One Instrument, Many Treatments: Instrumental Variables Identification of Multiple Causal Effects*

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July 2026

Abstract

Many instrumental variables applications specify a single Bernoulli treatment. But instruments may change outcomes through multiple pathways or by varying treatment intensity. Lottery instruments that boost charter school enrollment, for instance, may affect outcomes by lengthening time enrolled in a charter school and by moving students between charter schools of different types. We analyze the identification problem such scenarios present in a framework that generalizes the always-taker/never-taker/complier partition of treatment response types to cover a wide range of multinomial and ordered treatments with heterogeneous potential outcomes. This framework yields novel estimands in which a single randomly assigned instrument identifies (i) causal effects averaged over complier types and (ii) a causal conditional expectation function that captures effects for each element in a set of complier types. Three empirical applications demonstrate the utility of these results. The first extends an earlier analysis of the Head Start Impact Study allowing for multiple fallbacks. The second examines two causal channels for the impact of post-secondary financial aid on degree completion. The third estimates effects of additional births (an ordered treatment) on mothers' weeks worked.

*We thank Peng Ding, Avi Feller, and participants at the Berkeley and Harvard-MIT Econometrics Seminar for helpful comments; Pat Kline and Chris Walters for help with the Head Start data and for insightful comments; and Fatima Djalalova, Abigail Orbe, and Gabriel Solis for extraordinary research assistance.

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Many of the most compelling instrumental variables (IV) research designs exploit random assignment. For instance, IV studies of charter schools (surveyed in Chabrier et al. (2016)) use conditionally-random admissions lotteries as instruments for charter-school enrollment. Under plausible assumptions, IV estimates using randomized charter offers to instrument a charter enrollment dummy identify causal effects of charter attendance in the population induced to enroll by random assignment. Other lottery designs derive instruments for Medicaid (analyzed in Finkelstein et al. (2012)) and the draft lottery that determined eligibility for Vietnam-era conscription (analyzed in Angrist (1990)). In clinical trials with partial compliance, random assignment to treatment, also called intention-to-treat, can be used to instrument treatment received, identifying effects on trial participants who comply with the trial protocol (Angrist and Hull, 2023).

Random assignment alone does not guarantee instrument validity. Angrist and Imbens (1995) and Andresen and Huber (2021) note that quotidian implementation details such as recoding an ordered treatment as Bernoulli may compromise exclusion, thereby muddling IV estimates. Rose and Shem-Tov (2024) highlights this point in the context of the Oregon Medicaid lottery studied by Finkelstein et al. (2012), arguing that Medicaid exposure should be measured in years rather than as a dummy indicating any enrollment. Relying on an additive linear model for identification, Abdulkadiroglu et al. (2017) and other charter lottery studies similarly focus on years enrolled as a measure of treatment exposure. Angrist and Evans (1998), by contrast, uses dummies indicating multiple second births and same-sex sibships to construct instruments for a dummy indicating the presence of a third child. Exclusion fails here if these instruments shift childbearing at parities higher than three.

Exclusion failures can arise when a single instrument affects outcomes through multiple channels. For example, Angrist and Chen (2011) uses randomly assigned Vietnam-era draft eligibility as an instrument for schooling. This identification strategy is motivated by the GI Bill, a major source of post-secondary aid for Vietnam veterans and a possible explanation of earnings gaps by veteran status. But draft-eligibility also increases the likelihood of military service, arguably the dominant channel through which draft eligibility affects earnings and other long-run outcomes. Angrist and Chen (2011) accommodates this using additive constant-effects models with two treatments, identified by multiple instruments. However, two-stage least squares (2SLS) estimators with multiple treatments may fail to identify average causal effects in a world of heterogeneous potential outcomes (a point made in Behagel et al. (2013) and Kirkeboen et al. (2016)).

The interpretation of IV estimates is also complicated by heterogeneous counterfactual outcomes relevant for a given treatment. Abdulkadiroglu et al. (2017) notes, for instance, that a given charter school offer may pull applicants from other charter schools, possibly differing in type or quality, as well as increasing the odds of any-charter enrollment. Along the same lines, Kline and Walters (2016) uses the random assignment of Head Start seats and a structural model to distinguish causal effects of Head Start for children drawn into the program from home care and from competing preschools. Potential outcomes in these scenarios are determined by multinomial treatments, any one of which may be changed by the instrument at hand.

Some multinomial treatment variables are meant to *explain* causal effects. For example,

Angrist et al. (2022) analyzes data from a randomized trial that offered generous financial aid to college-bound high school seniors in the treatment group. This intervention boosted BA completion rates sharply. Angrist et al. (2022) argues that the resulting degree gains are caused by early four-year engagement as measured by college credits earned in freshman year. But other channels, such as a reduction in the burden of college debt, may matter as much or more than early engagement.

This paper develops identification and estimation strategies for a broad class of such problems. Specifically, we state assumptions and establish identification results for potential outcomes models that use a single instrument to identify a class of average causal effects in settings with ordered and multinomial treatments. We also extend the weighting strategy that Abadie (2003) uses to identify causal conditional expectation functions for Bernoulli treatments. Our results identify average causal effects in settings beyond those with Bernoulli treatments, while delineating a number of novel restrictions that deliver identification in these contexts.

We contribute to a large literature on IV identification focused on beyond-Bernoulli treatments. Like Heckman et al. (2006), Lee and Salanié (2018), Pinto (2021), Dutz et al. (2021), Goff (2024), Nibbering and Oosterveen (Forthcoming), and Imbens and Angrist (1994) and Angrist et al. (1996) among others, we use restrictions on choice behavior to identify causal parameters. Our analysis also complements Torgovitsky (2015), D'Haultfœuille and Février (2015), and Chernozhukov et al. (2024) which likewise study identification using discrete instruments, though exploiting assumptions unrelated to ours. In extensions of Imai et al. (2010), Frölich and Huber (2017), Kwon and Roth (2025), and Athey et al. (2025) explore the identifying power of IV-inspired restrictions on potential treatments like monotonicity in causal mediation analysis. Mogstad and Torgovitsky (2024) surveys alternative IV paradigms for analysis of models with heterogeneous potential outcomes.

The identification strategies developed here are closely related to the principal stratification approach to causal inference pioneered in Frangakis and Rubin (2002), applied in Hill et al. (2002), surveyed in Feller et al. (2017) and extended in Lu et al. (2026). Principal strata are conditioning sets defined by latent response types; conditional on these (and possibly on covariates), treatment is independent of potential outcomes. The next section starts with a restatement of this foundational result. Heckman and Pinto (2018) extends the principal strata idea to IV models with unordered treatments and discusses identification under various restrictions on latent response types. Unlike Heckman and Pinto (2018), we use limited homogeneity restrictions on potential outcomes to facilitate identification of average causal effects for individual response types. This leads to alternatives to the restrictions on response types considered in earlier work that may be easily verified in the context of any particular model.

We also build on Navjeevan et al. (2023), which gives high-level necessary and sufficient conditions for identification of causal parameters in a general class of potential outcomes models. We apply and extend Navjeevan et al. (2023) by introducing new causal parameters and novel, widely applicable identifying restrictions. Our results leverage unique features of models powered by a single Bernoulli instrument to obtain transparent, verifiable (in the context of a particular model) necessary and sufficient conditions for identification. Finally, we derive simple weighted least squares estimands for parameters identified by our extensions of Abadie

(2003), as well as straightforward IV estimands. Our results are unique in their derivation of a minimum mean-squared error approximation to underlying, possibly nonlinear, causal models for beyond-Bernoulli treatments. An appendix collects asymptotic results for estimators constructed as the sample analog of the estimands developed here. The utility of our theoretical results is demonstrated in empirical applications revisiting questions addressed in Kline and Walters (2016), Angrist et al. (2022), and Angrist and Evans (1998).

1 Econometric Framework

We begin by describing the causal models of interest. The data consist of an outcome $Y \in \mathbf{R}$, a discrete treatment $T \in \mathbf{T} \equiv \{t_1, \ldots, t_d\}$, a Bernoulli instrument $Z \in \mathbf{Z} \equiv \{0, 1\}$, and covariates $X \in \mathbf{X}$. Since d may exceed two, treatments can be ordered or multinomial. A vector of potential outcomes $Y^* \equiv (Y^*(t_1), \ldots, Y^*(t_d)) \in \mathbf{R}^d$ determines observed outcomes, $Y = Y^*(T)$. The observed treatment determines which potential outcome we see. Likewise, a pair of potential treatments $T^* \equiv (T^*(0), T^*(1)) \in \mathbf{T} \times \mathbf{T}$ determines observed treatments, $T = T^*(Z)$. Response type for each individual is defined by the pair of potential treatments, T^*. The instrument determines which of these we see.

The following assumption encapsulates this setup.

Assumption 1.1. *(i)* $Y^* \equiv (Y^*(t_1), \ldots, Y^*(t_d)) \in \mathbf{R}^d$, $Z \in \mathbf{Z} \equiv \{0, 1\}$, $X \in \mathbf{X}$, *and* $T^* = (T^*(0), T^*(1))$ *with* $T^*(z) \in \mathbf{T} \equiv \{t_1, \ldots, t_d\}$ *for* $z \in \{0, 1\}$*; (ii)* $T = T^*(Z)$ *and* $Y = Y^*(T)$*; (iii)* (Y^*, T^*) *is independent of* Z *conditional on* X*; (iv)* $E[|Y^*(t)|] < \infty$ *for all* $t \in \mathbf{T}$ *and* $P(Z = 1|X) \in (0, 1)$ *with probability one.*

Assumption 1.1(iii) says that instrumental variable Z is as good as randomly assigned in the sense of being independent of response types and potential outcomes (conditional on covariates). This assumption implies an exclusion restriction since, for a given treatment value, potential outcomes are presumed to be unrelated to the value of Z. Assumption 1.1(iv) also ensures that Z varies conditional on covariates and that average potential outcomes are well-defined.

Given Assumption 1.1, average causal effects are identified by restrictions on the manner in which an instrument affects treatment. Imbens and Angrist (1994) *monotonicity*, for example, says that Z moves a Bernoulli treatment in one direction only. Monotonicity is extended here to a wide class of restrictions suitable for ordered and multinomial treatments. These restrictions limit the set of possible response types. Specifically, T^* is presumed to belong to a known set $\mathbf{M}$, a subset of which is induced by an instrument to change treatment status. In formalizing this, the notation $t^* = (t^*(0), t^*(1))$ denotes possible values of $T^* \in \mathbf{M}$.

Assumption 1.2. *(i)* $P(T^* \in \mathbf{M}) = 1$ *for a set of response types* $\mathbf{M} \subseteq \mathbf{T} \times \mathbf{T}$*; (ii) For subset* $\mathbf{S} \equiv \{t^* \in \mathbf{M} : t^*(1) \neq t^*(0)\}$ *of* $\mathbf{M}$*,* $P(T^* \in \mathbf{S}) > 0$.

Assumption 1.2(ii) says that an assignment model characterized by $\mathbf{M}$ includes a non-empty subset of potential response types, denoted $\mathbf{S}$, induced by the instrument to *switch* from one treatment to another. Our results identify average causal effects for switchers, that is, for response types in $\mathbf{S}$.

The Imbens and Angrist (1994) setup without covariates illustrates this framework. In this case, $T \in \{0,1\}$ while monotonicity restricts T^* to satisfy $T^*(1) \geq T^*(0)$ with probability one. In our notation, monotonicity means $\mathbf{M} = \{t^* : t^*(1) \geq t^*(0)\}$. The subset of switchers, $\mathbf{S}$, contains a single type, compliers for whom $T^* = (0,1)$. Under monotonicity and Assumptions 1.1 and 1.2, a Wald-IV estimand identifies a local average treatment effect (LATE):

$$\frac{E[Y|Z=1] - E[Y|Z=0]}{E[T|Z=1] - E[T|Z=0]} = E[Y^*(1) - Y^*(0)|T^*(1) > T^*(0)] \equiv \tau_w.$$

The causal effect here is for a single switcher type, while the set of allowable response types includes compliers plus always-takers (with $T^*(1) = T^*(0) = 1$) and never-takers (with $T^*(1) = T^*(0) = 0$). The fact that causal effects are identified only for switchers induced to change treatment status by changing Z is key to the development that follows.

As a stepping stone to more general results, note that τ_w is the slope in a *causal conditional expectation function* (CCEF) that describes the expectation of potential outcomes conditional on potential assignments. This is a consequence of Assumption 1.1(iii) (independence of Z and (Y^*, T^*) conditional on X), Assumption 1.1(iv) ($P(Z=1|X) \in (0,1)$) implying compliers are treated with probability strictly between zero and one, and Assumption 1.2(ii) ($P(T^* \in \mathbf{S}) > 0$) requiring that the probability of compliance be positive. The CCEF for basic LATE is:

$$\begin{aligned} E[Y|T, T^*(1) > T^*(0)] &= E[Y^*(T)|T, T^*(1) > T^*(0)] \\ &= E[Y^*(0)|T^*(1) > T^*(0)] + \tau_w \times T. \end{aligned}$$

Because T disappears from the conditional expectation function for potential outcomes, the CCEF is generated by a notional randomized trial in which T is randomly assigned in the population of compliers. The CCEF is causal since it generates comparisons of mean potential outcomes for a fixed reference population, in this case, the population of compliers.

Heckman and Pinto (2018) note that this insight extends to a general potential-outcomes model with more elaborate treatment variables. Specifically, for an ordered or multinomial treatment, the conditional expectation of outcomes given treatment, response types, and covariates identifies average causal effects of treatment on outcome Y for treatment values and response types compatible with Assumption 1.2. In the terminology introduced in Frangakis and Rubin (2002), response types define principal strata. The following proposition expresses this result in our notation.[1]

Proposition 1.1 (CCEF Properties)**.** *If Assumptions 1.1, 1.2 hold, then for any $t^* \in \mathbf{M}$ with $P(T^* = t^*|X) > 0$ and $t \in \{T^*(0), T^*(1)\}$ the* CCEF*, $E[Y|T, T^* = t^*, X]$, satisfies:*

$$E[Y|T = t, T^* = t^*, X] = E[Y^*(t)|T^* = t^*, X].$$

Therefore, for any $t^* \in \mathbf{M}$ such that $P(T^* = t^*|X) > 0$, the CCEF *generates response-type-specific average treatment effects as follows:*

$$\begin{aligned} &E[Y^*(T^*(1)) - Y^*(T^*(0))|T^* = t^*, X] \\ &\qquad = E[Y|T = T^*(1), T^* = t^*, X] - E[Y|T = T^*(0), T^* = t^*, X]. \end{aligned} \tag{1}$$

[1] Appendix A.1 proves the theoretical results stated in Sections 1 and 2.

The causal contrasts in (1) are useful only for response types that *switch* from one treatment to another in response to the instrument—otherwise $T^*(0)$ equals $T^*(1)$ and (1) is zero. Proposition 1.1 implies that, for switchers, conditional-on-T^* comparisons of mean outcomes between groups with $T^*(1) = t$ and $T^*(0) = t'$ equal average causal effects of changing treatment from t' to t. We therefore seek to identify the CCEF and type-specific average causal effects:

$$\tau(T^*) \equiv E(E[Y^*(T^*(1)) - Y^*(T^*(0))|T^*, X]|T^*), \tag{2}$$

for $T^* \in \mathbf{S}$. For instance, with an ordered treatment in $\{0, 1, 2\}$ that satisfies monotonicity (a scenario considered in Angrist and Imbens (1995)), $\tau(T^*)$ takes up to three values according to whether T^* equals $(0, 1), (0, 2)$, or $(1, 2)$.

The basic LATE scenario illustrates a key feature of the CCEF that we aim to exploit. Suppose again that $T \in \{0, 1\}$, we can ignore covariates, and potential response types satisfy monotonicity. In this case, $\tau(T^*)$ equals τ_w, an average causal effect for the single response type (compliers). At the same time, τ_w is also the slope coefficient in a regression of Y on T for compliers. That is, τ_w equals β_1 in the regression

$$\begin{aligned}(\beta_0, \beta_1) &\equiv \arg\min_{b_0, b_1} E[(E[Y|T, T^*(1) > T^*(0)] - b_0 - b_1 T)^2 | T^*(1) > T^*(0)] \\ &= \arg\min_{b_0, b_1} E[(Y - b_0 - b_1 T)^2 1\{T^*(1) > T^*(0)\}].\end{aligned} \tag{3}$$

Moreover, while *individual* compliers are not identified (because one of $T^*(1), T^*(0)$ is counterfactual), Abadie (2003) shows that (3) is equivalent to a feasible weighted-least squares minimand. Specifically, the objective in (3) satisfies

$$E[(Y - b_0 - b_1 T)^2 1\{T^*(1) > T^*(0)\}] = E[(Y - b_0 - b_1 T)^2 \kappa(T, Z)],$$

where

$$\kappa(T, Z) = 1 - \frac{T(1 - Z)}{P(Z = 0)} - \frac{(1 - T)Z}{P(Z = 1)}. \tag{4}$$

Without covariates, the regression function obtained by κ-weighting is the basic LATE CCEF. With covariates, the Abadie (2003) κ-weighted minimand generates a best linear approximation to the CCEF.

The next section extends IV and κ-weighting identification strategies to accommodate ordered and multinomial treatments of the sort allowed by Assumption 1.1. In this more general setting, the set of switchers may contain more than one type. The generalization of the LATE Theorem established here identifies causal contrasts $\tau(T^*)$ averaged over switcher types when $\mathbf{S}$ contains multiple types. This parameter, which we call switcher LATE (SLATE), is defined by:

$$\tau(\mathbf{S}) \equiv E[\tau(T^*)|T^* \in \mathbf{S}].$$

SLATE generalizes an estimand defined in Kline and Walters (2016) for a trinomial treatment variable. We revisit this connection in Section 3.

For basic (Bernoulli treatment) LATE with no covariates, $\tau(\mathbf{S})$ coincides with the effect on compliers obtained using (3). Beyond this scenario, however, the CCEF delivers more than SLATE, distinguishing average causal effects for distinct switcher types. The CCEF is

characterized here by a model, denoted $F(T, Z, X; \beta)$, that depends on parameter vector β. Parameter vector β is defined by:

$$\begin{aligned}\beta &\equiv \arg\min_b E[(E[Y|T, T^*, X] - F(T, Z, X; b))^2 | T^* \in \mathbf{S}] \\ &= \arg\min_b E[(Y - F(T, Z, X; b))^2 1\{T^* \in \mathbf{S}\}].\end{aligned}$$

This minimization problem lays a foundation for identification of the response-type-specific average causal effects defined in (2). Perhaps surprisingly, while the CCEF does not depend on the instrument, models of the form $F(T, Z, X; b)$ allow us to identify a wider range of causal effects than do alternative models that do not depend on Z.

2 Identification of Causal Parameters

This section starts with necessary and sufficient conditions for identification of SLATE, deriving both IV and weighted-least squares SLATE estimands. We then construct IV and weighted least squares estimands for the CCEF. In contrast with SLATE identification, CCEF identification turns on restrictions on the model for potential outcomes as well as on the assignment restrictions embedded in $\mathbf{M}$.

2.1 SLATE

SLATE averages type-specific causal contrasts, $\tau(T^*)$, over the subset of switchers $\mathbf{S} \subseteq \mathbf{M}$ defined in Assumption 1.2. The following lemma shows that identification of the probability of being a switcher is key to identification of SLATE.

Lemma 2.1. *Suppose Assumptions 1.1 and 1.2 are satisfied. Then,*

$$E[\tau(T^*)1\{T^* \in \mathbf{S}\}] = E(E[Y|Z = 1, X] - E[Y|Z = 0, X])$$

and therefore

$$\tau(\mathbf{S}) = \frac{E(E[Y|Z = 1, X] - E[Y|Z = 0, X])}{P(T^* \in \mathbf{S})}.$$

This result raises the question of when $P(T^* \in \mathbf{S})$ is identified so that the lemma delivers an estimand for $\tau(\mathbf{S})$. The answer is surprisingly simple and determined by whether a particular set of linear equations has a solution. This is shown by supplementing Corollary 4.3 in Navjeevan et al. (2023) and Assumptions 1.1 and 1.2 with two regularity conditions.[2] In stating these conditions, P_{AB} and $P_{A|B}$ denote the joint and conditional distributions of any two random variables (A, B), while $P_A \ll P_B$ means that P_A is absolutely continuous with respect to P_B.[3]

Theorem 2.1 (Identification of $P(T^* \in \mathbf{S})$)**.** *Suppose Assumptions 1.1, 1.2 hold. Assume also that: (i) For some $\varepsilon > 0$, with probability one, $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ and $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$; (ii) $P_{Y^*|T^*X} \ll \{P_{Y|X}\}^d \ll P_{Y^*|T^*X}$ and $\tilde{P} \equiv P_{Y^*|X} P_{T^*ZX}$ satisfies $\tilde{P} \ll$*

[2] Heckman and Pinto (2018) gives sufficient conditions for identification of $P(T^* \in \mathbf{S})$.

[3] A distribution P_A is absolutely continuous with respect to a distribution P_B if any set C that has zero probability under P_B also has zero probability under P_A – i.e. $P_B(C) = 0$ implies $P_A(C) = 0$.

$P_{Y^*T^*ZX}$ with $d\tilde{P}/dP_{Y^*T^*ZX}$ bounded. Then $P(T^* \in \mathbf{S})$ is identified if and only if there are functions ν_0 and ν_1 of T that satisfy:

$$\nu_0(T^*(0)) + \nu_1(T^*(1)) = 1\{T^* \in \mathbf{S}\}, \tag{5}$$

with probability one.

The regularity conditions imposed by Theorem 2.1 require the instrument-assignment propensity score, $P(Z = 1|X)$, to be bounded away from zero and one, and the probability of response types allowed for by $\mathbf{M}$ to be bounded away from zero. The theorem also imposes absolute continuity requirements that restrict the joint support of the vector of potential outcomes in a manner that seems uncontroversial.[4]

Theorem 2.1 gives conditions under which restrictions on $\mathbf{M}$ are necessary and sufficient to identify $P(T^* \in \mathbf{S})$. The most important of these is equation (5). Importantly, because $T^* \in \mathbf{M}$ under Assumption 1.2, solutions to (5) depend on the restrictions embodied in $\mathbf{M}$ but not on the distribution of the data. We call (5) ISP as a reminder that this is equivalent to an *identified switcher probability*. ISP is formalized as:

Assumption ISP *There are functions ν_0 and ν_1 of T satisfying*

$$\nu_0(T^*(0)) + \nu_1(T^*(1)) = 1\{T^* \in \mathbf{S}\}, \tag{6}$$

with probability one over $T^* \in \mathbf{M}$.

Unsurprisingly, ISP is satisfied in the basic LATE scenario studied in Imbens and Angrist (1994). Recall that $\mathbf{M}$ in this case allows for three response types: $T^* = (T^*(0), T^*(1))$ equals $(0,0)$, $(1,1)$, or $(0,1)$. Setting $\nu_0(t) = -t$ and $\nu_1(t) = t$ for $t \in \{0,1\}$, we have:

$$\nu_0(T^*(0)) + \nu_1(T^*(1)) = T^*(1) - T^*(0),$$

which equals $1\{T^*(1) > T^*(0)\} = 1\{T^* \in \mathbf{S}\}$, as required by ISP. More generally, ISP generates a system of $|\mathbf{M}|$ linear equations that may or may not have a solution.

Given Assumptions 1.1 and 1.2, ISP is sufficient for a Wald-type IV estimand to identify $\tau(\mathbf{S})$, as shown by the following result.

Theorem 2.2 (IV for SLATE given ISP)**.** *Suppose Assumptions 1.1, 1.2, and* ISP *hold, and, for some $\varepsilon > 0$, $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ with probability one. Let*

$$\tilde{Z} \equiv \frac{Z - P(Z = 1|X)}{P(Z = 0|X)P(Z = 1|X)} \tag{7}$$

and, for functions ν_0 and ν_1 satisfying ISP*, let $V \equiv \nu_1(T)Z - \nu_0(T)(1 - Z)$. Then:*

$$P(T^* \in \mathbf{S}) = E(E[V|Z = 1, X] - E[V|Z = 0, X]) = E[\tilde{Z}V]$$

$$\tau(\mathbf{S}) = \frac{E(E[Y|Z = 1, X] - E[Y|Z = 0, X])}{E(E[V|Z = 1, X] - E[V|Z = 0, X])} = \frac{E[\tilde{Z}Y]}{E[\tilde{Z}V]}. \tag{8}$$

[4] These restrictions require, for instance, the support of Y^* conditional on T^* to be unrelated to T^* and are used to show the necessity part of the statement (i.e., that identification of $P(T^* \in \mathbf{S})$ implies that a solution to (5) exists). Theorem 2.2 below shows that the existence of a solution to (5) is sufficient for identification of $P(T^* \in \mathbf{S})$ without absolute continuity conditions.

This theorem says SLATE is an IV estimand in which treatment variable V is determined by functions ν_0 and ν_1 satisfying ISP. The first representation in (8) marginalizes over covariates in the manner described by Frölich (2007) for basic LATE. The second marginalization is derived using inverse propensity-score weighting (IPW), as suggested by Tan (2006).[5]

When does ISP fail? Consider a simplified ordered treatment model with $T \in \{0, 1, 2\}$, no covariates, and monotonicity. The set of response types $T^* = (T^*(0), T^*(1))$ allowed by $\mathbf{M}$ then consists of three types of stayers, $\{(0,0), (1,1), (2,2)\}$, and three types of switchers, $\{(0,1), (0,2), (1,2)\} = \mathbf{S}$. In this case, ISP fails because no functions ν_0 and ν_1 satisfy restriction (6). To see why, note first that evaluation of restriction (6) at staying response type $(T^*(0), T^*(1)) = (1,1)$ yields

$$\nu_0(1) + \nu_1(1) = 0. \tag{9}$$

Similarly, evaluating (6) at switching response types $T^* \in \{(0,1), (0,2), (1,2)\}$ gives

$$\nu_0(0) + \nu_1(1) = 1 \text{ for } T^* = (0,1) \tag{10}$$

$$\nu_0(0) + \nu_1(2) = 1 \text{ for } T^* = (0,2) \tag{11}$$

$$\nu_0(1) + \nu_1(2) = 1 \text{ for } T^* = (1,2). \tag{12}$$

Equations (9)-(12) have no solution (in ν_0, ν_1) because (10) and (11) imply $\nu_1(1) = \nu_1(2)$, which together with (12) yields $\nu_0(1) + \nu_1(1) = 1$. This contradicts (9). Since ISP does not hold, SLATE is unidentified without further restrictions on $\mathbf{M}$.[6]

The identifying power of ISP is highlighted by a model adding the Rose and Shem-Tov (2024) *extensive margin compliance only* (EMCO) assumption to our simple ordered-treatment story. EMCO says that an instrument induces shifts from no treatment to positive treatment, with no inframarginal changes in treatment intensity. With three treatment levels, EMCO allows three types of stayers, $\{(0,0), (1,1), (2,2)\}$, but only two types of switchers: $\mathbf{S} = \{(0,1), (0,2)\}$. Because EMCO rules out switcher type $T^* = (1,2)$, we can drop (12). Evaluating (6) for stayers, ISP requires ν_0 and ν_1 satisfy

$$\nu_0(t) + \nu_1(t) = 0 \text{ for all } t \in \{0, 1, 2\}. \tag{13}$$

The system of equations consisting of (10), (11), and (13) is solved by $\nu_1(t) = 1\{t > 0\}$ and $\nu_0(t) = -\nu_1(t)$. Hence, ISP holds and SLATE is identified. Specifically, since

$$V \equiv \nu_1(T)Z - \nu_0(T)(1 - Z) = 1\{T > 0\},$$

[5] An IPW estimand for the conditional reduced form in the first representation of $\tau(\mathbf{S})$ in (8) is

$$E\left[\frac{YZ}{P(Z=1|X)} - \frac{Y(1-Z)}{1-P(Z=1|X)}|X\right] = E[\tilde{Z}Y|X].$$

Marginalizing over X yields the numerator of the second representation in (8). A parallel operation yields the denominator of $\tau(\mathbf{S})$.

[6] Weighted average LATEs for *overlapping* groups of switchers may be identified without ISP, as shown in Angrist and Imbens (1995). Recently, Angrist et al. (2024) extends these results to identify dynamic treatment effects in randomized trials with time-varying compliance.

