Regression Discontinuity Aggregation, with an Application to Union Effects on Inequality

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Abstract

We extend the regression discontinuity (RD) design to settings where each unit's treatment status is an average or aggregate across multiple discontinuity events. Such situations arise in many studies where the outcome is measured at a higher level of spatial or temporal aggregation (e.g., by state with district-level discontinuities) or when spillovers from discontinuity events are of interest. We propose two novel estimation procedures — one at the level at which the outcome is measured and the other in the sample of RD events — and show that both identify a local average causal effect and possess attractive bias-variance properties under continuity assumptions similar to those of standard RD designs. We apply these ideas to study the effect of unionization on inequality in the United States. Using credible variation from close unionization elections at the establishment level, we show that a higher rate of newly unionized workers in a state-by-industry cell reduces wage inequality within the cell.

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1 Introduction

Regression discontinuity (RD) designs are a popular tool for causal inference. In canonical sharp RD designs, the treatment is determined by whether a "running variable" exceeds a fixed cutoff. For instance, one may study whether unionization affects establishment outcomes, leveraging the cutoff of 50% votes in a union election (DiNardo and Lee, 2004; Frandsen, 2021). A local causal parameter is typically estimated via local linear regression under the assumption that potential outcomes are continuous around the cutoff. For standard RD methods to apply, each unit has to be exposed to only one discontinuity.

In this paper, we extend the RD design to settings in which a unit is exposed to multiple discontinuity events—which we shorthand as "elections"—because the outcome is defined at a higher level of aggregation (e.g., in space or time) than the elections. For instance, one may be interested in the fraction of workers who have been unionized via elections in multiple establishments on regional labor market outcomes. The endogenous explanatory variable ("treatment") in such settings can be represented as a simple or weighted average or sum of the RD shocks across multiple elections. We propose two new estimators in these "RD aggregation" (RDA) settings. The "upper-level" estimator is based on an instrumental variable (IV) regression at the level at which the outcomes are measured, with a weighted average or sum of the RD shocks from narrow elections as the instrument, while controlling for similar weighted sums of the controls used in local linear estimation (e.g., of the running variables). The "lower-level," or "stacking," estimator is a fuzzy RD estimator in the sample of narrow elections, with the unusual feature that the same values of the outcome and endogenous variable are repeated for the elections affecting the same unit. In the unionization case, the regional fraction of unionized workers is instruments with either the regional fraction of workers who unionized in narrow wins or with the narrow win indicator at the establishment level, respectively.

We show that both estimators identify the same local average treatment effect and inherit attractive bias-reduction properties of conventional RD methods. Specifically, under the generalization of continuity and smoothness assumptions of conventional RD, the lower-level estimator has bias that shrinks with the bandwidth h as $O(h^2)$. The same is true for the upper-level estimator as long as the uniform kernel is used in the instrument construction. We characterize the asymptotic bias and variance of the two estimators, with neither of the estimators dominating the other in general. In this baseline theoretical analysis, we assume a random sample of (upper-level) units with the number of elections per unit not changing with the sample size, such that it is rare for a unit to have more than one close election. We therefore further investigate the robustness of our findings to two non-standard asymptotic sequences in which clusters of close elections are common even in large samples. In the first alternative asymptotic, we allow the number of elections per unit to grow with the sample size, finding that the bias of both estimators continues to be low (i.e., $O(h^2)$). In the second asymptotic, clusters of close elections arise from the running variables within the same unit becoming increasingly correlated; the two estimators also show some—albeit weaker—robustness. This asymptotic highlights that, even though the lower-level RDA specification looks like a special case of standard fuzzy RD, its asymptotic properties can be different; in particular, using a uniform kernel may be preferable.

We apply the RDA approach to study the union effects on earnings inequality. This question is one of the oldest in labor economics, dating back to at least Lewis (1963). It is also of central policy importance in modern days: Barack Obama, for instance, noted that *"as union membership has fallen, inequality has risen"* (Obama, 2016)— a correlation observed in the data by Card (2001). Yet, establishing a causal link between unionization and inequality has proved difficult. Recently, Fortin et al. (2021) point out that *"empirical evidence on the distributional impact of [union] threat effects is limited by the challenge of finding exogenous sources of variation in the rate of unionization."* Similarly, Jäger et al. (2025) emphasize "*the pressing need*

for more compelling causal analyses of collective bargaining coverage effects." We make progress by leveraging close union elections in an RDA design with outcomes measured at the level of state-by-industry cells.

Both estimators provide evidence that that a higher rate of unionization leads to a decline in various measures of inequality: the Gini index, the 90/10 ratio, the share of labor income by top 10% earners, and the variance in log wages without and with Mincer (1974) residualization. Surprisingly, this compression results primarily from the decline of wages at the top of the wage distribution rather than an increase at the bottom, with average log wages showing a *negative*, albeit marginally insignificant, effect. We suggest four mechanisms that can help explain these patterns. First, unions substantially reduce the wages of managers, who are legally exempt from union representation. Second, employment of top-earners (specifically, college graduates) drops in response to higher unionization rates, generating compositional changes. Third, pension coverage increases, providing a compensating differential. Finally, the overall effects hide interesting dynamics: average wages fall in the short-run (i.e., 1–5 years) while the increase in inequality arises in the longer-run (6–10 years) at which point the lower end of the earnings distribution benefits. Such dynamics are consistent with an initial period of contract negotiation and potential disruption from employer opposition (Wang and Young, 2022), followed by the realization of wage gains at the bottom once collective bargaining agreements take effect. In a back-of-the-envelope calculation, we find that around 40% of the growth in inequality within state-by-industry cells—the key component of overall income inequality—between 1970 and 2010 can be explained by the declining rates of unionization relative to the 1960s.

Beyond unionization effects, the RDA approach is applicable in—and yields practical insights for—many empirical settings in which the outcome is defined at the more aggregated level than the RD events, with the treatment variable aggregating across them. Out of over 50 such papers in economics and political science that we have identified, a large group considers legislatures that consist of politicians elected in single-seat constituencies. The researcher is interested in the impact of the seat share of politicians who possess a specific characteristic: e.g., gender (Clots-Figueras, 2011, 2012; Bhalotra and Clots-Figueras, 2014; Ben-Porath and Kolerman-Shemer, 2024), religion (Bhalotra et al., 2014, 2021), or affiliation with a specific party (Folke, 2014; Nellis et al., 2016; Nellis and Siddiqui, 2018; Ben-Porath and Kolerman-Shemer, 2024). Another group includes settings in which units are exposed to multiple RD events over time (Cellini et al. (2010); Dell and Querubin (2018); Biasi et al. (2024)); in the RDA approach, the researcher can specify the treatment as the total fraction of the period during which the unit is treated. Other studies aggregate discontinuities across investors in a region, university courses taken by a student, politicians that a firm is linked to, or destinations that a city is connected to by a direct flight.[1]

Our approach also extends naturally to estimating spillover effects from discontinuity events, such that each unit is exposed to multiple elections of their neighbors (in a spatial or network sense). Isen (2014), for instance, leverages close referenda on tax changes in Ohio counties to study the effects on neighboring counties. Similarly, Dechezleprêtre et al. (2023) study the effects of tax breaks, which have discontinuous eligibility rules, on technologically similar firms.

While the two estimators we propose are natural extensions of conventional RD to RDA settings, they differ from most *ad hoc* estimation approaches used in the existing empirical literature. Specifically, nearly all papers estimating upper-level specifications do not include all local linear controls aggregated in the way

[1]Specifically, Ater et al. (2021) construct a regional housing supply shock by aggregating across real estate investors. They leverage a discontinuity in the capital gains tax law that pushes some investors to sell their property. Tan (2023) considers the effects of the number of high letter grades of a university student, as aggregating across individual courses. Akey (2015) studies firms that donate to multiple politicians and measures the effects of a firm's political connectedness, as aggregating election outcomes of those politicians. Campante and Yanagizawa-Drott (2018) aggregate discontinuities in flight availability between city pairs with respect to distance to construct a shock to a city's overall connectivity.

suggested by our upper-level solution. As a result, they generally forfeit the low-bias advantages of local linear estimation that made it standard in conventional RD designs. Conversely, studies using lower-level specifications include local linear controls but, in many cases, use reduced-form estimation which does not define a coherent causal model for the upper-level outcome. Some of these papers also impose restrictions limiting the sample which is not necessary with the RDA approach.

Most strikingly, some literatures resort to difference-in-differences and other non-RD methods when analyzing aggregated outcomes despite preferring RD variation at the disaggregated level, sometimes even within the same paper. The literature on the effects of unionization on aggregated outcomes is a case in point: while several sources of identifying variation in union density and strength have have been proposed,[2] we are not aware of any previous attempts to use credible RD variation. The RDA approach resolves this tension, and our application to the effects on inequality and average wages provides an empirical blueprint; complementary work by Kolerman-Shemer (2024) used the RDA method to show that unionization makes local politics more Democratic in the US.

Methodologically, our paper bridges the literatures on shift-share instruments (Borusyak et al., 2022) and continuity-based RD designs (Hahn et al., 2001). Our theoretical analysis leverages the observation that the aggregated treatment can be viewed as a shift-share variable; thus, exogenous variation in the "shifts," i.e. the RD events, can be used to extract exogenous variation in the aggregated treatment. However, a theoretical challenge arises because, under the continuity assumption, the randomness of the RD shocks is only a heuristic and an approximation as the bandwidth converges to zero, necessitating a nonparametric analysis which we provide. While numerous empirical papers proposed *ad hoc* estimators trying to achieve that goal, with Cuñat et al. (2012) and Dube et al. (2019) coming closest to our upper- and lower-level estimators, respectively, we are the first to provide a formal theoretical analysis and bias-variance guarantees.

Our continuity assumption relaxes the assumptions in a recent literature that used the local randomization to RD for identification with instruments constructed from multiple RD shocks by a known, but potentially complex, formula or algorithm. As Borusyak and Hull (2023) pointed out, if within a fixed bandwidth around the cutoff the RD shocks are viewed as completely random, as in Cattaneo et al. (2015), one can compute the expectation of the constructed instrument over the distribution of RD shocks and control for it or "recenter" by it to isolate the exogenous variation, with no need for the analogs of local linear controls. Kott (2022) and Narita and Yata (2023) compute the expected instrument by simulating counterfactual realizations of RD shocks, while Abdulkadiroglu et al. (2022) and Angrist et al. (2024) compute it analytically. By replacing local randomization with the weaker continuity assumption and by restricting attention to linear (i.e., shift-share) instruments — which still covers a wide variety of empirical applications — we are able to leverage the local linear estimation approach to RD, develop estimators with better statistical properties, and derive inference results for them.[3]

The setting we study is distinct from "multi-score" RD settings which also feature multiple running variables per unit (see Cattaneo and Titiunik (2022) for a review). For instance, in spatial RD designs (e.g., Black, 1999; Dell, 2010), latitude and longitude are the two running variables, and a geographic boundary

[2]Specifically, some papers evaluated union effects by comparing neighboring counties across the border of right-to-work laws, which were mostly passed in the 1940s and 1950s and weakened unions: Holmes (1998), Feigenbaum et al. (2019), and Makridis (2019) studied the impacts on productivity, voting, and well-being, respectively. Other papers leveraged specific case studies that generate exogenous variation in union strength when looking at outcomes such as wages (Biasi, 2021) and the gender pay gap (Biasi and Sarsons, 2022). Another approach is taken by Baker et al. (2025) that leverage the staggered rollout of faculty unions across Canadian universities, using a difference-in-differences specification to assess the effect of unionization on the wage distribution.

[3]Note that, even if the treatment is a nonlinear function of the RD shocks, one can construct an instrument that is linear, albeit at an efficiency cost (see Borusyak et al. (2025a)). Then the RDA theory applies.

separates the treated and control areas. Another example is when a student is treated if several tests scores (e.g., for math and English) simultaneously exceed some cutoffs. Such designs feature a single binary treatment (or instrument, in the fuzzy case), and a simple transformation of the running variables reduces the problem to a standard RD, in which the signed distance to the boundary is the scalar running variable (although see Cattaneo et al. (2025) for a recent critique). In contrast, in our setting the instrument aggregates RD shocks. While still a function of the underlying running variables, it does not feature a single boundary, making existing methods inapplicable.

A recent literature has examined certain RD settings with interference. Dal Torrione et al. (2024) extend the multi-score approach, based on the minimum Euclidian distance from the set of neighbors' running variables to the boundary at which the spillover treatment, e.g. the number or share of treated neighbors, switches between two prespecified values. This allows them to identify more detailed spillover effects, e.g. of having all treated neighbors vs. all untreated neighbors conditionally on the unit itself being treated. We propose a very different estimation approach which identifies a causal spillover parameter in a simpler way, does not depend on the discreteness of the treatment, and applies to a variety of settings beyond just spillovers. Other papers have examined the case of interference occurring among units with similar values of the running variable, as may be natural in spatial RD designs (Keele and Titiunik (2018), Auerbach et al. (2024)). In our setting, even if the structure of spillovers is based on geography, each unit has its own running variable, leading to different issues and solutions.

Empirically, we contribute to the literature on the effects of declining unionization rates on U.S. average wages and inequality—two of the primary outcomes examined in the unionization literature. The standard approach for estimating the union wage premium regresses individual worker wages on union membership status while controlling for worker characteristics. Card (1996) finds a positive premium of 17%, and Farber et al. (2021, Figure 5) documents a remarkably stable premium of 10–20 log points across nine decades. A separate strand estimates union effects at the establishment level using RD designs based on close elections. DiNardo and Lee (2004) pioneered this approach and find close-to-zero wage effects. Frandsen (2021), using matched employer-employee data, finds small negative effects. Although our estimates that incorporate spillover effects are not directly comparable, our short-run results align with the RD estimates, while our long-run results are more consistent with the positive premia from worker-level studies.

A related literature examines the relationship between unions and wage inequality. Farber et al. (2021) address this question with a variety of methods including distributional decompositions following DiNardo et al. (1996), time series regressions, and state-year panel regressions. These estimates, each based on a form of conditional exogeneity of unionization, suggest that union decline accounts for 10–20% of the increase in the Gini coefficient between 1968 and 2014. Green et al. (2025) adopt a structural approach and, through counterfactual analysis holding union-related factors at 1980 levels, estimate that deunionization explains roughly one-third of the rise in the post-secondary wage premium between 1980 and 2010 (see their Figure 6). Both sets of estimates are broadly comparable to, though somewhat smaller than, our findings. Evidence from other settings includes Baker et al. (2025), who use a difference-in-differences design to study faculty unionization at Canadian universities and find substantial wage compression, and Derenoncourt et al. (2025), who develop a union-firm AKM framework using Brazilian administrative data but find only small union contributions to post-secondary wage premium based on an Oaxaca-Blinder decomposition (Table 2, Panel A).

The rest of this paper is organized as follows: in Section 2 we introduce our setting in the context of cross-sectional RD aggregation and propose the two estimators. Section 3 presents the theoretical analysis of

the estimand, asymptotic bias and variance. Section 4 briefly discusses extensions to settings with spillovers and temporal aggregation. Section 5 considers the practical implications of the RDA approach: we first discuss how our proposals differ from the current empirical practice and then apply them to estimate the impact of unions on U.S. inequality. Section 6 concludes.

2 Theoretical Framework

In Section 2.1, we introduce the RD aggregation setting and compare it to conventional RD designs. Section 2.2 puts forward two estimators and provides intuition why they may provide valid causal estimates.

2.1 Setting

We observe a sample of units $i = 1, \ldots, N$, which we refer to as the "upper level." For each unit, we observe a set $\mathcal{J}_i$ of "lower-level" subunits j (with $|\mathcal{J}_i| \equiv J_i \geq 0$) characterized by a running variable r_{ij}, treatment $z_{ij} = I(r_{ij} \geq 0)$ that happens whenever the running variable exceeds the cutoff normalized to zero, and an importance weight $s_{ij} > 0$.[4] To simplify notation, we write $(s_j, r_j)_{j \in \mathcal{J}_i}$ for $(s_{ij}, r_{ij})_{j \in \mathcal{J}_i}$ where $\mathcal{J}_i = \{(i,1), \ldots, (i, J_i)\}$.[5]

We observe a treatment variable X_i that depends on $\boldsymbol{z}_i \equiv \{z_j\}_{j \in \mathcal{J}_i}$, as formalized by potential outcomes $X_i(\boldsymbol{z}_i)$. The benchmark case, which generalizes sharp RD, is when X_i simply aggregates z_j across all relevant subunits,

$$X_i(\boldsymbol{z}_i) = \sum_{j \in \mathcal{J}_i} s_j z_j. \tag{1}$$

Generalizing this to aggregations of fuzzy RD, one can also consider

$$X_i(\boldsymbol{z}_i) = \sum_{j \in \mathcal{J}_i} s_j x_j(z_j), \tag{2}$$

for some potential treatments $x_j(\cdot)$ of which $x_j = x_j(z_j)$ is realized and observed.[6] The importance weights may or may not be the same for all $j \in \mathcal{J}_i$ and may or may not add up to one. In the election example, X_i may be the fraction of seats in the state i legislature represented by candidates from the focal party. Here z_j indicates the focal party's victory in district j and s_j measures the seat share of the district j in the legislature. In the common case of one seat per district, $s_j = 1/J_i$ is the inverse of the total number of seats.

We also observe an outcome Y_i. While we allow it to depend arbitrarily on $\boldsymbol{z}_i$, with potential outcomes $Y_i(\boldsymbol{z}_i)$, two benchmark cases are of interest. The first one imposes an exclusion restriction at the upper-level: that $\boldsymbol{z}_i$ only affects Y_i through X_i. Then one can represent potential outcomes as $Y_i(x)$:

$$Y_i(\boldsymbol{z}_i) = Y_i(X_i(\boldsymbol{z}_i)). \tag{3}$$

Second, under (2), in some settings it is plausible to assume the existence of latent subunit-level potential outcomes $y_j = y_j(x_j)$, which implicitly impose the lower-level exclusion of z_j (which is trivial in the sharp

[4] We use capital and small letters to denote unit- and subunit-level variables, respectively.

[5] We assume that the sets are non-overlapping to simplify notation. The setting naturally extends to the case where different units can be exposed to the same subunits, as when estimating the spillovers from RD events; see Section 4.1.

[6] While our theoretical results apply with unrestricted $X_i(\boldsymbol{z}_i)$, a structure like (1) or (2) is helpful for the estimators to be powerful and intuitive.

case of $x_j = z_j$) and no interference across subunits. The upper-level outcome in turn aggregates them:

$$Y_i(\boldsymbol{z}_i) = \sum_{j \in \mathcal{J}_i} s_j y_j(x_j(z_j)). \tag{4}$$

Both potential outcome formulations can be illustrated in the election example. If the outcome Y_i results from policies determined at the legislature level, the upper-level exclusion (3) may be plausible. If, conversely, the outcome is initially determined by district based on the policy adopted by the individual elected official of the district, the lower-level exclusion (4) is more natural. While in the former case the outcome is inherently defined at the upper-level, in the latter case aggregation is only useful when lower-level outcomes are unobserved by the researcher.[7]

Our setting nests conventional sharp RD which corresponds to the case where the outcomes are measured at the same level as the running variables, i.e. i and j are the same objects: $\mathcal{J}_i = \{i\}$, $J_i = s_j = 1$, and $X_i = z_j$. In that scenario, the standard local linear estimation approach restricts the sample to observations near the cutoff ($|r_j| \leq h$ for bandwidth h) and estimates by ordinary least squares (OLS) the specification

$$Y_i = \beta z_j + \gamma_0 + \gamma_1 r_j + \gamma_2 r_j^+ + \text{error}_j, \qquad j \equiv i, \tag{5}$$

where the controls $q_j = (1, r_j, r_j^+)$ include the intercept, the running variable, and its interaction with being on the right of the cutoff (i.e., $r_j^+ = r_j \cdot I(r_j \geq 0)$). Sometimes this regression is weighted by kernel weights that are larger when r_j is closer to the cutoff: $\kappa_h(r_j) \equiv \kappa(r_j/h)$ for a kernel function $\kappa(\cdot)$. The local linear regression identifies a local causal effect under assumptions on the continuity of potential outcomes (e.g., Hahn et al. (2001)).

But how can local variation in r_j be leveraged to estimate the relationship between X_i and Y_i in (3) or between x_j and the latent y_j in (4) when each unit i can be exposed to multiple elections, potentially with more than one near the cutoff? We next propose two strategies, one at the upper level and the other at the lower level, focusing on the intuitions before proceeding to formal analysis of consistency and asymptotic bias-variance tradeoffs.

2.2 The Two Estimators

Upper-level estimator. Our first proposal is a specification at the level of upper-level units i which involves instrumenting X_i with its component arising from the subunits whose running variables are near the cutoff, $\mathcal{C}_i = \{j \in \mathcal{J}_i \colon\ I(|r_j| \leq h)\} \subseteq \mathcal{J}_i$:

$$Z_i = \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) z_j. \tag{6}$$

Since the specification is at the i level, we include $\kappa_h(r_j)$ in the construction of Z_i rather than as regression weights. In the election example and with a uniform kernel $\kappa(x) = I(x \in [-1, 1])$, Z_i is the fraction of state legislature's seats represented by a candidate from the focal party who won in a close election, out of all seats.

We further propose to aggregate the vector of standard RD controls q_j to the i level in the same way as

[7] Intermediate cases could also be considered: e.g., that the latent lower-level outcome y_j is co-determined by the local treatment x_j and the upper-level treatment X_i which captures spillovers from other districts.

the instrument is constructed,

$$Q_i = \sum_{j \in \mathcal{C}_i} s_j q_j = \big(\sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j), \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j, \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j^+ \big)'. \tag{7}$$

This idea for these control variables arises from viewing Z_i as a shift-share variable constructed from shifts z_j and exposure shares $s_j \kappa_h(r_j)$.[8] Borusyak et al. (2022) show the usefulness of constructing shift-share aggregates of shift-level covariates, albeit in a setting in which the shifts are assumed conditionally random, which our RD setting does not impose.

We label the three local aggregate variables in Q_i the *RDA controls*. When the kernel is uniform, the first of them is the total weight of subunits near the cutoff. For instance, in the election example, it is the fraction of legislature seats determined in a close election. The second is the average running variable in narrow elections, rescaled by their total weight. And the last one is the average running variable in narrow wins, rescaled by the total weight of those narrow wins. Note that, when there are no close elections for unit i, both Z_i and Q_i are zero.

As a result, the upper-level RDA estimator corresponds to the instrumental variable (IV) specification

$$Y_i = \beta X_i + \gamma_0 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) + \gamma_1 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j + \gamma_2 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j^+ + \tilde{\gamma}' \tilde{W}_i + \text{error}_i, \tag{8}$$

instrumented by Z_i, where $\tilde{W}_i$ includes the i-level intercept and possibly additional variables that are continuous around the cutoff. Its reduced-form parallels (5):

$$Y_i = \beta_Y \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) z_j + \gamma_0 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) + \gamma_1 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j + \gamma_2 \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) r_j^+ + \tilde{\gamma}' \tilde{W}_i + \text{error}_i. \tag{9}$$

Formally, the upper-level IV estimator is defined as $\hat{\beta}_{\text{up}} = \hat{\beta}_{\text{up},Y} / \hat{\beta}_{\text{up},X}$, where the reduced-form $\hat{\beta}_{\text{up},Y}$ and first-stage $\hat{\beta}_{\text{up},X}$ solve, for $V = X, Y$,

$$\hat{\beta}_{\text{up},V} = \arg\min_{\beta} \sum_{i=1}^{N} \left(V_i - \beta Z_i - \gamma' Q_i - \tilde{\gamma}' \tilde{W}_i \right)^2.$$

Lower-level ("RD stacking") estimator. Our second proposal is to estimate a fuzzy RD local linear specification on the sample that stacks all subunits near the cutoff:

$$Y_{\mathbf{i}(j)} = \beta X_{\mathbf{i}(j)} + \gamma' q_j + \tilde{\gamma}' \tilde{W}_{\mathbf{i}(j)} + \text{error}_j, \qquad j \in \cup_i \mathcal{C}_i \tag{10}$$

where $\mathbf{i}(j)$ denotes the unit to which subunit j belongs, with $X_{\mathbf{i}(j)}$ instrumented by the subunit-level treatment z_j, and using $s_j \kappa_h(r_j)$ as importance weights.[9]

The lower-level estimator is therefore defined as $\hat{\beta}_{\text{lo}} = \hat{\beta}_{\text{lo},Y} / \hat{\beta}_{\text{lo},X}$, where the reduced-form $\hat{\beta}_{\text{lo},Y}$ and first-stage $\hat{\beta}_{\text{lo},X}$ solve, for $V = X, Y$,

$$\hat{\beta}_{\text{lo},V} = \arg\min_{\beta} \sum_{i=1}^{N} \sum_{j \in \mathcal{C}_i} s_j \kappa_h(r_j) \left(V_i - \beta Z_i - \gamma' q_j - \tilde{\gamma}' \tilde{W}_i \right)^2.$$

[8] More precisely, the shares are $s_j \kappa_h(r_j)$ for $j \in \mathcal{C}_i$ and zero for $j \notin \mathcal{C}_i$. While shift-share instruments usually involve shifts that are common to all observations, this is not a requirement (Borusyak et al., 2025b, Appendix A.6).

[9] While in many fuzzy RD designs the treatment is binary, this is not necessary (e.g., Almond et al., 2010, Section VIII).

Discussion. Should we expect good properties from either of the two proposed estimators? Conceptually, both of them seemingly have pros and cons. Since the lower-level estimator corresponds to a conventional fuzzy RD specification, it is clear it uses local variation around the cutoff. It also lends itself to conventional procedures for RD designs, such as RD plots for the first-stage and reduced-form (at the j level). At the same time, the lower-level specification is peculiar: equation (10) provides multiple expressions for Y_i of the same unit, and the instrument z_j varies among multiple observations — subunits of the same unit — that mechanically have the same value of the treatment $X_{\mathbf{i}(j)}$. The upper-level estimator avoids these conceptual shortcomings by specifying only one expression for each outcome Y_i. Aggregating both the instrument z_j and the covariates q_j in the same way is also perhaps intuitive. At the same time, it is less clear what variation that specification leverages.

We provide intuition for both estimators by establishing a tight connection between them. The *upper-level* estimator is numerically equivalent to a fuzzy RD estimator from a particularly transformed regression at the *subunit* level:

Proposition 1. *For any unit-level variable V_i let $V_i^{\perp}$ denote the residual from a projection of V_i on $W_i = \left(Q_i, \tilde{W}_i\right)'$. Then $\hat{\beta}_{up}$ equals the IV estimator of β from a subunit-level specification*

$$Y_{\boldsymbol{i}(j)}^{\perp} = \beta X_{\boldsymbol{i}(j)}^{\perp} + \gamma' q_j + error_j, \tag{11}$$

on the sample of $j \in \mathcal{C} \equiv \cup_i \mathcal{C}_i$, where $X_{\boldsymbol{i}(j)}^{\perp}$ is instrumented by z_j and the regression is weighted by $s_j \kappa_h(r_j)$.

Proof. Follows from Proposition 1 in Borusyak et al. (2022) by observing that Z_i is a shift-share instrument. The shift-level importance weight defined in that proposition, $\sum_i \sum_{j \in \mathcal{C}} s_j \kappa_h(r_j) I\left(j \in \mathcal{C}_i\right)$, simplifies to $s_j \kappa_h(r_j)$, and shift-level averages simplify to $\bar{V}_j = \sum_i s_j \kappa_h(r_j) I\left(j \in \mathcal{C}_i\right) V_i / \sum_i s_j \kappa_h(r_j) I\left(j \in \mathcal{C}_i\right) = V_{\mathbf{i}(j)}$. □

This representation of the upper-level estimator is useful in two ways. On one hand, it shows that the upper-level estimator also leverages local variation around the cutoff: it can be represented as a particular fuzzy RD specification with z_j as the instrument, standard RD controls q_j, and kernel weights. On the other hand, it shows that there is no conceptual problem with stacking: the lower-level specification only differs from (11), which is in turn equivalent to the coherent upper-level specification, by not residualizing the outcome and treatment on the upper-level covariates.

One may therefore conjecture that both estimators identify some local causal effect under assumptions similar to standard RD and have desirable bias-variance properties of local linear regressions. Still, the RDA setting is more complex than conventional fuzzy RD.

Before proceeding to the formal asymptotic analysis, it is also instructive to discuss intuitively when the two IV estimators may have powerful first-stages. The lower-level estimator requires that a typical j-level narrow victory moves the i-level treatment. This can be expected to hold as long as there are close elections with non-vanishing s_j. For instance, if $s_j = 1/J_{\mathbf{i}(j)}$, this requires the total number of elections per unit not to be too large. The best-case scenario is when each unit is exposed to a small (but non-zero) number of large elections, with substantial density of the running variables around the threshold. In contrast, having many small elections weakens the first stage despite increasing the number of close elections in the stacked sample. Having many small elections is a problem for the upper-level estimator, too, although the mechanics are different: with too many elections (whether narrow or not), the law of large numbers kills the variation in the upper-level instrument, which aggregates the random variation across the elections. Our main asymptotic regime will preclude this scenario but we will analyze it in one of the nonstandard asymptotics.

3 Asymptotic Analysis

In this section, we establish the asymptotic properties of both the upper- and lower-level estimators. We primarily focus on the standard asymptotic regime where the distribution of $(J_i, (s_j, r_j)_{j\in\mathcal{J}_i}, X_i, Y_i)$ is held fixed. After introducing some preliminaries in Section 3.1, Section 3.2 shows that, under conditions similar to those of standard fuzzy RD, both estimators identify a local average causal effect. Their asymptotic bias and variance are characterized in Sections 3.3 and 3.4, respectively. A limitation of the standard asymptotic regime is that a unit with one close election is unlikely to have another one. In Section 3.5, we address this issue by considering two non-standard asymptotic regimes that are more appropriate for settings in which some units contain clusters of close elections. All proofs are relegated to Appendix A.

3.1 Preliminaries

To describe the local parameter identified by the two RDA estimators we impose:

Assumption 1 (Random sampling of units)**.** *The data are a random sample of* $J_i, \boldsymbol{r}_i, \boldsymbol{z}_i, \boldsymbol{s}_i, \boldsymbol{q}_i, X_i, X_i(\cdot), \tilde{W}_i, Y_i, Y_i(\cdot)$ *where bold symbols indicate* J_i*-dimensional vectors containing similarly named subunit-level variables, and consistency conditions hold:* $z_j = I(r_j \geq 0)$*,* $X_i = X_i(\boldsymbol{z}_i)$*, and* $Y_i = Y_i(\boldsymbol{z}_i)$*.*

This sampling scheme in this standard asymptotic regime implies that the number of subunits $J_i = O_{\mathbb{P}}(1)$; we consider an alternative regime in Section 3.5. For simplicity, we abstract from additional covariates, setting $\tilde{W}_i = 0$; Calonico et al. (2019) show that adding predetermined covariates in a linearly separable way does not affect the RD estimand.

We further impose *irrelevance of labels*: for any $\bar{J}$ in the support of J_i, the distribution of $(\pi(\boldsymbol{r}_i), \pi(\boldsymbol{z}_i), \pi(\boldsymbol{s}_i), \pi(\boldsymbol{q}_i), X_i, X_i(\pi(\cdot)), Y_i, Y_i(\cdot)) \mid J_i = \bar{J}$ is the same for all permutations π of $\bar{J}$ elements. This is without loss of generality: one can always consider a random relabeling of indices within $\mathcal{J}_i$. Given such exchangeability, when we write the expectation of a term that involves both i and j objects, such as $\mathbb{E}[J_i s_j \mid r_j = r]$, this is interpreted as choosing a random $j \in \mathcal{J}_i$. We denote by f the marginal density of r_j and by u the joint density of (r_j, r_k), $j \neq k$, understood in the same way.

We write $\boldsymbol{z}_{\mathbf{i}(j)} \equiv (z_j, \boldsymbol{z}_{-j})$ and, for $V \in \{X, Y\}$, define

$$V_j^+ = V_i(1, z_{-j}), \quad V_j^- = V_i(0, z_{-j}), \quad \Delta V_j = V_j^+ - V_j^-.$$

Using this notation, we impose a first-stage condition:

Assumption 2 (First stage)**.** $f(0) > 0$*,* $\mathbb{E}[J_i s_j \Delta X_j \mid r_j = 0] \neq 0$*, and* $\mathbb{E}[J_i s_j^\ell \mid r_j = 0] \neq 0$ *for* $\ell \in \{1, 2\}$*.*

This assumption is expected to hold when the treatment is defined by equation (1), in which case $X_j^- = \sum_{k\neq j, k\in\mathcal{J}_i} s_k z_k$ and $\Delta X_j = s_j > 0$.

We assume that the kernel function κ is supported on $[-1, 1]$, symmetric, and non-negative, i.e.,

$$\kappa(x) = \kappa(-x) \geq 0, \ \ \forall x \in [-1, 1]; \quad \kappa(x) = 0, \ \ \forall x \notin [-1, 1].$$

Further, we define

$$M_k = \int_0^\infty x^k \kappa(x) dx, \quad \tilde{M}_k = \int_0^\infty x^k \kappa^2(x) dx \tag{12}$$

and, following the notation of Imbens and Kalyanaraman (2012),

$$C_{\text{bias}}(\kappa) = \frac{M_1 M_3 - M_2^2}{M_0 M_2 - M_1^2}, \quad C_{\text{var}}(\kappa) = \frac{M_2^2 \tilde{M}_0 - 2M_1 M_2 \tilde{M}_1 + M_1^2 \tilde{M}_2}{(M_0 M_2 - M_1^2)^2}. \tag{13}$$

For the upper-level estimator, we always assume that $\tilde{M}_2 M_0 \neq \tilde{M}_1 M_1$.

Finally, we denote by $\mathcal{C}^s$ the set of functions with L-Lipschitz s-th derivatives on a fixed neighborhood of zero $[-b, b]$ for some constants $L, b > 0$. We write q' for the first-order derivative of any univariate function q and $\partial_k q$ for the first-order derivative of a multivariate function q with respect to the k-th coordinate.

3.2 Identification of a Local Causal Effect

To show that both upper- and lower-estimators identify the same local causal effect, we impose:

Assumption 3 (Continuity)**.** *(a) Expected reweighted potential treatment* $\mathbb{E}\left[J_i s_j X_j^{\pm} \mid r_j = r\right]$*, outcome* $\mathbb{E}\left[J_i s_j Y_j^{\pm} \mid r_j = r\right]$*, and weights* $\mathbb{E}\left[J_i s_j \mid r_j = r\right]$ *are continuous in* r *at* $r = 0$*;*

(b) Marginal density $f(0)$ *and joint density* $u(0,0)$ *exist;*[10]

(c) For some fixed J^**,* $J_i \leq J^*$ *a.s.*

Part (a) is a generalization of the RD continuity assumption appropriate for the RDA setting. Part (b) rules out the case where multiple running variables are perfectly correlated. However, perfectly correlated elections can be combined into one without loss. Part (c) imposes an upper bound on the number of subunits per units. While it can be relaxed within the standard asymptotic regime, Section 3.5 tackles the case of large J_i directly.

We consider the estimands of the two estimators, defined as the expectations of the linearized estimators as a function of h (with this dependence suppressed for brevity):

$$\tilde{\beta}_{\text{lo}} = \frac{\tilde{\beta}_{\text{lo},Y}}{\tilde{\beta}_{\text{lo},X}}, \quad \tilde{\beta}_{\text{lo},V} = \arg\min_{\beta} \mathbb{E}\left[\sum_{j \in \mathcal{J}_i} \left(V_i - \beta z_j - \gamma' q_j\right)^2 s_j \kappa_h(r_j)\right], \tag{14}$$

and

$$\tilde{\beta}_{\text{up}} = \frac{\tilde{\beta}_{\text{up},Y}}{\tilde{\beta}_{\text{up},X}}, \quad \tilde{\beta}_{\text{up},V} = \arg\min_{\beta} \mathbb{E}\left[\sum_{j \in \mathcal{J}_i} \left(V_i - \beta Z_i - \gamma' Q_i\right)^2\right]. \tag{15}$$

The following proposition shows that both estimators have the same estimand as $h \to 0$:

Theorem 1. *Suppose Assumptions 1–3 hold. Then, for*

$$\beta_0 = \frac{\mathbb{E}\left[J_i s_j \Delta Y_j \mid r_j = 0\right]}{\mathbb{E}\left[J_i s_j \Delta X_j \mid r_j = 0\right]},$$

we have

$$\lim_{h \to 0} \tilde{\beta}_{up} = \lim_{h \to 0} \tilde{\beta}_{lo} = \beta_0. \tag{16}$$

To give β_0 a causal interpretation, we specialize it under the two exclusion restrictions, (3) and (4):

[10] This implicitly assumes that the density of r_j conditional on $J_i = \bar{J}$ is the same for all $\bar{J} \geq 1$, and similarly the joint density of (r_j, r_k) for $k \neq j$ conditional on $J_i = \bar{J}$ is the same for all $\bar{J} \geq 2$. We believe this restriction is inessential and will relax it in further drafts.

Corollary 1. *Assume monotonicity: that ΔX_j has weakly the same sign for all j (which is automatically satisfied in the sharp case (1)). Then we have:*

(a) Under the upper-level exclusion restriction (3),

$$\beta_0 = \frac{\mathbb{E}\left[J_i s_j \Delta X_j \cdot \frac{Y_i(X_j^- + \Delta X_j) - Y_i(X_j^-)}{\Delta X_j} \mid r_j = 0\right]}{\mathbb{E}\left[J_i s_j \Delta X_j \mid r_j = 0\right]}, \tag{17}$$

which is a convexly-weighted average of $\partial Y_i(x)/\partial x$ (or the discrete analog thereof when X_i is discrete) across i and x;[11]

(b) If additionally the assumptions of Theorem 1 hold, the causal effects are heterogeneous across units but linear in x, i.e. $Y_i(x) = Y_i(0) + \beta_i x$, and in the sharp case (1), we have

$$\beta_0 = \lim_{h \to 0} \frac{\mathbb{E}\left[\omega_i \beta_i\right]}{\mathbb{E}\left[\omega_i\right]}, \qquad \omega_i = \sum_{j \in \mathcal{C}_i} s_j^2. \tag{18}$$

(c) Under the treatment structure (2) and the lower-level exclusion restriction (4), $\Delta X_j = x_j(1) - x_j(0)$ and

$$\beta_0 = \frac{\mathbb{E}\left[J_i s_j \Delta X_j \cdot \frac{y_j(x_j(1)) - y_j(x_j(0))}{x_j(1) - x_j(0)} \mid r_j = 0\right]}{\mathbb{E}\left[J_i s_j \Delta X_j \mid r_j = 0\right]}, \tag{19}$$

which is a convexly-weighted average of $\partial y_j(x)/\partial x$ (or the discrete analog thereof when x_j is discrete) across j and x.

Parts (a) and (c) show that, under both potential outcome formulations, β_0 is interpretable as a proper local average causal effect. In both cases, the weights placed on unit i are given by $J_i s_j \Delta X_j$ given $r_j = 0$. Thus, they are higher for units with more numerous and more important subunits near the cutoff, as well as subunits with a stronger first-stage impact on the treatment.

Part (b) makes this result practical by characterizing the weights across units that define the effective sample (or "complier sample") of both RDA estimators in the sharp case. These weights can be computed in the data, and we report them in the application below. Like in the general case, the weight ω_i in (18) is higher for units that have more close elections and for which close elections have large s_j. Echoing the discussion of relevance in Section 2.2, both of these situations generate more variation in Z_i at the upper-level and ensure a strong first-stage for unit i at the lower level. For example, suppose $s_j = 1/J_i$, as when the treatment is the *share* of legislature seats won by the focal party. Then the effective weight is smaller in larger legislatures, as the law of large numbers eliminates much of the variation in the upper-level instrument, while at the lower level one close win shifts the treatment by less. If, however, $s_j = 1$, as when the treatment is the *number* of legislature seats with the focal party winning, the effective weight is instead larger in larger legislatures, as they will tend to have more individual seats with narrow elections. Since the effective sample can be computed in this case, both upper- and lower-level regressions can be reweighted to obtain other averages of causal effects that are of interest.

We finally note that the proof of Theorem 1 makes it clear that, among the covariates, only $\sum_{j \in \mathcal{C}_i} s_j$ in the upper-level specification and the intercept in the lower-level specification are necessary for identification.

[11] Without loss of generality, $\frac{Y_i(X_j^- + \Delta X_j) - Y_i(X_j^-)}{\Delta X_j}$ should be understood as zero when $\Delta X_j = 0$; same in the expression below.

A parallel to the local randomization approach to RD designs provides intuition for this conclusion: if the running variables are viewed as drawn from a uniform distribution around the cutoff, controlling for $\sum_{j\in\mathcal{C}_i} s_j$ is equivalent to controlling for the expected value of Z_i, as in the Borusyak and Hull (2023) recentering procedure. This control isolates a unit's "luck" in getting a higher-than-expected vs lower-than-expected value of the instrument. While our identification result (or the local randomization approach) do not provide a justification for including the aggregated local linear controls, the next subsection shows that they are important for reducing bias, as in standard RD designs based on the continuity assumption.

3.3 Asymptotic Bias

We next characterize the asymptotic bias of both estimators. We follow the convention in the fuzzy RD literature (e.g. Calonico et al., 2014, Section 3.2) to define the bias by comparing the estimands $\tilde{\beta}_{\text{lo}}, \tilde{\beta}_{\text{up}}$ with their target as $h \to 0$, β_0. We strengthen Assumption 3 to:

Assumption 4 (Smoothness)**.**

(a) $\mathbb{E}[J_i s_j V_j^{\pm} \mid r_j = r] \cdot f(r) \in \mathcal{C}^2$ *for* $V \in \{X, Y\}$*;*

(b) For $\ell = 1, 2$, $\mathbb{E}[J_i s_j^{\ell} \mid r_j = r] \cdot f(r) \in \mathcal{C}^2$*;*

(c) For $j, k \in \mathcal{J}_i$ *and* $k \neq j$, $\mathbb{E}[J_i(J_i - 1)s_j s_k \mid r_j = r, r_k = r'] \cdot u(r, r') \in \mathcal{C}^1$*;*

(d) For some fixed J^*, $J_i \leq J^*$ *a.s.*

We start by showing that the lower-level estimator has a second-order bias regardless of the kernel.

Theorem 2. *Under the standard asymptotic regime and Assumptions 1, 2, and 4, as* $h \to 0$*,*

$$\tilde{\beta}_{\text{lo}} - \beta_0 = h^2 \Gamma_2^{\text{lo}} + o(h^2),$$

where

$$\Gamma_2^{\text{lo}} = -\frac{C_{\text{bias}}(\kappa)}{2\mu_{X,\text{lo}}(0)} \left\{ \mu''_{Y,\text{lo}}(0) - \mu''_{X,\text{lo}}(0)\beta_0 \right\}$$

for

$$\mu_{V,\text{lo}}(r) = \frac{\mathbb{E}[J_i s_j \Delta V_j \mid r_j = r]}{\mathbb{E}[J_i s_j \mid r_j = r]}.$$

To draw the connection with standard fuzzy RD estimators, we define

$$\mathbb{E}^{(\ell)}[\Delta V_j \mid r_j = r] = \frac{\mathbb{E}[J_i s_j^{\ell} \Delta V_j \mid r_j = r]}{\mathbb{E}[J_i s_j^{\ell} \mid r_j = r]}. \tag{20}$$

Then $\mu_{V,\text{lo}}(r)$ can be reformulated as a reweighted conditional mean $\mathbb{E}^{(1)}[\Delta V_j \mid r_j = r]$ under a change of measure. In Appendix A.4, we prove that the bias formula in Theorem 2 can be equivalently expressed as the bias of a standard fuzzy RD estimator (e.g. Imbens and Kalyanaraman, 2012, Eq. (22)) applied to upper-level units under the above change of measure.

Next, we analyze the bias of the upper-level estimator. Interestingly, it has a first-order bias except when the kernel satisfies a certain condition.[12]

[12]The technical intuition is that the Gram matrix $\sum_i X_i^2$ involves terms with $\kappa_h(r_j)^2$ while $\sum_i X_i Y_i$ only has $\kappa_h(r_j)$. The biases resulting from these terms generate a mismatch that results in a first-order bias unless (21) holds.

Theorem 3. *Under the standard asymptotic regime and Assumptions 1, 2, and 4, as* $h \to 0$*,*

$$\tilde{\beta}_{\text{up}} - \beta_0 = h\Gamma_1^{\text{up}} + h^2\Gamma_2^{\text{up}} + O(h^3),$$

where Γ_1^{up} *and* Γ_2^{up} *are defined in Eq. (A30) in Appendix A.4. In particular,* $\Gamma_1^{\text{up}} = 0$ *without further restrictions if and only if*

$$\frac{\tilde{M}_2}{M_2} = \frac{\tilde{M}_1}{M_1} = \frac{\tilde{M}_0}{M_0}. \tag{21}$$

Under this condition, Γ_2^{up} *can be expressed as*

$$\Gamma_2^{\text{up}} = -\frac{C_{\text{bias}}(\kappa)}{2\mu_{X,\text{up}}(0)} \left\{ \mu''_{Y,\text{up}}(0) - \mu''_{X,\text{up}}(0)\beta_0 \right\}.$$

where

$$\mu_{V,\text{up}}(r) = \frac{\mathbb{E}[J_i s_j \Delta V_j \mid r_j = r]}{\mathbb{E}[J_i s_j^2 \mid r_j = r]}.$$

Note that the denominator of $\mu_{V,\text{up}}(r)$ involves the square of s_j, which is different from the numerator and all terms in $\mu_{V,\text{lo}}(r)$. Due to this mismatch, $\mu_{V,\text{up}}(r)$ cannot be interpreted as a conditional expectation under a change of measure.

For the uniform kernel, $M_k = \tilde{M}_k$ by definition (12) and hence the condition (21) holds. Surprisingly, we show that the uniform kernel is the only monotonic kernel that satisfies the condition:

Proposition 2. *Assume* κ *is a symmetric, non-negative kernel with support* $[-1, 1]$*. If* κ *is non-increasing on* $[0, 1]$ *and satisfies the condition* (21)*, it must be the uniform kernel.*

In particular, other commonly used kernels, such as the triangle kernel that is optimal for mean squared error (MSE; Cheng et al. 1997; Imbens and Kalyanaraman 2012), all produce a substantially larger bias for the upper-level estimator. This result highlights that the upper-level estimator has a different structure than the lower-level estimator, even though Γ_2^{up} takes a form similar to Γ_2^{lo} under the uniform kernel.

