces an 'assimilation effect', according to which a higher reference quantity pushes evaluations up. This is illustrated in the bottom panels of Figure 3.

Second, fixing beliefs, the average posterior mean of u_x *decreases* in the true utility of the reference point u_r. The reason is that u_r enters (4) by affecting *which* ordinal signal is observed, and a higher reference point makes it more likely that the comparative signal is negative, pushing down beliefs on average. As shown in the top panels of Figure 3, this 'contrast effect' exists locally around the reference point when the ordinal signal is noiseless, and globally once the ordinal signal is noisy and thus implicitly partly depends on the cardinal difference between u_x and u_r.

The overall effect of a change in the utility of the reference point u_r will be determined by the relative importance of these two effects. This in turn will depend on how beliefs $\tilde{u}_r$ depend on u_r. Depending on the parameterization of the problem either of the two channels could dominate. For example, if ordinal signals are noisy and $\tilde{u}_r$ is insensitive to u_r,[8] the contrast effect will dominate and an increase in the reference point will decrease the evaluation of x. On the other hand, if ordinal signals are noiseless and $\tilde{u}_r$ is very sensitive to u_r, then an increase in u_r that does not change the sign of $u_x - u_r$ will increase the evaluation of x as the assimilation effect will dominate.

Comparative statics with respect to uncertainty. Proposition 1 makes it clear that the reference-dependent component in our model also depends on (i) the degree of value uncertainty about the option, σ_x^2; (ii) uncertainty about the utility of the reference point, σ_r^2; and (iii) how noisy the ordinal information is, σ_o^2. This highlights a central implication of our approach, and a key point of difference with classic models of reference dependence—that the effect of the reference point depends on uncertainty.

Figures 2 and 3 illustrate how the magnitude of value uncertainty about the option modulates the effects of u_x and the reference point. A key insight is that the boost is bigger when x is less well-known. As a result, the degree of reference dependence—such as the strength of the contrast effect around the reference point—increases in value uncertainty.

As discussed above, our model typically predicts overestimation of u_x when it is right above the reference point and underestimation when it is far above. An increase in σ_x^2 has two effects on this over- and underestimation. First, as we have just seen, it will increase the boost. Second, the Bayesian shrinkage (and, as a result, behavioral attenuation) becomes more pronounced. As a result, an increase in value uncertainty will generally *push up* the posterior mean close to the reference point, yet *push it down* far away from the reference point. That is, it amplifies the contrast and attenuation effects. This effect can be seen by comparing the red and blue lines in the top panel of Figure 2.

The uncertainty in the knowledge about the reference point (σ_r^2) and in the ordinal signal

[8]For example due to a strong prior about the value of the reference point.

(σ_o^2) affect the boost in exactly the same way. If the ordinal signal is unexpected ($\tilde{u}_x^s \leq \tilde{u}_r$), the boost is decreasing in both—intuitively, more uncertainty gives smaller updating.

More surprisingly, when the ordinal signal is widely expected (the ordinal signal is positive and $\tilde{u}_x^s - \tilde{u}_r$ is large), then the boost can be *increasing* in both σ_r^2 and σ_o^2. As we explain in Appendix B.4, this is because, if $\tilde{u}_x^s$ is a long way above $\tilde{u}_r$ and there is no noise in the reference process, a positive signal is essentially uninformative—the DM already knows that x is better than r. However, the more r is imperfectly known, the more uncertainty there is about ex-ante values, and the more informative the signal is. (A similar logic applies to the role of σ_o^2).[9]

Formal statement of main comparative statics predictions. We conclude this section by summarizing and formally stating what we believe to be the key comparative statics predictions of our model for evaluations (such as willingness-to-pay). For ease of exposition, we focus here on what we believe to be the most interesting special cases and relegate a complete and general analysis of comparative statics to Appendix B.4. We focus on the case of $u_x > u_r$ as the opposite case is symmetric. The first proposition formalizes comparative statics with respect to the option and the uncertainty terms. The first part says that—relative to the full-information benchmark—the DM's evaluation is too sensitive to variation in u_x close to the reference point, and attenuated far away from it, where both of these effects are more pronounced the higher the DM's value uncertainty about the option. The second part says that if the ordinal noise is small, there is overestimation of u_x when it is close to the reference point and underestimation when it is far from it. The third part says that both over- and under-estimation increase in the uncertainty about u_x and decrease in the uncertainty about u_r.

The next proposition formalizes the main comparative statics with respect to the reference point. The first part says the posterior mean exhibits a contrast effect with respect to the reference point and an assimilation effect with respect to the reference evaluation, both of which increased in value uncertainty. The second part clarifies that, if only the reference point (or its underlying attributes) is observable, its observed net effect will...

3.5 Applying the Model: Average Posterior Means and Approximations

A difficulty in adopting the formulas in Proposition 1 in theoretical or empirical applications is that they involve Inverse Mills ratios, which, although relatively well understood, are hard to work with analytically. Fortunately, these admit a relatively simple approximation based on a

[9]When averaging over signal realizations an increase in the ordinal noise σ_o^2 has the additional effect of increasing the probability that contradictory signals are received. This has the effect of 'flattening out' the average boost,as can be seen when comparing panels (c) and (d) of Figure 2.

first-order Taylor approximation around $\psi(0) = \sqrt{\frac{2}{\pi}}$. Appendix D.2 shows that a good approximation for applications is given by the piece-wise linear function

$$\psi(t,o) \approx \psi^a(t,o) = \begin{cases} \psi^a_+(t) := \max\left\{\sqrt{\frac{2}{\pi}} - \frac{2}{\pi}t\ ;0\right\} & \text{if } o = + \\ \psi^a_-(t) := -\max\left\{\sqrt{\frac{2}{\pi}} + \frac{2}{\pi}t\ ;0\right\} & \text{if } o = -. \end{cases} \tag{7}$$

We also provide an approximation of the average boost across signal realizations, $\mathbb{E}_{o,s}\left[\mathbb{E}[u_x|o,s]|u_x,u_r\right]$, which is fully characterized in Proposition 8 in Appendix B. Let $t := \frac{\lambda u_x + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$. Then, the average posterior mean belief can be approximated by

$$\mathbb{E}[\tilde{u}_x^{s,o}] \approx \underbrace{\lambda u_x + (1-\lambda)\tilde{u}_x}_{\substack{\text{Bayesian shrinkage to} \\ \text{prior; Attenuation}}} + \underbrace{\frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}}_{\text{Weight of ref.-dep.}} \underbrace{\overline{\psi}^a(t,u_x,u_r,\sigma_o)}_{\text{Avg. Boost Approx.}}$$

$$\text{where} \quad \overline{\psi}^a(t,u_x,u_r,\sigma_o) := \underbrace{\left[\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\psi^a_+(t) + \left(1-\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)\right)\psi^a_-(t)\right]}_{\text{Gain / loss boosts weighted by prob. of correct ordinal}}. \tag{8}$$

When the ordinal signal is noiseless, the last part simplifies to the gain-loss component in the illustrative example in Section 2. Appendix D.3 shows simulations that illustrate how this is indeed a very effective approximation.

4 Applications of Reference Points as Information

4.1 Contrast, Assimilation and Differences with Prospect Theory

Differences with Prospect Theory. The mean perceived utility in our model resembles, in some ways, the structure of prospect theory's value function: steeper around the reference point and attenuated away from it. At the same time, there are several crucial differences.

First and foremost, our model is entirely information-based. We make no assumptions about preferences and instead simply study the beliefs of a Bayesian DM who receives comparative signals with respect to reference points. As a result, our model makes the distinctive comparative-statics prediction that people respond more to reference points when their value uncertainty is higher (see Figures 2 and 3). This is in line with Graeber et al. (2025), who show that market returns following earnings surprises are strongly inverse-S shaped around the reference point of

no earnings surprise, and that this effect is substantially more pronounced for firms for which valuation uncertainty is higher. Related, Graeber ⓡ Enke (2026) shows experimentally that reference point effects are more pronounced when a decision problem involves more uncertainty about how to map problem inputs into valuations and decisions.

Second, our model does not feature any gain-loss asymmetry. While below we show that it can generate several empirical patterns typically predicted by Prospect Theory through loss aversion, it does not, as stated, generate loss aversion itself (even in riskless choice) or the endowment effect. In Section 7 we discuss a natural extension of our model that does generate these predictions by introducing aversion to value uncertainty (caution).

Third, as we show next, our model explains various empirical regularities that are not explained by Prospect Theory, such as coherent arbitrariness, contrast, assimilation and decoy effects.

Contrast and assimilation. In prospect theory and its variants, a higher reference point always *decreases* the evaluation of an option. This captures the compelling intuition that if I make \$100 today, I am happier if I made \$10 rather than \$80 yesterday. This contrast effect with respect to the true attribute values (or the resulting utils) is present also in our model. Yet our model adds the prediction that evaluations increase in subjective beliefs about the reference point, holding fixed the true value of the reference point. For example, our model also predicts that if I make \$100 today, I am happier if I believe that my utility from making \$80 yesterday was 300 rather than 200.

In many cases. we would expect the true value of the reference point and its perceived utility to move together. Indeed, in most datasets, only one of these two is observable. However, a recent body of work both in economics and in perceptual psychology measures both and provides evidence that is consistent with our model's predictions.

Graeber ⓡ Enke (2026) propose that comparison points produce assimilation effects when they consist of decisions / evaluations, yet contrast effects when they are given by exogenous problem attributes. They implement experiments that induce both a reference point and a reference evaluation. For instance, subjects are asked to state their willingness-to-accept (WTA) for completing 10 real effort tasks, while being told that a different subject stated a WTA of \$8 for completing 20 tasks. Through the lens of our model, the reference point u_r is (some function of) the number of effort tasks the other subject was offered, while the evaluation of the reference point, $\tilde{u}_r$, is given by (some function of) the other subject's WTA. Graeber ⓡ Enke (2026) find that stated WTA increases in the other subject's WTA and simultaneously decreases in the other subject's offered workload, in line with the predictions of our model.

There is related evidence also in the perceptual domain in psychology. In experiments in which subjects sequentially estimate the magnitude of visual stimuli, a recent result is that subjects' estimates decrease in the reference stimulus (e.g. the previous stimulus) but increase in the subject's estimate of the reference stimulus (e.g. Moon and Kwon, 2022; Sadil et al., 2024). Thus, again, beliefs exhibit a contrast effect with respect to the true reference quantity (u_r) but increase in the evaluation of the reference quantity ($\tilde{u}_r$).

Finally, in terms of field evidence, Jin et al. (2024) find that physicians' evaluations of a patient increase in their evaluation of the previous patient (the reference evaluation) and—most importantly for our purposes—that this assimilation effect is stronger when medical uncertainty is higher or the physician is less experienced or more fatigued. This is exactly what our model predicts: that higher evaluative uncertainty (difficult case, less experience, more fatigue) increases the reliance on the reference evaluation.

4.2 Within- vs. Between-Subjects Experiments

It is well-understood in experimental economics that within-subjects designs potentially produce confounds that are immediately related to contrast effects. In reviews of the literature, Charness et al. (2012) and List (2025) suggest that subjects may be more responsive to treatments in within-subjects designs because "subjects have a reference or comparison point when responding to the second question" (Charness et al., 2012). Indeed, Schäfer and Schwarz (2019) report that the average effect size in within-subject psychology experiments is significantly larger than that in between-subject ones.

Consider one of the leading examples in Charness et al. (2012), a willingness-to-pay elicitation. Suppose that in the within-subjects experiment, the DM first evaluates x and then y (by generating signals s_x and s_y), where WLOG $u_x < u_y$; suppose the DM has identical priors between them. Then, the average difference in utility evaluations in the between-subjects design (without an ordinal signal) is

$$\mathbb{E}_{s_x,s_y}[\mathbb{E}[u_x|s_x] - \mathbb{E}[u_y|s_y]],$$

while that in the within-subjects design is

$$\mathbb{E}_{s_x,s_y,o}[\mathbb{E}[u_x|s_x] - \mathbb{E}[u_y|s_y, o]].$$

Proposition 10 shows that, under its assumption, the latter is smaller (more negative), because the ordinal signal boosts the evaluation of u_y away from u_r. Thus our model predicts stronger

within- than between-subject effects.

4.3 Over- and Underreaction

Suppose the DM's status quo consumption is r, and he now evaluates x. When will the DM over- or underreact to the difference between the two, relative to the true utility difference? More formally, assuming $x > r$, when is $\mathbb{E}_{s_x,o}[\mathbb{E}[u_x - u_r|s_x, o]] > u_x - u_r$?

Proposition **??** implies that the DM will overreact to small changes provided that the ordinal signal is not too noisy, and that he will underreact to large changes. The first effect is driven by the boost from the comparative signal and the second one by Bayesian attenuation. The dual pattern of overreaction to small and underreaction to large changes is a recurring one in the literature (e.g., Augenblick et al., 2025; Ba et al., 2025; Graeber et al., 2025; Meier et al., 2025).

4.4 Coherent Arbitrariness

In an influential paper, Ariely et al. (2003) make the observation that experimental subjects' initial valuations of an option are fairly arbitrary, yet their subsequent decisions are coherent in the sense that higher prices lead to lower demand, or higher quality to higher willingness-to-pay. Ariely et al. (2003) discuss these results as follows. "Suppose that the subject indicates, for whatever reason, that she would be willing to purchase the average bottle [of wine] for $25. If we were to ask her a moment later whether she would be willing to purchase the "rare" bottle [of wine] for $25, the answer would obviously be "yes," because from her perspective this particular "choice problem" has been solved and its solution is known: if an average wine is worth at least $25, then a rare wine must be worth more than $25!"

We believe that our model provides a natural interpretation of these results and intuitions. First, in the model, initial evaluations of u_x are noisy (and regressive to a prior) and thus, to some extent, arbitrary. Yet once the DM also observes an ordinal signal about how x compares to the previous option (the reference point), subsequent evaluations will be coherent in the sense that they will respect the ordinal ranking implied by the option attributes. In our model, highly noisy absolute evaluations and coherent responses to changes are entirely compatible with each other.

4.5 Income Targeting and Bunching

A canonical implication of reference dependence is bunching right at, or just above, a reference point, such as when workers' accumulated earnings bunch at an income target (Camerer et al., 1997; Abeler et al., 2011; DellaVigna et al., 2017, 2022) or when the performance of athletes

bunches at salient goals or reference points (e.g., Pope and Schweitzer, 2011; Allen et al., 2017). We now illustrate that our model also leads to bunching, focusing on the standard example of labor supply; the same intuition holds more generally.

A DM chooses hours a to work at wage $w > 0$. Utility is $U(a, w) = u(aw) - a^2/2$, where $u(aw) = \beta aw$. In general, the DM may face value uncertainty in both the earnings and effort-cost dimensions. We here focus on value uncertainty in the money dimension. This could be interpreted either as saying that there is more utility uncertainty about what to do with the money earned, or that the reference point is more precise in the earnings dimension than in the effort dimension.[10] Value uncertainty in the earnings dimension, $u(aw)$, can reflect uncertainty about β, the wage, or the mapping between them.

For each a, the DM has a prior about $u(aw)|a \sim \mathcal{N}(\overline{\beta w}a, \sigma_p^2)$ and receives cardinal signals, $s_a = u(aw) + \epsilon^a$, $\epsilon^a \sim \mathcal{N}(0, \sigma_\epsilon^2)$, where $\overline{\beta w}$ could be interpreted as average marginal benefit in the past. Note that, to be realistic, we assume the prior mean varies with the action so that, in the prior, working more hours delivers higher utility in the earnings dimension. The DM has a reference point given by earnings of r and knows that the utility of this earnings level is $u_r = \beta r$. For simplicity, we assume they receive a noiseless ordinal signal o_a about how each $u(aw)$ compares with the utility of the reference point. (Introducing ordinal noise leads to similar results, though 'bunching' is less sharp.)

Given that our DM understands monotonicity between the reference point and his earnings, it is further realistic to assume that the DM also receives ordinal signals about the dimension-specific utility ranking between different actions: he should know that working six hours delivers higher earnings than working five. This means that, for each action (except boundary actions), the DM receives two additional dimension-specific ordinal signals—comparing with the closest higher and the closest lower actions; without these, independent error realizations may lead to non-monotone posteriors, which is unrealistic. However, computing the Bayesian posterior mean for this rich signal structure is analytically intractable due to the interdependent truncations introduced by the many ordinal signals across actions. Fortunately, the posterior mean about the utility of each action can be well approximated using an alternative tractable model.

A simpler problem. Consider the problem above, but suppose instead that the error term in the cardinal signals is perfectly correlated across actions—$s'_a = u(aw)+\epsilon$ with $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$—yet the DM computes posterior beliefs ignoring this correlation. Because the error term in the cardinal signal is correlated, in this version, posterior beliefs always respect monotonicity. Crucially,

[10]Our results would reverse if we assumed the opposite—value uncertainty or a precise reference point only in the effort-cost dimension—just as standard loss-aversion models would predict effort targeting rather than income targeting when loss aversion applies to effort but not to money.

Appendix D.4 presents simulations to show that the distribution of posterior means is remarkably similar to that induced by the original model with all ordinal signals; we will therefore focus on this simpler version, which can be solved using standard tools.[11] For a given realization of ϵ, the DM's maximization problem is given by

$$\max_{a\in\mathbb{R}_+} \lambda(\beta w a+\epsilon)+(1-\lambda)\overline{\beta w}a+\sigma_x\,\psi\left(\frac{\lambda(\beta w a+\epsilon)+(1-\lambda)\overline{\beta w}a-\beta r}{\sigma_x},o_a\right)-\frac{a^2}{2} \tag{9}$$

where $o_a = +$ if $aw \geq r$, and $o_a = -$ otherwise.[12] Denote $a^*(w)$ its solution for any $w > 0$. This is a standard maximization problem except for the discontinuous, positive jump at $a = \frac{r}{w}$. It will be useful to compare its solution to the solution of the equivalent problem without reference point, call it $a^{\text{no rp}}(w)$.[13]

Proposition 2. *There exist* $0 < w_1 < w_2 < w_3$ *such that:*

1. *(Bunching and downward-sloping labor supply).* $a^*(w) = \dfrac{r}{w}$ *and* $\dfrac{\partial a^*(w)}{\partial w} = -\dfrac{r}{w^2} < 0$ *for all* $w \in (w_1, w_3)$.

2. *(Bunching of earnings).* $a^*(w)w$ *is strictly increasing for* $0 < w < w_1$*, is discontinuous at* w_1*, is constant and equal to* r *for* $w \in (w_1, w_3)$*, is discontinuous at* w_3*, and is strictly increasing for* $w > w_3$.

3. *(Comparison with reference-free).* $a^*(w) < a^{\text{no rp}}(w)$ *for* $0 < w < w_1$ *and* $w > w_2$*,* $a^*(w) > a^{\text{no rp}}(w)$ *for* $w_1 < w < w_2$.

Figure 4 illustrates these results. Without reference points, the optimal action is strictly increasing in w, with standard behavioral attenuation (Enke et al., 2024). Adding reference effects (i) decreases the expected value of earnings below r and (ii) increases it above that. This has several implications. When wages are low, the reference point reduces the perceived marginal benefit of earnings—recall that the boost is negative and decreasing as we approach the reference point from below, see Figure 2. Lower marginal benefits then imply a lower optimum given the convex cost of effort, and the DM *reduces* their effort compared to the case without

[11] To see why this is a good approximation, note that the models are identical except that the former has i.i.d. noise terms but ordinal signals across actions, and the latter has perfectly correlated noise but no ordinal signals across actions. While the ordinal signals of the former model yield monotone posteriors, the latter model achieves them automatically. But then, calibrating the noise terms will lead to a very similar distribution of posteriors outside boundary actions—precisely what the simulation shows.

[12] Note that this relies on the simplifying assumption that the DM receives a positive ordinal signal in the case of equality.

[13] That is, $a^{\text{no rp}}(w)$ is the solution to $\max_{a\in\mathbb{R}_+} \lambda(\beta w a+\epsilon)+(1-\lambda)\overline{\beta w}a-\frac{a^2}{2}$, meaning $a^{\text{no rp}}(w) = \lambda\beta w+(1-\lambda)\overline{\beta w}$.

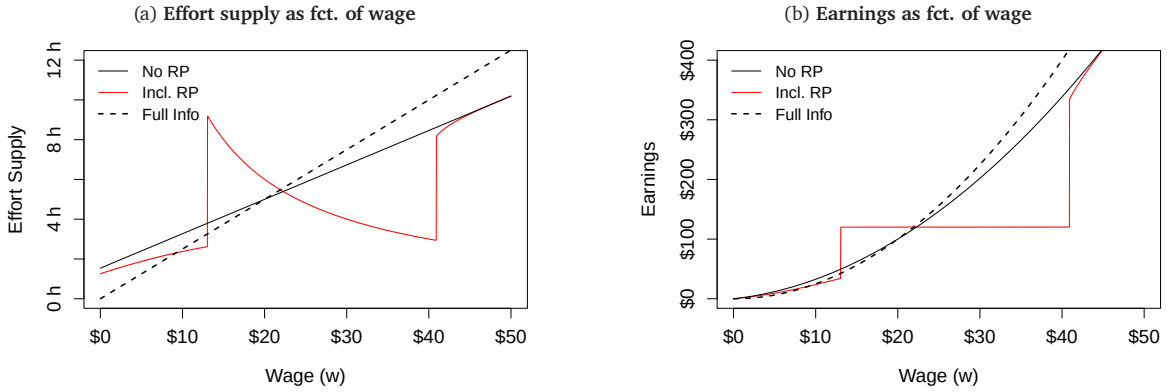


Figure 4: Effort supply and earnings as a function of the wage, with and without reference point. Parameters are $\underline{\beta} = 1$, $\beta = 1$, $\overline{w} = 20$, $\gamma = 4$, $\sigma_\epsilon = 80$, $\sigma_p = 120$, and reference earnings of \$120. Both panels show a single DM with noise realization $\epsilon = 0$.

reference points—a discouragement effect. This effect continues (and becomes stronger) until wages grow sufficiently high (w_1 in the proposition) that it is worth "pushing it" and reaching the reference point r: at that point, both action and earnings discontinuously jump to meet the earnings threshold as the DM seeks to avoid the negative boost. As wages further increase, however, optimal earnings stay put at r: as long as we are above the reference point, marginal benefits of effort drop and stay low. In turn, this means that a^* decreases as the wage increases (since wa^* stays constant). This continues until wages grow enough that increasing the action is again optimal—the action jumps discontinuously and converges (from below) to $a^{\text{no rp}}(w)$. While Figure 4 plots the actions and earnings of a single DM (with a given noise realization), averaging across multiple DMs with independent noise realizations would give smoother patterns.

5 A Theory of Endogenous Reference Points

Empirical studies have shown that many alternatives can serve as reference points, including the status quo, expectations, memory, or the outcomes of others. Our model offers a novel approach to predicting what the reference point will be in a given situation. Since we understand reference points as a source of information, it is natural to look at which reference point is *most informative*— i.e., reduces value uncertainty and thus leads to the best subsequent choice, in expectation. This provides a simple, tractable criterion for endogenizing the reference point, grounded in information theory.

We begin by making several observations about our approach. First, it may seem to demand an unrealistic level of sophistication to assume that people can "choose" the reference point. However, this need not be a conscious choice; rather, as is common in decision science, this

selection may be an unconscious background process. Much like it is widely accepted that people often unconsciously bring to mind pertinent memories for each situation, they may also make use of a reference point that is most relevant for a given choice situation. For instance, when we evaluate the utility from a high-quality wine, we arguably compare it with other high-quality wines we've had in the past, rather than with the worst wine we've ever tried. Moreover, well-documented patterns in selective and associative memory may in fact implement precisely this kind of relevance-based retrieval mechanism.

Second, we formulate a model in which a different reference point can be selected for each choice situation. While theoretically convenient, in practice it is more plausible that the same reference point is used across a certain *class* of choice situations. Our approach could easily be extended in this way.

Third, we assume that any reference point can be used at zero cognitive cost. This abstracts from the possibility that some reference points could exogenously be made transparently available at low or zero cost (making them more likely to get adopted), for example when an experimenter induces a reference point.

Fourth, we restrict ourselves to a single reference point. Of course, the logic of reference points as information suggests that people may compare the current choice option with many past experiences. Without explicitly modeling it, we assume that it is infeasible to select multiple reference points. Relaxing this assumption represents an interesting potential future extension.

Fifth, we are agnostic about the nature of the mental construct that may serve as a reference point. As we show below, a DM may wish to adopt a reference point that aligns with the expected outcome they will face. Most naturally, past experiences and memories can play this role; however, reference points may also be formed through mental simulation or imagination, even in the absence of direct experience. For example, a DM may never have lived in Los Angeles but may nonetheless form an imagined representation of life there and use it as a benchmark when evaluating other cities. As we detail below, for a reference point to be effective, it must (i) have utilities that are sufficiently well understood and (ii) be ordinally comparable to the object of interest. Thus, a DM may know that Los Angeles offers more seafood restaurants than Denver and may also have a clear sense of how much they value seafood restaurants. From a modeling perspective, we therefore allow for a broad class of potential reference points, with potentially infinite ordinal or reference point uncertainty to capture the idea that unexperienced or weakly imagined reference points are not very useful.