Theorem 2.2 implies:

$$\tau(\mathbf{S}) = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[1\{T>0\}|Z=1] - E[1\{T>0\}|Z=0]}, \tag{14}$$

a result obtained following a different path in Rose and Shem-Tov (2024).[7]

ISP is also satisfied by a latent-index ordered choice model like ordered probit. To see this, suppose ε is continuously distributed on the real line, and that treatment status is governed by:

$$T^*(z) = \begin{cases} 0 & \text{if } z\pi + \varepsilon \leq c_1 \\ 1 & \text{if } c_1 < z\pi + \varepsilon \leq c_2 \\ 2 & \text{if } c_2 < z\pi + \varepsilon, \end{cases} \tag{15}$$

where ε is independent of Z and $\pi > 0$. This model imposes restrictions on T^* that go beyond monotonicity. Suppose initially that $\pi \geq c_2 - c_1$. In this scenario, $T^*(0) = 1$ (i.e., $c_1 < \varepsilon \leq c_2$) implies $T^*(1) = 2$, because then $\pi + \varepsilon > c_2$. Hence, while there are up to three types of switchers, $\{(0,1),(0,2),(1,2)\}$, this restriction leaves only two types of stayers, $\{(0,0),(2,2)\}$.

Evaluating the ISP restrictions in (6) at these five possible response types yields:

$$\begin{aligned} \nu_0(0) + \nu_1(1) &= 1 \text{ for } T^* = (0,1) \\ \nu_0(0) + \nu_1(2) &= 1 \text{ for } T^* = (0,2) \\ \nu_0(1) + \nu_1(2) &= 1 \text{ for } T^* = (1,2) \\ \nu_0(0) + \nu_1(0) &= 0 \text{ for } T^* = (0,0) \\ \nu_0(2) + \nu_1(2) &= 0 \text{ for } T^* = (2,2). \end{aligned}$$

These equations are solved by $\nu_0(t) = -1\{t = 2\}$ and $\nu_1(t) = 1\{t > 0\}$. Therefore, ISP holds, and Theorem 2.2 implies SLATE is

$$\tau(\mathbf{S}) = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[1\{T>0\}|Z=1] - E[1\{T=2\}|Z=0]}. \tag{16}$$

Interestingly, EMCO and ordered selection yield different SLATE estimands for the same ordered treatment.

What happens if $\pi < c_2 - c_1$? ISP is satisfied even so. To see this, note first that, in this case, $T^*(0) = 0$ (i.e., $\varepsilon \leq c_1$) implies $T^*(1) < 2$, because then $\pi + \varepsilon < c_2$. So, when $\pi < c_2 - c_1$, there are no switchers with $T^* = (0,2)$. That leaves three stayer types, $\{(0,0),(1,1),(2,2)\}$, and two switcher types, $\{(0,1),(1,2)\}$. Evaluating the ISP restrictions at these five response types yields:

$$\begin{aligned} \nu_0(0) + \nu_1(1) &= 1 \text{ for } T^* = (0,1) \\ \nu_0(1) + \nu_1(2) &= 1 \text{ for } T^* = (1,2) \\ \nu_0(0) + \nu_1(0) &= 0 \text{ for } T^* = (0,0) \\ \nu_0(1) + \nu_1(1) &= 0 \text{ for } T^* = (1,1) \\ \nu_0(2) + \nu_1(2) &= 0 \text{ for } T^* = (2,2). \end{aligned}$$

[7] Rose and Shem-Tov (2024) also derives κ-weights that identify expectations of functions of observables for switchers under EMCO. As we explain in the next section, these weights are important, but not sufficient, for CCEF identification.

These equations are solved by $\nu_1(t) = t$ and $\nu_0(t) = -t$, again implying ISP holds and SLATE is identified. Applying Theorem 2.2 in this scenario yields:

$$\tau(\mathbf{S}) = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[T|Z=1] - E[T|Z=0]}, \tag{17}$$

so SLATE in this case is a Wald-IV estimand that instruments T. Thus, the construction and interpretation of SLATE turns on first-stage parameter values. Appendix A.2 shows that SLATE is identified in latent-index ordered-assignment models with more than three treatments, gives general formulas for ν_0 and ν_1, characterizes the set of switchers in these models, and gives conditions under which SLATE can be constructed by instrumenting for T.

SLATE by κ-Weighting

A novel restriction on allowable assignment types beyond ISP leads to new κ-weighted least squares estimands for both SLATE and the CCEF. This *in/out/stay* (IOS) assumption uses the following taxonomy of response types. For each possible treatment, t, define:

$$\begin{aligned}
\texttt{in}(\text{t}) &\equiv \{t^* \in \mathbf{M} : t^*(1) \neq t^*(0) \text{ and } t^*(1) = t\} \\
\texttt{out}(\text{t}) &\equiv \{t^* \in \mathbf{M} : t^*(1) \neq t^*(0) \text{ and } t^*(0) = t\} \\
\texttt{stay}(\text{t}) &\equiv \{t^* \in \mathbf{M} : t^*(0) = t^*(1) = t\}.
\end{aligned}$$

This taxonomy divides response types that may select treatment level t into three groups: (i) Switchers who select t when $Z = 1$, that is, they switch *in* to t when the instrument is turned on; (ii) Switchers who select t when $Z = 0$, that is, they switch *out* of t when the instrument is turned on; and (iii) Stayers who select t regardless of Z. In the basic LATE setup, for instance, $\texttt{stay}(0)$ contains never-takers, $\texttt{stay}(1)$ contains always-takers, $\texttt{out}(0)$ and $\texttt{in}(1)$ contain compliers, while $\texttt{in}(0)$ and $\texttt{out}(1)$ are empty.

Using this notation, IOS is formalized as:

Assumption IOS. *For all* $t \in \mathbf{T}$*, at least one of* $\texttt{in}(\text{t})$, $\texttt{out}(\text{t})$, *and* $\texttt{stay}(\text{t})$ *is empty.*

IOS says that there is no treatment level that is selected by some switchers when $Z = 1$, selected by other switchers when $Z = 0$, and at which some stay regardless of Z. In the context of a particular model, IOS is more easily checked than ISP since, for the former, no system of equations need be solved. Rather, for each treatment level, t, IOS is verified by examining response types in $\mathbf{M}$ that can choose t.

Under IOS, both a Wald-type IV estimand and κ-weighting identify SLATE. The probability T^* belongs to any set of types $\texttt{in}(\text{t})$ or $\texttt{out}(\text{t})$ and the distribution of (Y, T, X) conditional on $T^* \in \mathbf{S}$ are also identified under IOS. These results build on a partition of the set of possible treatment levels into four groups:

$$\begin{aligned}
\mathbf{T}_{\texttt{i}} &\equiv \{t \in \mathbf{T} : \texttt{in}(t) \neq \emptyset \text{ and } \texttt{out}(t) = \emptyset\} \\
\mathbf{T}_{\texttt{o}} &\equiv \{t \in \mathbf{T} : \texttt{in}(t) = \emptyset \text{ and } \texttt{out}(t) \neq \emptyset\} \\
\mathbf{T}_{\texttt{io}} &\equiv \{t \in \mathbf{T} : \texttt{in}(t) \neq \emptyset \text{ and } \texttt{out}(t) \neq \emptyset\} \\
\mathbf{T}_{\texttt{s}} &\equiv \{t \in \mathbf{T} : \texttt{in}(t) = \emptyset \text{ and } \texttt{out}(t) = \emptyset\}.
\end{aligned}$$

The set $\mathbf{T}_{\mathtt{i}}$ consists of *treatments* that response types may switch into, but not out of, when $Z = 1$. Set $\mathbf{T}_{\mathtt{o}}$ consists of treatments that response types may switch out of, but not into, when $Z = 0$. Switchers can switch in or out of treatments in $\mathbf{T}_{\mathtt{io}}$, while treatments in $\mathbf{T}_{\mathtt{s}}$ are never selected by switchers. In the basic LATE scenario, for example, $\mathbf{T} = \{0, 1\}$ and monotonicity implies $\mathbf{T}_{\mathtt{i}} = \{1\}$, $\mathbf{T}_{\mathtt{o}} = \{0\}$, and both $\mathbf{T}_{\mathtt{io}}$ and $\mathbf{T}_{\mathtt{s}}$ are empty.

The partition of $\mathbf{T}$ into these four subsets allows us to derive generalized κ-type weights, denoted $\kappa^\star$. Specifically, for $\tilde{Z}$ defined in (7), let

$$\kappa^\star(T, Z, X) \equiv \tilde{Z}(1\{T \in \mathbf{T}_{\mathtt{i}}\}P(Z = 1|X) - 1\{T \in \mathbf{T}_{\mathtt{o}}\}P(Z = 0|X)) + 1\{T \in \mathbf{T}_{\mathtt{io}}\}. \tag{18}$$

Given IOS, SLATE and the distribution of (Y, T, X) conditional on $T^* \in \mathbf{S}$ are identified via $\kappa^\star$-weighting. The IV estimand for SLATE also simplifies, obviating the need to pin down ν_0 and ν_1. These results are formalized by the following four-part theorem:

Theorem 2.3 (IV and $\kappa^\star$ for SLATE; the distribution of (Y, T, X) for switchers)**.** *Suppose Assumptions 1.1, 1.2, and* IOS *hold.*

(i) Then, ISP *holds with* $\nu_0(T) = -1\{T \in \mathbf{T}_{\mathtt{i}}\}$ *and* $\nu_1(T) = 1\{T \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{io}}\}$*. Therefore,* $\tau(\mathbf{S})$ *equals* IV *estimand* (8) *with* $V = 1\{T \in \mathbf{T}_{\mathtt{i}}\} + 1\{T \in \mathbf{T}_{\mathtt{io}}\}Z$*.*

(ii) Let $V = 1\{T \in \mathbf{T}_{\mathtt{i}}\} + 1\{T \in \mathbf{T}_{\mathtt{io}}\}Z$*. If* $\mathbf{T}_{\mathtt{s}} = \emptyset$*, then* $\kappa^\star$ *in* (18) *equals:*

$$\kappa^\star(T, Z, X) = 1 - \frac{V(1 - Z)}{P(Z = 0|X)} - \frac{(1 - V)Z}{P(Z = 1|X)}.$$

Suppose also that $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ *with probability one for some* $\varepsilon > 0$*. Then:*

(iii) For any g *satisfying* $E[|g(Y, T, X)|] < \infty$*,*

$$E[g(Y, T, X)\kappa^\star(T, Z, X)] = E[g(Y, T, X)1\{T^* \in \mathbf{S}\}]. \tag{19}$$

(iv) Let $h_0(X) \equiv 1/P(Z = 0|X)$ *and* $h_1(X) \equiv 1/[P(Z = 0|X)P(Z = 1|X)]$ *and suppose* $E[Y^2] < \infty$*. Also, define:*

$$(\alpha_0, \alpha_1) \equiv \arg\min_{(a_0, a_1)} E\left[\left(Y - a_0\frac{h_0(X)}{E[h_0(X)|T^* \in \mathbf{S}]} - a_1 V\frac{h_1(X)}{E[h_1(X)|T^* \in \mathbf{S}]}\right)^2 \kappa^\star(T, Z, X)\right].$$

Then, $\tau(\mathbf{S}) = \alpha_1$*.*

The first part of Theorem 2.3 says that IOS implies ISP. As we illustrate below, however, some models satisfy ISP only. Theorem 2.3(i) also says that, under IOS, the Wald-IV representation for SLATE obtained in Theorem 2.2 simplifies to IV with an easily-constructed Bernoulli treatment variable, $V = 1\{T \in \mathbf{T}_{\mathtt{i}}\} + 1\{T \in \mathbf{T}_{\mathtt{io}}\}Z$. The second part of the theorem says that in models in which every treatment value $t \in \mathbf{T}$ is selected by some switcher(s), so $\mathbf{T}_{\mathtt{s}} = \emptyset$, $\kappa^\star$-weights are of the form derived in Abadie (2003). Here, however, a conventional treatment dummy is replaced by the auxiliary treatment variable V.

Theorem 2.3(iii) says that $\kappa^\star$-weighting delivers identification of the expectation of any function of (Y, T, X) for switchers. Finally, Theorem 2.3(iv) gives a $\kappa^\star$-weighted least squares estimand for SLATE. In particular, without covariates, Theorem 2.3(iv) implies SLATE is the slope of a $\kappa^\star$-weighted regression on a constant and V. With covariates, SLATE can be

obtained by $\kappa^\star$-weighting a regression on a constant and V, weighted by scaled $h_0(X)$ and $h_1(X)$, respectively. The mean of these for scaling purposes can be obtained from part (iii). This result appears to be new even for the basic LATE model with covariates studied by Abadie (2003).[8]

To illustrate Theorem 2.3(i), return to $T \in \{0,1,2\}$ without covariates and the ordered latent-index model for T^* given by (15), a model that satisfies ISP. In this model $\texttt{in}(1)$ and $\texttt{out}(1)$ are non-empty. But when $\pi < c_2 - c_1$, $\texttt{stay}(1)$ is also non-empty since $c_1 < \varepsilon < c_2 - \pi$ generates response type $T^* = (1,1)$. IOS therefore fails even though ISP holds. If $\pi \geq c_2 - c_1$, however, $\texttt{stay}(1)$ is empty. Since $\texttt{in}(0)$ and $\texttt{out}(2)$ are also empty, IOS holds. Moreover, $\mathbf{T}_{\texttt{i}} = \{2\}$ and $\mathbf{T}_{\texttt{io}} = \{1\}$, so

$$V = 1\{T \in \mathbf{T}_{\texttt{i}}\} + 1\{T \in \mathbf{T}_{\texttt{io}}\}Z = 1\{T = 2\} + 1\{T = 1\}Z.$$

Theorem 2.3(i) therefore implies

$$\tau(\mathbf{S}) = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[1\{T>0\}|Z=1] - E[1\{T=2\}|Z=0]},$$

the Wald-IV estimand obtained in (16) using ISP and Theorem 2.2. Under IOS, however, we can find V using Theorem 2.3(i) rather than from the solution to equation (6).

Now, replace the latent-index model for T^* with EMCO. Recall that EMCO allows five response types, three stayers ($\{(0,0),(1,1),(2,2)\}$) and two switchers ($\{(0,1),(0,2)\}$). IOS holds here because $\texttt{in}(0)$, $\texttt{out}(1)$, and $\texttt{out}(2)$ are empty. Moreover, since $\mathbf{T}_{\texttt{i}} = \{1,2\}$ while $\mathbf{T}_{\texttt{io}}$ is empty, $V = 1\{T \in \mathbf{T}_{\texttt{i}}\} + 1\{T \in \mathbf{T}_{\texttt{io}}\}Z$ equals $1\{T > 0\}$. Theorem 2.3(i) therefore implies:

$$\tau(\mathbf{S}) = \frac{E[Y|Z=1] - E[Y|Z=0]}{E[1\{T>0\}|Z=1] - E[1\{T>0\}|Z=0]},$$

the EMCO estimand derived in Section 2.1.

SLATE is an easily-computed summary causal effect that is likely to be useful in many applications. For example, Kline and Walters (2016) interprets a conventional Wald estimand as SLATE in a scenario with a trinomial unordered treatment. The treatment variable in this case is defined by preschool sector, either Head Start, competing non-Head Start preschools, or home care. The instrument is a randomly assigned offer of Head Start placement. The Kline and Walters (2016) version of SLATE averages causal effects for switchers attracted to Head Start from each of the two alternative sectors. Section 3.1 explores this application further.

SLATE may be less useful for models without monotonicity. To see this, consider a Bernoulli-treatment scenario with neither monotonicity nor always-takers, so $T^* \in \{(0,1),(1,0),(0,0)\}$. IOS fails here because $\texttt{stay}(0)$, $\texttt{in}(0)$, and $\texttt{out}(0)$ are non-empty. ISP holds, however, with $\nu_0(t) = \nu_1(t) = t$. SLATE therefore equals:

$$\begin{aligned}\tau(\mathbf{S}) &= \frac{E[Y|Z=1] - E[Y|Z=0]}{E[T|Z=1] + E[T|Z=0]} \\ &= p_c E[Y^*(1) - Y^*(0)|T^*(1) > T^*(0)] + (1-p_c)E[Y^*(0) - Y^*(1)|T^*(1) < T^*(0)],\end{aligned}$$

[8] In a basic LATE scenario with covariates required for identification, Abadie (2003) minimizes $E[(Y - F(X,T;\beta))^2\kappa(T,Z,X)]$, where function $F(X,T;\beta)$ specifies a parametric approximation to the CCEF. The minimand in Theorem 2.3(ii), by contrast, requires only a model for the assignment propensity score.

where $p_c \equiv P[T^*(1) > T^*(0)|T^* \in \mathbf{S}]$. Suppose now that the causal effect of treatment is a constant, τ, for both switcher types. The usual Wald-IV estimand for treatment T identifies τ. However, $\tau(\mathbf{S})$ differs from τ because SLATE averages $Y(T^*(1)) - Y(T^*(0))$ for compliers, for whom $T^*(1) = 1$ and $T^*(0) = 0$, with the corresponding effects for defiers, for whom $T^*(1) = 0$ and $T^*(0) = 1$. Under constant effects, the first effect is τ while the second is $-\tau$; SLATE weights these by p_c and $1 - p_c$, respectively.[9]

Although SLATE does not distinguish causal effects for different types of switchers, IOS identifies features of the distribution of switcher types. Specifically, under IOS, the probabilities T^* belongs to $\texttt{in}(t)$ and $\texttt{out}(t)$ are identified as follows:

Lemma 2.2 (Probabilities of $\texttt{in}(t)$ and $\texttt{out}(t)$)**.** *Given Assumptions 1.1, 1.2, and* IOS*,*

$$P(T^* \in \texttt{in}(t)\,|X) = E[1\{T = t\}|Z = 1, X] - E[1\{T = t\}|Z = 0, X] \qquad \text{for all } t \in \mathbf{T}_{\texttt{i}},$$
$$P(T^* \in \texttt{out}(t)\,|X) = E[1\{T = t\}|Z = 0, X] - E[1\{T = t\}|Z = 1, X] \qquad \text{for all } t \in \mathbf{T}_{\texttt{o}},$$

and, for all $t \in \mathbf{T}_{\texttt{io}}$*,*

$$P(T^* \in \texttt{in}(t)\,|X) = E[1\{T = t\}|Z = 1, X]$$
$$\textit{and}\quad P(T^* \in \texttt{out}(t)\,|X) = E[1\{T = t\}|Z = 0, X].$$

Moreover, defining $\tilde{Z} \equiv (Z - P(Z = 1|X))/(P(Z = 1|X)P(Z = 0|X))$*, we have*

$$P(T^* \in \texttt{in}(t)) = E[1\{T = t\}\tilde{Z}] \qquad \text{for all } t \in \mathbf{T}_{\texttt{i}},$$
$$P(T^* \in \texttt{out}(t)) = -E[1\{T = t\}\tilde{Z}] \qquad \text{for all } t \in \mathbf{T}_{\texttt{o}},$$

and, for all $t \in \mathbf{T}_{\texttt{io}}$*,*

$$P(T^* \in \texttt{in}(t)) = E[1\{T = t\}Z\tilde{Z}]$$
$$\textit{and}\quad P(T^* \in \texttt{out}(t)) = -E[1\{T = t\}(1 - Z)\tilde{Z}].$$

This lemma links the shares of switcher types in $\texttt{in}(t)$ and $\texttt{out}(t)$ with first-stage relationships for treatment indicators of the form $1\{T = t\}$. For some models, the lemma also identifies the probability distribution of specific switcher types, T^*. This is illustrated in Section 3, below.

2.2 CCEF Identification

When are the underlying average causal effects defined by (2) identified? This question is answered by studying identification of the CCEF. Beyond basic LATE, CCEF identification typically requires stronger assumptions than does identification of SLATE. Our CCEF identification results combine IOS with a version of the *principal ignorability* assumption considered in Feller et al. (2017), Jiang and Ding (2021), and related work on principal stratification. This is formalized here as *partial switcher ignorability* (PSI).

[9] The de Chaisemartin (2017) compliers-defiers condition, which requires that there be a subpopulation of compliers, C_F, of the same size and with the same average treatment effect as defiers, yields $\tau(\mathbf{S}) = E[Y^*(1) - Y^*(0)|C_V]P(C_V|T^* \in \mathbf{S})$, where C_V is the complement of C_F in the set of compliers. This result also holds when there are more compliers than defiers conditional on the difference in potential outcomes.

Assumption PSI. *The CCEF for switchers satisfies*:

$$E[Y^*(t)|T^* = t^*, X] = E[Y^*(t)|T^* \in \texttt{in}(\text{t}), X] \quad \text{for all } t^* \in \texttt{in}(\text{t}) \tag{20}$$
$$E[Y^*(t)|T^* = t^*, X] = E[Y^*(t)|T^* \in \texttt{out}(\text{t}), X] \quad \text{for all } t^* \in \texttt{out}(\text{t})\,.$$

PSI differs from weak principal ignorability as defined in Feller et al. (2017) in that it leaves the CCEF for response types other than switchers unrestricted. PSI is "partial" because it restricts potential outcome means across a subset of switcher types. For treatment values chosen by a single switcher type, PSI is irrelevant. Returning to $T \in \{0, 1, 2\}$ under EMCO, for instance, the instrument moves some switchers from 0 to 1 and some from 0 to 2. PSI requires the mean of $Y^*(0)$ be the same for these two types, while leaving the means of $Y^*(1)$ and $Y^*(2)$ for switchers unrestricted.

Note also that, for a given $t \in \mathbf{T}_{\texttt{io}}$, PSI omits restrictions across switcher types belonging to $\texttt{in}(\text{t})$ and switcher types belonging to $\texttt{out}(\text{t})$. Suppose, for instance, $T \in \{0, 1, 2\}$ under monotonicity but not EMCO. Then $\texttt{in}(1) = \{(0, 1)\}$, $\texttt{out}(1) = \{(1, 2)\}$, and $\mathbf{T}_{\texttt{io}} = \{1\}$. Because $\texttt{out}(0) = \{(0, 1), (0, 2)\}$ and $\texttt{in}(2) = \{(0, 2), (1, 2)\}$, PSI requires

$$E[Y^*(0)|T^* = (0, 2)] = E[Y^*(0)|T^* = (0, 1)]$$
$$E[Y^*(2)|T^* = (0, 2)] = E[Y^*(2)|T^* = (1, 2)],$$

without imposing

$$E[Y^*(1)|T^* = (0, 1)] = E[Y^*(1)|T^* = (1, 2)]. \tag{21}$$

The following result shows that IOS is central to identification of a PSI-compliant CCEF.

Theorem 2.4 (The CCEF and IOS)**.** *Suppose Assumption 1.1, 1.2, and* PSI *hold, and: (i) For some $\varepsilon > 0$, with probability one, $\varepsilon \leq P(Z = 1|X) \leq 1-\varepsilon$, $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$, and $\text{Var}\{Y^*(t)|X\} \geq \varepsilon$; (ii) $Y^*(t)$ is bounded for all $t \in \mathbf{T}$; (iii) $P_{Y^*|T^*X} \ll \{P_{Y|X}\}^d \ll P_{Y^*|T^*X}$ and $\tilde{P} \equiv P_{Y^*(t_1)|X} \ldots P_{Y^*(t_d)|X} P_{T^*ZX}$ satisfies $\tilde{P} \ll P_{Y^*T^*ZX}$ with $d\tilde{P}/dP_{Y^*T^*ZX}$ bounded. Then, the* CCEF *is identified if and only if* IOS *holds.*

Remarkably, under the conditions of Theorem 2.4, IOS is both necessary and sufficient for CCEF identification.

PSI facilitates identification by limiting CCEF dependence on unobserved type T^* given observed T and X. This is formalized by the following corollary to Proposition 1.1.

Corollary 2.1 (Using PSI)**.** *Suppose Assumptions 1.1, 1.2, and PSI hold, and that $P(T^* = t^*|X) > 0$ for all $t^* \in \mathbf{S}$ with probability one.*

(i) There are functions $C_1(T, X)$, and $C_2(T, Z, X)$ such that

$$P(E[Y|T, T^*, X] = C_1(T, X)1\{T \in \mathbf{T}_{\texttt{i}} \cup \mathbf{T}_{\texttt{o}}\} + C_2(T, Z, X)1\{T \in \mathbf{T}_{\texttt{io}}\}|T^* \in \mathbf{S}) = 1.$$

(ii) Let $C(T, Z, X) \equiv C_1(T, X)1\{T \in \mathbf{T}_{\texttt{i}} \cup \mathbf{T}_{\texttt{o}}\} + C_2(T, Z, X)1\{T \in \mathbf{T}_{\texttt{io}}\}$. Covariate-specific causal effects for any $t^* = (t^*(0), t^*(1)) \in \mathbf{S}$ are given by

$$E[Y^*(T^*(1)) - Y^*(T^*(0))|T^* = t^*, X] = C(t^*(1), 1, X) - C(t^*(0), 0, X). \tag{22}$$

Instrument-treatment interactions distinguish between switcher types that select treatment levels $t \in \mathbf{T}_{\mathtt{io}}$. In the scenario where $T \in \{0,1,2\}$ under monotonicity without EMCO, treatment level $T = 1$ is chosen by switcher types $T^* = (0,1)$ and $T^* = (1,2)$. Conditional on $T^* \in \mathbf{S}$, the interaction term $1\{T = 1\}(1 - Z)$ indicates type $T^* = (1,2)$ since this type of switcher has treatment level 1 when the instrument is switched off. Likewise, the interaction term $1\{T = 1\}Z$ indicates a switcher of type $T^* = (0,1)$ since this type of switcher has treatment level 1 when the instrument is switched on.

Since the CCEF for switchers depends on the instrument only at treatment values in $\mathbf{T}_{\mathtt{io}}$, we focus on CCEF specifications consistent with Corollary 2.1(i). These can be written:

$$F(T,Z,X;\beta) = F_1(T,X;\beta)1\{T \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}\} + F_2(T,Z,X;\beta)1\{T \in \mathbf{T}_{\mathtt{io}}\}, \tag{23}$$

for functions F_1 and F_2. The combination of IOS and PSI identifies CCEF specifications satisfying (23) as follows.

Theorem 2.5 (CCEF Identification)**.** *Assume that for some $\varepsilon > 0$, $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ with probability one. Then, given Assumptions 1.1, 1.2, and* IOS*,*

$$E[(Y - F(T,Z,X;b))^2\kappa^\star(T,Z,X)] = E[(Y - F(T,Z,X;b))^2 1\{T^* \in \mathbf{S}\}] \tag{24}$$

provided $E[(Y - F(T,Z,X;b))^2] < \infty$ for all b. Moreover, given PSI*, the* CCEF *equals or is approximated by $F(T,Z,X;\beta)$, where β is the (assumed unique) solution to*

$$\begin{aligned}\beta &\equiv \arg\min_b E[(E[Y|T^*,T,X] - F(T,Z,X;b))^2|T^* \in \mathbf{S}] \\ &= \arg\min_b E[(Y - F(T,Z,X;b))^2\kappa^\star(T,Z,X)].\end{aligned} \tag{25}$$

Absent covariates, PSI and Corollary 2.1(i) ensure that the CCEF takes the form of a linear model saturated in T with a full set of T, Z interactions for treatment levels in $\mathbf{T}_{\mathtt{io}}$. Theorem 2.5 delivers identification of the parameters in such a specification, and hence of the CCEF, by $\kappa^\star$-weighted least squares. The first order conditions for the minimization problem in the theorem imply uniqueness of β as long as the probabilities in Lemma 2.2 are nonzero. With covariates, F is not guaranteed to equal the CCEF unless the covariates are discrete and F is fully saturated in X, including X interactions with T and TZ. When F is not the CCEF, Theorem 2.5 provides an MMSE approximation in the class of functions containing $F(T,Z,X;b)$.

2.3 IV and $\kappa^\star$ Estimands for the CCEF

Under PSI the CCEF can be constructed from a set of IV estimands. This result, which uses Theorem 2.5 and the first-order conditions for $\kappa^\star$-weighted minimization, is formalized below.