The second-order biases Γ_2^{lo} and Γ_2^{up} are generally hard to compare. However, there is a special case when the two coincide:

Proposition 3. *With uniform kernel* κ*, if* $s_j \perp\!\!\!\perp (r_j, J_i)$ *then* $\Gamma_2^{\text{lo}} = \Gamma_2^{\text{up}}$*.*

This in particular includes the case when all importance weights s_j are the same across units and subunits.

Our results in progress suggest that both Γ_2^{lo} and Γ_2^{up} can be estimated using the RDA generalization of a local quadratic regression, as Calonico et al. (2014) recommend for standard RD designs.

Lastly, we examine the benefit of local linear controls in q_j and Q_i. In the standard fuzzy RD setting, it is well known that using local linear regression reduces the bias relative to local constant regression, improving the leading bias term from first- to second-order (Imbens and Kalyanaraman, 2012; Calonico et al., 2014). In our RDA setting, we analogously consider the local constant version of the lower- and upper-level estimators, denoted by $\hat{\beta}_{\text{lo}}^{\text{const}}$ and $\hat{\beta}_{\text{up}}^{\text{const}}$, respectively. Following (14) and (15), we define $\tilde{\beta}_{\text{lo}}^{\text{const}}$ and $\tilde{\beta}_{\text{up}}^{\text{const}}$ by removing the local linear controls from q_j and Q_i. We show that both estimators have first-order bias regardless of the kernel choice.

Theorem 4. *Under the standard asymptotic regime and Assumptions 1, 2, and 4, as* $h \to 0$*,*

$$\tilde{\beta}_{\text{lo}}^{\text{const}} - \beta_0 = h\Gamma_1^{\text{const,lo}} + o(h), \quad \tilde{\beta}_{\text{up}}^{\text{const}} - \beta_0 = h\Gamma_1^{\text{const,up}} + o(h),$$

where $\Gamma_1^{\text{const,up}}$ *and* $\Gamma_1^{\text{const,up}}$ *are defined in Eq. (A33) and (A35) in Appendix A.4.5 and are non-zero without further restrictions.*

3.4 Asymptotic Normality and Variance

In this subsection, we analyze the asymptotic variance of the lower- and upper-level estimators under the standard asymptotic regime. We follow the convention that centers the estimators at their linearized estimands defined in (14) and (15). First, we derive the asymptotic normality and variance for the lower-level estimator without restricting the choice of kernel. For notational convenience, we write

$$U_j^{\pm} = Y_j^{\pm} - \beta_0 X_j^{\pm}. \tag{22}$$

For asymptotic normality, we assume the existence of higher moments:

Assumption 5. *For some* $\delta > 0$, $\mathbb{E}\left[|Y_i|^{2+\delta} \mid r_j = r\right]$ *and* $\mathbb{E}\left[|X_i|^{2+\delta} \mid r_j = r\right]$ *exist.*

Then we have for the lower-level estimator:

Theorem 5. *Under the standard asymptotic regime and Assumptions 1, 2, 4, and 5, as* $h \to 0$ *and* $Nh \to \infty$,

$$\sqrt{Nh}(\hat{\beta}_{\text{lo}} - \tilde{\beta}_{\text{lo}}) \xrightarrow{d} N\left(0, \sigma_{\text{lo}}^2\right),$$

where

$$\sigma_{\text{lo}}^2 = \frac{C_{\text{var}}(\kappa)}{f(0)} \frac{\mathbb{E}[J_i s_j^2 \mid r_j = 0]}{(\mathbb{E}[J_i s_j \Delta X_j \mid r_j = 0])^2} \left\{\mathbb{E}^{(2)}[(U_j^+ - \mathbb{E}^{(1)}[U_j^+])^2] + \mathbb{E}^{(2)}[(U_j^- - \mathbb{E}^{(1)}[U_j^-])^2]\right\},$$

and $\mathbb{E}^{(1)}$ *and* $\mathbb{E}^{(2)}$ *are defined in* (20).

The standard sharp RD estimator can be viewed as a special case with $J_i = 1, s_j = 1, X_i = I(r_i \leq 0)$ where we rewrite r_i for r_j because there is a single subunit. In this case, $X_j^+ = 1, X_j^- = 0$ and $\mathbb{E}^{(2)} = \mathbb{E}^{(1)} = \mathbb{E}$. Then the expression σ_{lo}^2 simplifies into

$$\sigma_{\text{lo}}^2 = \frac{C_{\text{var}}(\kappa)}{f(0)} \left\{\text{Var}(Y_j^+ \mid r_i = 0) + \text{Var}(Y_j^- \mid r_i = 0)\right\}.$$

This is identical to the variance formula in Imbens and Kalyanaraman (2012).

For the upper-level estimator, we focus on the uniform kernel as suggested by the asymptotic bias analysis in Theorem 3.

Theorem 6. *Under the standard asymptotic regime and Assumptions 1, 2, 4, and 5, if* κ *is the uniform kernel and as* $h \to 0$ *and* $Nh \to \infty$,

$$\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}) \xrightarrow{d} N\left(0, \sigma_{\text{up}}^2\right),$$

where $\zeta = \mathbb{E}[J_i s_j \mid r_j = 0]/\mathbb{E}[J_i s_j^2 \mid r_j = 0]$ *and*

$$\sigma_{\text{up}}^2 = \frac{C_{\text{var}}(\kappa)}{f(0)} \frac{\mathbb{E}[J_i s_j^2 \mid r_j = 0]}{(\mathbb{E}[J_i s_j \Delta X_j \mid r_j = 0])^2} \left\{\mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+]\right)^2\right] + \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-]\right)^2\right]\right\}.$$

To compare the asymptotic variances of the two estimators, we first establish an underappreciated fact that the uniform kernel is variance-optimal. This is in contrast to the commonly used triangle kernel which is MSE-optimal (Cheng et al., 1997; Imbens and Kalyanaraman, 2012).

Proposition 4. *For any symmetric, non-negative kernel κ with support $[-1,1]$, $C_{\text{bias}}(\kappa) \geq 4$ and the equality holds if κ is a uniform kernel.*

When the uniform kernel is used for both estimators, it remains to compare the terms inside the curly brackets. In general, neither dominates the other. Nevertheless, there are two special cases when a comparison can be made: when s_j is a constant, the two variances coincide (in addition to biases, per Proposition 3) while with exogenous number of subunits and importance weights the lower-level estimator has a weakly smaller variance.

Theorem 7. *In the setting of Theorem 6,*

(a) If s_j are the same across all units and subunits, $\sigma^2_{\text{up}} = \sigma^2_{\text{lo}}$;

(b) If $(J_i, (s_j, r_j)_{j\in\mathcal{J}_i})$ is independent of (U_j^+, U_j^-),

$$\sigma^2_{\text{up}} - \sigma^2_{\text{lo}} = \frac{4}{f(0)} \frac{\mathbb{E}[J_i s_j^2 \mid r_j = 0]}{(\mathbb{E}[J_i s_j \Delta X_j \mid r_j = 0])^2} \left(\text{CV}^{[1]}(s_j)\right)^4 \left\{ \left(\mathbb{E}^{[1]}[U_j^+]\right)^2 + \left(\mathbb{E}^{[1]}[U_j^-]\right)^2 \right\} \geq 0,$$

where $\text{CV}^{[1]}[\cdot]$ denotes the coefficient of variation, i.e., $\text{CV}^{(1)}[Z] = \sqrt{\text{Var}^{(1)}(Z)}/\mathbb{E}^{(1)}[Z]$.

Lastly, we discuss consistent variance estimators. Perhaps surprisingly, we find that both σ^2_{lo} (resp. σ^2_{up}) can be estimated directly by applying the `RDRobust` method based on the plug-in estimated residuals (Calonico et al., 2014, 2017) to $(r_j, Y_i)_{j\in\mathcal{J}_i, i\in[N]}$ (resp. $(r_j, Y_j^\perp)_{j\in\mathcal{J}_i, i\in[N]}$ per Proposition 1) with weights $(s_j)_{\in\mathcal{J}_i, i\in[N]}$ as if they are independent data points, without the need for clustering. We denote by $\hat{\sigma}^{\text{nc},2}_{\text{Rdr,lo}}$ and $\hat{\sigma}^{\text{nc},2}_{\text{Rdr,up}}$ these variance estimators. Furthermore, we also show the consistency of clustered `RDRobust` variance estimates, which we denote by $\hat{\sigma}^{\text{c},2}_{\text{Rdr,lo}}$ and $\hat{\sigma}^{\text{c},2}_{\text{Rdr,up}}$.

Theorem 8. *In the setting of Theorem 6, $\hat{\sigma}^{\text{c/nc},2}_{\text{Rdr,lo}} \xrightarrow{p} \sigma^2_{\text{lo}}$ and $\hat{\sigma}^{\text{c/nc},2}_{\text{Rdr,up}} \xrightarrow{p} \sigma^2_{\text{up}}$.*

Notably, this result does not conflict with the result of Bartalotti and Brummet (2017), which shows that clustering is required in the clustered RD setting. This is because, in our setting, the joint density $u(r, r')$ is assumed smooth at $(0, 0)$, while Bartalotti and Brummet (2017) assume all running variables have the same signs within the same cluster. Under our assumption, conditional on that $|r_j|, |r_k| \leq h$, r_j and r_k are approximately independent for $k \neq j$ with uniform marginals over $[-h, h]$. However, when the joint density has jumps on the axes, r_j and r_k are conditionally dependent, yielding non-negligible within-cluster dependence.

Given estimates of bias and variance for either estimator, one can estimate the MSE-optimal bandwidth and construct robust bias-corrected confidence intervals in the spirit of Calonico et al. (2014). We leave the details to future drafts.

3.5 Non-standard Asymptotic Regimes

Under the standard asymptotic regime, among upper-level units with at least one lower-level unit near the cutoff, the share that have more than one is on the order of $O(h)$, which is negligible. However, in

many applications, it is not uncommon to have multiple subunits near the cutoff. Indeed, in the empirical application of Section 5.2, a typical upper-level unit with at least one close election on average has 10 of them. This might render the standard asymptotic regime an inaccurate approximation.

To bridge the gap, we consider two non-standard asymptotic regimes. The first regime is the "growing-J regime" where J_i grows with N. For example, if $J_i \sim 1/h$, we would expect that a non-vanishing fraction of upper-level units have multiple subunits near the cutoff. With $J_i \gg 1/h$, most of them do. The second regime is the "growing-correlation regime" where the correlation among running variables of subunits within the same unit tends to 1 at certain rate. When the correlation approaches 1 at a rate proportional to h, a back-of-envelope calculation suggests that, conditional on having at least one subunit near the cutoff, the probability of having more than one remains bounded away from zero (i.e., is constant). This regime can also be approximately viewed as a relaxation of the cluster RD setting in Bartalotti and Brummet (2017), where all running variables have the same signs. We focus on estimation bias, leaving variance analyses to future drafts.[13]

Growing-J regime. In this regime, we assume

$$J_i = O_{\mathbb{P}}(J_N), \quad \text{for some deterministic sequence } J_N \to \infty.$$

Furthermore, to ease the exposition, we need to assume several functions, such as $\mathbb{E}[J_i s_j \mid r_j = r] \cdot f(r)$, do not vary with N. We leave these details to Appendix A.6.1. Our key finding is that the $O(h^2)$ bias of both estimators is robust to $J_N = O(1/h)$—a scenario which allows a non-vanishing fraction of upper-level units to have more than one close election.

More specifically, consider the lower-level estimator first. Interestingly, the critical regime is $J_N \sim 1/h^3$, below which it continues to be consistent. Moreover, the bias remains second-order in h if $J_N = O(1/h)$.

Theorem 9. *Under the growing-J regime defined in Appendix A.6.1 and Assumptions 1, 2, and 4, as $h \to 0$, we have the following results.*

(a) If $J_N = o(1/h^3)$,

$$\tilde{\beta}_{\text{lo}} - \beta_0 = (J_N h^3)\Gamma_1^{\text{lo},\mathcal{J}} + h^2\Gamma_2^{\text{lo},\mathcal{J}} + O(h^3 + J_N h^4),$$

where $\Gamma_1^{\text{lo},\mathcal{J}}$ and $\Gamma_2^{\text{lo},\mathcal{J}}$ are defined in Appendix A.6.1. In particular, when $J_N = O(1/h)$,

$$\tilde{\beta}_{\text{lo}} - \beta_0 = O(h^2).$$

(b) If $J_N = \xi/h^3$ for some constant $\xi > 0$, $\tilde{\beta}_{\text{lo}} \neq \beta_0$ without further restrictions.

For the upper-level estimator, similar to the result in the standard asymptotic regime, the order of bias depends on the choice of kernel. With a uniform kernel, the upper-level estimator is equally robust to large J_i as the lower-level estimator. For other kernels, a non-vanishing bias emerges when $J_N \sim 1/h$.

Theorem 10. *In the setting of Theorem 9, we have the following result.*

(a) If $J_N = o(1/h)$,

$$\tilde{\beta}_{\text{up}} - \beta_0 = (J_N h)\Gamma_1^{\text{up},\mathcal{J}} + O(h + J_N^2 h^2),$$

[13] We note that, as always, these asymptotics should not be interpreted literally, as the data-generating process changing with N. Rather, they aim to provide a better asymptotic approximations in the presence of non-trivial clusters of close elections.

where $\Gamma_1^{\mathrm{up},\mathcal{J}}$ *is defined in Appendix A.6.1. If the condition* (21) *does not hold for* κ, $\Gamma_1^{\mathrm{up},\mathcal{J}} \neq 0$ *in general.*

(b) If $J_N = \xi/h$ *for some* $\xi > 0$, *then* $\tilde{\beta}_{\mathrm{up}} \neq \beta_0$ *without further restrictions.*

(c) If κ *satisfies the condition* (21) *(e.g., uniform kernel) and* $J_N = o(1/h^3)$,

$$\tilde{\beta}_{\mathrm{up}} - \beta_0 = O(h^2 + J_N h^3).$$

(d) If κ *satisfies the condition* (21) *and* $J_N = \xi/h^3$ *for some constant* $\xi > 0$, $\tilde{\beta}_{\mathrm{up}} \neq \beta_0$ *without further restrictions.*

Growing-correlation regime. To quantify the effect of approximately perfect correlation, we consider a specific regime where

$$r_j = \sqrt{1-\rho_N^2}\bar{R}_i + \rho_N R_j, \quad \bar{R}_i, R_1, \ldots, R_{J_i} \mid J_i \overset{i.i.d.}{\sim} N(0,1). \tag{23}$$

for some $\rho_N \to 0$. We primarily consider the critical regime

$$\rho_N = ch, \quad \text{for some constant } c \in (0,\infty).$$

In this regime, as long as one election for a unit is narrow, a non-vanishing fraction of others are narrow, too. Due to the complication of close-to-perfect correlation, we make additional structural assumptions, similar to (4):

$$\mathbb{E}\left[V_i \mid \boldsymbol{s}_i, \boldsymbol{r}_i\right] = \sum_{j \in \mathcal{J}_i} s_j v(r_j), \quad v(r) = v^+(r)I(r>0) + v^-(r)I(r<0),$$

where $(V, v) \in \{(X,x),(Y,y)\}$ and $v^{\pm}(\cdot) \in \mathcal{C}^2$ are some fixed functions that do not vary with N. For example, when $x^+(r) = 1, x^-(r) = 0, y^+(r) = r^2, y^-(r) = 0$,

$$X_i = \sum_{j \in \mathcal{J}_i} s_j I(r_j \geq 0), \quad Y_i = \sum_{j \in \mathcal{J}_i} s_j r_j^2 I(r_j > 0),$$

up to idiosyncratic noise. Lastly, we fix the distribution of $(s_1, \ldots, s_{J_i}) \mid (r_1, \ldots, r_{J_i}), J_i$.

Our key results here are that *(i)* the $O(h^2)$ bias of the lower-level estimator is robust in the critical regime only if a uniform kernel is used and *(ii)* the $O(h^2)$ bias of the upper-level estimator breaks down in the critical regime but persists when the correlation of running variables grows more slowly.

More specifically, we find that the lower-level estimator generally has a first-order bias, except when the uniform kernel is used, in which case a second-order bias is retained. Notably, the uniform kernel is "special" for the lower-level estimator in this regime for a reason that is fundamentally different from why it is special for the upper-level estimator under the standard asymptotic regime. In particular, other kernels that satisfy the condition (21) do not yield a second-order bias in this case.

Theorem 11. *Under the growing-correlation regime described above and Assumptions 1, 2, and 4, as* $h \to 0$, *we have the following results.*

(a) $\tilde{\beta}_{\mathrm{lo}} - \beta_0 = h\Gamma_1^{\mathrm{lo,cor}}(c) + O(h^2)$, *where* $\Gamma_1^{\mathrm{lo,cor}}(c)$ *is defined in Appendix A.6.2.*

(b) When κ is the uniform kernel, $\Gamma_1^{\text{lo,cor}} = 0$ and

$$\tilde{\beta}_{\text{lo}} - \beta_0 = O(h^2).$$

This result highlights that, although the lower-level specification looks like a standard fuzzy RD specification, the RDA structure, in which multiple subunits belong to the same unit, can change the asymptotic properties of the estimator.

For the upper-level estimator, the uniform kernel surprisingly does not improve the order of bias in this regime. Instead, all kernels produce a first-order bias. Thus, the upper-level estimator is comparable to lower-level estimator when a non-uniform kernel is used but less robust to high correlation of running variables otherwise.

Theorem 12. *In the setting of Theorem 11,*

$$\tilde{\beta}_{\text{up}} - \beta_0 = h\Gamma_1^{\text{up,cor}}(c) + O(h^2),$$

where $\Gamma_1^{\text{up,cor}}(c)$ is defined in Appendix A.6.2. Furthermore, when κ is the uniform kernel, $\Gamma_1^{\text{up,cor}}(c) \neq 0$ for any $c \in (0, \infty)$ without further restrictions.

Results in progress suggest that the $O(h^2)$ bias of the upper-level estimator with a uniform kernel is preserved if the correlation among running variables for the same unit grows more slowly: $\rho_N = O(h^{-1/3})$.

4 Extensions: Spillovers and Temporal Aggregation

So far, we have considered settings in which the outcome is measured at a higher level of cross-sectional (for instance., spatial) aggregation than the RD events. In this section, we briefly discuss two related settings to which our RDA ideas should extend. Section 4.1 considers estimation of spillovers in RD settings, while Section 4.2 focuses on aggregating RD events over time.

4.1 Spillovers from RD Events

A number of studies consider spillovers from discontinuity events. Instead of units and subunits, we now have outcome units i and intervention units j.[14] In spillover settings, the exogenous ("contextual") peer effects model is often used, in which the endogenous variable X_i is the number or fraction of treated peers. When the intervention is determined by a cutoff rule, the resulting X_i satisfies equation (1), with $\mathcal{J}_i$ reinterpreted as the set of i's peers and s_{ij} equal to one or $1/J_i$ for the number and fraction of treated peers, respectively. The X_i corresponding to the endogenous linear-in-means peer effects model, i.e. the average outcome of the peers (Manski, 1993), has a similar structure.

Both of our estimators extend straightforwardly to this setting. For the upper-level estimator, the RDA instrument is still defined by (6), with $\mathcal{C}_i$ reinterpreted as the set of i's peers with narrow elections. The lower-level estimator can now take two equivalent forms. The stacked equation can be defined at the bilateral level: on a sample of *pairs* $\{(i, j) \colon j \in \mathcal{C}_i\}$, estimate

$$Y_i = \beta X_i + \gamma' \tilde{W}_i + \gamma' q_j + \text{error}_{ij}, \tag{24}$$

[14] This terminology is common when analyzing experiments on bipartite networks (e.g., Zigler and Papadogeorgou (2021)).

with z_j as the IV for X_i and $s_j\kappa_h(r_j)$ as weights. This is still a standard fuzzy RD specification with a continuous treatment, albeit with both i and j potentially observed multiple times. Absent additional i-level controls, i.e. if $\tilde{W}_i = 0$, one can also obtain the same estimate $\hat{\beta}$ by collapsing (24) to the level of intervention units j with narrow elections. This entails averaging across outcome units that are j's neighbors: i.e., estimating

$$\bar{Y_j} = \beta \bar{X}_j + \gamma' q_j + \text{error}_j, \tag{25}$$

where $\bar{Y_j} = \frac{1}{N_j}\sum_{i\in\mathcal{I}_j} Y_i$ and $\bar{X}_j = \frac{1}{N_j}\sum_{i\in\mathcal{I}_j} X_i$ for $\mathcal{I}_j = \{i\colon j \in \mathcal{J}_i\}$ and $N_j = |\mathcal{I}_j|$, instrumenting $\bar{X}_j$ with z_j and using $s_j\kappa_h(r_j)N_j$ as weights.[15]

While the insights of our paper should apply in these settings, network dependence of the data may require technical adjustments to our theoretical results, which we leave to future work.

4.2 RD Aggregation over Time

Another setting in which RDA ideas can be useful is when multiple RD events are aggregated over time. For instance, one may specify a state-level treatment as the share of years in the last decade when a Democratic governor is in power (in a cross-section of states or a panel with a rolling window). Our upper-level estimator would entail instrumenting this treatment with the share of years in which the current governor is a Democrat who won in a narrow election, and controlling for the share of years determined by narrow elections, along with the other RDA controls. In turn, the lower-level estimator would involve stacking all narrow elections of the previous decade, repeating the same outcome and aggregated treatment, and including standard RD controls.

Our formal results apply directly if earlier elections do not impact later ones. However, with temporal aggregation, this can be a strong assumption: for instance, it precludes incumbency advantage in the electoral context (Lee, 2008). We conjecture that this assumption can be relaxed and our estimators continue to be valid in this case, so long as the exclusion restriction (3) is maintained.[16]

5 RDA in Practice

To illustrate the practical implications of the RDA approach, we first compare our two estimators to the current empirical practice. We then apply the approach to estimate the effects of lower unionization rates on U.S. earnings inequality.

5.1 Comparison to Current Practice

Many published articles feature designs suitable for RDA. We identified over 50 such papers from economics and political science listed in Appendix Table A1. In this section, we summarize how the empirical strategies employed by those articles differ from our proposed estimators. Appendix B provides a detailed paper-by-paper comparison.

[15] Note that $\bar{X}_j$ is a convoluted treatment: it is the average treatment X_i of j's neighbors where X_i itself is an average or sum of its neighbors' RD shocks. The first-stage captures that, when j is treated, the average treatment exposure X_i of j's neighbors mechanically goes up.

[16] An even more complex version of this setting is when elections are not regular and the previous election's result can affect whether election is held in a future year (Cellini et al. (2010), Jackson and Mackevicius (2024)). We leave the analysis of such settings to future work.

We classify papers using RD with aggregated outcomes into three main groups. The first group gives up available RD variation entirely, even if the same variation is simultaneously used for outcomes defined at the level of RD events. These papers employ difference-in-differences or similar designs instead. While the underlying parallel trends assumption is often hard to justify *ex ante*, continuity assumptions underlying conventional RD and inherited by RDA are usually more convincing, yielding more credible estimates.

Papers in the second—and most populous—group estimate some specification at the upper level, with the main difference relative to our upper-level estimator in the set of included covariates. Most papers include the first RDA control, which makes those estimators consistent. While some variations on the second RDA control have been proposed (e.g., the average of all running variables), we are not aware of any paper that includes the weighted sum of running variables from narrow elections only. Moreover, we are not aware of any variant on the third RDA control, with the exception of Cuñat et al. (2012) who include polynomials on the left and on the right separately that take as an input the aggregation of all running variables. This is opposite from how the RDA controls from Section 2.2 are constructed, by computing (local) polynomials of one running variable at a time first and aggregating them after. Absent the three RDA controls, the upper-level estimators from the existing literature do not have a second-order bias guarantee (recall Theorem 4).

Papers in the third group, which estimate specifications at the lower level, also diverge from our lower-level estimator in at least one of two ways. First, some of the papers throw out useful variation, e.g. by focusing on units with one narrow election only. Second, many of these papers estimate a reduced-form rather than IV regression, which suffers from both conceptual and practical issues discussed in the Appendix.

Appendix B further discusses empirical practice for estimating RD spillovers. We have identified groups of papers implementing specifications at the level of outcome units i, intervention units j, and the pair level (i, j). Still, with the exception of Dube et al. (2019), none of those specifications coincide with those proposed in Section 4.1, with issues resembling those arising in RD aggregation settings.

5.2 Application: The Impact of Unions on U.S. Inequality

We now apply the RDA identification strategy to estimate the effect of unions on wage inequality within state-by-industry cells in the U.S.. Section 5.2.1 introduces the empirical setting and adapts the RDA design to this application. Section 5.2.2 describes the data construction and presents summary statistics. Section 5.2.3 validates our research design in a series of balance tests. Section 5.2.4 presents the empirical results, and Section 5.2.5 explores the mechanisms behind them. Section 5.2.6 concludes by discussing what these results mean for the contribution of declining unions to growing inequality in the U.S..

5.2.1 Empirical Setting and Specifications

Trade unions are significant labor market institutions in all Western countries, including the U.S.. Over the last 100 years, union density in the U.S. has followed an inverse U-shaped pattern: there was a sharp increase in unionization during the Roosevelt era (1930–45), a relative steady state of high unionization rates from 1945 to 1960, and a continuous decline since the 1960s (Farber et al., 2021, Figure 1). The strong negative correlation between this trend and the U-shaped pattern of U.S. income inequality (e.g., Piketty et al., 2018) contributes to the widespread perception of a relationship between the two phenomena.

Recent studies have explored the relationship between unionization and inequality (Callaway and Collins, 2018; Collins and Niemesh, 2019; Fortin et al., 2021; Farber et al., 2021; Baker et al., 2025). Although these studies report a significant negative association between unions and inequality in a cross-section or a panel,

a causal interpretation for most of those results requires strong assumptions, as illustrated by the quotes in the Introduction. Notably, none of these studies adopted the RD approach that is common in the analyses of the union effects on establishment-level outcomes (DiNardo and Lee, 2004; Sojourner et al., 2015; Frandsen, 2021).

Unionization in the U.S. occurs at the level of a bargaining unit, defined as "a group of two or more employees who share community of interest and may reasonably be grouped together for purposes of collective bargaining" (National Labor Relations Board, 1997, p.12). In most cases, a bargaining unit corresponds to a single establishment, representing on average close to half of the establishment's workforce (Frandsen, 2021).[17] The election typically offers two choices: for or against union formation. To form a union, a majority of 50% plus one out of the cast votes is necessary, lending itself to an RD design.

We exploit the discontinuities from narrow union elections to estimate union effects at the more aggregate level of state-industry cells. Specifically, we define the outcome ΔY_{sit} as the change in an economic outcome (e.g., a measure of wage inequality) in state s and industry i during decade t,[18] and the endogenous treatment NewUnions_{sit} as the share of newly unionized workers. The latter is measured as the ratio of the number of workers in the region-industry cell who join unions through unionization elections (both narrow and not) during the decade and the employment in the cell at the beginning of the decade.

Looking at outcomes beyond the establishment level has two advantages. First, it allows us to study the effects on measures, such as inequality, that are typically defined at more aggregated levels. Second, even for outcomes—such us average wages—that can be measured by establishment, our estimates capture the effects due to the spillovers of unionization on other establishments within the same cell.[19] Defining cells as a combination of state and industry follows the approach of Fortin et al. (2021, 2022). Establishments in the same state and industry often compete for the same workers and therefore may adjust wages or even enter or exit the market in response to another establishment's unionization.[20]

Applying our upper-level IV solution from Section 2.2 to this setting, we estimate the following structural equation and first-stage:

$$\Delta Y_{sit} = \beta \text{NewUnions}_{sit} + \gamma' Q_{sit} + FE_{st} + FE_{it} + \text{error}_{sit}, \tag{26}$$

$$\text{NewUnions}_{sit} = \pi Z_{sit} + \rho' Q_{sit} + FE_{st} + FE_{it} + \text{error}_{sit}. \tag{27}$$

Here the instrument Z_{sit} is the employment share of the bargaining units unionized in a narrow election in the state-industry employment. Formally, $Z_{sit} = \sum_{j \in \mathcal{C}_{sit}} s_j z_j$ where $\mathcal{C}_{sit}$ is the set of close elections in the si

[17] To be precise, Frandsen (2021, Table 1) reports an average of 93 votes and 254 total employees in his sample covering most unionization elections between 1980–2009. Using data from all unionization elections in this period, we found an average turnout rate of 85% (as a fraction of eligible voters), suggesting a typical bargaining unit size of approximately 109 employees (93/0.85). This indicates that bargaining units represented about 43% (109/254) of the total workforce in establishments holding elections.

[18] We use differenced outcome variables both to increase estimation power and as a way to reduce potential bias from manipulated unionization election results, as suggested by Frandsen (2017, 2021). Additionally, using differenced outcome variables aligns with our treatment—the rate of new unionization—representing much of the variation in the *change* in the share of unionized workers.

[19] These estimates should be interpreted as the sum of direct and spillover effects. When establishment-level data are available, one can isolate them from each other via the RDA approach. Working with publicly available data, we leave this to future research.

[20] Aggregating the analysis to purely geographic units — either states (Farber et al., 2021) or economic areas similar to commuting zones (such as Census State Economic Areas in Collins and Niemesh (2019)) — could help capture cross-industry spillovers, but such aggregation is not feasible for our study. State-level aggregation would substantially reduce statistical power, as typical cells would contain many close elections, and the law of large numbers would eliminate most identifying variation in our RDA approaches. Meanwhile, data limitations do not allow us to achieve spatial resolution that is finer than states. For metropolitan areas, Census public files report geographic units less aggregated than states. However, both the number of these units and their boundaries change each decade, hindering the construction of panel data and the use of differenced outcomes.

cell during decade t (precisely defined below), $z_j = I(r_j > 0)$ is the dummy that the vote share for the union strictly exceeds 50% in workplace j, for r_j defined as the union vote share minus 0.5, and s_j is the number of eligible voters in election j as a fraction of the cell employment at the beginning of the decade.[21] Given the theoretical advantage of using the uniform kernel (Theorem 3), we employ it throughout the application. The covariates Q_{sit} consist of the three RDA controls defined by (7); in this context they represent the share of employment involved in narrow elections ($\sum_{j \in \mathcal{C}_{sit}} s_j$), a rescaled average of the running variable r_j in close elections ($\sum_{j \in \mathcal{C}_{sit}} s_j r_j$), and a rescaled average of the running variable in close wins ($\sum_{j \in \mathcal{C}_{sit}} s_j r_j^+$). Finally, FE_{st} and FE_{it} are state-by-decade and industry-by-decade fixed effects, respectively, included solely to increase estimation power, as the RDA design alone should be sufficient to obtain causal estimates.[22] We also explore specifications with different sets of fixed effects. To obtain more efficient and economically meaningful estimates, we weight each observation by the cell employment at the beginning of the decade.

We also apply the stacking approach of Section 2.2, which involves estimating the following structural equation and first-stage in the sample of narrow unionization elections j:

$$\Delta Y_{s(j)i(j)t(j)} = \beta \text{NewUnions}_{s(j)i(j)t(j)} + \gamma' q_j + FE_{s(j)t(j)} + FE_{i(j)t(j)} + \text{error}_j, \tag{28}$$

$$\text{NewUnions}_{s(j)i(j)t(j)} = \pi z_j + \rho' q_j + FE_{s(j)t(j)} + FE_{i(j)t(j)} + \text{error}_j. \tag{29}$$

where $s(j), i(j), t(j)$ respectively indicate the state, industry, and decade in which election j took place and q_j consist of the standard RD controls: a constant, the vote margin, and that margin interacted with the union getting a majority. The structural equation is identical to the upper level specification, except for the controls and the error term. We weight observations by the number of eligible workers in each workplace j, which equals the product of the cell-level importance weight (i.e., beginning-of-period employment) and s_j.

5.2.2 Data

We collect data on unionization and labor market outcomes by state, industry, and decade over five decades, from 1960s to 2000s.[23] The data on establishment-level union elections is from the National Labor Review Board (NLRB), maintained by Henry Farber and John-Paul Ferguson. For each election, we observe the state, industry, and year, as well as the total number of eligible workers and the votes cast for and against. We construct the treatment variable NewUnions_{sit} using the results of all elections. The instrument, in turn, aggregates only the elections that satisfy three criteria. First, it only includes close elections, defined by the vote share for the union of 50 ± 10 percent (the smallest bandwidth employed in recently published papers that employ the RD strategy in the context of U.S. unions; Knepper (2020); Frandsen (2021)).[24] Second, we exclude tied elections and elections with a 1-vote margin, as there is evidence of manipulation in

[21] We distinguish between winning and obtaining a majority of the votes, as in a very small fraction of cases the final result was revised due to an appeal process. Our setting is "fuzzy" in this sense, and we handle it similarly to the standard fuzzy RD design: the instrument and the third RD control, r_j^+, are defined based on crossing the 50% cutoff, while the treatment is based on actual union victory.

[22] We do not include state-by-industry fixed effects both because the outcome is already measured in differences and because including such fixed effects would result in a substantial reduction of the sample: our industry definitions change after 2000, leading to only one post-2000 state-by-industry observation (see Section 5.2.2).

[23] This period was chosen based on the availability of unionization data, which is accessible from 1961 onwards and contains industry variables until the beginning of 2009.

[24] The 10p.p. bandwidth also closely aligns with the optimal bandwidth for local linear estimation. Applying the Calonico et al. (2014) procedure to our lower-level specification across our main six inequality outcomes, we find optimal bandwidths ranging from 9.1% to 13.9%, with an average of 11.2%. We also check robustness to using a 15% bandwidth which is also used by Knepper (2020) and Frandsen (2021).

such elections (Frandsen, 2021). Finally, we drop elections with less than 20 votes, as is customary in the unionization literature (DiNardo and Lee, 2004; Lee and Mas, 2012; Frandsen, 2021).[25]

Our wage inequality measures and other wage-related outcomes by state, industry, and year are based on the decennial population census samples from 1960 to 2000, supplemented by the American Community Survey for 2009–2013 that represents 2010. With some variation over the years, these samples represent roughly 5% of the U.S. population. We include observations from all 50 states, as well as from the District of Columbia, defined as an additional state. We only include workers with a strong attachment to the labor market, defined as working 50-52 weeks during the year and at least 20 hours per week on average.[26] Two of our main inequality outcomes are computed using total labor income: the Gini index and the top-10% income share. For the other measures, we calculate the average annual hourly pre-tax wage of each worker by dividing the total annual wage income by the product of the number of weeks employed and the average hours worked per week last year. We then compute the college wage premium, the log 90-10 wage ratio, the variance of log wages,and the variance of residual log wages from a Mincer (1974) regression controlling for education, experience, race, and gender.[27] The construction of the inequality measures, which follows Lemieux (2006), Autor et al. (2008), and Farber et al. (2021) with minor modifications, is detailed in the Data Appendix (Appendix C). For the inequality measures to be meaningful, we restrict the sample to include only cells with more than 1,000 workers after applying census projection weights.[28]

We additionally use the Current Population Survey (CPS) Monthly and Annual Social and Economic Supplement samples. We measure union density in each cell and year based on the self-reported union coverage question. We also construct two measures of worker benefits: indicators of a pension plan and employer-sponsored health insurance coverage. These questions are available from 1977–2014; to increase sample size we combine CPS data from several years to represent each decennial census year, as detailed in the Data Appendix. The cleaning process for the CPS data is otherwise the same as for the census data.

A challenge in the data construction process was to harmonize industry definitions over a long period and between different data sources. The NLRB data is based on the SIC and NAICS schemes, depending on the year, while the population census, ACS, and CPS use a different classification. We created two distinct coding schemes. Until 2000, our industry coding is based on the initial two digits of the SIC87 codes; from 2000 onwards, it relies on the initial three digits of the NAICS97 industry classification. In both cases, we merged a few categories, resulting in 70 industries categories pre-2000 and 73 (different) categories post-2000 that cover all private sectors.[29] To calculate differenced outcomes for each decade, we estimate all measures for the year 2000 based on both schemes.[30] In total, we were able to match 93% of unionization elections to

[25] During the period of our analysis, there are two types of unionization elections: certification elections, where workers attempt to form a new union, and decertification elections, where workers or employers try to dissolve an existing union. The rules for both types of elections are identical, as are the consequences: if the union receives a strict majority of votes, it will be certified or re-certified; if not, it will be decertified. The vast majority of elections are certification elections, accounting for 86.5% of elections and covering 89.7% of the workers. We do not distinguish between these two types of elections in our analysis; e.g., $NewUnions_{sit}$ counts workers in both newly certified and re-certified unions. Our results are robust to restricting the set of elections used for the treatment and instrument construction to certification elections only.

[26] We exclude from the sample workers who worked fewer than 50 weeks, as they have a higher likelihood of having switched industries during the past year, in which case we could attribute their annual income to the wrong industry.

[27] Residualization is done in each decade separately. Education is measured as five groups (less than high school, high school, some college, college, and more than college); experience enters as a quartic polynomial.

[28] This entails dropping 23.4% of the cells, but only 0.8% of total employment and 1.5% of close unionization elections. We verify there are no remaining cells with fewer than 10 raw observations. For measures based on smaller samples, such as for the average college wage, we further exclude cells with fewer than 10 relevant observations at the beginning or at the end of the decade.

[29] The NLRB data do not cover public sector unionization and, thus, those industries are excluded from our analysis. Our classification is more detailed than 10–11 industries used by Fortin et al. (2021, 2022).

[30] Since the 2000 census industry question was based on the NAICS industry scheme, we relied on the IPUMS crosswalk to match observations to the SIC-based census industries. While this could introduce noise in the 1990–2000 differences, we find it

our industry codes and 99% of individual census observations.

Table 1 presents summary statistics for the key variables at the state-industry-census year level, both in levels and first-differences. While columns 1 and 2 weight cells by beginning-of-period employment (representing the full sample), columns 3 and 4 focus on the population of compliers (representing the effective sample of our RDA estimates).[31] Panel A includes outcomes, while Panel B details other cell-level statistics. One can see, in particular, that all within-cell inequality measures show an increasing trend, mirroring national inequality trends. Comparing complier-weighted to standard averages reveals that cells in the effective sample of RDA are substantially smaller than in the full sample (average log employment of 9.6 versus 10.8, corresponding to 15,000 vs. 50,000 workers), have a higher share of male workers (67% versus 59%), a lower share of college-educated workers (13% versus 23%), and are heavily concentrated in manufacturing (74% versus 28%).

Panel C reports summary statistics for election-based variables. In an average cell and decade, there are 57 elections in which 5.8% of the workers are involved and $\bar{X} = 2.6\%$ unionize. Out of these elections, 8.4 are narrow, involving 1.6% workers and leading to $\bar{Z} = 0.7\%$ of workers unionizing. Unionization activity is almost an order of magnitude larger in the complier-weighted sample: while the average number of elections is slightly smaller (at 41), the share of workers involved in elections is 40.6%, with 15.4% unionizing. Narrow elections in this sample involve 21.4% of workers, leading to 7.8% unionizing. Appendix Tables A2 and A3 provide additional descriptive statistics at the cell and election levels, respectively. They show in particular that the complier-weighted sample is concentrated in the 1960s and 1970s when unionization activity was substantially higher, as also seen in the declining number of elections since 1970s and consistent with the negative trend in union membership reported in Panel A of Table 1.

While we only look at inequality within state-industry cells, this type of inequality is central to both the level and growth in overall inequality. To show this, Appendix Table A4 decomposes the overall variance of log wages into within- and between-cell components in each census year, broadly following Helpman et al. (2016). Throughout the period, the within component of the variance is the dominant one, accounting for 73–85% of total inequality. Moreover, the within component experienced a continuous rise, contributing significantly to the overall increase in inequality. In contrast, the between-group component remained relatively stable, decreasing in some periods and only experiencing a notable rise from 2000 to 2010.[32]

5.2.3 Balance

We now perform balance tests to illustrate how the RDA instrument and controls help isolate idiosyncratic variation in the treatment. Column 1 of Table 2 regresses the share of all newly unionized workers (i.e., our treatment of interest) on pre-determined covariates and lagged inequality measures from Panels A and B of Table 1 (excluding those that are not available for all decades). The treatment is correlated with many observables, as evidenced by the partial R^2 of 16.6% and a decisively rejecting F-test, which suggests that the treatment is also likely correlated with unobservables. In column 2 we estimate the same regression for

reassuring that employment changes do not exhibit substantially higher volatility than in the decade before or after.

[31] Following the characterization of RDA compliers in Section 3.2, the complier weight is proportional to beginning-of-period employment times $\sum_{j \in \mathcal{C}_{sit}} s_j^2$, capturing the fact that with more close elections and more concentrated close-election employment contribute more to the average of treatment effects estimated by RDA.

[32] A concern in this analysis is that some of the changes in the variance components are due to the changes in the definitions of industries in the year 2000 from SIC based to NAICS based. To address this, we conducted an identical analysis on with annual CPS samples in Appendix Figure A1. It indicates no discernible changes in the within and between components between 2002 and 2003, the year of the CPS industry scheme change, providing strong evidence that this transition does not drive the trends in the components. This analysis also replicates the findings from Appendix Table A4 using different data, confirming the key role of within-cell inequality.

Table 1: Summary Statistics

Variable	Mean	Mean(Δ)	Complier Mean	Complier Mean(Δ)	Data Source	Time Range
Panel A: Outcomes						
Log avg hourly wage	2.640	0.022	2.614	0.050	Census	1960-2010
	(0.273)	(0.111)	(0.224)	(0.116)		
Log college premium	0.505	0.051	0.454	0.017	Census	1960-2010
	(0.162)	(0.108)	(0.163)	(0.151)		
Log 90/10 ratio	1.325	0.074	1.140	0.053	Census	1960-2010
	(0.210)	(0.108)	(0.183)	(0.131)		
Gini index	0.354	0.025	0.290	0.018	Census	1960-2010
	(0.069)	(0.032)	(0.055)	(0.037)		
Top 10% share	0.287	0.018	0.240	0.011	Census	1960-2010
	(0.058)	(0.033)	(0.042)	(0.035)		
Var(log hourly wage)	0.316	0.034	0.232	0.018	Census	1960-2010
	(0.097)	(0.048)	(0.070)	(0.053)		
Share of union members	0.102	-0.043	0.142	-0.064	CPS	1980-2010
	(0.128)	(0.078)	(0.131)	(0.124)		
Pension coverage share	0.511	-0.012	0.592	0.013	CPS	1980-2010
	(0.189)	(0.117)	(0.171)	(0.164)		
Employer-sponsored insurance share	0.672	-0.058	0.756	-0.063	CPS	1980-2010
	(0.161)	(0.083)	(0.134)	(0.114)		
Panel B: Other Cell-Level Measures						
Log employment	10.823	0.164	9.582	0.253	Census	1960-2010
	(1.231)	(0.311)	(1.111)	(0.366)		
Share with college degree	0.226	0.041	0.128	0.027	Census	1960-2010
	(0.162)	(0.040)	(0.115)	(0.036)		
Share white	0.836	-0.026	0.866	-0.024	Census	1960-2010
	(0.119)	(0.042)	(0.123)	(0.047)		
Share male	0.592	-0.014	0.671	-0.037	Census	1960-2010
	(0.224)	(0.041)	(0.189)	(0.054)		
I (Manufacturing)	0.277		0.741		Census	1960-2010
	(0.448)		(0.438)			
Panel C: RDA Variables (bandwidth=10%)						
$100 \times \text{NewUnions}_{sit}$	2.595		15.357		NLRB	1960-2010
(Share newly unionized)	(5.172)		(14.403)			
$100 \times \text{Attempts}_{sit}$	5.846		40.638		NLRB	1960-2010
(Share involved in elections)	(9.716)		(27.523)			
$100 \times Z_{sit}$	0.663		7.832		NLRB	1960-2010
(Share newly unionized in close elections)	(1.642)		(9.738)			
$100 \times \sum_{j \in \mathcal{C}_{sit}} s_j$	1.602		21.385		NLRB	1960-2010
(Share involved in close elections)	(3.438)		(17.444)			
$100 \times \sum_{j \in \mathcal{C}_{sit}} s_j r_j$	-0.018		-0.421		NLRB	1960-2010
(Agg. vote share in close elections)	(0.121)		(1.074)			
$100 \times \sum_{j \in \mathcal{C}_{sit}} s_j r_j^+$	0.034		0.379		NLRB	1960-2010
(Agg. vote share in close wins)	(0.085)		(0.530)			
# of unionization elections	56.989		41.188		NLRB	1960-2010
	(85.498)		(57.099)			
# of close unionization elections	8.437		9.289		NLRB	1960-2010
	(14.946)		(11.004)			

Notes: The first column displays the averages and (in parentheses) standard deviations of the end-of-decade levels of the variables, while the second column reports these statistics for first differences over the decade. The statistics are weighted by employment at the beginning of the decade. The third and fourth columns report analogous statistics using complier weights, defined as beginning-of-period employment times $\sum_{j \in \mathcal{C}_{sit}} s_j^2$ where s_j is the employment share of workplace j in the cell and the sum is over close elections; these weights characterize the effective sample driving our RDA estimates. Observations are state-industry cells based on SIC industry codes pre-2000 and NAICS codes thereafter. Panel A summarizes the outcome variables used in the analysis. Panel B provides additional covariates. Panel C presents the RDA variables (scaled by a factor of 100) along with the average numbers of all and close unionization elections within each cell. "Close elections" refer to those used to construct the instrument and RDA controls: with a vote share for unionization of 40-60%, at least 20 eligible voters, and a vote margin of 2 votes or more. The sample is restricted to cells with more than 1,000 workers, using Census projection weights, both at the beginning and at the end of the decade.

Sources: Decennial Census, CPS Annual and Monthly Social and Economic Supplement, NLRB data, and authors' calculations.