5.1 The Informational Value of a Reference Point

We first endogenize the reference point in a setting in which the DM is asked to value a single option. We extend to a choice between two options in Section 6.5. A DM knows they will face option x with utility $u_x \sim \mathcal{N}(\tilde{u}_x, \sigma_p^2)$. They wish to understand the value of reference point r, of which they know $u_r \sim \mathcal{N}(\tilde{u}_r, \sigma_{p,r}^2)$. They also know that, at the time of evaluation, they will receive (*i*) an ordinal signal o about u_x vs. u_r with noise σ_o^2 and (*ii*) cardinal signals, $s_x = u_x + \epsilon_x$ and $s_r = u_r + \epsilon_r$, with $\epsilon_x \sim \mathcal{N}(0, \sigma_{\epsilon_x}^2)$, $\epsilon_r \sim \mathcal{N}(0, \sigma_{\epsilon_r}^2)$.[14]

After observing all signals, the DM infers u_x and chooses action a to minimize the Mean Square Prediction Error (MSPE). We can think of a as the agent's estimate of u_x. We can then define the benefit of reference point r as the difference in the expected MSPE with and without the ordinal signal it generates, that is,

$$\mathrm{V}(r) := \underbrace{\mathbb{E}_{s_x}[\min_{a\in\mathbb{R}} \mathbb{E}[(u_x - a)^2 | s_x]}_{\text{Expected MSPE without } r} - \underbrace{\mathbb{E}_{s_x,s_r,o}[\min_{a\in\mathbb{R}} \mathbb{E}[(u_x - a)^2 | s_x, s_r, o]}_{\text{Expected MSPE using } r}.$$

It turns out that this measure admits a particularly simple characterization and comparative statics. (We use the same notation as in previous sections: $\tilde{u}_x^{s_x}$ and σ_x^2 are the posterior mean and variance of the belief about u_x after observing s_x; the equivalent is true for $\tilde{u}_r^{s_r}$ and σ_r^2.)

Lemma 1. *The informational value of reference point r in evaluating x is*

$$V(r) = \frac{\sigma_x^4}{\sigma_x^2 + \sigma_r^2 + \sigma_o^2} \mathbb{E}_{s_x,s_r}\left[h\left(\frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right)\right] \tag{10}$$

where $h(x) := \frac{\phi^2(x)}{\Phi(x)(1-\Phi(x))}$.

The Lemma shows that the informational benefit of a reference point is captured by a simple function which, unsurprisingly, is formally related to that of our boosts—indeed, one may notice that h is the product of the boosts in both directions. Note that when no further information is expected (that is, $\sigma_{\epsilon_x}^2, \sigma_{\epsilon_r}^2 \to \infty$), Lemma 1 further simplifies to

$$V(r) = \frac{\sigma_x^4}{\sigma_x^2 + \sigma_r^2 + \sigma_o^2} h\left(\frac{\tilde{u}_r - \tilde{u}_x}{\sqrt{\sigma_x^2 + \sigma_r^2 + \sigma_o^2}}\right). \tag{11}$$

[14] We focus on one-dimensional objects for simplicity; extending to multiple dimensions is straightforward. By considering $\sigma_p^2, \sigma_{p,r}^2$ separately from $\sigma_{\epsilon_x}^2$ and $\sigma_{\epsilon_r}^2$, we allow a general treatment that allows for cases in which additional information is going to be revealed or not. Moreover, this allows for a general traetment of whether the evaluation of the reference point occurs before memories are accessed (high σ_x^2, σ_r^2, low $\sigma_{\epsilon_x}^2, \sigma_{\epsilon_r}^2$), or after they are ($\sigma_{\epsilon_x}^2 \sigma_{\epsilon_{p,r}}^2 \to \infty$).

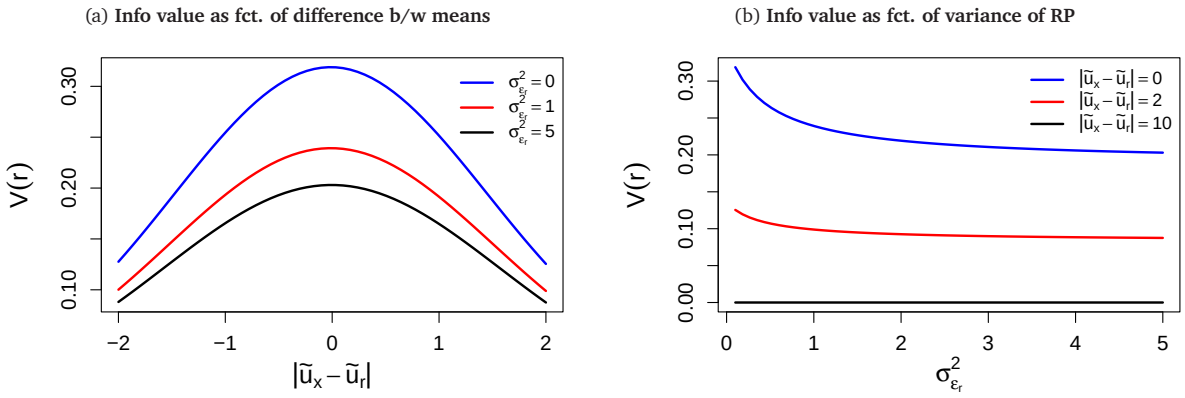


Figure 5: Comparative statics about the informational value of reference points. Left panel: $V(r)$ as a function of $|\tilde{u}_x - \tilde{u}_r|$. Right panel: $V(r)$ as a function of $\sigma^2_{\epsilon_r}$.

The most informative reference point and the bias-variance trade-off. The simple functional form in Lemma 1 leads to a first comparative static on the information benefits of a reference point.

Proposition 3. *If $\sigma_x^2 > 0$, $V(r)$ is strictly decreasing in $|\tilde{u}_r - \tilde{u}_x|$ and in $\sigma^2_{\epsilon_r}$.*

This result provides a simple rule of thumb to assess the information value of reference points. First, reference points are more informative the more the DM knows about them—a small $\sigma^2_{\epsilon_r}$. This is immediate: the more *precise* the DM's knowledge about r, the more informative any ordinal comparison is going to be. Second, reference points are more informative the closer they are to the *mean* of the distribution of u_x, that is, with a small *bias* $|\tilde{u}_r - \tilde{u}_x|$. To see why, note that when the reference point is centered exactly at $\tilde{u}_x$, we have an equal chance of either ordinal signal, and both ordinal signals are moderately informative. If, instead, $\tilde{u}_r$ is far above $\tilde{u}_x$, the ordinal signal $u_x < u_r$ is almost guaranteed and (correspondingly) not very informative, while the ordinal signal $u_x > u_r$ would be very informative, but also very unlikely.

Figure 5 visualizes these results. These two criteria in evaluating a reference point—how well-known and how well-centered it is—imply the existence of a bias-variance tradeoff. Yet when all potential reference points are equally well-known, the criteria also have a simple implication, which is that the expected outcome is the most informative reference point. This connects to theories in which the reference point is given by average outcomes or expectations, except that here it arises endogenously and purely for informational reasons.

Expected outcome vs. status quo. The tradeoff above has direct implications for when the status quo will be the most informative reference point. Consider a DM looking to buy a car. As a reference point, they can either use the car they currently own, the status quo (SQ). They

know this car well ($\sigma^2_{p,r_{\text{SQ}}} = 0$), but it is likely to be significantly worse than the ones in the store ($\tilde{u}_{r_{\text{SQ}}}$ is well below $\tilde{u}_x$). Alternatively, he can use a car that his friend recently bought (A), which he knows less well ($\sigma^2_{p,r_{\text{A}}} > 0$), but is more representative of the cars in the store ($\tilde{u}_{r_{\text{A}}} = \tilde{u}_x$).

The DM has a choice between a backward-looking reference point—the status quo, which is well known, but far away from what they expect to have to evaluate—or a forward-looking one—which is less well known but is selected to be close to the expected outcome. Our results give a precise rule for which is more informative. The optimal reference point is the status quo if it is not *too* far from what they expect to see in the store. There is an exact, unique threshold: Once $\tilde{u}_{r_{\text{SQ}}}$ and $\tilde{u}_x$ are sufficiently far apart, the friend's car A will be more informative. Similarly, the better the alternative reference point A is known (lower $\sigma^2_{\epsilon_{r_{\text{A}}}}$), the lower is this threshold. This can be formalized into a simple formula, a direct implication of Lemma 1.

Observation 1. *Consider r_{SQ} and r_{A} such that r_{SQ} is biased ($b_{\text{SQ}} := |\tilde{u}_{r_{\text{SQ}}} - \tilde{u}_x| > 0$) but perfectly known ($\sigma^2_{p,r_{\text{SQ}}} = 0$) while r_{A} is unbiased ($\tilde{u}_{r_{\text{A}}} = \tilde{u}_x$) but imperfectly known ($\sigma^2_{p,r_{\text{A}}} > 0$). Suppose for simplicity that no further information is expected and that there is no ordinal noise for either ($\sigma^2_{\epsilon_x}, \sigma^2_{\epsilon_{r_{\text{A}}}} \to \infty$, $\sigma^2_{o_{\text{SQ}}} = \sigma^2_{o_{\text{A}}} = 0$).*[15] *Then, r_{SQ} is a better reference point than r_{A} if and only if its bias is small enough relative to the uncertainty about x, that is,*

$$V(r_{\text{SQ}}) > V(r_{\text{A}}) \;\Leftrightarrow b_{\text{SQ}} < b^* \qquad \text{where} \quad b^* := \sigma_x h^{-1}\left(\frac{2}{\pi}\frac{\sigma_x^2}{\sigma_x^2 + \sigma_{r_{\text{A}}}^2}\right) > 0.$$

Moreover, b^* can be well-approximated by[16]

$$b^* \approx \frac{3}{2}\sigma_x\sqrt{\log\left(1 + \frac{5}{4}\cdot\frac{\sigma_{r_{\text{A}}}^2}{\sigma_x^2}\right)}.$$

Figure 6 plots the correct and approximated threshold.

Assuming that the status quo becomes better known over time, a direct implication of our model is that the "tenure" of an option as a status quo determines how likely it is to serve as a reference point. Central to this argument is the information value: a "just-adopted" status quo, that is less known, is also a less informative reference point.

Our results also speak to the experimental literature on reference points. A frequently made informal argument in this literature is that reference points (in particular expectations-based ones) need to "sink in" to affect behavior (see Heffetz, 2021, for a literature review). From our model's perspective, the simple explanation is that reference points are adopted only when the

[15] These results can be easily extended to the general case.

[16] Numerical simulations suggest errors are typically below 1.1% in the relevant range when $\sigma_{r_{\text{A}}} \leq 3\sigma_x$.

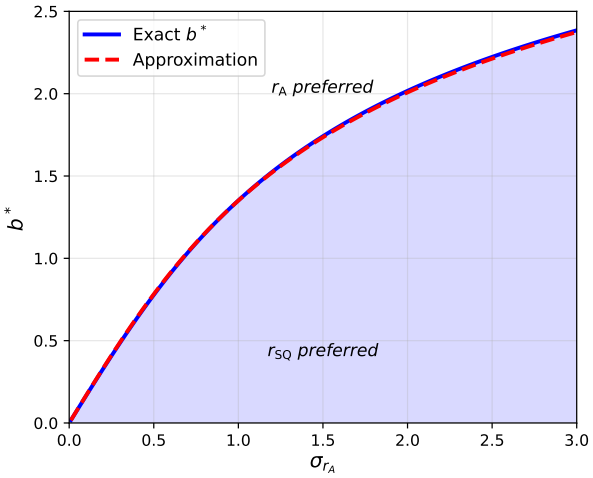


Figure 6: Bias-Variance tradeoff in most informative reference point. It assumes that r_{SQ} and r_A are such that r_{SQ} is biased ($b_{SQ} := |\tilde{u}_{r_{SQ}} - \tilde{u}_x| > 0$) but perfectly known ($\sigma^2_{p,r_{SQ}} = 0$) while r_A is unbiased ($\tilde{u}_{r_A} = \tilde{u}_x$) but imperfectly known ($\sigma^2_{p,r_A} > 0$). It also assumes $\sigma^2_p = 1$, no further information expected ($\sigma^2_{\epsilon_x}, \sigma^2_{\epsilon_{r_A}} \to \infty$), and no ordinal noise ($\sigma^2_{o_{SQ}} = \sigma^2_{o_A} = 0$).

DM has accumulated sufficient information about their utility values.

The role of ordinal noise. An additional comparative static result pertains to ordinal noise (and prior variance):

Proposition 4. *Suppose $\sigma^2_x > 0$. Then, there exists $\bar{\delta} > 0$ such that if $|\tilde{u}_r - \tilde{u}_x| < \bar{\delta}$, $V(r)$ is strictly decreasing in σ^2_o, while $V(r)$ is strictly increasing if $|\tilde{u}_r - \tilde{u}_x|$ is large enough. Similarly, $V(r)$ is strictly decreasing in $\sigma^2_{p,r}$ if $|\tilde{u}_r - \tilde{u}_x|$ is below a strictly positive threshold, while strictly increasing if $|\tilde{u}_r - \tilde{u}_x|$ is large enough.*

This shows that, as long as the bias is not too big, the DM prefers a reference point with smaller ordinal noise (and a smaller prior variance). This is intuitive, as less noise in the ordinal signal implies better information.[17] This means that reference points will be particularly useful when they allow precise ordinal comparisons, for example because they are nearby or because they are presented in similar formats and units. For these reasons, a DM may adopt a reference point that is easier to compare, even if it is less well-calibrated or less well-known.

[17]Notably, this holds only if the bias is not too large: when the bias is large, then the DM may actually prefer some ordinal noise or prior variance about r. This is because, when the bias is large the DM has little doubt about the ordinal signal they will get, meaning that they expect little information; however, adding some ordinal noise makes this expected ordinal signal more informative, because it lowers the (relative) chance of signals from values which are very close; this is the same intuition behind Proposition 10.(b).

5.2 Cues and Similarity-Based Memory Reference Points

A long literature in psychology formalizes memory through similarity-based recall (e.g. Tulving and Thomson, 1973; Shepard, 1987): essentially, the probability that a memory is recalled is proportional to its similarity to current circumstances. Building on this work, (e.g., Bordalo et al., 2020) model a DM whose reference point is given by similarity-based memory. Our model provides one potential microfoundation for this mechanism.

As noted above, when all potential reference points are equally well-known, the most informative one is given by the expected (average) outcome. Oftentimes, however, the DM may be unsure of the distribution of outcomes he faces and thus cannot directly use the expected utility outcome as a reference point. In this case, our DM tries to infer the average outcome from "cues" in the environment. For example, entering a new store, the DM may not know the average utility of clothes in this store, but he may observe contextual cues—how fancy is the store, are there mice on the floor? From these cues, the DM infers "what type of store" it is, and which reference point will be most informative to retrieve from memory. The idea is that the customer has different experiences stored in memory, and the cue provides information about which will one be most useful as a comparator. For instance, when the customer is served a glass of champagne upon entering a store, it is not particularly informative to retrieve as reference point the utility of the clothes from his college days.

Memory cues. Suppose the DM believes $u_x \sim N(\tilde{u}_x, \sigma_x^2)$, where the average outcome is uncertain itself, $\tilde{u}_x \sim N(\hat{\tilde{u}}_x, \sigma_\mu^2)$. Suppose that there is a cue c that is perfectly informative about the average outcome $\tilde{u}_x$ (this is just for simplicity and easily generalizes). The analyst observes this cue, while the DM observes it with some noise—he receives $c_o = c + e$, where $e \sim N(0, \sigma_e^2)$. The DM chooses a reference point from the set R of possibilities.

Imagine, for simplicity, that the DM has a wide set of potential reference points stored in memory. Given a cue c, what are the chances that the DM retrieves from memory the reference point $r = m$?

Corollary 1. *Suppose all potential reference points are available for a given noise level (that is, $\sigma_{p,r}^2$ and $\sigma_{e_r}^2$ are the same for all $r \in R$; and for each $z \in \mathbb{R}$, there exists $r \in R$ with $\tilde{u}_r = z$). Then, the probability density that m is the most informative reference point is*

$$\Pr(r = m|c) = \frac{a\, e^{-\tau(c-m)^2} \Pr(m)}{N} \tag{12}$$

where $a := \frac{1}{\sqrt{2\pi\, \sigma_{cond}^2}}$, $\tau := \frac{1}{2\, \sigma_{cond}^2}$, $\sigma_{cond}^2 := \frac{\sigma_x^2 \sigma_e^2}{\sigma_x^2 + \sigma_e^2}$ and N is a normalizing term.

This corollary follows immediately from our results.[18] It says that the probability of a particular memory becoming the reference point increases in its similarity to the cue. The most informative reference point is given by the expected mean of the object to be evaluated, conditional on the cue received. The probability of a given memory being used is therefore the probability of a cue that gives rise to such expectations, which is decreasing in the distance between the memory and the true situation.

Remarkably, the functional form in Eq. (12) is precisely the rule commonly used for recall or memory-based reference points in psychology (Kahana, 2012) and economics (Bordalo et al., 2020). The only difference is that here this is derived as optimal for a decision maker who has value uncertainty and uses the informational value of reference points.[19]

Summary. Viewing reference points through the lens of the most informative comparator for choice by reducing value uncertainty generates several natural predictions and links to earlier literature. We can summarize the results with four principles:

1. When the DM knows the distribution of outcomes and all potential reference points are equally well known, the most informative reference point is the one that aligns with the expected outcome (rational expectations).

2. When the DM knows the distribution of outcomes, yet some potential reference points are better known than others, a bias-variance tradeoff exists. If, for example, the status quo is known much better than the expected outcome, then it is the most informative reference point if it is not too biased.

3. In addition to the above, the DM will choose reference points that are easier to compare—for example, they are expressed in the same units, or in similar formats, or adjacent.

4. When the DM does not know the (mean of) the distribution of utility outcomes and all potential reference points are equally well known, the most informative reference point is determined by similarity-based recall.

[18] To see why, notice that, if all potential reference points are available and have the same noises, then by Lemma 1, the agent will choose the reference point r with expected utility equal to the posterior about $\tilde{u}_x$; that is, the probability that a given r is chosen is equal to the probability that the posterior about $\tilde{u}_x$ is equal to r. By Bayes' rule, the probability that the posterior is equal to r when the cue is c is equal to the probability that the cue was c given that the posterior is equal r times the prior of r, and normalized. But the probability that the cue is c given that the posterior is equal to r—thus that $\tilde{u} = c$—is simply the density of the posterior, $\frac{1}{\sqrt{2\pi\,\sigma^2_{\text{cond}}}} \exp\left(-\frac{(c-m))^2}{2\,\sigma^2_{\text{cond}}}\right)$, giving us the formula above.

[19] In this sense, our model relates to an earlier literature in psychology that emphasizes that similarity-based recall is sometimes optimal based on Bayesian principles (e.g., Anderson, 2013).

Finally, notice how our results could, in principle, also be used to study costly reference points. For instance, it may be cognitively costly to retrieve temporally distant experiences from memory, or it may be costly to conduct the ordinal comparisons that the reference point enables. Our machinery can be applied to this question because eq. (11) gives the informational value of a reference point, which can be compared with its cost.

6 Choice With Multiple Comparators

While our analysis thus far has focused on utility evaluations of a single option in the presence of a reference point, we can also apply the same technology to study choice between two or more options, and possibly a reference point as well.

In line with our approach, it is natural to assume that when two or more choice options are available, the decision-maker also receives ordinal signals on how they compare in each dimension (in addition to the comparative signals with respect to a potential reference point). This has an immediate implication: when there is no ordinal noise, the DM will never choose a dominated option. With ordinal noise, this is no longer true, both because, by construction, mistakes may occur and because the DM puts strictly positive weight on the noisy cardinal signals—but these mistakes will become increasingly unlikely as utility differences increase. This is, of course, in line with empirical evidence, which shows that, when dimensions are easy to ordinally compare—e.g., involve numbers with clear rankings—the probability of choosing dominated options vanishes; however, this is less true when ordinal comparisons are less transparent (e.g., Choi et al., 2007; Nosofsky, 1986; Tversky, 1969).

We proceed as follows. First, we study binary choice without a reference point. Next, we consider choices with more than two options and binary choices with a reference point. To this end, we will also extend our formal analysis of the "boost" to the case of multiple comparators. Finally, we consider informationally-optimal reference points for binary choice.

6.1 Binary Choice Without a Reference Point

The DM needs to choose between two options, x and y. The utilities are $U(x) = \sum u_{x,i}$ and $U(y) = \sum u_{y,i}$, the DM holds dimension-specific Normal priors for the utility in each dimension (which may or not coincide for the two options), and they receive a noisy cardinal signal $s_{a,i} \sim \mathcal{N}(u_{a,i}, \sigma^2_{\epsilon,a,i})$ for each i and $a \in \{x, y\}$. The key novelty is that the DM also receives *ordinal* signals o_i that compare $u_{x,i}$ and $u_{y,i}$ for each i, with noise term $\sigma^2_{o,i}$. Denote $D_x \in \{1, \ldots, n\}$ the dimensions where x dominates and D_y those in which y dominates. Akin to (4), we have that

for each dimension:

$$\mathbb{E}[u_{x,i}|o_i, s_{x,i}] = \sum_i \tilde{u}_{x,i}^{s_{x,i}} + \frac{\sigma_{x,i}^2}{\sqrt{\sigma_{x,i}^2 + \sigma_{y,i}^2 + \sigma_{o,i}^2}} \psi\left(\frac{\tilde{u}_{x,i}^{s_{x,i}} - \tilde{u}_{y,i}^{s_{y,i}}}{\sqrt{\sigma_{x,i}^2 + \sigma_{y,i}^2 + \sigma_{o,i}^2}}, o_i\right).$$

This gives the same formula as before, except that now the comparison is with another option, and the uncertainty about the reference point is replaced by that of the other option.

Consider the natural special case in which ordinal signals are noise-free and the DM has, in each dimension, the same prior about the two goods ($\tilde{u}_{x,i} = \tilde{u}_{y,i}, \sigma_{p,x,i}^2 = \sigma_{p,y,i}^2 =: \sigma_{p,i}^2 \ \forall i$) and the same noise variance ($\sigma_{\epsilon,x,i}^2 = \sigma_{\epsilon,y,i}^2 =: \sigma_{\epsilon,i}^2 \ \forall i$). Then, the posterior mean for the utility of x is

$$\mathbb{E}[u_{x,i}|s_i, o_i] = \underbrace{\sum_i \tilde{u}_{x,i}^{s_i}}_{\text{Belief after cardinal signals}} + \underbrace{\sum_{i \in D_x} \frac{\sigma_i}{\sqrt{2}} \psi\left(\frac{\lambda_i(s_{x,i} - s_{y,i})}{\sqrt{2}\sigma_i}, +\right)}_{\text{Positive comparisons}} + \underbrace{\sum_{i \in D_y} \frac{\sigma_i}{\sqrt{2}} \psi\left(\frac{\lambda_i(s_{x,i} - s_{y,i})}{\sqrt{2}\sigma_i}, -\right)}_{\text{Negative comparisons}},$$

where λ_i is the dimension-specific shrinkage weight and $\sigma_i := \frac{\sigma_{\epsilon,i}^2 \sigma_{p,i}^2}{\sigma_{\epsilon,i}^2 + \sigma_{p,i}^2}$ the dimension-specific posterior variance after the cardinal signal.

The evaluation of each option is given by a combination of the absolute cardinal assessment (comprising the cardinal signal and the prior) and a weighted sum of the boosts from the positive and negative pairwise comparisons in each dimension. The importance (or weight) of the boosts depends on the DM's uncertainty σ_i about a dimension.

Counting advantages. As value uncertainty grows, ordinal comparisons become more and more important. In the extreme case in which the DM can rely only on his prior and the ordinal comparisons, this leads to an even simpler formula based on counting advantages.[20]

Observation 2. *Suppose $\tilde{u}_{x,i} = \tilde{u}_{y,i}$ and $\sigma_{p,x,i}^2 = \sigma_{p,y,i}^2 =: \sigma_{p,i}^2 \ \forall i$ and $\sigma_{o,i}^2 = 0$. Then, for any $u_x, u_y \in \mathbb{R}^n$, there exists $\bar{\sigma}_\epsilon^2$ such that if $\sigma_{\epsilon,x_i}^2 = \sigma_{\epsilon,y_i}^2 > \bar{\sigma}_\epsilon^2$ for all i:*

1. $\mathbb{E}[\mathbb{E}[\tilde{u}_x^{s_i}|s_i, o_i]] > \mathbb{E}[\mathbb{E}[\tilde{u}_x^{s_i}|s_i, o_i]] \Leftrightarrow \sum_{i \in D_x} \sigma_{p,i}^2 > \sum_{i \in D_y} \sigma_{p,i}^2$
2. *If $\sigma_{p,i}^2 = \sigma_{p,j}^2 \ \forall i, j$, then* $\mathbb{E}[\mathbb{E}[\tilde{u}_x^{s_i}|s_i, o_i]] > \mathbb{E}[\mathbb{E}[\tilde{u}_x^{s_i}|s_i, o_i]] \Leftrightarrow |D_x| > |D_y|$.