Theorem 2.6 (CCEF IV)**.** *Suppose Assumptions 1.1, 1.2, and* IOS *hold. Assume also that, with probability one, for some $\varepsilon > 0$, $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ and $P(T^* = t^*|X) > 0$ for all $t^* \in \mathbf{S}$. Let $\tilde{Z}$ be as defined in (7).*

(i) Suppose PSI *holds. Then, for any* $t^* \in \mathbf{S}$ *and* $t \in \{t^*(0), t^*(1)\}$,

$$E[Y^*(t)|T^* = t^*, X] = \frac{E[\tilde{Z} \times 1\{T = t\}Y|X]}{E[\tilde{Z} \times 1\{T = t\}|X]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}$$

$$E[Y^*(t)|T^* = t^*, X] = \frac{E[\tilde{Z} \times 1\{T = t\}ZY|X]}{E[\tilde{Z} \times 1\{T = t\}Z|X]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{io}},\ t^* \in \mathtt{in}(\mathrm{t})$$

$$E[Y^*(t)|T^* = t^*, X] = \frac{E[\tilde{Z} \times 1\{T = t\}(1-Z)Y|X]}{E[\tilde{Z} \times 1\{T = t\}(1-Z)|X]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{io}},\ t^* \in \mathtt{out}(\mathrm{t})$$

(ii) Replace PSI *with the following unconditional analog:*

$$\begin{aligned} E[Y^*(t)|T^* = t^*] &= E[Y^*(t)|T^* \in \mathtt{in}(\mathrm{t})] \quad \text{for all } t^* \in \mathtt{in}(\mathrm{t}) \\ E[Y^*(t)|T^* = t^*] &= E[Y^*(t)|T^* \in \mathtt{out}(\mathrm{t})] \quad \text{for all } t^* \in \mathtt{out}(\mathrm{t})\,. \end{aligned} \tag{26}$$

Then, for any $t^* \in \mathbf{S}$ *and* $t \in \{t^*(0), t^*(1)\}$,

$$E[Y^*(t)|T^* = t^*] = \frac{E[\tilde{Z} \times 1\{T = t\}Y]}{E[\tilde{Z} \times 1\{T = t\}]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}$$

$$E[Y^*(t)|T^* = t^*] = \frac{E[\tilde{Z} \times 1\{T = t\}ZY]}{E[\tilde{Z} \times 1\{T = t\}Z]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{io}},\ t^* \in \mathtt{in}(\mathrm{t})$$

$$E[Y^*(t)|T^* = t^*] = \frac{E[\tilde{Z} \times 1\{T = t\}(1-Z)Y]}{E[\tilde{Z} \times 1\{T = t\}(1-Z)]} \qquad \text{if } t \in \mathbf{T}_{\mathtt{io}},\ t^* \in \mathtt{out}(\mathrm{t})$$

Part (i) of Theorem 2.6 shows how the CCEF can be constructed from IV estimands that condition on X. The first of these, identifying the CCEF at levels $t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}$ for all switchers that select t, uses $\tilde{Z}$ to instrument $1\{T = t\}$ in an estimating equation with dependent variable $1\{T = t\}Y$. The second, identifying the CCEF at levels $t \in \mathbf{T}_{\mathtt{io}}$ for response types $t^* \in \mathtt{in}(\mathrm{t})$, instruments $1\{T = t\}Z$ in an estimating equation with dependent variable $1\{T = t\}ZY$. The third, identifying the CCEF at levels $t \in \mathbf{T}_{\mathtt{io}}$ for response types $t^* \in \mathtt{out}(\mathrm{t})$, instruments $1\{T = t\}(1-Z)$ in an estimating equation with dependent variable $1\{T = t\}(1-Z)Y$.

The second part of the theorem lays the foundation for IV identification of unconditional (marginalized over covariates) average treatment effects of the form

$$\tau(t^*) = E[Y^*(T^*(1)) - Y^*(T^*(0))|T^* = t^*].$$

These effects (introduced in equation (2)) depend on the marginalized-over-X CCEF, evaluated at a particular value $T^* = t^*$:

$$E[Y^*(t)|T^* = t^*] = E(\underbrace{E[Y^*(t)|T^* = t^*, X]}_{\text{CCEF}}|T^* = t^*).$$

This marginalization weights the CCEF at each value of X by the distribution of X for a *particular* switcher type, t^*. However, Theorem 2.3(iii) identifies the distribution of X for the set of all switchers rather than for specific switcher types. This problem is solved by replacing PSI with (26) in Theorem 2.6(ii).

Back to $\kappa^\star$

Although covariate-mediated heterogeneity is often of substantive interest, IV estimators based on Theorem 2.6(ii) leave the role of covariates in the CCEF unspecified. In addition to generating interactions that may be of substantive interest, covariates in F can boost the precision of CCEF estimates, much as covariates in an OLS estimator boost precision by reducing residual variance. Moreover, neither the conditional nor the unconditional IV estimands in Theorem 2.6 lend themselves to imposition of parametric restrictions on the CCEF or to nonlinear models for the CCEF like probit or logit. As first noted in Abadie (2003), weighted least squares causal estimands readily accommodate nonlinear models for the CCEF.

In practice, $\kappa^\star$-weighted least squares estimation is complicated by the fact that $\kappa^\star$ can be negative, making the minimand non-convex. Theorem 2.6(i) motivates a computationally attractive solution to this problem. Define

$$\begin{aligned}\tilde{Y}_t &= \frac{\tilde{Z} \times 1\{T=t\}Y}{E[\tilde{Z} \times 1\{T=t\}|X]} \\ \tilde{Y}_{t,0} &= \frac{\tilde{Z} \times 1\{T=t\}(1-Z)Y}{E[\tilde{Z} \times 1\{T=t\}(1-Z)|X]} \\ \tilde{Y}_{t,1} &= \frac{\tilde{Z} \times 1\{T=t\}ZY}{E[\tilde{Z} \times 1\{T=t\}Z|X]}\end{aligned} \tag{27}$$

and positive weights

$$\begin{aligned}\omega_t(X) &= P(Z=1|X)(P(T=t|Z=1,X) - P(T=t|Z=0,X))_+ && \text{if } t \in \mathbf{T}_{\mathtt{i}} \\ \omega_t(X) &= P(Z=0|X)(P(T=t|Z=0,X) - P(T=t|Z=1,X))_+ && \text{if } t \in \mathbf{T}_{\mathtt{o}} \\ \omega_{t,z}(X) &= P(T=t, Z=z|X) && \text{if } t \in \mathbf{T}_{\mathtt{io}},\end{aligned} \tag{28}$$

where $a_+ \equiv \max(0, a)$ for any a. The following theorem shows that β solves an ω-weighted least squares problem.

Theorem 2.7. *Suppose the conditions of Theorems 2.5 and 2.6(i) are satisfied and that F is specified as in equation (23). Then,*

$$\beta = \arg\min_b \; E\left[\sum_{t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}} (\tilde{Y}_t - F_1(t, X; b))^2 \omega_t(X) + \sum_{t \in \mathbf{T}_{\mathtt{io}}, z \in \{0,1\}} (\tilde{Y}_{t,z} - F_2(t, z, X; b))^2 \omega_{t,z}(X)\right]$$

for parameter vector β defined in Theorem 2.5.

This result, which combines key relationships in our framework, is most easily seen when $\mathbf{T}_{\mathtt{io}}$ and $\mathbf{T}_{\mathtt{s}}$ are empty, so $\mathbf{T} = \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}$. Note first that Corollary 2.1 and Theorem 2.5 then imply

$$\begin{aligned}\beta &\equiv \arg\min_b E\left[(C_1(T,X) - F_1(T,X;b))^2 | T^* \in \mathbf{S}\right] \\ &= \arg\min_b E\left[\sum_{t \in \mathbf{T}} (C_1(t,X) - F_1(t,X;b))^2 E[1\{T=t\}\kappa^\star(t,Z,X)|X]\right].\end{aligned} \tag{29}$$

Using the $\kappa^\star$ representation from Theorem 2.3(ii) to evaluate $E[1\{T=t\}\kappa^\star(T,Z,X)|X]$ gives

$$E[1\{T=t\}\kappa^\star(T,Z,X)|X]$$
$$= \begin{cases} P(Z=1|X)[P(T=t|Z=1,X) - P(T=t|Z=0,X)] & \text{if } t \in \mathbf{T}_{\mathtt{i}} \\ P(Z=0|X)[P(T=t|Z=0,X) - P(T=t|Z=1,X)] & \text{if } t \in \mathbf{T}_{\mathtt{o}} \end{cases} \tag{30}$$

Since the two quantities on the right-hand side of (30) are positive by Lemma 2.2, this establishes $E[1\{T=t\}\kappa^\star(T,Z,X)|X] = \omega_t(X)$. Substituting for $E[1\{T=t\}\kappa^\star(T,Z,X)|X]$ in (29) gives

$$\beta = \arg\min_b E\left[\sum_{t\in\mathbf{T}}(C_1(t,X) - F_1(t,X;b))^2\omega_t(X)\right] \tag{31}$$
$$= \arg\min_b E\left[\sum_{t\in\mathbf{T}}(\tilde{Y}_t - F_1(t,X;b))^2\omega_t(X)\right], \tag{32}$$

where the final equality follows from the fact that Theorem 2.6(i) implies $E[\tilde{Y}_t|X] = C_1(t,X)$. The argument for the case where $\mathbf{T}_{\mathtt{io}}$ is not empty is similar, using $\omega_{t,z}(X) = E[1\{T=t, Z=z\}\kappa^\star(T,Z,X)|X]$ for any $t \in \mathbf{T}_{\mathtt{io}}$ to simplify the term involving $F_2(T,Z,X;b)1\{T \in \mathbf{T}_{\mathtt{io}}\}$.

Theorem 2.7 offers a novel solution to the problem of negative weights in an estimand for the full CCEF. This strategy can be compared with that in Abadie et al. (2002), which replaces κ-weights with theoretically non-negative $E[\kappa(T,Z,X)|T,X,Y]$ in a basic LATE setting. Nonparametric estimates of $E[\kappa(T,Z,X)|T,X,Y]$ can be sensitive to the choice of smoothing parameters when Y is continuous. Theorem 2.7, by contrast, requires only estimates of the distribution of (Z,T) given X. Since Z and T are discrete, this estimation is relatively straightforward when X is also discrete.

CCEF Identification Under Ordered Selection

Ordered selection model (15) illustrates CCEF identification strategies based on the results in this section. Assuming $\pi \geq c_2 - c_1$ so IOS holds, and that ε has large support, this model has two stayer types, $T^* \in \{(0,0),(2,2)\}$, and up to three switcher types, $T^* \in \{(0,1),(0,2),(1,2)\}$. By Corollary 2.1(i), PSI then requires

$$E[Y^*(0)|T^*=(0,1)] = E[Y^*(0)|T^*=(0,2)] \quad \text{(from (20) at } t=0)$$
$$E[Y^*(2)|T^*=(0,2)] = E[Y^*(2)|T^*=(1,2)] \quad \text{(from (20) at } t=2).$$

Since shared treatment level $t=1$ is in $\mathbf{T}_{\mathtt{io}}$, PSI leaves $E[Y^*(1)|T^*=(0,1)]$ and $E[Y^*(1)|T^*=(1,2)]$ unrestricted.

Given PSI, Corollary 2.1(i) says the CCEF can be written

$$F(T,Z;\beta) = \underbrace{1\{T=0\}\beta_0 + 1\{T=2\}\beta_2}_{F_1(T;\beta)1\{T\in\mathbf{T}_{\mathtt{i}}\cup\mathbf{T}_{\mathtt{o}}\}} + \underbrace{1\{T=1\}(\beta_1 + Z\beta_{11})}_{F_2(T,Z;\beta)1\{T\in\mathbf{T}_{\mathtt{io}}\}}, \tag{33}$$

since $\mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}} = \{0,2\}$ and $\mathbf{T}_{\mathtt{io}} = \{1\}$. Applying Corollary 2.1(ii), this specification generates three causal effects:

$$E[Y^*(2) - Y^*(0)|T^*=(0,2)] = F(2,1;\beta) - F(0,0;\beta) = \beta_2 - \beta_0 \tag{34}$$
$$E[Y^*(2) - Y^*(1)|T^*=(1,2)] = F(2,1;\beta) - F(1,0;\beta) = \beta_2 - \beta_1$$
$$E[Y^*(1) - Y^*(0)|T^*=(0,1)] = F(1,1;\beta) - F(0,0;\beta) = (\beta_1 + \beta_{11}) - \beta_0.$$

Note that effects for switchers from 0 to 1 and from 1 to 2 need not sum to the effect for switchers from 0 to 2.

Theorem 2.6 shows how to compute β using IV. Since $\mathbf{T}_{\mathtt{o}} = \{0\}$ and switcher types $T^* = (0,1)$ and $T^* = (0,2)$ can both select treatment level 0, we have:

$$E[Y^*(0)|T^* = (0,1)] = E[Y^*(0)|T^* = (0,2)] = \frac{E[(Z-p)1\{T=0\}Y]}{E[(Z-p)1\{T=0\}]} = \beta_0,$$

where $p = P(Z=1)$. Similarly, since $\mathbf{T}_{\mathtt{i}} = \{2\}$ and switcher types $T^* = (0,2)$ and $T^* = (1,2)$ can both select treatment level 2, we also have:

$$E[Y^*(2)|T^* = (0,2)] = E[Y^*(2)|T^* = (1,2)] = \frac{E[(Z-p)1\{T=2\}Y]}{E[(Z-p)1\{T=2\}]} = \beta_2.$$

Finally, $\mathbf{T}_{\mathtt{io}} = \{1\}$ with switcher type $T^* = (0,1)$ belonging to $\mathtt{in}(1)$, and switcher type $T^* = (1,2)$ belonging to $\mathtt{out}(1)$. Theorem 2.6 therefore implies

$$\begin{aligned} E[Y^*(1)|T^* = (1,2)] &= \frac{E[(Z-p)1\{T=1\}(1-Z)Y]}{E[(Z-p)1\{T=1\}(1-Z)]} = \beta_1 \\ E[Y^*(1)|T^* = (0,1)] &= \frac{E[(Z-p)1\{T=1\}ZY]}{E[(Z-p)1\{T=1\}Z]} = \beta_1 + \beta_{11}. \end{aligned}$$

These IV estimands are easily rearranged to generate the causal effects of interest.

Theorems 2.5 and 2.7 allow us to identify and estimate the CCEF and associated causal effects by $\kappa^\star$-weighting. Since $\mathbf{T}_{\mathtt{i}} = \{2\}$ and $\mathbf{T}_{\mathtt{io}} = \{1\}$, V as written in Theorem 2.3(ii) equals $V = 1\{T=2\} + 1\{T=1\}Z$. The relevant $\kappa^\star$-weights therefore simplify to

$$\begin{aligned} \kappa^\star(T,Z) &= 1 - \frac{V(1-Z)}{P(Z=0)} - \frac{(1-V)Z}{P(Z=1)} \\ &= 1 - \frac{1\{T=2\}(1-Z)}{P(Z=0)} - \frac{1\{T=0\}Z}{P(Z=1)}. \end{aligned} \tag{35}$$

Using these weights, Theorem 2.5 identifies β as the solution to

$$\beta = \arg\min_b E[(Y - F(T,Z;b))^2 \kappa^\star(T,Z)],$$

where F is as defined in (33). Theorem 2.7 provides a computationally attractive implementation of this minimand.

3 Applications

Three empirical applications illustrate our theoretical results. The first revisits the Head Start studies mentioned above. We compute both IV and $\kappa^\star$-weighted estimates of Head Start effects for two switcher types. The second analyzes data from the Angrist et al. (2022) post-secondary aid randomized trial. The effect of financial aid on degree completion is modeled as mediated by a pair of Bernoulli treatments. The third application examines effects of childbearing, an ordered treatment, on mothers' weeks worked. As in Angrist and Evans (1998), childbearing effects are identified using an indicator for multiple second births to instrument the number of children in households with at least two children. The family-size application features treatment levels

in $\mathbf{T}_{\mathtt{io}}$, so CCEF specifications that depend on Z are fruitful. The text skips most estimation issues; Appendix A.3 establishes consistency and asymptotic normality of estimators based on the identification results in Section 2.

3.1 From SLATE to the CCEF with an Unordered Treatment

Randomized offers of Head Start seats increase the probability of Head Start enrollment in the year after random assignment by around 68 percentage points. This is an estimate of the overall probability of switching, defined as Head Start enrollment as a consequence of a placement offer. The population of Head Start switchers contains two types: applicants who otherwise enroll in competing preschools and applicants who are otherwise in home care. This leads to a scenario with a trinomial unordered treatment variable that we represent by $T \in \{h, c, n\}$, where h denotes Head Start attendance, c denotes competing preschool attendance, and n denotes home care. All children fall into one of these three groups.

Kline and Walters (2016) and Feller et al. (2016) assume Head Start offers leave the choice between c and n unchanged since both options are available to everyone regardless of random assignment. In our notation, this restriction can be written:

$$T^*(1) \neq T^*(0) \Rightarrow T^*(1) = h. \tag{36}$$

This EMCO-like restriction leaves two types of switchers, $T^* = (n, h)$ and $T^* = (c, h)$. These switchers generate two Head Start effects, $E[Y^*(h) - Y^*(c)|T^* = (c, h)]$ for switchers out of competing preschools and $E[Y^*(h) - Y^*(n)|T^* = (n, h)]$ for switchers out of home care. Response types also include stayers at all three levels (some offers are declined while a few applicants not offered Head Start manage to find their way to a Head Start program anyway). Restriction (36) is sufficient for IOS since $\mathtt{out}(h)$, $\mathtt{in}(c)$, and $\mathtt{in}(n)$ are empty.

Ignoring covariates and applying Theorem 2.3(i), SLATE equals

$$\tau(\mathbf{S}) = \frac{E[Y|Z = 1] - E[Y|Z = 0]}{E[V|Z = 1] - E[V|Z = 0]},$$

where

$$\begin{aligned} V &= 1\{T \in \mathbf{T}_{\mathtt{i}}\} + 1\{T \in \mathbf{T}_{\mathtt{io}}\}Z \\ &= 1\{T = h\}, \end{aligned}$$

since $\mathbf{T}_{\mathtt{io}}$ is empty. In this case, therefore, $\tau(\mathbf{S})$ is a conventional Wald-IV estimand instrumenting $1\{T = h\}$. As noted in Kline and Walters (2016) and Feller et al. (2016), this IV estimand is a weighted average of $E[Y^*(h) - Y^*(n)|T^* = (n, h)]$ and $E[Y^*(h) - Y^*(c)|T^* = (c, h)]$. The weights are given by the share of each switcher type among switchers.

Our framework disentangles switcher-specific average causal effects by adding PSI to (36), thereby identifying the CCEF. Since $t = h$ is the only treatment level selected by both types of switchers, PSI with covariates requires

$$E[Y^*(h)|T^* = (c, h), X] = E[Y^*(h)|T^* = (n, h), X]. \tag{37}$$

This says that, conditional on covariates, average outcomes under the Head Start treatment are

the same for both switcher types. Counterfactual outcomes, however, are unrestricted. This contrasts with the two identification strategies deployed in Kline and Walters (2016). The first of these uses instrument-covariate interactions in the offer first stage to identify an additive causal model with treatment effects that are independent of covariates. The second relies on joint normality to obtain a control function that uses nonlinearity to distinguish CCEF variation with T, X from CCEF variation with T^*, X (Feller et al. (2016) also uses joint normality to identify type-specific Head Start effects).

By virtue of Corollary 2.1(i), PSI and the fact that $\mathbf{T}_{\mathtt{io}}$ is empty mean that the CCEF is a function of T only:

$$C(T, Z) = F(T; \beta) = 1\{T = h\}\beta_h + 1\{T = c\}\beta_c + 1\{T = n\}\beta_n.$$

Since IOS is satisfied, Theorem 2.5 implies $\beta \equiv (\beta_h, \beta_c, \beta_n)$ solves

$$\beta = \arg\min_b E[(Y - F(T; b))^2 \kappa^\star(T, Z)]. \tag{38}$$

Moreover, because $V = 1\{T = h\}$, Theorem 2.3(ii) implies that $\kappa^\star$-weights simplify to

$$\kappa^\star(T, Z) = 1 - \frac{1\{T = h\}(1 - Z)}{P(Z = 0)} - \frac{1\{T \neq h\}Z}{P(Z = 1)}.$$

These are Abadie (2003) κ-weights for Bernoulli treatment variable $1\{T = h\}$.

The CCEF identified by (38) generates two switcher-specific causal effects:

$$E[Y^*(h) - Y^*(c)|T^* = (c, h)] = \beta_h - \beta_c$$
$$E[Y^*(h) - Y^*(n)|T^* = (n, h)] = \beta_h - \beta_n.$$

PSI notwithstanding, this parameterization allows causal effects of Head Start to differ for switchers into Head Start from home and switchers into Head Start from competing preschools. Theorem 2.6 illuminates the information that identifies the CCEF in this application. Setting $p = P(Z = 1)$, the theorem implies

$$\beta_t = \frac{E[(Z - p)1\{T = t\}Y]}{E[(Z - p)1\{T = t\}]}, \tag{39}$$

for each $t \in \{h, c, n\}$. Equation (39) casts each β_t as an instrumental variable estimand with outcome $1\{T = t\}Y$ and treatment $1\{T = t\}$, instrumented by Z.

Given IOS, Lemma 2.2 ensures that the first stages in this context (the denominators of (39) at each t, divided by the variance of Z) identify the probability switchers select treatment level t for any $t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}}$. Applying the lemma to Head Start,

$$E[1\{T = h\}|Z = 1] - E[1\{T = h\}|Z = 0] = P(T^* \in \mathtt{in}(h)) = P(T^* \in \mathbf{S}),$$

where the second equality follows from the fact that all switchers switch in to h. Similarly,

$$E[1\{T = c\}|Z = 1] - E[1\{T = c\}|Z = 0] = -P(T^* \in \mathtt{out}(c)) = -P(T^* = (c, h)),$$

and likewise for $T^* = (n, h)$. Since $\mathtt{out}(c)$ and $\mathtt{out}(n)$ include only one type, the share of each type is identified. Note also that both switcher types must occur with positive probability for

the full set of β_t to be identified.

As in Kline and Walters (2016), our Head Start Impact Study (HSIS) sample includes 3,571 Head Start applicants aged 3 and 4 at the time of application.[10] The outcome variable is a composite test score derived from tests taken in the first year following random assignment. Test scores are standardized to be mean zero and standard deviation one in the random assignment control group, separately by application cohort and age. The HSIS design is such that interested families applied to one of 383 Head Start centers and were either offered a seat there or turned away. The probability applicants were offered a place in Head Start varies by center, but estimates that control for center dummies are similar to those reported here.[11]

Conventional 2SLS estimates of the effects of Head Start enrollment using random assignment to instrument a dummy for Head Start enrollment appear in the first two columns of Table 1. These show that randomly assigned offers of a Head Start seat boosted Head Start enrollment by 68 percentage points. The 2SLS estimate generated by this first stage suggests Head Start enrollment yielded an achievement gain of 0.25 standard deviations. As in Kline and Walters (2016), these estimates control for covariates in the form of dummies indicating applicant sex, race, Spanish language use, test language, age at application, teen mother, mother's marital status, mother's level of education, presence of both parents in the home, family size, special education status, income quartile dummies, urban status, a cubic function of baseline scores, and indicators for assigned center transportation and quality (gauged by center characteristics and practices).

In the context of our HSIS analysis, SLATE and 2SLS *estimators* differ in two ways: (i) SLATE instrument $\tilde{Z}$ is centered using a nonlinear model for $E[Z|X]$ (we specify probit), while 2SLS uses a linear model to control for covariates; (ii) SLATE marginalizes covariate-specific reduced-form and first-stage estimates in the numerator and denominator of (8) using inverse propensity score weighting, while 2SLS (with saturated covariate control) implicitly averages covariate-specific LATEs, weighting these by the conditional variance of first-stage fitted values given covariates. In other words, the 2SLS weighting function is given by the sample variance of $E[D|Z,X]$ (Angrist and Imbens, 1995).

Estimates of the overall switching probability and of SLATE, computed using Theorem 2.2 and reported in columns 3 and 4 of Panel A in Table 1, are virtually indistinguishable from the 2SLS estimates reported in the first two columns. The similarity here reflects the fact that $E[D|Z,X]$ and the assignment propensity score, $E[Z|X]$, vary little with X. This scenario makes the 2SLS conditional-variance weighting scheme over covariate-specific causal effects inconsequential.[12] $\kappa^\star$-based estimates of the overall switcher probability and SLATE, computed using Theorem 2.3 and reported in Panel B of Table 1, also equal the corresponding 2SLS estimates up to reported precision.

[10] These data are distributed by the Inter-university Consortium for Political and Social Research (ICPSR). See United States Department of Health and Human Services, Administration for Children and Families, Office of Planning, Research and Evaluation (2018) for documentation.

[11] Applicants are coded as enrolling in Head Start if they enrolled any time between assignment and test-taking. Competing preschool enrollment is determined from parent and provider interviews. Kline and Walters (2016) details the analysis sample further.

[12] 2SLS weights can be written $V(E[D|Z,X]|X)$, where $V(\cdot|X)$ denotes conditional variance. When the first stage is additive in X and Z, this is proportional to $V(Z|X)$. Blandhol et al. (2022) highlights the fact that convex 2SLS weighting requires either saturated covariate control or a correct model for the assignment propensity score.

Marginalized CCEF estimates computed from the sample analog of the IV estimands in Theorem 2.6(ii), reported in Panel A of Table 1, indicate children in Head Start have the highest achievement levels, with average test scores of 0.24 standard deviations. This is followed by the average achievement of children in competing preschools at 0.094, with children in home care scoring well below children enrolled in either type of preschool. These estimates appear (along with standard errors) in column 5 of Table 1. As can be seen in column 7 of the table, roughly 24% of Head Start applicants are switchers drawn from competing preschool programs, while 44% are switchers drawn from home care. These quantities amount to 35% and 65% of all switchers, respectively.

Estimated causal effects of Head Start enrollment differ markedly across switcher types. The estimates in column 8 of Panel A in Table 1 suggest Head Start boosts achievement by roughly 0.14 for children drawn from competing programs, while gains for children switching in to Head Start from home care are estimated to be over twice as large at 0.31. Estimated Head Start effects for both switcher types are significantly different from zero; these estimates are also significantly different from each other. While qualitatively similar to the estimates of switcher-specific effects reported in Kline and Walters (2016), our estimates of these parameters are much more precise. The Kline and Walters (2016) control function estimates of the effect of Head Start for children drawn from competing programs aren't significantly different from zero, while we obtain an estimate of roughly 0.14 with a standard error around 0.054 for this switcher type.[13]

IV estimates of SLATE and the CCEF are compatible in the sense that SLATE is a type-share weighted average of CCEF-implied causal effects for all response types. Appendix Lemma A.6 establishes this equivalence for SLATE and CCEF parameters, for the corresponding IV estimands, and for IV estimates of SLATE and the CCEF (assuming these are constructed using the same estimated propensity score). Consistent with this, the SLATE estimate of 0.25 in column 4 of Table 1 equals $.238 \times .136 + .443 \times .312$ divided by an overall probability of switching estimated at 0.68.

Estimates generated by minimizing the sample analog of the $\kappa^\star$-weighted least squares minimand defined in Theorem 2.7 with a specification of $F_1(T, X, \beta)$ that's additive in covariates are similar to the corresponding IV estimates. In particular, $\kappa^\star$-based estimation of the CCEF and the corresponding causal effects by switcher type, reported in Panel B of Table 1, generates much larger estimated Head Start gains for children drawn from home care than for children drawn from competing preschool programs.[14] The $\kappa^\star$-based treatment effect estimated for switchers from competing preschools exceeds the corresponding IV estimate, though only by about one standard error. Estimates of the CCEF in column 5 of Panel B are more precise than those in Panel A, while standard errors for the two pairs of estimated treatment effects

[13] Standard errors for estimates of $E[Y^*(h) - Y^*(c)|T^* = (c, h)]$ in Kline and Walters (2016) Table VIII, labeled "subLATEs", range from $0.15 - 0.47$. Feller et al. (2016) distinguishes switcher-specific effects using a parametric Bayesian framework, focusing on an outcome different from that examined here and in Kline and Walters (2016).

[14] To compute the weights and dependent variables used in Theorem 2.7, propensity score $E[Z|X]$ is estimated by probit and $P(T = t|Z, X)$ is estimated using multinomial probit. Probabilities $P(T = t, Z = z|X)$ and $P(T = t|X)$ used to construct the terms in (27) are obtained by manipulating these estimates. Both probit specifications are additive in explanatory variables. Standard errors and the p-value in Panel B come from a cluster bootstrap procedure (clustering on center), as discussed in Appendix A.3.

are close.

The $\kappa^\star$-weighted estimand in Theorem 2.7 facilitates estimation of CCEF models allowing for interactions between switcher type and covariates. These estimates can then be transformed into covariate-specific causal effects using Corollary 2.1. In the context of Head Start, for instance, we'd like to know whether differences in impact across covariate groups are similar for both switcher types and whether differences in impact by switcher type are similar across covariates. Such interactions are easily explored through restrictions on a CCEF approximation identified by $\kappa^\star$-weighting. IV estimates based on Theorem 2.6, by contrast, capture points in the support of the CCEF one at a time. In our application, $\kappa^\star$-weighting also yields a precision gain for estimates of the CCEF. This likely reflects the fact that our weighted least squares estimator models CCEF dependence on covariates while the IV estimator does not.