Table 2: Instrument Balance

	NewUnions	Z		
	(1)	(2)	(3)	(4)
Log avg hourly wage$_{t-1}$	0.613	-0.004	-0.151	0.005
	(0.563)	(0.109)	(0.064)	(0.027)
Log college premium$_{t-1}$	3.285	1.548	0.064	-0.006
	(0.406)	(0.134)	(0.079)	(0.034)
Log 90/10 ratio$_{t-1}$	3.516	0.790	-0.048	0.046
	(0.668)	(0.204)	(0.110)	(0.053)
Gini index$_{t-1}$	-66.693	-16.444	-0.837	0.208
	(6.288)	(1.480)	(0.859)	(0.387)
Top 10% share$_{t-1}$	48.512	10.937	0.415	-0.140
	(5.997)	(1.337)	(0.780)	(0.359)
Var(log hourly wage)$_{t-1}$	-8.299	-2.211	-0.011	-0.183
	(1.499)	(0.302)	(0.179)	(0.087)
Log employment$_{t-1}$	-0.149	-0.060	0.003	-0.004
	(0.054)	(0.016)	(0.009)	(0.004)
Share with college degree$_{t-1}$	-0.960	0.051	0.332	0.092
	(0.744)	(0.184)	(0.102)	(0.050)
Share white$_{t-1}$	-1.779	-0.458	-0.314	-0.024
	(0.577)	(0.183)	(0.100)	(0.041)
Share male$_{t-1}$	-0.812	-0.326	0.017	0.033
	(0.514)	(0.124)	(0.064)	(0.026)
I (Manufacturing)	1.900	0.749	-0.053	-0.009
	(0.197)	(0.049)	(0.030)	(0.013)
# significant coefficients	9	9	4	1
Partial R^2	0.166	0.129	0.001	0.000
Partial F-statistic	266.5	177.6	4.1	1.2
Partial F-test, p-value	0.000	0.000	0.000	0.252
RDA controls	None	None	$\sum_{j\in\mathcal{C}_{sit}} s_j$	All
Observations	12,459	12,459	12,459	12,459

Notes: Balance tests based on OLS estimates weighted by employment at the beginning of the decade. The outcome variable in the first column is the share of all newly unionized workers, while the next three columns focus on shares joining through close elections, with a bandwidth of 10% applied to define close elections. Column 3 additionally incorporates the first RDA control, $\sum_{j\in\mathcal{C}_{sit}} s_j$, and column 4 adds the remaining RDA variables, $\sum_{j\in\mathcal{C}_{sit}} s_j r_j$ and $\sum_{j\in\mathcal{C}_{sit}} s_j r_j^+$. Heteroskedasticity-robust standard errors are shown in parentheses.
Sources: Decennial Census, NLRB data, and authors' calculations.

our instrument: the share of workers who join a union through close elections; the conclusion is unchanged. In column 3, the inclusion of the first RDA control that represents the share of workers involved in narrow union elections vastly changes the statistics. The partial R^2 decreases massively to 0.1%, the F-statistic of the covariates' joint significance shrinks from 177.6 to 4.1, and most coefficients become orders of magnitude smaller. Yet, covariates are still jointly significant. When we finally add the two other RDA controls to the regression in column 4, the partial R^2 becomes essentially zero and, while one out of eleven covariates is individually significant, jointly they are not (p-value of 25.2%). The lack of correlation between the residual variation in our instrument and the observables speaks in favor of the validity of our IV strategy. In Appendix Table A5 we replicate this analysis with a 15% bandwidth defining close elections, and the same patterns emerge.

Appendix Table A6 performs a related pre-trend and placebo analysis, using predetermined covariates and lagged outcomes, both in levels and differences, as dependent variables when estimating the reduced-form

of our upper-level IV specification:

$$Y_{sit} = \rho Z_{sit} + \gamma' Q_{sit} + FE_{st} + FE_{it} + \text{error}_{sit}. \tag{30}$$

Out of the 15 coefficients, only the lagged ΔLog 90/10 coefficient is marginally significant at the 5% level when using conventional standard errors and none when employing robust bias-corrected confidence intervals of Calonico et al. (2014).

5.2.4 Baseline Results: Effects on Earnings Inequality

Table 3 examines the impact of unionization on six measures of income inequality: the college premium, the 90-10 wage ratio, the Gini coefficient, the top-10% income share, the variance of log hourly wages and the variance of residual log wages from a Mincer (1974) regression controlling for education, experience, race, and gender. Panel A shows the estimates using the upper-level approach, while Panel B shows the coefficients obtained at the lower level. Along with each estimate, we report robust bias-corrected confidence intervals following Calonico et al. (2014), obtained using the `RDrobust` package in `R`.[33]

The results reveal a negative effect of unionization on all measures of inequality, with 95% confidence intervals rejecting zero effects in columns 3–5 and with large but statistically insignificant negative effects for the college premium and the 90-10 wage ratio. Specifically, upper-level estimates suggest that an increase of 1 percentage point in new unionization reduces the college premium by 0.42 log-points, the 90-10 ratio by 0.50 log-points, the Gini coefficient by 0.0017, the top-10 income share by 0.14 percentage points, and the variance of log wages by 0.0026 (which is around 1% of its mean). Lower-level estimates are moderately smaller in magnitude but show the same statistical significance. Figure 1 visualizes these results with standard RD plots for the first-stage and reduced-form lower-level specifications (excluding the fixed effects to stay closer to the raw data).

Appendix Table A7 reports several robustness tests for these results, focusing on the upper-level analyses. First, we replace our baseline bandwidth of 10% with 15% bandwidth, a common *ad hoc* choice in the literature (e.g., Frandsen (2021); Knepper (2020)). Second, we exclude union decertification elections from the construction of the treatment, instrument, and RDA controls. Third, we replace state-year and industry-year fixed effects (FEs) with separate state, industry, and year FEs, or with no FEs at all. Fourth, we perform estimation without initial employment weights. Fifth, we exclude cells with very high unionization activity, defined as more than 25% of cell employees taking part in unionization attempts. Most of the results are similar to those from Table 3. Finally, we estimate the RDA version of a local constant regression that does not include the second and third RDA controls, i.e. aggregates of the running variables, as commonly done in the literature. The coefficients remain negative but are only marginally significant and much smaller, illustrating substantial bias of such specifications (recall Theorem 4).

We also explore effect heterogeneity, with the caveat that statistical power of these comparisons is limited. Appendix Table A8 splits the data by decade. While estimates are negative for all inequality measures in each decade, they are smaller in magnitude in the 1960s–1970s relative to the pooled estimates, attenuated further in the 1980s, and large in the 1990s.[34] Appendix Table A9 splits the data into manufacturing versus

[33] To obtain confidence intervals for our upper-level approach, we use the numerically equivalent lower-level specification, per Proposition 1. We do not use clustered inference (e.g., by state or by industry) following Theorem 8. We confirmed that clustered standard errors do not substantially exceed heteroskedasticity-robust ones.

[34] The effects are very large but very noisy in the 2000s as unionization activity declined. In unreported results we have verified that excluding this decade does not materially change the pooled estimates.

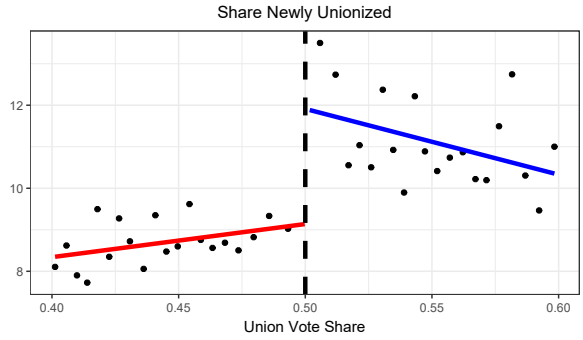
Table 3: The Impact of New Unionization on 10-Year Changes in Inequality

	ΔLog college premium (1)	ΔLog 90/10 (2)	ΔGini coeff. (3)	ΔTop 10% share (4)	ΔVar(log wage) (5)	ΔVar(res. log wage) (6)
Panel A: Upper-Level Estimator						
Share newly unionized	-0.423	-0.496	-0.169	-0.138	-0.263	-0.116
	(0.295)	(0.243)	(0.064)	(0.058)	(0.092)	(0.065)
	[-1.073,0.293]	[-1.057,0.103]	[-0.319,-0.013]	[-0.275,-0.000]	[-0.492,-0.042]	[-0.278,0.033]
Mean outcome	0.510	1.325	0.354	0.287	0.316	0.227
Mean Δoutcome	0.051	0.074	0.025	0.018	0.034	0.023
Mean treatment	2.51%	2.59%	2.59%	2.59%	2.59%	2.59%
RDA controls	✓	✓	✓	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓	✓	✓	✓
Observations	10,019	12,555	12,555	12,555	12,555	12,555
Panel B: Lower-Level Estimator						
Share newly unionized	-0.277	-0.277	-0.111	-0.094	-0.189	-0.085
	(0.270)	(0.176)	(0.049)	(0.046)	(0.069)	(0.049)
	[-0.920,0.362]	[-0.735,0.120]	[-0.250,-0.011]	[-0.221,0.000]	[-0.398,-0.053]	[-0.228,0.016]
Mean outcome	0.486	1.191	0.307	0.252	0.255	0.177
Mean Δoutcome	0.031	0.073	0.023	0.015	0.028	0.017
Mean treatment	9.14%	9.69%	9.69%	9.69%	9.69%	9.69%
RD controls	✓	✓	✓	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓	✓	✓	✓
Observations	29,183	31,313	31,313	31,313	31,313	31,313

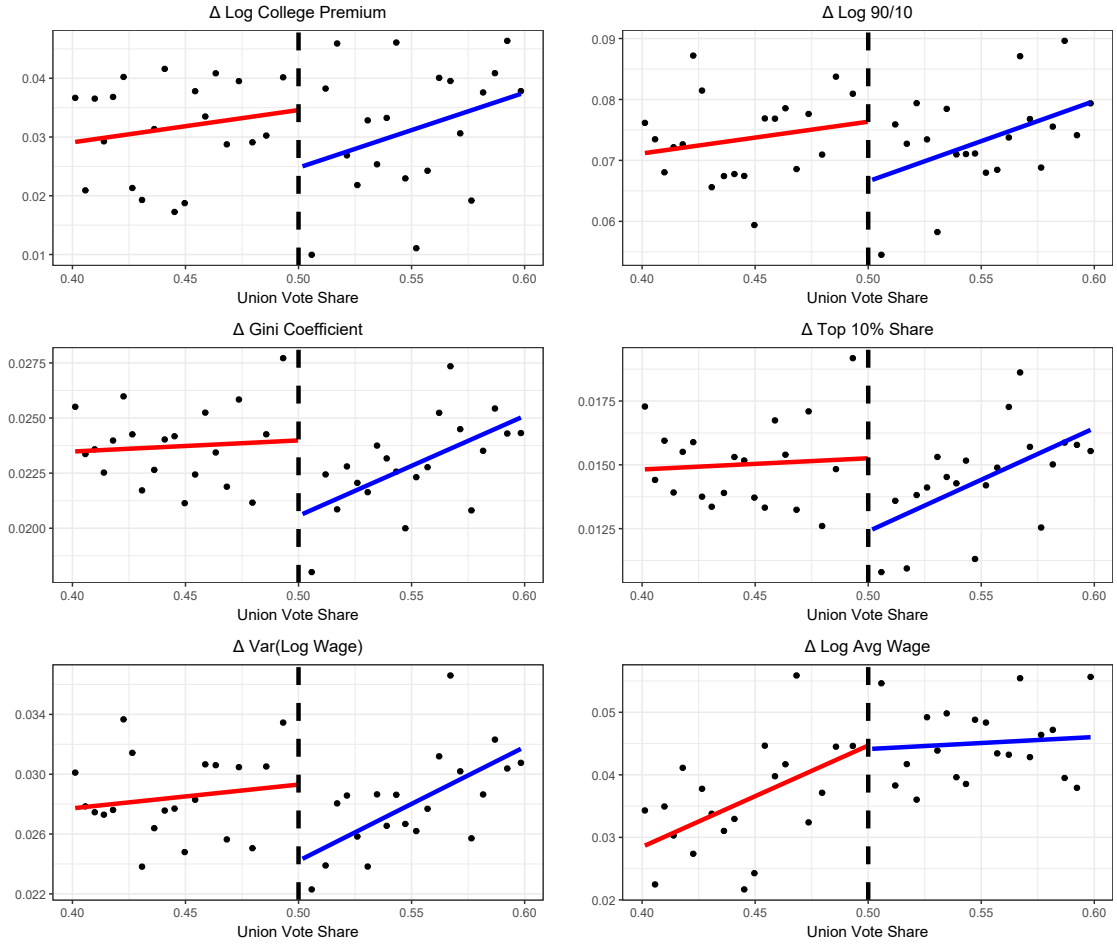
Notes: Panel A reports IV estimates of (26). Observations are defined as state-industry cells by decade (for 1960–2010). The endogenous variable is the share of workers unionized through unionization elections, relative to the beginning-of-decade cell employment. The instrument is the share of employment unionized through close elections, defined by the 10% bandwidth for the vote share around the cutoff of 50%. See Section 5.2.2 and the Data Appendix for outcome definitions and details of the sample construction. Observations are weighted by employment at the beginning of the decade. Panel B shows the coefficients obtained from the lower-level IV specification (28), in the sample of close elections. The specification includes standard RD controls and is weighted by the number of eligible workers. In both panels, heteroskedasticity-robust standard errors are in parentheses, and robust bias-corrected 95% confidence intervals of Calonico et al. (2014) are in brackets (in Panel A, constructed via the equivalent lower-level specification as in Proposition 1).
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Figure 1: Lower-Level Specifications: RD Plots

Panel A: First Stage



Panel B: Reduced Form



Notes: The first plot visualizes the first-stage specification (29) with no fixed effects. The other plots visualize the reduced-form version of (28) with no fixed effects for the outcomes included in columns 1–5 of Table 3 and column 1 of Table 5. The sample, weights, and outcome definitions are the same as in Panel B of the relevant tables. The x-axis represents the union vote share in a close union election, while the y-axis shows the outcome in the corresponding region-industry cell and year. The binned scatterplot shows 20 bins on each side of the cutoff, with each bin representing the average values for elections within specific ranges of the running variable. These ranges were selected so that, on each side of the cutoff, each point represents observations with the same sum of weights. The lines correspond to the local linear estimates. "Δ" stands for the first difference over a decade.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table 4: The Impact of New Unionization on 10-Year Changes in Union Density and Fringe Benefits

	Union density (1)	Pension coverage rate (2)	Health insurance coverage rate (3)
Panel A: Upper-Level Estimator			
Share newly unionized	1.059	1.814	0.279
	(0.654)	(0.801)	(0.503)
	[-0.386,2.599]	[-0.063,3.542]	[-0.866,1.234]
Mean outcome	0.098	0.511	0.672
Mean Δoutcome	-0.043	-0.012	-0.058
Mean treatment	1.08%	1.40%	1.40%
RDA controls	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓
Observations	7,860	6,461	6,505
Panel B: Lower-Level Estimator			
Share newly unionized	0.427	0.898	0.337
	(0.324)	(0.494)	(0.323)
	[-0.315,1.254]	[-0.215,2.150]	[-0.449,1.014]
Mean outcome	0.154	0.569	0.737
Mean Δoutcome	-0.066	-0.002	-0.063
Mean treatment	3.90%	4.42%	4.46%
RDA controls	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓
Observations	14,084	13,677	13,705

Notes: This table reports IV estimates from the same specifications as in Table 3 but for different outcomes: union density, pension coverage rate, and employment-sponsored insurance coverage rate. The outcomes are based on the CPS data and measured for the period from 1980–2010.
Sources: Decennial Census, CPS Annual and Monthly Social and Economic Supplement, NLRB unionization data; authors' calculations.

other industries and into right-to-work and other states (Holmes, 1998). All estimates are negative, with no notable differences by the right-to-work status. The effects in non-manufacturing industries are 2-5 times larger than those in manufacturing, though the non-manufacturing sample is much smaller and the estimators are much noisier. Smaller effects in manufacturing could perhaps be explained by weaker union power, as employers can relocate production to non-unionized establishments.

While we specified the treatment variable to be the rate of *new* unionization, the results would also be similar if we considered changes in union density, i.e. in the overall share of unionized workers. The latter measure would take into account differential employment growth of unionized and non-unionized establishments, as well as establishment entry and exit. Unfortunately, union density is only observed in the CPS since 1977. Thus, in the spirit of the first-stage for two-sample IV (Angrist and Pischke, 2009, Ch.4.3), we estimate the effect of new unionization rates (from the NLRB data) on union density (from the CPS monthly samples) for the shorter period of 1980–2010.[35] The coefficient is reported in the first column of Table 4. This coefficient may be smaller than one due to various types of measurement error: both on the right-hand side (discrepancies arising from out-of-state workers, errors in coding the industry of unionization,

[35]We informally think of this analysis as a first-stage, except that this is an IV specification leveraging close elections for the instrument as before.

or lower employment growth in unionized establishments) and on the left-hand side (misclassification of the union coverage variable in the CPS leading to a non-classical measurement error).[36] Conversely, if unionized workplaces have higher employment growth, the first-stage parameter might exceed one. In the data, we find estimates ranging from 0.43 to 1.06 (with a smaller lower-level estimate where the Table 3 coefficients are correspondingly smaller), suggesting that our main IV estimates are robust to using the change in the self-reported share of total unionized workers as the endogenous variable.

We next investigate what drives the negative union effects on inequality.

5.2.5 Mechanisms

Panel A of Figure 2 investigates the sources of the inequality reduction due to unionization, finding that it originates primarily from the *decline* in earnings among high-earners, rather than an increase at the bottom of the earnings distribution. Specifically, we report union effects across the earnings distribution: for each of twenty equally-sized quantile bins, it uses as the outcome the change in the average log wage within the bin. The effect of unionization is close to zero in the bottom half of the wage distribution, moderately negative between the 50th and 90th percentiles, and then declines sharply for the highest earners, with large and statistically significant negative effects concentrated in the top-5% bin. Consistent with these patterns, column 1 of Table 5 reports a negative, albeit marginally insignificant, effect on the average log wage.

The negative wage effects are puzzling in light of standard bargaining models that predict a positive union wage premium (Freeman and Medoff, 1984) and extensive empirical evidence supporting such effects (Card, 1996; Farber et al., 2021; Naidu and Schoefer, 2024). We next examine four mechanisms that may reconcile our findings with this prior evidence: wage drop for managers, changes in workforce composition, increases in fringe benefits, and dynamics of the effects.

Columns 2 and 3 of Table 5 reveals that much of the negative wage effect is driven by managers. Column 2 reports a sizable, albeit statistically insignificant, negative coefficient on managers' wages (-0.38 to -0.50), while Column 3 shows that the effect on average wages falls by 40% when managers are excluded. This pattern is consistent with unions being legally prohibited from representing managerial workers. We find similar effects for college vs. non-college workers wages in columns 4 and 5, potentially explained by differences in union representation, too.[37]

To understand how unionization affects the composition of workers — our second mechanism — Appendix Table A11 shows that unionization has unequal effects on employment across worker groups with different earnings potential. There is no clear negative effect on overall employment, nor on the employment of high-school graduates. In contrast, employment of workers with a college degree or more declines substantially in response to unionization. Similarly, employment effects are more negative for managers than non-managers, albeit with a smaller difference.

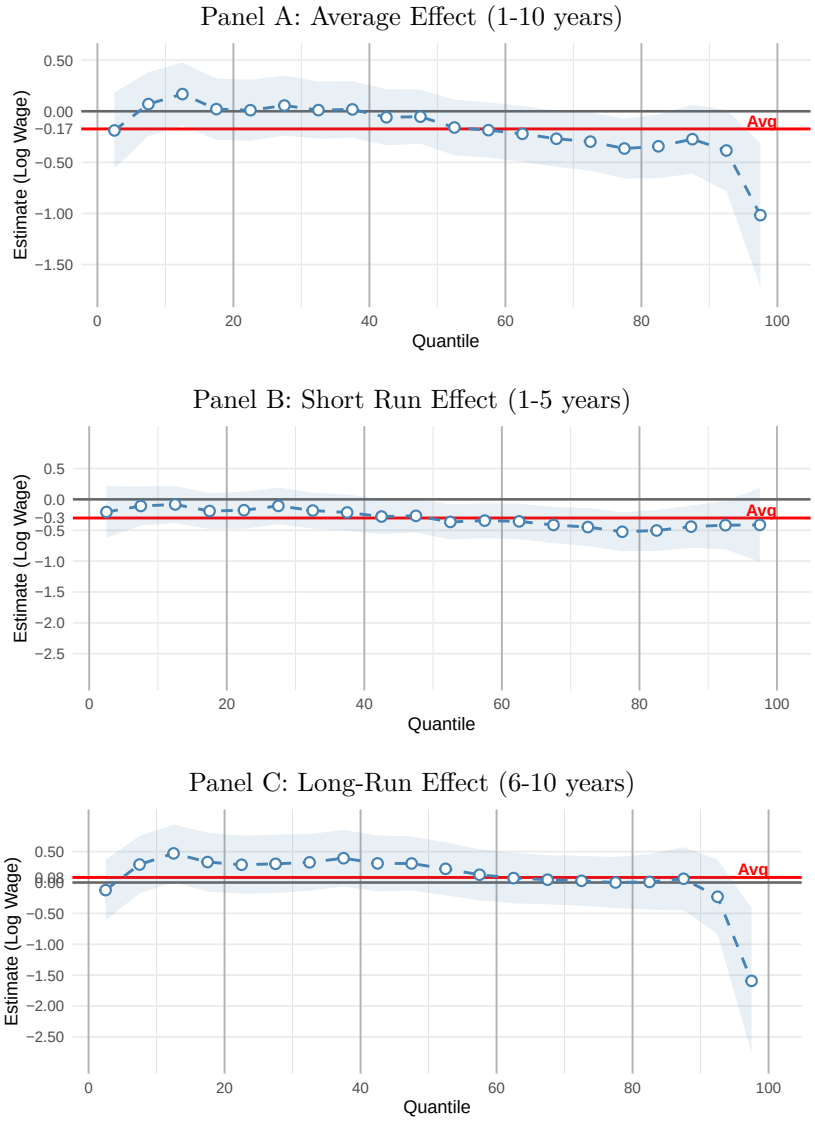
These compositional differences can account for some — but not all — of the negative wage impact of unions on inequality. This can be seen in Column 6 of Table 3 which estimates the impacts on the variance of log wages residualized on education, race, gender, an experience. The negative impact roughly halves relative to the effect on the variance of non-residualized log wages in column 5.

A third possible explanation for the negative effects on average wages is substitution toward fringe benefits. We examine this by estimating the effects of unionization on health insurance and pension coverage in columns

[36] Based on a 1977 validation survey, the estimate for the misclassification rate in the union variable is 2.7% (Card, 1996).

[37] Appendix Table A10 conducts several robustness checks for these results. The overall conclusion of a negative effect on average wages driven by managers and high earners is preserved (with the single exception of managers vs. non-managers in the specification without any fixed effects).

Figure 2: The Impact of New Unionization on 10-Year Changes of the Wage Distribution



Notes: Panel A reports upper-level IV estimates of the effect of new unionization on wages at different segments of the within-cell wage distribution. Each point represents a coefficient from specification (26), with the outcome defined as the decadal change in average log wages in a given 5-percentile range (0–5, 5–10, ..., 95–100). The shaded area indicates 95% confidence intervals. Panel B reports the short-run effect (1–5 years), by restricting variable definitions to close elections in years ending 5–9 in each decade. Panel C reports the long-run effect (6–10 years), by using elections from years ending 0–4. The horizontal red lines show the pooled effects estimated from column 1 of Table 5, Panel A, and column 1 of Appendix Table A13, Panels A and B.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table 5: The Impact of Unions on 10-Year Average Log Wage Changes

	All workers	Mgmt	Non-Mgmt	College	High school	All workers (residualized)
	(1)	(2)	(3)	(4)	(5)	(6)
Panel A: Upper-Level Estimator						
Share newly unionized	-0.173	-0.378	-0.103	-0.271	-0.063	-0.095
	(0.119)	(0.320)	(0.120)	(0.292)	(0.138)	(0.098)
	[-0.505,0.082]	[-1.129,0.416]	[-0.433,0.150]	[-0.915,0.489]	[-0.401,0.252]	[-0.368,0.119]
Mean outcome	2.640	3.017	2.582	3.023	2.498	0.013
Mean Δoutcome	0.022	0.041	0.015	0.013	-0.010	0.018
Mean treatment	2.59%	2.58%	2.59%	2.51%	2.60%	2.59%
RDA controls	✓	✓	✓	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓	✓	✓	✓
Observations	12,555	10,362	12,555	10,248	12,523	12,555
Panel B: Lower-Level Estimator						
Share newly unionized	-0.161	-0.496	-0.092	-0.242	-0.086	-0.079
	(0.099)	(0.288)	(0.098)	(0.286)	(0.107)	(0.082)
	[-0.466,0.007]	[-1.225,0.133]	[-0.389,0.075]	[-0.898,0.416]	[-0.367,0.147]	[-0.322,0.075]
Mean outcome	2.667	3.076	2.622	3.107	2.572	0.018
Mean Δoutcome	0.040	0.032	0.034	0.023	0.009	0.014
Mean treatment	9.69%	9.35%	9.69%	9.15%	9.69%	9.69%
RDA controls	✓	✓	✓	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓	✓	✓	✓
Observations	31,313	30,240	31,313	29,216	31,308	31,313

Notes: This table reports IV estimates from the same specifications as in Table 3 but for different outcomes. The outcome in column 1 is the change in log average wage in the sample of all workers. In columns 2–5, the outcome is the change in the log average wage among specific groups of workers: managerial occupations, non-managerial occupations, those with exactly a college degree, and those with exactly a high school degree. State-industry cells with fewer than ten observations for the relevant group at the beginning or end of the decade are excluded. In column 6, the outcome is the change in the average residual from a Mincer regression for log wages (see footnote 27). Observations are weighted by employment at the beginning of the decade in upper-level analyses and by the number of eligible voters at the beginning of the decade in lower-level analyses.

Sources: Decennial Census and NLRB unionization data; authors' calculations.

2 and 3 of Table 4. Since both the outcomes and the endogenous variable are measured in percentages, the coefficient is interpreted as the number of covered workers induced by one new union member. For health insurance, the effect is small and insignificant. However, for pension coverage, we estimate that each new union member corresponds to an increase of 1.81 pension holders (standard error of 0.80) using the upper-level estimator and 0.90 (SE of 0.49) when the lower-level estimator is employed. Although the coefficients are noisy, the fact that they are around one or even larger suggests a substantial spillover effect of unionization across establishments within state-industry cells, given that some of the unionized establishments had pension plans prior to unionization. This is consistent with the recent findings by Knepper (2020) that unionization of establishments within firms has significant impacts on employer pension contributions on other establishments of the same firm. Our complementary results suggest spillovers to other firms within the same state and industry.

Our final explanation concerns dynamics. We find that short-run wage effects are negative but relatively uniform, while the opposite happens in the long-run: unionization helps low-earners at the expense of high-earners without affecting average wages. To evaluate such dynamics, we split elections into those happening late in the decade (in years ending 5–9) and those happening early (years ending 0–4). For each subset, we construct the endogenous variable — the resulting rate of new unionization — as well as the RDA instrument and controls. Since our main outcomes are measured using decennial census data collected in round years, the effects of late elections are interpreted as the short-run (1–5 years after unionization) while the effects of early elections as the long-run effects (6–10 years after unionization).[38] In the short run (Panel B of Figure 2), unions appear to reduce wages across the entire distribution, moderately stronger above the median of the income distribution. Employment effects are also negative in the short-run (though marginally insignificant), driven by more strongly negative and statistically significant effects on college graduate employment (Appendix Table A13).

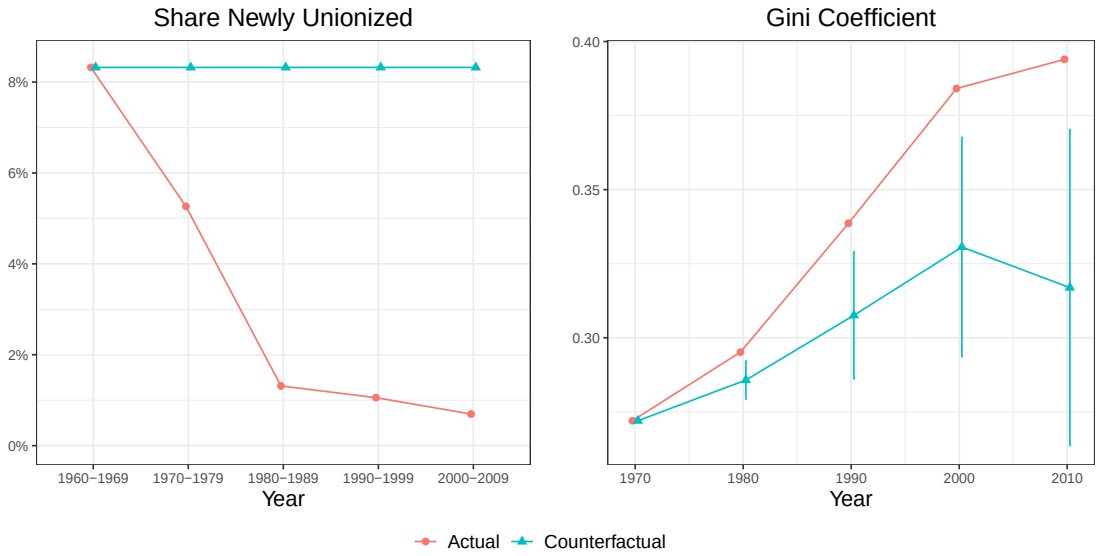
In the long run (Panel C of Figure 2), the pattern shifts markedly: wages rise in the lower half of the distribution (though these effects are statistically insignificant or only marginally significant), remain largely unchanged between the 60th and 90th percentiles, and decline sharply at the very top. Notably, the average long-run effect on wages is small but positive, driven by the gains in the lower half of the distribution. Thus, inequality effects from Table 3 arise from the long-run effects, while the short-run impacts are very small, as directly shown in Appendix Table A12. Long-run employment effects are small, positive, and statistically insignificant overall, with positive effects for high school graduates and negative effects for college graduates—though neither reaches statistical significance.

Several factors may explain these dynamics. In what we defined as the short-run, the union premium may not appear because many newly organized workplaces are still negotiating their first contract.[39] Moreover, Wang and Young (2022) show that employer opposition following unionization leads to significant short-run disruption, including the relocation of production by multi-establishment firms to non-unionized plants, which can explain the negative employment effects and could also lead to negative wage growth in the short-run. In line with our employment results, recent evidence based on establishment-level data also indicates that following unionization, employment of highly skilled incumbent workers shrinks (Frandsen, 2021; Kowarski and Wasser, 2025), creating negative composition effects in the short-run. In contrast, the union wage

[38] We implement the two analyses as two separate regressions since, the short-run and long-run RDA instruments should be uncorrelated after residualizing on the controls, which we verified in unreported results.

[39] According to Ferguson (2008) for 1999–2004, the average time from union election to reaching a first contract is approximately ten months, and there is also a non-negligible delay between the election and the NLRB's certification of the union. As a result, a substantial portion of newly organized workplaces remain without a finalized contract during the first years after certification. Our survey-based wage variable reflects income in the previous calendar year creates an additional delay.

Figure 3: The Counterfactual with Unionization Rates Remaining at the 1960s Level



Notes: The first panel presents the actual and counterfactual rates of newly unionized workers for each decade between 1960 and 2010. The actual rate is calculated as the share of workers joining unions through unionization elections minus the share leaving due to union decertification elections. The counterfactual rate assumes that the rate remained constant at its 1960s level. The second panel presents the actual and counterfactual Gini coefficients of income inequality within state-industry cells, averaged across cells (with beginning-of-decade employment weights). The counterfactual is calculated by adding to the actual Gini the product of the unionization effect estimated in Table 3, column 3, and the cumulative shortfall in the rate of new unionization between the actual and counterfactual scenarios. Confidence intervals for the counterfactual Gini are based on the 95% heteroskedasticity-robust confidence intervals for the treatment effect.

Sources: Decennial Census, NLRB unionization data; authors' calculations.

premium and positive wage spillovers, through both "threat effects" whereby non-union employers raise wages to prevent unionization (DiNardo and Lee, 2004) and improved outside options for non-unionized workers, may take longer to materialize, as they depend on the gradual diffusion of union wage standards across establishments. These long-run gains at the bottom may be accompanied with long-run losses at the top, consistent with evidence that unions compress managerial compensation, particularly reducing equity-based pay (DiNardo et al., 2000; Gomez and Tzioumis, 2006; Huang et al., 2017).

In sum, we show a surprising pattern that the inequality reduction arises from falling wages at the top rather than growing wages at the bottom. Part of this pattern originate in large negative effects on managers, who are legally excluded from union representation. Additionally, this pattern hides composition effects and dynamics in ways that align with suggested mechanisms for direct and indirect union effects. Falling wages are also partially offset by higher pension coverage rates.

5.2.6 How Much Did Declining Unions Contribute to Growing Inequality?

Rates of new unionization in the private sector in the U.S. experienced a massive and continuous decline since the 1970s, as can be seen in the first panel of Figure 3. In the 1960s, the net flow of unionized workers through unionization elections was 8.3%. This is the result of the share of workers joining unions through unionization elections (8.9%), minus some workers leaving unions through union decertification elections. The net flow declined consistently, reaching less than 1% in the 2000s.

Using the results in Table 3, columns 1–5, we conduct a back-of-the-envelope calculation of how much of

Table 6: Counterfactual Inequality Measures

Inequality measure	Actual			Counterfactual		Union contribution
	1970	2010	Change	2010	Change	as % of change
Log college premium	0.431	0.579	0.473	0.148	0.043	71.2%
			(0.074)			(49.6%)
Log 90/10 ratio	1.109	1.444	1.320	0.335	0.211	37.0%
			(0.061)			(18.2%)
Gini coefficient	0.272	0.394	0.352	0.122	0.080	34.6%
			(0.016)			(13.1%)
Top 10% share	0.227	0.317	0.282	0.090	0.055	38.5%
			(0.015)			(16.2%)
Var(Log wage)	0.218	0.371	0.305	0.152	0.087	43.1%
			(0.023)			(15.1%)

Notes: This table presents statistics regarding the five measures of wage inequality within state-industry cells used in Table 3, columns1–5. The 2010 counterfactual measures are calculated by adding to each actual inequality measure the product of the relevant treatment effect and the cumulative shortfall in new unionization resulting from the sharp decline since the 1960s. "Change" refers to the change from the actual measure in 1970 to the actual or counterfactual measure in 2010. The "Union Contribution" is calculated as one minus the counterfactual change in each measure divided by the actual change. The standard errors in parentheses are based on the standard errors for the corresponding estimates in Table 3.
Sources: Decennial Census, NLRB unionization data; authors' calculations.

the growth in U.S. inequality during the period of our analysis can be attributed to this sharp decline. To this end, we compute counterfactual inequality measures that may have occurred if new unionization rates had remained at their 1960s levels. Specifically, we add to each actual inequality measure the product of the relevant treatment effect and the cumulative shortfall in new unionization resulting from the sharp decline after the 1960s.

Like any back-of-the-envelope calculation, ours requires some caveats. The estimates from Table 3 ignore dynamics within the decade (as in Figure 2B,C) and composition effects (as in Appendix Table A11), and are identified using local variation from narrow elections. Moreover, it captures the effect of new unionization in the contemporaneous decade, while we extrapolate it by assuming that the effect on inequality at later dates is the same. Our analysis is also limited to assessing the effects of unions on inequality *within* state-industry cells and ignores spillovers across cells. However, as shown above, within-cell inequality accounts for the majority of overall inequality both in levels and in changes.[40]

With these caveats, the second panel of Figure 3 presents the actual and counterfactual average within-cell Gini coefficients by census year. The gap between the actual and counterfactual inequality is small in 1980 and 1990 but becomes substantial by 2000 and 2010. Appendix Figure A2 reports similar results for the other inequality measures.

Table 6 summarizes all results of this back-of-the-envelope exercise showing a sizable contribution of declining unions to the growing within-cell inequality between 1970 and 2010. Unions account for 35-43% of the changes in the Gini coefficient, the 90/10 ratio, the top 10% share, and the variance of log wages. The contribution to the log college premium is even higher (at 71.2%) but imprecisely estimated. These estimates are moderately larger than the Farber et al. (2021, Table 1) estimates of the contribution of unions to within-state inequality in the U.S. between 1968 and 2014.

[40] Another limitation is that we assess only the effect of new unionization, not the full impact of changes in union density, which may also result from the closure or growth of unionized establishments. However, this concern is mitigated by our earlier finding that the effect of new unionization on union density is approximately equal to 1.

6 Conclusion

We introduced a novel extension to the regression discontinuity design, termed Regression Discontinuity Aggregation (RDA), which aims at identifying local average causal effects in contexts where each unit may be exposed to multiple discontinuity events. The RDA approach is applicable when the outcome is defined at a higher level of aggregation (e.g., in space or time) than the discontinuity events or when spillovers of discontinuity events are of interest. We propose two estimators for causal effects in RDA settings that build on local linear estimation in conventional sharp and fuzzy RD designs and inherit their attractive bias-reduction properties. These solutions diverge from the estimators used—in a heuristic way—in current empirical practice, for which no such guarantees exist.

To illustrate the usefulness of this approach, we applied the proposed estimation techniques to the effects of unionization on earnings inequality in the U.S. at the level of state-by-industry cells. We found that higher rates of new unionization lead to significantly lower inequality, primarily by lowering top-tier wages rather than raising lower-end wages. As a result, the decline of unions since 1970s can explain a substantial fraction of increased inequality. We provide evidence on possible mechanisms related to managerial wage cuts, composition effects, compensating differential from pension coverage, and effect dynamics. Our findings provide causal evidence supporting common perceptions about the effects of unionization on inequality that were previously difficult to obtain as credibly.

Our paper has focused on *average* effects of treatments that aggregate RD shocks. However, it would be useful to trace the entire relationship between the aggregated treatment and the outcome and identify its nonlinearities. For instance, are the effects of unionization growing or attenuating with more unions, and are the effects of having more women in the local legislature most pronounced when women become the majority? Moreover, it would also be valuable to bring the attractive bias properties of the local linear approach to the general class *nonlinear* RD aggregations, as in Narita and Yata (2023). We leave these important extensions to future work.

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Appendix

A Proofs for Section 3

A.1 Two convenient reformulations

To exploit the symmetry, we redefine the independent variable (z_j, q_j) as $c_j \triangleq (c_j^+, c_j^-)$, where

$$c_j^+ = \Big(I\big(r_j > 0\big), r_j I\big(r_j > 0\big)\Big),\ c_j^- = \Big(I\big(r_j < 0\big), r_j I\big(r_j < 0\big)\Big).$$

Then it is easy to see that, for the lower-level regression,

$$\hat{\beta}_{\mathrm{lo},X} = \hat{\beta}^+_{\mathrm{lo},X} - \hat{\beta}^-_{\mathrm{lo},X}, \quad \hat{\beta}_{\mathrm{lo},Y} = \hat{\beta}^+_{\mathrm{lo},Y} - \hat{\beta}^-_{\mathrm{lo},Y}$$

where, for $V \in \{X, Y\}$,

$$\begin{aligned}
&(\hat{\beta}^+_{\mathrm{lo},V}, \hat{\beta}^-_{\mathrm{lo},V}, \hat{\gamma}^+_{\mathrm{lo},V}, \hat{\gamma}^-_{\mathrm{lo},V}) \\
&= \arg\min \sum_{i=1}^{N} \sum_{j \in \mathcal{J}_i} \Big(V_i - \beta^+ I\big(r_j > 0\big) - \beta^- I\big(r_j < 0\big) \\
&\qquad\qquad - \gamma^+ r_j I\big(r_j > 0\big) - \gamma^- r_j I\big(r_j < 0\big) \Big)^2 s_j \kappa_h(r_j). \qquad \text{(A1)}
\end{aligned}$$

Similarly, we redefine the aggregated variables (Z_i, Q_i) as

$$\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j.$$

Clearly, for the upper-level regression,

$$\hat{\beta}_{\mathrm{up},X} = \hat{\beta}^+_{\mathrm{up},X} - \hat{\beta}^-_{\mathrm{up},X}, \quad \hat{\beta}_{\mathrm{up},Y} = \hat{\beta}^+_{\mathrm{up},Y} - \hat{\beta}^-_{\mathrm{up},Y}$$

where, for $V \in \{X, Y\}$,

$$\begin{aligned}
&(\hat{\beta}^+_{\mathrm{up},V}, \hat{\beta}^-_{\mathrm{up},V}, \hat{\gamma}^+_{\mathrm{up},V}, \hat{\gamma}^-_{\mathrm{up},V}) \\
&= \arg\min \sum_{i=1}^{N} \Big(V_i - \beta^+ \sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) I\big(r_j > 0\big) - \beta^- \sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) I\big(r_j < 0\big) \\
&\qquad\qquad - \gamma^+ \sum_{j \in \mathcal{J}_i} s_j r_j \kappa_h(r_j) I\big(r_j > 0\big) - \gamma^- \sum_{j \in \mathcal{J}_i} s_j r_j \kappa_h(r_j) I\big(r_j < 0\big) \Big)^2. \qquad \text{(A2)}
\end{aligned}$$

Define the Gram matrices and inner product terms for both estimators, as well as their expectations, as follows.

$$\hat{\Sigma}_{\text{lo}}(h) \triangleq \hat{\mathbb{E}}_N \left[\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j c_j' \right], \quad \Sigma_{\text{lo}}(h) = \mathbb{E}[\hat{\Sigma}_{\text{lo}}(h)], \tag{A3}$$

$$\hat{\Sigma}_{\text{up}}(h) \triangleq \hat{\mathbb{E}}_N \left[\left(\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j \right) \left(\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j \right)' \right], \quad \Sigma_{\text{up}}(h) = \mathbb{E}[\hat{\Sigma}_{\text{up}}(h)], \tag{A4}$$

$$\hat{P}_V(h) = \hat{\mathbb{E}}_N \left[\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j V_i \right], \quad P_V(h) = \mathbb{E}[\hat{P}_V(h)]. \tag{A5}$$

Then

$$\begin{bmatrix} \hat{\beta}^+_{\text{lo},V} \\ \hat{\gamma}^+_{\text{lo},V} \\ \hat{\beta}^-_{\text{lo},V} \\ \hat{\gamma}^-_{\text{lo},V} \end{bmatrix} = \hat{\Sigma}_{\text{lo}}(h)^{-1} \hat{P}_V(h), \quad \begin{bmatrix} \hat{\beta}^+_{\text{up},V} \\ \hat{\gamma}^+_{\text{up},V} \\ \hat{\beta}^-_{\text{up},V} \\ \hat{\gamma}^-_{\text{up},V} \end{bmatrix} = \hat{\Sigma}_{\text{up}}(h)^{-1} \hat{P}_V(h).$$

For both estimators, we are only concerned with $\hat{\beta}^+_V - \hat{\beta}^-_V$. Here, we suppress the subscript lo or up when the argument applies to both estimators. Let

$$U = \begin{bmatrix} I_2 & -I_2 \\ -I_2 & -I_2 \end{bmatrix}.$$

Note that

$$U^{-1} = U/2.$$

Then

$$\begin{bmatrix} \hat{\beta}^+_V - \hat{\beta}^-_V \\ \hat{\gamma}^+_V - \hat{\gamma}^-_V \\ -\hat{\beta}^+_V - \hat{\beta}^-_V \\ -\hat{\gamma}^+_V - \hat{\gamma}^-_V \end{bmatrix} = U \begin{bmatrix} \hat{\beta}^+_V \\ \hat{\gamma}^+_V \\ \hat{\beta}^-_V \\ \hat{\gamma}^-_V \end{bmatrix} = (U\hat{\Sigma}(h)U^{-1})^{-1} U \hat{P}_V(h).$$

It turns out $\hat{\Sigma}(h)$ and $\hat{P}_V(h)$ are easier to derive by exploiting symmetry between both sides, while $U\hat{\Sigma}U^{-1}$ and $U\hat{P}_V$ have simpler expressions (in the limit).

Note that the above calculation is equivalent to replace the regressors c_j by Uc_j, up to a factor 2 which would be cancelled in the ratio $(\hat{\beta}^+_Y - \hat{\beta}^-_Y)/(\hat{\beta}^+_X - \hat{\beta}^-_X)$. In particular,

$$Uc_j = \begin{bmatrix} I(r_j > 0) - I(r_j < 0) \\ r_j I(r_j > 0) - r_j I(r_j < 0) \\ I(r_j > 0) + I(r_j < 0) \\ r_j I(r_j > 0) + r_j I(r_j < 0) \end{bmatrix} = \begin{bmatrix} \text{sign}(r_j) \\ |r_j| \\ 1 \\ r_j \end{bmatrix}.$$

This is slightly different from the original RDA controls $(z_j, 1, r_j, r_j^+)$.

Lastly, by (14) and (15), the pseudo-estimands can be expressed as

$$\begin{bmatrix} \tilde{\beta}^+_{\mathrm{lo},V}(h) \\ \tilde{\gamma}^+_{\mathrm{lo},V}(h) \\ \tilde{\beta}^-_{\mathrm{lo},V}(h) \\ \tilde{\gamma}^-_{\mathrm{lo},V}(h) \end{bmatrix} = \Sigma_{\mathrm{lo}}(h)^{-1} P_V(h), \quad \begin{bmatrix} \tilde{\beta}^+_{\mathrm{up},V}(h) \\ \tilde{\gamma}^+_{\mathrm{up},V}(h) \\ \tilde{\beta}^-_{\mathrm{up},V}(h) \\ \tilde{\gamma}^-_{\mathrm{up},V}(h) \end{bmatrix} = \Sigma_{\mathrm{up}}(h)^{-1} P_V(h).$$

Above, we append (h) onto all pseudo-estimands to highlight the dependence on the bandwidth. In particular,

$$\begin{bmatrix} \tilde{\beta}^+_V(h) - \tilde{\beta}^-_V(h) \\ \tilde{\gamma}^+_V(h) - \tilde{\gamma}^-_V(h) \\ -\tilde{\beta}^+_V(h) - \tilde{\beta}^-_V(h) \\ -\tilde{\gamma}^+_V(h) - \tilde{\gamma}^-_V(h) \end{bmatrix} = U \begin{bmatrix} \tilde{\beta}^+_V(h) \\ \tilde{\gamma}^+_V(h) \\ \tilde{\beta}^-_V(h) \\ \tilde{\gamma}^-_V(h) \end{bmatrix} = (U\Sigma(h)U^{-1})^{-1} U P_V(h). \tag{A6}$$

A.2 Notation

In this subsection we define quantities that will be repeatedly used in the results and proofs. Let

$$g(r) = \mathbb{E}[J_i s_j \mid r_j = r] \cdot f(r), \quad \tilde{g}(r) = \mathbb{E}[J_i s_j^2 \mid r_j = r] \cdot f(r),$$

$$a_k = \frac{g^{(k)}(0)}{k!}, \ \tilde{a}_k = \frac{\tilde{g}^{(k)}(0)}{k!},$$

and

$$G^{\pm}_V(r) = \mathbb{E}[J_i s_j V_j^{\pm} \mid r_j = r] \cdot f(r), \quad \Delta G_V(r) = \Delta G^+_V(r) - \Delta G^-_V(r),$$

$$a^+_{Vk} = \frac{(G^+_V)^{(k)}(0)}{k!}, \ a^-_{Vk} = \frac{(G^-_V)^{(k)}(0)}{k!}, \ \Delta a_{Vk} = a^+_{Vk} - a^-_{Vk}, \quad \Sigma a_{Vk} = a^+_{Vk} + a^-_{Vk}.$$

Clearly,

$$\beta_0 = \frac{\Delta a_{Y0}}{\Delta a_{X0}}.$$

Further, we define

$$v(r, r') = \mathbb{E}[J_i(J_i - 1) s_j s_k \mid r_j = r, r_k = r'] \cdot u(r, r'),$$

where u is the joint density of (r_j, r_k). Define

$$\nu_0 = w(0,0), \quad \nu_1 = v^{(1,0)}(0,0) = v^{(0,1)}(0,0).$$

Lastly, define

$$\eta_k = \begin{bmatrix} M_k \\ M_{k+1} \end{bmatrix}, \ A_k = \begin{bmatrix} M_k & M_{k+1} \\ M_{k+1} & M_{k+2} \end{bmatrix} = [\eta_k \ \ \eta_{k+1}], \ \tilde{A}_k = \begin{bmatrix} \tilde{M}_k & \tilde{M}_{k+1} \\ \tilde{M}_{k+1} & \tilde{M}_{k+2} \end{bmatrix}, \tag{A7}$$

and

$$D^+_h = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}, \ D^-_h = \begin{bmatrix} 1 & 0 \\ 0 & -h \end{bmatrix}, \ D_h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -h \end{bmatrix}, \quad U_h = U D_h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h \\ 0 & 0 & 1 & 0 \\ 0 & -h & 0 & 0 \end{bmatrix}.$$

It is easy to see that

$$U_h^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/h \\ 0 & 0 & 1 & 0 \\ 0 & -1/h & 0 & 0 \end{bmatrix}. \tag{A8}$$

We will express the k-th element of a vector v in two ways – $(v)_k$ and $e_k'v$ – depending upon which one is more convenience. Here, e_k denotes the k-th canonical basis in $\mathbb{R}^{\dim(v)}$.