When the noise in the cardinal signals is high enough, the DM simply "counts" the number of relative advantages and disadvantages of x and y, weighted by the prior uncertainty in that

[20]The observation follows by noting that, as $\sigma_{\epsilon,x_i}^2 \to \infty$, Bayesian posteriors after cardinal signals will get arbitrarily close to the priors, meaning that $\lambda_i(s_{x,i} - s_{y,i}) \to 0$, $\psi\left(\frac{\lambda_i(s_{x,i} - s_{y,i})}{\sqrt{2}\sigma_i}, +\right) \to \sqrt{\frac{2}{\pi}}$, $\psi\left(\frac{\lambda_i(s_{x,i} - s_{y,i})}{\sqrt{2}\sigma_i}, -\right) \to -\sqrt{\frac{2}{\pi}}$, and $\sigma_i \to \sigma_p$. The observation then follows from Eq. (6.1).

dimension. Intuitively, if priors are the same and no cardinal signals are given, then all the DM is left with is a collection of positive and negative boosts. If prior uncertainty is the same in each dimension, then the *optimal* choice of a fully rational Bayesian agent—not an approximation heuristic—boils down to counting which option has more advantages. When prior uncertainty varies across dimensions, this acts as a weight— if the DM knew that the variance in utilities in a given dimension was close to zero, for example, then he shouldn't attribute much information to the fact that one option is better than the other in that dimension.

Our model thus provides an intuitive sense in which the DM exhibits a preference for options with many advantages, and a rational foundation for a commonly used heuristic. It also suggests that such behavior should be more common when cardinal signals are very noisy—for example, when there are many dimensions, and the cardinal signal for each one is sufficiently noisy.

Sensitivity to changes in ordinal rankings. An almost mechanical implication of our model is the high sensitivity to changes in ordinal ranks. Consider x, y with n dimensions, and suppose $u_{x,i} < u_{y,i}$ for some i. Because of Bayesian attenuation, the effect of increasing $u_{x,i}$ on the probability of choosing x is going to be limited; however, this probability will increase substantially if $u_{x,i}$ is increased enough to go above $u_{y,i}$. Just like the strong effects of surpassing the reference point we documented earlier in the paper, our model also predicts strong effects of ordinally surpassing other options in a given dimension.[21]

Joint vs. separate evaluation. A widely-studied empirical regularity is that people's stated valuations for goods tend to be more different when multiple options are evaluated jointly rather than separately (e.g., Hsee, 1996; Hsee and Zhang, 2010). In each dimension, this is immediately predicted by our model, as the ordinal signal pushes posteriors further apart—we necessarily have $\mathbb{E}[u_x^s - u_y^s | u_x^s > u_y^s] > \mathbb{E}[u_x^s - u_y^s]$. To provide some intuition, suppose you evaluate the utility of a set of 10 plates and of a set of 15 (otherwise identical) plates. When evaluating each in isolation, the resulting valuations will be relatively close to each other—an instance of the behavioral attenuation generated by value uncertainty in separate-evaluation mode. When the two sets are evaluated jointly, however, the DM receives a clear ordinal signal that the 15-plate set should be valued more highly; this pushes the posterior for that option upward, producing more widely spaced valuations in joint evaluation.

[21] Consistent with this, empirical studies show strong discrete choice shifts when an ordinal dominance relation flips. These patterns are found in multi-attribute decisions (Campbell et al., 2007), laboratory inference tasks (Bröder, 2000) and risky choice (Brandstätter et al., 2006), showing limited sensitivity to small cardinal changes but large, discrete shifts when rankings change.

6.2 Multiple Comparators

Our analysis thus far focused on the case in which the DM receives only one ordinal signal per dimension. What if there are multiple ones? This may be the case with binary choice with a reference point, choices from more than two items, or multiple reference points: all of these may generate multiple ordinal signals per dimension. Indeed, one of the advantages of thinking about reference dependence from an informational perspective is that it naturally allows for multiple reference points to play a role. Before exploring these cases, we discuss how our "boosts" formulas extend to the case of more than one comparator.

Known Comparators. Suppose first that the DM gets ordinal signals with respect to multiple options—multiple reference points—that are perfectly known, and the ordinal signal is noiseless. Let C be the (finite) set of reference points and suppose, for simplicity, that there is only one dimension ($n = 1$). If, for some $r \in C$, $u_r = u_x$, then posteriors are degenerate at u_r. If $u_r < u_x$ or $u_r > u_x$ for all $r \in C$, then the situation collapses to that in which the DM has a single comparator equal to the largest or smallest u_r, respectively; we can then simply use the formulas derived in the previous section. Otherwise, let $u^* := \arg\min_{r \in C | u_r > u_x} u_r$ and $u_* := \arg\max_{r \in C | u_r < u_x} u_r$. Then, the posterior mean about u_x is simply that of a two-sided truncated Normal,

$$\mathbb{E}[u_x | o, s] = \tilde{u}_x^s + \sigma_x \frac{\phi\left(\frac{u_* - \tilde{u}_x}{\sigma_x}\right) - \phi\left(\frac{u^* - \tilde{u}_x^s}{\sigma_x}\right)}{\Phi\left(\frac{u^* - \tilde{u}_x^s}{\sigma_x}\right) - \Phi\left(\frac{u_* - \tilde{u}_x}{\sigma_x}\right)}.$$

Note how this "boost" can be positive or negative. If both u^* and u_* are below $\tilde{u}_x$, the boost is negative; if both are above $\tilde{u}_x$, the boost is positive. If $u^* > \tilde{u}_x > u_*$, the sign of the boost depends on whether $\tilde{u}_x$ is closer to u^* or u_*: negative in the former case, positive in the latter.

Comparators that are not perfectly known. When the utilities of comparators are not perfectly known or when ordinal information is noisy, formulas become more complex. In Balakrishnan and Dean (2025), we derive the solution for two comparators; we include the precise formulae in Appendix B.5 (see Theorem 1 for the case without ordinal noise; Corollary 5 includes ordinal noise.) These results show that two comparators that provide the same ordinal information (i.e., two positive or two negative boosts) reinforce each other, providing a bigger boost than either on its own. With conflicting information, the net boost depends on the mean distance between each comparator and the target and the relative variance of the two comparators. At the same time, independently of the specific formulas, posteriors with multiple comparators follow several regularities. The following result includes three of them, which will

be particularly useful in what follows.

Proposition 5. *Let u_x, u_y, u_z be independent and Normally distributed. Then*

$$\mathbb{E}\left[u_x - u_y | u_x > u_y\right] > \mathbb{E}\left[u_x - u_y | u_x > u_y > u_z\right]$$
$$\mathbb{E}\left[u_x - u_y | u_x > u_y\right] > \mathbb{E}\left[u_x - u_y | u_z > u_x > u_y\right]$$
$$\mathbb{E}\left[u_x - u_y | u_x > u_y\right] < \mathbb{E}\left[u_x - u_y | u_x > u_z > u_y\right].$$

Suppose that we are estimating the difference between u_x and u_y knowing that $u_x > u_y$. Then, the proposition shows how this estimate *shrinks* if we further learn that some u_z is above or below both ($u_x > u_y > u_z$ or $u_z > u_x > u_y$); intuitively, $u_x > u_y > u_z$ boosts u_y more than it does u_x, because u_y is necessarily closer to u_z, and we have already seen how this implies a bigger boost. On the other hand, this estimate grows if we learn that some u_z lies in between ($u_x > u_z > u_y$), as this pushes u_x upwards and u_y downwards. Most notably, this result holds independently of the priors on these random variables, provided they are Normal and independent.

6.3 Binary Choice with a Reference Point: Improvements versus Tradeoffs

A robust regularity is that, starting from a reference point—such as a status quo— , individuals exhibit a preference for options that imply improvements rather than tradeoffs relative to the reference point (e.g. Tversky and Kahneman, 1991). For instance, suppose that a DM chooses between two two-dimensional options, x and y. There are two potential reference points, r and r'. From r option x implies improvements in both dimensions, whereas option y implies tradeoffs (one improvement and one loss). Suppose also that from r' the pattern reverses. Then, the empirical regularity is that the DM is more likely to choose x when the reference point is r than when it is r'.

We now show how this follows immediately from our model. Denote o the complete collection of ordinal signals when the reference point is r and o' that when the reference point is r'. The following is an immediate corollary of Proposition 5.

Corollary 2. *Suppose $n = 2$, $u_{x,1} > u_{r,1} > u_{y,1} > u_{r',1}$ and $u_{y,2} > u_{r',2} > u_{x,2} > u_{r,2}$. Then,* $\mathbb{E}_s\left[\mathbb{E}[U(x) - U(y)|s, o]\right] > \mathbb{E}_s\left[\mathbb{E}[U(x) - U(y)|s, o']\right]$.

This result shows that, *for purely informational reasons*, a Bayesian DM who receives ordinal signals gives an advantage to the option that dominates the reference point in both dimensions rather than the one that implies a tradeoff with the reference point. Intuitively, this is because dominance over the reference point entails a strictly positive boost in both dimensions. In contrast, a tradeoff entails a positive boost in one dimension and a negative boost in the other.

Note how, in our model, this effect is modulated by value uncertainty. It will be stronger when value uncertainty is higher and disappear without it. Moreover, in our model, this should also hold in the absence of gain-loss asymmetries in preferences, such as in perceptual tasks, where preferences play no role. Indeed, in line with these predictions, Trueblood (2015) experimentally shows that people exhibit the improvement-versus-tradeoffs effect also in magnitude estimation problems in which the reference point cannot affect preferences.

6.4 Menu Effects

We can use our model to study a classic empirical pattern in choice: the asymmetric dominance, or decoy effect, a classic violation of WARP; see Soltani et al. (2012) for a review. In it, there are two dimensions, and subjects are first asked to choose between x and y, where x is better on dimension 2 and y is better on dimension 1. In a second choice set, subjects are asked to choose between x, y, and a third option z, which is dominated by x in both dimensions but better than y in dimension 2. The empirical observation is that adding z increases the fraction of subjects choosing x. In what follows, assume that DMs receive cardinal signals $s_{i,a}$ as we have been assuming thus far, as well as noiseless ordinal signals among available options; let s and o denote the vectors of all such signals, respectively. Our model naturally gives rise to this effect, as shown by the following result.

Corollary 3. *Suppose $n = 2$ and all ordinal signals are noiseless ($\sigma^2_{o,i} = 0\ \forall i$). Then,*

$$\mathbb{E}[U(x)-U(y)|u_{y,1} > u_{x,1}, u_{x,2} > u_{y,2}, s] < \mathbb{E}[U(x)-U(y)|u_{y,1} > u_{x,1} > u_{z,1}, u_{x,2} > u_{z,2} > u_{y,2}, s].$$

This result is an immediate corollary of Proposition 5. The addition of z boosts u_x relative to u_y in both dimensions: In dimension 1, u_z boosts both u_x and u_y upwards, but as u_x is closer to u_z than u_y, it receives the bigger boost. In dimension 2, u_z boosts u_x upwards and u_y downwards, once again increasing their difference. Thus, for every signal realization, the introduction of z strictly increases the assessment of x relative to y. Given that interim beliefs have full support, this means that the probability that x is chosen (when $\mathbb{E}[U(x)-U(y)|s, o] > 0$) strictly increases, in line with the evidence. (Also in line with the evidence, our model predicts that, because x dominates z, the latter is never chosen.)

Moreover, because our effect is based on information, it does not rely on z being chosable: the exact same effects would be at play if z is a observable but cannot be chosen. This is in line with the experimental evidence of the "phantom decoy" of Soltani et al. (2012).

Natenzon (2019) and subsequently Shubatt and Yang (2024) propose explanations for the asymmetric dominance effect that are conceptually very close. However, our approach does not

rely on x being better than z in all dimensions for the effect to work. Consider an extension to the example above in which there is a third dimension in which x and y are equally valued, but z is superior to both. The introduction of z would still increase the valuation of x relative to y: The effect on the first two dimensions would be as before, while the effect on the third dimension would be the same on both x and y. Indeed, our results hold almost by-dimension: the addition of an option z that is closer to asymmetric dominance increases the value of one option over the other, as shown by the following corollary (which also immediately follows from Proposition 5).

Corollary 4. *Let $z'_i = z_i$ in all $i \neq j$, $x_j > z'_j > y_j$ while either $x_j > y_j > z_j$ or $z_j > x_j > y_j$. Then $\mathbb{E}[U(x) - U(y)|s, o] < \mathbb{E}[U(x) - U(y)|s, o']$ where o and o' indicate ordinal signals of options $\{x, y, z\}$ and $\{x, y, z'\}$ respectively.*

Compromise effect? A second classic violation of the independence of irrelevant alternatives is the compromise effect. Here, if x is better on dimension 2 while y is better on dimension 1, adding a third option z that is even better than x on dimension 1 and even worse on dimension 2 tends to push people towards choosing z. Our model is consistent with either the compromise effect or its opposite, depending on the exact placement of the decoy; see Appendix B.6 for a discussion.

6.5 Endogenous Reference Point for Binary Choice

While in Section 5 we derived the most informative reference point for valuation, we can extend this analysis to the optimal reference point for binary choice. A decision maker knows they will have to choose one of $x, y \in \mathbb{R}^n$, where $u_{x,i} \sim N(\tilde{u}_{x,i}, \sigma_i^2)$ and $u_{y,i} \sim N(\tilde{u}_{y,i}, \sigma_i^2)$ for all i. (Note the same prior variance in each dimension.) Like in the rest of this section, the DM knows he will receive cardinal signals and dimension-by-dimension ordinal signals comparing the two options. In addition, he observes dimension-specific ordinal signals with respect to a reference point r, where for simplicity we assume that $u_{r,i}$ are known for all i and that ordinal signals are noise-free. What are the values of $u_{r,i}$ that maximize the expected value of this choice? Denote s the set of cardinal and o that of all ordinal signals. Then, the expected value of the choice is

$$U_{\max}(r) := \mathbb{E}\left[\max\{U(x), U(y)\}|s, o\right],$$

where we have emphasized the dependence on r. It turns out that the most informative r admits a simple characterization.

Proposition 6. *Let $u_{x,i} \sim N(\tilde{u}_{x,i}, \sigma_i^2)$ and $u_{y,i} \sim N(\tilde{u}_{y,i}, \sigma_i^2)$ for all i. Then $U_{\max}(r)$ is maximized by r such that $u_{r,i} = \frac{1}{2}\left(\tilde{u}_{x,i} + \tilde{u}_{y,i}\right)$ for all i.*

This proposition shows that the most informative reference point for binary choice lies exactly halfway between the prior means of the two options, in each dimension.[22] Again, this is broadly consistent with the idea that reference points are given by average outcomes, as is (exogenously) assumed in multiple models of reference dependence but here derived from value uncertainty and reference points as information.

7 Caution

Thus far, we have assumed that the DM is "risk neutral" with respect to his value uncertainty and chooses the option with the highest posterior mean. However, a growing body of evidence suggests that people are uncertainty averse with respect to their own value uncertainty (e.g., de Clippel et al., 2025). Intuitively, even if I think that x probably gives me a higher utility than y, I may pick y if I'm sure how happy it will make me, but I have lots of uncertainty about x. This type of uncertainty aversion is modeled in contributions on caution (Cerreia-Vioglio et al., 2015, 2024) and complexity aversion (Mononen, 2025; Puri, 2025; Gabaix, 2025), and it is empirically and theoretically related to standard ambiguity aversion.

To understand its implications in our model and maintain tractability, we assume that our DM has mean-variance utility with respect to his value uncertainty. The value of option x is given by:

$$v_x := \sum_{i=1}^{N} \mathbb{E}[u_{x,i}|s_{x,i}, o_i] - \alpha_i \, \mathbb{V}[u_{x,i}|s_{x,i}, o_i], \qquad \alpha_i > 0 \tag{13}$$

where the posterior variance $\mathbb{V}[u_{x,i}|s_{x,i}, o_i]$ is characterized in Proposition 7 in Appendix B. The key insight is that this posterior variance monotonically increases in the distance from the reference point, as illustrated in Appendix Figure 14. The reason is the presence of the ordinal signal— intuitively, if I know the utility of making \$100, then I will also have little uncertainty about the utility of making \$101, yet I might be very uncertain about the utility of making \$500.

Endowment effect. The dependence of the posterior variance on the distance to the reference point generates interesting effects once caution ($\alpha > 0$) is allowed for. Consider the endowment effect. If my reference point is to own a mug (and I thus have lower uncertainty about it, for example because I've had the item in my possession or because I thought about it for longer),

[22] The intuition is the following. Since $\mathbb{E}\left[\max\{U(x), U(y)\}\right] = \frac{1}{2}\left(\mathbb{E}[U(x)] + \mathbb{E}[U(y)]\right) + \frac{1}{2}\mathbb{E}[|U(x) - U(x)|]$ and $\mathbb{E}[U(x)]$ and $\mathbb{E}[U(y)]$ are independent of r, then the expected value of the choice is maximized by the r that maximizes the Expectation of $|U(x) - U(y)|$. This is highest when, in each dimension, $u_{r,i}$ lies between $u_{x,i}$ and $u_{y,i}$: Proposition 5 shows that, in this case, expected values are pushed apart, while the opposite happens when $u_{r,i}$ is above or below both. Since this is most likely to happen when $u_{r,i} = \frac{1}{2}\left(\tilde{u}_{x,i} + u_{y,i}\right)$, the proposition follows.

then my valuation v_x for owning a mug will mechanically be higher than if my reference point is to not own a mug. Thus the model trivially predicts the endowment effect.[23] The idea that value uncertainty and caution generate the endowment effect is similar to Cerreia-Vioglio et al. (2024), yet the analytical approach is different. Our model predicts that the endowment effect will be more pronounced when there is more value uncertainty. This is consistent with a wide range of evidence reviewed in Cerreia-Vioglio et al. (2024) that shows that the endowment effect is stronger for non-market than market goods and that it decreases with experience, expertise and familiarity.

Status quo bias. For very similar reasons, our model generates status quo bias. Suppose there's an option whose utility you know well because it's the status quo—you know what it's like to live in your hometown. Now you're considering moving away, but you're uncertain about the utility of living in another city. Then, our model says that the more dissimilar this other city is to the reference point of your hometown (the larger the difference between u_x and u_r dimension-by-dimension), the more uncertain you will be about its utility. Because of caution, you will then penalize the perceived utility from the other city, giving rise to status quo bias. According to this account, status quo bias should disappear in the absence of value uncertainty, and does not require any kind of gain-loss asymmetry.

8 Conclusion

This paper has argued that the combination of value uncertainty and reference points as information provides a framework for thinking about multiple phenomena at the core of behavioral economics. Building on the present paper, in Dean et al. (2026) we study how a similar mechanism—ordinal information from one's own past actions—gives rise to an information-based model of habit formation, status quo bias and reasoning by analogy ("internal herding"). We conclude by discussing limitations and potential extensions and applications.

Limitations. Our model has several limitations, and there are many behavioral economics phenomena and psychological mechanisms our model does not speak to. For instance, our model is silent on effects related to selective attention allocation such as salience (Bordalo et al., 2022) or rational inattention (Maćkowiak et al., 2023). Our model also does not allow for a role of memory cues that are entirely uninformative (or perceived to be entirely uninformative by the

[23] This is under the assumption that the DM has lower value uncertainty about mugs than about money, just like classic loss-aversion based explanations of the endowment effect require that people have more loss aversion for mugs than for money.

decision maker). Still in the same vein, we also do not capture categorization effects other than those that pertain to the gain-loss distinction (e.g. Mullainathan, 2002; Bordalo et al., 2024). We believe that it may be fruitful to combine our 'cognitive' approach to reference points with some of these other cognitive mechanisms. Finally, we emphasize that our approach does not rule out the existence of reference-dependent preferences, adaptation effects and/or loss aversion. For instance, it seems uncontroversial that lukewarm water feels more pleasant after cold than after hot water, a classic example of reference-dependent sensations. We believe that an interesting avenue for future research will be to study which of the effects of reference dependence reflect comparative information and which reflect preferences such as loss aversion.

Potential extensions and applications. We believe our model encourages several potential extensions, applications and empirical tests. A first potential extension involves social reference points. While introspection and evidence suggest that social reference points matter partly for preference-based reasons (e.g., emotions such as envy), it seems similarly intuitive that social reference points often contain information. For example, if I learn that a co-worker who is very similar to me makes twice as much money as I do, then this not only triggers envy in me but also suggests that I got screwed over by my boss, or that my marginal product is higher than I used to believe. Incorporating social reference points into our model might shed light on this.

A second useful extension might be to decision-making under risk and intertemporal choice. Here, the boost from the ordinal comparison may shed new light on familiar phenomena. For instance, when asked how much $100 in 6 months is worth to the decision maker today, he may be uncertain (Enke et al., 2025; Chakraborty, 2021), yet through simple ordinal reasoning, he will know that his present value ought to be less than $100, producing a downward boost in the evaluation of delayed payments. As a result of this ordinal information, a Bayesian decision maker *ought* to exhibit steeper discounting between today and tomorrow than between two future dates.

Third, because our information-theoretic model provides a novel way of thinking about reference points, it also offers a new perspective on the classic question of what the reference point is. We believe that the broad idea that the reference point is "optimal," in the sense that it reflects the most informative option from the perspective of reducing value uncertainty, can be fruitfully extended and applied in various different ways.

Finally, our model also encourages empirical applications and experimental tests. A natural first question is to what extent (and which) reference effects depend on value uncertainty. In the lab, experiments would need to separately vary value uncertainty about the option and the reference point. In field applications, one could study to what extent reference effects depend

on experience or choice complexity (as some prior work has done).

References

Abeler, Johannes, Armin Falk, Lorenz Goette, and David Huffman (2011) "Reference points and effort provision," *American Economic Review*, 101 (2), 470–492.

Agahi, Hamzeh (2015) "An elementary proof of the covariance inequality for Choquet integral," *Statistics & Probability Letters*, 106, 173–175.

Agranov, Marina and Pietro Ortoleva (2017) "Stochastic choice and preferences for randomization," *Journal of Political Economy*, 125 (1), 40–68.

Allen, Eric J, Patricia M Dechow, Devin G Pope, and George Wu (2017) "Reference-dependent preferences: Evidence from marathon runners," *Management Science*, 63 (6), 1657–1672.

Anderson, John R (2013) *The adaptive character of thought*: Psychology Press.

Ariely, Dan, George Loewenstein, and Drazen Prelec (2003) ""Coherent arbitrariness": Stable demand curves without stable preferences," *The Quarterly journal of economics*, 118 (1), 73–106.

Augenblick, Ned, Eben Lazarus, and Michael Thaler (2025) "Overinference from weak signals and underinference from strong signals," *The Quarterly Journal of Economics*, 140 (1), 335–401.

Ba, Cuimin, J Aislinn Bohren, and Alex Imas (2025) "Over-and underreaction to information," mimeo University of Pennsylvania.

Balakrishnan, Sivaraman and Mark Dean (2025) "Unknown," mimeo Columbia University.

Barberis, Nicholas C (2013) "Thirty years of prospect theory in economics: A review and assessment," *Journal of economic perspectives*, 27 (1), 173–196.

Bordalo, Pedro, Katherine Coffman, Nicola Gennaioli, and Andrei Shleifer (2016) "Stereotypes," *The Quarterly Journal of Economics*, 131 (4), 1753–1794.

Bordalo, Pedro, Nicola Gennaioli, Giacomo Lanzani, and Andrei Shleifer (2024) "A Cognitive Theory of Reasoning and Choice."

Bordalo, Pedro, Nicola Gennaioli, and Andrei Shleifer (2012) "Salience Theory of Choice under Risk," *The Quarterly Journal of Economics*, 127 (3), 1243–1285.

——— (2020) "Memory, attention, and choice," *The Quarterly journal of economics*, 135 (3), 1399–1442.

——— (2022) "Salience," *Annual Review of Economics*, 14 (1), 521–544.

Braida, Louis D and Nathaniel I Durlach (1972) "Intensity Perception. II. Resolution in One-Interval Paradigms," *The Journal of the Acoustical Society of America*, 51 (2B), 483–502.

Brandstätter, Eduard, Gerd Gigerenzer, and Ralph Hertwig (2006) "The priority heuristic: making choices without trade-offs.," *Psychological review*, 113 (2), 409.

Bröder, Arndt (2000) "Assessing the empirical validity of the" take-the-best" heuristic as a model of human probabilistic inference.," *Journal of Experimental Psychology: learning, memory, and Cognition*, 26 (5), 1332.

Bushong, Benjamin, Matthew Rabin, and Joshua Schwartzstein (2021) "A model of relative thinking," *The Review of Economic Studies*, 88 (1), 162–191.

Butler, David J and Graham C Loomes (2007) "Imprecision as an account of the preference reversal phenomenon," *American Economic Review*, 97 (1), 277–297.

Butler, David and Graham Loomes (2007) "Imprecision as an Account of the Preference Reversal Phenomenon," *American Economic Review*, 97 (1), 277–297.

——— (2011) "Imprecision as an account of violations of independence and betweenness," *Journal of Economic Behavior & Organization*, 80 (3), 511–522.

Camerer, Colin, Linda Babcock, George Loewenstein, and Richard Thaler (1997) "Labor supply of New York City cabdrivers: One day at a time," *The Quarterly Journal of Economics*, 112 (2), 407–441.

Campbell, Danny, W George Hutchinson, and Riccardo Scarpa (2007) "Lexicographic preferences in discrete choice experiments: Consequences on individual-specific willingness to pay estimates."

Cerreia-Vioglio, Simone, David Dillenberger, and Pietro Ortoleva (2015) "Cautious Expected Utility and the Certainty Effect," *Econometrica*, 83 (2), 693–728.

——— (2024) "Caution and Reference Effects," *Econometrica*, 92 (6), 2069–2103.

Chakraborty, Anujit (2021) "Present bias," *Econometrica*, 89 (4), 1921–1961.

Charness, Gary, Uri Gneezy, and Michael A Kuhn (2012) "Experimental methods: Between-subject and within-subject design," *Journal of economic behavior & organization*, 81 (1), 1–8.