Many evaluations of education interventions report differences in impact by sex.[15] CCEF estimates allowing differences in Head Start impact by sex suggest Head Start produces larger achievement gains for girls than for boys. These estimates appear in Panel A of Figure 1, which graphs estimated treatment effects separately by sex within switcher types. Sex differences in impact are around .16 for both types, though only the sex difference in impact for switchers from home care clears a 10% significance level.

Differences in impact by whether English is the main language at home, graphed in Panel B of Figure 1, are more mixed. Gains for switchers from home care are largest for English-speakers (a marginally significant gap of around 0.21). But estimates for switchers from competing programs are similar for the two language groups. These results imply that differences in impact by switcher type differ for English and non-English speakers. Paralleling overall differences in impact by switcher type, English speakers switching from home care see much larger gains than do English speakers switching from competing care. Among non-English speakers, by contrast, gains are similar for the two types of switchers.

3.2 Multiple Mediators of Financial Aid Effects

Does post-secondary financial aid boost degree completion or simply transfer resources to students who would earn degrees anyway? A large randomized trial that allocated full grant funding to public college and university students in Nebraska has recently revisited this much-studied question. The aid awarded through the trial comes from the Susan Thompson Buffett Foundation (STBF) Buffett Scholars program. Buffett Scholarships fund roughly 11% of Nebraska public university and college enrollment. A key outcome in this research, results of which are reported in Angrist et al. (2022), is BA completion.

Among STBF aid applicants who indicated a desire to attend four-year programs were they to receive aid, STBF awards boosted BA completion in the treatment group by around eight percentage points. Since awards were randomly assigned (within groups or strata), this reduced-form impact is easily estimated. It's harder to identify the mechanisms or mediators explaining why grant aid matters. On one hand, STBF awards might help students who've enrolled in a four-year degree program stay on track, perhaps by reducing the need to work

[15] For instance, Angrist and Lavy (2009) finds high school girls respond more than boys to financial incentives, while Gray-Lobe et al. (2023) reports larger long-term preschool effects for boys.

while in school. Grant aid also reduces student loans, thereby reducing distracting worries related to accumulating student debt. Alternatively, or in addition, grant aid might encourage or allow Buffett scholars who would otherwise attend two-year schools or remain unenrolled to embark immediately on a four-year program. Angrist et al. (2022) refers to this channel for aid effects as *early engagement*.

The early-engagement hypothesis is assessed here using a model that allows for two treatment variables that might mediate BA completion. The first treatment indicates students who borrowed relatively little in their first-year enrolled (STBF awards are roughly constant across years for students in good academic standing). The second indicates the set of STBF applicants strongly engaged with a four-year degree program in the first year after random assignment. STBF awards affect both channels, reducing loans and boosting early engagement. We'd like to know whether aid awards boost BA completion when loans are reduced without changing early four-year engagement, or whether increased early engagement is *necessary* for loan reduction to have an impact.[16]

For the purposes of this investigation, loan status is coded as a dummy variable, denoted by D, indicating students who borrowed less than \$2,000 as part of their first-year financial aid package. The analysis is limited to Pell-eligible STBF applicants who enrolled at any institution in the first year after random assignment. This is a relatively low-income group for whom aid is especially consequential. Roughly 51% of controls in this group borrowed less than \$2,000 in the first year after random assignment. STBF awards boosted this low-loan rate by around 31 percentage points. Early four-year engagement is coded as a dummy variable, denoted by S, indicating students who completed at least 24 credits at a four-year school (the STBF definition of a full course load) in the year following random assignment. STBF awards boosted this by roughly 11 percentage points, with a control-group base of 61%.[17]

This setting can be described by treatment vector $T = (D, S) \in \{0, 1\} \times \{0, 1\}$. Response types $T^*(z) = (D^*(z), S^*(z))$ are indexed against Bernoulli instrument Z indicating randomly assigned STBF awards. We distinguish D and S effects by restricting T^*. Specifically, we consider the identifying power of D-dominance, a restriction discussed in Navjeevan et al. (2023). In this context, D-dominance says that:

1. Awards weakly reduce student loans;

2. Awards weakly increase early engagement;

3. Awards increase early engagement only if they reduce student loans.

D-dominance consists of monotonicity restrictions for each underlying treatment while also making loan reduction necessary for early engagement. In other words, we assume that awards

[16] Angrist et al. (2022) explores an alternative version of the early engagement hypothesis, without allowing for direct loan effects.

[17] Our analysis is limited to STBF applicants who enrolled in the first year after random assignment and who listed a four-year school as their first choice on the STBF aid application. In this sample, mean first-year loan amounts are \$876 and \$3,304 for treatment and control applicants, respectively. We assume students with missing loan information have zero loans. Credits information for students enrolled outside Nebraska's public university system comes from National Student Clearinghouse (NSC) data where available. Otherwise, students missing credits data are assigned a record at random from a group matched on gender and dummies indicating at least one parent with a BA, white race, and Omaha residence, all interacted with award status.

without financial consequences do not change behavior. This restriction is consistent with the timing of STBF awards, which are announced before students commit to program type or course load.[18] In the notation of Assumption 1.2, D-dominance means:

$$\mathbf{M} \equiv \{T^* : D^*(1) \geq D^*(0),\ S^*(1) \geq S^*(0),\ \text{and } D^*(1) = D^*(0) \Rightarrow S^*(1) = S^*(0)\}.$$

Under D-dominance, IOS is satisfied. This can be seen by recoding the multinomial (D, S) treatment into a scalar treatment variable, T, as follows:

$$\begin{aligned} T &= 0 \textbf{ if } (D,S) = (0,0) \\ T &= 1 \textbf{ if } (D,S) = (1,0) \\ T &= 2 \textbf{ if } (D,S) = (0,1) \\ T &= 3 \textbf{ if } (D,S) = (1,1). \end{aligned}$$

In this representation, D-dominance allows 3 types of switchers, $T^* \in \{(0,1),(2,3),(0,3)\}$ and four types of stayers, one for each value of T. IOS is satisfied because the sets $\texttt{in}(0), \texttt{out}(1), \texttt{in}(2)$, and $\texttt{out}(3)$ are empty.

Because IOS holds, Theorem 2.3(i) identifies SLATE by instrumenting $V = 1\{T \in \mathbf{T}_\texttt{i}\} + 1\{T \in \mathbf{T}_\texttt{io}\}Z$ with $\tilde{Z}$. Since $\mathbf{T}_\texttt{io}$ is empty, auxiliary treatment variable V simplifies to

$$V = 1\{T \in \mathbf{T}_\texttt{i}\} = D.$$

The second equality here is a consequence of the fact that $\mathbf{T}_\texttt{i}$ contains treatment values $T = 1$ and $T = 3$, and these are observed if and only if $D = 1$. In this application, SLATE is a weighted average of causal effects for the three switcher types observed under D-dominance. Specifically, SLATE averages effects of loan reduction without early engagement, effects of loan reduction for students who engage early regardless of award status, and an effect of combined loan reduction and early engagement.

As can be seen in Table 2, the estimated (linear) first stage effect of Buffett awards on treatment variable D, computed with saturated controls for randomization strata, is around 0.31. The corresponding 2SLS estimate, reported in column 3 of Table 2, is roughly 0.26.[19] As noted in our discussion of 2SLS estimates for Head Start, with saturated controls for discrete strata (or when the assignment propensity score is additive), a conventional 2SLS estimate averages strata-specific conditional IV estimands using a conditional-variance weighting scheme.

Estimates of switching probabilities and SLATE, reported in columns 4-5 of Table 2, are almost the same as the corresponding linear first stage and 2SLS estimates. These estimates, all of which include saturated strata controls, are computed as detailed in Appendix A.3. Briefly, estimated SLATE is the sample analog of semiparametric IV estimand (8), with $E[Z|X]$ estimated by probit. Estimated $P(T^* \in \mathbf{S})$ is the denominator of this IV estimate. Standard errors for these estimates are computed using a generalized method of moments (GMM) procedure that accounts for sampling error in the estimated $E[Z|X]$. In principle, 2SLS estimates differ

[18] Nibbering and Oosterveen (Forthcoming) uses a version of D-dominance to bound the effect of going from $(0,0)$ to $(1,1)$ for all compliers.

[19] STBF awards were offered at different rates conditional on application year and the school applicants identified as the one they aspire to attend. These variables define randomization strata.

from the sample analog of (8) by virtue of the weights used to marginalize over X. In practice, the award propensity score is close to one-half across strata. This likely explains the similarity of 2SLS and SLATE estimates in Table 2.

SLATE for effects of STBF financial aid is arguably harder to interpret than SLATE for Head Start. This reflects the fact that SLATE in the aid application averages effects of two distinct treatments (D and S), while SLATE for Head Start averages effects of a common Head Start treatment relative to two possible counterfactuals. Because IOS identifies the CCEF, however, we can potentially distinguish switcher-specific causal effects. This turns in part on PSI, which in this case says that:

$$E[Y^*(0)|T^* = (0,1)] = E[Y^*(0)|T^* = (0,3)],$$
$$E[Y^*(3)|T^* = (2,3)] = E[Y^*(3)|T^* = (0,3)].$$

As with Head Start, these restrictions equate potential outcome means at shared levels for switcher types that access these levels. For instance, PSI requires average $Y^*(0)$ to be the same for switcher types $T^* = (0,1)$ and $T^* = (0,3)$ since both switch out of 0. At the same time, PSI allows causal effects to differ for all three switcher types.[20]

The distribution of T, reported in column 3 of Table 3, shows that most STBF applicants either had reduced loans or engaged early with a four-year degree program. For many, however, only one or the other of these mediators is switched on. CCEF estimates, computed using the IV estimands defined in Theorem 2.6, appear in column 4 of the table. Early engagement is associated with a high probability of four-year degree completion regardless of applicants' borrowing behavior. On the other hand, students who don't engage early are much less likely to earn a degree. The CCEF for $T = 1$ is estimated with low precision because aid awards have almost no effect on the likelihood of this treatment.

Roughly 11% of STBF applicants are induced by STBF awards to both reduce loans and engage early. Estimated switching probabilities appear in column 6 of Table 3. Switchers induced to borrow less while engaged in a BA program with or without awards are more prevalent, amounting to roughly 22% of the applicant population. On the other hand, loan reduction without engagement doesn't happen in practice, with a statistically zero switcher probability.

Differences in CCEF levels for each switcher type give switcher-specific causal effects. Causal effects of STBF awards, reported in column 7 of Table 3, are broadly consistent with the early-engagement hypothesis. Estimates for switcher type $T^* = (0,3)$ imply that STBF applicants induced by awards both to borrow less and to engage early saw their odds of degree completion increase by roughly 59 percentage points—relative to 0.22 when both channels are switched off. Induced loan reduction alone, however, did much less for switcher type $T^* = (2,3)$, the group of applicants who engage early with or without the benefit of an STBF award. The estimated effect of 0.06 for this group is not significantly different from zero.

[20]The two PSI restrictions can be benchmarked against the range of CCEF specifications possible in this context. Three switcher types with two potential outcome levels identified for each yields six distinct potential outcome means when unrestricted. The most restrictive homogeneity assumption equates all of these for a total of five restrictions and treatment effects constant at zero. Non-zero constant effects imposes two restrictions on a particular linear combination of CCEF levels.

3.3 Ordered Models for Childbearing Effects

Do falling fertility rates explain rising female labor force participation in the twentieth century? Births are strongly negatively correlated with mothers' work and earnings, but these correlations might reflect selection bias. Specifically, mothers in larger families may be less likely to work regardless of family size. Angrist and Evans (1998) tackles selection bias using instrumental variables, one of which indicates mothers who experience multiple second births. Twin births increase family size sharply for the almost 1% of mothers who have twins. If Assumption 1.1 is satisfied by a dummy indicating multiple second births, the twins instrument identifies causal effects of increasing the number of children beyond two.

Angrist and Evans (1998) codes treatment as a dummy variable indicating mothers with more than two children in a sample of mothers having at least two. Among women with more than two children, however, fertility instruments can shift the distribution of family size at parities above two. Evidence on this point includes Angrist et al. (2010), which shows that the event of a multiple second birth moves some Israeli mothers from three to more than three children as well as from two to more than two. As first shown in Angrist and Imbens (1995), such instrument-induced changes in the distribution of treatment make IV estimates that recode an ordered treatment as Bernoulli misleading.

These points lead us to consider identification strategies for effects of the number of children (T) on weeks worked, rather than a dummy indicating more than two. In this case, $T \in \mathbf{T} \equiv \{2, 3, 4\}$, where $T = 4$ for women with four or more children (few mothers in our sample have more than four). Potential treatment is assumed to satisfy monotonicity:

$$T^*(1) \geq T^*(0).$$

This restriction allows three possible switcher types:

$$\mathbf{S} = \{(2,3), (3,4), (2,4)\}.$$

With three treatment levels, conventional ordered probit for potential treatment assignment postulates

$$T^*(z) = \begin{cases} 2 & \text{if } z\pi + X'\gamma + \varepsilon \leq c_3 \\ 3 & \text{if } c_3 < z\pi + X'\gamma + \varepsilon \leq c_4 \\ 4 & \text{if } c_4 < z\pi + X'\gamma + \varepsilon \end{cases} \tag{40}$$

for $z \in \{0, 1\}$, with $\varepsilon | Z, X \sim N(0, 1)$ and $c_3 < c_4$. This model reduces the number of response types, thereby identifying SLATE and, possibly, the CCEF. When instrument Z indicates twin births, however, (40) is necessarily misspecified since $P(T = 2|X, Z = 1) = 0$, while ordered probit implies $P(T = t|X, Z = z) > 0$ for every t and z. We therefore also estimate a modified ordered model specifying

$$T^*(0) = \begin{cases} 2 & \text{if } X'\gamma + \varepsilon \leq c_3 \\ 3 & \text{if } c_3 < X'\gamma + \varepsilon \leq c_4 \\ 4 & \text{if } c_4 < X'\gamma + \varepsilon \end{cases} \quad T^*(1) = \begin{cases} 3 & \text{if } \pi + X'\gamma + \varepsilon \leq c_4 \\ 4 & \text{if } c_4 < \pi + X'\gamma + \varepsilon \end{cases}, \tag{41}$$

while maintaining $\varepsilon|Z,X \sim N(0,1)$ and $c_3 < c_4$. This model implies

$$\begin{aligned}
P(T=2|Z,X) &= (1-Z)\Phi(c_3 - X'\gamma) \\
P(T=3|Z,X) &= \Phi(c_4 - X'\gamma - \pi Z) - (1-Z)\Phi(c_3 - X'\gamma) \\
P(T=4|Z,X) &= 1 - \Phi(c_4 - X'\gamma - \pi Z),
\end{aligned}$$

imposing the constraint $P(T=2|X,Z=1)=0$.

As in the conventional ordered model, IOS holds in the modified model if $\pi > c_4 - c_3$, which rules out stayers at $T=3$. If $\pi < c_4 - c_3$, IOS fails but ISP holds with response types $(2,3),(3,4),(3,3),(4,4)$, two of which are switchers. ISP holds because the system

$$\begin{aligned}
\nu_1(3) + \nu_0(3) &= 0, \\
\nu_1(3) + \nu_0(2) &= 1, \\
\nu_1(4) + \nu_0(4) &= 0, \\
\nu_1(4) + \nu_0(3) &= 1,
\end{aligned}$$

is solved by $\nu_1(T) = -\nu_0(T) = T$.

Maximum likelihood estimates of conventional and modified ordered probit first stage parameters appear in Table 4. These estimates are for a sample of white women whose second child is age two or less.[21] By positioning the sample frame close to the time of mothers' second birth, we boost the twins first stage. As in Angrist and Evans (1998), first-stage and other estimates control for mother's age, age at first birth, ages of first two children, and indicators for male at first and second births. OLS estimates of a linear first stage with these controls indicate that twin births boost fertility by about one child. The corresponding estimate of π in the ordered probit specification is roughly 2.6. Since this exceeds $c_4 - c_3$, IOS is satisfied. As can be seen in column 2 of the first-stage table, the modified ordered model generates a smaller estimate of π equal to around 1.6. This is less than the estimated cutoff difference for the modified model, though not by much. The difference between estimated π and the IOS threshold in column 2 of Table 4 is only marginally statistically significant.

When IOS holds, SLATE equals an IV estimand with auxiliary treatment variable

$$\begin{aligned}
V &= 1\{T \in \mathbf{T}_{\mathtt{i}}\} + 1\{T \in \mathbf{T}_{\mathtt{io}}\}Z \\
&= 1\{T=4\} + 1\{T=3\}Z.
\end{aligned}$$

Without IOS, but given ISP, the auxiliary treatment variable is $V \equiv \nu_1(T)Z - \nu_0(T)(1-Z) = T$. Estimated SLATE assuming IOS appears in column 2 of Table 5 along with the associated switcher probability; the corresponding estimate under ISP appears in column 3. The two SLATE estimates are similar at -5.17 and -5.28, respectively. These can be compared with a conventional 2SLS estimate of the effect of number of children on mothers' weeks worked, reported in the first column of Table 5, equal to -5.49. This differs from SLATE under ISP only in that the latter uses (8) to average conditional effects over covariates. In general, SLATE averages causal effects over switcher types. When $\pi > c_4 - c_3$, these types consist of a mix of

[21] Our dataset is an extract from the Angrist and Fernandez-Val (2013) 1980 Census sample, which includes mothers aged 21-35 with at least two children whose second child is more than one year old, and whose first birth was at age 15 or later.

switchers induced to have 1-2 additional children, while without IOS, SLATE averages marginal effects of a single child.

The two estimates of SLATE in Table 5 differ only to the extent that the corresponding estimated switcher probabilities differ. As it happens, estimated switcher probabilities are similar, a result (also shown in the table) that explains why estimated SLATE under IOS and estimated SLATE under ISP differ little. Consistent with the statistical test of IOS reported at the bottom of Table 4, the modified ordered model seems to be close enough to IOS to justify CCEF estimation taking IOS as given. Assuming IOS holds, the CCEF is identified under PSI by Theorem 2.6 using IV or by $\kappa^\star$-weighting using Theorem 2.7.

With three switcher types accessing two treatment levels each, the CCEF has six points of support. Two switcher types are in $\texttt{out}(2)$ and two switcher types are in $\texttt{in}(4)$. PSI therefore imposes two pairwise restrictions on the CCEF at treatment values 2 and 4:

$$E[Y^*(2)|T^* = (2,3)] = E[Y^*(2)|T^* = (2,4)]$$
$$E[Y^*(4)|T^* = (2,4)] = E[Y^*(4)|T^* = (3,4)].$$

Although switcher types $T^* = (2,3)$ and $T^* = (3,4)$ both access $T = 3$, one type belongs to $\texttt{in}(3)$ and one type belongs to $\texttt{out}(3)$. PSI therefore leaves the difference in CCEF levels for this pair unrestricted.

Estimates of the CCEF, reported in columns 3-4 of Table 6, show monotonically declining weeks worked as a function of family size. It's noteworthy that CCEF estimates for those switching in and out of treatment level $T = 3$ are similar. The difference between $E[Y^*(3)|T^* = (2,3)]$ and $E[Y^*(3)|T^* = (3,4)]$ is less than a tenth of a week and is not significantly different from zero. It should be noted, however, that since few mothers have four children, estimates of the CCEF at $T = 4$ are imprecise.

Estimates of the causal effects of childbearing for each switcher type appear in the last column of Table 6. Types switching from two to three children and from three to four children are estimated to each experience a reduction of around 5.5 weeks worked as a result. The estimated causal effect of switching from two to four is close to the sum of these at -10.8, with an average per-child effect of -5.4. This concordance reflects the fact that the CCEF at $T = 3$ is similar for types $(2,3)$ and $(3,4)$. On balance, the estimates in Table 6 suggest effects of childbearing on weeks worked are linear in the number of children, at least in the sample used here.

4 Summary and Directions for Further Work

The theoretical framework developed here uses a single instrumental variable to identify average causal effects in diverse applications involving multifaceted treatments. Our identification results are founded in part on the concept of instrument-induced switchers, an extension of the ideas behind Angrist et al. (1996) compliers and Frangakis and Rubin (2002) principal strata. Switcher treatment status changes in response to instrument assignment. This includes changes in multinomial, multiple, and ordered treatments.

Our framework leverages restrictions on the distribution of switcher types to identify a

single average causal effect for all switchers, a parameter we've dubbed SLATE. We also use a combination of restrictions on switcher types and average potential outcomes to identify a general CCEF. The CCEF characterizes average potential outcomes and average causal effects for specific switcher types conditional on covariates. SLATE and CCEF parameters are shown to be identified both by IV estimands and by a weighting scheme that generalizes Abadie (2003) κ-weights for multiple treatments.

The utility of these results is demonstrated in three applications. Building on Kline and Walters (2016), we first consider average causal effects of Head Start preschool attendance on test scores in a setting with multinomial preschool treatments. As in Kline and Walters (2016), the estimates reported here suggest Head Start generates substantially larger achievement gains for children drawn into Head Start from home care than for children drawn into Head Start from competing programs. At the same time, our estimates of Head Start effects for children who switch into Head Start from competing programs are notably more precise than those reported in Kline and Walters (2016). Estimates of CCEF specifications allowing for covariate interactions are easily computed using $\kappa^{\star}$-weighting to estimate parsimonious approximations to a higher-dimensional CCEF. These show generally larger gains from Head Start for switchers from home care than for switchers from competing programs across covariate groups. Moreover, consistently across switcher types, girls benefit more from Head Start than do boys.

Our second application allows for two distinct pathways through which post-secondary grant aid boosts BA completion. The resulting estimates suggest early engagement with four-year college programs is the primary channel by which financial aid boosts degree completion; loan reduction alone does not seem to matter. The third application identifies and estimates the CCEF for effects of childbearing, an ordered treatment, on mothers' labor supply. Estimates of the CCEF for effects of childbearing on weeks worked show roughly constant and approximately linear reductions in labor supply of around five weeks per child, across switcher types.

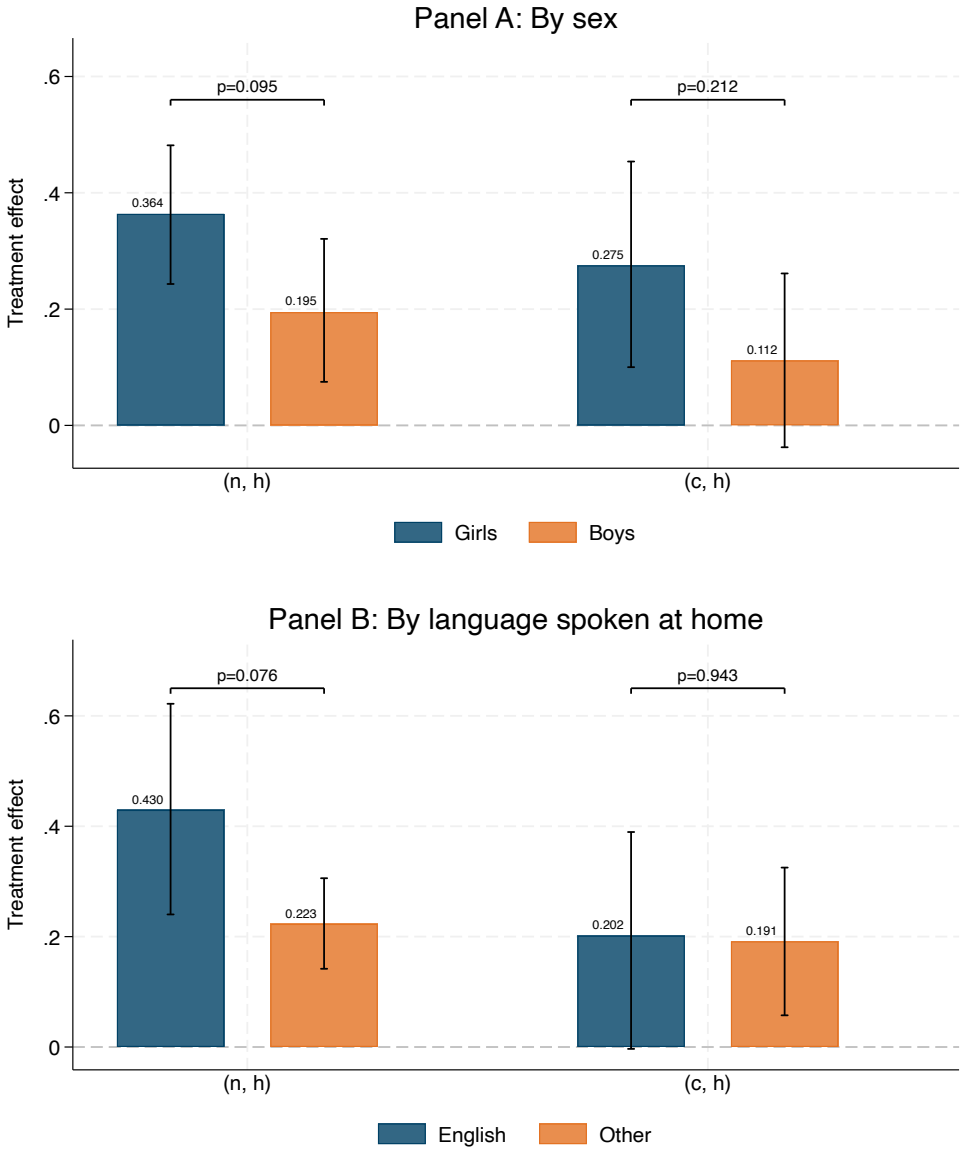
In future work, we plan to extend the framework developed here to exploit multiple instruments in models with multinomial and ordered treatments. Much work also remains to be done regarding estimation and specification testing in these settings. In addition to questions of efficiency, the homogeneity restriction PSI and restrictions on switcher types like D-dominance seem likely to lead to new tests of key identifying assumptions. These tests should be useful in a wide range of IV applications.

Table 1: Estimates of Head Start Effects on Test Scores

	2SLS		SLATE			Causal Effects		
Treatment	First Stage	Estimate	$P(T^* \in \mathbf{S})$	Estimate	CCEF	Switcher Types	$P(T^* = t^*)$	Estimate
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Panel A. IV estimates								
Head Start	0.679	0.250	0.680	0.250	0.230			
	(0.018)	(0.030)	(0.018)	(0.030)	(0.033)			
Competing preschools					0.094	(c, h)	0.238	0.136
					(0.057)		(0.015)	(0.054)
Home care					-0.081	(n, h)	0.443	0.312
					(0.048)		(0.020)	(0.039)
p-value for impact difference by switcher type								0.012
Panel B. $\kappa^\star$-weighting estimates								
Head Start			0.680	0.250	0.238			
			(0.018)	(0.030)	(0.024)			
Competing preschools					0.043	(c, h)	0.238	0.194
					(0.051)		(0.015)	(0.054)
Home care					-0.042	(n, h)	0.443	0.280
					(0.034)		(0.020)	(0.036)
p-value for impact difference by switcher type								0.166

Notes: This table reports 2SLS, SLATE, and CCEF estimates characterizing the effect of Head Start enrollment on test scores in the first year following random assignment. Parameters are identified by random assignment of Head Start offers. Treatment variable $T \in \{n, c, h\}$ denotes no preschool enrollment (n), other center enrollment (c), or Head Start (h). Column 1 reports conventional first stage estimates, with the corresponding 2SLS estimates instrumenting $1\{T = h\}$ shown in column 2. SLATE averages effects for the two switcher types (T^*) listed in column 6. SLATE (column 4) and the overall switching probability (column 3) in Panel A are estimated using Theorem 2.2 and setting $V = 1\{T = h\}$, as described in Appendix A.3. SLATE (column 4) and the overall switching probability (column 3) in Panel B are estimated using Theorem 2.3 as described in Appendix A.3. The CCEF (column 5) in Panel A is estimated by IV using Theorem 2.6, as described in Appendix A.3. The CCEF (column 5) in Panel B is estimated by $\kappa^\star$-weighting using Theorem 2.7, as described in Appendix A.3. Column 6 lists possible switcher types and column 7 shows the estimated probability of switcher types; these are computed using Lemma 2.2 as described in Appendix A.3. Column 8 reports estimated causal effects on test scores for each type, computed from the CCEF estimates in column 5. The p-value in Panel B is computed using the cluster bootstrap (clustering on center) as described in Appendix A.3. All models control for applicant sex, race, Spanish language use, language of testing, age cohort, teen mother, mother's marital status, mother's level of education, presence of both parents in the home, family size, special education status, income quartile dummies, urban status, a cubic function of baseline scores, and indicators for assigned center transportation and quality (quality measured by center characteristics and practices). CCEF models are linear on controls. The propensity score is estimated using probit. $P(T = t|Z, X)$ is estimated using multinomial probit. Standard errors in columns 1-2 (reported in parentheses) cluster on center. Standard errors in columns 3-5 and 7-8 (Panel A) are computed as described in Appendix A.3, also clustering on center. Standard errors in columns 3-5 and 8 in Panel B are computed using a cluster bootstrap (1,000 simulations) detailed in Appendix A.3. $N = 3,571$.