We conclude the subsection with several useful identities.

Proposition A1.

$$A_0^{-1}\eta_0 = e_1,\ A_0^{-1}A_1A_0^{-1}\eta_0 = A_0^{-1}\eta_1 = e_2,$$
$$(A_0^{-1}\eta_2)_1 = (A_0^{-1}A_1A_0^{-1}\eta_1)_1 = (A_0^{-1}A_1A_0^{-1}A_1A_0^{-1}\eta_0)_1 = (A_0^{-1}A_2A_0^{-1}\eta_0)_1 = -C_{\text{bias}}(\kappa)$$

Proof of Proposition A1. By definition, η_0 and η_1 are the first and second columns of A_0. Thus, $A_0^{-1}\eta_0 = e_1$ and $A_0^{-1}\eta_1 = e_2$. Since η_1 is the first column of A_1,

$$A_0^{-1}A_1A_0^{-1}\eta_0 = A_0^{-1}A_1e_1 = A_0^{-1}\eta_1 = e_2.$$

By a direct calculation, we have

$$(A_0^{-1}\eta_2)_1 = -C_{\text{bias}}(\kappa).$$

Moreover,

$$A_0^{-1}A_1A_0^{-1}A_1A_0^{-1}\eta_0 = A_0^{-1}A_1A_0^{-1}A_1e_1 = A_0^{-1}A_1A_0^{-1}\eta_1 = A_0^{-1}A_1e_2 = A_0^{-1}\eta_2.$$

Thus,

$$(A_0^{-1}A_1A_0^{-1}\eta_1)_1 = (A_0^{-1}A_1A_0^{-1}A_1A_0^{-1}\eta_0)_1 = -C_{\text{bias}}(\kappa).$$

Lastly, since η_2 is the first column of A_2, we have

$$A_0^{-1}A_2A_0^{-1}\eta_0 = A_0^{-1}A_2e_1 = A_0^{-1}\eta_2.$$

Thus,

$$(A_0^{-1}A_2A_0^{-1}\eta_0)_1 = -C_{\text{bias}}(\kappa).$$

□

A.3 Consistency

Proof of Theorem 1. Since $\beta_0 = \Delta a_{Y0}/\Delta a_{X0}$, it suffices to prove

$$\hat{\beta}_{\text{lo},X} = \frac{\Delta a_{X0}}{a_0} + o_{\mathbb{P}}(1), \quad \hat{\beta}_{\text{lo},Y} = \frac{\Delta a_{Y0}}{a_0} + o_{\mathbb{P}}(1),$$

and

$$\hat{\beta}_{\text{up},X} = \frac{\Delta a_{X0}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_0)_1 + o_{\mathbb{P}}(1), \quad \hat{\beta}_{\text{up},Y} = \frac{\Delta a_{Y0}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_0)_1 + o_{\mathbb{P}}(1).$$

We split the proof into the following two steps:

(a) $\tilde{\beta}_{\mathrm{lo},V} = \Delta a_{V0}/a_0 + o(1)$, $\tilde{\beta}_{\mathrm{up},V} = \Delta a_{V0}/\tilde{a}_0(\tilde{A}_0^{-1}\eta_0)_1 + o(1)$.

(b) $\hat{\beta}_{\mathrm{lo},V} = \tilde{\beta}_{\mathrm{lo},V} + o_{\mathbb{P}}(1)$, $\hat{\beta}_{\mathrm{up},V} = \tilde{\beta}_{\mathrm{up},V} + o_{\mathbb{P}}(1)$.

Proof of (a) for the lower-level estimator.

Fix $V \in \{X, Y\}$. Recall that $\{s_j\kappa_h(r_j)c_jc_j' : j \in \mathcal{J}_i\}$ are assumed to be exchangeable without loss of generality and hence they have identical marginal distributions. Thus,

$$\mathbb{E}\left[\sum_{j:j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jc_j'\right] = \mathbb{E}[J_i s_j\kappa_h(r_j)c_jc_j'].$$

Since $c_j^+ c_j^{-\prime} = 0$,

$$\mathbb{E}[J_i s_j\kappa_h(r_j)c_jc_j'] = \begin{bmatrix}\mathbb{E}[J_i s_j\kappa_h(r_j)c_j^+c_j^{+\prime}] & 0\\ 0 & \mathbb{E}[J_i s_j\kappa_h(r_j)c_j^-c_j^{-\prime}]\end{bmatrix} \tag{A9}$$

To simplify the other term, we note that

$$V_i = \begin{cases} V_i(1; z_{-j}) = V_j^+, & r_j > 0 \\ V_i(0; z_{-j}) = V_j^-, & r_j < 0 \end{cases}..$$

Thus,

$$s_j\kappa_h(r_j)c_jV_i = \begin{bmatrix} s_j\kappa_h(r_j)c_j^+V_i(1; z_{-j}) \\ s_j\kappa_h(r_j)c_j^-V_i(0; z_{-j})\end{bmatrix}.$$

Using the exchangeability assumption again, we have

$$\mathbb{E}\left[\sum_{j:j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jV_i\right] = \begin{bmatrix}\mathbb{E}\left[J_i s_j\kappa_h(r_j)c_j^+V_i(1; z_{-j})\right] \\ \mathbb{E}\left[J_i s_j\kappa_h(r_j)c_j^-V_i(0; z_{-j})\right]\end{bmatrix}. \tag{A10}$$

By Assumption 4 and Lemma A7, for $p \in \{0, 1\}$,

$$\begin{aligned}
\mathbb{E}[J_i s_j r_j^2\kappa_h(r_j)I\big((-1)^p r_j > 0\big)] &= h^3\left\{a_0M_2 + o(1)\right\},\\
\mathbb{E}[J_i s_j r_j\kappa_h(r_j)I\big((-1)^p r_j > 0\big)] &= (-1)^p h^2\left\{a_0M_1 + o(1)\right\},\\
\mathbb{E}[J_i s_j\kappa_h(r_j)I\big((-1)^p r_j > 0\big)] &= h\left\{a_0M_0 + o(1)\right\},\\
\mathbb{E}[J_i s_j r_j\kappa_h(r_j)I\big((-1)^p r_j > 0\big)V_i(1; z_{-j})] &= (-1)^p h^2\left\{a_{V0}^+M_1 + o(1)\right\},\\
\mathbb{E}[J_i s_j\kappa_h(r_j)I\big((-1)^p r_j > 0\big)V_i(1; z_{-j})] &= h\left\{a_{V0}^+M_0 + o(1)\right\}.
\end{aligned}$$

Putting all pieces into the matrix form, we have

$$\Sigma_{\mathrm{lo}}(h) = hD_h\left\{\begin{bmatrix}a_0A_0 & 0\\ 0 & a_0A_0\end{bmatrix} + o(1)\right\}D_h, \tag{A11}$$

and

$$P_V(h) = hD_h \left\{ \begin{bmatrix} a_{V0}^+ \eta_0 \\ a_{V0}^- \eta_0 \end{bmatrix} + o(1) \right\}. \tag{A12}$$

By Proposition A1,

$$D_h \begin{bmatrix} \tilde{\beta}_{\mathrm{lo},V}^+(h) \\ \tilde{\gamma}_{\mathrm{lo},V}^+(h) \\ \tilde{\beta}_{\mathrm{lo},V}^-(h) \\ \tilde{\gamma}_{\mathrm{lo},V}^-(h) \end{bmatrix} = D_h \Sigma_{\mathrm{lo}}(h)^{-1} P_V(h) = \begin{bmatrix} \frac{a_{V0}^+}{a_0} A_0^{-1} \eta_0 \\ \frac{a_{V0}^-}{a_0} A_0^{-1} \eta_0 \end{bmatrix} + o(1) = \begin{bmatrix} \frac{a_{V0}^+}{a_0} e_1 \\ \frac{a_{V0}^-}{a_0} e_1 \end{bmatrix} + o(1).$$

Then (a) is proved for $\tilde{\beta}_{\mathrm{lo},V}$ by extracting the first and third coordinates.

Proof of (a) for the upper-level estimator.

By exchangeability,

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j \right) \left(\sum_{j \in \mathcal{J}_i} s_j \kappa_h(r_j) c_j \right)' \right] \\ &= \mathbb{E}\left[\sum_{j \in \mathcal{J}_i} s_j^2 \kappa_h^2(r_j) c_j c_j' \right] + \mathbb{E}\left[\sum_{j \neq k : j,k \in \mathcal{J}_i} s_j s_k \kappa_h(r_j) \kappa_h(r_k) c_j c_k' \right] \\ &= \mathbb{E}[J_i s_j^2 \kappa_h^2(r_j) c_j c_j'] + \mathbb{E}[J_i (J_i - 1) s_j s_k \kappa_h(r_j) \kappa_h(r_k) c_j c_k'] \end{aligned} \tag{A13}$$

Replacing $J_i s_j$ by $J_i s_j^2$ in (A11), we obtain that

$$\mathbb{E}[J_i s_j^2 \kappa_h^2(r_j) c_j c_j'] = hD_h \left\{ \begin{bmatrix} \tilde{a}_0 \tilde{A}_0 & 0 \\ 0 & \tilde{a}_0 \tilde{A}_0 \end{bmatrix} + o(1) \right\} D_h \tag{A14}$$

Now we compute $\mathbb{E}[J_i(J_i - 1) s_j s_k \kappa_h(r_j) \kappa_h(r_k) c_j c_k']$ for $j \neq k$. By definition,

$$\begin{aligned} \kappa_h(r_j)\kappa_h(r_k) c_j c_k' &= \kappa_h(r_j)\kappa_h(r_k) \begin{bmatrix} c_j^+ c_k^{+\prime} & c_j^+ c_k^{-\prime} \\ c_j^- c_k^{+\prime} & c_j^- c_k^{-\prime} \end{bmatrix} \\ &= \kappa_h(r_j)\kappa_h(r_k) \begin{bmatrix} I(r_j > 0) I(r_k > 0) & I(r_j > 0) I(r_k < 0) \\ I(r_j < 0) I(r_k > 0) & I(r_j < 0) I(r_k < 0) \end{bmatrix} \otimes \begin{bmatrix} 1 & r_k \\ r_j & r_j r_k \end{bmatrix}. \end{aligned}$$

By Lemma A8,

$$\begin{aligned} \mathbb{E}[J_i(J_i - 1) s_j s_k \kappa_h(r_j)\kappa_h(r_k) c_j^+ c_k^{+\prime}] &= D_h \left\{ O(h^2) \right\} D_h \\ \mathbb{E}[J_i(J_i - 1) s_j s_k \kappa_h(r_j)\kappa_h(r_k) c_j^- c_k^{-\prime}] &= D_h \left\{ O(h^2) \right\} D_h \\ \mathbb{E}[J_i(J_i - 1) s_j s_k \kappa_h(r_j)\kappa_h(r_k) c_j^+ c_k^{-\prime}] &= D_h \left\{ O(h^2) \right\} D_h \\ \mathbb{E}[J_i(J_i - 1) s_j s_k \kappa_h(r_j)\kappa_h(r_k) c_j^+ c_k^{-\prime}] &= D_h \left\{ O(h^2) \right\} D_h. \end{aligned}$$

As a result,

$$\mathbb{E}[J_i(J_i-1)s_j s_k \kappa_h(r_j)\kappa_h(r_k)c_j c_k^{'}] = D_h\left\{O(h^2)\right\}D_h. \tag{A15}$$

Combining (A14) and (A15), we arrive at

$$\Sigma_{\text{up}}(h) = hD_h\left\{\begin{bmatrix}\tilde{a}_0\tilde{A}_0 & 0\\ 0 & \tilde{a}_0\tilde{A}_0\end{bmatrix} + o(1)\right\}D_h.$$

By (A12),

$$D_h\begin{bmatrix}\tilde{\beta}^{+}_{\text{up},V}(h)\\ \tilde{\gamma}^{+}_{\text{up},V}(h)\\ \tilde{\beta}^{-}_{\text{up},V}(h)\\ \tilde{\gamma}^{-}_{\text{up},V}(h)\end{bmatrix} = D_h\Sigma_{\text{up}}(h)^{-1}P_V(h) = \begin{bmatrix}\frac{a^{+}_{V0}}{a_0}\tilde{A}_0^{-1}\eta_0\\ \frac{a^{-}_{V0}}{a_0}\tilde{A}_0^{-1}\eta_0\end{bmatrix} + o(1) = \begin{bmatrix}\frac{a^{+}_{V0}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_0)\\ \frac{a^{-}_{V0}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_0)\end{bmatrix} + o(1).$$

Then (a) is proved for $\tilde{\beta}_{\text{up},V}$ by extracting the first and third coordinates.

Proof of (b) for the lower-level estimator.

Let

$$\tilde{c}_j = D_h^{-1}c_j, \quad B_j = s_j\kappa_h(r_j)\tilde{c}_j\tilde{c}_j', \quad \Lambda_{\text{lo},i} = \frac{1}{h}\sum_{j\in\mathcal{J}_i}B_j.$$

Then

$$\frac{1}{h}D_h^{-1}(\hat{\Sigma}_{\text{lo}}(h) - \tilde{\Sigma}_{\text{lo}}(h))D_h^{-1} = \frac{1}{N}\sum_{i=1}^{N}\{\Lambda_{\text{lo},i} - \mathbb{E}[\Lambda_{\text{lo},i}]\}.$$

Note that

$$\|B_j\|_{\text{op}} = s_j\kappa_h(r_j)\|\tilde{c}_j\tilde{c}_j'\|_{\text{op}} = s_j\kappa_h(r_j)\|\tilde{c}_j\|_2^2,$$

where the last line uses the fact that $s_j \in [0,1]$. By definition,

$$\|\tilde{c}_j\|_2^2 = (I(r_j>0))^2 + (I(r_j<0))^2 + \left(\frac{r_j}{h}I(r_j>0)\right)^2 + \left(\frac{r_j}{h}I(r_j<0)\right)^2 = 1 + \frac{r_j^2}{h^2}. \tag{A16}$$

Since κ is supported on $[-1,1]$,

$$\kappa_h(r_j) = \kappa_h(r_j)\cdot I(|r_j|\le h).$$

Thus,

$$\|B_j\|_{\text{op}} \le s_j\kappa_h(r_j)\|\tilde{c}_j\|_2\cdot I(|r_j|\le h) \le 2s_j\kappa_h(r_j). \tag{A17}$$

This implies

$$\|B_j\|_{\text{op}} \le L, \quad \text{where } L = 2\left(\max_x \kappa(x)\right)^2.$$

By triangle inequality,

$$\|B_j - \mathbb{E}[B_j]\|_{\text{op}} \le 2L.$$

By triangle inequality and boundedness of J_i,

$$\|\Lambda_{\mathrm{lo},i}\|_{\mathrm{op}} \leq \frac{J_i L}{h} \Longrightarrow \|\Lambda_{\mathrm{lo},i} - \mathbb{E}[\Lambda_{\mathrm{lo},i}]\|_{\mathrm{op}} \leq \frac{2J_i L}{h} \leq \frac{2J^* L}{h} = O\left(\frac{1}{h}\right). \tag{A18}$$

We can bound the second moment of $\|\Lambda_{\mathrm{lo},i}\|_{\mathrm{op}}$ as follows:

$$\begin{aligned}
&\mathbb{E}\left[\Big\| \sum_{j\in\mathcal{J}_i} (B_j - \mathbb{E}[B_j])\Big\|_{\mathrm{op}}^2\right] \\
&\leq \mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} \|(B_j - \mathbb{E}[B_j])\|_{\mathrm{op}}\right)^2\right] && \text{(by triangle inequality)} \\
&\leq \mathbb{E}\left[J_i \sum_{j\in\mathcal{J}_i} \|(B_j - \mathbb{E}[B_j])\|_{\mathrm{op}}^2\right] && \text{(by AM-GM inequality)} \\
&= \mathbb{E}\left[J_i^2 \|(B_j - \mathbb{E}[B_j])\|_{\mathrm{op}}^2\right] && \text{(by exchangeability)} \\
&\leq 16\mathbb{E}[J_i^2 s_j^2 \kappa_h^2(r_j)] && \text{(by (A17))} \\
&\leq 16J^*\mathbb{E}[J_i s_j^2 \kappa_h^2(r_j)] && \text{(by boundedness of } J_i\text{)} \\
&= O(h) && \text{(by Assumption 3 and Lemma A6).}
\end{aligned}$$

By definition,

$$\mathbb{E}\left[\|\Lambda_{\mathrm{lo},i} - \mathbb{E}[\Lambda_{\mathrm{lo},i}]\|_{\mathrm{op}}^2\right] = O\left(\frac{1}{h}\right). \tag{A19}$$

By (A19), (A18), and matrix Bernstein's inequality (Tropp et al., 2015, Theorem 6.6.1),

$$\left\|\frac{1}{h} D_h^{-1}(\hat{\Sigma}_{\mathrm{lo}}(h) - \tilde{\Sigma}_{\mathrm{lo}}(h)) D_h^{-1}\right\|_{\mathrm{op}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}} + \frac{1}{Nh}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right) = o_{\mathbb{P}}(1). \tag{A20}$$

To deal with $\hat{P}_V(h)$, we define

$$\omega_{V,i} = \frac{1}{h} \sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) \tilde{c}_j V_i.$$

Then by (A16),

$$\begin{aligned}
\mathbb{E}[\|\omega_{V,i}\|_2^2] &\leq \frac{1}{h^2}\mathbb{E}\left[J_i \sum_{j\in\mathcal{J}_i} s_j^2 \kappa_h^2(r_j) \|\tilde{c}_j\|_2^2 V_i^2\right] && \text{(by AM-GM inequality)} \\
&= \frac{1}{h^2}\mathbb{E}\left[J_i^2 s_j^2 \kappa_h^2(r_j) \|\tilde{c}_j\|_2^2 V_i^2\right] && \text{(by exchangeability)} \\
&\leq \frac{2}{h^2}\mathbb{E}\left[J_i^2 s_j^2 \kappa_h^2(r_j) V_i^2\right] && \text{(by (A16))} \\
&\leq \frac{2J^*}{h^2}\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) V_i^2\right] && \text{(by Assumption 3)} \\
&\leq \frac{4J^*}{h^2}\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j)(V_i^{+2} + V_i^{-2})\right] && \text{(by AM-GM inequality)} \\
&= O(h). && \text{(by Assumption 3 and Lemma A6)}
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\|D_h^{-1}\hat{P}_V(h)\|_2^2\right] = \frac{1}{N^2h^2}\sum_{i=1}^{N}\mathbb{E}[\|\omega_{V,i}\|_2^2] = O\left(\frac{1}{Nh}\right).$$

By Chebyshev's inequality,

$$D_h^{-1}\hat{P}_V(h) - D_h^{-1}P_V(h) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right). \tag{A21}$$

Combining (A20) and (A21) together, we conclude that

$$\begin{bmatrix}\hat{\beta}^+_{\mathrm{lo},V}\\ \hat{\gamma}^+_{\mathrm{lo},V}\\ \hat{\beta}^-_{\mathrm{lo},V}\\ \hat{\gamma}^-_{\mathrm{lo},V}\end{bmatrix} - \begin{bmatrix}\tilde{\beta}^+_{\mathrm{lo},V}\\ \tilde{\gamma}^+_{\mathrm{lo},V}\\ \tilde{\beta}^-_{\mathrm{lo},V}\\ \tilde{\gamma}^-_{\mathrm{lo},V}\end{bmatrix} = \left(h^{-1}D_h^{-1}\hat{\Sigma}_{\mathrm{lo}}(h)D_h^{-1}\right)^{-1}\left(D_h^{-1}\hat{P}_V(h) - D_h^{-1}P_V(h)\right) = o_{\mathbb{P}}(1).$$

Proof of (b) for the upper-level estimator.

Using the same argument for (A20),

$$\left\|\frac{1}{h}D_h^{-1}\left(\hat{\mathbb{E}}_N[s_j^2\kappa_h(r_j)c_jc_j'] - \mathbb{E}[s_j^2\kappa_h(r_j)c_jc_j']\right)D_h^{-1}\right\|_{\mathrm{op}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right) = o_{\mathbb{P}}(1). \tag{A22}$$

Now we prove

$$\begin{aligned}&\left\|\frac{1}{h}D_h^{-1}\left(\hat{\mathbb{E}}_N\left[\sum_{j\neq k:j,k\in\mathcal{J}_i}s_js_k\kappa_h(r_j)\kappa_h(r_k)c_jc_k'\right] - \mathbb{E}\left[\sum_{j\neq k:j,k\in\mathcal{J}_i}s_js_k\kappa_h(r_j)\kappa_h(r_k)c_jc_k'\right]\right)\right\|_{\mathrm{op}}\\ &= O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right) = o_{\mathbb{P}}(1).\end{aligned} \tag{A23}$$

Let

$$B_{j,k} = s_js_k\kappa_h(r_j)\kappa_h(r_k)\tilde{c}_j\tilde{c}_k', \quad \Lambda_{\mathrm{up},i} = \sum_{j\neq k:j,k\in\mathcal{J}_i}B_{j,k}.$$

By Cauchy-Schwarz inequality,

$$\|B_{j,k}\|_{\mathrm{op}} \leq \|s_j\kappa_h(r_j)\tilde{c}_j\|_2\|s_k\kappa_h(r_k)\tilde{c}_k\|_2.$$

Similar to (A18),

$$\|B_{j,k}\|_{\mathrm{op}} = O(1).$$

Following the same logic,

$$\|\Lambda_{\mathrm{up},i} - \mathbb{E}[\Lambda_{\mathrm{up},i}]\|_{\mathrm{op}} = O\left(\frac{1}{h}\right).$$

Using the same argument for (A19), we can also show that

$$\mathbb{E}\left[\|\Lambda_{\mathrm{up},i} - \mathbb{E}[\Lambda_{\mathrm{up},i}]\|_{\mathrm{op}}^2\right] = O\left(\frac{1}{h}\right).$$

By the matrix Bernstein's inequality, we conclude that (Tropp et al., 2015, Theorem 6.6.1),

$$\mathbb{E}\left[\left\|\frac{1}{h}D_h^{-1}\left(\hat{\mathbb{E}}_N\left[\sum_{j\neq k: j,k\in\mathcal{J}_i} s_j s_k \kappa_h(r_j)\kappa_h(r_k)c_j c_k'\right] - \mathbb{E}\left[\sum_{j\neq k: j,k\in\mathcal{J}_i} s_j s_k \kappa_h(r_j)\kappa_h(r_k)c_j c_k'\right]\right)\right\|_{\text{op}}^2\right]$$
$$= O_{\mathbb{P}}\left(\frac{1}{Nh}\right).$$

Then (A23) is proved by Markov's inequality.
Piecing (A22) and (A23) together,

$$\left\|\frac{1}{h}D_h^{-1}(\hat{\Sigma}_{\text{up}}(h) - \tilde{\Sigma}_{\text{up}}(h))D_h^{-1}\right\|_{\text{op}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}} + \frac{1}{Nh}\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right) = o_{\mathbb{P}}(1). \tag{A24}$$

By (A21),

$$\begin{bmatrix}\hat{\beta}^+_{\text{up},V}\\ \hat{\gamma}^+_{\text{up},V}\\ \hat{\beta}^-_{\text{up},V}\\ \hat{\gamma}^-_{\text{up},V}\end{bmatrix} - \begin{bmatrix}\tilde{\beta}^+_{\text{up},V}\\ \tilde{\gamma}^+_{\text{up},V}\\ \tilde{\beta}^-_{\text{up},V}\\ \tilde{\gamma}^-_{\text{up},V}\end{bmatrix} = \left(h^{-1}D_h^{-1}\hat{\Sigma}_{\text{up}}(h)D_h^{-1}\right)^{-1}\left(D_h^{-1}\hat{P}_V(h) - D_h^{-1}P_V(h)\right) = o_{\mathbb{P}}(1).$$

□

Proof of Corollary 1. Equations (17) and (19) follow immediately from Theorem 1 by plugging in the corresponding exclusion restrictions. They can be interpreted as convex weighted averages of $\partial Y_i(x)/\partial x$ and $\partial y_j(x)/\partial x$ (or discrete analogs thereof) by standard arguments, e.g. Yitzhaki (1996).

In part (b), $\Delta Y_j/\Delta X_j = \beta_i$ and $\Delta X_j = s_j$.Thus, by Theorem 1

$$\beta_0 = \frac{\mathbb{E}\left[J_i s_j^2 \beta_i \mid r_j = 0\right]}{\mathbb{E}\left[J_i s_j^2 \mid r_j = 0\right]} = \lim_{h\to 0}\frac{\mathbb{E}\left[\beta_i \sum_{j\in\mathcal{C}_i} s_j^2\right]}{\mathbb{E}\left[\sum_{j\in\mathcal{C}_i} s_j^2\right]},$$

as required. □

A.4 Asymptotic bias analysis in the standard asymptotic regime

A.4.1 Key lemma

Lemma A1. *Under Assumptions 1, 2, and 4,*

$$\Sigma_{\mathrm{lo}}(h) = hD_h\left\{\begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0\end{bmatrix} + h\begin{bmatrix} a_1A_1 & 0 \\ 0 & -a_1A_1\end{bmatrix} + h^2\begin{bmatrix} a_2A_2 & 0 \\ 0 & a_2A_2\end{bmatrix} + O\left(h^3\right)\right\}D_h$$

$$\begin{aligned}\Sigma_{\mathrm{up}}(h) = hD_h\Bigg\{&\begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0\end{bmatrix} + h\begin{bmatrix} \tilde{a}_1\tilde{A}_1 + \nu_0\eta_0\eta_0' & \nu_0\eta_0\eta_0' \\ \nu_0\eta_0\eta_0' & -\tilde{a}_1\tilde{A}_1 + \nu_0\eta_0\eta_0'\end{bmatrix} \\ &+ h^2\begin{bmatrix} \tilde{a}_2\tilde{A}_2 + \nu_1(\eta_1\eta_0' + \eta_0\eta_1') & \nu_1(\eta_1\eta_0' - \eta_0\eta_1') \\ -\nu_1(\eta_1\eta_0' - \eta_0\eta_1') & \tilde{a}_2\tilde{A}_2 - \nu_1(\eta_1\eta_0' + \eta_0\eta_1')\end{bmatrix} + O\left(h^3\right)\Bigg\}D_h\end{aligned}$$

$$P_V(h) = hD_h\left\{\begin{bmatrix} a_{V0}^+\eta_0 \\ a_{V0}^-\eta_0\end{bmatrix} + h\begin{bmatrix} a_{V1}^+\eta_1 \\ -a_{V1}^-\eta_1\end{bmatrix} + h^2\begin{bmatrix} a_{V2}^+\eta_2 \\ a_{V2}^-\eta_2\end{bmatrix} + O(h^3)\right\}.$$

As a result,

$$U\Sigma_{\mathrm{lo}}(h)U^{-1} = hU_h\left\{\begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0\end{bmatrix} + h\begin{bmatrix} 0 & -a_1A_1 \\ -a_1A_1 & 0\end{bmatrix} + h^2\begin{bmatrix} a_2A_2 & 0 \\ 0 & a_2A_2\end{bmatrix} + O\left(h^3\right)\right\}U_h$$

$$U\Sigma_{\mathrm{up}}(h)U^{-1} = hU_h\left\{\begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0\end{bmatrix} + h\begin{bmatrix} 0 & -\tilde{a}_1\tilde{A}_1 \\ -\tilde{a}_1\tilde{A}_1 & 2\nu_0\eta_0\eta_0'\end{bmatrix} + h^2\begin{bmatrix} \tilde{a}_2\tilde{A}_2 & -2\nu_1\eta_1\eta_0' \\ -2\nu_1\eta_0\eta_1' & \tilde{a}_2\tilde{A}_2\end{bmatrix} + O\left(h^3\right)\right\}U_h$$

$$UP_V(h) = hU_h\left\{\begin{bmatrix} \Delta a_{V0}\eta_0 \\ -\Sigma a_{V0}\eta_0\end{bmatrix} + h\begin{bmatrix} \Sigma a_{V1}\eta_1 \\ -\Delta a_{V1}\eta_1\end{bmatrix} + h^2\begin{bmatrix} \Delta a_{V2}\eta_2 \\ -\Sigma a_{V2}\eta_2\end{bmatrix} + O(h^3)\right\},$$

Proof of Lemma A1. We apply Lemma A7 to analyze (A9) and (A10). By Assumption 4 and Lemma A7, for $p \in \{0,1\}$,

$$\begin{aligned}\mathbb{E}[J_is_jr_j^2\kappa_h(r_j)I\big((-1)^pr_j > 0\big)] &= h^3\left\{a_0M_2 + (-1)^pa_1M_3h + a_2M_4h^2 + O(h^3)\right\},\\ \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big((-1)^pr_j > 0\big)] &= (-1)^ph^2\left\{a_0M_1 + (-1)^pa_1M_2h + a_2M_3h^2 + O(h^3)\right\},\\ \mathbb{E}[J_is_j\kappa_h(r_j)I\big((-1)^pr_j > 0\big)] &= h\left\{a_0M_0 + (-1)^pa_1M_1h + a_2M_2h^2 + O(h^3)\right\},\\ \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big((-1)^pr_j > 0\big)V_i(1;z_{-j})] &= (-1)^ph^2\left\{a_{V0}^+M_1 + (-1)^pa_{V1}^+M_2h + a_{V2}^+M_3h^2 + O(h^3)\right\},\\ \mathbb{E}[J_is_j\kappa_h(r_j)I\big((-1)^pr_j > 0\big)V_i(1;z_{-j})] &= h\left\{a_{V0}^+M_0 + (-1)^pa_{V1}^+M_1h + a_{V2}^+M_2h^2 + O(h^3)\right\}.\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[J_is_j\kappa_h(r_j)c_j^+c_j^{+\prime}] &= \begin{bmatrix}\mathbb{E}[J_is_j\kappa_h(r_j)I\big(r_j > 0\big)] & \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big(r_j > 0\big)] \\ \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big(r_j > 0\big)] & \mathbb{E}[J_is_jr_j^2\kappa_h(r_j)I\big(r_j > 0\big)]\end{bmatrix}\\ &= hD_h^+\left\{a_0A_0 + ha_1A_1 + h^2a_1A_2 + O(h^3)\right\}D_h^+ \qquad \text{(A25)}\\ \mathbb{E}[J_is_j\kappa_h(r_j)c_j^-c_j^{-\prime}] &= \begin{bmatrix}\mathbb{E}[J_is_j\kappa_h(r_j)I\big(r_j < 0\big)] & \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big(r_j < 0\big)] \\ \mathbb{E}[J_is_jr_j\kappa_h(r_j)I\big(r_j < 0\big)] & \mathbb{E}[J_is_jr_j^2\kappa_h(r_j)I\big(r_j < 0\big)]\end{bmatrix}\\ &= hD_h^-\left\{a_0A_0 - ha_1A_1 + h^2a_2A_2 + O(h^3)\right\}D_h^-. \qquad \text{(A26)}\end{aligned}$$

Combining the two terms, we obtain the expression of $\Sigma_{\text{lo}}(h)$:

$$\Sigma_{\text{lo}}(h) = hD_h \left\{ \begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0 \end{bmatrix} + h \begin{bmatrix} a_1A_1 & 0 \\ 0 & -a_1A_1 \end{bmatrix} + h^2 \begin{bmatrix} a_2A_2 & 0 \\ 0 & a_2A_2 \end{bmatrix} + O\left(h^3\right) \right\} D_h.$$

Next, we prove the expression for $\Sigma_{\text{up}}(h)$. Replacing $J_i s_j$ by $J_i s_j^2$ in (A25) and (A26), we obtain that

$$\begin{aligned} &\mathbb{E}[J_i s_j^2 \kappa_h^2(r_j) c_j c_j'] \\ &= hD_h \left\{ \begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0 \end{bmatrix} + h \begin{bmatrix} \tilde{a}_1\tilde{A}_1 & 0 \\ 0 & -\tilde{a}_1\tilde{A}_1 \end{bmatrix} + h^2 \begin{bmatrix} \tilde{a}_2\tilde{A}_2 & 0 \\ 0 & \tilde{a}_2\tilde{A}_2 \end{bmatrix} + O\left(h^3\right) \right\} D_h \end{aligned} \tag{A27}$$

Now we apply Lemma A9 to (A15). Let

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_0 = \eta_0\eta_0' = \begin{bmatrix} M_0^2 & M_0M_1 \\ M_0M_1 & M_1^2 \end{bmatrix},$$

$$C_1 = \eta_1\eta_0' + \eta_0\eta_1' = \begin{bmatrix} 2M_1M_0 & M_1^2 + M_2M_0 \\ M_1^2 + M_2M_0 & 2M_2M_1 \end{bmatrix},$$

and

$$C_2 = \eta_1\eta_0' - \eta_0\eta_1' = \begin{bmatrix} 0 & M_1^2 - M_2M_0 \\ -(M_1^2 - M_2M_0) & 0 \end{bmatrix}.$$

By Lemma A9,

$$\begin{aligned} \mathbb{E}[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_j^+c_k^{+'}] &= h^2D_h^+\left\{\nu_0C_0 + h\nu_1C_1 + O\left(h^2\right)\right\}D_h^+ \\ \mathbb{E}[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_j^-c_k^{-'}] &= h^2D_h^+\left\{\nu_0SC_0S - h\nu_1SC_1S + O\left(h^2\right)\right\}D_h^+ \\ \mathbb{E}[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_j^+c_k^{-'}] &= h^2D_h^+\left\{\nu_0C_0S + h\nu_1C_2S + O\left(h^2\right)\right\}D_h^+ \\ \mathbb{E}[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_j^+c_k^{-'}] &= h^2\tilde{a}_0D_h^+\left\{\nu_0SC_0 - h\nu_1SC_2 + O\left(h^2\right)\right\}D_h^+. \end{aligned}$$

As $D_h^- = SD_h^+ = D_h^+S$, we can concatenate the above results to arrive at

$$\begin{aligned} &\mathbb{E}[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_jc_k^{'}] \\ &= hD_h\left\{h\nu_0\begin{bmatrix} C_0 & C_0 \\ C_0 & C_0 \end{bmatrix} + h^2\nu_1\begin{bmatrix} C_1 & C_2 \\ -C_2 & -C_1 \end{bmatrix} + O\left(h^3\right)\right\}D_h. \end{aligned} \tag{A28}$$

Combining (A27) and (A28), we reach at an expression for the Gram matrix:

$$\Sigma_{\text{up}}(h) = hD_h\Bigg\{ \begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0 \end{bmatrix} + h \begin{bmatrix} \tilde{a}_1\tilde{A}_1 + \nu_0 C_0 & \nu_0 C_0 \\ \nu_0 C_0 & -\tilde{a}_1\tilde{A}_1 + \nu_0 C_0 \end{bmatrix} + h^2 \begin{bmatrix} \tilde{a}_2\tilde{A}_2 + \nu_1 C_1 & \nu_1 C_2 \\ -\nu_1 C_2 & \tilde{a}_2\tilde{A}_2 - \nu_1 C_1 \end{bmatrix} + O\left(h^3\right) \Bigg\} D_h.$$

Lastly, we derive the expression of $P_V(h)$. By Lemma A7 and Assumption 4,

$$\begin{aligned} \mathbb{E}\left[J_i s_j \kappa_h(r_j) c_j^+ V_i(1; z_{-j})\right] &= hD_h^+ \{a_{V0}^+\eta_0 + ha_{V1}^+\eta_1 + h^2 a_{V2}^+\eta_2 + O(h^3)\} \\ \mathbb{E}\left[J_i s_j \kappa_h(r_j) c_j^- V_i(0; z_{-j})\right] &= hD_h^- \{a_{V0}^-\eta_0 - ha_{V1}^-\eta_1 + h^2 a_{V2}^-\eta_2 + O(h^3)\}. \end{aligned}$$

Thus,

$$P_V(h) = hD_h \left\{ \begin{bmatrix} a_{V0}^+\eta_0 \\ a_{V0}^-\eta_0 \end{bmatrix} + h \begin{bmatrix} a_{V1}^+\eta_1 \\ -a_{V1}^-\eta_1 \end{bmatrix} + h^2 \begin{bmatrix} a_{V2}^+\eta_2 \\ a_{V2}^-\eta_2 \end{bmatrix} + O(h^3) \right\}.$$

The expressions of $U\Sigma_{\text{lo}}(h)U^{-1}, U\Sigma_{\text{up}}(h)U^{-1}$, and $UP_V(h)$ can be derived by direct calculation. □

A.4.2 Bias analysis for the lower-level estimator

We restate Theorem 2 into two pieces with a formal definition of Γ_2^{lo}. The first part establishes a different-looking expression based on the notation in Appendix A.2. In the second part we prove it is equivalent to the expression in Theorem 2.

Theorem A1. *Let*

$$\Gamma_{2,V}^{\text{lo}} = -C_{\text{bias}}(\kappa)\left(\Delta a_{V2} - \frac{a_1}{a_0}\Delta a_{V1}\right). \tag{A29}$$

Under Assumptions 1, 2, and 4,

$$\tilde{\beta}_{\text{lo},V}(h) := \tilde{\beta}_{\text{lo},V}^+(h) - \tilde{\beta}_{\text{lo},V}^-(h) = \frac{1}{a_0}\left\{\Delta a_{V0} + h^2\left[\Gamma_{2,V}^{\text{lo}} + C_{\text{bias}}(\kappa)\Delta a_{V0}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right] + O(h^3)\right\},$$

and

$$\tilde{\beta}_{\text{lo}}(h) = \beta_0 + h^2\Gamma_2^{\text{lo}} + O\left(h^3\right),$$

where

$$\Gamma_2^{\text{lo}} = \frac{1}{\Delta a_{X0}}\left(\Gamma_{2,Y}^{\text{lo}} - \beta_0\Gamma_{2,X}^{\text{lo}}\right).$$

Proof of Theorem A1. By (A6),

$$\tilde{\beta}_{\text{lo},V}(h) = \tilde{\beta}_{\text{lo},V}^+(h) - \tilde{\beta}_{\text{lo},V}^-(h) = e_1'(U\Sigma_{\text{lo}}(h)U^{-1})^{-1}UP_V(h),$$

where $e_1 = (1,0,0,0)'$. By Lemma A1,

$$
\begin{aligned}
&\tilde{\beta}_{\mathrm{lo},V}(h) \\
&= e_1' U_h^{-1} \left\{ \begin{bmatrix} a_0 A_0 & 0 \\ 0 & a_0 A_0 \end{bmatrix} + h \begin{bmatrix} 0 & -a_1 A_1 \\ -a_1 A_1 & 0 \end{bmatrix} + h^2 \begin{bmatrix} a_2 A_2 & 0 \\ 0 & a_2 A_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&\qquad \cdot \left\{ \begin{bmatrix} \Delta a_{V0} \eta_0 \\ -\Sigma a_{V0} \eta_0 \end{bmatrix} + h \begin{bmatrix} \Sigma a_{V1} \eta_1 \\ -\Delta a_{V1} \eta_1 \end{bmatrix} + h^2 \begin{bmatrix} \Delta a_{V2} \eta_2 \\ -\Sigma a_{V2} \eta_2 \end{bmatrix} + O(h^3) \right\} \\
&= e_1' U_h^{-1} \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h \begin{bmatrix} 0 & -(a_1/a_0) A_0^{-1} A_1 \\ -(a_1/a_0) A_0^{-1} A_1 & 0 \end{bmatrix} + h^2 \begin{bmatrix} (a_2/a_0) A_0^{-1} A_2 & 0 \\ 0 & (a_2/a_0) A_0^{-1} A_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&\qquad \cdot \left\{ \begin{bmatrix} (\Delta a_{V0}/a_0) A_0^{-1} \eta_0 \\ -(\Sigma a_{V0}/a_0) A_0^{-1} \eta_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/a_0) A_0^{-1} \eta_1 \\ -(\Delta a_{V1}/a_0) A_0^{-1} \eta_1 \end{bmatrix} + h^2 \begin{bmatrix} (\Delta a_{V2}/a_0) A_0^{-1} \eta_2 \\ -(\Sigma a_{V2}/a_0) A_0^{-1} \eta_2 \end{bmatrix} + O(h^3) \right\}.
\end{aligned}
$$

Applying Lemma A9 with

$$
A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & -(a_1/a_0) A_0^{-1} A_1 \\ -(a_1/a_0) A_0^{-1} A_1 & 0 \end{bmatrix} + h \begin{bmatrix} (a_2/a_0) A_0^{-1} A_2 & 0 \\ 0 & (a_2/a_0) A_0^{-1} A_2 \end{bmatrix} + O\left(h^2\right),
$$

we have

$$
\begin{aligned}
&\left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h \begin{bmatrix} 0 & -(a_1/a_0) A_0^{-1} A_1 \\ -(a_1/a_0) A_0^{-1} A_1 & 0 \end{bmatrix} + h^2 \begin{bmatrix} (a_2/a_0) A_0^{-1} A_2 & 0 \\ 0 & (a_2/a_0) A_0^{-1} A_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - h \begin{bmatrix} 0 & -(a_1/a_0) A_0^{-1} A_1 \\ -(a_1/a_0) A_0^{-1} A_1 & 0 \end{bmatrix} \\
&\qquad + h^2 \begin{bmatrix} (a_1/a_0)^2 A_0^{-1} A_1 A_0^{-1} A_1 - (a_2/a_0) A_0^{-1} A_2 & 0 \\ 0 & (a_1/a_0)^2 A_0^{-1} A_1 A_0^{-1} A_1 - (a_2/a_0) A_0^{-1} A_2 \end{bmatrix} + O(h^3).
\end{aligned}
$$

By (A8),

$$
e_1' U_h^{-1} = e_1'.
$$

Thus,

$$
\begin{aligned}
&\tilde{\beta}_{\mathrm{lo},V}(h) \\
&= \left\{ e_1' - h[0 \quad -(a_1/a_0) e_1' A_0^{-1} A_1] + h^2 [(a_1/a_0)^2 e_1' A_0^{-1} A_1 A_0^{-1} A_1 - (a_2/a_0) e_1' A_0^{-1} A_2 \quad 0] + O(h^3) \right\} \\
&\qquad \cdot \left\{ \begin{bmatrix} (\Delta a_{V0}/a_0) A_0^{-1} \eta_0 \\ -(\Sigma a_{V0}/a_0) A_0^{-1} \eta_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/a_0) A_0^{-1} \eta_1 \\ -(\Delta a_{V1}/a_0) A_0^{-1} \eta_1 \end{bmatrix} + h^2 \begin{bmatrix} (\Delta a_{V2}/a_0) A_0^{-1} \eta_2 \\ -(\Sigma a_{V2}/a_0) A_0^{-1} \eta_2 \end{bmatrix} + O(h^3) \right\} \\
&= \frac{\Delta a_{V0}}{a_0} (A_0^{-1} \eta_0)_1 + h \left\{ \frac{\Sigma a_{V1}}{a_0} (A_0^{-1} \eta_1)_1 - \frac{\Sigma a_{V0} a_1}{a_0^2} (A_0^{-1} A_1 A_0^{-1} \eta_0)_1 \right\} \\
&\qquad + h^2 \left\{ \frac{\Delta a_{V2}}{a_0} (A_0^{-1} \eta_2)_1 - \frac{\Delta a_{V1} a_1}{a_0^2} (A_0^{-1} A_1 A_0^{-1} \eta_1)_1 + \frac{\Delta a_{V0} a_1^2}{a_0^3} (A_0^{-1} A_1 A_0^{-1} A_1 A_0^{-1} \eta_0)_1 \right. \\
&\qquad\qquad \left. - \frac{\Delta a_{V0} a_2}{a_0^2} (A_0^{-1} A_2 A_0^{-1} \eta_0)_1 \right\} + O(h^3).
\end{aligned}
$$

By Proposition A1,

$$
(A_0^{-1} \eta_0)_1 = 1, \quad (A_0^{-1} \eta_1)_1 = (A_0^{-1} A_1 A_0^{-1} \eta_0)_1 = 0,
$$

and

$$(A_0^{-1}\eta_2)_1 = (A_0^{-1}A_1A_0^{-1}\eta_1)_1 = (A_0^{-1}A_1A_0^{-1}A_1A_0^{-1}\eta_0)_1 = (A_0^{-1}A_2A_0^{-1}\eta_0)_1 = -C_{\text{bias}}(\kappa).$$

Thus,

$$\begin{aligned}
&\tilde{\beta}_{\text{lo},V}(h) \\
&= \frac{1}{a_0}\left\{\Delta a_{V0} - h^2 C_{\text{bias}}(\kappa)\left[\Delta a_{V2} - \frac{a_1}{a_0}\Delta a_{V1} - \Delta a_{V0}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right] + O(h^3)\right\} \\
&= \frac{1}{a_0}\left\{\Delta a_{V0} + h^2\left[\Gamma_{2,V}^{\text{lo}} + C_{\text{bias}}(\kappa)\Delta a_{V0}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right]\right\} + O(h^3).
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\beta}_{\text{lo}}(h) - \beta_0 &= \frac{\tilde{\beta}_{\text{lo},Y}(h)}{\tilde{\beta}_{\text{lo},X}(h)} - \beta_0 \\
&= \frac{\Delta a_{Y0}\left\{1 + h^2 C_{\text{bias}}(\kappa)\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right\} + h^2\Gamma_{2,Y}^{\text{lo}} + O(h^3)}{\Delta a_{X0}\left\{1 + h^2 C_{\text{bias}}(\kappa)\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right\} + h^2\Gamma_{2,X}^{\text{lo}} + O(h^3)} - \frac{\Delta a_{Y0}}{\Delta a_{X0}} \\
&= \frac{h^2\Gamma_2^{\text{lo}} + O(h^3)}{\Delta a_{X0}\left\{1 + h^2 C_{\text{bias}}(\kappa)\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right\} + h^2\Gamma_{2,X}^{\text{lo}} + O(h^3)} \\
&= \frac{h^2\Gamma_2^{\text{lo}}(\Delta a_{X0}) + O(h^3)}{\Delta a_{X0} + O(h^2)} \\
&= \frac{h^2\Gamma_2^{\text{lo}}(\Delta a_{X0}) + O(h^3)}{\Delta a_{X0}}(1 + O(h^2)) \\
&= h^2\Gamma_2^{\text{lo}} + O(h^3).
\end{aligned}$$

□

Theorem A2. *The expression of* Γ_2^{lo} *in Theorem A1 is equivalent to that in Theorem 2.*

Proof of Theorem A2. Recalling the definition of $\mu_{V,\text{lo}}$, we have

$$\mu_{V,\text{lo}}(r) = \frac{\Delta G_V(r)}{g(r)}.$$

Through tedious algebra, we obtain that

$$\begin{aligned}
\mu_{V,\text{lo}}^{''}(0) &= \frac{1}{g(0)}\left\{\Delta G_V^{''}(0) - \frac{2g'(0)}{g(0)}\Delta G_V^{'}(0) + \left(\frac{2g'^2(0)}{g^2(0)} - \frac{g''(0)}{g(0)}\right)\Delta G_V(0)\right\} \\
&= \frac{1}{a_0}\left\{2\Delta a_{V2} - \frac{2a_1}{a_0}\Delta a_{V1} + \left(\frac{2a_1^2}{a_0^2} - \frac{2a_2}{a_0}\right)\Delta a_{V0}\right\} \\
&= \frac{2}{a_0}\left\{\Gamma_{2,V}^{\text{lo}} + \Delta a_{V0}\left(\frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}\right)\right\},
\end{aligned}$$

and

$$\Delta\mu_X(0) = \frac{\Delta G_X(0)}{g(0)} = \frac{\Delta a_{X0}}{a_0}.$$

As a result,

$$\frac{\mu''_{Y,\text{lo}}(0) - \mu''_{X,\text{lo}}(0)\beta_0}{2\mu_{X,\text{lo}}(0)} = \frac{1}{\Delta a_{X0}} \left(\Gamma^{\text{lo}}_{2,Y} - \beta_0 \Gamma^{\text{lo}}_{2,X}\right).$$

□

A.4.3 Bias analysis for the upper-level estimator

We restate Theorem 3 into three pieces with a formal definition of Γ_1^{up} and Γ_2^{up}. The first result is about the general case.