Choi, Syngjoo, Raymond Fisman, Douglas Gale, and Shachar Kariv (2007) "Consistency and heterogeneity of individual behavior under uncertainty," *American economic review*, 97 (5), 1921–1938.

de Clippel, Geoffroy, Paola Moscariello, Pietro Ortoleva, and Kareen Rozen (2025) "Caution in the Face of Complexity," mimeo, Princeton University.

Cubitt, Robin P, Daniel Navarro-Martinez, and Chris Starmer (2015) "On preference imprecision," *Journal of Risk and Uncertainty*, 50 (1), 1–34.

Dean, Mark, Benjamin Enke, Thomas Graeber, and Pietro Ortoleva (2026) "Internal Herding," mimeo Columbia University.

DellaVigna, Stefano, Jörg Heining, Johannes F Schmieder, and Simon Trenkle (2022) "Evidence on job search models from a survey of unemployed workers in germany," *The Quarterly Journal of Economics*, 137 (2), 1181–1232.

DellaVigna, Stefano, Attila Lindner, Balázs Reizer, and Johannes F Schmieder (2017) "Reference-dependent job search: Evidence from Hungary," *The Quarterly Journal of Economics*, 132 (4), 1969–2018.

Enke, Benjamin and Thomas Graeber (2023) "Cognitive uncertainty," *The Quarterly Journal of Economics*, 138 (4), 2021–2067.

Enke, Benjamin, Thomas Graeber, and Ryan Oprea (2025) "Complexity and time," *Journal of the European Economic Association*, 23 (5), 1838–1867.

Enke, Benjamin, Thomas Graeber, Ryan Oprea, and Jeffrey Yang (2024) "Behavioral Attenuation," mimeo Harvard University.

Gabaix, Xavier (2014) "A sparsity-based model of bounded rationality," *The Quarterly Journal of Economics*, 129 (4), 1661–1710.

——— (2019) "Behavioral inattention," in *Handbook of behavioral economics: Applications and foundations 1*, 2, 261–343: Elsevier.

——— (2025) "A Theory of Complexity Aversion," mimeo Harvard University.

Garner, Wendell R (1953) "An informational analysis of absolute judgments of loudness.," *Journal of experimental psychology*, 46 (5), 373.

Gershman, Samuel (2021) *What makes us smart: The computational logic of human cognition*: Princeton University Press.

Gilboa, Itzhak and Massimo Marinacci (2011) "Ambiguity and the Bayesian paradigm," mimeo Bocconi University.

Graeber, Thomas Ⓣ Benjamin Enke (2026) "Cognitive Comparison Effects," mimeo Harvard University.

Graeber, Thomas, Christopher Roth, and Marco Sammon (2025) "Coarse Categories Compete in a Complex World," mimeo HBS.

Griffiths, Thomas, Nick Chater, and Joshua B Tenenbaum (2024) *Bayesian models of cognition*.

Gul, Faruk (1991) "A theory of disappointment aversion," *Econometrica*, 59 (3), 667–686.

Halevy, Yoram, David Walker-Jones, and Lanny Zrill (2023) *Difficult decisions*: University of Toronto, Department of Economics.

Heckman, James J (1979) "Sample selection bias as a specification error," *Econometrica: Journal of the econometric society*, 153–161.

Heffetz, Ori (2021) "Are reference points merely lagged beliefs over probabilities?" *Journal of Economic Behavior & Organization*, 181, 252–269.

Hollands, JG and Brian P Dyre (2000) "Bias in proportion judgments: the cyclical power model.," *Psychological review*, 107 (3), 500.

Hsee, Christopher K (1996) "The evaluability hypothesis: An explanation for preference reversals between joint and separate evaluations of alternatives," *Organizational behavior and human decision processes*, 67 (3), 247–257.

Hsee, Christopher K and Jiao Zhang (2010) "General evaluability theory," *Perspectives on Psychological Science*, 5 (4), 343–355.

Jin, Lawrence, Rui Tang, Han Ye, Junjian Yi, and Songfa Zhong (2024) "Path dependency in physician decisions," *Review of Economic Studies*, 91 (5), 2916–2953.

Kahana, Michael Jacob (2012) *Foundations of human memory*: OUP USA.

Kahneman, Daniel and Amos Tversky (1979) "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, 47 (2), 263–292.

Karlin, Samuel and Yosef Rinott (1980) "Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions," *Journal of Multivariate Analysis*, 10 (4), 467–498.

Khaw, Mel Win, Ziang Li, and Michael Woodford (2021) "Cognitive imprecision and small-stakes risk aversion," *The review of economic studies*, 88 (4), 1979–2013.

Kőszegi, Botond and Matthew Rabin (2006) "A Model of Reference-Dependent Preferences," *Quarterly Journal of Economics*, 121 (4), 1133–1165.

Kőszegi, Botond and Adam Szeidl (2013) "A model of focusing in economic choice," *The Quarterly journal of economics*, 128 (1), 53–104.

Laming, Donald (1984) "The relativity of 'absolute'judgements," *British Journal of Mathematical and Statistical Psychology*, 37 (2), 152–183.

——— (2009) "Weber's law," in *Inside Psychology: A science over 50 years*, 179–191: Oxford Univ. Press London, UK.

Laming, Donald Richard John (1997) *The measurement of sensation* (30): Oxford University Press.

List, John A (2025) "The Experimentalist Looks Within: Toward an Understanding of Within-Subject Experimental Designs,"Technical report, National Bureau of Economic Research.

Loomes, Graham and Robert Sugden (1986) "Disappointment and Dynamic Consistency in Choice under Uncertainty," *The Review of Economic Studies*, 53 (2), 271–282.

Luce, R Duncan and David M Green (1974) "The response ratio hypothesis for magnitude estimation," *Journal of Mathematical Psychology*, 11 (1), 1–14.

Maćkowiak, Bartosz, Filip Matějka, and Mirko Wiederholt (2023) "Rational inattention: A review," *Journal of Economic Literature*, 61 (1), 226–273.

Masatlioglu, Yusufcan and Efe A. Ok (2005) "Rational Choice with Status-Quo Bias," *Journal of Economic Theory*, 121, 1–29.

——— (2014) "A Canonical Choice Model with Initial Endowment," *Review of Economic Studies*, 81 (2), 851–883.

Meier, Pascal Flurin, Raphael Flepp, and Egon Franck (2025) "Expectational reference points and belief formation: Field evidence from financial analysts," *Journal of Economic Behavior & Organization*, 229, 106788.

Mononen, Lasse (2025) "On preference for simplicity and probability weighting," University of Bielefeld.

Moon, Jongmin and Oh-Sang Kwon (2022) "Attractive and repulsive effects of sensory history concurrently shape visual perception," *BMC biology*, 20 (1), 247.

Mullainathan, Sendhil (2002) "Thinking through categories,"Technical report, Working Paper, Harvard University.

Natenzon, Paulo (2019) "Random choice and learning," *Journal of Political Economy*, 127 (1), 419–457.

Nishimura, Hiroki and Efe A Ok (2021) "Preference Structures," mimeo, NYU.

Nosofsky, Robert M (1986) "Attention, similarity, and the identification–categorization relationship.," *Journal of experimental psychology: General*, 115 (1), 39.

Oaksford, Mike and Nick Chater (2007) *Bayesian rationality: The probabilistic approach to human reasoning*: Oxford University Press, USA.

Ok, Efe, Pietro Ortoleva, and Gil Riella (2011) "Revealed (P)Reference Theory," Mimeo California Institute of Technology.

Parducci, Allen (1965) "Category judgment: a range-frequency model.," *Psychological review*, 72 (6), 407.

Pope, Devin G and Maurice E Schweitzer (2011) "Is Tiger Woods loss averse? Persistent bias in the face of experience, competition, and high stakes," *American Economic Review*, 101 (1), 129–157.

Puri, Indira (2025) "Simplicity and risk," *The Journal of Finance*, 80 (2), 1029–1080.

Sadil, Patrick, Rosemary A Cowell, and David E Huber (2024) "The push–pull of serial dependence effects: Attraction to the prior response and repulsion from the prior stimulus," *Psychonomic Bulletin & Review*, 31 (1), 259–273.

Sagi, Jacob (2006) "Anchored Preference Relations," *Journal of Economic Theory*, 130, 283–295.

Sampford, Michael R (1953) "Some inequalities on Mill's ratio and related functions," *The Annals of Mathematical Statistics*, 24 (1), 130–132.

Schäfer, Thomas and Marcus A Schwarz (2019) "The meaningfulness of effect sizes in psychological research: Differences between sub-disciplines and the impact of potential biases," *Frontiers in psychology*, 10, 813.

Shepard, Roger N (1987) "Toward a universal law of generalization for psychological science," *Science*, 237 (4820), 1317–1323.

Shubatt, Cassidy and Jeffrey Yang (2024) "Tradeoffs and Comparison Complexity," arXiv preprint arXiv:2401.17578.

Soltani, Alireza, Benedetto De Martino, and Colin Camerer (2012) "A range-normalization model of context-dependent choice: a new model and evidence," *PLoS computational biology*, 8 (7), e1002607.

Stewart, Neil, Gordon DA Brown, and Nick Chater (2005) "Absolute identification by relative judgment.," *Psychological review*, 112 (4), 881.

Stewart, Neil, Nick Chater, and Gordon DA Brown (2006) "Decision by sampling," *Cognitive psychology*, 53 (1), 1–26.

Sugden, Robert (2003) "Reference-dependent subjective expected utility," *Journal of economic theory*, 111 (2), 172–191.

Trueblood, Jennifer S (2015) "Reference point effects in riskless choice without loss aversion.," *Decision*, 2 (1), 13.

Tulving, Endel and Donald M Thomson (1973) "Encoding specificity and retrieval processes in episodic memory.," *Psychological review*, 80 (5), 352.

Tversky, Amos (1969) "Intransitivity of preferences.," *Psychological review*, 76 (1), 31.

Tversky, Amos and Daniel Kahneman (1991) "Loss Aversion in Riskless Choice: A Reference-Dependent Model," *Quarterly Journal of Economics*, 106 (4), 1039–1061.

Vieider, Ferdinand M (2024) "Decisions under uncertainty as bayesian inference on choice options," *Management Science*, 70 (12), 9014–9030.

Villas-Boas, J Miguel (2024) "Toward an information-processing theory of loss aversion," *Marketing Science*, 43 (3), 523–541.

Woodford, Michael (2012) "Inattentive Valuation and Reference-Dependent Choice," mimeo, Columbia University.

——— (2020) "Modeling imprecision in perception, valuation, and choice," *Annual Review of Economics*, 12 (1), 579–601.

Wu, Keyu (2024) "Context-dependent perceptions and decision-making," mimeo University of Zurich.

ONLINE APPENDIX

A Evidence on Comparative and Ordinal Reasoning

A core component of our model is that, when people are uncertain how to judge or value objects, they rely on comparators as a source of information. This section reviews evidence that suggests that people indeed often rely on comparators (in a partially ordinal way) when they are uncertain what the right decision is.

The broad idea that decision making is often comparative in nature is one of the central and earliest insights of behavioral economics, as evidenced in prospect theory. A classic example is that warm water feels warmer after we stick our hand into cold rather than hot water. However, the comparative nature of cognition and decision making is not limited to reference-dependent preferences. Kahneman and Tversky (1979) already note that "the perceptual apparatus is attuned to the evaluation of changes or differences rather than to the evaluation of absolute magnitudes". Indeed, cognitive psychology has long emphasized that people are more adept at judging differences between stimuli than estimating absolute magnitudes.

Decades of psychophysical research suggest that performance on absolute judgment tasks – evaluating the magnitude of a stimulus – is substantially poorer than on tasks that require comparisons between two stimuli – evaluating which one is brighter, longer, heavier etc. This supports the view that human perception is optimized for detecting differences rather than encoding absolute magnitudes (Laming, 2009, 1984; Braida and Durlach, 1972; Garner, 1953). Empirically, the precision of the ordering of two stimuli implied by cardinal estimates is about two orders of magnitude lower than the precision of direct comparisons between the same stimuli (Laming, 1997). Moreover, responses in the classic psychological absolute judgment tasks (magnitude estimation and absolute identification) are more strongly shaped by contextual factors than in the comparative task of ordinal stimulus discrimination (Luce and Green, 1974; Parducci, 1965; Laming, 1997), suggesting that comparisons are more stable.

Much research suggests that such comparative thinking is partly ordinal in nature. It is generally easier to assess whether one stimulus is greater than, less than, or equal to another ones than it is to make cardinal judgments. The importance of ordinal judgments has been most prominent in perceptual tasks but also heavily influenced work on decision-making. For example, it is intuitively easier to realize that six apples are better than four apples, compared to assessing how much better they are. One clear illustration of this in the perceptual domain is Hollands and Dyre (2000) who show that in magnitude estimation tasks with an exogenously-provided reference point of known value, subjects correctly recognize whether a stimulus is above or below

the reference point but are insufficiently sensitive to how much, the two hallmark characteristics of partiall ordinal reasoning.

Importantly, much evidence suggests that people partly implicitly rely on ordinal comparisons even when they produce cardinal estimates. A classic reference is Braida and Durlach (1972), who showed that magnitude estimation (assigning a number proportional to intensity) and absolute identification (judging which stimulus is larger) yield essentially the same pattern of responses. Stewart et al. (2005) review this and related empirical observations, and propose a psychological model of absolute identification through relative judgment. Relatedly, the influential line of work on decision-by-sampling essentially asserts that people make decisions by ordinally comparing options with comparators retrieved from memory, and counting the number of advantages (Stewart et al., 2006, and references therein).

More recently, some contributions in economics suggested that relying on comparative thinking may partly reflect a simplification strategy and / or incomplete information about how to solve a problem. In studying belief updating, Augenblick et al. (2025) model a DM who knows in which direction to update (ordinally) but not by how much. Graeber et al. (2025) and Graeber ⓡ Enke (2026) show empirically that higher processing difficulty or uncertainty lead to stronger reliance on comparison points, and that such comparative thinking appears to be partly ordinal in nature.

In summary, when people exhibit uncertainty about how to estimate the value or magnitude of a stimulus, they partially rely on comparisons, be they explicitly provided or retrieved from memory. Given its low cost and simplicity, these comparisons are partly ordinal in nature. Unlike some of the more extreme hypotheses and models in psychology, we do not posit that people only rely on comparative or ordinal information. Instead, our model features a standard absolute, cardinal signal, augmented with a comparative, ordinal one.[24]

B Additional Results

B.1 Posterior Variances After Ordinal Signals and Average Posterior Means

We now present two additional results: the characterization of the posterior variances after the ordinal signal as well as the characterization of the *average* posterior mean that the agent will have—averaging over the realizations of the (noisy) cardinal and ordinal signals.

[24] A notable caveat is that while humans generally perform better at, and tend to rely on, ordinal judgments, these judgments are not perfect, either. In particular, they break down when stimulus differences fall below the so-called "just noticeable difference" threshold, which we abstract away from in this paper.

Proposition 7 (Posterior Variances). *The following is true:*

$$\mathbb{V}[u_x|o,s] = \frac{\sigma_x^2}{\sigma_x^2+\sigma_r^2+\sigma_o^2}\left[\sigma_o^2+\sigma_r^2+\sigma_x^2\left(1-\chi\left(\frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}},o\right)\right)\right]$$

$$\text{where}\quad \chi(t,o) = \begin{cases} t\frac{\phi(t)}{\Phi(t)} + \left(\frac{\phi(t)}{\Phi(t)}\right)^2 & \text{if } o=+ \\ -t\frac{\phi(t)}{1-\Phi(t)} + \left(\frac{\phi(t)}{1-\Phi(t)}\right)^2 & \text{if } o=-. \end{cases}$$

Proposition 8 (Average Posterior Means). *For all $s\in\mathbb{R}$, let $z_s := \frac{\tilde{u}_x^s-\tilde{u}_r}{\sqrt{\tilde{\sigma}_x^2+\sigma_r^2+\sigma_o^2}}$. Then,*

$$\mathbb{E}[\tilde{u}_x^{s,o}] = \lambda u_x + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\mathbb{E}_s\left[\left(\Phi\left(\frac{u_x-u_r}{\sigma_o}\right)-\Phi(z_s)\right)\frac{\phi(z_s)}{\Phi(z_s)\big(1-\Phi(z_s)\big)}\right]. \tag{14}$$

B.2 An equivalent formulation with comparative cardinal signals

A seemingly strong assumption of our model is that the comparative signal is only ordinal in nature – the DM may observe that u_x is better than u_r but has no idea by how much. Yet in some settings, the DM may also have a sense of *how much* better or worse u_x is. We now show that our setting (a cardinal signal about u_x and an ordinal signal about u_x-u_r) is informationally equivalent to a setting in which the DM receives a cardinal signal about u_x-u_r that is especially precise about ordinal relationships.

In the *Original Setting*, parametrized by $(\sigma_{\varepsilon_x}^2,\sigma_{\varepsilon_r}^2,\sigma_v^2)$, the DM holds independent priors $u_x\sim\mathcal{N}(\tilde{u}_x,\sigma_x^2)$, $u_r\sim\mathcal{N}(\tilde{u}_r,\sigma_r^2)$ and observes cardinal signals $s_x=u_x+\varepsilon_x$ and $s_r=u_r+\varepsilon_r$, with $\varepsilon_x\sim\mathcal{N}(0,\sigma_{\varepsilon_x}^2)$ and $\varepsilon_r\sim\mathcal{N}(0,\sigma_{\varepsilon_r}^2)$, and an ordinal signal $o=\text{sign}(u_x-u_r-v)$ where $v\sim\mathcal{N}(0,\sigma_v^2)$.[25] All random variables are independent.

In the *Alternative Setting*, parametrized by $(\sigma_{e_x}^2,\sigma_{e_d}^2,\sigma_{e_r}^2,\sigma_{v'}^2)$, the DM holds the same priors and receives cardinal signals $s'_x=u_x+e_x$ and $s'_r=u_r+e_r$, an ordinal signal $o'=\text{sign}(u_x-u_r-v')$ where $v'\sim\mathcal{N}(0,\sigma_{v'}^2)$ (independent of everything else), and a *cardinal comparative signal* $s_d=u_x-u_r+e_d$. Assume that the error vector (e_x,e_d,e_r) is independent of (u_x,u_r) and distributed as $\mathcal{N}(\mathbf{0},\Sigma)$, where

$$\Sigma=\begin{pmatrix}\sigma_{e_x}^2 & \sigma_{e_x}^2 & 0\\ \sigma_{e_x}^2 & \sigma_{e_d}^2 & \sigma_{e_r}^2\\ 0 & \sigma_{e_r}^2 & \sigma_{e_r}^2\end{pmatrix},$$

[25] The DM's knowledge of u_r can be interpreted as arising from a prior together with a cardinal signal about u_r, left implicit above.

with $\sigma^2_{e_d} > \sigma^2_{e_x} + \sigma^2_{e_r}$. This covariance structure implies that we can write $e_d = e_x + e_r + t$, where $t \sim \mathcal{N}(0, \sigma^2_{e_d} - \sigma^2_{e_x} - \sigma^2_{e_r})$ is independent of e_x and e_r: intuitively, the comparative signal s_d is subject to the noise from the individual components plus additional independent noise.

Despite their different formulations, these settings are informationally equivalent: for every Alternative Setting, there exists a corresponding Original Setting, and vice versa, that generates the same distribution of posterior beliefs about (u_x, u_r). This equivalence is formalized in the following observation (proved for completeness in the appendix).

Observation 3 (Equivalence of Settings)**.** *The Alternative and Original Settings are informationally equivalent: they generate the same distribution of posterior beliefs over (u_x, u_r) up to a reparametrization. Specifically:*

1. ***From Alternative to Original:*** *Any Alternative Setting $(\sigma^2_{e_x}, \sigma^2_{e_d}, \sigma^2_{e_r}, \sigma^2_{v'})$ generates the same distribution of posterior beliefs as the Original Setting $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$ such that*

$$\sigma^2_{\varepsilon_x} = \sigma^2_{e_x}, \qquad \frac{1}{\sigma^2_{\varepsilon_r}} = \frac{1}{\sigma^2_{e_r}} + \frac{4}{\sigma^2_{e_d} - \sigma^2_{e_x} - \sigma^2_{e_r}}, \qquad \sigma^2_v = \sigma^2_{v'}.$$

2. ***From Original to Alternative:*** *Conversely, any Original Setting $(\sigma^2_{\varepsilon_x}, \sigma^2_{\varepsilon_r}, \sigma^2_v)$ generates the same distribution of posterior beliefs as any Alternative Setting such that $\sigma^2_{e_x} = \sigma^2_{\varepsilon_x}$, $\sigma^2_{v'} = \sigma^2_v$, any $\sigma^2_{e_r} > \sigma^2_{\varepsilon_r}$, and*

$$\sigma^2_{e_d} = \sigma^2_{\varepsilon_x} + \sigma^2_{e_r} + 4\left(\frac{1}{\sigma^2_{\varepsilon_r}} - \frac{1}{\sigma^2_{e_r}}\right)^{-1}.$$

B.3 A model with an unrestricted comparative cardinal signal

B.4 Comparative Statics of Reference Dependence

We now formalize the comparative statics observations made informally above. Readers who are more interested in the model's applications may wish to skip to Section 4.

We analyze comparative statics both for specific signal realizations and in expectation. Let $B(+, s)$ denote the average effect of the ordinal signal + given cardinal signal s, and $C(u_x, u_r)$ the average effect of the ordinal signal given true values of u_x and u_r aggregating across possible ordinal and cardinal signals. That is

$$B(+, s) := \mathbb{E}[u_x|+, s] - \tilde{u}^s_x \quad \text{and} \quad C(u_x, u_r) := \mathbb{E}_{o,s}\left[\mathbb{E}[u_x|o, s]|u_x, u_r\right] - \mathbb{E}_s\left[\mathbb{E}[u_x|s]|u_x\right].$$

Proposition 9. *If $\sigma^2_x > 0$, then $B(s, +) > 0$ and*

(a) *$B(s,+)$ is strictly increasing in σ_x^2.*

(b) *The effects of σ_r^2 and σ_o^2 on $B(s,+)$ are identical, that is $\frac{\partial B(s,+)}{\partial \sigma_r^2} = \frac{\partial B(s,+)}{\partial \sigma_o^2}$. Moreover, if $\tilde{u}_x^s \leq \tilde{u}_r$, $B(s,+)$ is strictly decreasing in σ_r^2 and σ_o^2. If $\tilde{u}_x^s > \tilde{u}_r$, $B(s,+)$ is decreasing in σ_r^2 and σ_o^2 if $\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$ is small, and strictly increasing otherwise.*[26]

(c) *$B(s,+)$ is strictly decreasing in $\tilde{u}_x^s - \tilde{u}_r$.*

Parts (a) and (c) show how the boost is strictly *increasing* in σ_x^2 and *decreasing* in $\tilde{u}_x^s - \tilde{u}_r$. Both are intuitive: The more uncertain he is about u_x, captured by σ_x^2, the more the DM will rely on the ordinal information. Similarly, the larger $\tilde{u}_x^s - \tilde{u}_r$, the smaller the surprise of a + ordinal signal, thus the smaller its effect.

More subtle are the effects of σ_r^2 and σ_o^2, discussed in Part (b). Intuitively, one may expect the two noises to have a *negative* effect on the boost—at the end of the day, both make the ordinal information less precise. Indeed, this is the case when $\tilde{u}_x^s \leq \tilde{u}_r$ or when $\tilde{u}_x^s - \tilde{u}_r$ is positive but small enough. However, if $\tilde{u}_x^s - \tilde{u}_r$ is very large, an increase in σ_r^2 or σ_o^2 may *increase* the boost. To see why, suppose that $\tilde{u}_x^s - \tilde{u}_r$ is extremely large and suppose first there is no noise in the ordinal signal. Then, a + signal is not very informative—the DM expected this to be the case already. However, crucially, it will be at least a bit more informative when σ_r^2 is higher—because even if $\tilde{u}_r$ is much below $\tilde{u}_x^s$, a high σ_r^2 increases the chance that u_r is above $\tilde{u}_r$ and therefore makes the signal more relevant. Similarly, suppose that $\tilde{u}_x^s - \tilde{u}_r$ is extremely large and $\sigma_r^2 = 0$. Again, adding a bit of noise to the ordinal signal may (paradoxically) make it *more* informative because in our model, the chances of a correct ordinal signal depend on the distance between the realized u_x and u_r; therefore, noise in the ordinal signal makes the + signal more meaningful—because, with noise, a − signal could have been possible, but it didn't realize, increasing the chances that u_x is well above u_r. Similar comparative statics also hold for the average boost.

Mark: I think for part b it should be $\frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$

Proposition 10. *Suppose $u_x > u_r = \tilde{u}_r = \tilde{u}_x$, $\sigma_p^2, \sigma_\epsilon^2 > 0$, and $\sigma_r^2 + \sigma_o^2 > 0$. Then, $C(u_x, u_r) > 0$. Moreover*

(a) *C is strictly increasing in σ_x^2.*

(b) *C is decreasing in σ_r^2 and σ_o^2 as long as $u_x - u_r$ is not too big. If it is big enough, C may be*

[26] Specifically, let $z^\star$ be the unique root of $g(z) = 1 - z^2 - z\frac{\phi(z)}{\Phi(z)}$ on $(0,\infty)$, where $z^\star \approx 0.84$. Then $\frac{\partial B(s,+)}{\partial \sigma_r^2} = \frac{\partial B(s,+)}{\partial \sigma_o^2} > 0 \Leftrightarrow \frac{\tilde{u}_x^s - \tilde{u}_r}{\sqrt{\sigma_x^2+ \sigma_r^2+\sigma_o^2}} > z^\star$.

increasing in σ_r^2 and σ_o^2.[27]

(c) If $\sigma_o = 0$, C is strictly decreasing in $u_x - u_r$. If $\sigma_o > 0$, there exists $\Delta_* \in (0, \infty)$ such that C is strictly increasing in $u_x - u_r$ if $0 < u_x - u_r < \Delta_*$ and strictly decreasing if $u_x - u_r > \Delta_*$.