Figure 1: Estimates of Head Start Effects on Test Scores by Sex and Primary Language



Notes: This figure graphs estimated causal effects of Head Start on test scores by switcher type and sex (in Panel A) and by switcher type and language spoken at home (in Panel B). Causal effects are computed from a CCEF estimated by $\kappa^{\star}$-weighting using Theorem 2.7, as described in Appendix A.3. Controls are the same as in Table 1 and enter the CCEF additively. P-values are for tests of differences in impact by demographic group within switcher types. Confidence intervals and p-values are computed using a cluster bootstrap (clustering on center), as discussed in Appendix A.3. The propensity score is estimated using probit; $P(T = t|Z, X)$ is estimated using multinomial probit. $N = 3,571$.

Table 2: Estimates of Financial Aid Effects on BA Completion Pooling Switchers

	2SLS (instrumenting D)		SLATE ($V = D$)	
Switcher Types	First Stage	Estimate	$P(T^* \in \mathbf{S})$	Estimate
(1)	(2)	(3)	(4)	(5)
$0 \to 1$, $2 \to 3$, $0 \to 3$	0.307	0.256	0.309	0.253
	(0.013)	(0.046)	(0.013)	(0.046)

Notes: This table reports 2SLS and SLATE estimates for the effects of financial aid on BA completion among Pell-eligible STBF applicants who enrolled at any institution in the first year after random assignment. Parameters are identified by the random assignment of STBF aid awards. Treatment $T \in \{0, 1, 2, 3\}$ recodes the pair of Bernoulli treatments $(D, S) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ into four levels, where D indicates reduced loans and S indicates early engagement. 2SLS estimates use aid offers to instrument D. SLATE averages effects for the three switcher types (T^*) listed in column 1. SLATE (column 5) and overall switching probability (column 4) are estimated using Theorem 2.2, as described in Appendix A.3, setting $V = D$. The propensity score is estimated using probit; controls are randomization strata indicators. 2SLS standard errors, reported in parentheses, are robust. Standard errors in columns 4-5 are computed as described in Appendix A.3. $N = 4,934$.

Table 3: Estimates of the CCEF for Aid Effects on BA Completion

				Causal Effects		
Treatment Level	(D, S)	$P(T = t)$	CCEF	Switcher Type	$P(T^* = t^*)$	IV Estimate
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	(0, 0)	0.091	0.219	$0 \to 1$	-0.020	-0.107
			(0.041)		(0.012)	(0.383)
1	(1, 0)	0.251	0.112	$2 \to 3$	0.221	0.055
			(0.380)		(0.012)	(0.036)
2	(0, 1)	0.257	0.756	$0 \to 3$	0.108	0.592
			(0.027)		(0.014)	(0.047)
3	(1, 1)	0.401	0.811			
			(0.023)			

Notes: This table reports the estimated CCEF and corresponding treatment effects of reduced loans (D) and early engagement (S) on BA completion among Pell-eligible STBF applicants who enrolled at any institution in the first year after random assignment. Parameters are identified by the random assignment of STBF aid awards. Column 3 gives sample proportions at each treatment level. Column 4 shows the CCEF, estimated by IV using Theorem 2.6, as described in Appendix A.3. Column 5 lists possible switcher types under D-dominance and column 6 shows the estimated probability of switcher types; these are computed as described in Appendix A.3. Column 7 reports estimated causal effects on BA completion for each type, computed from the CCEF estimates in column 4. The propensity score is estimated using probit; controls are randomization strata indicators. Standard errors for the estimates in columns 4, 6, and 7 are computed as described in Appendix A.3. $N = 4,934$.

Table 4: Ordered Probit First-Stage Estimates for Twins

Parameter	FS Model	
	OP	Modified OP
	(1)	(2)
π	2.59	1.59
	(0.055)	(0.129)
c_3	1.45	1.46
	(0.095)	(0.096)
c_4	3.49	3.20
	(0.124)	(0.107)
$c_4 - c_3$	2.04	1.75
	(0.081)	(0.050)
$H_0 : \pi > c_4 - c_3$	1	0.097

Notes: This table reports estimates of the ordered probit and modified ordered probit models detailed in Section 3.3. The dependent variable is the number of children censored at 4 for mothers of two or more children. The last row reports the p-value for a one-sided t-test testing IOS. Data are from the Angrist and Fernandez-Val (2013) 1980 Census sample restricted to white mothers with second child aged 2 or less. Controls are age, age at first birth, age of first two children (in quarters), and indicators for boy 1st and boy 2nd. Robust standard errors are reported in parentheses. $N = 34,941$.

Table 5: Estimates of Childbearing Effects Pooling Switchers

		Modified OP		
	2SLS	IOS	ISP	
	(1)	(2)	(3)	col 2−col 3
V	T	$1\{T=4\}$ $+1\{T=3\}Z$	T	–
Switcher Types	–	$(2,3),(3,4)$ $(2,4)$	$(2,3),(3,4)$	–
FS/$P(T^* \in \mathbf{S})$	0.977	0.998	0.976	0.022
	(0.014)	(0.002)	(0.015)	(0.015)
2SLS/SLATE	-5.49	-5.17	-5.28	0.115
	(0.958)	(0.942)	(0.959)	(0.081)

Notes: This table reports 2SLS and SLATE estimates of effects of childbearing on weeks worked. Parameters are identified using the twins instruments. Treatment $T \in \{2,3,4\}$ is the number of children censored at 4. 2SLS estimates use the twins dummy to instrument T. SLATE averages effects for switcher types listed in the second row; switcher types and the formula for V in column 2 follow from the modified ordered probit model under the null hypothesis $\pi > c_4 - c_3$ (IOS) or point estimates in Table 4 (ISP). SLATE and overall switching probability $P(T^* \in \mathbf{S})$ are estimated using Theorem 2.2, as described in Appendix A.3. Covariates and sample are as for Table 4. The propensity score is estimated using probit. 2SLS standard errors, reported in parentheses, are robust. Standard errors for the probability of switching and SLATE are computed as described in Appendix A.3. $N = 34,941$.

Table 6: The CCEF for Childbearing Effects on Weeks Worked

		CCEF			Causal Effects	
T	$P(T=t)$	Switcher Types	$\texttt{in}(t)$	$\texttt{out}(t)$	Switcher Type	Estimates
	(1)	(2)	(3)	(4)	(5)	(6)
2	0.907	$(2,3),(2,4)$	$\emptyset$	14.7	$(2,3)$	-5.25
				(0.11)		(0.991)
3	0.092	$(2,3),(3,4)$	9.49	9.58	$(3,4)$	-5.66
			(0.984)	(0.307)		(2.076)
4	0.002	$(2,4),(3,4)$	3.92	$\emptyset$	$(2,4)$	-10.8
			(2.054)			(2.06)

Notes: This table reports the estimated CCEF and causal effects of childbearing on weeks worked. Parameters are identified using the twins instrument. Treatment $T \in \{2,3,4\}$ is the number of children censored at 4. Column 1 gives sample proportions at each treatment level. Column 2 reports switcher types getting into or out of t. Columns 3 and 4 report CCEF estimates for switchers in the sets $\texttt{in}(t)$ and $\texttt{out}(t)$; these sets follow from the modified ordered probit model detailed in Section 3.3 under the null hypothesis $\pi > c_4 - c_3$. CCEF is estimated by IV using Theorem 2.6, as described in Appendix A.3. Column 6 reports the implied causal effect estimate for the switcher type in column 5. The propensity score is estimated using probit. Covariates and sample are as for Table 4. Standard errors for the estimates in columns 3, 4, and 6 are computed as described in Appendix A.3. $N = 34,941$.

A Appendix

Identification and estimation results are proved in separate appendices.

A.1 Proofs for Sections 1 and 2

A.1.1 Proof of Proposition 1.1

Using that $Y = Y^*(T)$ and $T = T^*(Z)$, we obtain that

$$\begin{aligned} E[Y1\{T=t\}1\{T^*=t^*\}|X] &= E[Y^*(t)1\{t^*(Z)=t\}1\{T^*=t^*\}|X] \\ &= E[Y^*(t)1\{T^*=t^*\}|X]P(t^*(Z)=t|X), \end{aligned} \tag{42}$$

for any $t \in \{t^*(0), t^*(1)\}$, where the second equality follows from the law of iterated expectations and Lemma A.1 in Navjeevan et al. (2023). Furthermore, by the same arguments we also have

$$E[1\{T^*=t^*\}1\{T=t\}|X] = E[1\{T^*=t^*\}|X]P(t^*(Z)=t|X). \tag{43}$$

Therefore, the theorem follows from dividing (42) by (43) and employing that $P(t^*(Z)=t|X) > 0$ for any $t \in \{t^*(0), t^*(1)\}$ due to $0 < P(Z=1|X) < 1$. ■

A.1.2 Proof of Lemma 2.1

First note that since $P(T^* \in \mathbf{M}) = 1$ by Assumption 1.2(i), for any $z \in \{0,1\}$ it follows

$$\begin{aligned} E[Y|Z=z,X] &= \sum_{t^* \in \mathbf{M}} E[Y1\{T^*=t^*\}|Z=z,X] \\ &= \sum_{t^* \in \mathbf{M}} E[Y^*(t^*(z))1\{T^*=t^*\}|Z=z,X] = \sum_{t^* \in \mathbf{M}} E[Y^*(t^*(z))1\{T^*=t^*\}|X] \end{aligned} \tag{44}$$

where the second equality follows from Assumption 1.1(ii) and the third from (Y^*, T^*) being independent of Z conditionally on X. Therefore, since $t^*(1) = t^*(0)$ for any $t^* \in \mathbf{M} \setminus \mathbf{S}$, result (44) and the law of iterated expectations yield

$$\begin{aligned} E[E[Y|Z=1,X] - E[Y|Z=0,X]] &= \sum_{t^* \in \mathbf{S}} E[(Y^*(t^*(1)) - Y^*(t^*(0)))1\{T^*=t^*\}] \\ &= E[\tau(T^*)1\{T^* \in \mathbf{S}\}], \end{aligned} \tag{45}$$

which establishes the first claim of the lemma. The second claim is immediate by dividing both sides of (45) by $P(T^* \in \mathbf{S})$, which is positive by Assumption 1.2(ii). ■

A.1.3 Proof of Theorem 2.1

First define the function $\ell(T^*) \equiv 1\{T^* \in \mathbf{S}\}$ and observe that by definition we have

$$P(T^* \in \mathbf{S}) = E[\ell(T^*)]. \tag{46}$$

Corollary 4.3 in Navjeevan et al. (2023) provides necessary and sufficient conditions for the identification of moments with the structure in (46). In order to apply said corollary, let $U_{\mathbf{M}}$ denote the uniform distribution on $\mathbf{M}$ and set $\mu \equiv \{P_{Y|X}\}^d U_{\mathbf{M}} P_Z P_X$. By Lemma A.1,

$P_{Y^*T^*ZX} \ll \mu$ and therefore Assumption 1.1 implies that Assumptions 2.1 and 2.2 of Navjeevan et al. (2023) are satisfied with $\mathcal{Q} = \mathbf{Q} = L^1(\mu)$. Moreover, since Lemma A.1 further establishes that $\mu \ll P_{Y^*T^*ZX}$, we may set $\bar{Q} = P_{Y^*T^*ZX}$. Condition (ii) of the Theorem therefore implies that $\bar{Q}^{\mathtt{it}} \ll \bar{Q}$ with $d\bar{Q}^{\mathtt{it}}/d\bar{Q}$ bounded as required by Corollary 4.3 in Navjeevan et al. (2023). Moreover, by definition of μ, we have that $\mu_{Z|X} = P_Z$ and therefore we obtain that

$$\frac{dP_{Z|X}}{d\mu_{Z|X}} = \frac{dP_{Z|X}}{dP_Z} \geq \varepsilon$$

because $P(Z = z|X) \geq \varepsilon$ for some $\varepsilon > 0$ and all $z \in \{0, 1\}$ by assumption. Finally, we note that since ℓ is bounded, condition (i) of the Theorem implies all the remaining requirements of Corollary 4.3(b) in Navjeevan et al. (2023) are satisfied. Therefore, Corollary 4.3(b) in Navjeevan et al. (2023) implies that $P(T^* \in \mathbf{S})$ is identified if and only if there exists a function f of (T, Z, X) satisfying $E[|f(T, Z, X)|] < \infty$ and the equality

$$\mu(\sum_{t \in \mathbf{T}} E_{\mu_{Z|X}}[f(t, Z, X)1\{T^*(Z) = t)] = \ell(T^*)) = 1 \tag{47}$$

where $E_{\mu_{Z|X}}$ denotes an expectation taken with respect to the conditional distribution of Z given X under μ. Next, note that $P(Z = 1) \in (0, 1)$, $\mu_{Z|X} = P_Z$, and $\mu_{T^*X} = P_{T^*X}$ by definition of μ imply that requirement (47) is equivalent to

$$P_{T^*X}(\sum_{t \in \mathbf{T}} \sum_{z \in \{0,1\}} f(t, z, X)1\{T^*(z) = t\} = \ell(T^*)) = 1 \tag{48}$$

holding for some function f of (T, Z, X). Moreover, since $P_{T^*}P_X \ll P_{T^*X}$ due to $P(T^* = t^*|X) \geq \varepsilon > 0$ for all $t^* \in \mathbf{M}$, condition (48) is itself equivalent to

$$P_{T^*}P_X(\sum_{t \in \mathbf{T}} \sum_{z \in \{0,1\}} f(t, z, X)1\{T^*(z) = t\} = \ell(T^*)) = 1. \tag{49}$$

Next, note that for any $(t, z) \in \mathbf{T} \times \{0, 1\}$, we obtain from T^* being independent of Z conditionally on X and $P(Z = z|X) \geq \varepsilon$ for some $\varepsilon > 0$ that

$$\begin{aligned} P(T = t, Z = z|X) &= P(T^*(z) = t|X)P(Z = z|X) \asymp P(T^*(z) = t|X) \\ &= \sum_{t^* \in \mathbf{M}: t^*(z) = t} P(T^* = t^*|X) \asymp \#\{t^* \in \mathbf{M} : t^*(z) = t\} \asymp P(T^*(z) = t), \end{aligned} \tag{50}$$

where the fourth result follows from $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$, and the final result from $P(T^*(z) = t) = 0$ if and only if $t^*(z) \neq t$ for all $t^* \in \mathbf{M}$ due to $P(T^* = t^*|X) \geq \varepsilon$. Moreover, by the law of iterated expectations and result (50) we also obtain

$$P(T = t, Z = z) = E[P(T = t, Z = z|X)] \asymp P(T^*(z) = t) \asymp P(T = t, Z = z|X). \tag{51}$$

In particular, note that $P(T = t, Z = z) \asymp P(T^*(z) = t)$ for all $(t, z) \in \mathbf{T} \times \{0, 1\}$ implies that condition (49), and hence identification of (46), is equivalent to

$$P_{T^*}P_X\left(\sum_{z \in \{0,1\}} \sum_{t: P(T=t, Z=z) > 0} f(t, z, X)1\{T^*(z) = t\} = \ell(T^*)\right) = 1. \tag{52}$$

To conclude, note that if (5) holds, then (52), and hence identification of $P(T^* \in \mathbf{S})$, holds due to $T^* \in \mathbf{M}$ with probability one by Assumption 1.2 and $P(T = t, Z = z) \asymp P(T^*(z) = t)$ for all $(t, z) \in \mathbf{T} \times \{0, 1\}$. For the converse, suppose that (52) holds for some f satisfying $E[|f(T, Z, X)|] < \infty$. Then note that for any set $E \subseteq \mathbf{X}$, Bayes Rule and result (51) imply for any $(t, z) \in \mathbf{T} \times \{0, 1\}$ satisfying $P(T = t, Z = z) > 0$ that

$$P(X \in E) = \frac{P(X \in E|T = t, Z = z)P(T = t, Z = z)}{P(T = t, Z = z|X \in E)} \asymp P(X \in E|T = t, Z = z). \tag{53}$$

In particular, result (53) implies that $E[|f(t, z, X)|] \lesssim E[|f(T, Z, X)|]$ for any (t, z) satisfying $P(T = t, Z = z) > 0$. Since $E[|f(T, Z, X)|] < \infty$ we may therefore define

$$\nu(t, z) = \begin{cases} E[f(t, z, X)] & \text{if } P(T = t, Z = z) > 0 \\ 0 & \text{if } P(T = t, Z = z) = 0 \end{cases}.$$

Finally, note that condition (52) implies that with P_{T^*} probability one we have that

$$\ell(T^*) = \sum_{z \in \{0,1\}} \sum_{t: P(T=t, Z=z)>0} \nu(t, z)1\{T^*(z) = t\} = \sum_{z \in \{0,1\}} \sum_{t \in \mathbf{T}} \nu(t, z)1\{T^*(z) = t\} \tag{54}$$

where the final equality follows from $P(T = t, Z = z) \asymp P(T^*(z) = t)$ by (51). Since $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$, result (54) implies that (5) holds with $\nu_0(t) = \nu(t, 0)$ and $\nu_1(t) = \nu(t, 1)$, and therefore that the identification of $P(T^* \in \mathbf{S})$ implies (5). ■

A.1.4 Proof of Theorem 2.2

To establish the result, we first use the law of iterated expectations to show that

$$\begin{aligned} &E[E[\nu_0(T)|Z = 0, X] + E[\nu_1(T)|Z = 1, X]] \\ &\quad = E[\nu_0(T)\frac{(1 - Z)}{P(Z = 0|X)} + \nu_1(T)\frac{Z}{P(Z = 1|X)}] \\ &\quad = \sum_{t \in \mathbf{T}} E[\nu_0(t)\frac{1\{T = t, Z = 0\}}{P(Z = 0|X)} + \nu_1(t)\frac{1\{T = t, Z = 1\}}{P(Z = 1|X)}]. \end{aligned} \tag{55}$$

Moreover, using the fact that response types T^* are independent of Z conditional on X yields

$$\begin{aligned} &\sum_{t \in \mathbf{T}} E[\nu_0(t)\frac{1\{T = t, Z = 0\}}{P(Z = 0|X)} + \nu_1(t)\frac{1\{T = t, Z = 1\}}{P(Z = 1|X)}] \\ &\quad = \sum_{t \in \mathbf{T}} E[\nu_0(t)P(T^*(0) = t|X)\frac{P(Z = 0|X)}{P(Z = 0|X)} + \nu_1(t)P(T^*(1) = t|X)\frac{P(Z = 1|X)}{P(Z = 1|X)}] \\ &\quad = \sum_{t \in \mathbf{T}} E[\nu_0(t)1\{T^*(0) = t\} + \nu_1(t)1\{T^*(1) = t\}]. \end{aligned} \tag{56}$$

Since $T^* \in \mathbf{M}$ with probability one, ISP, results (55) and (56), and the definition of V imply

$$P(T^* \in \mathbf{S}) = E[E[V|Z = 1, X] - E[V|Z = 0, X]]. \tag{57}$$

Next, note that for any scalar random variable W satisfying $E[|W|] < \infty$ we have that

$$E[W\tilde{Z}] = E[\frac{WZ}{P(Z = 1|X)} - \frac{W(1 - Z)}{P(Z = 0|X)}] = E[E[W|Z = 1, X] - E[W|Z = 0, X]]. \tag{58}$$

Therefore, the first claim of the theorem follows from (57) and applying (58) with $W = V$. The second claim follows from Lemma 2.1 and results (57) and (58). ■

A.1.5 Proof of Theorem 2.3

To establish part (i) note that for $\nu_0(T) = -1\{T \in \mathbf{T}_\mathtt{i}\}$ and $\nu_1(T) = 1\{T \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{io}\}$,

$$\sum_{t \in \mathbf{T}} \nu_0(t) 1\{t^*(0) = t\} + \nu_1(t) 1\{t^*(1) = t\}$$
$$= \sum_{t \in \mathbf{T}} (-1\{t \in \mathbf{T}_\mathtt{i}\} 1\{t^*(0) = t\} + 1\{t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{io}\} 1\{t^*(1) = t\}) = 1\{t^* \in \mathbf{S}\},$$

where the second equality follows by noting that if $t^* \in \mathbf{M} \setminus \mathbf{S}$ then $t^*(0) = t^*(1) \notin \mathbf{T}_\mathtt{io}$ by definition of $\mathbf{T}_\mathtt{io}$ and IOS, while if $t^* \in \mathbf{S}$ then $1\{t^*(1) = t\} = 1$ for some $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{io}$ and $1\{t^*(0) = t\} = 0$ for all $t \in \mathbf{T}_\mathtt{i}$. Hence, IOS implies ISP as claimed.

To establish part (ii) of the theorem, first note that by definition of $\tilde{Z}$ we have that

$$\tilde{Z}(1\{T \in \mathbf{T}_\mathtt{io}, Z = 1\} P(Z = 1|X) - 1\{T \in \mathbf{T}_\mathtt{io}, Z = 0\} P(Z = 0|X))$$
$$= 1\{T \in \mathbf{T}_\mathtt{io}, Z = 1\} + 1\{T \in \mathbf{T}_\mathtt{io}, Z = 0\} = 1\{T \in \mathbf{T}_\mathtt{io}\}. \quad (59)$$

Also note that $\mathbf{T} = \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o} + \mathbf{T}_\mathtt{io}$ because $\mathbf{T}_\mathtt{s}$ is empty by hypothesis. Therefore, since $V = 1\{T \in \mathbf{T}_\mathtt{i}\} + 1\{\mathbf{T} \in \mathbf{T}_\mathtt{io}, Z = 1\}$ it follows that $1 - V = 1\{T \in \mathbf{T}_\mathtt{o}\} + 1\{T \in \mathbf{T}_\mathtt{io}, Z = 0\}$. Hence, the definition of $\kappa^\star$ (see (18)) and result (59) yield that

$$\kappa^\star(T, Z, X) = V \tilde{Z} P(Z = 1|X) - (1 - V) \tilde{Z} P(Z = 0|X). \quad (60)$$

Part (ii) of the theorem then follows from (60) and the definition of $\tilde{Z}$.

Part (iii) of the theorem follows from the more general result in Theorem A.1. Finally, to establish part (iv) of the theorem, note that Theorem A.1 implies that

$$(\alpha_0, \alpha_1) \equiv \arg \min_{(a_0, a_1)} E[(Y - a_0 \frac{h_0(X)}{E[h_0(X)|T^* \in \mathbf{S}]} - a_1 \frac{V h_1(X)}{E[h_1(X)|T^* \in \mathbf{S}]})^2 \kappa^\star(T, Z, X)]$$
$$= E[(Y - a_0 \frac{h_0(X)}{E[h_0(X)|T^* \in \mathbf{S}]} - a_1 \frac{V h_1(X)}{E[h_1(X)|T^* \in \mathbf{S}]})^2 1\{T^* \in \mathbf{S}\}].$$

Next, note that $1\{T^*(Z) \in \mathbf{T}_\mathtt{i}, T^* \in \mathbf{S}\} = 1\{T^*(1) \in \mathbf{T}_\mathtt{i}, T^* \in \mathbf{S}\} Z$ by definition of $\mathbf{T}_\mathtt{i}$. Therefore, using that $V = 1\{T \in \mathbf{T}_\mathtt{i}\} + 1\{T \in \mathbf{T}_\mathtt{io}, Z = 1\}$ we are able to conclude that

$$V 1\{T^* \in \mathbf{S}\} = 1\{T^*(1) \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{io}\} Z 1\{T^* \in \mathbf{S}\} = Z 1\{T^* \in \mathbf{S}\}. \quad (61)$$

Therefore, result (61) and Z being independent of T^* conditional on X yields that

$$E[V h_0(X) h_1(X) 1\{T^* \in \mathbf{S}\}] = E[\frac{1\{T^* \in \mathbf{S}\}}{P^2(Z = 0|X)}]. \quad (62)$$

Combining result (62) with the law of iterated expectations then enables us to conclude

$$\frac{E[V h_0(X) h_1(X) 1\{T^* \in \mathbf{S}\}] / E[h_1(X)|T^* \in \mathbf{S}] E[h_0(X)|T^* \in \mathbf{S}])}{E[h_0^2(X) 1\{T^* \in \mathbf{S}\}] / (E[h_0(X)|T^* \in \mathbf{S}])^2} = \frac{E[h_0(X)|T^* \in \mathbf{S}]}{E[h_1(X)|T^* \in \mathbf{S}]} \quad (63)$$

Thus, the Frisch-Waugh-Lovell theorem yields that α_1 is the minimizer to the problem

$$\begin{aligned}\alpha_1 &= \arg\min_{a_1} E[(Y - a_1(\frac{Vh_1(X)}{E[h_1(X)|T^* \in \mathbf{S}]} - \frac{h_0(X)}{E[h_1(X)|T^* \in \mathbf{S}]}))^2 1\{T^* \in \mathbf{S}\}] \\ &= \arg\min_{a_1} E[(Y - \frac{a_1}{E[h_1(X)|T^* \in \mathbf{S}]}(Vh_1(X) - h_0(X)))^2 1\{T^* \in \mathbf{S}\}].\end{aligned} \tag{64}$$

Next observe that result (61), Z being independent of (Y^*, T^*) conditional on X, the definitions of h_0 and h_1, and the law of iterated expectations imply

$$\begin{aligned}&E[Y(Vh_1(X) - h_0(X))1\{T^* \in \mathbf{S}\}] \\ &\quad = E[(YZ + Y(1-Z))(Zh_1(X) - h_0(X))1\{T^* \in \mathbf{S}\}] \\ &\quad = E[(Y^*(T^*(1))Z(h_1(X) - h_0(X)) - Y^*(T^*(0))(1-Z)h_0(X))1\{T^* \in \mathbf{S}\}] \\ &\quad = E[(Y^*(T^*(1)) - Y^*(T^*(0)))1\{T^* \in \mathbf{S}\}].\end{aligned} \tag{65}$$

Moreover, again using result (61) and the law of iterated expectations further implies

$$\begin{aligned}&E[(Vh_1(X) - h_0(X))^2 1\{T^* \in \mathbf{S}\}] \\ &\quad = E[(Zh_1^2(X) - Zh_1(X)h_0(X) + h_0^2(X))1\{T^* \in \mathbf{S}\}] = E[h_1(X)1\{T^* \in \mathbf{S}\}].\end{aligned} \tag{66}$$

Therefore, by combining results (64), (65), and (66) we are able to conclude that

$$\begin{aligned}\alpha_1 &= \frac{E[Y(Vh_1(X) - h_0(X))1\{T^* \in \mathbf{S}\}]/E[h_1(X)|T^* \in \mathbf{S}]}{E[(Vh_1(X) - h_0(X))^2 1\{T^* \in \mathbf{S}\}]/(E[h_1(X)|T^* \in \mathbf{S}])^2} \\ &= \frac{E[(Y^*(T^*(1)) - Y^*(T^*(0)))1\{T^* \in \mathbf{S}\}]E[h_1(X)|T^* \in \mathbf{S}]}{E[h_1(X)1\{T^* \in \mathbf{S}\}]} \\ &= \tau(\mathbf{S}),\end{aligned} \tag{67}$$

which establishes the final claim of the theorem. ■

A.1.6 Proof of Lemma 2.2

The claim for $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$ follows from Lemma A.3 and the law of iterated expectations. The claim for $t \in \mathbf{T}_\mathtt{io}$ follows by noting that if $t \in \mathbf{T}_\mathtt{io}$, then $1\{Z = 1\}1\{T = t\} = 1\{Z = 1\}1\{T^* \in \mathtt{in(t)}\}$ and $1\{Z = 0\}1\{T = t\} = 1\{T^* \in \mathtt{out(t)}\}$. Therefore, we have

$$\begin{aligned}E[1\{T = t\}|Z = 1, X] &= E[1\{T^* \in \mathtt{in(t)}\}|Z = 1, X] = P(T* \in \mathtt{in(t)}\,|X) \\ E[1\{T = t\}|Z = 0, X] &= E[1\{T^* \in \mathtt{out(t)}\}|Z = 0, X] = P(T* \in \mathtt{out(t)}\,|X)\end{aligned}$$

because T^* is independent of Z conditional on X by Assumption 1.1(iii). Marginalization over the distribution of X proceeds as in the end of the proof of Theorem 2.2. ■