Theorem A3. *Assume that*

$$\tilde{M}_2 \tilde{M}_0 \neq \tilde{M}_1 M_1.$$

Let

$$\begin{aligned}
\Gamma^{\text{up}}_{1,V} &= \Sigma a_{V1} \frac{(\tilde{A}_0^{-1}\eta_1)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \Sigma a_{V0} \frac{\tilde{a}_1}{\tilde{a}_0} \frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} \\
\Gamma^{\text{up}}_{2,V} &= \Delta a_{V2} \frac{(\tilde{A}_0^{-1}\eta_2)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \Delta a_{V1} \frac{\tilde{a}_1}{\tilde{a}_0} \frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_1)_1}{(\tilde{A}_0^{-1}\eta_0)_1} \\
&\quad + \Sigma a_{V0} (\eta_0' \tilde{A}_0^{-1}\eta_0) \left\{ -\frac{2\nu_1}{\tilde{a}_0} \frac{(\tilde{A}_0^{-1}\eta_1)_1}{(\tilde{A}_0^{-1}\eta_0)_1} + \frac{2\tilde{a}_1\nu_0}{\tilde{a}_0^2} \frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} \right\}.
\end{aligned}$$

Under Assumptions 1, 2, and 4,

$$\begin{aligned}
\tilde{\beta}_{\text{up},V}(h) = \frac{(\tilde{A}_0^{-1}\eta_0)_1}{\tilde{a}_0} \Bigg\{ & \Delta a_{V0} + h\Gamma^{\text{up}}_{1,V} \\
& + h^2 \left[\Gamma^{\text{up}}_{2,V} + \Delta a_{V0} \left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2} \frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0} \frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} \right) \right] + O(h^3) \Bigg\},
\end{aligned}$$

and

$$\tilde{\beta}_{\text{up}}(h) = \beta_0 + h\Gamma^{\text{up}}_1 + h^2\Gamma^{\text{up}}_2 + O\left(h^3\right),$$

where

$$\Gamma^{\text{up}}_1 = \frac{1}{\Delta a_{X0}} \left(\Gamma^{\text{up}}_{1,Y} - \beta_0 \Gamma^{\text{up}}_{1,X}\right), \quad \Gamma^{\text{up}}_2 = \frac{1}{\Delta a_{X0}} \left(\Gamma^{\text{up}}_{2,Y} - \beta_0 \Gamma^{\text{up}}_{2,X} - \Gamma^{\text{up}}_{1,X}\Gamma^{\text{up}}_1\right). \tag{A30}$$

Proof of Theorem A3. It is easy to see that $\tilde{a}_0 \neq 0$ iff $\tilde{M}_2 M_0 \neq \tilde{M}_1 M_1$. By (A6),

$$\tilde{\beta}_{\text{up},V}(h) = \tilde{\beta}^+_{\text{up},V}(h) - \tilde{\beta}^-_{\text{up},V}(h) = e_1'(U\Sigma_{\text{up}}(h)U^{-1})^{-1}UP_V(h),$$

By Lemma A1,

$$
\begin{aligned}
&\tilde{\beta}_{\mathrm{up},V}(h) \\
&= e_1' U_h^{-1} \left\{ \begin{bmatrix} \tilde{a}_0 \tilde{A}_0 & 0 \\ 0 & \tilde{a}_0 \tilde{A}_0 \end{bmatrix} + h \begin{bmatrix} 0 & -\tilde{a}_1 \tilde{A}_1 \\ -\tilde{a}_1 \tilde{A}_1 & 2\nu_0 \eta_0 \eta_0' \end{bmatrix} + h^2 \begin{bmatrix} \tilde{a}_2 \tilde{A}_2 & -2\nu_1 \eta_1 \eta_0' \\ -2\nu_1 \eta_0 \eta_1' & \tilde{a}_2 \tilde{A}_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&\qquad \cdot \left\{ \begin{bmatrix} \Delta a_{V0} \eta_0 \\ -\Sigma a_{V0} \eta_0 \end{bmatrix} + h \begin{bmatrix} \Sigma a_{V1} \eta_1 \\ -\Delta a_{V1} \eta_1 \end{bmatrix} + h^2 \begin{bmatrix} \Delta a_{V2} \eta_2 \\ -\Sigma a_{V2} \eta_2 \end{bmatrix} + O(h^3) \right\} \\
&= e_1' \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h \begin{bmatrix} 0 & -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 \\ -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 & 2(\nu_0/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix} + h^2 \begin{bmatrix} (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 & -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \eta_0' \\ -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0' \eta_1 & (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&\qquad \cdot \left\{ \begin{bmatrix} (\Delta a_{V0}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \\ -(\Sigma a_{V0}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \\ -(\Delta a_{V1}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \end{bmatrix} + h^2 \begin{bmatrix} (\Delta a_{V2}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_2 \\ -(\Sigma a_{V2}/\tilde{a}_0) \tilde{A}_0^{-1} \eta_2 \end{bmatrix} + O(h^3) \right\},
\end{aligned}
$$

where the last line uses (A8). Applying Lemma A9 with

$$
A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 \\ -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 & 2(\nu_0/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix} + h \begin{bmatrix} (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 & -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \eta_0' \\ -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0' \eta_1 & (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 \end{bmatrix} + O\left(h^2\right),
$$

we have

$$
\begin{aligned}
&e_1' \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h \begin{bmatrix} 0 & -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 \\ -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 & 2(\nu_0/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix} + h^2 \begin{bmatrix} (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 & -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \eta_0' \\ -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0' \eta_1 & (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 \end{bmatrix} + O\left(h^3\right) \right\}^{-1} \\
&= e_1' \Bigg\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - h \begin{bmatrix} 0 & -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 \\ -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 & 2(\nu_0/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix} \\
&\qquad - h^2 \begin{bmatrix} (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 & -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_1 \eta_0' \\ -2(\nu_1/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0' \eta_1 & (\tilde{a}_2/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_2 \end{bmatrix} + h^2 \begin{bmatrix} 0 & -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 \\ -(\tilde{a}_1/\tilde{a}_0) \tilde{A}_0^{-1} \tilde{A}_1 & 2(\nu_0/\tilde{a}_0) \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix}^2 + O(h^3) \Bigg\} \\
&= e_1' - h[0 \quad -(\tilde{a}_1/\tilde{a}_0) e_1' \tilde{A}_0^{-1} \tilde{A}_1] - h^2 [(\tilde{a}_2/\tilde{a}_0) e_1' \tilde{A}_0^{-1} \tilde{A}_2 \quad -2(\nu_1/\tilde{a}_0) e_1' \tilde{A}_0^{-1} \eta_1 \eta_0'] \\
&\qquad + h^2 \begin{bmatrix} (\tilde{a}_1/\tilde{a}_0)^2 e_1' \tilde{A}_0^{-1} \tilde{A}_1 \tilde{A}_0^{-1} \tilde{A}_1 & -2(\nu_0 \tilde{a}_1/\tilde{a}_0^2) e_1' \tilde{A}_0^{-1} \tilde{A}_1 \tilde{A}_0^{-1} \eta_0 \eta_0' \end{bmatrix} + O(h^3) \\
&= e_1' + h[0 \quad (\tilde{a}_1/\tilde{a}_0) e_1' \tilde{A}_0^{-1} \tilde{A}_1] \\
&\qquad + h^2 \begin{bmatrix} (\tilde{a}_1/\tilde{a}_0)^2 e_1' \tilde{A}_0^{-1} \tilde{A}_1 \tilde{A}_0^{-1} \tilde{A}_1 - (\tilde{a}_2/\tilde{a}_0) e_1' \tilde{A}_0^{-1} \tilde{A}_2 & 2(\nu_1/\tilde{a}_0)(\tilde{A}_0^{-1} \eta_1)_1 \eta_0' - 2(\nu_0 \tilde{a}_1/\tilde{a}_0^2)(\tilde{A}_0^{-1} \tilde{A}_1 \tilde{A}_0^{-1} \eta_0)_1 \eta_0' \end{bmatrix} \\
&\qquad + O(h^3).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\tilde{\beta}_{\mathrm{up},V}(h) \\
&= \frac{\Delta a_{V0}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_0)_1 + h\left\{\frac{\Sigma a_{V1}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_1)_1 - \frac{\Sigma a_{V0}\tilde{a}_1}{\tilde{a}_0^2}(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1\right\} \\
&\quad + h^2\Bigg\{\frac{\Delta a_{V2}}{\tilde{a}_0}(\tilde{A}_0^{-1}\eta_2)_1 - \frac{\Delta a_{V1}\tilde{a}_1}{\tilde{a}_0^2}(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_1)_1 + \frac{\Delta a_{V0}\tilde{a}_1^2}{\tilde{a}_0^3}(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1 - \frac{\Delta a_{V0}\tilde{a}_2}{\tilde{a}_0^2}(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1 \\
&\qquad - 2\frac{\Sigma a_{V0}\nu_1}{\tilde{a}_0^2}(\tilde{A}_0^{-1}\eta_1)_1(\eta_0'\tilde{A}_0^{-1}\eta_0) + 2\frac{\Sigma a_{V0}\nu_0\tilde{a}_1}{\tilde{a}_0^3}(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_1)_1(\eta_0'\tilde{A}_0^{-1}\eta_0)\Bigg\} \\
&= \frac{(\tilde{A}_0^{-1}\eta_0)_1}{\tilde{a}_0}\Bigg\{\Delta a_{V0} + h\Gamma_{1,V}^{\mathrm{up}} \\
&\quad + h^2\left[\Gamma_{2,V}^{\mathrm{up}} + \Delta a_{V0}\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right] + O(h^3)\Bigg\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\tilde{\beta}_{\mathrm{up}}(h) - \beta_0 = \frac{\tilde{\beta}_{\mathrm{up},Y}(h)}{\tilde{\beta}_{\mathrm{up},X}(h)} - \beta_0 \\
&= \frac{\Delta a_{Y0} + h\Gamma_{1,Y}^{\mathrm{up}} + h^2\left[\Gamma_{2,Y}^{\mathrm{up}} + \Delta a_{Y0}\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right] + O(h^3)}{\Delta a_{X0} + h\Gamma_{1,X}^{\mathrm{up}} + h^2\left[\Gamma_{2,X}^{\mathrm{up}} + \Delta a_{X0}\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right] + O(h^3)} - \frac{\Delta a_{Y0}}{\Delta a_{X0}} \\
&= \frac{\Delta a_{Y0}\left\{1 + h^2\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right\} + h\Gamma_{1,Y}^{\mathrm{up}} + h^2\Gamma_{2,Y}^{\mathrm{up}} + O(h^3)}{\Delta a_{X0}\left\{1 + h^2\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right\} + h\Gamma_{1,X}^{\mathrm{up}} + h^2\Gamma_{2,X}^{\mathrm{up}} + O(h^3)} - \frac{\Delta a_{Y0}}{\Delta a_{X0}} \\
&= \frac{h\Gamma_1^{\mathrm{up}}(\Delta a_{X0}) + h^2\{\Gamma_2^{\mathrm{up}} + \Gamma_1^{\mathrm{up}}(\Gamma_{1,X}^{\mathrm{up}}/\Delta a_{X0})\}(\Delta a_{X0}) + O(h^3)}{\Delta a_{X0}\left\{1 + h^2\left(\frac{\tilde{a}_1^2}{\tilde{a}_0^2}\frac{(\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1} - \frac{\tilde{a}_2}{\tilde{a}_0}\frac{(\tilde{A}_0^{-1}\tilde{A}_2\tilde{A}_0^{-1}\eta_0)_1}{(\tilde{A}_0^{-1}\eta_0)_1}\right)\right\} + h\Gamma_{1,X}^{\mathrm{up}} + h^2\Gamma_{2,X}^{\mathrm{up}} + O(h^3)}. \qquad \text{(A31)}
\end{aligned}
$$

This entails that

$$
\tilde{\beta}_{\mathrm{up}}(h) - \beta_0 = \frac{h\Gamma_1^{\mathrm{up}} + h^2\{\Gamma_2^{\mathrm{up}} + \Gamma_1^{\mathrm{up}}(\Gamma_{1,X}^{\mathrm{up}}/\Delta a_{X0})\} + O(h^3)}{1 + h(\Gamma_{1,X}^{\mathrm{up}}/\Delta a_{X0})}(1 + O(h^2)) = h\Gamma_1^{\mathrm{up}} + h^2\Gamma_2^{\mathrm{up}} + O(h^3).
$$

□

The second piece is about the special case under the moment conditions on κ.

Theorem A4. $\Gamma_1^{\mathrm{up}} = 0$ *for all* $(\tilde{a}_0, \tilde{a}_1, a_{X0}^+, a_{X0}^-)$ *iff*

$$
\frac{\tilde{M}_2}{M_2} = \frac{\tilde{M}_1}{M_1} = \frac{\tilde{M}_0}{M_0}.
$$

Under this condition, Γ_2^{up} *in the second-order bias term can be simplified as*

$$
\Gamma_2^{\mathrm{up}} = \frac{1}{\Delta a_{X0}}\left(\Gamma_{2,Y}^{\mathrm{up}} - \beta_0\Gamma_{2,X}^{\mathrm{up}}\right)
$$

where

$$
\Gamma_{2,V}^{\mathrm{up}} = -C_{\mathrm{bias}}(\kappa)\left(\Delta a_{V2} - \frac{\tilde{a}_1}{\tilde{a}_0}\Delta a_{V1}\right).
$$

Proof of Theorem A4. By definition, $\Gamma_1^{\text{up}} = 0$ for all $(\tilde{a}_0, \tilde{a}_1, a_{X0}^+, a_{X0}^-)$ iff

$$(\tilde{A}_0^{-1}\eta_1)_1 = (\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1 = 0.$$

The first condition is equivalent to the collinearity between η_1 and the second column of $\tilde{A}_0$, i.e., $(M_1, M_2)' \propto (\tilde{M}_1, \tilde{M}_2)'$. Similarly, the second condition is equivalent to $\tilde{A}_1\tilde{A}_0^{-1}\eta_0 \propto (\tilde{M}_1, \tilde{M}_2)'$. Since $(\tilde{M}_1, \tilde{M}_2)$ is the first column of $\tilde{A}_1$, it can be equivalently written as $(\tilde{A}_0^{-1}\eta_0)_2 = 0$. Using the same argument as in the first condition, this is equivalent to the collinearity between η_0 and the first column of $\tilde{A}_0$, i.e., $(M_0, M_1)' \propto (\tilde{M}_0, \tilde{M}_1)'$. The simplified expression of Γ_2^{up} can be easily derived by noting that $\tilde{A}_k \propto A_k$. □

The last result is the equivalence between the expressions of Γ_2^{up}.

Theorem A5. *The expression of Γ_2^{up} in Theorem A4 is equivalent to that in Theorem 3.*

Proof of Theorem A5. Recalling the definition of $\mu_{V,\text{lo}}$, we have

$$\mu_{V,\text{up}}(r) = \frac{\Delta G_V(r)}{\tilde{g}(r)}.$$

Applying the same argument as in the proof of Theorem A2 with a_k replaced by $\tilde{a}_k$, we conclude that

$$\frac{\mu_{Y,\text{up}}''(0) - \mu_{X,\text{up}}''(0)\beta_0}{2\mu_{X,\text{up}}(0)} = \frac{1}{\Delta a_{X0}}\left(\Gamma_{2,Y}^{\text{up}} - \beta_0\Gamma_{2,X}^{\text{up}}\right).$$

□

Next, we prove Proposition 2, which states that the uniform kernel is the only monotonic kernel that satisfies the moment condition. (21).

Proof of Proposition 2. By definition, $\tilde{M}_1/M_1 = \tilde{M}_0/M_0$ implies

$$\begin{aligned}
&\frac{\int_0^1 x\kappa^2(x)dx}{\int_0^1 x\kappa(x)dx} = \frac{\int_0^1 \kappa^2(x)dx}{\int_0^1 \kappa(x)dx} \\
&\Longrightarrow \left(\int_0^1 x\kappa^2(x)dx\right)\left(\int_0^1 \kappa(x)dx\right) = \left(\int_0^1 x\kappa(x)dx\right)\left(\int_0^1 \kappa^2(x)dx\right) \\
&\Longrightarrow \int_0^1\int_0^1 \left\{x\kappa^2(x)\kappa(y) + y\kappa^2(y)\kappa(x) - x\kappa(x)\kappa^2(y) - y\kappa(y)\kappa^2(x)\right\}dxdy = 0 \\
&\Longrightarrow \int_0^1\int_0^1 \kappa(x)\kappa(y)(\kappa(x) - \kappa(y))(x - y)dxdy = 0.
\end{aligned}$$

Since κ is monotonic on $[0, 1)$, $(\kappa(x) - \kappa(y))(x - y)$ has the same sign across all (x, y). Thus, the equality holds iff $\kappa(x)$ is a constant on $[0, 1]$.

On the other hand, if κ is a uniform kernel, it is clear that $\tilde{M}_k \propto M_k$ for all k. □

A.4.4 More about the bias formulae

We first prove Proposition 3, which gives sufficient conditions under which the two estimators have the same second-order bias when equipped with a uniform kernel.

Proof of Proposition 3. From the definitions of $\Gamma_{2,V}^{\text{lo}}$ in Theorem A1 and $\Gamma_{2,V}^{\text{up}}$ in Theorem A4, we see that

$$\Gamma_2^{\text{lo}} = \Gamma_2^{\text{up}} \Longleftarrow \frac{\tilde{a}_1}{\tilde{a}_0} = \frac{a_1}{a_0}.$$

When $s_j \perp\!\!\!\perp (J_i, r_j)$,

$$g(r) = \mathbb{E}[s_j]\mathbb{E}[J_i \mid r_j = r]f(r), \quad \tilde{g}(r) = \mathbb{E}[s_j^2]\mathbb{E}[J_i \mid r_j = r]f(r).$$

Then

$$\frac{\tilde{a}_1}{\tilde{a}_0} = \frac{a_1}{a_0} = \frac{(d/dr)\mathbb{E}[J_i \mid r_j = r]f(r)|_{r=0}}{\mathbb{E}[J_i \mid r_j = 0]f(0)}.$$

□

Next, we calculate the first-order bias for the upper-level estimator with commonly used non-uniform kernels.

Example 1. (Triangular kernel) When $\kappa(x) = (1 - |x|)_+$,

$$M_k = \int_0^1 x^k(1-x)dx = \frac{\Gamma(k+1)\Gamma(2)}{\Gamma(k+3)} = \frac{1}{(k+2)(k+1)},$$

and

$$\tilde{M}_k = \int_0^1 x^k(1-x)^2dx = \frac{\Gamma(k+1)\Gamma(3)}{\Gamma(k+4)} = \frac{2}{(k+3)(k+2)(k+1)}.$$

Then

$$(\tilde{A}_0^{-1}\eta_0)_1 = 2/3, \quad (\tilde{A}_0^{-1}\eta_1)_1 = -1/3, \quad (\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1 = -2/9.$$

As a result,

$$\tilde{\Gamma}_{1,X} = -\frac{1}{2}\frac{\Sigma a_{X1}}{\tilde{a}_0} + \frac{1}{3}\frac{\tilde{a}_1}{\tilde{a}_0}\frac{\Sigma a_{X0}}{\tilde{a}_0}, \quad \tilde{\Gamma}_{1,Y} = -\frac{1}{2}\frac{\Sigma a_{Y1}}{\tilde{a}_0} + \frac{1}{3}\frac{\tilde{a}_1}{\tilde{a}_0}\frac{\Sigma a_{Y0}}{\tilde{a}_0}$$

and the first-order bias is

$$h \cdot \frac{1}{\tilde{a}_0}\left\{-\frac{1}{2}\left(\Sigma a_{Y1} - \beta_0 \Sigma a_{X1}\right) + \frac{1}{3}\frac{\tilde{a}_1}{\tilde{a}_0}\left(\Sigma a_{Y0} - \beta_0 \Sigma a_{X0}\right)\right\}.$$

Example 2. (Epanechnikov kernel) When $\kappa(x) = (1 - x^2)_+$,

$$M_k = \int_0^1 x^k(1-x^2)dx = \int_0^1 (x^k - x^{k+2})dx = \frac{2}{(k+3)(k+1)},$$

and

$$\tilde{M}_k = \int_0^1 x^k(1-x^2)^2dx = \int_0^1 (x^k - 2x^{k+2} + x^{k+4})dx = \frac{8}{(k+5)(k+3)(k+1)}.$$

Then

$$(\tilde{A}_0^{-1}\eta_0)_1 = 0.710, \quad (\tilde{A}_0^{-1}\eta_1)_1 = -0.247, \quad (\tilde{A}_0^{-1}\tilde{A}_1\tilde{A}_0^{-1}\eta_0)_1 = -0.153.$$

As a result, the first-order bias is

$$h \cdot \frac{1}{\tilde{a}_0}\left\{-0.348\left(\Sigma a_{Y1} - \beta_0 \Sigma a_{X1}\right) + 0.216\frac{\tilde{a}_1}{\tilde{a}_0}\left(\Sigma a_{Y0} - \beta_0 \Sigma a_{X0}\right)\right\}.$$

A.4.5 Bias analysis without local linear controls

Without local linear controls, we can define the controls as

$$c_j^{(const)} = (I(r_j > 0), I(r_j < 0))'.$$

The pseudo-estimands for both estimators can be expressed as follows.

- (Lower-level estimator) $\tilde{\beta}_{\text{lo},V}^{(const)}(h) = \tilde{\beta}_{\text{lo},V}^{+,(const)}(h) - \tilde{\beta}_{\text{lo},V}^{-,(const)}(h)$ where

$$\begin{aligned}
&(\tilde{\beta}_{\text{lo},V}^{+,(const)}(h), \tilde{\beta}_{\text{lo},V}^{-,(const)}(h))' \\
&= \mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j^{(const)} c_j^{(const)'}\right]^{-1} \mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j^{(const)} V_i\right] \\
&\triangleq \Sigma_{\text{lo}}^{(const)}(h)^{-1} P_V^{(const)}(h).
\end{aligned}$$

- (Upper-level estimator) $\tilde{\beta}_{\text{up},V}^{(const)}(h) = \tilde{\beta}_{\text{up},V}^{+,(const)}(h) - \tilde{\beta}_{\text{up},V}^{-,(const)}(h)$ where

$$\begin{aligned}
&(\tilde{\beta}_{\text{up},V}^{+,(const)}(h), \tilde{\beta}_{\text{up},V}^{-,(const)}(h))' \\
&= \mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j^{(const)}\right)\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j^{(const)}\right)'\right]^{-1} \mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j^{(const)} V_i\right] \\
&\triangleq \Sigma_{\text{up}}^{(const)}(h)^{-1} P_V^{(const)}(h).
\end{aligned}$$

Clearly, $\Sigma_{\text{lo/up}}^{(const)}(h)$ is the submatrix of $\Sigma_{\text{lo/up}}(h)$ corresponding to the first and third rows and columns, and $P_V^{(const)}(h)$ is the subvector of $P_V(h)$ corresponding to the first and third elements. Thus, the expansions of the Gram matrix and the cross-product terms can be obtained by symbolically replacing A_k, $\tilde{A}_k, \eta_k\eta_\ell'$ with their top-left element and η_k with its first element. [41]

Lemma A2.

$$\begin{aligned}
\Sigma_{\text{lo}}^{(const)}(h) &= h\left\{\begin{bmatrix} a_0 M_0 & 0 \\ 0 & a_0 M_0 \end{bmatrix} + h\begin{bmatrix} a_1 M_1 & 0 \\ 0 & -a_1 M_1 \end{bmatrix} + O\left(h^2\right)\right\} \\
\Sigma_{\text{up}}^{(const)}(h) &= h\left\{\begin{bmatrix} \tilde{a}_0 \tilde{M}_0 & 0 \\ 0 & \tilde{a}_0 \tilde{M}_0 \end{bmatrix} + h\begin{bmatrix} \tilde{a}_1 \tilde{M}_1 + \nu_0 M_0^2 & \nu_0 M_0^2 \\ \nu_0 M_0^2 & -\tilde{a}_1 \tilde{A}_1 + \nu_0 M_0^2 \end{bmatrix} + O\left(h^2\right)\right\} D_h \\
P_V^{(const)}(h) &= h\left\{\begin{bmatrix} a_{X0}^+ M_0 \\ a_{X0}^- M_0 \end{bmatrix} + h\begin{bmatrix} a_{X1}^+ M_1 \\ -a_{X1}^- M_1 \end{bmatrix} + O(h^2)\right\}.
\end{aligned}$$

[41] We could prove Lemma A2 under a weaker smoothness assumption that replaces $\mathcal{C}^2$ by $\mathcal{C}^1$ in Assumption 2. We do not pursue it as it is not our primary focus.

Let $U^{(const)} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$. *Then*

$$U^{(const)}\Sigma_{\text{lo}}^{(const)}(h)\left(U^{(const)}\right)^{-1} = h\left\{\begin{bmatrix} a_0M_0 & 0 \\ 0 & a_0M_0 \end{bmatrix} + h\begin{bmatrix} 0 & -a_1M_1 \\ -a_1M_1 & 0 \end{bmatrix} + O\left(h^2\right)\right\}$$

$$U^{(const)}\Sigma_{\text{up}}^{(const)}(h)\left(U^{(const)}\right)^{-1} = h\left\{\begin{bmatrix} \tilde{a}_0\tilde{M}_0 & 0 \\ 0 & \tilde{a}_0\tilde{M}_0 \end{bmatrix} + h\begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1 \\ -\tilde{a}_1\tilde{M}_1 & 2\nu_0M_0^2 \end{bmatrix} + O\left(h^2\right)\right\}$$

$$U^{(const)}P_V^{(const)}(h) = h\left\{\begin{bmatrix} \Delta a_{X0}M_0 \\ -\Sigma a_{X0}M_0 \end{bmatrix} + h\begin{bmatrix} \Sigma a_{X1}M_1 \\ -\Delta a_{X1}M_1 \end{bmatrix} + O(h^2)\right\},$$

With Lemma A2, we can follow the proof of Theorem A1 and A3 to derive the asymptotic bias of the lower- and upper-level estimators with local linear controls.

Theorem A6. *Let*

$$\Gamma_{1,V}^{(const),\text{lo}} = \frac{M_1}{M_0}\left(\Sigma a_{X1} - \frac{a_1}{a_0}\Sigma a_{X0}\right). \tag{A32}$$

Under Assumptions 1, 2, and 4,

$$\tilde{\beta}_{\text{lo},V}^{(const)}(h) = \frac{1}{a_0}\left\{\Delta a_{X0} + h\Gamma_{1,V}^{(const),\text{lo}} + O(h^2)\right\},$$

and

$$\tilde{\beta}_{\text{lo}}^{(const)}(h) = \beta_0 + h\Gamma_1^{(const),\text{lo}} + O\left(h^2\right),$$

where

$$\Gamma_1^{(const),\text{lo}} = \frac{1}{\Delta a_{X0}}\left(\Gamma_{1,Y}^{(const),\text{lo}} - \beta_0\Gamma_{1,X}^{(const),\text{lo}}\right). \tag{A33}$$

Proof of Theorem A6. By definition

$$\tilde{\beta}_{\text{lo},V}(h) = e_1'(U\Sigma_{\text{lo}}^{(const)}(h)U^{-1})^{-1}UP_V^{(const)}(h),$$

where $e_1 = (1,0,0,0)'$. By Lemma A2,

$$\begin{aligned}
&\tilde{\beta}_{\text{lo},V}(h) \\
&= e_1'\left\{\begin{bmatrix} a_0M_0 & 0 \\ 0 & a_0M_0 \end{bmatrix} + h\begin{bmatrix} 0 & -a_1M_1 \\ -a_1M_1 & 0 \end{bmatrix} + O\left(h^2\right)\right\}^{-1}\left\{\begin{bmatrix} \Delta a_{V0}M_0 \\ -\Sigma a_{V0}M_0 \end{bmatrix} + h\begin{bmatrix} \Sigma a_{V1}M_1 \\ -\Delta a_{V1}M_1 \end{bmatrix} + O(h^2)\right\} \\
&= e_1'\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h\begin{bmatrix} 0 & -a_1M_1/a_0M_0 \\ -a_1M_1/a_0M_0 & 0 \end{bmatrix} + O\left(h^2\right)\right\}^{-1} \\
&\qquad \cdot\left\{\begin{bmatrix} \Delta a_{V0}/a_0 \\ -\Sigma a_{V0}/a_0 \end{bmatrix} + h\begin{bmatrix} (\Sigma a_{V1}/a_0)(M_1/M_0)) \\ -(\Delta a_{V1}/a_0)(M_1/M_0)) \end{bmatrix} + O(h^2)\right\}.
\end{aligned}$$

Applying Lemma A9 with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & -a_1 M_1 / a_0 M_0 \\ -a_1 M_1 / a_0 M_0 & 0 \end{bmatrix} + O(h),$$

we have

$$\begin{aligned} &\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} 0 & -a_1 M_1 / a_0 M_0 \\ -a_1 M_1 / a_0 M_0 & 0 \end{bmatrix} + O\left(h^2\right) \right\}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - h \begin{bmatrix} 0 & -a_1 M_1 / a_0 M_0 \\ -a_1 M_1 / a_0 M_0 & 0 \end{bmatrix} + O\left(h^2\right). \end{aligned}$$

Combining the above two pieces, we obtain that

$$\begin{aligned} &\tilde{\beta}_{\text{lo},V}^{(const)}(h) \\ &= \left\{ e_1' + h[0 \;\; a_1 M_1 / a_0 M_0] + O(h^2) \right\} \left\{ \begin{bmatrix} \Delta a_{V0} / a_0 \\ -\Sigma a_{V0} / a_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1} / a_0)(M_1 / M_0)) \\ -(\Delta a_{V1} / a_0)(M_1 / M_0)) \end{bmatrix} + O(h^2) \right\} \\ &= \frac{\Delta a_{V0}}{a_0} + h \left\{ \frac{\Sigma a_{V1}}{a_0} \frac{M_1}{M_0} - \frac{\Sigma a_{V0} a_1}{a_0^2} \frac{M_1}{M_0} \right\} + O(h^2) \\ &= \frac{\Delta a_{V0}}{a_0} + h \frac{\Gamma_{1,V}^{(const),\text{lo}}}{a_0} + O(h^2). \end{aligned}$$

Then

$$\begin{aligned} &\tilde{\beta}_{\text{lo}}^{(const)}(h) - \beta_0 = \frac{\tilde{\beta}_{\text{lo},Y}^{(const)}(h)}{\tilde{\beta}_{\text{lo},X}^{(const)}(h)} - \beta_0 \\ &= \frac{\Delta a_{Y0} + h\Gamma_{1,Y}^{(const),\text{lo}} + O(h^2)}{\Delta a_{X0} + h\Gamma_{1,X}^{(const),\text{lo}} + O(h^2)} - \frac{\Delta a_{Y0}}{\Delta a_{X0}} = \frac{h\Gamma_1^{(const),\text{lo}}(\Delta a_{X0}) + O(h^2)}{\Delta a_{X0} + h\Gamma_{1,X}^{(const),\text{lo}} + O(h^2)} \\ &= \frac{h\Gamma_1^{(const),\text{lo}}(\Delta a_{X0}) + O(h^2)}{\Delta a_{X0}}(1 + O(h)) = h\Gamma_1^{(const),\text{lo}} + O(h^2). \end{aligned}$$

□

Theorem A7. *Let*

$$\Gamma_{1,V}^{(const),\text{up}} = \frac{M_1}{M_0} \Sigma a_{X1} - \frac{\tilde{a}_1}{\tilde{a}_0} \frac{\tilde{M}_1}{\tilde{M}_0} \Sigma a_{X0}. \tag{A34}$$

Under Assumptions 1, 2, and 4,

$$\tilde{\beta}_{\text{up},V}^{(const)}(h) = \frac{1}{\tilde{a}_0} \frac{M_0}{\tilde{M}_0} \left\{ \Delta a_{V0} + h\Gamma_{1,V}^{(const),\text{up}} + O(h^2) \right\},$$

and

$$\tilde{\beta}_{\text{up}}^{(const)}(h) = \beta_0 + h\Gamma_1^{(const),\text{up}} + O\left(h^2\right),$$

where

$$\Gamma_1^{(const),\mathrm{up}} = \frac{1}{\Delta a_{X0}} \left(\Gamma_{1,Y}^{(const),\mathrm{up}} - \beta_0 \Gamma_{1,X}^{(const),\mathrm{up}} \right). \tag{A35}$$

Proof of Theorem A7. By definition

$$\tilde{\beta}_{\mathrm{up},V}(h) = e_1'(U\Sigma_{\mathrm{up}}^{(const)}(h)U^{-1})^{-1}UP_V^{(const)}(h).$$

By Lemma A2,

$$\begin{aligned}
&\tilde{\beta}_{\mathrm{up},V}(h) \\
&= e_1' \left\{ \begin{bmatrix} \tilde{a}_0\tilde{M}_0 & 0 \\ 0 & \tilde{a}_0\tilde{M}_0 \end{bmatrix} + h \begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1 \\ -\tilde{a}_1\tilde{M}_1 & 2\nu_0 M_0^2 \end{bmatrix} + O\left(h^2\right) \right\}^{-1} \left\{ \begin{bmatrix} \Delta a_{V0} M_0 \\ -\Sigma a_{V0} M_0 \end{bmatrix} + h \begin{bmatrix} \Sigma a_{V1} M_1 \\ -\Delta a_{V1} M_1 \end{bmatrix} + O(h^2) \right\} \\
&= e_1' \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 \\ -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 & 2\nu_0 M_0^2/\tilde{M}_0 \end{bmatrix} + O\left(h^2\right) \right\}^{-1} \\
&\qquad \cdot \left\{ \begin{bmatrix} \Delta a_{V0}/\tilde{a}_0 \\ -\Sigma a_{V0}/\tilde{a}_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/\tilde{a}_0)(M_1/M_0)) \\ -(\Delta a_{V1}/\tilde{a}_0)(M_1/M_0)) \end{bmatrix} + O(h^2) \right\}.
\end{aligned}$$

Applying Lemma A9 with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 \\ -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 & 2\nu_0 M_0^2/\tilde{M}_0 \end{bmatrix} + O\left(h\right),$$

we have

$$\begin{aligned}
&\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 \\ -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 & 2\nu_0 M_0^2/\tilde{M}_0 \end{bmatrix} + O\left(h^2\right) \right\}^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - h \begin{bmatrix} 0 & -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 \\ -\tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0 & 2\nu_0 M_0^2/\tilde{M}_0 \end{bmatrix} + O\left(h^2\right).
\end{aligned}$$

Combining the above two pieces, we obtain that

$$\begin{aligned}
&\tilde{\beta}_{\mathrm{up},V}^{(const)}(h) \\
&= \left\{ e_1' + h[0 \;\; \tilde{a}_1\tilde{M}_1/\tilde{a}_0\tilde{M}_0] + O(h^2) \right\} \left\{ \begin{bmatrix} \Delta a_{V0}/\tilde{a}_0 \\ -\Sigma a_{V0}/\tilde{a}_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/\tilde{a}_0)(M_1/M_0)) \\ -(\Delta a_{V1}/\tilde{a}_0)(M_1/M_0)) \end{bmatrix} + O(h^2) \right\} \\
&= \frac{\Delta a_{V0}}{\tilde{a}_0} + h \left\{ \frac{\Sigma a_{V1}}{\tilde{a}_0} \frac{M_1}{M_0} - \frac{\Sigma a_{V0}\tilde{a}_1}{\tilde{a}_0^2} \frac{\tilde{M}_1}{\tilde{M}_0} \right\} + O(h^2) \\
&= \frac{\Delta a_{V0}}{\tilde{a}_0} + h \frac{\Gamma_{1,V}^{(const),\mathrm{up}}}{\tilde{a}_0} + O(h^2).
\end{aligned}$$

Then

$$\begin{aligned}
&\tilde{\beta}_{\text{up}}^{(const)}(h) - \beta_0 = \frac{\tilde{\beta}_{\text{up},Y}^{(const)}(h)}{\tilde{\beta}_{\text{up},X}^{(const)}(h)} - \beta_0 \\
&= \frac{\Delta a_{Y0} + h\Gamma_{1,Y}^{(const),\text{up}} + O(h^2)}{\Delta a_{X0} + h\Gamma_{1,X}^{(const),\text{up}} + O(h^2)} - \frac{\Delta a_{Y0}}{\Delta a_{X0}} = \frac{h\Gamma_1^{(const),\text{up}}(\Delta a_{X0}) + O(h^2)}{\Delta a_{X0} + h\Gamma_{1,X}^{(const),\text{up}} + O(h^2)} \\
&= \frac{h\Gamma_1^{(const),\text{up}}(\Delta a_{X0}) + O(h^2)}{\Delta a_{X0}}(1 + O(h)) = h\Gamma_1^{(const),\text{up}} + O(h^2).
\end{aligned}$$

□

A.5 Asymptotic variance analysis in the standard asymptotic regime

A.5.1 A key lemma

Lemma A3. *For $V \in \{X, Y\}$, let*

$$\tau_{\text{lo},V}(h) = (\tilde{\beta}_{\text{lo},V}^{+}(h), \tilde{\gamma}_{\text{lo},V}^{+}(h), \tilde{\beta}_{\text{lo},V}^{-}(h), \tilde{\gamma}_{\text{lo},V}^{-}(h))', \quad \tau_{\text{up},V}(h) = (\tilde{\beta}_{\text{up},V}^{+}(h), \tilde{\gamma}_{\text{up},V}^{+}(h), \tilde{\beta}_{\text{up},V}^{-}(h), \tilde{\gamma}_{\text{up},V}^{-}(h))'.$$

Under Assumptions 1, 2, and 4,

$$\tau_{\text{lo},V}(h) = \begin{bmatrix} a_{V0}^{+}/a_0 \\ -a_{V0}^{+}a_1/a_0^2 + a_{V1}^{+}/a_0 \\ a_{V0}^{-}/a_0 \\ a_{V0}^{-}a_1/a_0^2 - a_{V1}^{-}/a_0 \end{bmatrix} + O(h) = O(1), \quad \tau_{\text{up},V}(h) = \begin{bmatrix} a_{V0}^{+}/\tilde{a}_0 \\ -a_{V0}^{+}\tilde{a}_1/\tilde{a}_0^2 + a_{V1}^{+}/\tilde{a}_0 \\ a_{V0}^{-}/\tilde{a}_0 \\ a_{V0}^{-}\tilde{a}_1/\tilde{a}_0^2 - a_{V1}^{-}/\tilde{a}_0 \end{bmatrix} + O(h) = O(1).$$

Proof of Lemma A3. Truncating the expansions to $O(h^2)$ in Lemma A1, we have

$$\begin{aligned}
U\Sigma_{\text{lo}}(h)U^{-1} &= hU_h \left\{ \begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0 \end{bmatrix} + h \begin{bmatrix} 0 & -a_1A_1 \\ -a_1A_1 & 0 \end{bmatrix} + O\left(h^2\right) \right\} U_h \\
U\Sigma_{\text{up}}(h)U^{-1} &= hU_h \left\{ \begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0 \end{bmatrix} + h \begin{bmatrix} 0 & -\tilde{a}_1\tilde{A}_1 \\ -\tilde{a}_1\tilde{A}_1 & 2\nu_0\eta_0\eta_0' \end{bmatrix} + O\left(h^2\right) \right\} U_h \\
UP_V(h) &= hU_h \left\{ \begin{bmatrix} \Delta a_{V0}\eta_0 \\ -\Sigma a_{V0}\eta_0 \end{bmatrix} + h \begin{bmatrix} \Sigma a_{V1}\eta_1 \\ -\Delta a_{V1}\eta_1 \end{bmatrix} + O(h^2) \right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\tau_{\text{lo},V}(h) &= U^{-1}\left(U\Sigma_{\text{lo}}(h)U^{-1}\right)^{-1}UP_V(h) \\
&= U^{-1}U_h^{-1}\left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h \begin{bmatrix} 0 & -(a_1/a_0)A_0^{-1}A_1 \\ -(a_1/a_0)A_0^{-1}A_1 & 0 \end{bmatrix} + O(h^2) \right\}^{-1} \\
&\quad \cdot \left\{ \begin{bmatrix} (\Delta a_{V0}/a_0)A_0^{-1}\eta_0 \\ -(\Sigma a_{V0}/a_0)A_0^{-1}\eta_0 \end{bmatrix} + h \begin{bmatrix} (\Sigma a_{V1}/a_0)A_0^{-1}\eta_1 \\ -(\Delta a_{V1}/a_0)A_0^{-1}\eta_1 \end{bmatrix} + O(h^2) \right\}.
\end{aligned}$$

By Lemma A10,

$$\left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + h\begin{bmatrix} 0 & -(a_1/a_0)A_0^{-1}A_1 \\ -(a_1/a_0)A_0^{-1}A_1 & 0 \end{bmatrix} + O(h^2)\right\}^{-1}$$
$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - h\begin{bmatrix} 0 & -(a_1/a_0)A_0^{-1}A_1 \\ -(a_1/a_0)A_0^{-1}A_1 & 0 \end{bmatrix} + O(h^2).$$

Thus,

$$U_h U\tau_{\mathrm{lo},V}(h) = \begin{bmatrix} (\Delta a_{V0}/a_0)A_0^{-1}\eta_0 \\ -(\Sigma a_{V0}/a_0)A_0^{-1}\eta_0 \end{bmatrix} + h\left\{\begin{bmatrix} -(\Sigma a_{V0}a_1/a_0^2)A_0^{-1}A_1A_0^{-1}\eta_0 \\ (\Delta a_{V0}a_1/a_0^2)A_0^{-1}A_1A_0^{-1}\eta_0 \end{bmatrix} + \begin{bmatrix} (\Sigma a_{V1}/a_0)A_0^{-1}\eta_1 \\ -(\Delta a_{V1}/a_0)A_0^{-1}\eta_1 \end{bmatrix}\right\} + O(h^2).$$

By Proposition A1,

$$U_h U\tau_{\mathrm{lo},V}(h) = \begin{bmatrix} (\Delta a_{V0}/a_0)e_1 \\ -(\Sigma a_{V0}/a_0)e_1 \end{bmatrix} + h\begin{bmatrix} (-\Sigma a_{V0}a_1/a_0^2 + \Sigma a_{V1}/a_0)e_2 \\ (\Delta a_{V0}a_1/a_0^2 - \Delta a_{V1}/a_0)e_2 \end{bmatrix} + O(h^2)$$
$$\implies UD_h\tau_{\mathrm{lo},V}(h) = \begin{bmatrix} \Delta a_{V0}/a_0 \\ (-\Sigma a_{V0}a_1/a_0^2 + \Sigma a_{V1}/a_0)h \\ -\Sigma a_{V0}/a_0 \\ (\Delta a_{V0}a_1/a_0^2 - \Delta a_{V1}/a_0)h \end{bmatrix} + O(h^2)$$
$$\implies D_h\tau_{\mathrm{lo},V}(h) = U^{-1}\begin{bmatrix} \Delta a_{V0}/a_0 \\ (-\Sigma a_{V0}a_1/a_0^2 + \Sigma a_{V1}/a_0)h \\ -\Sigma a_{V0}/a_0 \\ (\Delta a_{V0}a_1/a_0^2 - \Delta a_{V1}/a_0)h \end{bmatrix} + O(h^2)$$
$$= \frac{1}{2}\begin{bmatrix} (\Delta a_{V0} + \Sigma a_{V0})/a_0 \\ (-\Sigma a_{V0}a_1/a_0^2 + \Sigma a_{V1}/a_0 - \Delta a_{V0}a_1/a_0^2 + \Delta a_{V1}/a_0)h \\ (\Sigma a_{V0} - \Delta a_{V0})/a_0 \\ (-\Delta a_{V0}a_1/a_0^2 + \Delta a_{V1}/a_0 + \Sigma a_{V0}a_1/a_0^2 - \Sigma a_{V1}/a_0)h \end{bmatrix} + O(h^2)$$
$$= \begin{bmatrix} a_{V0}^+/a_0 \\ (-a_{V0}^+a_1/a_0^2 + a_{V1}^+/a_0)h \\ a_{V0}^-/a_0 \\ (a_{V0}^-a_1/a_0^2 - a_{V1}^-/a_0)h \end{bmatrix} + O(h^2)$$
$$\implies \tau_{\mathrm{lo},V}(h) = \begin{bmatrix} a_{V0}^+/a_0 \\ -a_{V0}^+a_1/a_0^2 + a_{V1}^+/a_0 \\ a_{V0}^-/a_0 \\ a_{V0}^-a_1/a_0^2 - a_{V1}^-/a_0 \end{bmatrix} + O(h) = O(1).$$

Using a similar argument, we obtain that

$$\begin{aligned}\tau_{\mathrm{up},V}(h) &= U^{-1}\left(U\Sigma_{\mathrm{up}}(h)U^{-1}\right)^{-1}UP_V(h)\\ &= U^{-1}U_h^{-1}\left\{\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} - h\begin{bmatrix}0 & -(\tilde{a}_1/\tilde{a}_0)\tilde{A}_0^{-1}\tilde{A}_1\\ -(\tilde{a}_1/\tilde{a}_0)\tilde{A}_0^{-1}\tilde{A}_1 & 2(\nu_0/\tilde{a}_0)\tilde{A}_0^{-1}\eta_0\eta_0'\end{bmatrix} + O(h^2)\right\}\\ &\quad\cdot\left\{\begin{bmatrix}(\Delta a_{V0}/\tilde{a}_0)\tilde{A}_0^{-1}\eta_0\\ -(\Sigma a_{V0}/\tilde{a}_0)\tilde{A}_0^{-1}\eta_0\end{bmatrix} + h\begin{bmatrix}(\Sigma a_{V1}/\tilde{a}_0)\tilde{A}_0^{-1}\eta_1\\ -(\Delta a_{V1}/\tilde{a}_0)\tilde{A}_0^{-1}\eta_1\end{bmatrix} + O(h^2)\right\}.\end{aligned}$$

When the uniform kernel is used, $\tilde{A}_k = A_k$. By Proposition A1,

$$\begin{aligned}U_hU\tau_{\mathrm{up},V}(h) &= \begin{bmatrix}(\Delta a_{V0}/\tilde{a}_0)A_0^{-1}\eta_0\\ -(\Sigma a_{V0}/\tilde{a}_0)A_0^{-1}\eta_0\end{bmatrix} + h\left\{\begin{bmatrix}-(\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2)A_0^{-1}A_1A_0^{-1}\eta_0\\ (\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2)A_0^{-1}A_1A_0^{-1}\eta_0\end{bmatrix} + \begin{bmatrix}(\Sigma a_{V1}/\tilde{a}_0)A_0^{-1}\eta_1\\ -(\Delta a_{V1}/\tilde{a}_0)A_0^{-1}\eta_1\end{bmatrix}\right.\\ &\qquad\left. + \begin{bmatrix}0\\ 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)A_0^{-1}\eta_0\eta_0'A_0^{-1}\eta_0\end{bmatrix}\right\} + O(h^2)\\ &= \begin{bmatrix}(\Delta a_{V0}/\tilde{a}_0)e_1\\ -(\Sigma a_{V0}/\tilde{a}_0)e_1\end{bmatrix} + h\begin{bmatrix}(-\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Sigma a_{V1}/\tilde{a}_0)e_2\\ (\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 - \Delta a_{V1}/\tilde{a}_0)e_2 + 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)(\eta_0'A_0^{-1}\eta_0)e_1\end{bmatrix} + O(h^2)\\ &= \begin{bmatrix}(\Delta a_{V0}/\tilde{a}_0)e_1\\ -(\Sigma a_{V0}/\tilde{a}_0)e_1\end{bmatrix} + h\begin{bmatrix}(-\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Sigma a_{V1}/\tilde{a}_0)e_2\\ (\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 - \Delta a_{V1}/\tilde{a}_0)e_2 + 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0e_1\end{bmatrix} + O(h^2)\\ &= \begin{bmatrix}\Delta a_{V0}/\tilde{a}_0\\ (-\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Sigma a_{V1}/\tilde{a}_0)h\\ -\Sigma a_{V0}/\tilde{a}_0 + 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 - \Delta a_{V1}/\tilde{a}_0)h\end{bmatrix} + O(h^2).\end{aligned}$$

This entails that

$$\begin{aligned}D_h\tau_{\mathrm{up},V}(h) &= U^{-1}\begin{bmatrix}\Delta a_{V0}/\tilde{a}_0\\ (-\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Sigma a_{V1}/\tilde{a}_0)h\\ -\Sigma a_{V0}/\tilde{a}_0 + 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 - \Delta a_{V1}/\tilde{a}_0)h\end{bmatrix} + O(h^2)\\ &= \frac{1}{2}\begin{bmatrix}(\Delta a_{V0} + \Sigma a_{V0})/\tilde{a}_0 - 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (-\Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Sigma a_{V1}/\tilde{a}_0 - \Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Delta a_{V1}/\tilde{a}_0)h\\ (\Sigma a_{V0} - \Delta a_{V0})/\tilde{a}_0 - 2(\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (-\Delta a_{V0}\tilde{a}_1/\tilde{a}_0^2 + \Delta a_{V1}/\tilde{a}_0 + \Sigma a_{V0}\tilde{a}_1/\tilde{a}_0^2 - \Sigma a_{V1}/\tilde{a}_0)h\end{bmatrix} + O(h^2)\\ &= \begin{bmatrix}a_{V0}^+/\tilde{a}_0 - (\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (-a_{V0}^+\tilde{a}_1/\tilde{a}_0^2 + a_{V1}^+/\tilde{a}_0)h\\ a_{V0}^-/\tilde{a}_0 - (\Sigma a_{V0}\nu_0/\tilde{a}_0^2)M_0h\\ (a_{V0}^-\tilde{a}_1/\tilde{a}_0^2 - a_{V1}^-/\tilde{a}_0)h\end{bmatrix} + O(h^2)\\ \Longrightarrow \tau_{\mathrm{up},V}(h) &= \begin{bmatrix}a_{V0}^+/\tilde{a}_0\\ -a_{V0}^+\tilde{a}_1/\tilde{a}_0^2 + a_{V1}^+/\tilde{a}_0\\ a_{V0}^-/\tilde{a}_0\\ a_{V0}^-\tilde{a}_1/\tilde{a}_0^2 - a_{V1}^-/\tilde{a}_0\end{bmatrix} + O(h) = O(1).\end{aligned}$$

□

A.5.2 Variance analysis for the lower-level estimator

We restate Theorem 5.