Notice that the average boost C may be *negative* even if $u_x > u_r$ when $\tilde{u}_x$ is much above $\tilde{u}_r$.[28]

B.5 Beliefs with Two Uncertain Comparison Points

Theorem 1. *Let x be a choice alternative and r and t two comparators, with beliefs about utilities along a given dimension given by*

$$\begin{aligned} u(x) &\sim N(\tilde{u}(x), \sigma_x^2) \\ u(r) &\sim N(\tilde{u}(r), \sigma_r^2) \\ u(t) &\sim N(\tilde{u}(t), \sigma_t^2) \end{aligned}$$

[27]Specifically, there exists $\Delta_r^* \in (0, \infty]$ and $\Delta_o^* \in (0, \infty]$ such that

$$\frac{\partial C}{\partial \sigma_r^2} < 0 \text{ for } 0 < u_x - u_r < \Delta_r^*, \qquad \frac{\partial C}{\partial \sigma_r^2} > 0 \text{ for } u_x - u_r > \Delta_r^*.$$

and

$$\frac{\partial C}{\partial \sigma_o^2} < 0 \text{ for } 0 < u_x - u_r < \Delta_o^*, \qquad \frac{\partial C}{\partial \sigma_o^2} > 0 \text{ for } u_x - u_r > \Delta_o^*.$$

Note that Δ_r^* or Δ_0^* may be ∞, in which case C is decreasing in the respective variances for all u_x, u_r with $u_x > u_r$.

[28]For example, if $r = \mu_r = 0$ and $\mu_x = 2,\ x = 0.1,\ \sigma_o = 1,\ \sigma_r^2 = 0,\ \sigma_p^2 = 0.04,\ \sigma_e^2 = 4$. Then $C < 0$ (numerically ≈ -0.0402).

distributed independently. Then, if the DM learns that the utility of x is greater than that of r, which in turn is greater than that of t we have

$$\begin{aligned}
E(u(x)|u(x) &> u(r) > u(t)) \\
&= \tilde{u}(x) + \frac{\sigma_x^2}{\sqrt{\left(\sigma_x^2+\sigma_r^2\right)}} \\
&\quad \times\phi\left(\frac{\tilde{u}(x)-\tilde{u}(r)}{\sqrt{\left(\sigma_x^2+\sigma_r^2\right)}}\right) \\
&\quad \times\Phi\left(\left(\frac{\left(\tilde{u}(r)-\tilde{u}(x)\right)\sigma_x}{\sigma_r\sqrt{\sigma_x^2+\sigma_r^2}} - \left(\frac{\tilde{u}(t)-\tilde{u}(x)}{\sigma_x}\right)\left(\frac{\sqrt{\sigma_x^2+\sigma_r^2}}{\sigma_r}\right)\right)\frac{1}{\sqrt{1+\left(\frac{1}{\sigma_r^2}+\frac{1}{\sigma_x^2}\right)\sigma_t^2}}\right) \\
&\quad \times\frac{1}{\Phi_2\left(\frac{-(\tilde{u}(r)-\tilde{u}(x))}{\sqrt{\sigma_r^2+\sigma_x^2}}, \frac{-(\tilde{u}(t)-\tilde{u}(r))}{\sqrt{\sigma_t^2+\sigma_r^2}}; \rho(\sigma_x,\sigma_r,\sigma_t)\right)} \\
&\equiv \mu_x + T((\tilde{u}(x),\sigma_x),(\tilde{u}(r),\sigma_r),(\tilde{u}(t),\sigma_t))
\end{aligned}$$

where

$$\rho = \frac{-1}{\sqrt{\left(1+\frac{\sigma_x^2}{\sigma_r^2}\right)\left(1+\frac{\sigma_t^2}{\sigma_r^2}\right)}}$$

If instead the DM learns that x is less than t which is in turn is less than r we have

$$\begin{aligned}
E(x|r &> t > x) \\
&= \tilde{u}(x) - \frac{\sigma_x^2}{\sqrt{\left(\sigma_x^2+\sigma_t^2\right)}} \\
&\quad \times\phi\left(\frac{\tilde{u}(x)-\tilde{u}(t)}{\sqrt{\sigma_x^2+\sigma_t^2}}\right) \\
&\quad \times\Phi\left(\left(\frac{\left(\tilde{u}(x)-\tilde{u}(t)\right)\sigma_x}{\sigma_t\sqrt{\sigma_x^2+\sigma_t^2}} - \left(\frac{\left(\tilde{u}(x)-\tilde{u}(r)\right)}{\sigma_x}\right)\left(\frac{\sqrt{\sigma_t^2+\sigma_x^2}}{\sigma_t}\right)\right)\frac{1}{\sqrt{1+\left(\frac{1}{\sigma_t^2}+\frac{1}{\sigma_x^2}\right)\sigma_r^2}}\right) \\
&\quad \times\frac{1}{\Phi_2\left(\frac{-(\tilde{u}(t)-\tilde{u}(r))}{\sqrt{\sigma_t^2+\sigma_r^2}}, \frac{-(\tilde{u}(x)-\tilde{u}(t))}{\sqrt{\sigma_x^2+\sigma_r^2}}; \rho(\sigma_r,\sigma_t,\sigma_x)\right)} \\
&\equiv \mu_z - B((\tilde{u}(r),\sigma_r),(\tilde{u}(t),\sigma_t),(\tilde{u}(x),\sigma_x))
\end{aligned}$$

Finally, if x is less than r but greater than t we have

$$\begin{aligned} E(x|r > x > t) &= \tilde{u}(x) \\ &+B((\tilde{u}(r),\sigma_r),(\tilde{u}(x),\sigma_x),(\tilde{u}(t),\sigma_t)) \\ &-T((\tilde{u}(r),\sigma_r),(\tilde{u}(x),\sigma_x),(\tilde{u}(t),\sigma_t)) \end{aligned}$$

Proof. See Balakrishnan and Dean (2025). □

Corollary 5. *To extend theorem 1 to the case of imperfect ordinal information, consider the case in which the DM learns that*

$$u(x) > u(r) + \varepsilon > u(t) + \mu$$

Where

$$\begin{aligned} \varepsilon &\sim N(0,\sigma_\varepsilon^2) \\ \mu &\sim N(0,\sigma_\mu^2) \end{aligned}$$

are Probit errors distributed independently from each other and $\tilde{u}(x)$, $\tilde{u}(r)$ and $\tilde{u}(t)$. Note that this is equivalent to learning that

$$u(x) > v(r) > v(t)$$

where

$$\begin{aligned} v(r) &\sim N(\tilde{u}(r),\sigma_r^2+\sigma_\varepsilon^2) \\ v(t) &\sim N(\tilde{u}(t),\sigma_t^2+\sigma_\mu^2) \end{aligned}$$

Thus, the formulae from theorem 1 can be applied directly, with σ_r^2 replaced with $\sigma_r^2+\sigma_\varepsilon^2$ and σ_t^2 replaced with $\sigma_t^2+\sigma_\mu^2$

Theorem 2. *Let $x \sim N(\mu_x,\sigma_x^2)$, $y \sim N(\mu_y,\sigma_y^2)$, and $z \sim N(\mu_z,\sigma_z^2)$ be independent. Then*

$$E(x-y|y>x>z) > E(x-y|y>x).$$

Proof. Note that, for any $\bar{x}$

$$\begin{aligned} E(y|y &> \bar{x}) = \mu_y + \sigma_y \frac{\phi\left(\frac{\mu_y - \bar{x}}{\sigma_y}\right)}{\Phi\left(\frac{\mu_y - \bar{x}}{\sigma_y}\right)} \\ &= \mu_y + \sigma_y \lambda\left(\frac{\mu_y - \bar{x}}{\sigma_y}\right) \end{aligned}$$

where $\phi(.)$ and $\Phi(.)$ are the PDF and CDF of a standard normal and $\lambda(.) = \frac{\phi(.)}{\Phi(.)}$ is the inverse mills ratio. Define

$$\begin{aligned} h(\bar{x}) &= \bar{x} - E(y|y > \bar{x}) \\ &= \bar{x} - \mu_y - \sigma_y \lambda\left(\frac{\mu_y - \bar{x}}{\sigma_y}\right) \end{aligned}$$

This implies

$$h'(\bar{x}) = 1 + \lambda'\left(\frac{\mu_y - \bar{x}}{\sigma_y}\right)$$

Next note that we can bound $\lambda'(t)$ above -1. To do so, first note that

$$\begin{aligned} \lambda'(t) &= \frac{\Phi(t)\phi'(t) - \phi(t)^2}{\Phi(t)^2} \\ &= \frac{-t\phi(t)\Phi(t) - \phi(t)^2}{\Phi(t)^2} \\ & \quad -\lambda(t)(t + \lambda(t)) \end{aligned}$$

Second, consider the function

$$z_t(u) = \frac{e^{-\frac{1}{2}u^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv}$$

is a PDF on $[-\infty, x]$, and has a variance equal to

$$\frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \left[\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} du\right] - \left(\frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv}\right)^2 \left(\int_{-\infty}^{t} u e^{-\frac{1}{2}u^2} du\right)^2$$

Setting $dv = ue^{-\frac{1}{2}u^2}$ and $w = u$ and applying integration by parts gives

$$\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} = \left[-ue^{-\frac{1}{2}u^2}\right]_{-\infty}^{t} + \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du \tag{15}$$

The bracketed term evaluated at $-\infty$ is zero, so we have

$$\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} = -te^{-\frac{1}{2}t^2} + \int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du$$

and so

$$\begin{aligned} & \frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \left[\int_{-\infty}^{t} u^2 e^{-\frac{1}{2}u^2} du \right] \\ = & \frac{-te^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} + \frac{\int_{-\infty}^{t} e^{-\frac{1}{2}u^2} du}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \\ = & -t\lambda(t) + 1 \end{aligned}$$

where we use the fact that we can rewrite the inverse mills ratio as $\lambda(t) = \frac{e^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv}$

Next, note that

$$\left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(\left[-e^{-\frac{1}{2}u^2} \right]_{-\infty}^{t} \right)^2$$

again, the limit as $u = -\infty$ is equal to zero, so that gives

$$\left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(-e^{-\frac{1}{2}t^2} \right)^2$$

and

$$\left(\frac{1}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \right)^2 \left(\int_{-\infty}^{t} ue^{-\frac{1}{2}u^2} du \right)^2 = \left(\frac{-e^{-\frac{1}{2}t^2}}{\int_{-\infty}^{t} e^{-\frac{1}{2}v^2} dv} \right)^2 = \lambda(t)^2$$

And so the variance of $z_t(u)$ is given by

$$1 - t\lambda(t) - \lambda(t)^2 > 0$$

Using the fact that $\lambda'(t) = -\lambda(t)(t + \lambda(t))$ gives

$$1 + \lambda'(t) > 0 \implies \lambda(t) > -1$$

Implying that $h'(\bar{x}) > 0$.

Next, we find an expression for $E(x - y | y > x)$. Define

$$p_A(\bar{x}) = p(x = \bar{x}, y > x) = \frac{1}{\sigma_x} \phi \left(\frac{\bar{x} - \mu_x}{\sigma_x} \right) \left(1 - \Phi \left(\frac{\bar{x} - \mu_y}{\sigma_y} \right) \right).$$

Then

$$\begin{aligned} E(x-y|y &> x) = \int E(\bar{x}-y|y>\bar{x})p(x=\bar{x}|y>x)d(\bar{x}) \\ &= \int h(\bar{x})\frac{p(x=\bar{x},y>x)}{p(y>x)}d(\bar{x}) \\ &= \int h(\bar{x})\frac{p_A(\bar{x})}{\int p_A(x)d(x)}d(\bar{x}) \\ &= E(h(\bar{x})|y>x), \end{aligned}$$

noting that $p(x=\bar{x}|y>x) = \frac{p_A(\bar{x})}{\int p_A(x)d(x)}$.

Next we find an expression for $E(x-y|y>x>z)$. First note that

$$p(x=\bar{x},y>x>z) = p_A(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)$$

as z is independent of x and y. In turn,

$$p(y>x>z) = \int p_A(x)\Phi\left(\frac{x-\mu_z}{\sigma_z}\right)d(x).$$

We can then write

$$\begin{aligned} E(x-y|y>x>z) &= \int E(\bar{x}-y|y>\bar{x}>z)p(x=\bar{x}|y>x>z)d(\bar{x}) \\ &= \int E(\bar{x}-y|y>\bar{x}>z)\frac{p(x=\bar{x},y>x>z)}{p(y>x>z)}d(\bar{x}) \\ &= \int E(\bar{x}-y|y>\bar{x}>z)\frac{p_A(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)}{\int p_A(x')\Phi\left(\frac{x'-\mu_z}{\sigma_z}\right)d(x')}d(\bar{x}). \end{aligned}$$

Dividing through top and bottom by $\int p_A(x)d(x)$ gives

$$\begin{aligned} E(x-y|y>x>z) &= \frac{\int E(\bar{x}-y|y>\bar{x}>z)\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)\frac{p_A(\bar{x})}{\int p_A(x)d(x)}d(\bar{x})}{\int \frac{p_A(x')}{\int p_A(x)d(x)}\Phi\left(\frac{x'-\mu_z}{\sigma_z}\right)d(x')} \\ &= \frac{\int h(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)p(\bar{x}|y>x)d(\bar{x})}{\int \Phi\left(\frac{x'-\mu_z}{\sigma_z}\right)p(x'|y>x)d(x')} \\ &= \frac{E\left(h(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)}{E\left(\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)}. \end{aligned}$$

Recall that we have shown that h is strictly increasing; $\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)$ is also a strictly increasing function of $\bar{x}$. Then, Chebyshev's covariance inequality (see e.g. Agahi (2015)) guarantees that

$$cov\left(h(\bar{x}),\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)>0.$$

, with the strict inequality guaranteed by the fact that that h and ϕ are strictly increasing and $\bar{x}$ is not almost surely constant. This means

$$cov\left(h(\bar{x}),\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)=E\left(h(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)-E(h(\bar{x})|y>x)E\left(\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)>0$$

thus (since $E\left(\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)>0$)

$$\frac{E\left(h(\bar{x})\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)}{E\left(\Phi\left(\frac{\bar{x}-\mu_z}{\sigma_z}\right)|y>x\right)}>E(h(\bar{x})|y>x)$$

hence $E(x-y|y>x>z)>E(x-y|y>x)$, completing the proof. □

Theorem 3. *Let $x\sim N(\mu_x,\sigma_x^2)$, $y\sim N(\mu_y,\sigma_y^2)$, and $z\sim N(\mu_z,\sigma_z^2)$ be independent. Then*

$$E(x-y|x>z>y)>E(x-y|x>y).$$

Proof. Let

$$h(x,y):=x-y \quad \text{and} \quad q(x,y):=p(z\in(y,x))=\Phi\left(\frac{x-\mu_z}{\sigma_z}\right)-\Phi\left(\frac{y-\mu_z}{\sigma_z}\right).$$

Now define $h(x,y) = x - y$ and note that, conditional on $x > y$, $h(x,y)$ and $q(x,y) > 0$ are both are strictly increasing in their first and strictly decreasing in their second argument. This in turn means that both are strictly increasing functions of $(x, -y)$. We wish to show that this implies that that the covariance of the two functions must be positive conditional on $x > y$. To do so, first note also that, as x and y are independent, their the joint unconditional distribution is Multivariate Totally Positive of Order 2 (MTP_2) (see Karlin and Rinott (1980) Proposition 3.1). We need to show that this property is preserved following conditioning on $x > y$. To do so, first define the indicator function

$$I(x,y) = \begin{cases} 1 \text{ if } x > y \\ 0 \text{ otherwise} \end{cases}$$

It is easy to note that I is also MTP_2.[29] We can write the distribution of x and y conditional on $x > y$ as

$$f(x,y|x>y) = \frac{p(x,y \text{ and } x>y)}{p(x>y)} = \frac{f(x,y)I(x,y)}{p(x>y)}$$

As we have established that both f and I are MTP_2, by Remark 2.1 of Karlin and Rinott (1980), we can conclude that $f(x,y|x>y)$ is MTP_2, and so, by equation 1.9 in the same paper

$$cov(h(x,y), q(x,y)|x>y) > 0$$

Mark: need to confirm strictness

Now note that

$$p(x=\bar{x}, y=\bar{y}|x>z>y) = \frac{p(x=\bar{x}, y=\bar{y}|x>y)q(\bar{x},\bar{y})|x>y}{p(x>z>y|x>y)}.$$

[29] A function f is MTP_2 if, for any (x_1, y_1), (x_2, y_2)

$$\begin{aligned} & f(\max(x_1,x_2), \max(y_1,y_2)) f(\min(x_1,x_2), \min(y_1,y_2)) \\ \geq \; & f(x_1,y_1) f(x_2,y_2) \end{aligned}$$

Note that this will be trivially satisfied for I unless $x_1 > y_1$ and $x_2 > y_2$ in which case the RHS of the above expression equals 1. WLOG assume $x_1 > x_2$, implying $x_1 > y_1$ and $x_1 > y_2$, and so $\max(x_1,x_2) > \max(y_1,y_2)$. Again WLOG assume $y_1 \leq y_2$, meaning that $y_1 < x_1$ and $y_1 < x_2$ and so $\min(x_1,x_2) > \min(y_1,y_2)$ Together this implies the LHS of the equation is also equal to 1, and so I is MTP_2.

So

$$\begin{aligned} E(x-y|x>z>y) &= E(h(x,y)|x>z>y) = \int h(\bar{x},\bar{y})p(x=\bar{x},y=\bar{y}|x>z>y)d(\bar{x},\bar{y}) \\ &= \int h(\bar{x},\bar{y})\frac{p(x=\bar{x},y=\bar{y}|x>y)q(\bar{x},\bar{y}|x>y)}{p(x>z>y|x>y)}d(\bar{x},\bar{y}) \\ &= \int h(\bar{x},\bar{y})\frac{p(x=\bar{x},y=\bar{y}|x>y)q(\bar{x},\bar{y}|x>y)}{\int p(x=x',y=y'|x>y)q(x',y'|x>y)d(x',y')}d(\bar{x},\bar{y}) \\ &= \frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)}. \end{aligned}$$

Note that

$$\begin{aligned} cov(h(x,y),q(x,y)|x>y) &> 0 \quad \Rightarrow \\ E(h(x,y)q(x,y)|x>y) - E(h(x,y)|x>y)E(q(x,y)|x>y) &> 0 \quad \Rightarrow \\ \frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)} &> E(h(x,y)|x>y) \end{aligned}$$

since $E(q(x,y)|x>y) > 0$. But then

$$E(x-y|x>z>y) = \frac{E(h(x,y)q(x,y)|x>y)}{E(q(x,y)|x>y)} > E(h(x,y)|x>y) = E(x-y|x>y),$$

completing the proof. □

B.6 Further Details on Compromise Effect

Our model can generate either the compromise effect or the opposite, depending on where the decoy is placed.

As a concrete example, consider a case in which interim beliefs are such that the means of x are {2,2} on dimensions 1 and 2 respectively, while the means of y are {3,1}. Assume that standard deviations are 1 for each alternative on each dimension. If we add a phantom decoy z with means {1,3} (and standard deviations also equal to 1) then this will have no effect on the probability of choice between x and y due to the symmetry of the problem.

If instead the z has mean {1,4} then our model predicts the decoy effect. This is because moving z further away in dimension 2 has a bigger effect on the closer option. In dimension 1, x receives 0 boost, as the positive boost from z and the negative boost from y cancel each other out. Using the formula from Balakrishnan and Dean (2025), y receives a positive boost of about 0.34. In dimension 2, x receives a positive boost of 0.18: the positive boost from y is smaller

than the negative boost from z because the former is closer. y receives a negative boost of 0.31, meaning that x is favored overall.

If instead we place the decoy at {0,3} the pattern is reversed. Now x receives a negative boost in dimension 1 of -0.18 and a 0 boost in dimension 2, while y receives a positive boost of 0.31 in dimension 1 and a negative boost of 0.34 in dimension 2, meaning that overall the addition of z favors y.

C Proofs

C.1 Proof of Observation 3

Fix the common priors $u_x \sim \mathcal{N}(\tilde{u}_x, \sigma_x^2)$ and $u_r \sim \mathcal{N}(\tilde{u}_r, \sigma_r^2)$, $\sigma_v^2 = \sigma_{v'}^2$, and recall that v and v' are independent of all other random variables. Since o and o' have the same conditional distribution given (u_x, u_r), it suffices to show that the cardinal signals alone induce the same posterior over (u_x, u_r).

Step 1: Reduce the Alternative Setting by an invertible transformation. In the Alternative Setting, define the residual error $t := e_d - e_x - e_r$. Under the stated covariance restrictions, $t \sim \mathcal{N}(0, \sigma_t^2)$ with $\sigma_t^2 = \sigma_{e_d}^2 - \sigma_{e_x}^2 - \sigma_{e_r}^2 > 0$, and t is independent of (e_x, e_r). Define the transformed signal

$$z := s'_x - s_d = (u_x + e_x) - (u_x - u_r + e_d) = u_r - e_r - t.$$

The mapping $(s'_x, s'_r, s_d) \mapsto (s'_x, s'_r, z)$ is invertible (since $s_d = s'_x - z$), so it preserves the agent's information set. Conditional on the states (u_x, u_r), the signals are

$$s'_x = u_x + e_x, \qquad (s'_r, z) = (u_r + e_r,\ u_r - e_r - t).$$

Since e_x is independent of (e_r, t), the belief update regarding u_x (via s'_x) is independent of the update regarding u_r (via s'_r and z).

Step 2: Extract an equivalent single Gaussian signal about u_r. The agent holds two signals about u_r: s'_r and z. Consider the further invertible transformation to the pair $(s'_r, \bar{s}_r)$, where

$$\bar{s}_r := \frac{s'_r + z}{2} = \frac{(u_r + e_r) + (u_r - e_r - t)}{2} = u_r - \frac{t}{2}.$$

Conditional on u_r, the signal s'_r has error e_r and the signal $\bar{s}_r$ has error $-t/2$. Because e_r and t are independent, the agent effectively holds two independent Gaussian signals about u_r with

independent errors with precisions $1/\sigma_{e_r}^2$ and $1/(\sigma_t^2/4) = 4/\sigma_t^2$, respectively. By sufficiency, these are equivalent to a single Gaussian signal $s_r^* = u_r + \varepsilon_r^*$ with precision

$$\frac{1}{\sigma_{\varepsilon_r}^2} = \frac{1}{\sigma_{e_r}^2} + \frac{4}{\sigma_t^2} = \frac{1}{\sigma_{e_r}^2} + \frac{4}{\sigma_{e_d}^2 - \sigma_{e_x}^2 - \sigma_{e_r}^2}.$$

Step 3: Conclude Alternative → Original. Combining Steps 1 and 2, the Alternative Setting yields the same information about (u_x, u_r) as observing

$$s_x' = u_x + e_x \quad \text{and} \quad s_r^* = u_r + \varepsilon_r^*,$$

with independent errors $e_x \sim \mathcal{N}(0, \sigma_{e_x}^2)$ and $\varepsilon_r^* \sim \mathcal{N}(0, \sigma_{\varepsilon_r}^2)$. Together with the ordinal signal, this induces the same posterior as the Original Setting with parameters defined in item 1 of the Observation.

Step 4: Conclude Original → Alternative. Conversely, fix an Original Setting $(\sigma_{\varepsilon_x}^2, \sigma_{\varepsilon_r}^2, \sigma_v^2)$. Set $\sigma_{e_x}^2 = \sigma_{\varepsilon_x}^2$ and $\sigma_{v'}^2 = \sigma_v^2$. Choose any $\sigma_{e_r}^2 > \sigma_{\varepsilon_r}^2$ and define

$$\sigma_t^2 := 4\left(\frac{1}{\sigma_{\varepsilon_r}^2} - \frac{1}{\sigma_{e_r}^2}\right)^{-1} > 0, \qquad \sigma_{e_d}^2 := \sigma_{e_x}^2 + \sigma_{e_r}^2 + \sigma_t^2.$$

Let (e_x, e_r, t) be independent Gaussians with these variances and set $e_d = e_x + e_r + t$. Then (e_x, e_d, e_r) satisfies the covariance structure of the Alternative Setting. By the equivalence established in Steps 1–3, this Alternative Setting yields the same posterior beliefs as the Original Setting. □

C.2 Proof of Observation ??.

Mark: do we need this proof here or is it now contained in some other proofs or the Bala paper?

The aim is to show that if $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, distributed independently, then

$$E(X|X > Y) = \mu_X + \frac{\sigma_X^2}{\sqrt{\sigma_X^2 + \sigma_Y^2}} \frac{\phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)}{\Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)}$$

where ϕ is the pdf and Φ is the cdf of the normal distribution.