A.1.7 Proof of Theorem 2.4

We first show that if the CCEF is identified, then IOS holds. We establish this direction by showing the converse: If IOS fails, then the CCEF is not identified. To this end, suppose IOS

fails so there exists a $t_0 \in \mathbf{T}$ such that $\mathtt{in}(t_0)$, $\mathtt{out}(t_0)$, and $\mathtt{stay}(t_0)$ are not empty. Let

$$\begin{aligned}\gamma_{\mathtt{i}}(X) &\equiv 1/P(T^* \in \mathtt{in}(t_0)|X)\\ \gamma_{\mathtt{o}}(X) &\equiv P(T^* \in \mathtt{in}(t_0)|X)/P(T^* \in \mathtt{out}(t_0)|X)\\ \gamma_{\mathtt{s}}(X) &\equiv -P(T^* \in \mathtt{in}(t_0)|X)/P(T^* \in \mathtt{stay}(t_0)|X)\end{aligned}$$

and note that $\gamma_{\mathtt{i}}$, $\gamma_{\mathtt{o}}$, and $\gamma_{\mathtt{s}}$ are all bounded by $1/\varepsilon$ because $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$ and $\mathtt{in}(t_0)$, $\mathtt{out}(t_0)$, and $\mathtt{stay}(t_0)$ are not empty by hypothesis. Further set

$$f(T^*, X) \equiv 1\{T^* \in \mathtt{in}(t_0)\}\gamma_{\mathtt{i}}(X) + 1\{T^* \in \mathtt{out}(t_0)\}\gamma_{\mathtt{o}}(X) + 1\{T^* \in \mathtt{stay}(t_0)\}\gamma_{\mathtt{s}}(X) \tag{68}$$

and note that $1\{T^*(0) = t_0\} = 1\{T^* \in \mathtt{stay}(t_0) \cup \mathtt{out}(t_0)\}$, Assumption 1.1(iii), and (68) imply

$$\begin{aligned}&E_P[f(T^*, X)1\{T^*(Z) = t_0\}|X, Z = 0]\\ &\qquad = E_P[1\{T^* \in \mathtt{out}(t_0)\}\gamma_{\mathtt{o}}(X) + 1\{T^* \in \mathtt{stay}(t_0)\}\gamma_{\mathtt{s}}(X)|X] = 0,\end{aligned} \tag{69}$$

where the subscript P in E_P is meant to indicate that the expectation is taken with respect to the distribution P of (Y^*, T^*, Z, X) generating the data. Moreover, by similar arguments

$$E_P[f(T^*, X)1\{T^*(Z) = t_0\}|X, Z = 1] = 0.$$

For $\tilde{P} \equiv P_{Y^*(t_1)|X} \ldots P_{Y^*(t_d)|X} P_{T^*ZX}$ and $d\tilde{P}/dP$ its density with respect to P, next set

$$s(Y^*, T^*, X) \equiv \frac{Y^*(t_0) - E_P[Y^*(t_0)|X]}{\mathrm{Var}_P\{Y^*(t_0)|X\}} \times f(T^*, X) \times \frac{d\tilde{P}}{dP}(Y^*, T^*, Z, X)$$

and note that s is bounded because f is bounded, and $Y^*(t_0)$, $1/\mathrm{Var}\{Y^*(t_0)|X\}$ and $d\tilde{P}/dP$ are bounded. Moreover, for any $t \neq t_0$ and integrable function g of (Y, Z, X) we have

$$\begin{aligned}&E_P[s(Y^*, T^*, X)g(Y, Z, X)1\{T = t\}]\\ &\quad = E_P[s(Y^*, T^*, X)g(Y^*(t), Z, X)1\{T^*(Z) = t\}]\\ &\quad = E_{\tilde{P}}[E_P[\frac{Y^*(t_0) - E_P[Y^*(t_0)|X]}{\mathrm{Var}_P\{Y^*(t_0)|X\}}|X]g(Y^*(t), Z, X)f(T^*, X)1\{T^*(Z) = t\}]\\ &\quad = 0,\end{aligned} \tag{70}$$

where the second equality holds by definition of s, $(Y^*(t), Z, T^*)$ being independent of $Y^*(t_0)$ conditional on X under $\tilde{P}$, and the joint distribution of $(Y^*(t_0), X)$ being the same under $\tilde{P}$ and P. Moreover, by similar arguments and results (68) and (69) we can conclude that

$$\begin{aligned}&E_P[s(Y^*, T^*, X)g(Y, Z, X)1\{T = t_0\}]\\ &\quad = E_{\tilde{P}}[\frac{Y^*(t_0) - E_P[Y^*(t_0)|X]}{\mathrm{Var}_P\{Y^*(t_0)|X\}}g(Y^*(t_0), Z, X)E_P[f(T^*, X)1\{T^*(Z) = t_0\}|Z, X]]\\ &\quad = 0.\end{aligned} \tag{71}$$

Hence, since the function g was arbitrary, results (70) and (71) allow us to conclude that

$$E_P[s(Y^*, T^*, X)|Y, T, Z, X] = 0.$$

By Lemma A.2, it therefore follows that for some $\eta > 0$ the measure P' defined by

$$P'((Y^*, T^*, Z, X) \in A) \equiv P((Y^*, T^*, Z, X) \in A) + \eta E_P[s(Y^*, T^*, X) 1\{(Y^*, T^*, Z, X) \in A\}] \quad (72)$$

(for any A in the domain of P) satisfies Assumptions 1.1(iii), 1.2(i), and is observationally equivalent to P. Moreover, $Y^*(t)$ being bounded under P, $P' \ll P$, and P' being observationally equivalent to P implying $P(Z = 1|X) = P'(Z = 1|X)$, allow us to conclude that P' also satisfies Assumption 1.1(iv). Further note that by definition of s we also have

$$E_P[s(Y^*, T^*, X) 1\{T^* \in \mathbf{S}\}] = E_{\tilde{P}}[E_P[\frac{Y^*(t_0) - E_P[Y^*(t_0)|X]}{\mathrm{Var}_P\{Y^*(t_0)|X\}}|X] f(T^*, X) 1\{T^* \in \mathbf{S}\}] = 0,$$

which together with (72) implies $P(T^* \in \mathbf{S}) = P'(T^* \in \mathbf{S})$ and therefore that P' additionally satisfies Assumption 1.2(ii). We next show that P' also satisfies PSI. To this end, note that for any $t \neq t_0$, $t^* \in \mathbf{M}$, and integrable functions g_0 of $(Y^*(t), X)$ we have

$$E_P[g_0(Y^*(t), X) 1\{T^* = t^*\} s(Y^*, T^*, X)] = E_{\tilde{P}}[E_P[\frac{Y^*(t_0) - E_P[Y^*(t_0)|X]}{\mathrm{Var}_P\{Y^*(t_0)|X\}}|X] g_0(Y^*(t), X) 1\{T^* = t^*\} f(T^*, X)] = 0. \quad (73)$$

Therefore, for any integrable function g_1 of X, $t \neq t_0$, and $t^* \in \mathbf{M}$, we can conclude that

$$\begin{aligned} E_{P'}[Y^*(t) 1\{T^* = t^*\} g_1(X)] &= E_P[Y^*(t) 1\{T^* = t^*\} g_1(X)] \\ &= E_P[E_P[Y^*(t)|T^* = t^*, X] 1\{T^* = t^*\} g_1(X)] \\ &= E_{P'}[E_P[Y^*(t)|T^* = t^*, X] 1\{T^* = t^*\} g_1(X)], \end{aligned} \quad (74)$$

where the first equality follows from the definition of P' and evaluating (73) at $g_0(Y^*(t), X) = Y^*(t) g_1(X)$, the second from the law of iterated expectations, and the third from evaluating (73) at $g_0(Y^*(t), X) = E_P[Y^*(t)|T^* = t^*, X] g_1(X)$. Since (74) holds for any g_1, we obtain

$$E_{P'}[Y^*(t)|T^* = t^*, X] = E_P[Y^*(t)|T^* = t^*, X] \quad (75)$$

for any $t \neq t_0$. On the other hand, for any $t^* \in \mathbf{M}$ and integrable function g_1 of X we have

$$\begin{aligned} &E_P[Y^*(t_0) 1\{T^* = t^*\} g_1(X) s(Y^*, T^*, X)] \\ &\quad = E_{\tilde{P}}[E_P[\frac{Y^*(t_0)(Y^*(t_0) - E_P[Y^*(t_0)|X])}{\mathrm{Var}_P\{Y^*(t_0)|X\}}|X] f(T^*, X) 1\{T^* = t^*\} g_1(X)] \\ &\quad = E_P[f(T^*, X) 1\{T^* = t^*\} g_1(X)], \end{aligned} \quad (76)$$

where the first equality follows by definition of s, and the second by the law of iterated expectations. Hence, the definitions of P' and f, result (76), and P satisfying PSI yield that

$$E_{P'}[Y^*(t_0) 1\{T^* = t^*\} g_1(X)] = \begin{cases} E_P[(E_P[Y^*(t_0)|T^* \in \mathtt{in}(t_0), X] + \eta\gamma_{\mathtt{i}}(X)) 1\{T^* = t^*\} g_1(X)] & \text{if } t^* \in \mathtt{in}(t_0) \\ E_P[(E_P[Y^*(t_0)|T^* \in \mathtt{out}(t_0), X] + \eta\gamma_{\mathtt{o}}(X)) 1\{T^* = t^*\} g_1(X)] & \text{if } t^* \in \mathtt{out}(t_0) \end{cases}. \quad (77)$$

Moreover, setting $g_0(Y^*(t), X) = (E_P[Y^*(t_0)|T^* \in \mathtt{in}(t_0), X] + \gamma_{\mathtt{i}}(X))g_1(X)$ in (73) yields that

$$E_P[(E_P[Y^*(t_0)|T^* \in \mathtt{in}(t_0), X] + \eta\gamma_{\mathtt{i}}(X))1\{T^* = t^*\}g_1(X)] \\ = E_{P'}[(E_P[Y^*(t_0)|T^* \in \mathtt{in}(t_0), X] + \eta\gamma_{\mathtt{i}}(X))1\{T^* = t^*\}g_1(X)], \quad (78)$$

while setting $g_0(Y^*(t), X) = (E_P[Y^*(t_0)|T^* \in \mathtt{out}(t_0), X] + \gamma_{\mathtt{o}}(X))g_1(X)$ in (73) implies that

$$E_P[(E_P[Y^*(t_0)|T^* \in \mathtt{out}(t_0), X] + \eta\gamma_{\mathtt{o}}(X))1\{T^* = t^*\}g_1(X)] \\ = E_{P'}[(E_P[Y^*(t_0)|T^* \in \mathtt{out}(t_0), X] + \eta\gamma_{\mathtt{o}}(X))1\{T^* = t^*\}g_1(X)]. \quad (79)$$

Therefore, it follows from results (77), (78), and (79) holding for any integrable function g_1 that

$$E_{P'}[Y^*(t_0)|T^* = t^*, X] = \begin{cases} E_P[Y^*(t_0)|T^* \in \mathtt{in}(t_0), X] + \eta\gamma_{\mathtt{i}}(X) & \text{if } t^* \in \mathtt{in}(t_0) \\ E_P[Y^*(t_0)|T^* \in \mathtt{out}(t_0), X] + \eta\gamma_{\mathtt{o}}(X) & \text{if } t^* \in \mathtt{out}(t_0) \end{cases}. \quad (80)$$

Thus, we can conclude from (75), (80), and P satisfying PSI that P' satisfies PSI as well. We have thus shown that P' is observationally equivalent to P while satisfying all our maintained assumptions. At the same time, result (80) and $\eta > 0$, $\gamma_{\mathtt{i}}(X) \neq 0$, and $\gamma_{\mathtt{o}}(X) \neq 0$ imply that the CCEF under P and P' are different and therefore that the CCEF is not identified. We thus conclude that if the CCEF is identified, then IOS must hold.

The reverse direction that IOS implies the CCEF is identified follows from Theorem 2.6. ■

A.1.8 Proof of Corollary 2.1

For the first claim, note that $P(T = t, T^* = t^*|X) > 0$ for any $t^* \in \mathbf{S}$ and $t \in \{t^*(0), t^*(1)\}$ by Assumption 1.1(iv) and $P(T^* = t^*|X) > 0$ for all $t^* \in \mathbf{S}$. Moreover, Proposition 1.1 and $1\{T = t, T^* = t^*, T^* \in \mathbf{S}\}$ equaling zero whenever $t^* \notin \mathtt{in(t)} \cup \mathtt{out(t)}$ imply that

$$E[Y|T,T^*,X]1\{T^* \in \mathbf{S}\} \quad (81) \\ = \sum_{t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{o}} \cup \mathbf{T}_{\mathtt{io}}} \sum_{t^* \in \mathtt{in(t)} \cup \mathtt{out(t)}} E[Y^*(t)|T^* = t^*, X]1\{T = t, T^* = t^*\}$$

with probability one, and where we used $\mathtt{in(t)} = \emptyset$ and $\mathtt{out(t)} = \emptyset$ for all $t \in \mathbf{T}_{\mathtt{s}}$. Therefore, if PSI holds, then result (81) implies that with probability one we must have

$$E[Y|T,T^*,X]1\{T^* \in \mathbf{S}\} = \sum_{t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{io}}} E[Y^*(t)|T^* \in \mathtt{in(t)}, X]1\{T = t, T^* \in \mathtt{in(t)}\} \\ + \sum_{t \in \mathbf{T}_{\mathtt{o}} \cup \mathbf{T}_{\mathtt{io}}} E[Y^*(t)|T^* \in \mathtt{out(t)}, X]1\{T = t, T^* \in \mathtt{out(t)}\}.$$

Moreover, using that $1\{T = t, T^* \in \mathtt{in(t)}\} = 1\{T = t, Z = 1, T^* \in \mathbf{S}\}$ and similarly $1\{T = t, T^* \in \mathtt{out(t)}\} = 1\{T = t, Z = 0, T^* \in \mathbf{S}\}$ allows us to conclude that

$$E[Y|T,T^*,X]1\{T^* \in \mathbf{S}\} = \sum_{t \in \mathbf{T}_{\mathtt{i}} \cup \mathbf{T}_{\mathtt{io}}} E[Y^*(t)|T^* \in \mathtt{in(t)}, X]1\{T = t, Z = 1\}1\{T^* \in \mathbf{S}\} \\ + \sum_{t \in \mathbf{T}_{\mathtt{o}} \cup \mathbf{T}_{\mathtt{io}}} E[Y^*(t)|T^* \in \mathtt{out(t)}, X]1\{T = t, Z = 0\}1\{T^* \in \mathbf{S}\}. \quad (82)$$

However, note that for any $t \in \mathbf{T}_{\mathtt{i}}$ we have $\mathtt{out}(\mathrm{t}) = \emptyset$ and therefore $1\{T = t, Z = 0, T^* \in \mathbf{S}\} = 0$. Similarly, for any $t \in \mathbf{T}_{\mathtt{o}}$ we have $1\{T = t, Z = 1, T^* \in \mathbf{S}\} = 0$. Therefore setting

$$C_1(t, X) = \begin{cases} E[Y^*(t)|T^* \in \mathtt{in}(\mathrm{t}), X] & \text{if } t \in \mathbf{T}_{\mathtt{i}} \\ E[Y^*(t)|T^* \in \mathtt{out}(\mathrm{t}), X] & \text{if } t \in \mathbf{T}_{\mathtt{o}} \end{cases} \tag{83}$$

and

$$C_2(t, Z, X) = E[Y^*(t)|T^* \in \mathtt{in}(\mathrm{t}), X]1\{Z = 1\} + E[Y^*(t)|T^* \in \mathtt{out}(\mathrm{t}), X]1\{Z = 0\} \tag{84}$$

for any $t \in \mathbf{T}_{\mathtt{io}}$ establishes the first claim of the corollary. The second claim of the corollary is immediate from the first part and definitions in (83) and (84). ■

A.1.9 Proof of Theorem 2.5

The result follows by setting $g(Y, T, Z, X) = (Y - F(T, Z, X; b))^2$ in Theorem A.1 and noting that $h = g$ in the notation of that theorem due to (23). ■

A.1.10 Proof of Theorem 2.6

We first derive some results that are helpful for establishing both parts (i) and (ii). Note that for any $t \in \mathbf{T}_{\mathtt{i}}$ we have from Lemma A.3 and the law of iterated expectations that

$$E[f(Y^*(t))1\{T^* \in \mathtt{in}(\mathrm{t})\}|X] = E[f(Y)1\{T = t\}\tilde{Z}|X]. \tag{85}$$

Similarly, for any $t \in \mathbf{T}_{\mathtt{o}}$, Lemma A.3 and the law of iterated expectations imply that

$$E[f(Y^*(t))1\{T^* \in \mathtt{out}(\mathrm{t})\}|X] = -E[f(Y)1\{T = t\}\tilde{Z}|X]. \tag{86}$$

Finally, Assumption 1.1(iii), and $1\{T = t, Z = 1\} = 1\{Z = 1, T^* \in \mathtt{in}(\mathrm{t})\}$ and $1\{T = t, Z = 0\} = 1\{Z = 0, T^* \in \mathtt{out}(\mathrm{t})\}$ for any $t \in \mathbf{T}_{\mathtt{io}}$ imply for any function f that

$$\begin{aligned} E[f(Y^*(t))1\{T^* \in \mathtt{in}(\mathrm{t})\}|X] &= E[f(Y)\frac{1\{T = t, Z = 1\}}{P(Z = 1|X)}|X] \\ E[f(Y^*(t))1\{T^* \in \mathtt{out}(\mathrm{t})\}|X] &= E[f(Y)\frac{1\{T = t, Z = 0\}}{P(Z = 0|X)}|X] \end{aligned} \tag{87}$$

for any $t \in \mathbf{T}_{\mathtt{io}}$. Also note $Z(Z - P(Z = 1|X)) = Z(1 - P(Z = 1|X))$ implies that

$$E[f(Y)\frac{1\{T = t, Z = 1\}}{P(Z = 1|X)}|X] = E[f(Y)1\{T = t\}Z\tilde{Z}|X]. \tag{88}$$

Similarly, since $(1 - Z)(Z - P(Z = 1|X)) = (1 - Z)(-P(Z = 1|X))$ we have that

$$E[f(Y)\frac{1\{T = t, Z = 0\}}{P(Z = 0|X)}|X] = -E[f(Y)1\{T = t\}(1 - Z)\tilde{Z}|X]. \tag{89}$$

We can now establish part (i) of the theorem. Note that $P(T^* \in \mathtt{in}(\mathrm{t})\,|X) > 0$ for any $t \in \mathbf{T}_{\mathtt{i}}$ and $P(T^* \in \mathtt{out}(\mathrm{t})\,|X) > 0$ for any $t \in \mathbf{T}_{\mathtt{o}}$ because $P(T^* = t^*|X) > 0$ for any $t^* \in \mathbf{S}$ by

assumption. Therefore, evaluating results (85) and (86) at $f(Y) = Y$ and $f(Y) = 1$ implies

$$\frac{E[\tilde{Z} \times 1\{T = t\}Y|X]}{E[\tilde{Z} \times 1\{T = t\}|X]} = \begin{cases} E[Y^*(t)|T^* \in \texttt{in}(\texttt{t}), X] & \text{if } t \in \mathbf{T}_\texttt{i} \\ E[Y^*(t)|T^* \in \texttt{out}(\texttt{t}), X] & \text{if } t \in \mathbf{T}_\texttt{o} \end{cases} . \tag{90}$$

Similarly, evaluating results (87), (88), and (89) at $f(Y) = Y$ and $f(Y) = 1$ implies that

$$\begin{aligned} E[Y^*(t)|T^* \in \texttt{in}(\texttt{t}), X] &= \frac{E[\tilde{Z} \times 1\{T = t\}ZY|X]}{E[\tilde{Z} \times 1\{T = t\}Z|X]} \\ E[Y^*(t)|T^* \in \texttt{out}(\texttt{t}), X] &= \frac{E[\tilde{Z} \times 1\{T = t\}(1-Z)Y|X]}{E[\tilde{Z} \times 1\{T = t\}(1-Z)|X]} \end{aligned} \tag{91}$$

for any $t \in \mathbf{T}_\texttt{io}$. The first part of the theorem then follows from PSI, (90) and (91).

For the second part of the theorem, we take expectations (over X) of results (85) and (86) evaluated at both $f(Y) = Y$ and $f(Y) = 1$ to obtain that

$$\frac{E[\tilde{Z} \times 1\{T = t\}Y]}{E[\tilde{Z} \times 1\{T = t\}]} = \begin{cases} E[Y^*(t)|T^* \in \texttt{in}(\texttt{t})] & \text{if } t \in \mathbf{T}_\texttt{i} \\ E[Y^*(t)|T^* \in \texttt{out}(\texttt{t})] & \text{if } t \in \mathbf{T}_\texttt{o} \end{cases} . \tag{92}$$

Similarly, evaluating results (87), (88), and (89) at $f(Y) = Y$ and $f(Y) = 1$ implies that

$$\begin{aligned} E[Y^*(t)|T^* \in \texttt{in}(\texttt{t})] &= \frac{E[\tilde{Z} \times 1\{T = t\}ZY]}{E[\tilde{Z} \times 1\{T = t\}Z]} \\ E[Y^*(t)|T^* \in \texttt{out}(\texttt{t})] &= \frac{E[\tilde{Z} \times 1\{T = t\}(1-Z)Y]}{E[\tilde{Z} \times 1\{T = t\}(1-Z)]} \end{aligned} \tag{93}$$

for any $t \in \mathbf{T}_\texttt{io}$. Part (ii) of the theorem then follows from (92), (93), and condition (26). ■

A.1.11 Proof of Theorem 2.7

By Corollary 2.1(i), under PSI the CCEF $E[Y|T^*, T, X]$ for switchers can be written as

$$C(T, Z, X) \equiv C_1(T, X)1\{T \in \mathbf{T}_\texttt{i} \cup \mathbf{T}_\texttt{o}\} + C_2(T, Z, X)1\{T \in \mathbf{T}_\texttt{io}\}. \tag{94}$$

Therefore, since F satisfies (23) by hypothesis, Theorem A.1 allows us to conclude that

$$\begin{aligned} \beta &= \arg\min_b E[(C(T, Z, X) - F(T, Z, X; b))^2 1\{T^* \in \mathbf{S}\}] \\ &= \arg\min_b E[(C(T, Z, X) - F(T, Z, X; b))^2 \kappa^\star(T, Z, X)] . \end{aligned} \tag{95}$$

Using the fact that $\kappa^\star(t, Z, X) = 0$ for any $t \in \mathbf{T}_\texttt{s}$ and the law of iterated expectations yields

$$\begin{aligned} &E[(C(T, Z, X) - F(T, Z, X; b))^2 \kappa^\star(T, Z, X)] \\ &= \sum_{t \in \mathbf{T} \setminus \mathbf{T}_\texttt{s}} \sum_{z \in \{0,1\}} E[(C(t, z, X) - F(t, z, X; b))^2 1\{T = t, Z = z\}\kappa^\star(t, z, X)] \\ &= \sum_{t \in \mathbf{T} \setminus \mathbf{T}_\texttt{s}} \sum_{z \in \{0,1\}} E[(C(t, z, X) - F(t, z, X; b))^2 P(T = t, Z = z|X)\kappa^\star(t, z, X)] . \end{aligned} \tag{96}$$

Note also that for any $t \in \mathbf{T}_\mathtt{i}$ we have that $1\{T^*(1) = t\} = 1\{T^* \in \mathtt{stay}(\mathrm{t}) \cup \mathtt{in}(\mathrm{t})\}$ and $1\{T^*(0) = t\} = 1\{T^* \in \mathtt{stay}(\mathrm{t})\}$. Therefore, Assumption 1.1(iii) implies

$$\begin{aligned} P(T = t|Z = 1,X) - P(T = t|Z = 0, X) \\ = P(T^*(1) = t|X) - P(T^*(0) = t|X) = P(T^* \in \mathtt{in}(\mathrm{t})\,|X) > 0\ . \end{aligned} \tag{97}$$

Likewise, for any treatment level $t \in \mathbf{T}_\mathtt{o}$,

$$P(T = t|Z = 0, X) - P(T = t|Z = 1, X) = P(T^* \in \mathtt{out}(\mathrm{t})\,|X) > 0\ . \tag{98}$$

Hence, definition (94) and F satisfying (23) imply that, for any $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$, we have

$$\begin{aligned} &\sum_{z\in\{0,1\}} E[(C(t,z,X) - F(t,z,X;b))^2 P(T = t, Z = z|X)\kappa^\star(t,z,X)] \\ &\quad = E[(C_1(t,X) - F_1(t,X;b))^2 \sum_{z\in\{0,1\}} P(T = t, Z = z|X)\kappa^\star(t,z,X)] \\ &\quad = E[(C_1(t,X) - F_1(t,X;b))^2 \omega_t(X)]\ , \end{aligned} \tag{99}$$

where the final equality follows from (97) and (98), the definition of $\kappa^\star$, and direct calculation. Similarly, since $\kappa^\star(t,z,X) = 1$ when $t \in \mathbf{T}_\mathtt{io}$, we have that for any $t \in \mathbf{T}_\mathtt{io}$

$$\begin{aligned} &\sum_{z\in\{0,1\}} E[(C(t,z,X) - F(t,z,X;b))^2 P(T = t, Z = z|X)] \\ &\quad = \sum_{z\in\{0,1\}} E[(C_2(t,z,X) - F_2(t,z,X;b))^2 \omega_{t,z}(X)]\ . \end{aligned} \tag{100}$$

Hence, combining (95), (96), (99), and (100), and using $\mathbf{T} \setminus \mathbf{T}_\mathtt{s} = \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o} \cup \mathbf{T}_\mathtt{io}$, yields

$$\begin{aligned} \beta = \arg\min_b \Big\{ &E[\sum_{t\in\mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}} (C_1(t,X) - F_1(t,X;b))^2 \omega_t(X)] \\ &+ E[\sum_{t\in\mathbf{T}_\mathtt{io}} \sum_{z\in\{0,1\}} (C_2(t,z,X) - F_2(t,z,X;b))^2 \omega_{t,z}(X)] \Big\}\ . \end{aligned} \tag{101}$$

Finally, note that Corollary 2.1, the definition of $\tilde{Y}_t$, and Theorem 2.6(i) imply that

$$\begin{aligned} C_1(t,X) = E[Y^*(t)|T^* = t^*, X] = E[\tilde{Y}_t|X] \text{ for any } t \in \mathbf{T}_\mathtt{i},\ t^* \in \mathtt{in}(\mathrm{t}) \\ C_1(t,X) = E[Y^*(t)|T^* = t^*, X] = E[\tilde{Y}_t|X] \text{ for any } t \in \mathbf{T}_\mathtt{o},\ t^* \in \mathtt{out}(\mathrm{t})\ . \end{aligned} \tag{102}$$

Similarly, Corollary 2.1, the definition of $\tilde{Y}_{t,z}$, and Theorem 2.6(i) together imply

$$\begin{aligned} C_2(t,0,X) = E[Y^*(t)|T^* = t^*, X] = E[\tilde{Y}_{t,0}|X] \text{ for any } t \in \mathbf{T}_\mathtt{io},\ t^* \in \mathtt{out}(\mathrm{t}) \\ C_2(t,1,X) = E[Y^*(t)|T^* = t^*, X] = E[\tilde{Y}_{t,1}|X] \text{ for any } t \in \mathbf{T}_\mathtt{io},\ t^* \in \mathtt{in}(\mathrm{t})\,. \end{aligned} \tag{103}$$

The theorem then follows from (101), (102), and (103). ■

A.1.12 Supporting Results

Lemma A.1. *Suppose Assumptions 1.1, 1.2 hold,* $P_{Y^*|T^*X} \ll \{P_{Y|X}\}^d \ll P_{Y^*|T^*X}$, *and that for some* $\varepsilon > 0$, *with probability one, we have* $P(T^* = t^*|X) \geq \varepsilon$ *for all* $t^* \in \mathbf{M}$ *and* $\varepsilon \leq P(Z =$

$1|X) \leq 1-\varepsilon$. *Letting* $U_{\mathbf{M}}$ *be the uniform distribution on* $\mathbf{M}$ *and setting* $\mu \equiv \{P_{Y|X}\}^d U_{\mathbf{M}} P_Z P_X$, *it then follows that* $P_{Y^*T^*ZX} \ll \mu \ll P_{Y^*T^*ZX}$.