Theorem A8. *Under Assumptions 1, 2, 4, and 5,*

$$\sqrt{Nh}(\hat{\beta}_{\mathrm{lo}} - \tilde{\beta}_{\mathrm{lo}}(h)) \xrightarrow{d} N\left(0, \sigma^2_{\mathrm{lo}}\right),$$

where

$$\sigma^2_{\mathrm{lo}} = \frac{C_{\mathrm{var}}(\kappa)\tilde{a}_0}{(\Delta a_{X0})^2}\left\{\mathbb{E}^{(2)}[(U_j^+ - \mathbb{E}^{(1)}[U_j^+])^2] + \mathbb{E}^{(2)}[(U_j^- - \mathbb{E}^{(1)}[U_j^-])^2]\right\}.$$

This expression is equivalent to the one in Theorem 5.

***Proof of Theorem** A.5.3.*
Step 1. By (A20)

$$(hU_h)^{-1}(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1} - U\tilde{\Sigma}_{\mathrm{lo}}(h)U^{-1})U_h^{-1} = (hD_h)^{-1}(\hat{\Sigma}_{\mathrm{lo}}(h) - \tilde{\Sigma}_{\mathrm{lo}}(h))D_h^{-1} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right).$$

By Lemma A1,

$$(hU_h)^{-1}(U\tilde{\Sigma}_{\mathrm{lo}}(h)U^{-1})U_h^{-1} = \begin{bmatrix} a_0 A_0 & 0 \\ 0 & a_0 A_0 \end{bmatrix} + O(h).$$

Using the fact that $U_h = UD_hU^{-1}$, and combining the above two pieces, we have

$$(hU_h)^{-1}(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1})U_h^{-1} = \begin{bmatrix} a_0 A_0 & 0 \\ 0 & a_0 A_0 \end{bmatrix} + o_{\mathbb{P}}(1) = \Omega_{\mathbb{P}}(1). \tag{A36}$$

Step 2. Let

$$U_{\mathrm{lo},i} = Y_i - \tilde{\beta}_{\mathrm{lo}}(h)X_i, \quad U^{\pm}_{\mathrm{lo},j} = Y_j^{\pm} - \tilde{\beta}_{\mathrm{lo}}(h)X_j^{\pm},$$

and

$$\tilde{\beta}_{\mathrm{lo},U}(h) = \tilde{\beta}_{\mathrm{lo},Y}(h) - \tilde{\beta}_{\mathrm{lo}}(h)\tilde{\beta}_{\mathrm{lo},X}(h), \quad \tilde{\gamma}_{\mathrm{lo},U}(h) = \tilde{\gamma}_{\mathrm{lo},Y}(h) - \tilde{\beta}_{\mathrm{lo}}(h)\gamma_{\mathrm{lo},X}(h),$$

and

$$\tau_{\mathrm{lo},U}(h) = \tau_{\mathrm{lo},Y}(h) - \tilde{\beta}_{\mathrm{lo}}(h)\tau_{\mathrm{lo},X}(h).$$

Then $\hat{P}_Y(h) - \tilde{\beta}_{\mathrm{lo}}(h)\hat{P}_X(h) = \hat{P}_{U_{\mathrm{lo}}}(h)$, and

$$\hat{P}_{U_{\mathrm{lo}}}(h) - \hat{\Sigma}_{\mathrm{lo}}(h)\tau_{\mathrm{lo},U}(h) = \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h)).$$

Moreover, since $e_1'U = e_1' - e_3'$,

$$e_1'U\tau_{\mathrm{lo},U}(h) = (e_1 - e_3)'\tau_{\mathrm{lo},U}(h) = (\tilde{\beta}^+_{\mathrm{lo},Y}(h) - \tilde{\beta}^-_{\mathrm{lo},Y}(h)) - \tilde{\beta}_{\mathrm{lo}}(h)(\tilde{\beta}^+_{\mathrm{lo},X}(h) - \tilde{\beta}^-_{\mathrm{lo},X}(h)) = 0 \tag{A37}$$

Note that

$$\mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h))\right] = P_Y(h) - \Sigma_{\mathrm{lo}}(h)\tau_{\mathrm{lo},U}(h) = 0. \tag{A38}$$

Step 3. Now we bound the $(2+\delta)$-th moment of $\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h))$. By (A16),

$$\|s_j\kappa_h(r_j)c_j\| \le \sqrt{2}\|s_j\kappa_h(r_j)\| \le L, \tag{A39}$$

where L is defined in (A17). Then

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jU_{\mathrm{lo},i}\right\|^{2+\delta}\right] \\
&\le \mathbb{E}\left[J_i^{1+\delta}\sum_{j\in\mathcal{J}_i}\|s_j\kappa_h(r_j)c_jU_{\mathrm{lo},i}\|^{2+\delta}\right] && \text{(by AM-GM inequality)} \\
&= \mathbb{E}\left[J_i^{2+\delta}\|s_j\kappa_h(r_j)c_jU_{\mathrm{lo},i}\|^{2+\delta}\right] && \text{(by exchangeability)} \\
&= (2J^*L)^{2+\delta}\,\mathbb{E}\left[|s_jU_{\mathrm{lo},i}|^{2+\delta}\kappa_h(r_j)\right] && \text{(by boundedness of } J_i \text{ and (A39))} \\
&\le (2J^*L)^{2+\delta}\,\mathbb{E}\left[|U_{\mathrm{lo},i}|^{2+\delta}\kappa_h(r_j)\right] && (s_j \le 1) \\
&= (2J^*L)^{2+\delta}\left(\mathbb{E}\left[|U_{\mathrm{lo},i}|^{2+\delta} \mid r_j = 0\right] + o(1)\right) && \text{(by Assumption 5 and Lemma A6).}
\end{aligned}$$

For a sufficiently small h,

$$\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jU_{\mathrm{lo},i}\right\|^{2+\delta}\right] \le 2\,(2J^*L)^{2+\delta}\,\mathbb{E}\left[|U_{\mathrm{lo},i}|^{2+\delta} \mid r_j = 0\right]. \tag{A40}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jc_j'\tau_{\mathrm{lo},U}\right\|^{2+\delta}\right] \\
&\le \|\tau_{\mathrm{lo},U}\|^{2+\delta}\cdot\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jc_j'\right\|_{\mathrm{op}}^{2+\delta}\right] \\
&\le \|\tau_{\mathrm{lo},U}\|^{2+\delta}\cdot\mathbb{E}\left[J_i^{1+\delta}\sum_{j\in\mathcal{J}_i}\left\|s_j\kappa_h(r_j)c_jc_j'\right\|_{\mathrm{op}}^{2+\delta}\right] && \text{(by AM-GM inequality)} \\
&= \|\tau_{\mathrm{lo},U}\|^{2+\delta}\cdot\mathbb{E}\left[J_i^{2+\delta}\left\|s_j\kappa_h(r_j)c_jc_j'\right\|_{\mathrm{op}}^{2+\delta}\right] && \text{(by exchangeability)} \\
&\le (2J^*L)^{2+\delta}\,\|\tau_{\mathrm{lo},U}\|^{2+\delta} && \text{(by boundedness of } J_i \text{ and (A17)).}
\end{aligned}$$

By AM-GM inequality, for a sufficiently small h,

$$\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U})\right\|^{2+\delta}\right] \le 2\,(2J^*L)^{2+\delta}\left(\mathbb{E}\left[|U_{\mathrm{lo},i}|^{2+\delta} \mid r_j = 0\right] + \|\tau_{\mathrm{lo},U}\|^{2+\delta}\right) = O(1),$$

where the last claim stems from Assumption 5 and Lemma A3.

Step 4. By Lemma A11,

$$\begin{aligned}
&\sqrt{N}\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right)^{-1/2}\cdot\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right]\\
&=\sqrt{N}\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right)^{-1/2}\\
&\quad\cdot\left\{\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right]-\mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right]\right\}\\
&\xrightarrow{d} N(0,I_4) \qquad\qquad (A41)
\end{aligned}$$

Next, we compute the limiting covariance matrix. By definition,

$$\begin{aligned}
&U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h)\\
&=U_{\mathrm{lo},i}-\tilde{\beta}^+_{\mathrm{lo},U}(h)I(r_j>0)-\tilde{\beta}^-_{\mathrm{lo},U}(h)I(r_j<0)-\tilde{\gamma}^+_{\mathrm{lo},U}(h)r_jI(r_j>0)-\tilde{\gamma}^-_{\mathrm{lo},U}(h)r_jI(r_j<0)\\
&=\underbrace{(U^+_{\mathrm{lo},j}-\tilde{\beta}^+_{\mathrm{lo},U}(h))I(r_j>0)+(U^-_{\mathrm{lo},j}-\tilde{\beta}^-_{\mathrm{lo},U}(h))I(r_j<0)}_{\triangleq B_{j1}}\\
&\quad-\underbrace{\{\tilde{\gamma}^+_{\mathrm{lo},U}(h)r_jI(r_j>0)+\tilde{\gamma}^-_{\mathrm{lo},U}(h)r_jI(r_j<0)\}}_{\triangleq B_{j2}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right)\\
&=\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right)^2\right] \qquad \text{(because the term is mean-zero)}\\
&=\mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j^2\kappa_h^2(r_j)c_jc_j'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))^2\right]\\
&\quad+\mathbb{E}\left[\sum_{j,k\in\mathcal{J}_i,j\neq k} s_js_k\kappa_h(r_j)\kappa_h(r_k)c_jc_k'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))(U_{\mathrm{lo},i}-c_k'\tau_{\mathrm{lo},U}(h))\right]\\
&=\mathbb{E}\left[J_is_j^2\kappa_h^2(r_j)c_jc_j'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))^2\right]\\
&\quad+\mathbb{E}\left[J_i(J_i-1)s_js_k\kappa_h(r_j)\kappa_h(r_k)c_jc_k'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))(U_{\mathrm{lo},i}-c_k'\tau_{\mathrm{lo},U}(h))\right] \qquad \text{(by exchangeability).}\\
&\qquad\qquad (A42)
\end{aligned}$$

We first consider the second term of (A42). Applying Lemma A9 with

$$V=J_i(J_i-1)s_js_kc_jc_k'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))(U_{\mathrm{lo},i}-c_k'\tau_{\mathrm{lo},U}(h)).$$

Since J_i, s_j, s_k are bounded, Assumption 5 implies that $\mathbb{E}[V \mid r_j=0, r_k=0]$ is bounded. Following the same

argument for (A15), we obtain that

$$\mathbb{E}\left[J_i(J_i-1)s_j s_k \kappa_h(r_j)\kappa_h(r_k)c_j c_k'(U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))(U_{\mathrm{lo},i}-c_k'\tau_{\mathrm{lo},U}(h))\right] = D_h\{O(h^2)\}D_h. \tag{A43}$$

Now we return to the first term of (A42). We decompose it as follows:

$$\begin{aligned}
&\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' (U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))^2\right] \\
&= \mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' (B_{j1}-B_{j2})^2\right] \\
&= \mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j1}^2\right] - 2\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j1}B_{j2}\right] + \mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j2}^2\right].
\end{aligned}$$

By Lemma A7, similar to (A25) and (A26), we have

$$\begin{aligned}
&\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j1}^2\right] \\
&= hD_h\left\{\begin{bmatrix} \mathbb{E}[J_i s_j^2 (U_{\mathrm{lo},j}^+ - \tilde\beta_{\mathrm{lo},U}^+(h))^2 \mid r_j=0]\tilde A_0 & 0 \\ 0 & \mathbb{E}[J_i s_j^2 (U_{\mathrm{lo},j}^- - \tilde\beta_{\mathrm{lo},U}^-(h))^2 \mid r_j=0]\tilde A_0 \end{bmatrix} + O(h)\right\} D_h,
\end{aligned}$$

$$\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j1}B_{j2}\right] = \mathbb{E}\left[J_i s_j^2 r_j \kappa_h^2(r_j) c_j c_j' B_{j1}(\tilde\gamma_{\mathrm{lo},U}^+(h)I(r_j>0) + \tilde\gamma_{\mathrm{lo},U}^-(h)I(r_j<0))\right] = hD_h\{O(h)\}D_h,$$

and

$$\mathbb{E}\left[J_i s_j^2 \kappa_h^2(r_j) c_j c_j' B_{j2}^2\right] = \mathbb{E}\left[J_i s_j^2 r_j^2 \kappa_h^2(r_j) c_j c_j' (\tilde\gamma_{\mathrm{lo},U}^+(h)I(r_j>0) + \tilde\gamma_{\mathrm{lo},U}^-(h)I(r_j<0))^2\right] = hD_h\{O(h^2)\}D_h.$$

Combining the above three pieces and (A43), we conclude that

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j) c_j (U_{\mathrm{lo},i}-c_j'\tau_{\mathrm{lo},U}(h))\right) \\
&= hD_h\left\{\begin{bmatrix} \mathbb{E}[J_i s_j^2 (U_{\mathrm{lo},j}^+ - \tilde\beta_{\mathrm{lo},U}^+(h))^2 \mid r_j=0]\tilde A_0 & 0 \\ 0 & \mathbb{E}[J_i s_j^2 (U_{\mathrm{lo},j}^- - \tilde\beta_{\mathrm{lo},U}^-(h))^2 \mid r_j=0]\tilde A_0 \end{bmatrix} + O(h)\right\} D_h \\
&= hD_h\tilde a_0\left\{\begin{bmatrix} \mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^+ - \tilde\beta_{\mathrm{lo},U}^+(h))^2]\tilde A_0 & 0 \\ 0 & \mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^- - \tilde\beta_{\mathrm{lo},U}^-(h))^2]\tilde A_0 \end{bmatrix} + O(h)\right\} D_h,
\end{aligned}$$

where the last line follows from Lemma A3 and the definition of $\mathbb{E}^{(1)}$. By Lemma A3,

$$\tilde\beta_{\mathrm{lo},U}^{\pm}(h) = \frac{\Delta a_{Y0}^{\pm} - \tilde\beta_{\mathrm{lo}}(h)\Delta a_{X0}^{\pm}}{a_0} + o(1) = \mathbb{E}^{(1)}[U_{\mathrm{lo},j}^+] + o(1).$$

By Assumption 5,

$$\mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^+ - \tilde\beta_{\mathrm{lo},U}^+(h))^2] = \mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^+ - \mathbb{E}^{(1)}[U_{\mathrm{lo},j}^+] + o(1))^2] = \mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^+ - \mathbb{E}^{(1)}[U_{\mathrm{lo},j}^+])^2] + o(1).$$

Note that

$$U_{\mathrm{lo},j}^+ = U_j + (\tilde\beta_{\mathrm{lo}}(h) - \beta_0)X_j^{\pm}.$$

By Theorem 2 and Assumption 5,

$$\mathbb{E}^{(2)}[(U_{\mathrm{lo},j}^+ - \mathbb{E}^{(1)}[U_{\mathrm{lo},j}^+])^2] = \mathbb{E}^{(2)}[(U_j^+ - \mathbb{E}^{(1)}[U_j^+])^2] + o(1).$$

Similarly,

$$\mathbb{E}^{(2)}[(U^-_{\mathrm{lo},j} - \tilde{\beta}^-_{\mathrm{lo},U}(h))^2] = \mathbb{E}^{(2)}[(U^-_j - \mathbb{E}^{(1)}[U^-_j])^2] + o(1).$$

Therefore,

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h))\right) \\
&= hD_h\tilde{a}_0\left\{\begin{bmatrix}\mathbb{E}^{(2)}[(U^+_j - \mathbb{E}^{(1)}[U^+_j])^2]\tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}[(U^-_j - \mathbb{E}^{(1)}[U^-_j])^2]\tilde{A}_0\end{bmatrix} + O(h)\right\}D_h.
\end{aligned}$$

Multiplying the sum by $h^{-1/2}D_h^{-1}$, we obtain that

$$\begin{aligned}
&\mathrm{Var}\left(h^{-1/2}D_h^{-1}\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h))\right) \\
&= \tilde{a}_0\begin{bmatrix}\mathbb{E}^{(2)}[(U^+_j - \mathbb{E}^{(1)}[U^+_j])^2]\tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}[(U^-_j - \mathbb{E}^{(1)}[U^-_j])^2]\tilde{A}_0\end{bmatrix} + o(1).
\end{aligned}$$

Similar to (A38),

$$\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{lo},i} - c_j'\tau_{\mathrm{lo},U}(h))\right] = \hat{P}_{U_{\mathrm{lo}}}(h) - \hat{\Sigma}_{\mathrm{lo}}(h)\tau_{\mathrm{lo},U}(h), \tag{A44}$$

By (A41),

$$\begin{aligned}
&\sqrt{N}\tilde{a}_0^{-1/2}\begin{bmatrix}\left(\mathbb{E}^{(2)}[(U^+_j - \mathbb{E}^{(1)}[U^+_j])^2]\right)\tilde{A}_0 & 0 \\ 0 & \left(\mathbb{E}^{(2)}[(U^-_j - \mathbb{E}^{(1)}[U^-_j])^2]\right)\tilde{A}_0\end{bmatrix}^{-1/2} \\
&\qquad\cdot h^{-1/2}D_h^{-1}\left(\hat{P}_{U_{\mathrm{lo}}}(h) - \hat{\Sigma}_{\mathrm{lo}}(h)\tau_{\mathrm{lo},U}\right) \\
&\xrightarrow{d} N(0, I_4)
\end{aligned}$$

We can rewrite it as:

$$\begin{aligned}
&\sqrt{\frac{N}{h}}D_h^{-1}\left(\hat{P}_{U_{\mathrm{lo}}}(h) - \hat{\Sigma}_{\mathrm{lo}}(h)\tau_{\mathrm{lo},U}\right) \\
&\xrightarrow{d} N\left(0, \tilde{a}_0\begin{bmatrix}\mathbb{E}^{(2)}[(U^+_j - \mathbb{E}^{(1)}[U^+_j])^2]\tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}[(U^-_j - \mathbb{E}^{(1)}[U^-_j])^2]\tilde{A}_0\end{bmatrix}\right)
\end{aligned} \tag{A45}$$

Step 5. By definition,

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\mathrm{lo}} - \tilde{\beta}_{\mathrm{lo}}(h)) \\
&= \sqrt{Nh}\left\{\frac{e_1'(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1})^{-1}U\hat{P}_Y(h)}{e_1'(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1})^{-1}U\hat{P}_X(h)} - \tilde{\beta}_{\mathrm{lo}}(h)\right\} \\
&= \sqrt{Nh}\frac{e_1'(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1})^{-1}U\hat{P}_{U_{\mathrm{lo}}}(h)}{e_1'(U\hat{\Sigma}_{\mathrm{lo}}(h)U^{-1})^{-1}U\hat{P}_X(h)}.
\end{aligned}$$

We have proved in Theorem 1 that

$$e_1'(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})^{-1}U\hat{P}_X(h) = \frac{\Delta a_{X0}}{a_0} + o_{\mathbb{P}}(1) = \frac{\Delta a_{X0}}{a_0}(1 + o_{\mathbb{P}}(1)). \tag{A46}$$

By (A37),

$$e_1'(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})^{-1}U\hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U} = e_1'U\tau_{\text{lo},U} = 0.$$

Thus,

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{lo}} - \tilde{\beta}_{\text{lo}}(h)) \\
&= \sqrt{Nh}\frac{e_1'(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})^{-1}U\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}}{e_1'(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})^{-1}U\hat{P}_X(h)} \\
&= \sqrt{Nh}\cdot\frac{a_0}{\Delta a_{X0}}\cdot\{e_1'(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})^{-1}U\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}\cdot(1 + o_{\mathbb{P}}(1)) \\
&= \frac{a_0}{\Delta a_{X0}}\cdot e_1'\left\{(hU_h)^{-1}(U\hat{\Sigma}_{\text{lo}}(h)U^{-1})U_h^{-1}\right\}^{-1}\cdot\sqrt{Nh}(hU_h)^{-1}U\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}\cdot(1 + o_{\mathbb{P}}(1)),
\end{aligned}$$

where the last line uses the fact that $e_1'U_h^{-1} = e_1'$. By (A36),

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{lo}} - \tilde{\beta}_{\text{lo}}(h)) \\
&= \frac{a_0}{\Delta a_{X0}}\cdot e_1'\left\{(hU_h)^{-1}(U\tilde{\Sigma}_{\text{lo}}(h)U^{-1})U_h\right\}^{-1}\cdot\sqrt{Nh}(hU_h)^{-1}U\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}\cdot(1 + o_{\mathbb{P}}(1)) \\
&= \frac{a_0}{\Delta a_{X0}}\cdot e_1'\left\{(hU_h)^{-1}(U\tilde{\Sigma}_{\text{lo}}(h)U^{-1})U_h\right\}^{-1}\cdot U\cdot\sqrt{\frac{N}{h}}D_h^{-1}\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}\cdot(1 + o_{\mathbb{P}}(1)).
\end{aligned}$$

where the last line uses the fact that $U_h^{-1}U = (UD_hU^{-1})^{-1}U = UD_h^{-1}$. By (A36),

$$(hU_h)^{-1}(U\tilde{\Sigma}_{\text{lo}}(h)U^{-1})U_h = \begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0 \end{bmatrix} + o(1).$$

Thus,

$$e_1'\left\{(hU_h)^{-1}(U\tilde{\Sigma}_{\text{lo}}(h)U^{-1})U_h\right\}^{-1} = e_1'\begin{bmatrix} a_0A_0 & 0 \\ 0 & a_0A_0 \end{bmatrix}^{-1} + o(1) = \frac{1}{a_0}[e_1'A_0^{-1} \quad 0]\cdot(1 + o(1)).$$

Then, we have

$$\sqrt{Nh}(\hat{\beta}_{\text{lo}} - \tilde{\beta}_{\text{lo}}(h)) = \frac{1}{\Delta a_{X0}}[e_1'A_0^{-1} \quad -e_1'A_0^{-1}]\cdot\sqrt{\frac{N}{h}}D_h^{-1}\left\{\hat{P}_{U_{\text{lo}}}(h) - \hat{\Sigma}_{\text{lo}}(h)\tau_{\text{lo},U}\right\}\cdot(1 + o_{\mathbb{P}}(1)).$$

By (A45), we have

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{lo}} - \tilde{\beta}_{\text{lo}}(h)) \\
&\xrightarrow{d} N\left(0, \frac{\tilde{a}_0}{(\Delta a_{X0})^2}\cdot e_1'A_0^{-1}\tilde{A}_0A_0^{-1}e_1\cdot\left\{\mathbb{E}^{(2)}[(U_j^+ - \mathbb{E}^{(1)}[U_j^+])^2] + \mathbb{E}^{(2)}[(U_j^- - \mathbb{E}^{(1)}[U_j^-])^2]\right\}\right).
\end{aligned} \tag{A47}$$

The proof is then completed by

$$e_1'A_0^{-1}\tilde{A}_0A_0^{-1}e_1 = C_{\text{var}}(\kappa).$$

□

A.5.3 Variance analysis for the upper-level estimator

We first present two lemmas to facilitate the proof.

Lemma A4. *Let A_n and B_n be two sequence of random vectors. Assume $A_n = \rho_n(A + o(1))$ for some constant invertible matrix A and scalar sequence ρ_n. Further, assume $\|\mathrm{Var}(B_n)\|_{\mathrm{op}} = o(\rho_n)$. Then*

$$\mathrm{Var}(A_n + B_n) = \mathrm{Var}(A_n) \cdot (1 + o(1)).$$

Proof. By definition,

$$\mathrm{Var}(A_n + B_n) = \mathrm{Var}(A_n) + \mathrm{Var}(B_n) + \mathrm{Cov}(A_n, B_n).$$

By Cauchy-Schwarz inequality,

$$\|\mathrm{Cov}(A_n, B_n)\|_{\mathrm{op}} \le \|\mathrm{Var}(A_n)\|_{\mathrm{op}}^{1/2} \|\mathrm{Var}(B_n)\|_{\mathrm{op}}^{1/2} = o(\rho_n).$$

Since A is invertible,

$$\mathrm{Var}(A_n + B_n) = \mathrm{Var}(A_n)(1 + o(1)).$$

□

Lemma A5. *Let $L \ge 2$ and V be a random variable such that $\mathbb{E}[|V| \mid R_1 = r_1, R_2 = r_2] \cdot u(r_1, r_2)$ is continuous in (r_1, r_2). Then*

$$\mathbb{E}[|V|\kappa_h(R_1)\kappa_h(R_2)\cdots\kappa_h(R_L)] = O(h^2).$$

Proof. Since κ is bounded,

$$\mathbb{E}[|V|\kappa_h(R_1)\kappa_h(R_2)\cdots\kappa_h(R_L)] \le \mathbb{E}[|V|\kappa_h(R_1)\kappa_h(R_2)] \cdot (\max_x \kappa(x))^{L-2}.$$

The proof is then completed by Lemma A9. □

Next, we restate Theorem 6.

Theorem. *Under Assumptions 1, 2, 4, and 5,*

$$\sqrt{Nh}(\hat{\beta}_{\mathrm{up}} - \tilde{\beta}_{\mathrm{up}}(h)) \xrightarrow{d} N\left(0, \sigma_{\mathrm{up}}^2\right),$$

where

$$\sigma_{\mathrm{up}}^2 = \frac{C_{\mathrm{var}}(\kappa)\tilde{a}_0}{(\Delta a_{X0})^2} \left\{ \mathbb{E}^{(2)}[(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+])^2] + \mathbb{E}^{(2)}[(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-])^2] \right\}.$$

This expression is equivalent to the one in Theorem 6.

***Proof of Theorem** A.5.3.*
Step 1. By (A24)

$$(hU_h)^{-1}(U\hat{\Sigma}_{\mathrm{up}}(h)U^{-1} - U\tilde{\Sigma}_{\mathrm{up}}(h)U^{-1})U_h^{-1} = (hD_h)^{-1}(\hat{\Sigma}_{\mathrm{up}}(h) - \tilde{\Sigma}_{\mathrm{up}}(h))D_h^{-1} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{Nh}}\right),$$

By Lemma A1,

$$(hU_h)^{-1}(U\tilde{\Sigma}_{\mathrm{up}}(h)U^{-1})U_h^{-1} = \begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0 \end{bmatrix} + O(h)$$

Using the fact that $U_h = UD_hU^{-1}$, and combining the above two pieces, we have

$$(hU_h)^{-1}(U\hat{\Sigma}_{\mathrm{up}}(h)U^{-1})U_h^{-1} = \begin{bmatrix} \tilde{a}_0\tilde{A}_0 & 0 \\ 0 & \tilde{a}_0\tilde{A}_0 \end{bmatrix} + o_{\mathbb{P}}(1) = \Omega_{\mathbb{P}}(1). \tag{A48}$$

Step 2. Let

$$U_{\mathrm{up},i} = Y_i - \tilde{\beta}_{\mathrm{up}}(h)X_i, \quad U^{\pm}_{\mathrm{up},j} = Y^{\pm}_j - \tilde{\beta}_{\mathrm{up}}(h)X^{\pm}_j,$$

and

$$\tilde{\beta}_{\mathrm{up},U}(h) = \tilde{\beta}_{\mathrm{up},Y}(h) - \tilde{\beta}_{\mathrm{up}}(h)\tilde{\beta}_{\mathrm{up},X}(h), \quad \tilde{\gamma}_{\mathrm{up},U}(h) = \tilde{\gamma}_{\mathrm{up},Y}(h) - \tilde{\beta}_{\mathrm{up}}(h)\tilde{\gamma}_{\mathrm{up},X}(h),$$

$$\tau_{\mathrm{up},U}(h) = \tau_{\mathrm{up},Y}(h) - \tilde{\beta}_{\mathrm{up}}(h)\tau_{\mathrm{up},X}(h).$$

Then $\hat{P}_Y(h) - \tilde{\beta}_{\mathrm{up}}(h)\hat{P}_X(h) = \hat{P}_{U_{\mathrm{up}}}(h)$, and

$$\hat{P}_{U_{\mathrm{up}}}(h) - \hat{\Sigma}_{\mathrm{up}}(h)\tau_{\mathrm{up},U}(h) = \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \left\{ U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \right)' \tau_{\mathrm{up},U}(h) \right\}.$$

Moreover, since $e_1'U = e_1' - e_3'$,

$$e_1'U\tau_{\mathrm{up},U}(h) = (e_1 - e_3)'\tau_{\mathrm{up},U}(h) = (\tilde{\beta}^+_{\mathrm{up},Y}(h) - \tilde{\beta}^-_{\mathrm{up},Y}(h)) - \tilde{\beta}_{\mathrm{up}}(h)(\tilde{\beta}^+_{\mathrm{up},X}(h) - \tilde{\beta}^-_{\mathrm{up},X}(h)) = 0 \tag{A49}$$

Note that

$$\mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \left(U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \right)' \tau_{\mathrm{up},U}(h) \right) \right] = P_Y(h) - \Sigma_{\mathrm{up}}(h)\tau_{\mathrm{up},U}(h) = 0. \tag{A50}$$

Step 3. Next, we bound the $(2+\delta)$-th moment of $\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \left(U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \right)' \tau_{\mathrm{up},U}(h) \right)$. Replacing $U_{\mathrm{lo},i}$ by $U_{\mathrm{up},i}$ in (A40), we obtain that

$$\mathbb{E}\left[\left\| \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_jU_{\mathrm{up},i} \right\|^{2+\delta} \right] \leq 2\left(2J^*L\right)^{2+\delta} \mathbb{E}\left[|U_{\mathrm{up},i}|^{2+\delta} \mid r_j = 0 \right]. \tag{A51}$$

Note that

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}\right\|^{2+\delta}\right] \\
&\leq \|\tau_{\mathrm{up},U}\|^{2+\delta}\cdot\mathbb{E}\left[\left\|\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\right\|_{\mathrm{op}}^{2+\delta}\right] \\
&= \|\tau_{\mathrm{up},U}\|^{2+\delta}\cdot\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right\|^{4+\delta}\right] \\
&\leq \|\tau_{\mathrm{up},U}\|^{2+\delta}\cdot\mathbb{E}\left[J_i^{3+\delta}\sum_{j\in\mathcal{J}_i}\|s_j\kappa_h(r_j)c_j\|^{4+\delta}\right] && \text{(by AM-GM inequality)} \\
&= \|\tau_{\mathrm{up},U}\|^{2+\delta}\cdot\mathbb{E}\left[J_i^{4+\delta}\|s_j\kappa_h(r_j)c_j\|^{4+\delta}\right] && \text{(by exchangeability)} \\
&\leq (2J^*L)^{4+\delta} && \text{(by boundedness of } J_i \text{ and (A17)).}
\end{aligned}
$$

Combining this with (A51), the AM-GM inequality implies that,

$$
\mathbb{E}\left[\left\|\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j(U_{\mathrm{up},i}-c_j'\tau_{\mathrm{up},U})\right\|^{4+\delta}\right] \leq 2\left(2J^*L\right)^{2+\delta}\left(\mathbb{E}\left[|U_{\mathrm{up},i}|^{2+\delta}\mid r_j=0\right]+\|\tau_{\mathrm{up},U}\|^{2+\delta}\right)=O(1),
$$

for a sufficiently small h, where the last claim stems from Assumption 5 and Lemma A3.

Step 4. By Lemma A11,

$$
\begin{aligned}
&\sqrt{N}\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i}-\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right)^{-1/2} \\
&\quad\cdot\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i}-\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right] \\
&=\sqrt{N}\mathrm{Var}\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i}-\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right)^{-1/2} \\
&\quad\cdot\left(\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i}-\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right]\right. \\
&\quad\left.-\mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i}-\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right]\right) \\
&\xrightarrow{d} N(0,I_4) && \text{(A52)}
\end{aligned}
$$

Next, we compute the limiting covariance matrix. Recall $\tilde{c}_j = D_h^{-1} c_j$ in (A16). We split the term into two:

$$\begin{aligned}
&\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \left\{ U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j \right)' \tau_{\mathrm{up},U}(h) \right\} \\
&= D_h \cdot \underbrace{\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \left\{ U_{\mathrm{up},i} - s_j\kappa_h(r_j)c_j'\tau_{\mathrm{up},U}(h) \right\}}_{Q_{i1}} \\
&\qquad - D_h \cdot \underbrace{\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \left(\sum_{\ell\in\mathcal{J}_i,\ell\neq j} s_\ell\kappa_h(r_\ell)c_\ell \right)' \tau_{\mathrm{up},U}(h)}_{Q_{i2}} . \qquad \text{(A53)}
\end{aligned}$$

We first compute $\mathrm{Var}(Q_{i1})$. By definition,

$$\begin{aligned}
Q_{i1} &= \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \Big\{ U_{\mathrm{up},i} - \tilde{\beta}^+_{\mathrm{up},U}(h)s_j\kappa_h(r_j)I(r_j>0) - \tilde{\beta}^-_{\mathrm{up},U}(h)s_j\kappa_h(r_j)I(r_j<0) \\
&\qquad\qquad - \tilde{\gamma}^+_{\mathrm{up},U}(h)s_j\kappa_h(r_j)r_jI(r_j>0) - \tilde{\gamma}^-_{\mathrm{up},U}(h)s_j\kappa_h(r_j)r_jI(r_j<0) \Big\} \\
&= \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \left\{ \underbrace{(U^+_{\mathrm{up},j} - \tilde{\beta}^+_{\mathrm{up},U}(h)s_j\kappa_h(r_j))I(r_j>0) + (U^-_{\mathrm{up},j} - \tilde{\beta}^-_{\mathrm{up},U}(h)s_j\kappa_h(r_j))I(r_j<0)}_{\triangleq \tilde{B}_{j1}} \right\} \\
&\quad - \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \left\{ \underbrace{\tilde{\gamma}^+_{\mathrm{up},U}(h)s_j\kappa_h(r_j)r_jI(r_j>0) + \tilde{\gamma}^-_{\mathrm{up},U}(h)s_j\kappa_h(r_j)r_jI(r_j<0)}_{\triangleq \tilde{B}_{j2}} \right\} . \qquad \text{(A54)}
\end{aligned}$$

Then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j\tilde{B}_{j1}\right)\left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j\tilde{B}_{j1}\right)'\right] \\
&= \mathbb{E}\Bigg[\Bigg\{\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^+_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)I(r_j>0) \\
&\quad + \sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^-_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^-_{\mathrm{up},U}(h)\right)I(r_j<0)\Bigg\} \\
&\quad \Bigg\{\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^+_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)I(r_j>0) \\
&\quad + \sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^-_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^-_{\mathrm{up},U}(h)\right)I(r_j<0)\Bigg\}'\Bigg] \\
&= \mathbb{E}\left[\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^+_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)\left(s_j\kappa_h(r_j)\tilde{c}_jU^+_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)'I(r_j>0)\right] \\
&\quad + \mathbb{E}\left[\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)\tilde{c}_jU^-_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^-_{\mathrm{up},U}(h)\right)\left(s_j\kappa_h(r_j)\tilde{c}_jU^-_{\mathrm{up},j} - s_j^2\kappa_h^2(r_j)\tilde{c}_j\tilde{\beta}^-_{\mathrm{up},U}(h)\right)'I(r_j<0)\right] \\
&\quad + \mathbb{E}\Bigg[\sum_{j_1,j_2\in\mathcal{J}_i, j_1\neq j_2}\Bigg\{\left(s_{j_1}\kappa_h(r_{j_1})\tilde{c}_{j_1}U^+_{\mathrm{up},j_1} - s_{j_1}^2\kappa_h^2(r_{j_1})\tilde{c}_{j_1}\tilde{\beta}^+_{\mathrm{up},U}(h)\right)I(r_{j_1}>0) \\
&\qquad\qquad + \left(s_{j_1}\kappa_h(r_{j_1})\tilde{c}_{j_1}U^-_{\mathrm{up},j_1} - s_{j_1}^2\kappa_h^2(r_{j_1})\tilde{c}_{j_1}\tilde{\beta}^-_{\mathrm{up},U}(h)\right)I(r_{j_1}<0)\Bigg\} \\
&\qquad\qquad \cdot\Bigg\{\left(s_{j_2}\kappa_h(r_{j_2})\tilde{c}_{j_2}U^+_{\mathrm{up},j_2} - s_{j_2}^2\kappa_h^2(r_{j_2})\tilde{c}_{j_2}\tilde{\beta}^+_{\mathrm{up},U}(h)\right)I(r_{j_2}>0) \\
&\qquad\qquad + \left(s_{j_2}\kappa_h(r_{j_2})\tilde{c}_{j_2}U^-_{\mathrm{up},j_2} - s_{j_1}^2\kappa_h^2(r_{j_2})\tilde{c}_{j_2}\tilde{\beta}^-_{\mathrm{up},U}(h)\right)I(r_{j_2}<0)\Bigg\}'\Bigg]
\end{aligned}
\tag{A55}
$$

By (A16), for any j,

$$\|s_j\kappa_h(r_j)\tilde{c}_j\| \le 2\max_x \kappa(x) = 2,$$

where the last line uses the condition that κ is the uniform kernel. Moreover, Assumption 5 implies that $U^{\pm}_{\mathrm{up},j}$ satisfies the condition of Lemma A5. Therefore, the third expectation in (A55) is $O(h^2)$.

To compute the first two expectations, we write $\tilde{c}_j$ as $\begin{bmatrix}\tilde{c}_j^+\\ \tilde{c}_j^-\end{bmatrix}$, where

$$\tilde{c}_j^+ = \left(I(r_j>0), \frac{r_j}{h}I(r_j>0)\right)', \quad \tilde{c}_j^- = \left(I(r_j<0), -\frac{r_j}{h}I(r_j<0)\right)'.$$

Then

$$\tilde{c}_jI(r_j>0) = \begin{bmatrix}\tilde{c}_j^+\\ 0\end{bmatrix}, \quad \tilde{c}_jI(r_j<0) = \begin{bmatrix}0\\ \tilde{c}_j^-\end{bmatrix}.$$

Then the first expectation is a 4-by-4 matrix with zero third and fourth rows and columns. Since κ is the

uniform kernel, the top-left 2-by-2 block is given by

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)U^+_{\mathrm{up},j}-s_j^2\kappa_h^2(r_j)\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\tilde{c}_j^+(\tilde{c}_j^+)'\right]\\
&=\mathbb{E}\left[\sum_{j\in\mathcal{J}_i}\left(s_j\kappa_h(r_j)U^+_{\mathrm{up},j}-s_j^2\kappa_h(r_j)\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\tilde{c}_j^+(\tilde{c}_j^+)'\right]\\
&=\mathbb{E}\left[\sum_{j\in\mathcal{J}_i}s_j^2\kappa_h(r_j)\left(U^+_{\mathrm{up},j}-s_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\tilde{c}_j^+(\tilde{c}_j^+)'\right]\\
&=\mathbb{E}\left[J_is_j^2\left(U^+_{\mathrm{up},j}-s_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\kappa_h(r_j)\tilde{c}_j^+(\tilde{c}_j^+)'\right],
\end{aligned}$$

where the last line is due to exchangeability. By the same argument as in (A11), we have

$$\begin{aligned}
&\mathbb{E}\left[J_is_j^2\left(U^+_{\mathrm{up},j}-s_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\kappa_h(r_j)\tilde{c}_j^+(\tilde{c}_j^+)'\right]\\
&=h\mathbb{E}\left[J_is_j^2\left(U^+_{\mathrm{up},j}-s_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\mid r_j=0\right]A_0. \qquad \text{(A56)}
\end{aligned}$$

Note that

$$U^+_{\mathrm{up},j}-U_j^+=X_j^{\pm}\cdot(\tilde{\beta}_{\mathrm{up}}(h)-\beta_0).$$

By Theorem 1,

$$\tilde{\beta}_{\mathrm{up}}(h)=\beta_0+o(1).$$

Then,

$$\mathbb{E}^{(1)}[U^+_{\mathrm{up},j}]-\mathbb{E}^{(1)}[U_j^+]=\mathbb{E}[J_is_j\mid r_j=0]\cdot(\tilde{\beta}_{\mathrm{up}}(h)-\beta_0)=o(1).$$

By Lemma A3,

$$\tilde{\beta}^{\pm}_{\mathrm{up},U}(h)=\frac{\Delta a^{\pm}_{Y0}-\tilde{\beta}_{\mathrm{up}}(h)\Delta a^{\pm}_{X0}}{\tilde{a}_0}+o(1)=\frac{a_0}{\tilde{a}_0}\mathbb{E}^{(1)}[U^+_{\mathrm{up},j}]+o(1)=\zeta\mathbb{E}^{(1)}[U_j^+]+o(1).$$

Expanding the square term in (A56), we obtain

$$\begin{aligned}
&\mathbb{E}\left[J_is_j^2\left(U^+_{\mathrm{up},j}-s_j\tilde{\beta}^+_{\mathrm{up},U}(h)\right)^2\mid r_j=0\right]\\
&\mathbb{E}\left[J_is_j^2\left(U^+_{\mathrm{up},j}-\zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2\mid r_j=0\right]+o(1)\\
&=\mathbb{E}\left[J_is_j^2\left(\tilde{\beta}_{\mathrm{up}}(h)-\beta_0+U_j^+-\zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2\mid r_j=0\right]+o(1)\\
&=\mathbb{E}\left[J_is_j^2\left(U_j^+-\zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2\mid r_j=0\right]+o(1)\\
&=\tilde{a}_0\mathbb{E}^{(2)}\left[\left(U_j^+-\zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2\mid r_j=0\right]+o(1).
\end{aligned}$$

Therefore, the first expectation in (A55) is given by

$$h\tilde{a}_0 \begin{bmatrix} \mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+]\right)^2 \mid r_j = 0\right] \tilde{A}_0 & 0 \\ 0 & 0 \end{bmatrix} + o(h).$$

By symmetry, the second expectation in (A55) is given by

$$h\tilde{a}_0 \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-]\right)^2 \mid r_j = 0\right] \tilde{A}_0 \end{bmatrix} + o(h).$$

Putting pieces together, we arrive at

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j1}\right)\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j1}\right)'\right] \\ &= h\tilde{a}_0 \begin{bmatrix} \mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+]\right)^2 \mid r_j = 0\right] \tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-]\right)^2 \mid r_j = 0\right] \tilde{A}_0 \end{bmatrix} + o(h). \end{aligned} \tag{A57}$$

Next, we move to the second term of (A54).