First, note that we can rewrite

$$\begin{aligned} X &= \sigma_X U + \mu_X \\ Y &= \sigma_Y V + \mu_Y \end{aligned}$$

Where U and V and standard normals, and so

$$X > Y \Leftrightarrow U > aV - b$$

where

$$\begin{aligned} a &= \frac{\sigma_Y}{\sigma_X} \\ b &= \frac{\mu_X - \mu_Y}{\sigma_X} \end{aligned}$$

And so

$$E(X|X > Y) = \sigma_X E(U|U > aV - b) + \mu_X \tag{16}$$

Focusing on

$$E(U|U > aV - b) \tag{17}$$

Note first that

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and so

$$\phi'(x) = -x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\phi(x).$$

As

$$\begin{aligned} E(U|U &> aV - b) \\ &= \frac{1}{P(U > aV - b)} \int_{-\infty}^{\infty} \int_{av-b}^{\infty} u\phi(u)du\phi(v)dv \\ &= \frac{1}{P(U > aV - b)} \int_{-\infty}^{\infty} \int_{av-b}^{\infty} -\phi'(u)du\phi(v)dv \end{aligned}$$

Using the fundamental theorem of calculus we have

$$\begin{aligned} &\int_{av-b}^{\infty} -\phi'(u)du \\ &\phi(av - b) \end{aligned}$$

and so

$$E(U|U > aV - b) = \frac{1}{P(U > aV - b)} \int_{-\infty}^{\infty} \phi(av - b)\phi(v)dv \tag{18}$$

Note that

$$\begin{aligned} & \phi(av - b)\phi(v) \\ = & \frac{1}{2\pi} \exp\left\{\frac{-(av - b)^2 - v^2}{2}\right\} \\ = & \frac{1}{2\pi} \exp\left\{\frac{-\left(a^2 + 1\right)v^2 - b^2 + 2avb}{2}\right\} \end{aligned}$$

Defining $c = \sqrt{1 + a^2}$ and $w = cv$ this becomes

$$\begin{aligned} & \frac{1}{2\pi} \exp\left\{\frac{-w^2 - b^2 + \frac{2ab}{c}w}{2}\right\} \\ = & \frac{1}{2\pi} \exp\left\{\frac{-\left(w - \frac{ab}{c}\right)^2 - b^2 + \left(\frac{ab}{c}\right)^2}{2}\right\} \end{aligned}$$

And so

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(av + b)\phi(v)dv \\ = & \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{\frac{-\left(w - \frac{ab}{c}\right)^2 - b^2 + \left(\frac{ab}{c}\right)^2}{2}\right\} \frac{1}{c} dw \end{aligned}$$

using the fact that $\frac{dw}{dv} = c$. The above expression can then be rewritten as

$$\frac{1}{c} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-b^2 + \left(\frac{ab}{c}\right)^2}{2}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-\left(w - \frac{ab}{c}\right)^2}{2}\right\} dw$$

The term inside the integral is the pdf of a normal with a mean of $\frac{ab}{c}$, and so the integral

equals 1 leaving

$$\begin{aligned}
& \frac{1}{c}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-b^2+\left(\frac{ab}{c}\right)^2}{2}\right\} \\
= & \frac{1}{c}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-b^2(1-\left(\frac{a}{c}\right)^2)}{2}\right\} \\
= & \frac{1}{c}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-b^2(1-\frac{a^2}{1+a^2})}{2}\right\} \\
= & \frac{1}{c}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-\left(\frac{b}{c}\right)^2}{2}\right\} \\
= & \frac{1}{c}\phi(\frac{b}{c})
\end{aligned}$$

Substituting back into equation 18 gives

$$E(U|U > aV - b) = \frac{1}{P(U > aV - b)}\frac{1}{c}\phi(\frac{b}{c})$$

Notice that

$$\begin{aligned}
P(U & > aV - b) \\
& = P(b > aV - U)
\end{aligned}$$

Where $aV - U$ is a normal with mean zero and variance $a^2 + 1 = c^2$ and so

$$P(U > aV - b) = \Phi\left(\frac{b}{c}\right)$$

finally giving

$$E(U|U > aV - b) = \frac{\phi(\frac{b}{c})}{c\Phi\left(\frac{b}{c}\right)}$$

Plugging this into equation 16 gives

$$E(X|X > Y) = \sigma_X\frac{\phi(\frac{b}{c})}{c\Phi\left(\frac{b}{c}\right)} + \mu_X$$

and noting that

$$\begin{aligned}\frac{b}{c} &= \frac{\frac{\mu_X-\mu_Y}{\sigma_X}}{\sqrt{1+\left(\frac{\sigma_Y}{\sigma_X}\right)^2}} \\ &= \frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\end{aligned}$$

and

$$\begin{aligned}c &= \sqrt{1+a^2} \\ &= \frac{\sqrt{\sigma_X^2+\sigma_Y^2}}{\sigma_X}\end{aligned}$$

gives

$$E(X|X>Y) = \mu_X + \frac{\sigma_X^2}{\sqrt{\sigma_X^2+\sigma_Y^2}} \frac{\phi\left(\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)}{\Phi\left(\frac{\mu_X-\mu_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)} \tag{19}$$

C.3 Proof of Observation 2

When $\sigma_{\epsilon,i}^2 \to \infty$, then $\frac{\lambda_i}{\bar{\sigma}_i} = \frac{\sigma_{p,i}}{\sigma_{\epsilon,i}\sqrt{\sigma_{\epsilon,i}^2+\sigma_{p,i}^2}} \to 0$. Now notice that $\psi(0) = \frac{\bar{\sigma}_i}{\sqrt{2}}\sqrt{2/\pi} = \frac{\bar{\sigma}_i}{\sqrt{\pi}}$.

Moreover, as $\sigma_{\epsilon,i}^2 \to \infty$, we get $\bar{\sigma}_i \to \sigma_{p,i}^2$

Thus, the boost in that dimension is simply $\frac{\sigma_{p,i}^2}{\sqrt{\pi}}$. If $\sigma_{\epsilon,i}^2 \to \infty$ for all i, we obtain that the perceived utility is

$$E[u(x)|s_i, ord_i] = \sum_i \mu_i^x + \frac{\sigma_{p,i}^2}{\sqrt{\pi}}\left(|D_x| - |D_y|\right).$$

When the priors are the same on each dimensions of the two objects, and if the DM has the same prior variance across dimensions, he values x above y if and only if $|D_x| > |D_y|$. This will also be the case when the prior is uninformative, $\sigma_{p,i}^2 \to \infty$.

C.4 Proof of Proposition 2

Let

$$\mu(w) := a^{\text{no rp}}(w) \;=\; \lambda\beta w + (1-\lambda)\,\overline{\beta w}, \qquad t(a,w) := \frac{\mu(w)a + \lambda\epsilon - \beta r}{\sigma_x}.$$

Then, if we define

$$f(a;w) \;:=\; \mu(w)\,a - \frac{a^2}{2} \;+\; \sigma_x\,\psi\big(t(a,w), o_a\big) \;+\; \lambda\epsilon, \tag{20}$$

our maximization problem is equivalent to $\max_{a\geq 0} f(a,w)$.

Step 1: Basic Facts We begin by establishing a few basic facts. Let

$$m(t) := \frac{\phi(t)}{\Phi(t)}, \qquad n(t) := \frac{\phi(t)}{1-\Phi(t)}.$$

Then, a well-known property of Mill's ratios (easily verified) is that

$$m'(t) = -m(t)\big(t + m(t)\big) \in (-1,0), \qquad m''(t) > 0,$$

$$n'(t) = n(t)\big(n(t) - t\big) \in (0,1), \qquad n''(t) > 0,$$

and by Normal symmetry $m(t) = n(-t)$ and $n'(t) = -m'(-t)$.

Notice that for $a \neq r/w$ we have

$$\frac{\partial t}{\partial a} = \frac{\mu(w)}{\sigma_x}, \qquad f_a(a;w) = \mu(w) - a + \mu(w)\,\psi'\big(t(a,w), o_a\big).$$

Thus, the left and right derivatives at $a = r/w$ are

$$\begin{aligned} D_-(w) &:= \lim_{a\uparrow r/w} f_a(a;w) = \mu(w) - \frac{r}{w} - \mu(w)\,n'\big(t(r/w,w)\big), \\ D_+(w) &:= \lim_{a\downarrow r/w} f_a(a;w) = \mu(w) - \frac{r}{w} + \mu(w)\,m'\big(t(r/w,w)\big). \end{aligned} \tag{21}$$

Given the properties above, $D_\pm(\cdot)$ are continuous in w, with

$$\lim_{w\downarrow 0} D_\pm(w) = -\infty, \qquad \lim_{w\to\infty} D_\pm(w) = +\infty.$$

Hence there exist $w_-, w_+ > 0$ such that $D_-(w_-) = 0$ and $D_+(w_+) = 0$. Moreover, using $m(t) = n(-t)$ and $m'' > 0$ one checks that $D_-(w) - D_+(w)$ is strictly increasing in w, giving $w_- < w_+$.

Next, we establish the concavity of the problems on the left and right. For $a < r/w$,

$$f_{aa}(a;w) = -1 - \frac{\mu(w)^2}{\sigma_x}\,n''\big(t(a,w)\big) < -1,$$

so $f(\cdot; w)$ is strictly concave on $(-\infty, r/w)$ and $f_a(\cdot; w)$ is strictly decreasing there. For $a > r/w$,

$$f_{aa}(a; w) = -1 + \frac{\mu(w)^2}{\sigma_x}\, m''\big(t(a, w)\big),$$

so interior right-side stationary points that satisfy the second-order condition (SOC)

$$-1 + \frac{\mu(w)^2}{\sigma_x}\, m''(t) < 0$$

are strict local maxima.

Finally, notice that we have a positive jump of the objective function at the point of discontinuity: because $\psi(t, -) = -n(t)$ and $\psi(t, +) = m(t)$,

$$f\big(\tfrac{r}{w}{}^{+}; w\big) - f\big(\tfrac{r}{w}{}^{-}; w\big) = \sigma_x\big(m(t(r/w, w)) + n(t(r/w, w))\big) \;>\; 0.$$

Step 2. Solving the Left and Right Problems. For each w, consider the left problem $\max_{a \le r/w} f(a; w)$ and the right problem $\max_{a \ge r/w} f(a; w)$. On the left, strict concavity implies a unique maximizer $a^-(w)$: either $a^-(w) = r/w$ if $D_-(w) \ge 0$, or else $a^-(w) \in (0, r/w)$ solves

$$0 = \mu(w) - a - \mu(w)\, n'\big(t(a, w)\big).$$

On the right, any interior maximizer $a^+(w) > r/w$ satisfies

$$0 = \mu(w) - a + \mu(w)\, m'\big(t(a, w)\big), \qquad -1 + \frac{\mu(w)^2}{\sigma_x}\, m''\big(t(a, w)\big) < 0.$$

If instead $D_+(w) \le 0$, the right-side maximum is attained at the boundary $a = r/w$.

Let $V^-(w)$ and $V^+(w)$ denote the left- and right-side value functions. Because $f(\frac{r}{w}{}^{+}; w) - f(\frac{r}{w}{}^{-}; w) > 0$, evaluating at the discontinuity $a = \frac{r}{w}$ from the right weakly dominates evaluating from the left. Hence

$$V^+(w) \;\ge\; f\big(\tfrac{r}{w}{}^{+}; w\big) \;=\; f\big(\tfrac{r}{w}{}^{-}; w\big) + \sigma_x\big(m(t(r/w, w)) + n(t(r/w, w))\big) \;>\; f\big(\tfrac{r}{w}{}^{-}; w\big)\,.$$

Step 3: Existence of w_1 and w_3 and bunching on (w_1, w_3). Consider the difference

$$\Delta_1(w) := V^+(w) - V^-(w).$$

As $w \downarrow 0$, $r/w \to \infty$ and (by the quadratic term) $f(\frac{r}{w}^+; w) \to -\infty$, hence $\Delta_1(w) < 0$ for small w. As $w \to \infty$, $\mu(w) \sim \lambda\beta w \to \infty$ and the right side dominates, so $\Delta_1(w) > 0$ for large w. By continuity, there exists $w_1 > 0$ with $\Delta_1(w_1) = 0$ and $\Delta_1(w) > 0$ for $w > w_1$ in a neighborhood of w_1.

Independently, by (21) and the limits above there exists $w_3 > w_1$ with $D_+(w_3) = 0$ and $D_+(w) < 0$ for $w < w_3$. Therefore, for every $w \in (w_1, w_3)$: (i) the right-side problem attains its maximum at the boundary $a = r/w$ (since it is decreasing just to the right of the boundary), and (ii) $\Delta_1(w) \geq 0$ so the right-side value weakly exceeds the left-side value. Hence the global maximizer is

$$a^*(w) = \frac{r}{w} \qquad \text{for all } w \in (w_1, w_3),$$

and $a^{*\prime}(w) = -r/w^2 < 0$. This proves item (1).

Step 4: Bunching of earnings. On (w_1, w_3) we have $a^*(w)w = r$, i.e. earnings are constant. At w_1 and w_3 there are upward jumps: at w_1 the optimizer switches from an interior left solution $a^-(w) < r/w$ to $a = \frac{r}{w}$, and at w_3 the right derivative D_+ turns nonnegative so the optimizer moves to an interior right solution $a^+(w) > r/w$. For $w < w_1$ or $w > w_3$, the solution is interior and satisfies the respective FOCs. Differentiating the FOC implicitly yields (using $\mu'(w) = \lambda\beta > 0$ and the properties of the Mills Ratio discussed in Step 1)

$$a'_w = \mu'(1 + \psi') \frac{1 + \frac{\mu^2}{\sigma_x}\psi''}{1 - \frac{\mu^2}{\sigma_x}\psi''},$$

where $\psi' = -n' \in (-1, 0)$ and $\psi'' = -n'' < 0$ on the left, and $\psi' = m' \in (-1, 0)$, $\psi'' = m'' > 0$ on the right (with SOC ensuring the denominator > 0 on the right). In both cases $a'_w > 0$, hence

$$\frac{d}{dw}\big(a^*(w)\, w\big) = a^*(w) + w\, a'_w \;>\; 0$$

outside (w_1, w_3). This proves item (2).

Step 4: Comparison with the reference-free benchmark and existence of w_2. For $w < w_1$ (left interior), the FOC gives $a^*(w) = \mu(w)\big(1 - n'(t)\big) < \mu(w) = a^{\text{no rp}}(w)$. For $w > w_3$ (right interior with SOC), the FOC gives $a^*(w) = \mu(w)\big(1 + m'(t)\big) < \mu(w) = a^{\text{no rp}}(w)$. For $w \in (w_1, w_3)$ we have $a^*(w) = r/w$; since $w \mapsto r/w$ is strictly decreasing and $w \mapsto \mu(w)$ strictly increasing, there exists a unique $w_2 > 0$ solving $r/w_2 = \mu(w_2)$, namely

$$w_2 := \frac{-(1-\lambda)\,\overline{\beta w} + \sqrt{(1-\lambda)^2 \overline{\beta w}^{\,2} + 4\lambda\beta r}}{2\lambda\beta}.$$

At $w = w_3$, $D_+(w_3) = 0$ implies $\frac{r}{w_3} = \mu(w_3)\big(1 + m'(t(r/w_3, w_3))\big) < \mu(w_3)$, hence $w_2 < w_3$. At $w = w_1$, by the definition of w_1 via the value crossing and the monotonicity of Δ_1, one has $\frac{r}{w_1} > \mu(w_1)$, hence $w_1 < w_2$. Therefore $w_1 < w_2 < w_3$, and the inequalities in item (3) follow. □

C.5 Proof of Lemma 1

Note that for any information set $\mathcal{I}$, we have $\min_a \mathbb{E}[(u_x - a)^2|\mathcal{I}] = \mathbb{V}[u_x|\mathcal{I}]$. Then $V(r)$ can be rewritten as the expected reduction in the posterior variance of u_x provided by the ordinal signal o, that is,

$$V(r) = \mathbb{E}_s\left[\mathbb{V}[u_x|s] - \mathbb{E}_{o|s}\left[\mathbb{V}[u_x|s,o]\right]\right],$$

where s denotes the cardinal signals s_x, s_r. Recall the Law of Total Variance, which specifies that for any random variables X, Y, $\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$, thus $\mathbb{V}[Y] - \mathbb{E}[\mathbb{V}[Y|X]] = \mathbb{V}[\mathbb{E}[Y|X]]$. Applying it to $Y = u_x$ and $X = o$, we obtain

$$V(r) = \mathbb{E}_s\left[\mathbb{V}_{o|s}\left[\mathbb{E}[u_x \mid s,o]\right]\right].$$

Let $z_s := \frac{\tilde{u}_x^s - \tilde{u}_r^s}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$ and note that, conditional on s, the ordinal signal o is a binary random variable with $o = +$ with probability $p := P(u_x - u_r + \epsilon_o \geq 0 \mid s) = \Phi(z_s)$; and $o = -$ with probability $1 - p = 1 - \Phi(z_s)$.

For each s, denote by M the update to the posterior mean given by the ordinal signal, that is, $M := \mathbb{E}[u_x \mid s,o] - \tilde{u}_x^s$. Note that $\mathbb{V}_{o|s}[\mathbb{E}[u_x|s,o]] = \mathbb{V}_{o|s}[\tilde{u}_x^s + M] = \mathbb{V}_{o|s}[M|s]$, since $\tilde{u}_x^s$ is constant conditional on s. It follows that we have $V(r) = \mathbb{E}_s\left[\mathbb{V}_{o|s}[M|s]\right]$.

From Proposition 1, M is a binary variable that takes values:

$$m_+ = \frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{\Phi(z_s)} \quad \text{and} \quad m_- = -\frac{\sigma_x^2}{\sqrt{\Sigma}}\frac{\phi(z_s)}{1-\Phi(z_s)}.$$

Moreover, note how we must have $\mathbb{E}[M \mid s] = 0$.[30] Since M is a binary variable with mean zero,

[30] To see why, $\mathbb{E}[M \mid s] = p \cdot m_+ + (1-p) \cdot m_- = \frac{\sigma_x^2}{\sqrt{\Sigma}}\left[\Phi(z_s)\frac{\phi(z_s)}{\Phi(z_s)} - (1-\Phi(z_s))\frac{\phi(z_s)}{1-\Phi(z_s)}\right] = 0.$

its variance is the expected squared value $\mathbb{E}[M^2 \mid s]$. This means

$$\begin{aligned}
\mathbb{V}_{o|s}[M|s] &= p \cdot (m_+)^2 + (1-p) \cdot (m_-)^2 \\
&= \Phi(z_s) \left[\frac{\sigma_x^2}{\sqrt{\Sigma}} \frac{\phi(z_s)}{\Phi(z_s)} \right]^2 + (1 - \Phi(z_s)) \left[-\frac{\sigma_x^2}{\sqrt{\Sigma}} \frac{\phi(z_s)}{1-\Phi(z_s)} \right]^2 \\
&= \frac{\sigma_x^4}{\Sigma} \phi(z_s)^2 \left[\frac{\Phi(z_s)}{\Phi(z_s)^2} + \frac{1-\Phi(z_s)}{(1-\Phi(z_s))^2} \right] \\
&= \frac{\sigma_x^4}{\Sigma} \phi(z_s)^2 \left[\frac{1}{\Phi(z_s)} + \frac{1}{1-\Phi(z_s)} \right] = \frac{\sigma_x^4}{\Sigma} \left[\frac{\phi(z_s)^2}{\Phi(z_s)(1-\Phi(z_s))} \right] = \frac{\sigma_x^4}{\Sigma} h(z_s).
\end{aligned}$$

Since we proved $V(r) = \mathbb{E}_s \left[\mathbb{V}_{o|s}[M|s] \right]$, taking the expectation of the expression above over s completes the proof. □

C.6 Proof of Proposition 3

In what follows, let $\Sigma := \sigma_x^2 + \sigma_r^2 + \sigma_o^2$ and $b = \tilde{u}_x - \tilde{u}_r$. Recall from Lemma 1 that

$$V(r) = \frac{\sigma_x^4}{\Sigma} \mathbb{E}_{s_x, s_r} \left[h\left(\frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{\sqrt{\Sigma}} \right) \right].$$

Step 1: Monotonicity in Bias. Note that $\tilde{u}_x^s$ and $\tilde{u}_r^s$ are normally distributed random variables (the posterior means after receiving cardinal signals). The normalized difference $D_b = \frac{\tilde{u}_x^s - \tilde{u}_r^s}{\sqrt{\Sigma}}$ is therefore normally distributed with mean $\frac{b}{\sqrt{\Sigma}}$ and variance $\sigma_D^2 = \frac{\mathrm{Var}(\tilde{u}_x^s) + \mathrm{Var}(\tilde{u}_r^s)}{\Sigma}$.

Note how the function h is even.[31] Similarly, the distribution of D when the bias is b is the mirror image of the distribution when the bias is $-b$: D_b has the same distribution of $-D_{-b}$. It follows that

$$\mathbb{E}[h(D_b)] = \mathbb{E}[h(-D_{-b})] = \mathbb{E}[h(D_{-b})],$$

and thus $V(r)$ is an even function of b.

We are left with proving that V is strictly decreasing in $|b|$. Because h is even, differentiable, and strictly decreasing on $(0, \infty)$, its derivative h' is odd with $h'(t) < 0$ for $t > 0$ and $h'(t) > 0$ for $t < 0$.

Recall that we can write $D_b = \frac{b}{\sqrt{\Sigma}} + \sigma_D Z$ with $Z \sim \mathcal{N}(0,1)$. By differentiation under the integral sign (justified by dominated convergence)

$$\frac{\partial}{\partial b} \mathbb{E}[h(D_b)] = \mathbb{E}\left[h'(D_b) \frac{\partial D_b}{\partial b} \right] = \frac{1}{\sqrt{\Sigma}} \mathbb{E}[h'(D_b)].$$

[31] Indeed, $h(-t) = \frac{\phi(-t)^2}{\Phi(-t)(1-\Phi(-t))} = \frac{\phi(t)^2}{(1-\Phi(t))\Phi(t)} = h(t)$, using $\phi(-t) = \phi(t)$ and $\Phi(-t) = 1 - \Phi(t)$.

For $b > 0$, the distribution of D_b places more mass on positive values than on negative values, and h' is negative on $(0, \infty)$ and positive on $(-\infty, 0)$, so $\mathbb{E}[h'(D_b)] < 0$ for all $b > 0$, and strictly so because the distribution of D_b has full support.

Thus for $b > 0$, $\frac{\partial}{\partial b}\mathbb{E}[h(D_b)] < 0$, and, since $V(r)$ is a positive scalar multiple of $\mathbb{E}[h(D_b)]$, $\frac{\partial V}{\partial b}(b) < 0$ for $b > 0$. By evenness, V is strictly decreasing in $|b|$.

Step 2: Monotonicity in Signal Noise $\sigma^2_{\epsilon,r}$. Note that for any information structure $\mathcal{I}$, we have $\min_a \mathbb{E}[(u_x - a)^2|\mathcal{I}] = \mathbb{V}[u_x|\mathcal{I}]$. Then $V(r)$ can be rewritten as the expected reduction in the posterior variance of u_x provided by the ordinal signal o, that is,

$$V(r) = \mathbb{E}\left[\mathbb{V}[u_x|s]\right] - \mathbb{E}\left[\mathbb{V}[u_x|s, o]\right],$$

where s denotes the cardinal signals s_x, s_r. Recall the Law of Total Variance, which specifies that for any random variables X, Y, $\mathbb{V}[Y] = \mathbb{E}[\mathbb{V}[Y|X]] + \mathbb{V}[\mathbb{E}[Y|X]]$, thus $\mathbb{E}[\mathbb{V}[Y|X]] = \mathbb{V}[Y] - \mathbb{V}[\mathbb{E}[Y|X]]$. Applying it to the formulas above and simplifying the common term (the prior variance), yields

$$V(r) = \mathrm{Var}(\mathbb{E}[u_x \mid s, o]) - \mathrm{Var}(\mathbb{E}[u_x \mid s]).$$

Note that the second term does not vary in $\sigma^2_{\epsilon,r}$. Therefore, to show that that $V(r)$ decreases in $\sigma^2_{\epsilon,r}$ it is sufficient to show that $\mathrm{Var}(\mathbb{E}[u_x \mid s, o])$ decreases in $\sigma^2_{\epsilon,r}$.

Consider two noise levels $0 < \sigma^2_{\varepsilon,r,L} < \sigma^2_{\varepsilon,r,H}$. Write $s_r^L = u_r + \varepsilon_r^L$ with $\varepsilon_r^L \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,L})$ and $s_r^H = u_r + \varepsilon_r^H$ with $\varepsilon_r^H \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,H})$, and assume $\varepsilon_r^H = \varepsilon_r^L + \eta$ where $\eta \sim \mathcal{N}(0, \sigma^2_{\varepsilon,r,H} - \sigma^2_{\varepsilon,r,L})$ is independent of everything else. Let

$$\mathcal{I}_L := (s_x, s_r^L, o), \qquad \mathcal{I}_H := (s_x, s_r^H, o).$$

By construction, the conditional distribution of $\mathcal{I}_H$ given the state (u_x, u_r) is obtained from that of $\mathcal{I}_L$ by adding the independent noise η to s_r^L. Thus $\mathcal{I}_H$ is a garbling of $\mathcal{I}_L$ in the sense of Blackwell.