Proof. First note that because (Y^*, T^*) is independent of Z conditionally on X by Assumption 1.1(iii) and $T^* \in \mathbf{M}$ with probability one by Assumption 1.2 we obtain that

$$P_{Y^*T^*ZX} = P_{Y^*|T^*ZX} P_{T^*|ZX} P_{ZX} \ll P_{Y^*|T^*X} U_{\mathbf{M}} P_{ZX} \ll \mu, \tag{104}$$

where the final result follows from the definition of μ, $P_{ZX} \ll P_Z P_X$, and $P_{Y^*|T^*X} \ll \{P_{Y|X}\}^d$ by hypothesis. Next, note that $P(Z = z|X) \geq \varepsilon$ for some $\varepsilon > 0$ and all $z \in \{0, 1\}$ implies that $P_Z P_X \ll P_{ZX}$. Therefore, since $\{P_{Y|X}\}^d \ll P_{Y^*|T^*X}$ by assumption and $P(T^* = t^*|X) \geq \varepsilon$ for all $t^* \in \mathbf{M}$ we also obtain that

$$\mu \equiv \{P_{Y|X}\}^d U_{\mathbf{M}} P_Z P_X \ll P_{Y^*|T^*X} P_{T^*|X} P_{ZX} = P_{Y^*T^*ZX}, \tag{105}$$

where the final result follows from (Y^*, T^*) being independent of Z conditionally on X by Assumption 1.1(iii). The lemma therefore follows from (104) and (105). ■

Lemma A.2. *Let Assumptions 1.1, 1.2 hold,* $P_{Y^*|T^*X} \ll \{P_{Y|X}\}^d \ll P_{Y^*|T^*X}$, *and suppose that for some* $\varepsilon > 0$, *with probability one, we have* $P(T^* = t^*|X) \geq \varepsilon$ *for all* $t^* \in \mathbf{M}$ *and* $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$. *If there is a bounded function* s *of* (Y^*, T^*, X) *satisfying*

$$E_P[s(Y^*, T^*, X)|Y, T, Z, X] = 0,$$

then it follows that for some constant $\eta > 0$ *the measure* P' *of* (Y^*, T^*, Z, X) *defined by*

$$P'((Y^*, T^*, Z, X) \in A) \equiv P((Y^*, T^*, Z, X) \in A) + \eta E_P[s(Y^*, T^*, X) 1\{(Y^*, T^*, Z, X) \in A\}]$$

(for any A *in the domain of* P*) satisfies Assumptions 1.1(iii), 1.2(i), and is observationally equivalent to* P.

Proof. First note that Assumption 1.1 implies Assumptions 2.1 and 2.2(i) of Navjeevan et al. (2023), while Lemma A.1 implies we may set $\mu \equiv \{P_{Y|X}\}^d U_{\mathbf{M}} P_Z P_X$ to verify Assumption 2.2(ii) in Navjeevan et al. (2023). Setting $\mathbf{Q} = \mathcal{Q} = L^1(\mu)$ in Assumptions 2.2(iii) of Navjeevan et al. (2023) then implies that Assumptions 2.2(iii) and 4.1(iii) in Navjeevan et al. (2023) are also satisfied. The claim of the lemma therefore follows from Lemma A.3 in Navjeevan et al. (2023) after noting that $\mathcal{S} = L^\infty(\mu)$ and that $Q^{\mathtt{a}}$ can be set to equal P because $\mathcal{Q} = L^1(\mu)$. ■

Theorem A.1. *Suppose Assumptions 1.1, 1.2, and* IOS *hold, and that, for some* $\varepsilon > 0$, $\varepsilon \leq P(Z = 1|X) \leq 1 - \varepsilon$ *with probability one. For any function* g *of* (Y, T, Z, X) *define*

$$h(Y, t, z, X) = \begin{cases} g(Y, t, 1, X) & \text{for any } t \in \mathbf{T}_{\mathtt{i}} \text{ and } z \in \{0, 1\} \\ g(Y, t, 0, X) & \text{for any } t \in \mathbf{T}_{\mathtt{o}} \text{ and } z \in \{0, 1\} \\ g(Y, t, z, X) & \text{otherwise} \end{cases}. \tag{106}$$

If h *satisfies* $E[|h(Y, T, Z, X)|] < \infty$, *then it follows that*

$$E[h(Y, T, Z, X)\kappa^\star(T, Z, X)] = E[g(Y, T, Z, X) 1\{T^* \in \mathbf{S}\}]. \tag{107}$$

Proof. First, note the definition of $\kappa^\star$ in (18) and $P(Z = z|X) \geq \varepsilon > 0$ for $z \in \{0, 1\}$

by hypothesis, imply that $\kappa^\star \in L^\infty(P)$. Hence, it follows that $\kappa^\star h \in L^1(P)$ because $h \in L^1(P)$ by hypothesis. Next, observe that Lemma A.3(i), the law of iterated expectations, and $h(Y^*(t), t, 1, X) = h(Y^*(t), t, 0, X)$ when $t \in \mathbf{T}_\mathtt{i}$ yield

$$\begin{aligned}\sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y,T,Z,X)1\{T=t\}\tilde{Z}P(Z=1|X)] &= \sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y^*(t),t,Z,X)1\{T^*\in \mathtt{in}(t)\}P(Z=1|X)] \\ &= \sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y,T,Z,X)1\{T^*\in \mathtt{in}(t)\}1\{T=t\}],\end{aligned} \tag{108}$$

where the second equality follows from Lemma A.3(ii). Moreover, by similar arguments

$$\begin{aligned}\sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y,T,Z,X)1\{T=t\}(-\tilde{Z})P(Z=0|X)] &= \sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y^*(t),t,Z,X)1\{T^*\in \mathtt{out}(t)\}P(Z=0|X)] \\ &= \sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y,T,Z,X)1\{T^*\in \mathtt{out}(t)\}1\{T=t\}],\end{aligned} \tag{109}$$

where the first and second equalities again rely on Lemmas A.3(i) and A.3(ii). On the other hand, by IOS, any $t \in \mathbf{T}_\mathtt{io}$ satisfies $\mathtt{stay}(t) = \emptyset$, and hence

$$\begin{aligned}1\{T=t\}1\{Z=0\} &= 1\{T=t\}1\{T^*\in\mathtt{out}(t)\} \\ 1\{T=t\}1\{Z=1\} &= 1\{T=t\}1\{T^*\in\mathtt{in}(t)\}\end{aligned} \tag{110}$$

for any $t \in \mathbf{T}_\mathtt{io}$. Therefore, we can conclude from result (110) and $Z \in \{0,1\}$ that

$$\begin{aligned}\sum_{t\in\mathbf{T}_\mathtt{io}} E[h(Y,T,Z,X)1\{T=t\}] &= \sum_{t\in\mathbf{T}_\mathtt{io}} E[h(Y,T,Z,X)1\{T=t\}(1\{Z=0\}+1\{Z=1\})] \\ &= \sum_{t\in\mathbf{T}_\mathtt{io}} E[h(Y,T,Z,X)1\{T=t\}1\{T^*\in\mathtt{out}(t)\cup\mathtt{in}(t)\}].\end{aligned} \tag{111}$$

Next, note that for any $t \in \mathbf{T}_\mathtt{i}$ and $t^* \in \mathbf{S}$, $1\{T^*=t^*\}1\{T=t\} = 1$ implies that $T^* \in \mathtt{in}(t)$ due to $\mathtt{out}(t) = \emptyset$ when $t \in \mathbf{T}_\mathtt{i}$. Therefore, using that $\mathtt{in}(t) \subseteq \mathbf{S}$, we obtain

$$\begin{aligned}\sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y,T,Z,X)1\{T^*\in\mathtt{in}(t)\}1\{T=t\}] &= \sum_{t\in\mathbf{T}_\mathtt{i}}\sum_{t^*\in\mathbf{S}} E[h(Y,T,Z,X)1\{T^*\in\mathtt{in}(t)\}1\{T^*=t^*\}1\{T=t\}] \\ &= \sum_{t\in\mathbf{T}_\mathtt{i}}\sum_{t^*\in\mathbf{S}} E[h(Y,T,Z,X)1\{T^*=t^*\}1\{T=t\}].\end{aligned} \tag{112}$$

Similarly, since for any $t \in \mathbf{T}_\mathtt{o}$ and $t^* \in \mathbf{S}$ it follows that $1\{T^*=t^*\}1\{T=t\} = 1$ implies

$T^* \in \texttt{out}(t)$, we can conclude from the set inclusion $\texttt{out}(t) \subseteq \mathbf{S}$ that

$$\sum_{t \in \mathbf{T_o}} E[h(Y,T,Z,X)1\{T^* \in \texttt{out}(t)\}1\{T=t\}] = \sum_{t \in \mathbf{T_o}} \sum_{t^* \in \mathbf{S}} E[h(Y,T,Z,X)1\{T^*=t^*\}1\{T=t\}]. \tag{113}$$

Furthermore, because $1\{T^*=t^*\}1\{T=t\}=1$ implies $T^* \in \texttt{in}(t) \cup \texttt{out}(t)$ for any $t \in \mathbf{T_{io}}$ and $t^* \in \mathbf{S}$, it also follows from the set inclusion $\texttt{in}(t) \cup \texttt{out}(t) \subseteq \mathbf{S}$ that

$$\sum_{t \in \mathbf{T_{io}}} E[h(Y,T,Z,X)1\{T=t\}1\{T^* \in \texttt{out}(t) \cup \texttt{in}(t)\}] = \sum_{t \in \mathbf{T_{io}}} \sum_{t^* \in \mathbf{S}} E[h(Y,T,Z,X)1\{T^*=t^*\}1\{T=t\}]. \tag{114}$$

To conclude, note $1\{T=t\}1\{T^*=t^*\}=0$ for any $t^* \in \mathbf{S}$ and $t \in \mathbf{T_s}$. Since $\mathbf{T_s}$, $\mathbf{T_i}$, $\mathbf{T_o}$, and $\mathbf{T_{io}}$ form a partition of $\mathbf{T}$, we can combine the definition of $\kappa^\star$ in (18) with results (108), (109), (111), (113), and (114) together with Lemma A.4 to obtain that

$$\begin{aligned} E[h(Y,T,Z,X)\kappa^\star(T,Z,X)] &= \sum_{t \in \mathbf{T}} \sum_{t^* \in \mathbf{S}} E[h(Y,T,Z,X)1\{T^*=t^*\}1\{T=t\}] \\ &= \sum_{t^* \in \mathbf{S}} E[g(Y,T,Z,X)1\{T^*=t^*\}], \end{aligned} \tag{115}$$

which verifies the claim of the theorem. ■

Lemma A.3. *If Assumptions 1.1(i)-(iii) hold, then: (i) The following equalities hold:*

$$\begin{aligned} E[1\{T=t\}\tilde{Z}|Y^*,T^*,X] &= 1\{T^* \in \texttt{in}(t)\} \text{ for all } t \in \mathbf{T_i} \\ -E[1\{T=t\}\tilde{Z}|Y^*,T^*,X] &= 1\{T^* \in \texttt{out}(t)\} \text{ for all } t \in \mathbf{T_o}; \end{aligned}$$

and (ii) For any function h of (Y,T,Z,X) satisfying $h(Y,t,0,X)=h(Y,t,1,X)$ for any $t \in \mathbf{T_i} \cup \mathbf{T_o}$ and $E[|h(Y,T,Z,X)|]<\infty$ it follows that

$$\begin{aligned} &\sum_{t \in \mathbf{T_i}} E[1\{T^* \in \texttt{in}(t)\}(1\{T=t\}h(Y,T,Z,X) - P(Z=1|X)h(Y^*(t),t,Z,X)] = 0 \\ &\sum_{t \in \mathbf{T_o}} E[1\{T^* \in \texttt{out}(t)\}(1\{T=t\}h(Y,T,Z,X) - P(Z=0|X)h(Y^*(t),t,Z,X)] = 0. \end{aligned}$$

Proof. To establish (i), first note $\tilde{Z} \in L^1(P)$ by the law of iterated expectations. Letting $E_{Z|X}$ denote expectations with respect to the conditional distribution of Z given X, it then follows from Corollary A.1 in Navjeevan et al. (2023) that for any $t \in \mathbf{T_i}$

$$\begin{aligned} E[1\{T=t\}\tilde{Z}|Y^*,T^*,X] &= E_{Z|X}[\frac{Z-P(Z=1|X)}{P(Z=1|X)P(Z=0|X)}1\{T^*(Z)=t\}] \\ &= 1\{T^*(1)=t\} - 1\{T^*(0)=t\} = 1\{T^* \in \texttt{in}(t)\}, \end{aligned} \tag{116}$$

where the first equality holds by definition of $\tilde{Z}$, and the final equality follows from $\texttt{out}(t)=\emptyset$ for any $t \in \mathbf{T_i}$. Moreover, again relying on Corollary A.1 in Navjeevan et al. (2023) and the

definition of $\tilde{Z}$ yields, for any $t \in \mathbf{T}_\mathtt{o}$, that

$$-E[1\{T=t\}\tilde{Z}|Y^*,T^*,X] = E_{Z|X}[\frac{P(Z=1|X)-Z}{P(Z=1|X)P(Z=0|X)}1\{T^*(Z)=t\}] \tag{117}$$
$$= 1\{T^*(0)=t\} - 1\{T^*(1)=t\} = 1\{T^* \in \mathtt{out}(t)\},$$

where in the final equality we used that $\mathtt{in}(t) = \emptyset$ for any $t \in \mathbf{T}_\mathtt{o}$. Therefore, part (i) of the lemma follows from results (116) and (117).

In order to establish part (ii) of the lemma, first note that for any $t \in \mathbf{T}$ we have

$$1\{T=t\}1\{T^* \in \mathtt{in}(t)\} = 1\{Z=1\}1\{T^* \in \mathtt{in}(t)\} \tag{118}$$
$$1\{T=t\}1\{T^* \in \mathtt{out}(t)\} = 1\{Z=0\}1\{T^* \in \mathtt{out}(t)\}$$

by definition of the sets $\mathtt{in}(t)$ and $\mathtt{out}(t)$. Hence, by employing (118) we can conclude

$$\sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y,T,Z,X)1\{T=t\}1\{T^* \in \mathtt{in}(t)\}] \tag{119}$$
$$= \sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y^*(t),t,1,X)1\{T^* \in \mathtt{in}(t)\}1\{Z=1\}]$$
$$= \sum_{t\in\mathbf{T}_\mathtt{i}} E[h(Y^*(t),t,Z,X)1\{T^* \in \mathtt{in}(t)\}P(Z=1|X)],$$

where the second equality follows from the law of iterated expectations, (Y^*,T^*) being independent of Z conditionally on X by Assumption 1.1(iii), and $h(Y^*(t),t,1,X) = h(Y^*(t),t,0,X)$ for any $t \in \mathbf{T}_\mathtt{i}$ by hypothesis. Similarly, we also have

$$\sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y,T,Z,X)1\{T=t\}1\{T^* \in \mathtt{out}(t)\}] \tag{120}$$
$$= \sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y^*(t),t,0,X)1\{T^* \in \mathtt{out}(t)\}1\{Z=0\}]$$
$$= \sum_{t\in\mathbf{T}_\mathtt{o}} E[h(Y^*(t),t,Z,X)1\{T^* \in \mathtt{out}(t)\}P(Z=0|X)]$$

and therefore part (ii) of the lemma follows from results (119) and (120). ■

Lemma A.4. *Let Assumptions 1.1, 1.2 hold. For any function g of (Y,T,Z,X) set*

$$h(Y,t,z,X) = \begin{cases} g(Y,t,1,X) & \text{for any } t \in \mathbf{T}_\mathtt{i} \text{ and } z \in \{0,1\} \\ g(Y,t,0,X) & \text{for any } t \in \mathbf{T}_\mathtt{o} \text{ and } z \in \{0,1\} \\ g(Y,t,z,X) & \text{otherwise} \end{cases}.$$

Then, it follows that g and h satisfy the following equality with probability one

$$\sum_{t^*\in\mathbf{S}} h(Y,T,Z,X)1\{T^*=t^*\} = \sum_{t^*\in\mathbf{S}} g(Y,T,Z,X)1\{T^*=t^*\}.$$

Proof. First note that $\mathbf{T}_\mathtt{s}, \mathbf{T}_\mathtt{io}, \mathbf{T}_\mathtt{i}$, and $\mathbf{T}_\mathtt{o}$ form a partition of the set of possible treatments $\mathbf{T}$.

Therefore, by definition of h we can conclude that

$$\sum_{t^* \in \mathbf{S}} g(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{io}} \cup \mathbf{T}_{\mathtt{s}}, T^* = t^*\} = \sum_{t^* \in \mathbf{S}} h(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{io}} \cup \mathbf{T}_{\mathtt{s}}, T^* = t^*\}. \tag{121}$$

Moreover, note that if $t \in \mathbf{T}_{\mathtt{i}}$, then $\mathtt{out}(\mathrm{t}) = \emptyset$, which implies $1\{T \in \mathbf{T}_{\mathtt{i}}, T^* = t^*\} = 1\{T \in \mathbf{T}_{\mathtt{i}}, T^* = t^*, Z = 1\}$ for any $t^* \in \mathbf{S}$. Hence, we can conclude that

$$\begin{aligned} \sum_{t^* \in \mathbf{S}} g(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{i}}, T^* = t^*\} &= \sum_{t^* \in \mathbf{S}} g(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{i}}, T^* = t^*, Z = 1\} \\ &= \sum_{t^* \in \mathbf{S}} h(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{i}}, T^* = t^*\}, \end{aligned} \tag{122}$$

where the final equality holds by definition of h. By similar arguments we also obtain

$$\sum_{t^* \in \mathbf{S}} g(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{o}}, T^* = t^*\} = \sum_{t^* \in \mathbf{S}} h(Y,T,Z,X) 1\{T \in \mathbf{T}_{\mathtt{o}}, T^* = t^*\}, \tag{123}$$

and therefore the lemma follows from (121), (122), (123), and $\mathbf{T}_{\mathtt{s}}, \mathbf{T}_{\mathtt{io}}, \mathbf{T}_{\mathtt{i}}$, and $\mathbf{T}_{\mathtt{o}}$ forming a partition of $\mathbf{T}$. ■

A.2 Ordered Choice, ISP, and IOS

This appendix shows that latent-index ordered choice models generalizing (15) necessarily satisfy ISP and that these models satisfy IOS under additional, easily-verified restrictions. The auxiliary treatment variable V is defined here as in Theorem 2.2, while ε denotes an arbitrary scalar random variable. Although the lemma below holds regardless, when combined with ISP or IOS in pursuit of causal effects, ε is assumed to be independent of Z so that $T^*(z)$ is independent of Z.

Lemma A.5. *Assume discrete ordered treatment $T \in \mathbf{T} = \{J_l, \ldots, J^u\}$ and that treatment status is determined according to*

$$T^*(z) = \begin{cases} J_l & \text{if } z\pi + \varepsilon \leq c_{J_l+1} \\ J_l + 1 & \text{if } c_{J_l+1} < z\pi + \varepsilon \leq c_{J_l+2} \\ \vdots & \\ J^u & \text{if } c_{J^u} < z\pi + \varepsilon \end{cases}$$

with $\pi > 0$ and $c_{J_l+1} < c_{J_l+2} < \ldots < c_{J^u}$. Also set $c_{J_l} = -\infty$ and $c_{J^u+1} = +\infty$ and define the set $\mathcal{A} \equiv \{J_l \leq t \leq J^u : P(c_t < \varepsilon \leq c_{t+1} - \pi) > 0\}$. Given this choice model:

(i) ISP *holds with functions ν_0 and ν_1 given by $\nu_1(t) = |\{s \in \mathcal{A} : s < t\}|$ and*

$$\nu_0(t) = \begin{cases} 1 - \nu_1(t) & \text{if } t \notin \mathcal{A} \\ -\nu_1(t) & \text{if } t \in \mathcal{A} \end{cases}. \tag{124}$$

(ii) If $c_{t+1} - c_t \leq \pi$ for all $J_l + 1 \leq t \leq J^u - 1$, then IOS *holds. If in addition ε is continuously*

distributed with support including the interval $[(c_{J_l+1} \wedge (c_{J_l+2} - \pi)) - \delta, c_{J^u} + \delta]$ *for some* $\delta > 0$*, then* IOS *holds if and only if* $c_{t+1} - c_t \leq \pi$ *for all* $J_l + 1 \leq t \leq J^u - 1$*, in which case*

$$V = 1\{T \in \mathbf{T}_\mathtt{i}\} + 1\{T \in \mathbf{T}_\mathtt{io}\}Z = 1\{T = J^u\} + 1\{J_l < T < J^u\}Z.$$

(iii) If Assumption 1.1 holds, $\pi \leq c_{t+1} - c_t$ *for all* $J_l + 1 \leq t \leq J^u - 1$ *and the conditional distribution of* ε *given* X *is continuous with support including* $[(c_{J_l+1} - \pi) - \delta, c_{J^u} + \delta]$ *for some* $\delta > 0$ *(with probability one over* X*), then*

$$\tau(\mathbf{S}) = \frac{E(E[Y|Z=1,X] - E[Y|Z=0,X])}{E(E[T|Z=1,X] - E[T|Z=0,X])}.$$

(iv) If ε *is continuously distributed with support including the interval* $[(c_{J_l+1} \wedge (c_{J_l+2} - \pi)) - \delta, (c_{J^u-1} \vee (c_{J^u} - \pi)) + \delta]$ *for some* $\delta > 0$*, then*

$$\mathbf{S} = \{(t, t') \in \mathbf{T} \times \mathbf{T} : c_{t'} - c_{t+1} < \pi < c_{t'+1} - c_t\}$$

and $P(T^* = t^*) > 0$ *for all* $t^* \in \mathbf{S}$*.*

Proof. First, note that for any $J_l \leq t \leq J^u$ it follows that $(T^*(0), T^*(1)) = (t, t)$ if and only if $c_t < \varepsilon$ and $\varepsilon \leq c_{t+1} - \pi$. Therefore, the ordered model is such that $\mathtt{stay}(\mathrm{t}) \neq \emptyset$ if and only if $P(c_t < \varepsilon \leq c_{t+1} - \pi) > 0$. Hence, by definition of $\mathcal{A}$ it follows that

$$\mathcal{A} = \{t \in \mathbf{T} : \mathtt{stay}(\mathrm{t}) \neq \emptyset\}. \tag{125}$$

Also observe that any switching type $T^* \in \mathbf{S}$ with $T^* = (T^*(0), T^*(1))$, corresponds to an ε satisfying $\varepsilon \leq c_{T^*(0)+1}$ and $\varepsilon + \pi > c_{T^*(1)}$ implying $\pi > c_{T^*(1)} - c_{T^*(0)+1}$. Since $c_{J_l+1} < c_{J_l+2} < \ldots < c_{J^u}$, the definition of $\mathcal{A}$ implies that for any switcher $T^* \in \mathbf{S}$ we have

$$\{s \in \mathcal{A} : T^*(0) < s < T^*(1)\} = \emptyset. \tag{126}$$

To verify ISP here, consider $T^* \notin \mathbf{S}$. Any such T^* satisfies $(T^*(0), T^*(1)) = (t, t)$ for some $t \in \mathcal{A}$ by result (125). Therefore, the definition of ν_0 in (124) implies

$$\nu_0(T^*(0)) + \nu_1(T^*(1)) = 0 \text{ for any } T^* \notin \mathbf{S}. \tag{127}$$

Next, consider any $T^* \in \mathbf{S}$, which implies $T^*(0) < T^*(1)$. If $T^*(0) \notin \mathcal{A}$, then (126) and the definition of ν_1 imply $\nu_1(T^*(0)) = \nu_1(T^*(1))$. Since $\nu_0(T^*(0)) = 1 - \nu_1(T^*(0))$ because $T^*(0) \notin \mathcal{A}$, it follows that $\nu_0(T^*(0)) + \nu_1(T^*(1)) = 1$. On the other hand, if $T^*(0) \in \mathcal{A}$, then $\nu_1(T^*(1)) = \nu_1(T^*(0)) + 1$ by (126) and the definition of ν_1. Since $\nu_0(T^*(0)) = -\nu_1(T^*(0))$ because $T^*(0) \in \mathcal{A}$, we have $\nu_0(T^*(0)) + \nu_1(T^*(1)) = 1$ and

$$\nu_0(T^*(0)) + \nu_1(T^*(1)) = 1 \text{ for any } T^* \in \mathbf{S}, \tag{128}$$

Together, results (127) and (128) verify ISP holds and hence part (i) follows.

To establish part (ii), note that $\pi > 0$ implies $\mathtt{in}(J_l) = \emptyset$ and $\mathtt{out}(J^u) = \emptyset$. Therefore, IOS holds if and only if for every $J_l < t < J^u$ at least one of $\mathtt{in}(\mathrm{t})$, $\mathtt{out}(\mathrm{t})$, or $\mathtt{stay}(\mathrm{t})$ is empty. However, $c_{t+1} - c_t \leq \pi$ for all $J_l + 1 \leq t \leq J^u - 1$ implies $P(c_t < \varepsilon \leq c_{t+1} - \pi) = 0$ for all $J_l + 1 \leq t \leq J^u - 1$. Therefore, result (125) implies $\mathtt{stay}(\mathrm{t}) = \emptyset$ for all $J_l + 1 \leq t \leq J^u - 1$, and

therefore IOS holds.

Now, suppose ε is continuously distributed and its support includes $[(c_{J_l+1} \wedge (c_{J_l+2} - \pi)) - \delta, c_{J^u} + \delta]$ for some $\delta > 0$. For any $J_l + 1 \leq t \leq J^u - 1$, then note that a $T^* \in \texttt{out}(\mathrm{t})$ corresponds to an ε satisfying $c_t < \varepsilon \leq c_{t+1}$ and $\varepsilon > c_{t+1} - \pi$. Hence, since

$$P(c_t \vee (c_{t+1} - \pi) < \varepsilon \leq c_{t+1}) > 0 \tag{129}$$

for all $J_l + 1 \leq t \leq J^u - 1$ due to our assumption on the support of ε, we obtain that $\texttt{out}(\mathrm{t}) \neq \emptyset$ for all $J_l + 1 \leq t \leq J^u - 1$. Similarly, since any $T^* \in \texttt{in}(\mathrm{t})$ corresponds to an ε satisfying $c_t < \varepsilon + \pi \leq c_{t+1}$ and $\varepsilon \leq c_t$, and our assumptions on ε imply

$$P(c_t - \pi < \varepsilon \leq c_t \wedge (c_{t+1} - \pi)) > 0 \tag{130}$$

for any $J_l + 1 \leq t \leq J^u - 1$, it follows that $\texttt{in}(\mathrm{t}) \neq \emptyset$ for any $J_l + 1 \leq t \leq J^u - 1$. Therefore, IOS holds if and only if $\texttt{stay}(\mathrm{t}) = \emptyset$ for all $J_l + 1 \leq t \leq J^u - 1$. By result (125) and our assumptions on ε, this condition is equivalent to $c_{t+1} - c_t \leq \pi$ for all $J_l + 1 \leq t \leq J^u - 1$, and therefore the lemma follows.

For part (iii), note that the fact that ε is continuously distributed with support including $[(c_{J_l+1} - \pi) - \delta, c_{J^u} + \delta]$ for some $\delta > 0$ implies that $\mathcal{A} = \{J_l, \ldots, J^u\}$. By part (i), therefore, $\nu_1(t) = t - J_l$ and $\nu_0(t) = -t + J_l$. Equivalently, $\tilde{\nu}_1(t) = -\tilde{\nu}_0(t) = t$ also satisfy ISP, so we can drop the additive shift, J_l. Part (iii) then follows from Theorem 2.2.

Finally, to establish part (iv) of the lemma, consider type $t^* = (t, t') \in \mathbf{T} \times \mathbf{T}$ with $t \neq t'$. This type has ε satisfying $c_t < \varepsilon \leq c_{t+1}$ and $c_{t'} < \pi + \varepsilon \leq c_{t'+1}$. Under the support condition in the statement of part (iv), $\max\{c_t, c_{t'} - \pi\} < \min\{c_{t'+1} - \pi, c_{t+1}\}$ if, and only if, $P(T^* = t^*) > 0$. Given $c_{J_l+1} < \cdots < c_{J^u}$, $\max\{c_t, c_{t'} - \pi\} < \min\{c_{t'+1} - \pi, c_{t+1}\}$ if, and only if, $c_t < c_{t'+1} - \pi$ and $c_{t'} - \pi < c_{t+1}$. ■

A.3 Estimation and Inference

Section 3 discusses estimates of SLATE and the CCEF constructed using the IV estimands in Theorems 2.2 and 2.6, respectively, $\kappa^\star$-based estimation of SLATE, as well as the CCEF weighted least squares estimand in Theorem 2.7. These are two-step estimates in which the first step is to estimate a (possibly parametric) model of the conditional distribution of (T, Z) (or Z alone) given X. Although the causal estimands of interest here are novel, the properties of the estimators they generate follow from standard results on the distribution of generalized method of moments (GMM) estimators. These properties are discussed below.