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j2}\right)\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j2}\right)'\right] \\ &= \mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j^2 \kappa_h(r_j) r_j^2 \left\{\tilde{\gamma}^+_{\mathrm{up},U}(h) I(r_j > 0) + \tilde{\gamma}^-_{\mathrm{up},U}(h) I(r_j < 0)\right\}^2 \tilde{c}_j \tilde{c}_j'\right] && (\kappa \text{ is uniform}) \\ &\leq h^2 \mathbb{E}\left[\sum_{j\in\mathcal{J}_i} s_j^2 \kappa_h(r_j) \left\{\tilde{\gamma}^+_{\mathrm{up},U}(h) I(r_j > 0) + \tilde{\gamma}^-_{\mathrm{up},U}(h) I(r_j < 0)\right\}^2 \tilde{c}_j \tilde{c}_j'\right] && (|\kappa_h(r_j) r_j| \leq h) \\ &= h^2 \mathbb{E}\left[J_i s_j^2 \kappa_h(r_j) \left\{\tilde{\gamma}^+_{\mathrm{up},U}(h) I(r_j > 0) + \tilde{\gamma}^-_{\mathrm{up},U}(h) I(r_j < 0)\right\}^2 \tilde{c}_j \tilde{c}_j'\right] && (|\kappa_h(r_j) r_j| \leq h). \end{aligned}$$

By Lemma A3, $\tilde{\gamma}^{\pm}_{\mathrm{up},U}(h) = O(1)$. By (A16) and boundedness of J_i,

$$\mathbb{E}\left[\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j2}\right)\left(\sum_{j\in\mathcal{J}_i} s_j \kappa_h(r_j)\tilde{c}_j \tilde{B}_{j2}\right)'\right] = O(h^3) = o(h). \tag{A58}$$

By Lemma A4, (A57), and (A58),

$$\mathrm{Var}(Q_{i1}) = h\tilde{a}_0 \begin{bmatrix} \mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+]\right)^2 \mid r_j = 0\right] \tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-]\right)^2 \mid r_j = 0\right] \tilde{A}_0 \end{bmatrix} + o(h). \tag{A59}$$

Lastly, we analyze the variance of Q_{i2}, defined in (A53).

$$\begin{aligned} Q_{i2} &= \sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)\tilde{c}_j \left(\sum_{\ell\in\mathcal{J}_i,\ell\neq j} s_\ell\kappa_h(r_\ell)c_\ell\right)' \tau_{\mathrm{up},U}(h) \\ &= \sum_{j,\ell\in\mathcal{J}_i,j\neq\ell} s_j s_\ell\kappa_h(r_j)\kappa_h(r_\ell)(c_\ell'\tau_{\mathrm{up},U}(h))\tilde{c}_j. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathbb{E}[Q_{i2}Q_{i2}'] \\ &= \mathbb{E}\left[\sum_{j_1,j_2,\ell_1,\ell_2\in\mathcal{J}_i,j_1\neq\ell_1,j_2\neq\ell_2} s_{j_1}s_{\ell_1}s_{j_2}s_{\ell_2}\kappa_h(r_{j_1})\kappa_h(r_{\ell_1})\kappa_h(r_{j_2})\kappa_h(r_{\ell_2})(c_{\ell_1}'\tau_{\mathrm{up},U}(h))(c_{\ell_2}'\tau_{\mathrm{up},U}(h))\tilde{c}_{j_1}\tilde{c}_{j_2}'\right] \\ &= \mathbb{E}\left[\begin{bmatrix}J_i\\2\end{bmatrix}^2 s_{j_1}s_{\ell_1}s_{j_2}s_{\ell_2}\kappa_h(r_{j_1})\kappa_h(r_{\ell_1})\kappa_h(r_{j_2})\kappa_h(r_{\ell_2})(c_{\ell_1}'\tau_{\mathrm{up},U}(h))(c_{\ell_2}'\tau_{\mathrm{up},U}(h))\tilde{c}_{j_1}\tilde{c}_{j_2}'\right]. \end{aligned}$$

By boundedness of J_i and Lemma A5,

$$\mathrm{Var}(Q_{i2}) \preceq \mathbb{E}[Q_{i2}Q_{i2}'] = O(h^2) = o(h). \tag{A60}$$

By Lemma A4, (A59), (A60), and (A53)

$$\begin{aligned} &\mathrm{Var}\left(h^{-1/2}D_h^{-1}\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right) \\ &= \tilde{a}_0\begin{bmatrix}\mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2 \mid r_j=0\right]\tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j\mathbb{E}^{(1)}[U_j^-]\right)^2 \mid r_j=0\right]\tilde{A}_0\end{bmatrix} + o(1). \end{aligned} \tag{A61}$$

Similar to (A38),

$$\hat{\mathbb{E}}_N\left[\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\left\{U_{\mathrm{up},i} - \left(\sum_{j\in\mathcal{J}_i} s_j\kappa_h(r_j)c_j\right)'\tau_{\mathrm{up},U}(h)\right\}\right] = \hat{P}_{U_{\mathrm{lo}}}(h) - \hat{\Sigma}_{\mathrm{up}}(h)\tau_{\mathrm{up},U}(h), \tag{A62}$$

By (A52),

$$\begin{aligned} &\sqrt{N}\tilde{a}_0^{-1/2}\begin{bmatrix}\mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2 \mid r_j=0\right]\tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j\mathbb{E}^{(1)}[U_j^-]\right)^2 \mid r_j=0\right]\tilde{A}_0\end{bmatrix}^{-1/2} \\ &\quad\cdot h^{-1/2}D_h^{-1}\left(\hat{P}_{U_{\mathrm{up}}}(h) - \hat{\Sigma}_{\mathrm{up}}(h)\tau_{\mathrm{up},U}\right) \\ &\xrightarrow{d} N(0, I_4) \end{aligned}$$

We can rewrite it as:

$$\begin{aligned}
&\sqrt{\frac{N}{h}} D_h^{-1} \left(\hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right) \\
&\xrightarrow{d} N \left(0, \tilde{a}_0 \begin{bmatrix} \mathbb{E}^{(2)} \left[\left(U_j^+ - \zeta s_j \mathbb{E}^{(1)}[U_j^+] \right)^2 \mid r_j = 0 \right] \tilde{A}_0 & 0 \\ 0 & \mathbb{E}^{(2)} \left[\left(U_j^- - \zeta s_j \mathbb{E}^{(1)}[U_j^-] \right)^2 \mid r_j = 0 \right] \tilde{A}_0 \end{bmatrix} \right) \quad \text{(A63)}
\end{aligned}$$

Step 5. By definition,

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}(h)) \\
&= \sqrt{Nh} \left\{ \frac{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_Y(h)}{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_X(h)} - \tilde{\beta}_{\text{up}}(h) \right\} \\
&= \sqrt{Nh} \frac{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_{U_{\text{up}}}(h)}{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_X(h)}.
\end{aligned}$$

We have proved in Theorem 1 that

$$e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_X(h) = \frac{\Delta a_{X0}}{\tilde{a}_0} + o_{\mathbb{P}}(1) = \frac{\Delta a_{X0}}{\tilde{a}_0}(1 + o_{\mathbb{P}}(1)). \quad \text{(A64)}$$

By (A49),

$$e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} = e_1' U \tau_{\text{up},U} = 0.$$

Thus,

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}(h)) \\
&= \sqrt{Nh} \frac{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \left\{ \hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right\}}{e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \hat{P}_X(h)} \\
&= \sqrt{Nh} \cdot \frac{\tilde{a}_0}{\Delta a_{X0}} \cdot \{ e_1'(U\hat{\Sigma}_{\text{up}}(h)U^{-1})^{-1} U \left\{ \hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right\} \cdot (1 + o_{\mathbb{P}}(1)) \\
&= \frac{\tilde{a}_0}{\Delta a_{X0}} \cdot e_1' \left\{ (hU_h)^{-1} (U\hat{\Sigma}_{\text{up}}(h)U^{-1}) U_h^{-1} \right\}^{-1} \cdot \sqrt{Nh} (hU_h)^{-1} U \left\{ \hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right\} \cdot (1 + o_{\mathbb{P}}(1)),
\end{aligned}$$

where the last line uses the fact that $e_1' U_h^{-1} = e_1'$. By (A48),

$$\begin{aligned}
&\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}(h)) \\
&= \frac{\tilde{a}_0}{\Delta a_{X0}} \cdot e_1' \left\{ (hU_h)^{-1} (U\tilde{\Sigma}_{\text{up}}(h)U^{-1}) U_h \right\}^{-1} \cdot \sqrt{Nh} (hU_h)^{-1} U \left\{ \hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right\} \cdot (1 + o_{\mathbb{P}}(1)) \\
&= \frac{\tilde{a}_0}{\Delta a_{X0}} \cdot e_1' \left\{ (hU_h)^{-1} (U\tilde{\Sigma}_{\text{up}}(h)U^{-1}) U_h \right\}^{-1} \cdot U \cdot \sqrt{\frac{N}{h}} D_h^{-1} \left\{ \hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h) \tau_{\text{up},U} \right\} \cdot (1 + o_{\mathbb{P}}(1)).
\end{aligned}$$

where the last line uses the fact that $U_h^{-1} U = (U D_h U^{-1})^{-1} U = U D_h^{-1}$. By (A48),

$$(hU_h)^{-1} (U\tilde{\Sigma}_{\text{up}}(h)U^{-1}) U_h = \begin{bmatrix} \tilde{a}_0 \tilde{A}_0 & 0 \\ 0 & \tilde{a}_0 \tilde{A}_0 \end{bmatrix} + o(1).$$

Thus,

$$e_1'\left\{(hU_h)^{-1}(U\tilde{\Sigma}_{\text{up}}(h)U^{-1})U_h\right\}^{-1} = e_1'\begin{bmatrix}\tilde{a}_0\tilde{A}_0 & 0\\ 0 & \tilde{a}_0\tilde{A}_0\end{bmatrix}^{-1} + o(1) = \frac{1}{\tilde{a}_0}[e_1'\tilde{A}_0^{-1} \quad 0]\cdot(1+o(1)).$$

Then, we have

$$\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}(h)) = \frac{1}{\Delta a_{X0}}[e_1'\tilde{A}_0^{-1} \quad -e_1'\tilde{A}_0^{-1}]\cdot\sqrt{\frac{N}{h}}D_h^{-1}\left\{\hat{P}_{U_{\text{up}}}(h) - \hat{\Sigma}_{\text{up}}(h)\tau_{\text{up},U}\right\}\cdot(1+o_{\mathbb{P}}(1)).$$

By (A63), we have

$$\begin{aligned}&\sqrt{Nh}(\hat{\beta}_{\text{up}} - \tilde{\beta}_{\text{up}}(h))\\ &\xrightarrow{d} N\left(0, \frac{\tilde{a}_0}{(\Delta a_{X0})^2}\cdot e_1'A_0^{-1}\tilde{A}_0A_0^{-1}e_1\cdot\left\{\mathbb{E}^{(2)}\left[\left(U_j^+ - \zeta s_j\mathbb{E}^{(1)}[U_j^+]\right)^2\right] + \mathbb{E}^{(2)}\left[\left(U_j^- - \zeta s_j\mathbb{E}^{(1)}[U_j^-]\right)^2\right]\right\}\right).\end{aligned} \tag{A65}$$

Again, the proof is then completed by

$$e_1'A_0^{-1}\tilde{A}_0A_0^{-1}e_1 = C_{\text{var}}(\kappa).$$

□

A.6 Asymptotic bias analysis in non-standard asymptotic regimes

A.6.1 Growing-J regime

TBA.

A.6.2 Growing-correlation regime

TBA.

A.7 Miscellaneous results

Lemma A6. *For a bivariate random vector (V,R), let $g(r) = \mathbb{E}[V \mid R=r]\cdot f(r)$ where $f(r)$ be the density function of R. Assume g is continuous. Then, for any $k \geq 0$ and $p \in \{0,1\}$, as $h \to 0$,*

$$\mathbb{E}\left[VR^k\kappa_h(R)I((-1)^pR > 0)\right] = (-1)^{pk}h^{k+1}\cdot\left\{g(0)M_k + o(1)\right\},$$

Proof of Lemma A6. We first prove the case of $p=0$. By law of iterated expectation,

$$\begin{aligned}&\mathbb{E}\left[VR^k\kappa_h(R)I(R>0)\right] = \mathbb{E}\left[\mathbb{E}[V\mid R]R^k\kappa_h(R)I(R>0)\right]\\ &= \int_0^\infty g(r)r^k\kappa_h(r)dr \overset{x=hr}{=} h^{k+1}\int_0^\infty g(hx)x^k\kappa(x)dx\\ &= h^{k+1}\int_0^1 g(hx)x^k\kappa(x)dx,\end{aligned} \tag{A66}$$

where the last line is because κ is supported on $[-1,1]$. By Dominated Convergence Theorem,

$$\lim_{h\to 0}\int_0^1 g(hx)x^k\kappa(x)dx = \int_0^1 g(0)x^k\kappa(x)dx = g(0)M_k.$$

To prove the second claim, we rewrite the LHS as $(-1)^k\mathbb{E}[V(-R)^k I(-R\in(0,h))]$. The proof is completed by noting that $\mathbb{E}[V\mid -R=r]\cdot f(-r) = g(-r)$ where $f(-r)$ is the density of $-R$. □

Lemma A7. *In the setting of Lemma A6, assume that $g\in\mathcal{C}^s$ and let $g^{(t)}(r)$ denote the t-th order derivative of g for any $t\le s$. Then, for any $k\ge 0$ and $p\in\{0,1\}$, as $h\to 0$,*

$$\mathbb{E}\left[VR^k\kappa_h(R)I((-1)^pR>0)\right] = (-1)^{pk}h^{k+1}\cdot\left\{\sum_{t=0}^{s}\frac{(-1)^{pt}g^{(t)}(0)M_{k+t}}{t!}h^t + O\left(\frac{Lh^{s+1}M_{k+s+1}}{(s+1)!}\right)\right\},$$

where $O(\cdot)$ only hides absolute constants.

Proof of Lemma A7. We first prove the case of $p=0$. Let

$$b(r) = g(r) - \sum_{t=0}^{s}\frac{g^{(t)}(0)}{t!}r^t.$$

Since $g\in\mathcal{C}_L^s(\mathbb{R})$, $|b(r)|\le L|r|^{s+1}/(s+1)!$. Then

$$\begin{aligned}
&\left|\int_0^1 g(hx)x^k\kappa(x)dx - \sum_{t=0}^{s-1}h^t\cdot\frac{g^{(t)}(0)}{t!}\int_0^1 x^{k+t}\kappa(x)dx\right| \le \int_0^\infty b(hx)x^k\kappa(x)dx\\
&\Longrightarrow\left|\int_0^1 g(hx)x^k\kappa(x)dx - \sum_{t=0}^{s}\frac{g^{(t)}(0)M_{k+t}}{t!}h^t\right| \le h^{s+1}\frac{L}{(s+1)!}\cdot\int_0^\infty x^{k+s}\kappa(x)dx\\
&= \frac{LM_{k+s}}{(s+1)!}h^{s+1}.
\end{aligned}$$

By (A66), we obtain the result. The second claim follows from the same argument at the end of the proof of Lemma A6. □

Lemma A8. *For a trivariate random vector (V,R_1,R_2), let $v(r_1,r_2) = \mathbb{E}[V\mid R_1=r_1,R_2=r_2]\cdot u(r_1,r_2)$, where $u(r_1,r_2)$ be the density function of (R_1,R_2). Assume that v is continuous. For any non-negative integers t_1,t_2, write $v^{(t_1,t_2)}(r_1,r_2)$ for $\frac{\partial^{t_1+t_2}}{\partial r_1^{t_1}\partial r_2^{t_2}}v(r_1,r_2)$. Then, for any $k,\ell\ge 0$ and $p,q\in\{0,1\}$, as $h\to 0$,*

$$\mathbb{E}\left[VR_1^kR_2^\ell\kappa_h(R_1)\kappa_h(R_2)I\Big((-1)^pR_1>0\Big)I((-1)^qR_2>0)\right] = (-1)^{pk+q\ell}h^{k+\ell+2}\Big\{v^{(0,0)}(0,0)M_kM_\ell + o(1)\Big\}.$$

Proof of Lemma A8. We first prove the case with $p=q=0$. The proof strategy closely mirrors that of

Lemma A7. By law of iterated expectation,

$$\begin{aligned}
&\mathbb{E}\left[VR_1^kR_2^\ell\kappa_h(R_1)\kappa_h(R_2)I(R_1>0)I(R_2>0)\right]\\
&=\int_0^\infty\int_0^\infty v(r_1,r_2)r_1^kr_2^\ell\kappa\left(\frac{r_1}{h}\right)\kappa\left(\frac{r_2}{h}\right)dr_1dr_2\\
&\overset{x_1=r_1/h,x_2=r_2/h}{=}h^{k+\ell+2}\int_0^\infty\int_0^\infty v(hx_1,hx_2)x_1^kx_2^\ell\kappa\left(x_1\right)\kappa\left(x_2\right)dx_1dx_2\\
&=h^{k+\ell+2}\int_0^1\int_0^1 v(hx_1,hx_2)x_1^kx_2^\ell\kappa\left(x_1\right)\kappa\left(x_2\right)dx_1dx_2
\end{aligned}$$

where the last line is because κ is supported on $[-1,1]$. By Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{h\to0}\int_0^1\int_0^1 v(hx_1,hx_2)x_1^kx_2^\ell\kappa\left(x_1\right)\kappa\left(x_2\right)dx_1dx_2\\
&=\int_0^1\int_0^1 v(0,0)x_1^kx_2^\ell\kappa\left(x_1\right)\kappa\left(x_2\right)dx_1dx_2\\
&=v(0,0)M_kM_\ell.
\end{aligned}$$

In general, we can replace by (R_1,R_2) by $((-1)^pR_1,(-1)^qR_2)$. Then $v(r_1,r_2)$ should be replaced by $v((-1)^pr_1,(-1)^qr_2)$. Thus, $v^{(t_1,t_2)}(0,0)$ becomes $(-1)^{pt_1+qt_2}v^{(t_1,t_2)}(0,0)$. □

Lemma A9. *In the setting of Lemma A8, assume that $v\in\mathcal{C}^s$. Then, for any $k,\ell\geq0$ and $p,q\in\{0,1\}$, as $h\to0$,*

$$\begin{aligned}
&\mathbb{E}\left[VR_1^kR_2^\ell\kappa_h(R_1)\kappa_h(R_2)I\Big((-1)^pR_1>0\Big)I((-1)^qR_2>0\Big)\right]\\
&=(-1)^{pk+q\ell}h^{k+\ell+2}\Bigg\{\sum_{t_1+t_2\leq s}(-1)^{pt_1+qt_2}v^{(t_1,t_2)}(0,0)\frac{M_{k+t_1}M_{\ell+t_2}}{t_1!t_2!}h^{t_1+t_2}\\
&\qquad+O\left(\frac{L2^{s+1}}{(s+1)!}(M_{k+s+1}M_\ell+M_{\ell+s+1}M_k)h^{s+1}\right)\Bigg\}.
\end{aligned}$$

In particular, when $s=1$, $k,\ell=O(1)$, and $M_{\max\{k,\ell\}+2}<\infty$, then

$$\begin{aligned}
&\mathbb{E}\left[VR_1^kR_2^\ell\kappa_h(R_1)\kappa_h(R_2)I\Big((-1)^pR_1>0\Big)I((-1)^qR_2>0\Big)\right]\\
&=(-1)^{pk+q\ell}h^{k+\ell+2}\Bigg\{v(0,0)M_kM_\ell+\Big((-1)^pv^{(1,0)}(0,0)M_{k+1}M_\ell+(-1)^qv^{(0,1)}(0,0)M_kM_{\ell+1}\Big)h\\
&\qquad+O\left(Lh^2\right)\Bigg\}.
\end{aligned}$$

Proof of Lemma A9. We first prove the case with $p=q=0$. Let

$$b(r_1,r_2)=v(r_1,r_2)-\sum_{t_1+t_2\leq s}\frac{v^{(t_1,t_2)}(0,0)}{t_1!t_2!}r_1^{t_1}r_2^{t_2}.$$

Since $v \in \mathcal{C}_L^s(\mathbb{R}^2)$,

$$b(r_1, r_2) \leq \sum_{t_1+t_2=s+1} \frac{L}{t_1!t_2!}|r_1|^{t_1}|r_2|^{t_2} \leq \sum_{t_1+t_2=s+1} \frac{L}{t_1!t_2!}(|r_1|^{s+1} + |r_2|^{s+1}) \leq$$
$$= \frac{L2^{s+1}}{(s+1)!}(|r_1|^{s+1} + |r_2|^{s+1}).$$

Then

$$\left| \int_0^\infty \int_0^\infty v(hx_1, hx_2)x_1^k x_2^\ell \kappa(x_1)\kappa(x_2)\, dx_1 dx_2 \right.$$
$$\left. - \sum_{t_1+t_2\leq s} h^{t_1+t_2} \frac{v^{(t_1,t_2)}(0,0)}{t_1!t_2!} \int_0^\infty \int_0^\infty x_1^{k+t_1} x_2^{\ell+t_2} \kappa(x_1)\kappa(x_2) dx_1 dx_2 \right|$$
$$\leq h^{s+1} \frac{L2^{s+1}}{(s+1)!} \int_0^\infty \int_0^\infty \left(x_1^{k+s+1}x_2^\ell + x_1^k x_2^{\ell+s+1}\right) \kappa(x_1)\kappa(x_2) dx_1 dx_2.$$

This is equivalent to

$$\left| \int_0^\infty \int_0^\infty v(hx_1, hx_2)x_1^k x_2^\ell \kappa(x_1)\kappa(x_2)\, dx_1 dx_2 - \sum_{t_1+t_2\leq s} h^{t_1+t_2} \frac{v^{(t_1,t_2)}(0,0)}{t_1!t_2!} M_{k+t_1} M_{\ell+t_2} \right|$$
$$\leq h^{s+1} \frac{L2^{s+1}}{(s+1)!} \left(M_{k+s+1}M_\ell + M_k M_{\ell+s+1}\right).$$

The result with other (p, q) can be obtained by the same argument at the end of the proof of Lemma A8. □

Lemma A10. *For any matrices A, Ψ, assume that A is invertible and*

$$h\|\Psi\|_{\text{op}}\|A^{-1}\|_{\text{op}} < 1/2.$$

Then, for any $s \geq 0$,

$$(A + h\Psi)^{-1} = \sum_{t=0}^{s} h^t \cdot A^{-1}(\Psi A^{-1})^t + O\left(h^{s+1}\|A^{-1}\|_{\text{op}}^{s+2}\|\Psi\|_{\text{op}}^{s+1}\right), \tag{A67}$$

where $O(\cdot)$ only hides universal constants.

Proof of Lemma A10. We have

$$(A + h\Psi)^{-1} = A^{-1/2}(I + hA^{-1/2}\Psi A^{-1/2})^{-1}A^{-1/2}.$$

Our condition implies $\|hA^{-1/2}\Psi A^{-1/2}\|_{\text{op}} < 1$. As a result,

$$\left(I + hA^{-1/2}\Psi A^{-1/2}\right)^{-1} = \sum_{t=0}^{\infty} h^t (A^{-1/2}\Psi A^{-1/2})^t,$$

and, for any $s \geq 0$,

$$\begin{aligned}
&\left\| \left(I + hA^{-1/2}\Psi A^{-1/2}\right)^{-1} - \sum_{t=0}^{s} h^t (A^{-1/2}\Psi A^{-1/2})^t \right\| \\
&\leq \sum_{t>s} h^t \|A^{-1/2}\Psi A^{-1/2}\|_{\text{op}}^t \leq 2h^{s+1}\|A^{-1/2}\Psi A^{-1/2}\|_{\text{op}}^{s+1} \\
&= 2h^{s+1}\|A^{-1}\Psi\|_{\text{op}}^{s+1} \leq 2h^{s+1}\|A^{-1}\|_{\text{op}}^{s+1}\|\Psi\|_{\text{op}}^{s+1}.
\end{aligned}$$

□

Lemma A11. *Let $\xi_1, \ldots, \xi_N$ be i.i.d. random vectors in $\mathbb{R}^d$ with $\mathbb{E}[\|\xi_i\|_2^{2+\delta}] < \infty$ and $\text{Var}(\xi_i)$ is strictly positive definite. Then*

$$d_W\left((\text{Var}(\xi_i))^{-1/2} \cdot \sqrt{N}(\hat{\mathbb{E}}_N[\xi_i] - \mathbb{E}[\xi_i]), N(0, I_d)\right) \leq \frac{C}{N^{\delta/2}} \left\{ \mathbb{E}\left[\|\xi_i\|_2^{2+\delta}\right] + \left(\mathbb{E}\left[\|\xi_i\|_2^{2+\delta}\right]\right)^{\delta/(2+\delta)} \right\},$$

where d_W denotes the Wasserstein-1 distance.

Proof of Lemma A11. By AM-GM inequality and Jensen inequality,

$$\mathbb{E}\left[\|\xi_i - \mathbb{E}[\xi_i]\|_2^{2+\delta}\right] \leq 2^\delta \left(\mathbb{E}\left[\|\xi_i\|_2^{2+\delta}\right] + \left(\|\mathbb{E}[\xi_i]\|_2\right)^{2+\delta} \right) \leq 2^{1+\delta}\mathbb{E}\left[\|\xi_i\|_2^{2+\delta}\right]$$

By Theorem 1.1 of Gallouët et al. (2018),

$$d_W\left((\text{Var}(\xi_i))^{-1/2} \cdot \sqrt{N}(\hat{\mathbb{E}}_N[\xi_i] - \mathbb{E}[\xi_i]), N(0, I_d)\right) \leq \frac{C}{N^{\delta/2}} \left\{ \mathbb{E}\left[\|\xi_i\|_2^{2+\delta}\right] + \mathbb{E}\left[\|\xi_i\|_2^{\delta}\right] \right\}.$$

The proof is then completed by Hölder's inequality. □

B Detailed Comparison to Current Practice

Many published articles feature designs suitable for RDA. In this section, we describe the empirical strategies employed by the authors of those articles. We detail the main differences between their approaches and ours, highlighting the potential gains from using the estimators from Section 2.2. Section B.1 discusses studies that, in the absence of the RDA methodology, did not use RD variation at all. We then turn to papers that used some RD specifications at the aggregated or disaggregated levels in Sections B.2 and B.3, respectively. Finally, we discuss papers that estimated spillovers from RD events, as in Section 4.1.

B.1 Difference-in-Differences Specifications in RDA Settings

The first group of studies, listed in Panel A of Appendix Table A1, uses difference-in-differences and similar designs in settings where RDA estimation is feasible and could provide more credible identification.

Besley and Case (2003, Tables 8–10) consider the effects of legislature composition — one of the prime applications of RDA. To estimate the effects on policy choices of the fractions of female and Democratic legislators in U.S. state houses, where most seats are elected in single-seat races, they employ the two-way fixed effects OLS specification with a time-varying treatment:

$$Y_{it} = \beta X_{it} + \tilde{\gamma}'\tilde{W}_{it} + FE_i + FE_t + \text{error}_{it}.$$

A credible instrument for X_{it} could be obtained by leveraging the structure of the treatment as aggregating discontinuity events.

Similar examples are found in difference-in-differences studies of the effects of political connections on industries and firms. Akcigit et al. (2023, Table 10) estimate how a larger fraction of politically connected firms in an industry — defined as firms whose employee has been elected in a local council — affects industry outcomes. They use OLS estimation, despite using conventional RD for firm-level outcomes. Cooper et al. (2010) similarly consider the association between the number of politicians supported by a firm who get elected and firm performance.

Finally, Saez et al. (2019) estimate the wage effects of a payroll tax cut that applied to workers below the age of 25. Their difference-in-differences specification compares firms with different shares of young workers, under a parallel trends assumption. The RDA approach would instead leverage the age cutoff, recognizing that a firm's share of young workers is an aggregate of worker-level discontinuities. The RDA strategy would thus entail comparing firms with workers age 25 or just above to firms with workers just below 25, providing credible RD variation.

B.2 Upper-Level Outcome and Upper-Level Specification

We now consider 19 papers from Panel B of Appendix Table A1 where RD variation is used in some way and the regression analysis is conducted at the upper-level (i-level in the notation of Section 2). These specifications can generally be represented as

$$Y_i = \beta X_i + \tilde{\psi}'\tilde{Q}_i + \tilde{\gamma}'\tilde{W}_i + \text{error}, \tag{B68}$$

where X_i is constructed from all subunits and is instrumented by $Z_i = \sum_{j\in\mathcal{C}_i} s_j z_j$ constructed from the subunits near the cutoff only. Furthermore, $\tilde{Q}_i$ represents controls constructed in some way from subunit-level variables and $\tilde{W}_i$ are other controls and fixed effects.

The key difference between our proposal and what is found in the literature is in the set of controls $\tilde{Q}_i$. We discuss the three RDA controls Q_i in equation (8) in turn. The majority of papers include the first control: the total weight of narrow elections.[42] One exception is Valentim and Dinas (2024) who instead control for the weight of narrow elections *below* the cutoff; this produces a biased coefficient.[43]

The second RDA control — the aggregated running variable in narrow elections — is not found in any of the papers we have reviewed. While some papers have no analog to it (e.g., Campante and Yanagizawa-Drott (2018); Nellis and Siddiqui (2018)), several variations have been employed. Azoulay et al. (2019) take an unweighted average of running variables near the cutoff, while the instrument is a sum of RD shocks weighted by some dollar amounts. The same is true for Folke (2014) and Meriläinen (2022) whose average additionally includes non-close elections. Clots-Figueras (2012) and Bhalotra et al. (2014, 2021) use a different strategy: they control for the running variables of each subunit (including with non-narrow elections) separately. This requires ordering the subunits in an arbitrary way (such that changing the order would change the IV estimate) and filling missing values as zeros for units that have fewer subunits than others. Neither of these strategies appears to have a clear justification.

[42] Instead of controlling for it, some papers (e.g., Folke (2014), Azoulay et al. (2019)), in the language of Borusyak and Hull (2023), recenter the instrument Z_i by subtracting 0.5 times the total weight of narrow elections.

[43] To see this bias, notice that $\beta\sum_{j\in\mathcal{C}_i} s_j z_j + \gamma_0 \sum_{j\in\mathcal{C}_i} s_j = (\beta+\gamma_0)\sum_{j\in\mathcal{C}_i} s_j z_j + \gamma_0 \sum_{j\in\mathcal{C}_i} s_j(1-z_j)$. The left-hand side here corresponds to (8), while the right-hand side to the specification of Valentim and Dinas (2024). The reduced-form coefficient therefore identifies $\beta+\gamma_0$ instead of β. Akey (2015) (Table 7, column 2) employs a similar specification, except he interprets the second coefficient as a causal effect of losing.

Finally, we are not aware of any study that included any version of the running variable aggregated among narrow wins only. Absent the second and third RDA variables, the estimators used in the literature do not inherit the properties of standard RD and therefore may not enjoy the low bias advantages of local linear estimation.

We conclude by noting that, while most of the papers estimate (B68) by IV, Akey (2015) considers its reduced-form instead. We see several disadvantages to this approach. It is a less coherent specification than IV since the explanatory variable depends on the bandwidth choice, and thus is not an economic variable with independent meaning. Practically speaking, if the fraction of narrow victories is correlated with the fraction of non-narrow victories (as may happen when the bandwidth is not infinitesimal), the reduced-form specification is biased. And even if the correlation is small or absent, the reduced-form specification is likely less efficient, as the error term includes the effects of non-narrow victories.

B.3 Upper-Level Outcome and Lower-Level Specification

The final group of 8 papers (Panel C of Appendix Table A1) considers outcomes measured at the more aggregated level than the discontinuity events but estimates certain regressions at the lower level. To structure the discussion, recall that the stacking approach of Section 2.2 involves the sample of all narrow elections and a fuzzy RD specification with the reduced-form

$$Y_{\mathbf{i}(j)} = \rho z_j + \gamma' q_j + \text{error}, \tag{B69}$$

with $X_{\mathbf{i}(j)}$ as the endogenous variable in the structural equation, and $s_j \kappa_h(r_j)$ as weights. The current practice deviates in terms of the sample restrictions and the use of OLS vs IV.

First, while some papers (e.g., de Benedictis-Kessner and Warshaw (2020); de Benedictis-Kessner et al. (2024)) keep the full stacked sample of narrow elections, others restrict the sample in different ways. Beach and Jones (2017) only keep one narrow election for each unit, which has the running variable closest to the cutoff (although this does not affect too many units in their setting). Dell and Querubin (2018, Section III) only keep the narrow election that happened chronologically first. These approaches limit the amount of identifying variation, which our paper shows not to be necessary.

Second, while some papers (Carozzi et al. (2022) and Dell and Querubin (2018)) clearly introduce the upper-level treatment and use the IV specification (10), the majority of papers estimate the reduced-form (B69) by OLS. This has both conceptual and practical issues. Conceptually, (B69) is not a coherent causal model: it provides multiple, mutually inconsistent, expressions for the outcome, and the Stable Unit Treatment Value Assumption (SUTVA) is clearly violated. The causal effects of an extra election victory are also expected to vary substantially with s_j, limiting external validity of the OLS coefficient. For instance, de Benedictis-Kessner and Warshaw (2020) find the effect of an extra Democrat to be much larger in small legislatures. This issue would be avoided by an IV specification with the share of Democrats in the legislature as the treatment. Practically, in finite samples where the bandwidth is non-negligible, z_j may be non-trivially correlated with the treatment statuses of other subunits (both narrow and others). This potentially results in a form of asymptotic bias that doesn't arise in IV analyses, although we do not perform a formal analysis.

B.4 RD Spillovers

Section 4.1 proposes three RDA estimators in the context of spillovers: specifications at the level of outcome units i, at the bilateral level i, j, and by intervention unit j. Versions of all three approaches have been used

previously; yet, like in Sections B.2 and B.3, none of them exactly coincides with our proposal.

Surprisingly, we have found only one paper that performs estimation at the i level (Mora-Garćia and Rau, 2023). This stands in contrast not only with the many papers using the upper-level specification in the aggregation setting of Section B.2, but more importantly with how peer effects are usually estimated in economics, via either exogenous or endogenous peer effects models (Manski, 1993). Ultimately, i-level specifications are, in our view, the most natural and coherent way to study i-level outcomes. Similar to papers in Section B.2, Mora-Garćia and Rau (2023) do not include the third RDA control of equation (9) and only include the second RDA control as a robustness check.[44]

All other papers use specifications at the bilateral level or averaged by intervention unit j, and they are subject to the issues raised in Section B.3. The first issue involves unnecessarily restricting the sample. We find this in Dahl et al. (2014) who entirely drop all units with more than one neighbor near the cutoff. The second issue is that most papers consider reduced-form OLS regressions on z_j, rather than introducing the treatment at the level of outcome units (or perhaps as in (25)). The only exception we are aware of is Dube et al. (2019) who estimate a bilateral IV specification similar to what we propose, with the average status of the peers as the endogenous variable. Other papers — e.g., Baskaran and Hessami (2018) and Dechezleprêtre et al. (2023) at the bilateral level and Santoleri et al. (2022) and Bhalotra et al. (2018) at the j level — are subject to the conceptual and practical concerns of reduced-form specifications discussed in Section B.3.[45]

C Data Appendix

Our analysis is at the level of state-industry cells for each decennial census year from 1960–2010 (i.e., each year that divides by ten). We begin by discussing the data construction steps for our three data sources separately: decennial censuses, CPS, the unionization elections data. We then turn to how we merge them into a consistent cell-level panel; this involves creating a harmonized industry classification and computing cell-level variables, such as inequality measures.

Decennial census data. Census public-use micro-data were downloaded from the IPUMS USA website. For 1960, 1980, 1990, and 2000, we used the 5% "state" samples. For 1970, we combined the 2% "state" and 2% "metro" samples. To represent 2010, we combine the American Community Survey (ACS) samples for 2009–2013. All of these samples are nationally representative for the US population.

We assign each individual to their state of residence. While state of employment could be more relevant for our analysis, it is not available in some of the samples. The state of residence is observed in most samples; the only exception is the 1970 metro samples which do not report the state for the 17% of individuals living in metropolitan areas and other county groups that cross state boundaries. We duplicate observations for those individuals and adjust their projection weights to equal the original weight multiplied by each state's proportion of the metropolitan area or county group's total population.[46] We discuss the assignment of workers to industries below.

[44] In addition, they limit the sample of outcome units to those outside the bandwidth and, in the main analysis, define the treatment by averaging only across the peers within the bandwidth. Our theory would apply even with the full sample and a more natural definition of the treatment based on all peers.

[45] These concerns also apply to papers that use IV estimation with a j-level treatment, rather than our proposed i-level treatment. Isen (2014), for instance, estimates the effect of Y_j on the average outcome of j's neighbors. This is in contract to a conventional endogenous peer effects model that would estimate the effect of the average outcome of i's neighbors on Y_i. See Clark (2009) and Dahl et al. (2014) for related examples.

[46] We used IPUMS USA (2024) to identify the county composition of each metropolitan area and county group, and obtained the 1970 county populations from IPUMS NHGIS.

For each worker, we observe their pre-tax annual labor income (variable `INCWAGE`). Top-coded income values are multiplied by 1.5 (Autor et al., 2008).[47] We then compute the hourly wage as the income dividing by the weeks of work and the average number of hours per week.[48]

We measure workers' education and potential work experience using the most detailed education question (`EDUCD`). The groupings defined by this question vary across samples, sometimes closer aligned with the years of schooling and other times with the highest degree obtained. In all cases, it is sufficiently detailed to determine whether the individual has exactly a high school degree, exactly a college degree, or other education levels above and below. It is also sufficient to measure years of schooling, at least approximately, which we use to define potential work experience as the age minus 6 and minus the years of schooling.

We classify a worker as a manager if their occupation (`OCC1990`) belongs to the IPUMS categories "Executive, Administrative, and Managerial" and "Management Related" .

We impose several sample restrictions. First, we focus on 18–65 year-old adults. Second, we only keep workers who have been employed for 50-52 weeks (the largest weeks-worked category) during the year preceding the survey and worked at least 20 hours a week on average. Individuals who worked fewer weeks likely changed jobs within the year, making it more difficult to correctly assign them to an industry. Third, we require that workers earn at least half of the effective minimum hourly wage (obtained from U.S. Department of Labor (2025)).[49] Fourth, we exclude workers in the public administration and armed forces industries, as well as self-employed workers, retaining only those eligible to unionize through NLRB representation elections. [50]

CPS data. To capture variables not available in the census—unionization rates and fringe benefits—we leverage CPS samples, downloaded from IPUMS CPS.[51] To measure unionization rates, we use CPS MORG (Merged Outgoing Rotation Group) samples from 1983–2015, along with the May supplement to the CPS in 1977–1980. The surveys ask workers whether they are union members (`UNION` variable). [52] For fringe benefits, we use CPS ASEC (Annual Social and Economic Supplements) from 1980–2014, along with the May supplement in 1979. The workers are asked if they are covered by an employer-sponsored pension plan (`PENSION` variable) and whether they enrolled in their employer's group health plan during the previous year (`INCLUGH` variable). We use the same four individual-level sample restrictions as in the decennial census. Respondents with imputed values for the union, pension, or health-insurance variables were excluded from the construction of the corresponding measures.

To increase sample size when measuring changes in the rates of unionization, pension, and insurance coverage between adjacent census years, we pool data from multiple years of the CPS. Where possible, we compare the five years after the end of the decade to five years before the beginning of the decade. For instance, the differenced outcome for the decade of the 2000s is calculated as the change between 1986–90 and 2000–04. We deviate from this rule in the 1980s because of data availability; Appendix Table A14

[47] Until 1980, Census income data used uniform national top-coding thresholds. From 1990 onward, top-coding is state-specific. During the full period we multiply top-coded values by 1.5.

[48] Due to data availability, we measure working hours using the IPUMS `HRSWORK2` variable for 1960 and 1970 and the `UHRSWORK` variable from 1980 onward. The former is categorical and measures "hours worked last week", while the latter is continuous and measure "usual hours worked per week [in the last year]". For each category of `HRSWORK2`, we assign the midpoint value; for the top category (60+), we assign 65. Weeks worked is based on the categorized `WKSWORK2` variable which is available for all years and measures: "Weeks worked last year".

[49] The effective minimum wage is the maximum of the state and federal minimum wages in each state-year. State-level minimum wage data are unavailable for 1960; we therefore use the federal minimum wage of $1 for all states in that year.

[50] We filter based on industry classification rather than self-reported government employment, as the latter may not fully align with the National Labor Relations Act's definition of covered employees

[51] With the exception of the May supplement for 1979, which was downloaded from NBER.

[52] The CPS union status question requires respondents to select one of three options: "No union coverage," "Member of labor union," or "Covered by union but not a member." We define unionized workers as those who select the second option only.

provides the details.

Unionization elections. We use the NLRB (National Labor Relations Board) unionization election data maintained by Hank Farber and John-Paul Ferguson. [53] We keep observations from 1961—the earliest available year—until the beginning of 2009, after which industry variables are no longer available. The original dataset includes 250,719 elections.

The raw data contain a small number of duplicated records of the same election. We remove these in two steps. First, among elections with the same case number, unit type, and number of eligible voters, we drop duplicates that share either the same election date or the same employer name. Second, among elections with the same employer name, unit type, and number of eligible voters, we drop duplicates that match on at least two of the following three characteristics: county, election date, and votes in favor of the union.

Each election is attributed to a geographic location (directly including the state), an industry, and a decade. We generally define decades as starting in the year that ends with 0 and finishing in the year which ends with 9, e.g. 1980–1989. The earliest and latest decades are slightly shorter because, as just mentioned, the data start in 1961 and end early in 2009. We discuss industry classifications below.

We observe the number of eligible voters in the bargaining unit, the number of votes cast for and against the union, as well as whether the union has been formed. In most cases, the union is formed when the vote share (out of the total votes cast) strictly exceeds 50%, but in a small fraction of cases this is not the case, perhaps because of the reversal during the appeal process. We further observe whether the election is a union decertification election. In the main analysis, we do not distinguish between certification and decertification elections, counting workers in re-certified unions as newly unionized; we make other choices in a robustness check, as well as in Section 5.2.6.

While we use the full set of union elections to construct the treatment variable,[54] our instrument is constructed only from the elections that satisfy several criteria. Of course, it only includes close elections, defined by the vote share in favor of the union falling within a bandwidth around the cutoff of 50%. We make two additional restrictions. First, following common practice in the unionization RD literature, elections with fewer than 20 votes were excluded (DiNardo and Lee, 2004; Lee and Mas, 2012; Frandsen, 2021). Second, we drop elections that ended in a tie. Although by the National Labor Relations Act a tie is equivalent to a union loss, including those elections in the sample may lead to bias. Elections with an even number of votes will mechanically have a much higher chance of resulting in a close union loss than in a close union win. For example, consider an election with 20 votes given bandwidth of 10%. There are there outcomes resulting in a close loss (8-12, 9-11, and 10-1) but only two outcomes resulting in a close win (12-8 and 11-9). Excluding the tie, we prevent a negative mechanical correlation between the election result and workplace size.[55]

Third, following Frandsen (2017, 2021), the instrument construction also excludes elections that ended with a one-vote margin in either direction. Frandsen (2017, 2021) documented the imbalance of covariates and unequal frequency of union wins and losses in such elections. Figure A3 confirms this in our dataset, showing approximately 18,000 "missing" workers in elections that ended with exactly a one-vote margin in favor of the unions. A presumed explanation is that published unionization result may in some cases report vote counts post appeals and other legal process, and that the tendency to appeal and the acceptance rate

[53] For elections from 1961–1990, we use Ferguson's "nlrb_old_rcases" dataset, available on his GitHub repository. For elections from 1991–2009, we use data files provided directly by Hank Farber.

[54] Except a small fraction that we cannot attribute to an industry. This explains the slightly smaller total number of elections of 232,267 in Table A3. We include the entire set of 250,063 elections in the counterfactual analysis of Section 5.2.6.

[55] In support of this argument, Wang and Young (2022) provide empirical evidence that tied elections are very different from elections where the union lost or won by a slightly larger margin.

of those appeals is heterogeneous between close wins and close losses.

Table A3 presents summary statistics for the full set of unionization elections, as well as the restricted set for the instrument construction. The average vote share for the unions and the share of union wins are close to 50% in both samples. The average number of votes in the treatment (instrument) sample is 54.9 (98.3), and the share of manufacturing establishments is 43% (52%). In both samples, over half of the elections occurred in the 1960s and 1970s, which constitute only 40% of the study period, reflecting the decline in unionization since 1980.