Blackwell's theorem then implies that $\mathcal{I}_L$ is more informative than $\mathcal{I}_H$ for any decision problem with state (u_x, u_r). In particular, consider the estimation problem for u_x under squared loss. For such a problem, a standard corollary of Blackwell dominance is that the more informative information structure yields a posterior mean with larger variance:

$$\mathbb{V}\big(\mathbb{E}[u_x \mid \mathcal{I}_L]\big) \;\geq\; \mathbb{V}\big(\mathbb{E}[u_x \mid \mathcal{I}_H]\big),$$

with strict inequality unless the two experiments are equivalent for this problem. This shows that $\mathrm{Var}(\mathbb{E}[u_x \mid s, o])$ is strictly decreasing in $\sigma^2_{\epsilon,r}$, yielding the claim. □

C.7 Proof of Proposition 4

Let $\sigma_T^2 := \sigma_x^2 + \sigma_r^2 + \sigma_o^2$, $d := \frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{\sqrt{\sigma_T^2}}$, and $s := (s_x, s_r)$. Moreover, denote $V_s(r) := V(r \mid s_x, s_r)$ and recall from Lemma 1 that

$$V_s(r) = \frac{\sigma_x^4}{\sigma_T^2} h(d).$$

Note that, when we vary σ_r^2 or σ_o^2, for fixed (s_x, s_r), the posterior means $\tilde{u}_x^{s_x}, \tilde{u}_r^{s_r}$ and σ_x^2 are constant; only σ_T^2 changes. We can then differentiate V_s with respect to σ_T^2. By chain rule,

$$\frac{\partial V_s}{\partial \sigma_T^2} = \frac{\partial}{\partial \sigma_T^2}\left(\frac{\sigma_x^4}{\sigma_T^2} h(d)\right) = -\frac{\sigma_x^4}{(\sigma_T^2)^2} h(d) + \frac{\sigma_x^4}{\sigma_T^2} h'(d)\, \frac{\partial d}{\partial \sigma_T^2}.$$

By the definition of d and since $\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}$ does not depend on σ_T^2,

$$\frac{\partial d}{\partial \sigma_T^2} = \left(\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}\right)\left(-\frac{1}{2}\right)(\sigma_T^2)^{-3/2} = -\frac{\tilde{u}_r^{s_r} - \tilde{u}_x^{s_x}}{2(\sigma_T^2)^{3/2}} = -\frac{d}{2\sigma_T^2}.$$

where the last step uses again the definition of d. Substituting into the expression for $\partial V_s / \partial \sigma_T^2$ gives

$$\frac{\partial V_s}{\partial \sigma_T^2} = -\frac{\sigma_x^4}{(\sigma_T^2)^2} h(d) + \frac{\sigma_x^4}{\sigma_T^2} h'(d)\left(-\frac{d}{2\sigma_T^2}\right) = -\frac{\sigma_x^4}{(\sigma_T^2)^2}\left[h(d) + \tfrac{1}{2} d h'(d)\right].$$

Using $\partial \sigma_T^2 / \partial \sigma_r^2 = \partial \sigma_T^2 / \partial \sigma_o^2 = 1$ and the chain rule,

$$\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_T^2}\frac{\partial \sigma_T^2}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_T^2}, \qquad \frac{\partial V_s}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2}\frac{\partial \sigma_T^2}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2}.$$

Thus

$$\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_o^2} = -\frac{\sigma_x^4}{(\sigma_x^2 + \sigma_r^2 + \sigma_o^2)^2}\left[h(d) + \tfrac{1}{2} d h'(d)\right].$$

Let $q(d) := h(d) + \frac{1}{2} d h'(d)$ and note that the sign of $\frac{\partial V_s}{\partial \sigma_T^2}$ is the opposite of the sign of $q(d)$. Thus, all we are left with is to determine the sign of $q(d)$. We do so by analyzing its values at zero, its asymptotic behavior, and its roots.

Since h is even, $h'(0) = 0$. At $d = 0$, using $\Phi(0) = 1/2$ and $\phi(0) = 1/\sqrt{2\pi}$, we have

$$q(0) = h(0) = \frac{(1/\sqrt{2\pi})^2}{1/2 \cdot 1/2} = \frac{2}{\pi} > 0.$$

As $d \to \infty$, we use the standard Mill's ratio approximation $1 - \Phi(d) \approx \frac{\phi(d)}{d}$. This implies

$h(d) \approx d\phi(d)$ and $h'(d) \approx \phi(d)(1-d^2)$. Substituting these into $q(d)$ yields

$$q(d) \approx d\phi(d) + \frac{1}{2}d\left[\phi(d)(1-d^2)\right] \approx -\frac{1}{2}d^3\phi(d).$$

Thus, for sufficiently large d, $q(d) < 0$. By the Intermediate Value Theorem, there exists at least one zero $z^\star$.

Next, we show that $z^\star$ is unique, that is, that q admits a unique root. Consider $S(d) := \frac{q(d)}{h(d)} = 1 + \frac{d}{2}(\ln h(d))'$. Since $h(d) > 0$, the roots of $q(d)$ are identical to the roots of $S(d)$. To prove that q admits a unique root it is therefore sufficient to prove that S has a unique root. Since we have already established that $q(0) > 0$ and $q(d) < 0$ for d large enough and the same must be true for S, then to prove that S has a unique root, it is sufficient to show that S is strictly decreasing on $(0, \infty)$.

Let $r(x) := \frac{\phi(x)}{1-\Phi(x)}$. Since $h(d) = \frac{\phi(d)^2}{\Phi(d)\Phi(-d)}$, then $\ln h(d) = \ln r(d) + \ln r(-d)$. Differentiating with respect to d and using the property $r'(x) = r(x)(r(x) - x)$, we obtain:

$$\frac{\partial \ln h(d)}{\partial d} = r(d) - r(-d) - 2d.$$

Let $g(d) := r(d) - r(-d) - 2d$. The derivative is $g'(d) = r'(d) + r'(-d) - 2$. Note that r is a well-known function and it is known that $0 < r'(x) < 1$ for all real x (see, e.g., Sampford, 1953). It follows that $g'(d) < 0$. Since $g(0) = 0$, it follows that $g(d)$ is negative and strictly decreasing for $d > 0$. Substituting back into $S(d)$, we obtain

$$S(d) = 1 + \frac{1}{2}d \cdot g(d).$$

Since $d > 0$ is increasing and $g(d) < 0$ and decreasing, the product $d \cdot g(d)$ is strictly decreasing. Therefore, $S(d)$ is strictly decreasing. It follows that S has a unique root and so does q.

We have therefore established that $q(d) > 0$ for $|d| < z^\star$ and $q(d) < 0$ for $|d| > z^\star$, and that q admits a unique root. Since the sign of $\frac{\partial V_s}{\partial \sigma_T^2}$ is the opposite of the sign of $q(d)$, we have $\frac{\partial V_s}{\partial \sigma_r^2} = \frac{\partial V_s}{\partial \sigma_o^2} = \frac{\partial V_s}{\partial \sigma_T^2} < 0$ for d small enough and > 0 otherwise. Note also that the sign of $\frac{\partial V_s}{\partial \sigma_{p,r}^2}$ matches that of $\frac{\partial V_s}{\partial \sigma_r^2}$.

We are now left to derive the corresponding properties for $V(r)$, where $V(r) = \mathbb{E}_s[V_s(r)]$. We begin with the comparative statics w.r.t. σ_o^2. Since the distribution of s is independent of σ_o^2, we have $\frac{\partial V}{\partial \sigma_o^2} = \mathbb{E}_s\left[\frac{\partial V_s}{\partial \sigma_o^2}\right]$. We established that $\frac{\partial V_s}{\partial \sigma_o^2} < 0$ if and only if $|d| < z^*$. If the prior bias $|\tilde{u}_r - \tilde{u}_x|$ is small, the distribution of the random variable d is concentrated around 0. By continuity, for sufficiently small bias, the probability mass of d contained within $(-z^*, z^*)$ is sufficiently large such that $\mathbb{E}_s\left[\frac{\partial V_s}{\partial \sigma_o^2}\right] < 0$. Conversely, for a large bias, the mass of d shifts to the tails where the

derivative is positive. This proves the statement for σ_o^2.

Finally, consider the comparative static with respect to the prior variance $\sigma_{p,r}^2$. The parameter $\sigma_{p,r}^2$ affects both σ_T^2 and the variance of the distribution of d. At zero bias, the distribution of d is centered at 0, so most mass lies in $|d| < z^*$, where the derivative is negative. Furthermore, increasing $\sigma_{p,r}^2$ increases the variance of d, spreading mass from the center (where V_s is maximized) to the tails (where V_s is lower), which reinforces the negative effect. Thus, by continuity, for sufficiently small bias, $V(r)$ is strictly decreasing in $\sigma_{p,r}^2$. Conversely, as bias grows to ∞, the probability mass of d concentrates entirely in the region where $|d| > z^*$ (indeed, $\Pr(|d| > z^*) \to 1$), where $\frac{\partial V_s}{\partial \sigma_T^2}$ is strictly positive. Thus, for sufficiently large bias, $V(r)$ is strictly increasing in $\sigma_{p,r}^2$. □

C.8 Proof of Proposition 6

Let $\mathcal{I}$ denote the collection of all (cardinal and ordinal) signals the DM receives before choosing between x and y, and write

$$\hat{U}_a := E[U(a) \mid \mathcal{I}], \qquad a \in \{x, y\}.$$

Step 1: Decomposition of the objective. Given $\mathcal{I}$, the DM chooses the option with the larger posterior mean, so his ex–ante expected utility is

$$U_{\max}(r) := E\big[\max\{\hat{U}_x, \hat{U}_y\}\big].$$

Using $\max\{a, b\} = \frac{1}{2}(a + b) + \frac{1}{2}|a - b|$, we obtain

$$U_{\max}(r) = \frac{1}{2}E[\hat{U}_x + \hat{U}_y] + \frac{1}{2}E\big[|\hat{U}_x - \hat{U}_y|\big].$$

Since $E[\hat{U}_a] = E[U(a)]$ for $a \in \{x, y\}$ and since these quantities depend only on the priors, not on r, then maximizing $U_{\max}(r)$ is equivalent to maximizing

$$E\big[|\hat{U}_x - \hat{U}_y|\big].$$

Step 2: Separability and mean–preserving spreads. Since the utility is additive, $U(a) = \sum_{i=1}^{n} u_{a,i}$

and, because priors and signals factor by dimension,

$$\hat{U}_a = \sum_{i=1}^{n} \hat{u}_{a,i}, \qquad \hat{u}_{a,i} := E[u_{a,i} \mid \mathcal{I}_i],$$

where $\mathcal{I}_i$ collects the signals in dimension i. Define the posterior difference in dimension i by

$$D_i(r_i) := \hat{u}_{x,i} - \hat{u}_{y,i},$$

and the total posterior difference by $D(r) := \hat{U}_x - \hat{U}_y = \sum_{i=1}^{n} D_i(r_i)$. For each i, r_i affects only $D_i(r_i)$, and the vectors $\big(D_i(r_i)\big)_{i=1}^{n}$ are independent across dimensions.

Recall that $|\cdot|$ is convex. We will use the following simple fact. Let X' be a mean–preserving spread of X and let Y be independent of (X, X'). Then, for every convex φ, $E[\varphi(X' + Y)] \geq E[\varphi(X + Y)]$. (This follows from conditioning on $Y = y$ and using the fact that $x \mapsto \varphi(x + y)$ is convex for each y.)

Applied with $\varphi(z) = |z|$, this implies that if we replace $D_i(r_i)$ by a mean–preserving spread $D_i'(r_i)$, holding all other dimensions fixed, then $E[|D(r)|]$ strictly increases. Therefore, to maximize $U_{\max}(r)$ it suffices to choose each r_i so that $D_i(r_i)$ is as dispersed (in the sense of mean–preserving spreads) as possible. We henceforth focus on a single dimension i (and drop the index from all subscripts).

Step 3: The reference point and mean–preserving spreads. Because ordinal signals with respect to r are noiseless, the realization of ordinal signals partitions the state space into three events:

$$\begin{aligned} A^- &:= \{u_x > u_y > u_r\} \cup \{u_y > u_x > u_r\}, \\ A^+ &:= \{u_r > u_x > u_y\} \cup \{u_r > u_y > u_x\}, \\ A^0 &:= \{u_y < u_r < u_x\} \cup \{u_x < u_r < u_y\}. \end{aligned}$$

Conditional on all other (cardinal and pairwise-ordinal) signals in this dimension, (u_x, u_y, u_r) are independent Normal random variables. Applying Proposition 5 to (u_x, u_y, u_r), first on the event $\{u_x > u_y\}$ and then on $\{u_y > u_x\}$ (swapping the roles of x and y), yields the strict inequalities

$$E\big[|u_x - u_y| \mid A^{\pm}\big] \;<\; E\big[|u_x - u_y| \mid u_x \neq u_y\big] \;<\; E\big[|u_x - u_y| \mid A^0\big].$$

Since $D(r)$ is the conditional expectation of $u_x - u_y$ given all signals, the same ordering holds

for $|D(r)|$:

$$E\big[|D(r)| \mid A^{\pm}\big] \;<\; E\big[|D(r)| \mid u_x \neq u_y\big] \;<\; E\big[|D(r)| \mid A^{0}\big].$$

Moreover, the unconditional mean $E[D(r)] = E[u_x - u_y]$ is independent of r, because the reference point affects only the information structure, not the prior.

Thus, as we vary u_r, moving probability mass from $A^{\pm}$ (where $|D(r)|$ is relatively small) to A^0 (where $|D(r)|$ is relatively large), while keeping $E[D(r)]$ fixed, yields a (strict) mean–preserving spread of $D(r)$. By Step 2, this strictly increases $E|D(r)|$ and hence $U_{\max}(r)$. Consequently, in each dimension, we want to choose u_r so as to maximize

$$P_{\text{split}}(u_r) := P(u_y < u_r < u_x) + P(u_x < u_r < u_y),$$

the probability that u_r lies between u_x and u_y ("splits" the two options).

Step 4: Maximizing the splitting probability. Fix a dimension and write $\tilde{u}_x, \tilde{u}_y$ for the prior means and σ^2 for the common prior variance. Let $X \sim N(\tilde{u}_x, \sigma^2)$ and $Y \sim N(\tilde{u}_y, \sigma^2)$ be independent. Then

$$P_{\text{split}}(u_r) = P(X > u_r)P(Y < u_r) + P(X < u_r)P(Y > u_r),$$

since the events for X and Y are independent. Let Φ and ϕ denote the standard Normal cdf and pdf, and set

$$z_x := \frac{u_r - \tilde{u}_x}{\sigma}, \qquad z_y := \frac{u_r - \tilde{u}_y}{\sigma}.$$

Then

$$\begin{aligned} P_{\text{split}}(u_r) &= [1 - \Phi(z_x)]\Phi(z_y) + \Phi(z_x)[1 - \Phi(z_y)] \\ &= \Phi(z_x) + \Phi(z_y) - 2\,\Phi(z_x)\Phi(z_y). \end{aligned}$$

Differentiating with respect to u_r gives

$$P'_{\text{split}}(u_r) = \frac{1}{\sigma}\phi(z_y)\big[1 - 2\Phi(z_x)\big] + \frac{1}{\sigma}\phi(z_x)\big[1 - 2\Phi(z_y)\big].$$

Write $\Delta := \tilde{u}_x - \tilde{u}_y \neq 0$ and assume, without loss of generality, that $\tilde{u}_x > \tilde{u}_y$. Consider the midpoint

$$u_r^* := \frac{\tilde{u}_x + \tilde{u}_y}{2}.$$

At this point, $z_x = -z_y$ and therefore $\phi(z_x) = \phi(z_y)$ and $\Phi(z_x) + \Phi(z_y) = 1$. Substituting into

the derivative,

$$P'_{\text{split}}(u_r^*) = \frac{\phi(z_x)}{\sigma}\Big[(1 - 2\Phi(z_x)) + (1 - 2\Phi(z_y))\Big]$$
$$= \frac{\phi(z_x)}{\sigma}\big[2 - 2(\Phi(z_x) + \Phi(z_y))\big] = 0.$$

A second derivative calculation (omitted for brevity) shows that $P''_{\text{split}}(u_r^*) < 0$, whereas $P_{\text{split}}(u_r) \to 0$ as $u_r \to \pm\infty$. Hence u_r^* is the unique maximizer of P_{split} in this dimension:

$$u_r^* = \frac{1}{2}\big(\tilde{u}_x + \tilde{u}_y\big).$$

Since the dimensions are independent and utility is additive, the argument applies dimension by dimension. Thus $U_{\max}(r)$ is maximized by choosing, for every i,

$$u_{r,i} = \frac{1}{2}\big(\tilde{u}_{x,i} + \tilde{u}_{y,i}\big).$$

Step 5: Sufficiency of the Splitting Probability via Symmetry. It remains to show that maximizing $P_{\text{split}}(u_r)$ maximizes the total objective $U_{\max}(r)$. Let $M(u_r) = \mathbb{E}[|\hat{U}_x - \hat{U}_y|]$. Note that the problem is symmetric around the midpoint $m = (\tilde{u}_x + \tilde{u}_y)/2$. Specifically, consider a reference point $u_r = m + \delta$. Due to the symmetry of the Normal priors and the boost function $\psi(\cdot)$, the distribution of the posterior difference $\hat{U}_x - \hat{U}_y$ under $u_r = m + \delta$ is identical to the distribution of $\hat{U}_y - \hat{U}_x$ under $u_r = m - \delta$. Since the objective function depends on the *absolute* difference $|\hat{U}_x - \hat{U}_y|$, it is invariant to this reflection: $M(m + \delta) = M(m - \delta)$.

This symmetry implies that $u_r = m$ is a critical point of the objective function. To confirm it is a maximum, recall that the total expectation is the sum of the expected differences conditional on splitting and not splitting. We established in Step 4 that the probability of the splitting event (where the ordinal signals force the posteriors apart, $\hat{U}_x > \hat{U}_y$) is strictly concave and uniquely maximized at $u_r = m$. Conversely, the probabilities of the non-splitting events (where boosts are in the same direction and partially cancel out, reducing $|\hat{U}_x - \hat{U}_y|$) are minimized at $u_r = m$. Since the "split" event generates the largest ordinal separation, the maximization of the probability weight on this event, combined with the symmetry of the conditional magnitudes around m, ensures that $u_r = m$ globally maximizes the ex-ante expected utility. □

D Additional Results on Convenient Approximations

D.1 Approximations of Gain-Loss Component

D.2 Approximation of Inverse Mills Ratio

We compare the approximation $\psi^a(\cdot)$, as defined in (7), with the true inverse Mill's ratio (IMR) function $\psi(\cdot)$, as defined in (5). In order to do so, we simulate the posterior mean $\mathbb{E}[\tilde{u}_x^{s,o}]$ (averaged over signal realizations) when the agent receives an ordinal signal under both models. Formally, we compute

$$\text{Full Model:} \quad \frac{1}{N}\sum_{i=1}^{N}\left[\lambda\, s + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\psi\left(\frac{\lambda\, s + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}, o\right)\right]$$

$$\text{Approximation:} \quad \frac{1}{N}\sum_{i=1}^{N}\left[\lambda\, s + (1-\lambda)\tilde{u}_x + \frac{\sigma_x^2}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}\,\psi^a\left(\frac{\lambda\, s + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}, o\right)\right]$$

Where cardinal and ordinal errors are drawn independently for $i = 1, 2, ..., N$. We let $N = 10,000$ and fix parameters $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and $\sigma_p = \sigma_r = 1$. We compare the simulation average across $u_x \in [-2, 2]$ for all combinations of $\sigma_\epsilon \in \left\{\frac{1}{2}, 1, 2\right\}$ and $\sigma_0 \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. Figure 8 displays the simulated means under the full and approximated model. The approximation $\psi^a(\cdot)$ performs well, and is virtually indistinguishable from the full model.

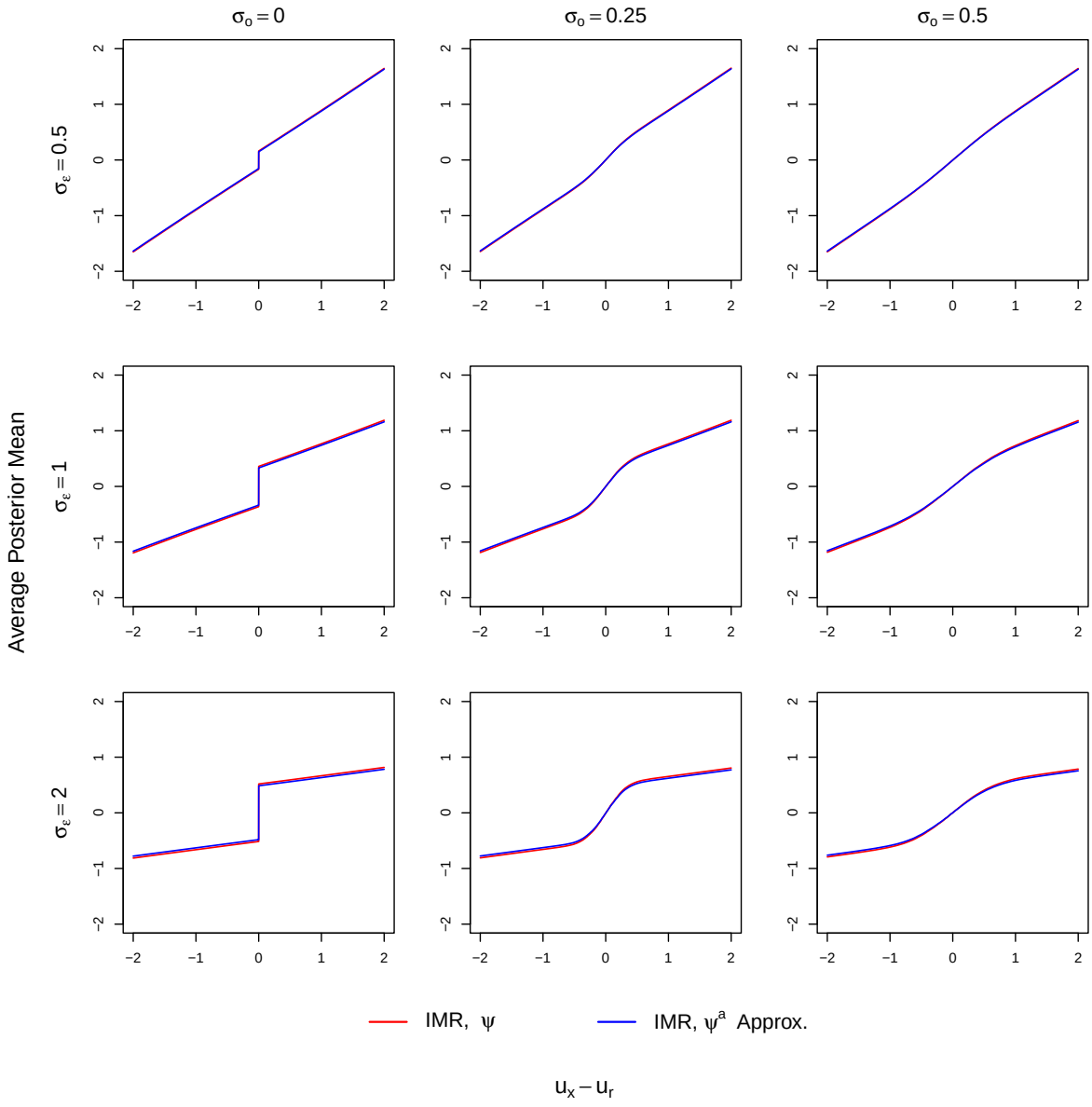


Figure 7: Simulation of posterior mean using IMR and its approximation $\psi^a(\cdot)$, averaged in both cases over 10,000 error draws. Prior and reference point parameters are set to $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and utility is given by $u_x = x$, which we vary between [-2,2].

D.3 Approximation of Average Inverse Mills Ratio

Intuitively, the approximation of the *average* Inverse Mills ratio $\overline{\psi}^a(\cdot)$ is given by weighting the approximation $\psi^a(\cdot)$ (which depends on the realization of the ordinal signal) by the relative likelihood of positive and negative ordinal signals $o_{x,i}$ under the assumed noise distributions. Suppose the true u_x and u_r are constant across agents. Then, the likelihood of left- and right-sided truncations is given by

$$P(\text{Truncation Below}) = \Phi\left(\frac{u_x - u_r}{\sigma_o}\right); \qquad P(\text{Truncation Above}) = 1 - \Phi\left(\frac{u_x - u_r}{\sigma_o}\right)$$

For any given side of truncation, the average IMR "boost" component, across realizations of the cardinals signals, can be approximated by $\psi^a_+(t)$ and $\psi^a_-(t)$ with $t := \frac{\lambda u_x + (1-\lambda)\tilde{u}_x - \tilde{u}_r}{\sqrt{\sigma_x^2+\sigma_r^2+\sigma_o^2}}$. The argument, t, represents the expectation across cardinal signal realizations. The reason why the expectation of the IMR (over cardinal signals) can be approximated as IMR of the expected cardinal signal is that $\psi^a(\cdot)$ is linear in the cardinal signals whenever the $\tilde{u}^s_x$ is not too far from u_r. This yields the following simple approximation of the average posterior mean in (7):

$$\overline{\psi}^a(t, u_x, u_r, \sigma_o) := \left[\Phi\left(\frac{u_x - u_r}{\sigma_o}\right)\psi^a_+(t) + \left(1 - \Phi\left(\frac{u_x - u_r}{\sigma_o}\right)\right)\psi^a_-(t)\right]$$

We simulate the average posterior mean $\mathbb{E}[\tilde{u}^{s,o}_x]$ for the full model as in Appendix D.2 by averaging across 10,000 realizations of cardinal and ordinal signals. We compare this simulation average to the approximation of (8) (computed once per u_x) for all combinations of $\sigma_\epsilon \in \left\{\frac{1}{2}, 1, 2\right\}$ and $\sigma_0 \in \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$. In all simulations, we set $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and $\sigma_p = \sigma_r = 1$. Figures 8 illustrates the simulation results.We find that the approximation computed using $\overline{\psi}^a(\cdot)$ closely tracks the true average.

D.4 Approximation of Bayesian Continuous Action Choice Model

We use simulations to assess the quality of the quasi-Bayesian model of continuous action choice in Section 4.5. The DM evaluates the utilities of different levels of an action, a. Given the additive separability of our model, we here focus on a single dimension.

Bayesian model. The DM has a prior belief about the dimension-specific utility of action a, $u_a|a \sim \mathcal{N}(\tilde{u}_a, \sigma_p^2)$, where the prior mean may depend on the action. As laid out in Section 4.5, the DM receives two types of signals. First, for each action, a cardinal signal with a noise term

that is independent across a.

$$s^a_{\text{Bayesian}} \sim \mathcal{N}(u_a, \sigma^2_{\epsilon,\text{Bayesian}}).$$

Second, the DM receives noiseless ordinal signals that tell him, for each action and associated utility outcome u_a, whether $u_a \gtrless u_{a'}$ for all $u_a, u_{a'}$. In other words, the DM understands monotonicity not just with respect to the reference point but also with respect to the outcomes induced by his actions. For example, he may not know his earnings utility from working 8 hours but he does not that his earnings utility from working 8 hours is higher than that from working 7 hours. These ordinal signals ensure that the posterior mean about u_a is monotonic in the action. However, the interdependent truncations arising from the many ordinal signals make the Bayesian posterior mean intractable.

Computationally, we explicitly enforce knowledge of monotonicity through a constrained posterior estimation process. Specifically, we employ Hamiltonian Monte Carlo No-U-Turn Sampler (NUTS sampler) from the `PyMC` package with an ordered transformation constraining the parameter space, resulting in a posterior distribution that is monotonic but analytically not tractable.

Quasi-Bayesian model. We approximate this model by making use of the idea that – just like independent noise terms in combination with ordinal signals ensure monotonic posteriors – correlated noise terms also ensure monotonicity. Suppose there is a single error draw $\epsilon_0 \sim \mathcal{N}(0, \sigma^2_{\epsilon,\text{quasi-Bayesian}})$, and that the DM receives signals

$$s^a_{\text{quasi-Bayesian}} = u_a + \epsilon_0.$$

Further suppose that the DM updates his prior belief from these cardinal signals by fully ignoring the correlation among them. Then, the quasi-Bayesian posterior mean is given by

$$\mathbb{E}[\, u_a \mid s^a_{\text{quasi-Bayesian}} \,] = \lambda\, s^a_{\text{quasi-Bayesian}} + (1-\lambda)\, \tilde{u}_a \tag{22}$$

where λ is the usual shrinkage weight.

The thought behind our approximated model is that, by picking a noise variance, the quasi-Bayesian posterior mean from the model with perfectly correlated noise terms approximates the Bayesian posterior of the model with independent noise and ordinal signals. We thus pick the variance $\sigma^2_{\epsilon,\text{quasi-Bayesian}}$ that minimizes the distance between the two posterior means through nested interval optimization. Specifically, given the prior mean, prior variance and the Bayesian DM's signal variance, we identify the optimal quasi-Bayesian variance that minimizes the Mean Squared Error (MSE) between the posterior means of the quasi-Bayesian and constrained Bayesian

models.

Simulation. We simulate both models over a range of parameter combinations using Python's `PyMC` package with the Hamiltonian Monte Carlo No-U-Turn Sampler (NUTS). Figure 9 displays the average posterior mean as a function of a for both the Bayesian and quasi-Bayesian DMs. The Bayesian posterior mean is computed by explicitly enforcing monotonicity via an ordered transformation, whereas the quasi-Bayesian posterior mean utilizes the analytical formula given by equation (22). The results indicate that the quasi-Bayesian approximation closely replicates the average posterior means produced by the fully Bayesian model, with deviations observed near the boundaries of the simulated range.

Quality of approximation with a reference point. The simulations above suggest that the posterior means of the two models are close to each other when there is no reference point. We now extend the analysis to a reference point, r. Specifically, the DM now additionally receives ordinal signals o_a that indicate whether $u_a \gtrless u_r$. Because this introduces an additional layer of truncation in addition to the monotonicity constraints among the different actions, simulating the fully Bayesian model for a large number of actions becomes computationally prohibitive. As a proof-of-concept, we instead conduct a brute-force simulation over ten randomly sampled values of $u_a \in (0, 1000)$, but constrain them to include at least two values close to the reference point. For each iteration, we first draw a noisy signal about each u_a, from which the DM creates a posterior distribution $u_a|s^a \sim \mathcal{N}(\lambda s^a, \sigma_x^2)$ (where $\sigma_\epsilon = \sigma_p = \tilde{u}_a = 0$). We truncate the posterior distributions on $u_r = 500 \gtrless u_a$. We then sample a candidate utility path from the sequence of truncated posterior distributions, and accept this path only if it is strictly monotonic across all outcomes. This process is repeated until a sufficient number of valid monotonic paths are collected, and their average forms our "fully Bayesian" benchmark. Figure 10 plots this path along with our quasi-Bayesian model.

E Figures

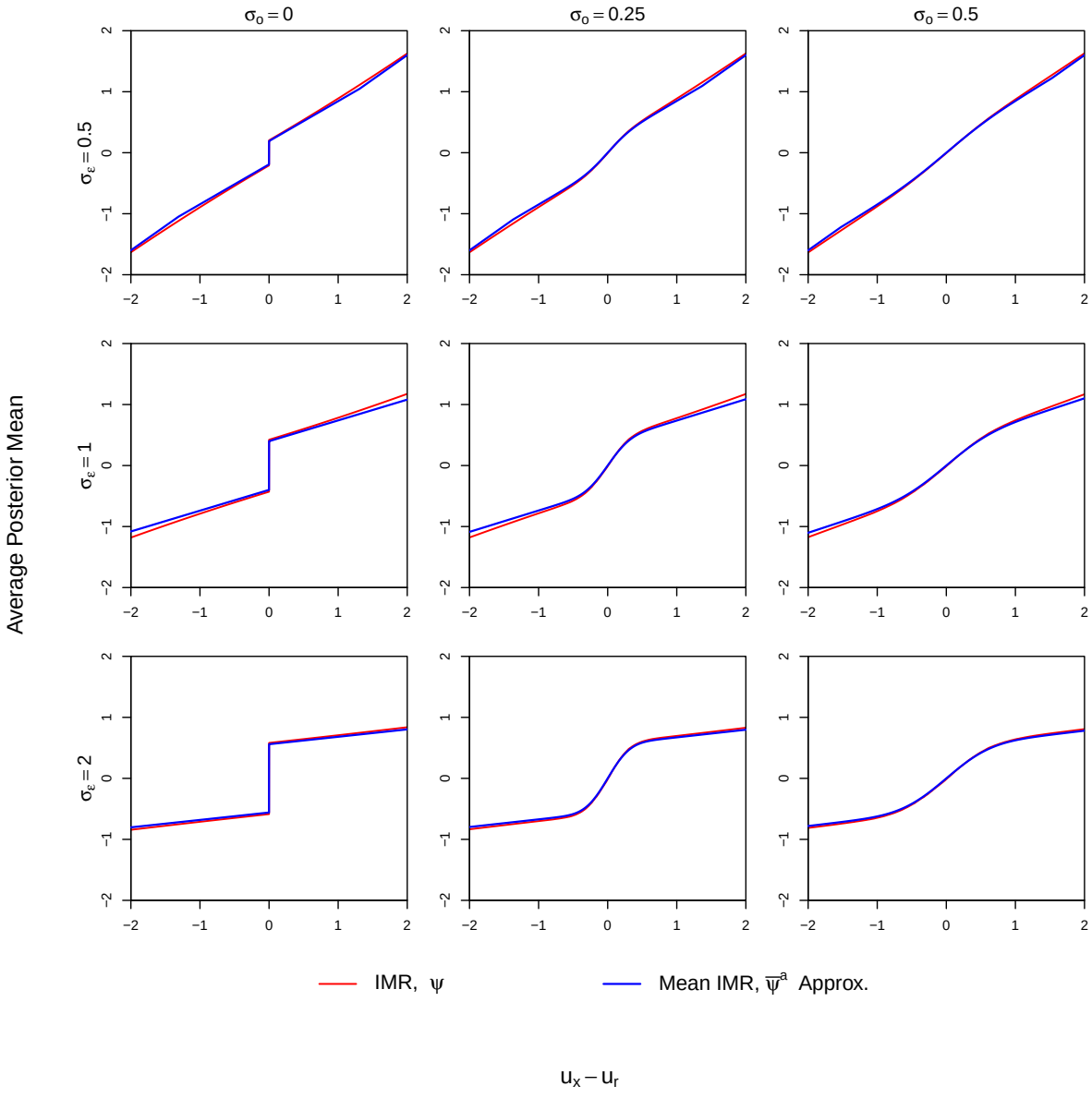


Figure 8: Simulation of Posterior Mean using IMR (average over 10,000 error draws) and its approximation $\overline{\psi}^{a}(\cdot)$. Prior and reference point parameters are set to $\tilde{u}_x = \tilde{u}_r = u_r = 0$ and utility is given by $u_x = x$, which we vary between [-2,2].

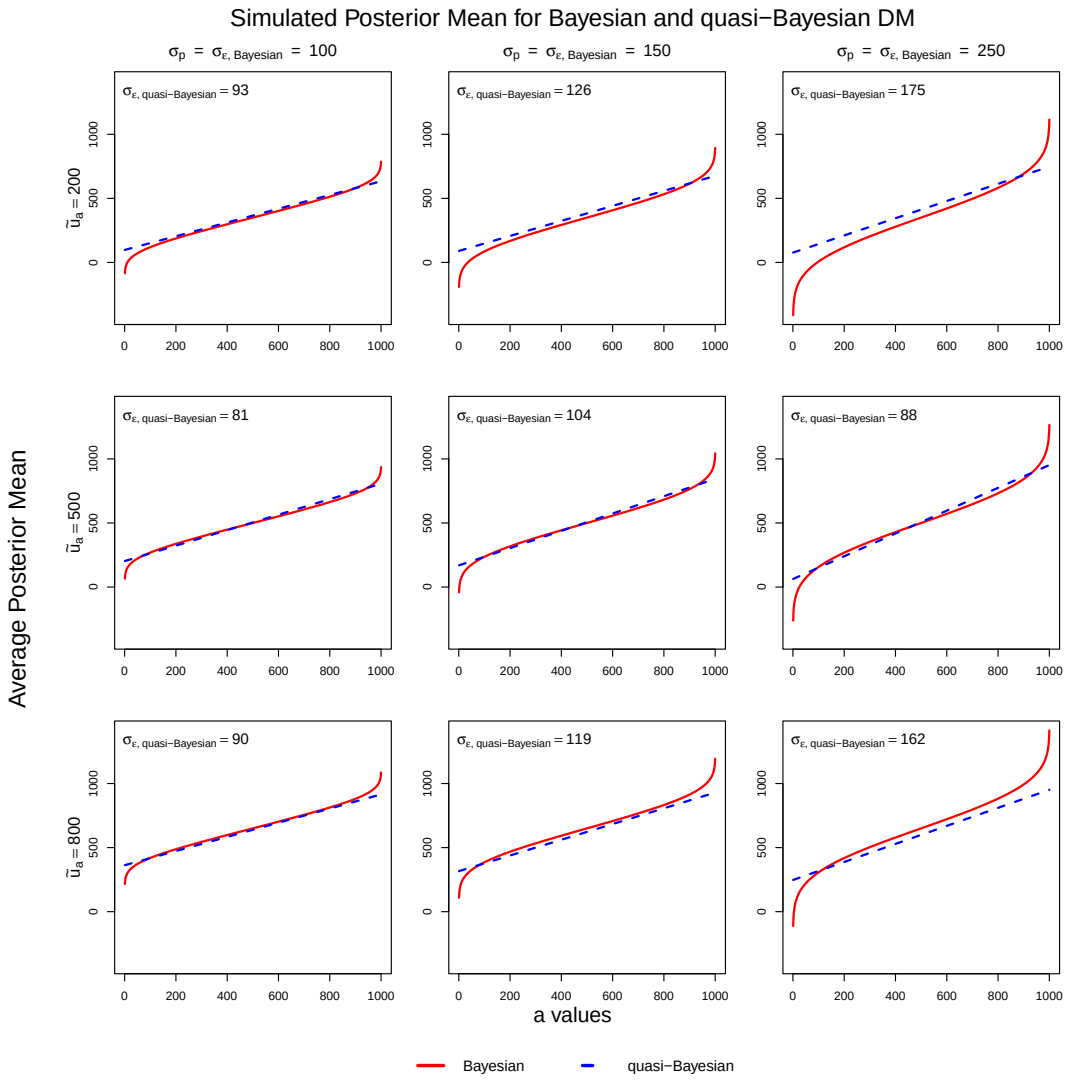


Figure 9: Simulated posterior mean utilities for a fully Bayesian agent enforcing monotonicity (solid line) and a quasi-Bayesian agent using the single-error approximation (dashed line). Each panel reports averages across 200 simulation iterations. Rows differ by prior mean $\tilde{u}_a$ (left label), and columns differ by the prior standard deviation, set equal to the Bayesian agent's signal standard deviation (top label). For each panel, the quasi-Bayesian signal standard deviation minimizing the MSE relative to the Bayesian posterior mean is indicated in the top-left corner. The continuous action variable a ranges from 1 to 1000 with unit increments for simulation purposes.

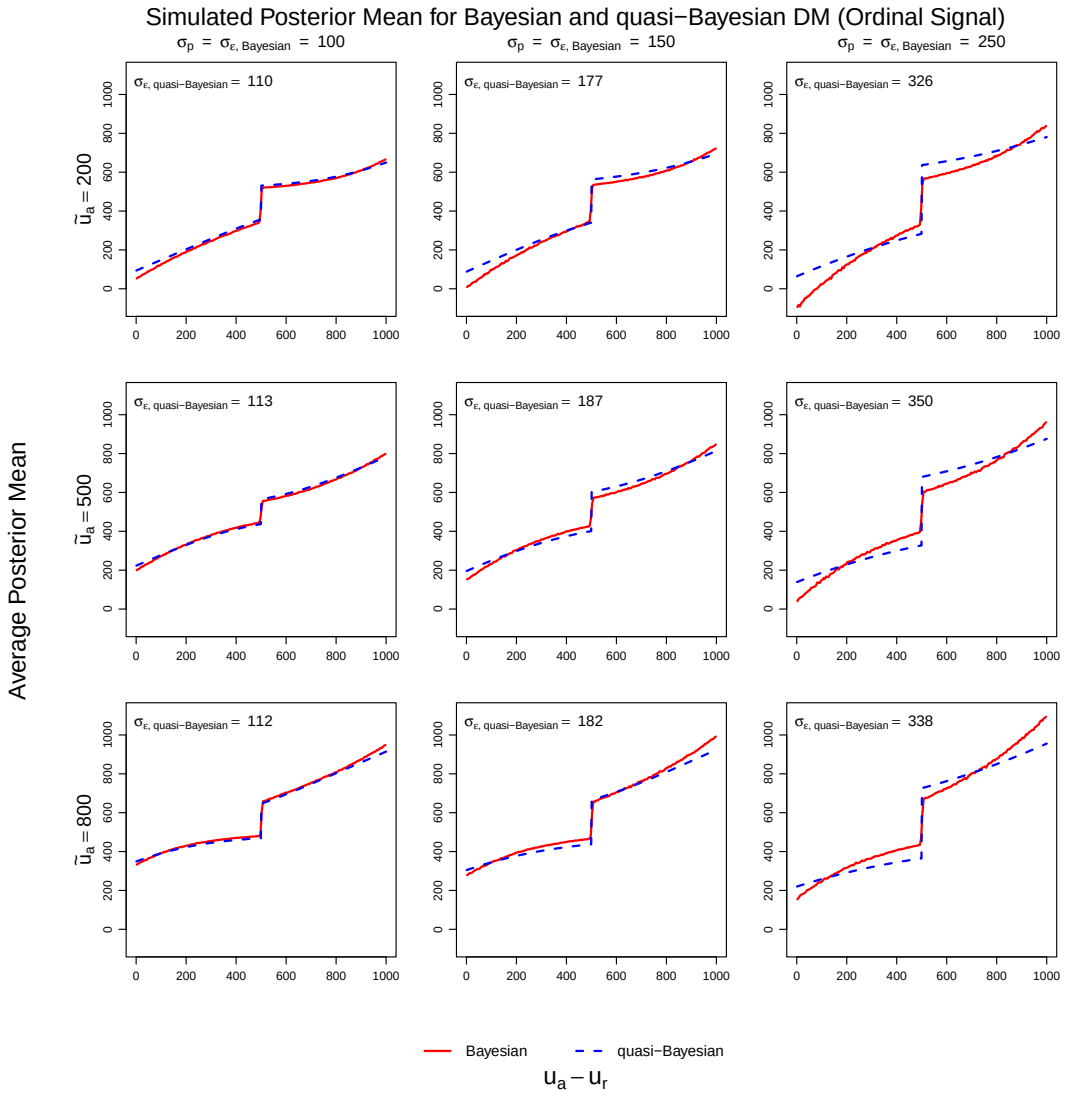


Figure 10: Comparison of posterior means for a reference-dependent DM. The solid line represents the Bayesian benchmark, while the dashed line shows the quasi-Bayesian model. Each panel reports the average of 50,000 accepted monotonic sample paths, generated via a brute-force acceptance-rejection method with posteriors truncated at the reference point ($u_r = 500$ across all panels). The quasi-Bayesian approximation is computed as the average over 1,000 draws of signal error $\epsilon_0 \sim \mathcal{N}(0, \sigma_\epsilon^2)$, where the quasi-Bayesian signal standard deviation $\sigma_{\epsilon,\text{quasi-Bayesian}}$ that minimizes the MSE relative to the Bayesian posterior mean is indicated in the top-left corner. Rows differ by prior mean $\tilde{u}_a$ (left label), and columns differ by the prior standard deviation, set equal to the Bayesian agent's signal standard deviation.

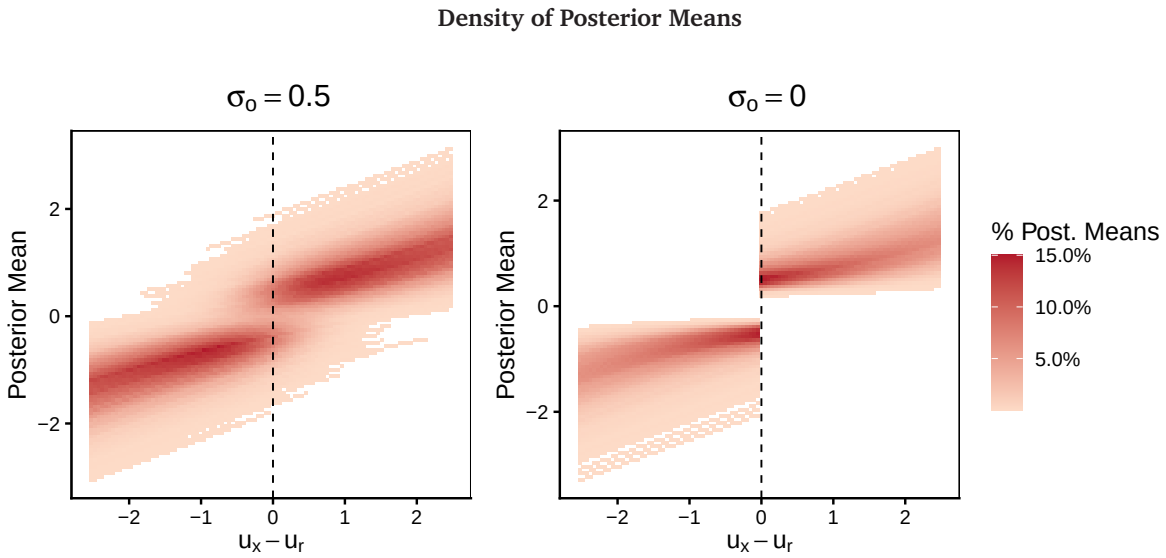


Figure 11: Distribution of posterior means $\mathbb{E}[u_x \mid s, o]$ for $\tilde{u}_x = \tilde{u}_r = 0$, and $\sigma_\epsilon = \sigma_p = 1$. The left panel shows the case in which the ordinal signal is noisy, $\sigma_o = 0.5$. The right panel shows the case the ordinal signal is noiseless, $\sigma_o = 0$. Each panel shows, for 100 steps of $u_x \in [-2.5, 2.5]$, the average over 10,000 draws of $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$ and $\nu \sim \mathcal{N}(0, \sigma_o^2)$.

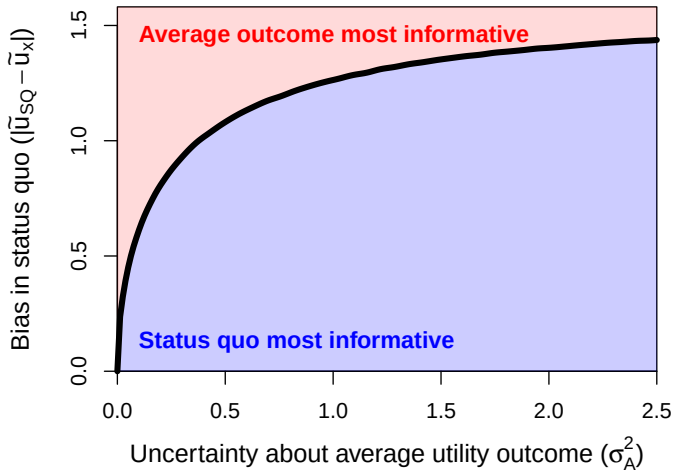


Figure 12: Bias-variance tradeoff in whether the most informative reference point is the status quo or the average outcome (rational expectations). The x-axis shows the uncertainty in the knowledge about the average outcome and the y-axis the bias in the status quo.

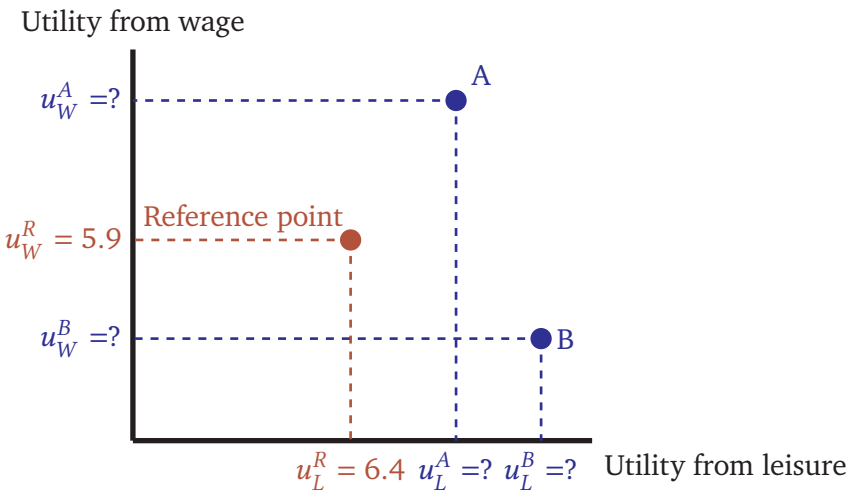


Figure 13: Illustration of the asymmetric dominance or improvements-vs.-tradeoffs effect

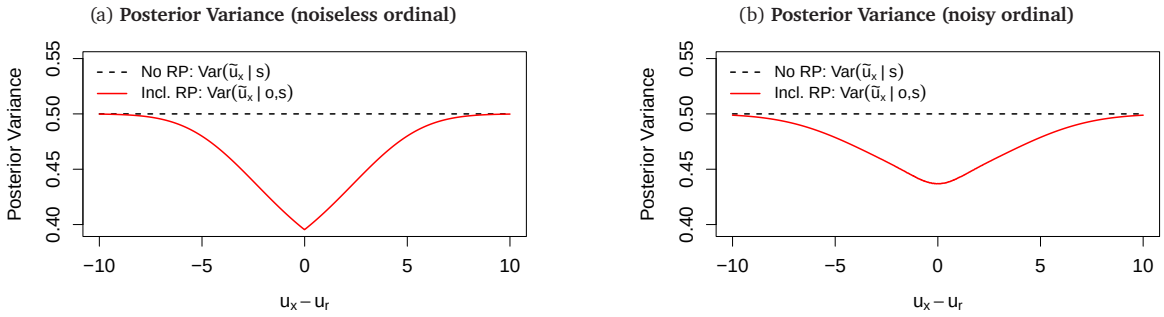


Figure 14: Posterior variance with and without ordinal information. Both panels show the reference-free posterior variance (black) for comparison. Parameters are held fixed across panels with $\sigma_p, \sigma_e, \sigma_r = 1$; the left panel uses $\sigma_o = 0$ (noiseless), while the right panel uses $\sigma_o = 1$ (noisy ordinal).