Industry classification. To construct both labor market outcomes and new unionization rates by state-industry cells, we had to create an industry classification harmonized both across data sources and over time. The classifications in all our data sources are based, in one way or another, on SIC codes before around 2000 and on NAICS codes thereafter. Thus, we also created two harmonized classifications for the two subperiods, resulting in 70 industries based on 2-digit SIC87 codes and 73 industries based on 3-digit NAICS97 codes covering the entire private sector of the economy. We use our SIC-based classification for the 1960s–1990s and the NAICS-based classification for the 2000s.

For union elections filed before the end of 1998 (which covers most elections that took place before the end of 1999), the NLRB data report the 3-digit SIC code (with some changes in the classification over time discussed below). The decennial censuses up until 1990 and the CPS up until 2002 use a different classification—variable `IND1990`—that was harmonized over time by IPUMS.[56] This classification is loosely based on on 3-digit SIC87 codes. To minimize measurement error, we convert `IND1990` industries to 2-digit SIC87 codes by leveraging crosswalks created by David Dorn (Autor et al., 2019). Specifically, he published many-to-one crosswalks from `IND1990` to a slightly aggregated classification developed by him called `IND1990ddx`, as well as from 4-digit SIC87 codes to `IND1990ddx`. While the resulting mapping from `IND1990` to the 3-digit SIC87 codes that we observe in the NLRB data would be many-to-many, the mapping to 2-digit SIC87 codes is mostly many-to-one. [57] In the rare cases where this is not the case, we merge those 2-digit SIC87 codes together, resulting in the 70 industry codes in our classification.

We make adjustments for the changes in the SIC classifications over time in the NLRB data. While SIC87 codes are reported since 1987, the SIC72 classification is used between 1972 and 1986. In those years, we only keep the elections in industries for which over 95% of national employment mapped to one of our (significantly more aggregated) codes.[58] Elections prior to 1972 use even earlier SIC schemes (SIC57 and SIC67). We drop all elections that did not align with a single code in our classification using the crosswalk used by Eckert et al. (2021).

The NAICS-based concordance is more straightforward. For union elections filed since 1999, the NLRB data report the 3-digit NAICS code, also with some changes in the classification over time. The decennial

[56]Note that the harmonization process conducted by IPUMS follows a majority rule, assigning each original industry code to the harmonized industry that the majority of observations would have been assigned to. This process may lead to some inconsistencies between the census and the unionization data in earlier decades.

[57]The census industry classification scheme includes some "not specified" industries within broad industry categories (e.g., IND1990 code 122 "Food industries, not specified"). In David Dorn's crosswalk, these industries were excluded from matching to SIC industries, presumably because they would need to be matched to too many 4-digit SIC industries. We took advantage of our more aggregate 2-digit SIC classification and manually matched these industries to single 2-digit SIC codes where possible. Our manual matches include: IND1990 code 122 "Food industries, not specified" to SIC87 code 20 "Food and Kindred Products"; IND1990 code 332 "Machinery, not specified." to SIC87 code 35 "Industrial and Commercial Machinery and Computer Equipment"; IND1990 code 350 "Electrical machinery, equipment, and supplies, not specified" to SIC87 code 36 "Electronic & Other Electrical Equipment & Components"; IND1990 code 472 "Utilities, not specified" to SIC87 code 49 "Electric, Gas and Sanitary Services"; IND1990 code 863 "Family child care homes" to SIC87 code 83 "Social Services."

[58]This calculation relies on the economic censuses of 1987 (Census, 1993), which report total employment data according to both SIC72 and SIC87.

censuses of 2000 and 2010 and the CPS starting in 2003 report the variable `INDNAICS` which is closely based on NAICS97 codes and harmonized over time. Specifically, its first three digits coincide with 3-digit NAICS97, except for "not specified" industries which we drop and a few other industries which we merge together.[59] We also merge a few codes where the NAICS codes changed over time in the decennial censuses and CPS in a way that is not many-to-one from the newer codes to NAICS97, ultimately resulting in our 73-industry classification. In total, across the entire period of our study, we were able to match 92.9% of unionization elections and 99% of census records to our industry codes.

The transition from SIC to NAICS industries creates additional complications when measuring outcomes in 10-year changes. For decennial census variables, this affects the change from 1990 to 2000. The 2000 census, however, includes the `IND1990` variable converted from `INDNAICS` using a crosswalk designed by IPUMS. We therefore compute the 1990 to 2000 change using our SIC-based classification. While the conversion could introduce noise in differenced outcomes, we find it reassuring that employment changes do not exhibit substantially higher volatility in the 1990s than in the decade before or after.[60]

Industry transitions in the CPS, which happened between 2002 and 2003, require more care. This affects differenced outcomes not only in the 1990s (measured as 2000–2004 minus 1986–1990) but also in the 2000s (measured as 2010–2014 minus 1996–2000; see Table A14). The conversion from `INDNAICS` codes to `IND1990` for 2003–04 was done by IPUMS, and we use it to compute the changes over the 1990s in our SIC-based classification. In the other direction, we apply the IPUMS crosswalk ourselves, converting `IND1990` to `INDNAICS` for 1996–2000 to compute the changes over the 2000s using our NAICS-based classification.

State-by-industry panel and variable construction. Once all of our datasets have harmonized state and industry information and are assigned to a decade (in the NLRB data) or a census year (in the decennial censuses and the CPS), we compute differenced outcomes in the panel of state-level cells by decade. Since our industry classification changes in 2000, we compute all outcomes for the year 2000 using both classifications.

The computation of the treatment, instrument, and the RDA controls is described in detail in the main text. We drop a small number of cell-decades in which there were more workers eligible to vote in unionization elections than the total number of workers at the beginning of the decade according to the census, as those cells probably contain miscoding of the industry variable or inconsistency between the census and NLRB coding.

Most outcomes are straightforward. The union density, pension coverage, and insurance coverage rates are fractions of the corresponding answers in the CPS, using projection weights. In the decennial censuses, the Gini index and the top-10% income share are calculated based on the total labor income. The log 90-10 wage ratio and the variance of log wages, along with average wages for different workforce segments, are computed based from log hourly wages. All measures are weighted by projection weights. Employment is measured as the sum of weights within each cell.

To estimate the college wage premium by cell and year, we broadly follow Autor et al. (2008) and Farber et al. (2021). We select the sample or workers with either exactly a high-school degree or at least a college degree (or more). We then estimate the following regression by OLS with projection weights:

[59] We dropped codes 3MS "Not specified industries" and 4MS "Not specified retail trade." An example of industries that we merged is construction, which has only one `INDNAICS` code (code 22) but multiple 3-digit NAICS codes.

[60] A small issue also arises from the fact that about one-third of the unionization elections that happened in 1999 are recorded with NAICS codes. We exclude these elections from the RDA variables construction.

$$\log Wage_o = \gamma_{s(o)i(o)t(o)} College_o + \beta'_{s(o)t(o)} X_o + \text{error}_o \tag{C70}$$

where o stands for an individual observation that belongs to state $s(o)$, industry $i(o)$, and year $t(o)$, $College_o$ is the dummy of having at least a college degree, and X_o include the following set of demographics variables: a female dummy, a non-white dummy, a quartic term in experience, and interaction terms of the female dummy with both the non-white dummy and the quartic in experience term. Like Farber et al. (2021), we allow the demographic coefficients to vary by state and decade. However, we restrict them to be the same across industries within each state-decade because the sample size would be insufficient for a fully-flexible specification. The estimates of γ_{sit} are the estimated college wage premiums.

For the inequality variables (except the college wage premium), overall wages and employment variables, and all CPS variables, we replace a cell-decade observation as missing if it represents less than 1,000 workers after applying projection weights or is based on less than 10 raw observations, either at the beginning or at the end of the decade. For census variables based on a subgroup, we require to have 10 raw observations for that subgroup: e.g., we only compute the average wage of college-educated workers in a cell if there are at least 10 college-educated respondents in that cell. For the college wage premium, we require at least 10 observations for both high school graduates and workers with a college degree or more.

Appendix Tables

Table A1: Published Papers Suitable for the RDA Design

Citation	Journal	Equations	Tables
Panel A: Difference-in-Differences Specifications			
Besley and Case (2003)	Journal of Economic Literature		
Cooper et al. (2010)	The Journal of Finance		
Saez et al. (2019)	American Economic Review		
Akcigit et al. (2023)	Econometrica		
Panel B: Aggregated Outcomes, Upper-Level Specifications			
Clots-Figueras (2011)	Journal of Public Economics	3,4	7
Clots-Figueras (2012)	American Economic Journal: Applied Economics	1,2	2,3
Bhalotra and Clots-Figueras (2014)	American Economic Journal: Economy Policy	1,2	2
Cuñat et al. (2012)	The Journal of Finance	6,7	4,5
Bhalotra et al. (2014)	Journal of Economic Behavior & Organization	1,2	2,3
Folke (2014)	Journal of the European Economic Association	4,5	3
Akey (2015)	The Review of Financial Studies	–	7
Nellis et al. (2016)	Quarterly Journal of Political Science	2,3	3
Nellis and Siddiqui (2018)	American Political Science Review	2,3	2
Campante and Yanagizawa-Drott (2018)	Quarterly Journal of Economics	7	4
Hyytinen et al. (2018)	American Political Science Review	1,4	3
Azoulay et al. (2019)	The Review of Economic Studies	3.3, 3.4	8
Priyanka (2020)	Labour Economics	1,2	3
Bhalotra et al. (2021)	Journal of Development Economics	1,2	2
Meriläinen (2022)	The Economic Journal	2,4	3
Bahar et al. (2023)	Management Science	6,7	8
Sørensen (2023)	Journal of Economic Behavior & Organization	1,2	2
Baskaran et al. (2024b)	Journal of Economic Behavior & Organization	1,2	1
Geys et al. (2024)	The Journal of Politics	1,4	Fig. 2
Valentim and Dinas (2024)	British Journal of Political Science	–	3
Panel C: Aggregated Outcomes, Lower-Level Specifications			
Ferreira and Gyourko (2014)	Journal of Public Economics	4	11
Beach and Jones (2017)	American Economic Journal: Economy Policy	1	3
Dell and Querubin (2018)	Quarterly Journal of Economics	1	2,3
de Benedictis-Kessner and Warshaw (2020)	Journal of Politics	–	Fig. 7
Carozzi et al. (2022)	American Economic Journal: Applied Economics	2,3	2
Baskaran and Hessami (2023)	The Review of Economics and Statistics	2	3
Tan (2023)	Journal of Labor Economics	2	2–4
de Benedictis-Kessner et al. (2024)	American Journal of Political Science	–	Fig. 4
Panel D: Spillovers			
Clark (2009)	Journal of Political Economy	2	4
Dahl et al. (2014)	American Economic Review	3–5	1
Isen (2014)	Journal of Public Economics	5,6	3–5
Baskaran and Hessami (2018)	American Economic Journal: Economic Policy	2	9
Bhalotra et al. (2018)	The Economic Journal	1	3
Dube et al. (2019)	American Economic Review	9	3
Fu and Gregory (2019)	Econometrica	4	Fig. 4
Johnson (2020)	American Economic Review	2	2
Santoleri et al. (2022)	The Review of Economics and Statistics	1	7
Jones et al. (2022)	American Economic Review	3	7,8
Mora-Garćia and Rau (2023)	The Review of Economics and Statistics	3,4	1
Baskaran et al. (2024a)	Journal of Economic Growth	1	4
Dechezleprêtre et al. (2023)	American Economic Journal: Economic Policy	5–7	6
Panel E: RD Aggregation over Time			
Cellini et al. (2010)	Quarterly Journal of Economics	12	4
Darolia (2013)	Journal of Public Economics	4,5	4–8
Martorell et al. (2016)	Journal of Public Economics	3	3
Lang (2018)	Journal of Public Economics	11	3
Baron (2022)	American Economic Journal: Economy Policy	2	5
Kuliomina (2021)	European Journal of Political Economy	–	4
Biasi et al. (2024)	NBER Working Paper	4	3

Notes: This table lists published papers featuring research designs suitable for RDA. We have organized these papers into the categories as in Appendix B, plus the RDA in time setting discussed in Section 4.2, although some papers may belong to more than one category. The "Equations" column specifies the regression specification most suitable for the RDA analysis, while the "Tables" column references the tables and figures containing the main results from these specifications. There are no references to equations and tables in Panel A since those papers do not use RDA-type specifications in settings where that could be possible.

Table A2: Additional Summary Statistics

Variable	Mean	Compliers Mean	Data Source	Time Range
Union Attempts 0–25%	0.954	0.387	NLRB	1960-2010
	(0.210)	(0.487)		
Union Attempts 25–50%	0.037	0.290	NLRB	1960-2010
	(0.189)	(0.454)		
Union Attempts 50–75%	0.007	0.151	NLRB	1960-2010
	(0.082)	(0.358)		
Union Attempts 75–100%	0.003	0.172	NLRB	1960-2010
	(0.051)	(0.378)		
$100 \times \sum_{j \in \mathcal{C}_{sit}} s_j^2$	0.035	3.025	NLRB	1960-2010
	(0.325)	(5.913)		
I (1960s)	0.104	0.293	Census	—
	(0.306)	(0.455)		
I (1970s)	0.136	0.390	Census	—
	(0.343)	(0.488)		
I (1980s)	0.199	0.117	Census	—
	(0.399)	(0.322)		
I (1990s)	0.256	0.121	Census	—
	(0.436)	(0.326)		
I (2000s)	0.305	0.079	Census	—
	(0.461)	(0.270)		

Notes: Same as Table 1.
Sources: Decennial Census, CPS Annual and Monthly Social and Economic Supplement, NLRB data, and authors' calculations.

Table A3: Unionization Data Summary Statistics

Variable	Treatment sample			Instrument sample		
	All	Union wins	Union losses	All	Union wins	Union losses
Number of elections	233307	117008	116299	35005	15599	19406
Union win	0.500	0.995	0.001	0.444	0.996	0.001
Union vote share	0.538	0.780	0.295	0.493	0.558	0.441
Number of votes	54.808	45.932	63.738	115.360	106.754	122.277
Number of eligible voters	64.644	57.739	71.591	128.480	119.471	135.721
Years 1961–1969	0.226	0.261	0.191	0.246	0.259	0.235
Years 1970–1979	0.327	0.318	0.337	0.323	0.313	0.331
Years 1980–1989	0.199	0.172	0.225	0.181	0.175	0.185
Years 1990–1999	0.149	0.140	0.159	0.156	0.154	0.158
Years 2000–2009	0.098	0.108	0.088	0.094	0.099	0.090
Manufacturing industry	0.431	0.411	0.450	0.518	0.497	0.536
Decertification	0.135	0.082	0.188	0.119	0.129	0.112

Notes: The table reports mean characteristics for union elections included in our sample. The set of elections used to construct the treatment variable ("treatment sample") contains all elections, while the set used for the instrument construction ("instrument sample") contains only elections with the vote share for the union of $50 \pm 10\%$, at least 20 votes, and vote margin of at least 2 votes. Union wins are defined as strictly more than 50% of votes in favor of the union.
Sources: NLRB data.

Table A4: Within-Between Variance Decomposition of Log Wages, Census Data

Year	Var(log hourly wage)			Industry
	Total	Within cells	Between cells	classification
1960	0.271	0.198	0.074	SIC
1970	0.288	0.225	0.063	SIC
1980	0.293	0.237	0.057	SIC
1990	0.377	0.308	0.069	SIC
2000	0.416	0.352	0.064	SIC
2000	0.416	0.347	0.069	NAICS
2010	0.475	0.373	0.101	NAICS

Notes: The table presents a variance decomposition of log wages, broadly following Helpman et al. (2016). For each census year, we decompose the overall variance in log wages among workers in our Census sample into within and between (state-industry) cell components. In 2000, we perform the decomposition twice, using two industry coding frameworks: SIC-based (used in our main analysis until 2000) and NAICS-based (used from 2000 onwards). Census projection weights are used.
Sources: Decennial Census; authors' calculations.

Table A5: Instrument Balance, 15% Bandwidth

	NewUnions	Z		
	(1)	(2)	(3)	(4)
Log avg hourly wage$_{t-1}$	0.613	-0.009	-0.212	-0.030
	(0.563)	(0.144)	(0.088)	(0.036)
Log college premium$_{t-1}$	3.285	1.964	-0.027	0.042
	(0.406)	(0.173)	(0.095)	(0.044)
Log 90/10 ratio$_{t-1}$	3.516	1.149	-0.004	-0.012
	(0.668)	(0.260)	(0.134)	(0.066)
Gini index$_{t-1}$	-66.693	-22.968	-1.467	0.147
	(6.288)	(1.959)	(1.104)	(0.481)
Top 10% share$_{t-1}$	48.512	15.633	0.744	-0.117
	(5.997)	(1.767)	(1.002)	(0.446)
Var(log hourly wage)$_{t-1}$	-8.299	-2.975	0.131	-0.154
	(1.499)	(0.399)	(0.227)	(0.111)
Log employment$_{t-1}$	-0.149	-0.094	0.000	0.000
	(0.054)	(0.020)	(0.011)	(0.005)
Share with college degree$_{t-1}$	-0.960	-0.063	0.410	0.154
	(0.744)	(0.223)	(0.125)	(0.063)
Share white$_{t-1}$	-1.779	-0.811	-0.536	-0.058
	(0.577)	(0.254)	(0.142)	(0.053)
Share male$_{t-1}$	-0.812	-0.497	-0.040	0.049
	(0.514)	(0.166)	(0.084)	(0.036)
I (Manufacturing)	1.900	0.996	-0.126	-0.006
	(0.197)	(0.064)	(0.038)	(0.017)
# significant controls	9	11	4	1
Partial R^2	0.166	0.141	0.002	0.000
Partial F-statistic	266.5	214.9	7.5	0.9
Partial F-test, p-value	0.000	0.000	0.000	0.558
RDA controls	None	None	$\sum_{j \in \mathcal{C}_{sit}} s_j$	All
Observations	12,459	12,459	12,459	12,459

Notes: Same as Table 2, except the instrument Z is constructed from close elections defined by the bandwidth of 15%, i.e. the union vote share of 50 ± 15 percent.
Sources: Decennial Census, NLRB data, and authors' calculations.

Table A6: Placebo Tests: The Impact of New Unionization on Predetermined Outcomes

Panel A: Lagged Outcomes in Levels

	College premium	Log 90/10	Gini coefficient	Top 10% share	Var(Log Wage)
Z_{sit}	0.097	0.149	-0.002	-0.005	0.051
	(0.198)	(0.218)	(0.055)	(0.043)	(0.073)
	[-0.566,0.663]	[-0.488,0.842]	[-0.183,0.172]	[-0.148,0.130]	[-0.164,0.287]
Observations	10,422	12,555	12,555	12,555	12,555

Panel B: Lagged Outcomes in Differences

	ΔCollege premium	ΔLog 90/10	ΔGini coefficient	ΔTop 10% share	ΔVar(Log Wage)
Z_{sit}	-0.311	0.526	0.049	-0.003	0.134
	(0.329)	(0.224)	(0.056)	(0.051)	(0.081)
	[-1.637,0.453]	[-0.053,1.426]	[-0.122,0.238]	[-0.176,0.146]	[-0.089,0.423]
Observations	6,283	7,487	7,487	7,487	7,487

Panel C: Covariates

	log(Wage)	log(Emp.)	(%)Coll.	(%)White	(%)Male
Z_{sit}	-0.034	-0.177	-0.021	-0.083	0.078
	(0.160)	(1.140)	(0.058)	(0.108)	(0.089)
	[-0.462,0.534]	[-3.785,3.860]	[-0.173,0.168]	[-0.442,0.222]	[-0.139,0.443]
Observations	12,555	12,555	12,555	12,555	12,555

Notes: Placebo tests based on OLS estimates corresponding to the reduced-form of our main upper-level IV specification as in Table 3, Panel A. See notes to Table 3 for details about data and variables construction. The outcomes are measured at the beginning of the decade in which the instrument is measured in Panels A and C and as a 10-year lagged difference in Panel B. *Sources:* Decennial Census and NLRB unionization data; authors' calculations.

Table A7: The Impact of New Unionization on 10-Year Changes in Inequality: Robustness Checks

	ΔLog college premium (1)	ΔLog 90/10 (2)	ΔGini coeff. (3)	ΔTop 10% share (4)	ΔVar (log wage) (5)	ΔVar(res. log wage) (6)
Panel A: Baseline						
Share newly unionized	-0.423	-0.496	-0.169	-0.138	-0.263	-0.116
	(0.295)	(0.243)	(0.064)	(0.058)	(0.092)	(0.065)
	[-1.073,0.293]	[-1.057,0.103]	[-0.319,-0.013]	[-0.275,-0.000]	[-0.492,-0.042]	[-0.278,0.033]
Panel B: 15% Bandwidth						
Share newly unionized	-0.373	-0.291	-0.097	-0.074	-0.191	-0.070
	(0.213)	(0.160)	(0.043)	(0.039)	(0.061)	(0.044)
	[-0.914,0.050]	[-0.672,0.075]	[-0.200,0.000]	[-0.172,0.010]	[-0.334,-0.047]	[-0.168,0.035]
Panel C: Excluding Union Decertification Elections						
Share newly unionized	-0.504	-0.535	-0.173	-0.134	-0.269	-0.112
	(0.352)	(0.264)	(0.069)	(0.063)	(0.100)	(0.070)
	[-1.242,0.353]	[-1.157,0.117]	[-0.340,-0.007]	[-0.286,0.013]	[-0.520,-0.032]	[-0.291,0.047]
Panel D: State, Year, Industry Fixed Effects						
Share newly unionized	-0.503	-0.503	-0.160	-0.129	-0.223	-0.098
	(0.293)	(0.260)	(0.067)	(0.059)	(0.100)	(0.073)
	[-1.098,0.265]	[-1.115,0.109]	[-0.315,0.005]	[-0.266,0.017]	[-0.464,0.013]	[-0.275,0.065]
Panel E: No Fixed Effects						
Share newly unionized	-0.522	-0.519	-0.171	-0.136	-0.250	-0.130
	(0.355)	(0.284)	(0.071)	(0.063)	(0.109)	(0.082)
	[-1.289,0.403]	[-1.202,0.161]	[-0.339,0.000]	[-0.287,0.013]	[-0.521,0.011]	[-0.333,0.059]
Panel F: Unweighted						
Share newly unionized	-0.460	-0.559	-0.244	-0.185	-0.357	-0.192
	(0.348)	(0.264)	(0.083)	(0.080)	(0.119)	(0.082)
	[-1.532,0.621]	[-1.164,0.339]	[-0.392,0.048]	[-0.351,0.065]	[-0.647,-0.022]	[-0.416,0.036]
Panel G: Less than 25% unionization attempts						
Share newly unionized	-0.145	-0.974	-0.268	-0.239	-0.407	-0.185
	(0.353)	(0.390)	(0.106)	(0.098)	(0.155)	(0.110)
	[-0.979,0.695]	[-1.908,0.011]	[-0.523,-0.019]	[-0.482,-0.018]	[-0.809,-0.045]	[-0.481,0.071]
Panel H: No Running Variable Controls						
Share newly unionized	-0.152	-0.080	-0.037	-0.032	-0.063	-0.014
	(0.095)	(0.075)	(0.019)	(0.018)	(0.028)	(0.022)

Notes: Panel A reports upper-level IV estimates of the effects of new unionization rates on wage inequality, replicating Panel A of Table 3. The following columns deviate from Panel A in different ways, one at a time. Panel B redefines the instrument and RDA controls using a bandwidth of 15% instead of 10%. Panel C reports results excluding decertification elections from the unionization elections sample when constructing the treatment, instrument, and RDA controls. Panel D includes separate state, industry, and year fixed effects instead of state-year and industry-year fixed effects. Panel E reports estimates without any fixed effects. Panel F presents unweighted estimation results, instead of weighting state-industry-decade observations by beginning-of-decade employment. Panel G restricts the sample to state-industry-decade cells where less than 25% of employment was in establishments that held unionization elections. Panel H excludes the RDA running variable controls ($\sum_{j\in\mathcal{C}_{sit}} s_j r_j$, $\sum_{j\in\mathcal{C}_{sit}} s_j r_j^+$) from the specification.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A8: The Impact of New Unionization on 10-Year Changes in Inequality: Heterogeneity by Decades

	ΔLog college premium (1)	ΔLog 90/10 (2)	ΔGini coeff. (3)	ΔTop 10% share (4)	ΔVar(log wage) (5)	ΔVar(res. log wage) (6)	ΔMean(Log Wage) (7)
Panel A: 1960s							
Share Newly Unionized	-0.455	-0.332	-0.084	-0.040	-0.120	-0.056	-0.092
	(0.530)	(0.303)	(0.068)	(0.057)	(0.088)	(0.073)	(0.138)
	[-1.564,0.880]	[-1.025,0.417]	[-0.228,0.107]	[-0.157,0.115]	[-0.320,0.132]	[-0.221,0.140]	[-0.433,0.264]
Panel B: 1970s							
Share Newly Unionized	-0.210	-0.502	-0.105	-0.066	-0.154	-0.043	-0.262
	(0.452)	(0.377)	(0.089)	(0.072)	(0.123)	(0.080)	(0.210)
	[-1.163,0.984]	[-1.428,0.389]	[-0.329,0.110]	[-0.242,0.113]	[-0.470,0.141]	[-0.250,0.138]	[-0.807,0.179]
Panel C: 1980s							
Share Newly Unionized	-0.019	0.090	-0.019	-0.036	-0.180	-0.102	-0.336
	(0.674)	(0.615)	(0.143)	(0.134)	(0.244)	(0.192)	(0.400)
	[-1.518,1.497]	[-1.196,1.503]	[-0.306,0.278]	[-0.290,0.223]	[-0.696,0.370]	[-0.537,0.352]	[-1.264,0.746]
Panel D: 1990s							
Share Newly Unionized	-0.540	-0.664	-0.393	-0.339	-0.437	-0.234	-0.094
	(0.778)	(0.662)	(0.243)	(0.240)	(0.302)	(0.232)	(0.355)
	[-2.456,1.167]	[-1.992,0.947]	[-0.946,0.195]	[-0.886,0.237]	[-1.126,0.278]	[-0.763,0.325]	[-0.893,0.717]
Panel E: 2000s							
Share Newly Unionized	-2.620	-4.411	-1.610	-1.774	-2.786	-1.576	-0.924
	(2.472)	(4.471)	(1.632)	(1.709)	(2.815)	(1.771)	(1.269)
	[-9.912,2.555]	[-16.992,5.463]	[-6.119,1.672]	[-6.662,1.685]	[-10.922,3.101]	[-6.591,2.013]	[-5.272,1.291]

Notes: Each panel reports upper-level IV estimates from the baseline specification (Panel A of Table 3), estimated separately by decade.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A9: The Impact of New Unionization on 10-Year Changes in Inequality: Heterogeneity Analysis

	ΔLog college premium (1)	ΔLog 90/10 (2)	ΔGini coeff. (3)	ΔTop 10% share (4)	ΔVar(log wage) (5)	ΔVar(res. log wage) (6)	ΔMean(Log Wage) (7)
Panel A: Manufacturing							
Share newly unionized	-0.326	-0.199	-0.121	-0.093	-0.168	-0.057	-0.181
	(0.344)	(0.245)	(0.067)	(0.061)	(0.086)	(0.060)	(0.127)
	[-1.067,0.559]	[-0.724,0.410]	[-0.274,0.041]	[-0.228,0.049]	[-0.365,0.043]	[-0.199,0.090]	[-0.511,0.107]
Panel B: Non Manufacturing							
Share newly unionized	-0.829	-0.941	-0.265	-0.268	-0.476	-0.247	-0.270
	(0.659)	(0.592)	(0.155)	(0.151)	(0.252)	(0.166)	(0.282)
	[-2.333,0.555]	[-2.417,0.453]	[-0.637,0.111]	[-0.636,0.091]	[-1.113,0.106]	[-0.679,0.122]	[-1.074,0.376]
Panel C: Right-to-Work State							
Share newly unionized	-0.152	-0.486	-0.160	-0.122	-0.226	-0.093	-0.280
	(0.401)	(0.252)	(0.077)	(0.075)	(0.100)	(0.082)	(0.148)
	[-1.006,0.902]	[-0.995,0.142]	[-0.343,0.010]	[-0.304,0.039]	[-0.458,0.002]	[-0.297,0.084]	[-0.593,0.059]
Panel D: Non Right-to-Work State							
Share newly unionized	-0.575	-0.385	-0.154	-0.142	-0.247	-0.096	-0.121
	(0.411)	(0.394)	(0.097)	(0.086)	(0.148)	(0.095)	(0.175)
	[-1.571,0.363]	[-1.346,0.566]	[-0.379,0.095]	[-0.340,0.075]	[-0.621,0.119]	[-0.329,0.139]	[-0.622,0.276]

Notes: Each panel reports upper-level IV estimates from the baseline specification, estimated separately by subsample: manufacturing industries (Panel A), non-manufacturing industries (Panel B), right-to-work states (defined as in Holmes (1998); Panel C), and non-right-to-work states (Panel D). During the sample period, three states adopted right-to-work laws: Louisiana (1976), Idaho (1985), and Oklahoma (2001). To maintain balanced panels, Louisiana is coded as right-to-work throughout and Idaho and Oklahoma as non-right-to-work.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A10: The Impact of New Unionization on 10-Year Wage Changes: Robustness Checks

	All workers	Mgmt	Non-Mgmt	College	High school	All workers (residualized)
	(1)	(2)	(3)	(4)	(5)	(6)
Panel A: Baseline						
Share newly unionized	-0.173	-0.378	-0.103	-0.271	-0.063	-0.095
	(0.119)	(0.320)	(0.120)	(0.292)	(0.138)	(0.098)
	[-0.505,0.082]	[-1.129,0.416]	[-0.433,0.150]	[-0.915,0.489]	[-0.401,0.252]	[-0.368,0.119]
Panel B: 15% Bandwidth						
Share newly unionized	-0.087	-0.128	-0.044	-0.128	0.022	-0.013
	(0.082)	(0.229)	(0.082)	(0.211)	(0.094)	(0.069)
	[-0.265,0.122]	[-0.648,0.373]	[-0.218,0.169]	[-0.697,0.308]	[-0.165,0.271]	[-0.167,0.157]
Panel C: Excluding Union Decertification Elections						
Share newly unionized	-0.218	-0.713	-0.135	-0.374	-0.156	-0.123
	(0.130)	(0.377)	(0.130)	(0.345)	(0.147)	(0.106)
	[-0.580,0.059]	[-1.586,0.216]	[-0.493,0.139]	[-1.121,0.508]	[-0.527,0.172]	[-0.426,0.105]
Panel D: State, Year, Industry Fixed Effects						
Share newly unionized	-0.074	-0.166	-0.005	-0.262	0.112	-0.011
	(0.186)	(0.340)	(0.190)	(0.291)	(0.213)	(0.154)
	[-0.566,0.336]	[-0.949,0.665]	[-0.511,0.405]	[-0.862,0.526]	[-0.433,0.586]	[-0.440,0.313]
Panel E: No Fixed Effects						
Share newly unionized	-0.174	-0.097	-0.131	-0.293	0.045	-0.058
	(0.306)	(0.469)	(0.313)	(0.539)	(0.317)	(0.180)
	[-0.885,0.535]	[-1.141,1.062]	[-0.862,0.587]	[-1.408,1.074]	[-0.673,0.771]	[-0.515,0.341]
Panel F: Unweighted						
Share newly unionized	-0.080	-0.415	-0.024	-0.435	0.043	-0.063
	(0.106)	(0.394)	(0.108)	(0.309)	(0.136)	(0.094)
	[-0.624,0.208]	[-1.607,0.398]	[-0.571,0.262]	[-1.338,0.783]	[-0.519,0.414]	[-0.504,0.178]
Panel G: Less than 25% unionization attempts						
Share newly unionized	-0.173	-0.042	-0.023	0.008	-0.008	-0.108
	(0.209)	(0.403)	(0.211)	(0.363)	(0.230)	(0.163)
	[-0.766,0.284]	[-0.975,0.900]	[-0.610,0.425]	[-0.825,0.921]	[-0.564,0.557]	[-0.573,0.257]
Panel H: No Running Variable Controls						
Share newly unionized	0.012	-0.110	0.035	-0.027	0.039	0.044
	(0.042)	(0.111)	(0.042)	(0.102)	(0.046)	(0.039)

Notes: This table performs the robustness checks of Table A7 for the outcomes of Table 5. Please refer to those tables for the specifications and the data construction.

Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A11: The Impact of New Unionization on 10-Year Employment Changes

	All	Mgmt	Non-Mgmt	College	HS
	(1)	(2)	(3)	(4)	(5)
Panel A: Upper-Level Estimator					
Share newly unionized	-0.402	-0.639	-0.306	-2.058	0.272
	(0.462)	(0.668)	(0.464)	(0.883)	(0.478)
Mean outcome	0.164	0.216	0.155	0.404	0.155
Mean treatment	10.823	8.692	10.668	8.766	9.809
RDA controls	2.59%	2.60%	2.59%	2.59%	2.59%
Industry-decade and state-decade FE	✓	✓	✓	✓	✓
Observations	✓	✓	✓	✓	✓
	12,555	12,388	12,555	12,555	12,371
Panel B: Lower-Level Estimator					
Share newly unionized	0.032	-0.100	0.092	0.032	-1.300
	(0.370)	(0.559)	(0.373)	(0.370)	(0.781)
Mean outcome	10.609	8.221	10.501	10.609	8.149
Mean treatment	9.69%	9.68%	9.69%	9.69%	9.68%
RDA controls	✓	✓	✓	✓	✓
Industry-decade and state-decade FE	✓	✓	✓	✓	✓
Observations	31,313	31,306	31,313	31,313	31,226

Notes: This table reports IV estimates from the same specifications as in Table 3 but for different outcomes. The outcome in column 1 is the 10-year change in the total log employment in the state-industry cell. The other columns consider changes in log employment for specific groups: workers in management positions (column 2), in non-management positions (column 3), those with exactly a college degree (column 4), and those with exactly a high-school degree (column 5). As in the main analyses, upper-level observations are weighted by total employment at the beginning of the decade (not limited to the subgroup); lower level observations are weighted by number of eligible voters in the beginning of the decade.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A12: The Impact of New Unionization on 10-Year Inequality Changes, Dynamic Analysis

	ΔLog college premium (1)	ΔLog 90/10 (2)	ΔGini coeff. (3)	ΔTop 10% share (4)	ΔVar (log wage) (5)	ΔVar(res. log wage) (6)
Panel A: Short-Run Effects (Elections in Years Ending 5-9)						
Share Newly	0.081	-0.409	-0.033	0.013	-0.114	-0.067
Unionized	(0.328)	(0.204)	(0.054)	(0.050)	(0.073)	(0.060)
(late)	[-0.782,1.134]	[-1.086,0.199]	[-0.185,0.127]	[-0.122,0.170]	[-0.347,0.097]	[-0.243,0.101]
Panel B: Long-Run Effects (Elections in Years Ending 0-4)						
Share Newly	-1.072	-0.459	-0.309	-0.310	-0.394	-0.149
Unionized	(0.480)	(0.385)	(0.109)	(0.102)	(0.153)	(0.105)
(early)	[-2.750,0.326]	[-1.779,0.666]	[-0.793,0.015]	[-0.781,-0.014]	[-1.059,0.044]	[-0.546,0.130]

Notes: Panel A estimates short-run effects of unionization using elections held in years ending 5–9 of each decade (1–5 years before the end of the decade). Panel B estimates long-run effects using elections held in years ending 0–4 (6–10 years before the end of the decade). Each panel uses the baseline upper-level specification from Table 3 with treatment, instrument, and RDA controls constructed from the respective election subsample separately.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A13: The Impact of New Unionization on 10-Year Employment Changes, Dynamic Analysis

	ΔMean(Log wage) (1)	ΔLog(Emp) (2)	ΔLog(HS Emp) (3)	ΔLog(College Emp) (4)
Panel A: Short-Run Effects (Elections in Years Ending 5-9)				
Share Newly	-0.302	-0.831	-0.117	-2.388
Unionized	(0.125)	(0.525)	(0.534)	(1.001)
(late)	[-0.753,0.077]	[-2.779,0.749]	[-1.919,1.328]	[-6.145,0.701]
Panel B: Long-Run Effects (Elections in Years Ending 0-4)				
Share Newly	0.082	0.245	0.622	-1.099
Unionized	(0.183)	(0.668)	(0.698)	(1.184)
(early)	[-0.561,0.537]	[-1.702,2.112]	[-1.336,2.706]	[-5.078,1.801]

Notes: Same as Table A12.
Sources: Decennial Census and NLRB unionization data; authors' calculations.

Table A14: Definitions of Decades

Decade	NLRB	Decennial Census and ACS	CPS MORG (for union density)	CPS ASEC (for fringe benefits)
	(1)	(2)	(3)	(4)
1960s	1961–69	1970 minus 1960	n/a	n/a
1970s	1970–79	1980 minus 1970	n/a	n/a
1980s	1980–89	1990 minus 1980	1990–1994 minus 1977–1980[b]	1990–1994 minus 1979–1981[b]
1990s	1990–99	2000 minus 1990	2000–2004 minus 1986–1990	2000–2004 minus 1986–1990
2000s	2000–09[a]	2009–2013 minus 2000	2010–2014 minus 1996–2000	2010–2014 minus 1996–2000

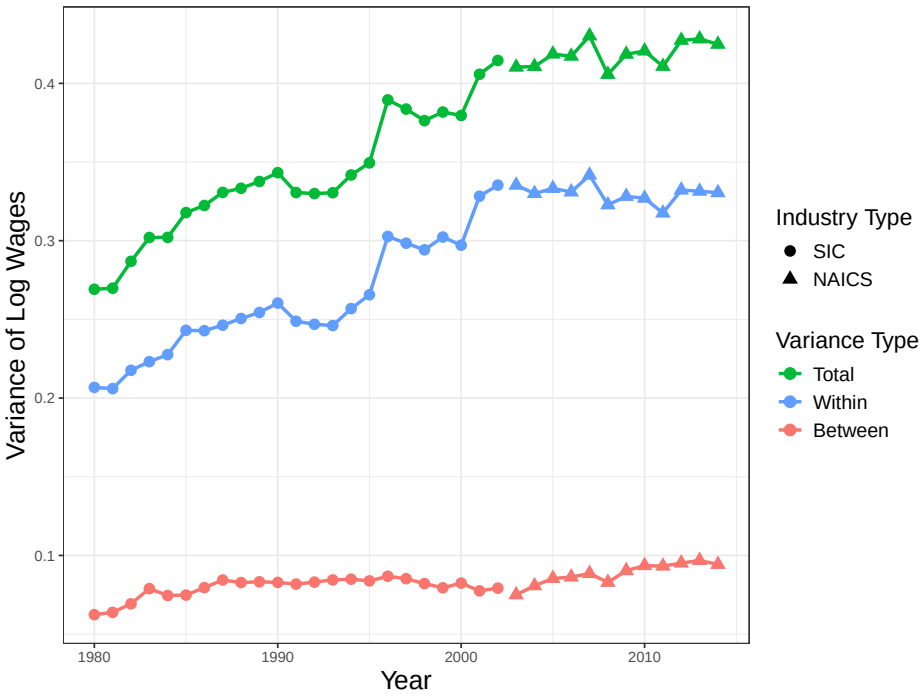
Notes: For each decade in the data, column 1 reports the years of NLRB data used to compute the new unionization rate. Columns 2–4 report the years in the decennial census (plus ACS) data, CPS MORG, and CPS ASEC data used to compute differenced outcomes. For example, the change over 1990s in the CPS MORG is computed as the average of 2000–2004 minus the average of 1986–1990.

[a] The NLRB data for the year of 2009 cover only the beginning of year.

[b] The May supplement to the CPS replaces CPS MORG for years 1977–1980 and replaces CPS ASEC for the year of 1979.

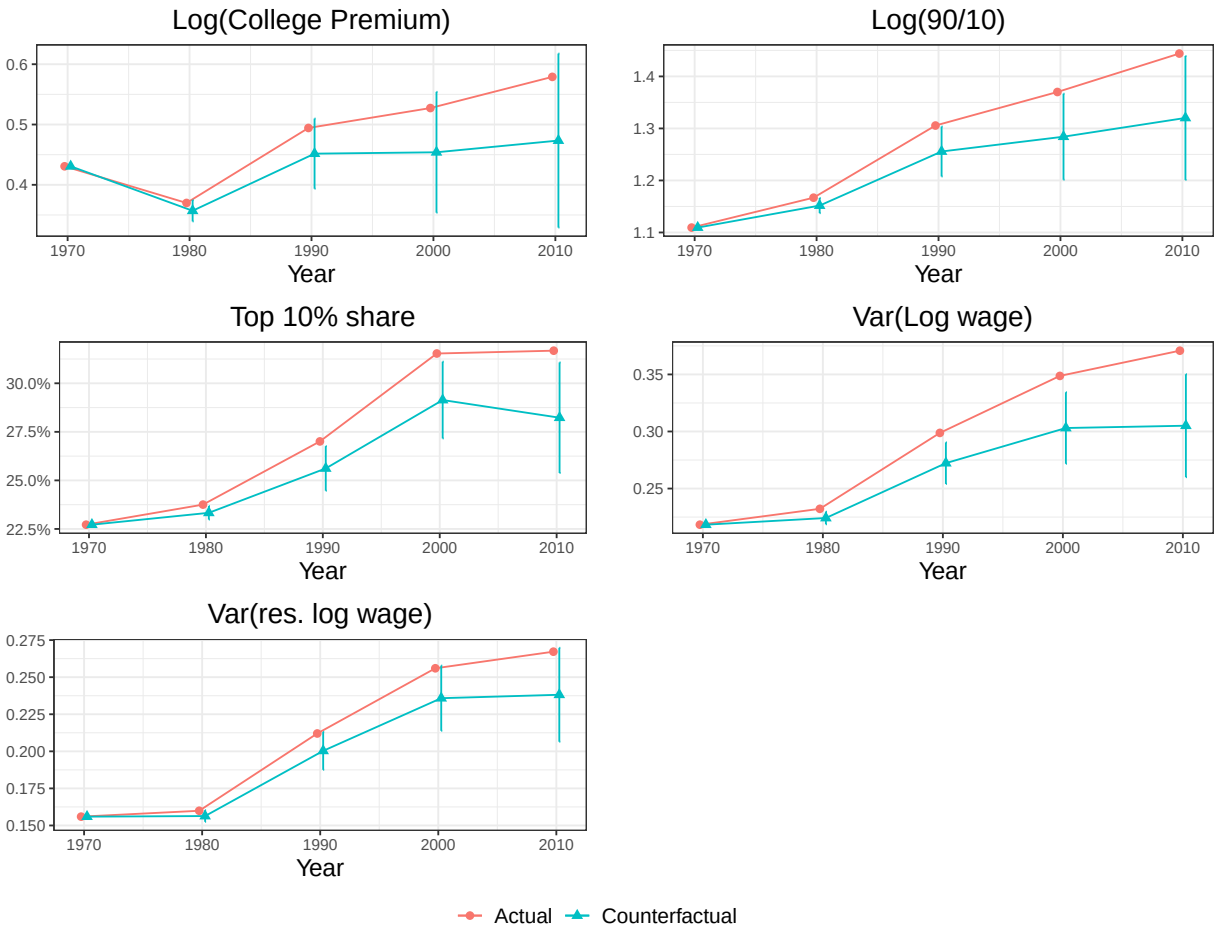
Appendix Figures

Figure A1: Within-Between Variance Decomposition of Log Wages, CPS Data



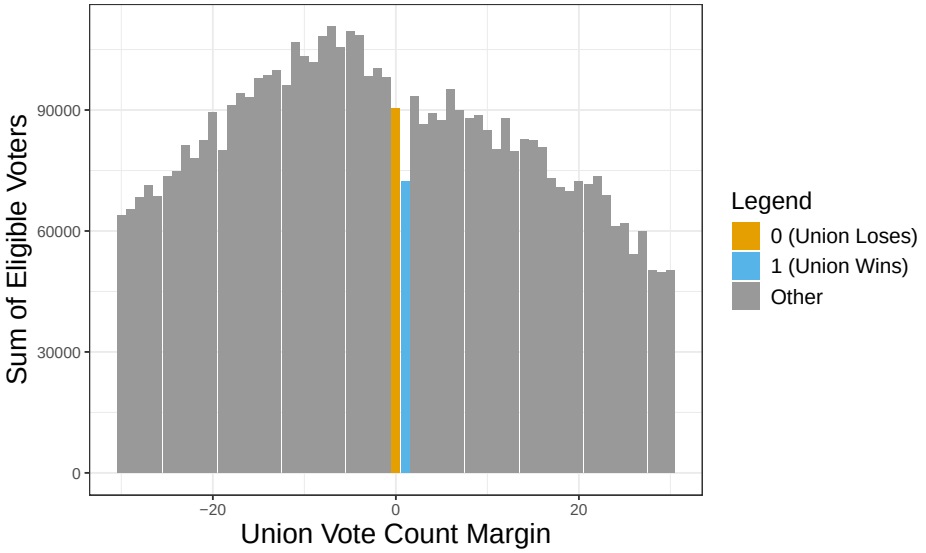
Notes: The plot presents a variance decomposition of log wages, broadly following Helpman et al. (2016), using CPS ASEC samples. In each year from 1980–2014, we decompose the overall variance in log wages among workers into within and between (state-industry) cell components, using projection weights. The industry coding framework shifts from SIC-based to NAICS-based in 2003.
Sources: CPS ASEC samples; authors' calculations.

Figure A2: The Counterfactual with Unionization Remaining at the 1960s Level: Other Inequality Outcomes



Notes: Each panel presents a measure of actual within (state-industry) cell inequality (averaged across cells with beginning-of-decade employment weights) by census year from 1970–2010, as well as its counterfactual value had the rate of new unionization stayed at the 1960s level. The counterfactual is calculated by adding to the actual measure the product of the treatment effect estimated in Table 3 and the cumulative shortfall in the rate of new unionization between the actual and counterfactual scenarios (presented in the first panel of Figure 3). Confidence intervals for the counterfactuals are based on the 95% heteroskedasticity-robust confidence intervals for the corresponding treatment effects.
Sources: Decennial Census, NLRB unionization data; authors' calculations.

Figure A3: Distribution of Total Eligible Voters by Union Vote Count Margin



Notes: For each margin between -30 and 30 votes, the histogram shows the total number of eligible workers in workplaces where unionization elections ended with this vote margin (i.e., the gap between the votes in favor and against the union). Tied elections and elections in which the union wins by one vote are highlighted. The time rage is 1961-2009.
Sources: NLRB unionization data; authors' calculations.