SLATE Estimators Using Theorems 2.2 and 2.3

The IV estimand for SLATE in Theorem 2.2 is

$$\tau(\mathbf{S}) = \frac{E[\tilde{Z}Y]}{E[\tilde{Z}V]}.$$

As for the estimates in Section 3, the propensity score is assumed to be estimated by a parametric model like probit or logit. This model is written $P_\alpha(Z = 1|X)$ to denote dependence on

parameter vector α. Estimates of α, denoted $\hat{\alpha}$, may be obtained by maximum likelihood, non-linear least squares, or covariate balancing as described in Imai and Ratkovic (2014).

Given a first-step estimate of $\hat{\alpha}$, and defining

$$R_i(\hat{\alpha}) \equiv \frac{Z_i - P_{\hat{\alpha}}(Z=1|X_i)}{P_{\hat{\alpha}}(Z=1|X_i)(1-P_{\hat{\alpha}}(Z=1|X_i))},$$

$\tilde{Z}$ is consistently estimated by

$$\tilde{Z}_i(\hat{\alpha}) \equiv R_i(\hat{\alpha}) - \frac{1}{N}\sum_{j=1}^{N} R_j(\hat{\alpha}). \tag{131}$$

When the sample mean of $R_i(\hat{\alpha})$ is zero, $\tilde{Z}_i(\hat{\alpha})$ equals $R_i(\hat{\alpha})$ and second-step estimates are invariant to additive shifts in Y or V.[22]

Setting $V_i \equiv \nu_1(T_i)Z_i - \nu_0(T_i)(1-Z_i)$ generates second-step estimators of SLATE and the probability of switching that can be written:

$$\hat{\tau}(\mathbf{S}) \equiv \frac{\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})Y_i}{\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})V_i} \qquad \text{and} \qquad \hat{P}(T^* \in \mathbf{S}) \equiv \frac{1}{N}\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})V_i. \tag{132}$$

The asymptotic properties of these estimators follow from the fact that they have a GMM representation. Specifically, $\hat{\tau}(\mathbf{S})$ is the value of b_1 that solves

$$\frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} Y_i - b_0 - b_1 V_i \\ (Y_i - b_0 - b_1 V_i)R_i(\hat{\alpha}) \end{pmatrix} = 0. \tag{133}$$

When $\hat{\alpha}$ also has a GMM representation, we can account for first-step estimation of $\hat{\alpha}$ by adding the relevant moment conditions to (133). Similarly, the value of b_1 that solves

$$\frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} R_i(\hat{\alpha}) - b_0 \\ V_i(R_i(\hat{\alpha}) - b_0) - b_1 \end{pmatrix} = 0, \tag{134}$$

is a GMM estimator of $\hat{P}(T^* \in \mathbf{S})$. The GMM framework easily accommodates clustered data, as in our Head Start application.

Under IOS, Theorem 2.3(iv) identifies SLATE as the solution to a weighted least squares problem, with weights equal to $\kappa^\star$. Estimated SLATE in Panel B of Table 1 uses $\kappa^\star$ constructed by plugging (131) into (18). The corresponding switcher probability is estimated as the sample mean of $\kappa^\star$. The terms $E[h_0(X)|T^* \in \mathbf{S}]$ and $E[h_1(X)|T^* \in \mathbf{S}]$ in Theorem 2.3(iv) are then obtained by applying Theorem 2.3(iii). Finally, SLATE is estimated by solving the first order conditions for the sample analog of the minimization problem in Theorem 2.3(iv). Standard errors and p-values are obtained by bootstrapping the procedure sketched in this paragraph.

[22] Maximum likelihood does not impose a sample mean of $R_i(\hat{\alpha})$ equal to zero. Słoczyński et al. (2025) discusses normalization of weighting estimators in the context of basic LATE.

CCEF IV Estimators Using Theorem 2.6(ii)

Under the assumptions of Theorem 2.6(ii), the CCEF depends only on the treatment level and whether $T^* \in \texttt{in}(\text{t})$ or $T^* \in \texttt{out}(\text{t})$. Denote these values of the CCEF by

$$\begin{aligned}
\gamma_t &\equiv E[Y^*(t)|T^* = t^*] && \text{if } t \in \mathbf{T}_{\texttt{i}} \cup \mathbf{T}_{\texttt{o}} \\
\gamma_{t,\texttt{o}} &\equiv E[Y^*(t)|T^* = t^*] && \text{if } t \in \mathbf{T}_{\texttt{io}},\ t^* \in \texttt{out}(\text{t}) \\
\gamma_{t,\texttt{i}} &\equiv E[Y^*(t)|T^* = t^*] && \text{if } t \in \mathbf{T}_{\texttt{io}},\ t^* \in \texttt{in}(\text{t})\,.
\end{aligned}$$

Theorem 2.6(ii) shows that these parameters are identified by the IV estimands:

$$\begin{aligned}
\gamma_t &= \frac{E[\tilde{Z} \times 1\{T = t\}Y]}{E[\tilde{Z} \times 1\{T = t\}]} && \text{for any } t \in \mathbf{T}_{\texttt{i}} \cup \mathbf{T}_{\texttt{o}} \\
\gamma_{t,\texttt{o}} &= \frac{E[\tilde{Z} \times 1\{T = t\}(1 - Z)Y]}{E[\tilde{Z} \times 1\{T = t\}(1 - Z)]} && \text{for any } t \in \mathbf{T}_{\texttt{io}} \\
\gamma_{t,\texttt{i}} &= \frac{E[\tilde{Z} \times 1\{T = t\}ZY]}{E[\tilde{Z} \times 1\{T = t\}Z]} && \text{for any } t \in \mathbf{T}_{\texttt{io}}.
\end{aligned} \tag{135}$$

As with SLATE, the CCEF is estimated using sample analogs of the corresponding IV estimands with $\tilde{Z}_i(\hat{\alpha})$ (as defined in (131)) as instrument. These CCEF estimators are:

$$\begin{aligned}
\hat{\gamma}_t &= \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}Y_i}{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}} && \text{for any } t \in \mathbf{T}_{\texttt{i}} \cup \mathbf{T}_{\texttt{o}} \\
\hat{\gamma}_{t,\texttt{o}} &= \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}(1 - Z_i)Y_i}{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}(1 - Z_i)} && \text{for any } t \in \mathbf{T}_{\texttt{io}} \\
\hat{\gamma}_{t,\texttt{i}} &= \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}Z_iY_i}{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = t\}Z_i} && \text{for any } t \in \mathbf{T}_{\texttt{io}}.
\end{aligned}$$

Similarly, $P(T^* \in \texttt{in}(\text{t}))$ and $P(T^* \in \texttt{out}(\text{t}))$ in Lemma 2.2 are consistently estimated by the corresponding sample analogs of these parameters:

$$\hat{P}(T^* \in \texttt{in}(\text{t})) = \begin{cases} \frac{1}{N}\sum_{i=1}^N 1\{T_i = t\}\tilde{Z}_i(\hat{\alpha}) & \text{if } t \in \mathbf{T}_{\texttt{i}} \\ \frac{1}{N}\sum_{i=1}^N 1\{T_i = t\}Z_i\tilde{Z}_i(\hat{\alpha}) & \text{if } t \in \mathbf{T}_{\texttt{io}} \end{cases} \tag{136}$$

and

$$\hat{P}(T^* \in \texttt{out}(\text{t})) = \begin{cases} -\frac{1}{N}\sum_{i=1}^N 1\{T_i = t\}\tilde{Z}_i(\hat{\alpha}) & \text{if } t \in \mathbf{T}_{\texttt{o}} \\ -\frac{1}{N}\sum_{i=1}^N 1\{T_i = t\}(1 - Z_i)\tilde{Z}_i(\hat{\alpha}) & \text{if } t \in \mathbf{T}_{\texttt{io}}. \end{cases} \tag{137}$$

Limiting distributions again follow from the GMM representation of IV estimation. This includes inference for causal effects, since these are linear combinations of CCEF levels.

CCEF and SLATE are compatible in the sense that SLATE can be obtained as a weighted average of the elements of the CCEF. The next lemma shows this holds both for parameters, estimands, and for the IV estimators detailed in this appendix.

Lemma A.6. SLATE *and* CCEF *parameters and estimators are related as follows.*

(i) Suppose Assumptions 1.1 and 1.2 hold. Then SLATE *and the* CCEF *satisfy:*

$$\tau(\mathbf{S}) = \sum_{t \in \mathbf{T_i} \cup \mathbf{T_{io}}} E[Y^*(t) | T^* \in \mathtt{in}(t)] \times \frac{P(T^* \in \mathtt{in}(t))}{P(T^* \in \mathbf{S})} - \sum_{t \in \mathbf{T_o} \cup \mathbf{T_{io}}} E[Y^*(t) | T^* \in \mathtt{out}(t)] \times \frac{P(T^* \in \mathtt{out}(t))}{P(T^* \in \mathbf{S})}. \quad (138)$$

(ii) Suppose Assumptions 1.1, 1.2 hold, and $E[V\tilde{Z}] \neq 0$. Then (138) holds replacing SLATE *by the corresponding estimand in Theorem 2.2, replacing the* CCEF *by the corresponding estimand in Theorem 2.6(ii), and replacing switcher probabilities by the corresponding estimands in Lemma 2.2. In terms of estimands, we have:*

$$\frac{E[Y\tilde{Z}]}{E[V\tilde{Z}]} = \sum_{t \in \mathbf{T_i}} \frac{E[\tilde{Z}1\{T=t\}Y]}{E[V\tilde{Z}]} + \sum_{t \in \mathbf{T_{io}}} \frac{E[\tilde{Z}1\{T=t\}ZY]}{E[V\tilde{Z}]} + \sum_{t \in \mathbf{T_o}} \frac{E[\tilde{Z}1\{T=t\}Y]}{E[V\tilde{Z}]} + \sum_{t \in \mathbf{T_{io}}} \frac{E[\tilde{Z}1\{T=t\}(1-Z)Y]}{E[V\tilde{Z}]}. \quad (139)$$

(iii) Suppose $\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha}) \neq 0$ and $\sum_{i=1}^N Y_i\tilde{Z}_i(\hat{\alpha})1\{T_i \in \mathbf{T_s}\} = 0$ (for example, when $\mathbf{T_s} = \emptyset$). Then (139) holds for estimators in the form of sample analogs:

$$\frac{\sum_{i=1}^N Y_i\tilde{Z}_i(\hat{\alpha})}{\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha})} = \sum_{t \in \mathbf{T_i}} \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i=t\}Y_i}{\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha})} + \sum_{t \in \mathbf{T_{io}}} \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i=t\}Z_iY_i}{\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha})} + \sum_{t \in \mathbf{T_o}} \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i=t\}Y_i}{\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha})} + \sum_{t \in \mathbf{T_{io}}} \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i=t\}(1-Z_i)Y_i}{\sum_{i=1}^N V_i\tilde{Z}_i(\hat{\alpha})}. \quad (140)$$

Part (i) of this lemma aligns CCEF and SLATE parameters. Part (ii) shows the equality holds for estimands as well as parameters. This result requires neither ISP nor IOS, and thus holds under weaker conditions than the results that justify the estimands. Part (iii) says that our IV estimators of SLATE and the CCEF are similarly compatible under minimal assumptions. This result requires only a non-zero first stage for treatment V and that $\sum_{i=1}^N Y_i\tilde{Z}_i(\hat{\alpha})1\{T_i \in \mathbf{T_s}\} = 0$. The latter condition assuredly holds whenever $\mathbf{T_s}$ is empty.

Some specifications of $\mathbf{M}$ facilitate identification of the distribution of response types T^*. In this case, Lemma A.6(iii) implies that the SLATE estimator equals a weighted average of switcher-specific effects. In the Head Start application, for instance, $\mathbf{T_i} = \{h\}$, $\mathbf{T_o} = \{n, c\}$, and $\mathbf{T_{io}}$ and $\mathbf{T_s}$ are empty. Since $\mathtt{out}(c) = \{(c,h)\}$ and $\mathtt{out}(n) = \{(n,h)\}$, we have

$$P(T^* = (c,h)) = P(T^* \in \mathtt{out}(c)) \qquad P(T^* = (n,h)) = P(T^* \in \mathtt{out}(n)). \quad (141)$$

By Lemma 2.2 the probability of each of these switcher type is identified. Using (137), set

$$\hat{P}(T^* = (c,h)) = -\frac{1}{N}\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = c\} \qquad \hat{P}(T^* = (n,h)) = -\frac{1}{N}\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha})1\{T_i = n\}$$

and note that

$$\begin{aligned}\hat{P}(T^* \in \texttt{in}(h)) &= \frac{1}{N}\sum_{i=1}^{N} \tilde{Z}_i(\hat{\alpha}) 1\{T_i = h\} \\ &= \frac{1}{N}\sum_{i=1}^{N} \tilde{Z}_i(\hat{\alpha})(1 - 1\{T_i = c\} - 1\{T_i = n\}) \\ &= \hat{P}(T^* = (n,h)) + \hat{P}(T^* = (c,h)),\end{aligned}$$

because the mean of $\tilde{Z}_i(\hat{\alpha})$ is zero by construction. Lemma A.6(iii) therefore implies

$$\begin{aligned}\frac{\sum_{i=1}^N Y_i \tilde{Z}_i(\hat{\alpha})}{\sum_{i=1}^N V_i \tilde{Z}_i(\hat{\alpha})} &= \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha}) 1\{T_i = h\} Y_i}{\sum_{i=1}^N V_i \tilde{Z}_i(\hat{\alpha})} + \sum_{t \in \{c,n\}} \frac{\sum_{i=1}^N \tilde{Z}_i(\hat{\alpha}) 1\{T_i = t\} Y_i}{\sum_{i=1}^N V_i \tilde{Z}_i(\hat{\alpha})} \\ &= \hat{\gamma}_h \frac{\hat{P}(T^* \in \texttt{in}(h))}{\hat{P}(T^* \in \mathbf{S})} - (\hat{\gamma}_c \frac{\hat{P}(T^* \in \texttt{out}(c))}{\hat{P}(T^* \in \mathbf{S})} + \hat{\gamma}_n \frac{\hat{P}(T^* \in \texttt{out}(n))}{\hat{P}(T^* \in \mathbf{S})}) \\ &= (\hat{\gamma}_h - \hat{\gamma}_n) \frac{\hat{P}(T^* = (n,h))}{\hat{P}(T^* \in \mathbf{S})} + (\hat{\gamma}_h - \hat{\gamma}_c) \frac{\hat{P}(T^* = (c,h))}{\hat{P}(T^* \in \mathbf{S})}.\end{aligned}$$

This explains why estimated SLATE for Head Start in Section 3.1 equals a weighted average of the corresponding CCEF estimates.

Proof of Lemma A.6. In order to establish part (i), we first equivalently express SLATE as

$$\tau(\mathbf{S}) = \frac{1}{P(T^* \in \mathbf{S})} \times \sum_{t^* \in \mathbf{S}} E[(Y^*(t^*(1)) - Y^*(t^*(0)))1\{T^* = t^*\}]. \tag{142}$$

Since for any response type $t^* \in \mathbf{S}$ we must have $t^*(1) \in \mathbf{T_i} \cup \mathbf{T_{io}}$, it therefore follows that

$$\begin{aligned}&\sum_{t^* \in \mathbf{S}} E[Y^*(t^*(1))1\{T^* = t^*\}] = \sum_{t^* \in \mathbf{S}} \sum_{t \in \mathbf{T_i} \cup \mathbf{T_{io}}} E[Y^*(t)1\{T^* = t^*\}1\{t^*(1) = t\}] \\ &= \sum_{t \in \mathbf{T_i} \cup \mathbf{T_{io}}} E[Y^*(t)(\sum_{t^* \in \mathbf{S}} 1\{T^* = t^*\}1\{t^*(1) = t\})] = \sum_{t \in \mathbf{T_i} \cup \mathbf{T_{io}}} E[Y^*(t)1\{T^* \in \texttt{in(t)}\}],\end{aligned} \tag{143}$$

where the final equality follows by definition of $\texttt{in(t)}$. By similar arguments we have that

$$\sum_{t^* \in \mathbf{S}} E[Y^*(t^*(0))1\{T^* = t^*\}] = \sum_{t \in \mathbf{T_o} \cup \mathbf{T_{io}}} E[Y^*(t)1\{T^* \in \texttt{out(t)}\}] \tag{144}$$

and therefore the first claim of the lemma follows from results (142)-(144).

To establish part (ii) note that since $\mathbf{T_i}$, $\mathbf{T_o}$, $\mathbf{T_{io}}$ and $\mathbf{T_s}$ partition $\mathbf{T}$,

$$\begin{aligned}&\sum_{t \in \mathbf{T_i} \cup \mathbf{T_o}} E[\tilde{Z}1\{T = t\}Y] + \sum_{t \in \mathbf{T_{io}}} (E[\tilde{Z}1\{T = t\}ZY] + E[\tilde{Z}1\{T = t\}(1 - Z)Y]) \\ &= E[\tilde{Z}Y] - \sum_{t \in \mathbf{T_s}} E[\tilde{Z}1\{T = t\}Y] \\ &= E[\tilde{Z}Y] - \sum_{t \in \mathbf{T_s}} E[\tilde{Z}1\{T^* \in \texttt{stay(t)}\}Y^*(t)] \\ &= E[\tilde{Z}Y],\end{aligned} \tag{145}$$

where the second equality follows from $1\{T = t\} = 1\{T^* \in \texttt{stay(t)}\}$, and the third from $E[\tilde{Z}|X] = 0$ and $\tilde{Z}$ being independent of (T^*, Y^*) conditionally on X by Assumption 1.1(iii).

Part (ii) of the lemma therefore follows from (145) and $E[V\tilde{Z}] \neq 0$.

Finally, to establish part (iii) we follow the same manipulations as in (145) to obtain

$$\sum_{i=1}^{N}\Big(\sum_{t\in\mathbf{T_i}\cup\mathbf{T_o}}\tilde{Z}_i(\hat{\alpha})1\{T_i=t\}Y_i+\sum_{t\in\mathbf{T_{io}}}(\tilde{Z}_i(\hat{\alpha})1\{T_i=t\}Z_iY_i+\tilde{Z}_i(\hat{\alpha})1\{T_i=t\}(1-Z_i)Y_i)\Big)$$
$$=\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})Y_i-\sum_{i=1}^{N}\sum_{t\in\mathbf{T_s}}\tilde{Z}_i(\hat{\alpha})Y_i1\{T_i=t\}=\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})Y_i, \quad (146)$$

where the final equality follows from $\sum_{i=1}^{N}Y_i\tilde{Z}_i(\hat{\alpha})1\{T_i\in\mathbf{T_s}\}=0$. The third claim of the lemma therefore follows from (146) and $\sum_{i=1}^{N}\tilde{Z}_i(\hat{\alpha})V_i\neq 0$. ■

CCEF Estimators Based on Theorem 2.7

Theorem 2.7 identifies the CCEF as the solution to a population weighted least squares problem. Distribution theory for estimators based on this result is more involved than for SLATE and CCEF estimates using IV.

Note first that weights and outcomes in the least squares problem here depend on the joint conditional probability $P(T=t,Z=z|X)$ rather than the instrument-assignment propensity score. As with the propensity score, this conditional probability is estimated using a parametric model, $P_\alpha(T=t,Z=z|X)$, that depends on parameter vector α. Again, let $\hat{\alpha}$ denote the corresponding first-step estimator.

First-step estimator $\hat{\alpha}$ is used to construct plug-in estimators $\tilde{Y}_t(\hat{\alpha})$ for $\tilde{Y}_t$, $\omega_t(X;\hat{\alpha})$ for $\omega_t(X)$ (for all $t\in\mathbf{T_i}\cup\mathbf{T_o}$), $\tilde{Y}_{t,z}(\hat{\alpha})$ for $\tilde{Y}_{t,z}$, and $\omega_{t,z}(X;\hat{\alpha})$ for $\omega_{t,z}(X)$ (for all $t\in\mathbf{T_{io}}$ and $z\in\{0,1\}$). Specifically, factoring $P_{\hat{\alpha}}(T=t,Z=z|X)$, we set

$$\tilde{Y}_t(\hat{\alpha})\equiv\frac{(Z-P_{\hat{\alpha}}(Z=1|X))1\{T=t\}Y}{P_{\hat{\alpha}}(Z=1,T=t|X)-P_{\hat{\alpha}}(Z=1|X)P_{\hat{\alpha}}(T=t|X)}$$
$$\tilde{Y}_{t,0}(\hat{\alpha})\equiv\frac{1\{T=t\}(1-Z)Y}{P_{\hat{\alpha}}(T=t,Z=0|X)}$$
$$\tilde{Y}_{t,1}(\hat{\alpha})\equiv\frac{1\{T=t\}ZY}{P_{\hat{\alpha}}(T=t,Z=1|X)}$$

and

$$\omega_t(X;\hat{\alpha})=P_{\hat{\alpha}}(Z=1|X)(P_{\hat{\alpha}}(T=t|Z=1,X)-P_{\hat{\alpha}}(T=t|Z=0,X))_+ \quad \text{if } t\in\mathbf{T_i}$$
$$\omega_t(X;\hat{\alpha})=P_{\hat{\alpha}}(Z=0|X)(P_{\hat{\alpha}}(T=t|Z=0,X)-P_{\hat{\alpha}}(T=t|Z=1,X))_+ \quad \text{if } t\in\mathbf{T_o}$$
$$\omega_{t,z}(X;\hat{\alpha})=P_{\hat{\alpha}}(T=t,Z=z|X) \quad \text{if } t\in\mathbf{T_{io}}.$$

CCEF estimator $\hat{\beta}$ minimizes objective function Q_N over parameter space $\mathcal{B}$, where

$$Q_N(b)\equiv\frac{1}{N}\sum_{i=1}^{N}\Big(\sum_{t\in\mathbf{T_i}\cup\mathbf{T_o}}(\tilde{Y}_{t,i}(\hat{\alpha})-F_1(t,X_i;b))^2\omega_t(X_i;\hat{\alpha})$$
$$+\sum_{t\in\mathbf{T_{io}},z\in\{0,1\}}(\tilde{Y}_{t,z,i}(\hat{\alpha})-F_2(t,z,X_i;b))^2\omega_{t,z}(X_i;\hat{\alpha})\Big).$$

Under conditions stated below, $\hat{\beta}$ is consistent for the β that minimizes Q over $\mathcal{B}$, where

$$Q(b) \equiv E\Big[\sum_{t\in\mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}} (\tilde{Y}_t(\alpha) - F_1(t,X;b))^2\omega_t(X;\alpha) + \sum_{t\in\mathbf{T}_\mathtt{io},z\in\{0,1\}} (\tilde{Y}_{t,z}(\alpha) - F_2(t,z,X;b))^2\omega_{t,z}(X;\alpha)\Big].$$

When the model for the conditional probability $P(T=t,Z=z|X)$ is correctly specified, $\tilde{Y}_t(\alpha)$, $\tilde{Y}_{t,z}(\alpha)$, $\omega_t(X;\alpha)$, and $\omega_{t,z}(X;\alpha)$ equal $\tilde{Y}_t$, $\tilde{Y}_{t,z}$, $\omega_t(X)$, and $\omega_{t,z}(X)$, respectively. In this case, minimizer β of Q minimizes the objective function given in Theorem 2.7.

The asymptotic properties of first-step estimator $\hat{\alpha}$ are derived under standard conditions for GMM estimators. Specifically, let $W \equiv (Y,T,Z,X)$ denote the vector of observed variables, and suppose the first-step estimator $\hat{\alpha}$ is chosen to solve

$$\frac{1}{N}\sum_{i=1}^{N} m_\alpha(W_i,\hat{\alpha}) = 0, \tag{147}$$

with probability approaching one, where $m_\alpha(W,\hat{\alpha})$ is of the same dimension as $\hat{\alpha}$. For any matrix Ω, also let $\|\Omega\|_o$ denote the square root of the largest eigenvalue of $\Omega'\Omega$ (i.e., the largest singular value of Ω). Estimator $\hat{\alpha}$ is then assumed to satisfy:

Assumption A.1. *(i) $\hat{\alpha} \xrightarrow{p} \alpha$; (ii) $E[m_\alpha(W,\alpha)] = 0$ and $E[\|m_\alpha(W,\alpha)\|^2] < \infty$; (iii) With probability approaching one (147) holds; (iv) There is a neighborhood $\mathcal{N}_\alpha$ of α such that $m_\alpha(w,a)$ is differentiable in $a \in \mathcal{N}_\alpha$ and $E[\sup_{a\in\mathcal{N}_\alpha}\|\nabla_a m_\alpha(W,a)\|_o] < \infty$; (v) The matrix $E[\nabla_a m_\alpha(W,\alpha)]$ is invertible.*

Assumption A.1(i) requires $\hat{\alpha}$ to be consistent for α. Assumptions A.1(ii)-(v) are standard regularity conditions that guarantee asymptotic normality of $\hat{\alpha}$.

Consistency of $\hat{\beta}$ follows from Assumption A.1(i) and the following two assumptions:

Assumption A.2. *(i) β is the unique minimizer of Q in $\mathcal{B}$; (ii) $\mathcal{B}$ is compact; (iii) $F_1(t,X;\cdot)$ is continuous and $E[\sup_{b\in\mathcal{B}} F_1^2(t,X;b)] < \infty$ for all $t \in \mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}$; (iv) $F_2(t,z,X;\cdot)$ is continuous and $E[\sup_{b\in\mathcal{B}} F_2^2(t,z,X;b)] < \infty$ for all $t \in \mathbf{T}_\mathtt{io}$, $z \in \{0,1\}$.*

Assumption A.3. *(i) There is a neighborhood $\mathcal{N}_\alpha$ of α and $M < \infty$ with $\omega_t(X;a)$ and $\tilde{Y}_t(a)$ continuous in $a \in \mathcal{N}_\alpha$, $E[\sup_{a\in\mathcal{N}_\alpha}\tilde{Y}_t^2(a)] < \infty$, $P(\sup_{a\in\mathcal{N}_\alpha}|\omega_t(X;a)| \le M) = 1$ for all $t \in \mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}$; (ii) $\omega_{t,z}(X;a)$ and $\tilde{Y}_{t,z}(a)$ are continuous in $a \in \mathcal{N}_\alpha$ and satisfy $E[\sup_{a\in\mathcal{N}_\alpha}\tilde{Y}_{t,z}^2(a)] < \infty$ and $P(\sup_{a\in\mathcal{N}_\alpha}|\omega_{t,z}(X;a)| \le M) = 1$ for all $t \in \mathbf{T}_\mathtt{io}$, $z \in \{0,1\}$.*

Assumption A.3 requires the weights $\omega_t(X;\alpha)$ (for $t \in \mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}$) and $\omega_{t,z}(X;\alpha)$ (for $t \in \mathbf{T}_\mathtt{io}$) to be bounded. This condition is sure to be satisfied in first-step specifications for which $P_\alpha(T=t,Z=z|X)$ belongs to $[0,1]$.

Assumptions yielding asymptotic normality of $\hat{\beta}$ use the notation

$$m_\beta(W,b,a) \equiv \sum_{t\in\mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}} (\tilde{Y}_t(a) - F_1(t,X;b))\omega_t(X;a)\nabla_b F_1(t,X;b) + \sum_{t\in\mathbf{T}_\mathtt{io},z\in\{0,1\}} (\tilde{Y}_{t,z}(a) - F_2(t,z,X;b))\omega_{t,z}(X;a)\nabla_b F_2(t,z,X;b),$$

so that $E[m_\beta(W,b,\alpha)]$ equals the derivative of Q at b. We then require:

Assumption A.4. *(i) There is a neighborhood $\mathcal{N}_\beta$ of β with $\mathcal{N}_\beta \subset \mathcal{B}$; (ii) $F_1(t,X;b)$ is differentiable in $b \in \mathcal{N}_\beta$, $E[\|\nabla_b F_1(t,X;\beta)\|^2] < \infty$ for all $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$; (iii) $F_2(t,z,X;b)$ is differentiable in $b \in \mathcal{N}_\beta$, $E[\|\nabla_b F_2(t,z,X;\beta)\|^2] < \infty$ for all $t \in \mathbf{T}_\mathtt{io}$, $z \in \{0,1\}$; (iv) $m_\beta(W,b,a)$ is differentiable in $b \in \mathcal{N}_\beta$ for all $a \in \mathcal{N}_\alpha$, $E[\sup_{b\in\mathcal{N}_\beta, a\in\mathcal{N}_\alpha} \|\nabla_b m_\beta(W,b,a)\|_o] < \infty$; (v) $E[\|m_\beta(W,\beta,\alpha)\|^2] < \infty$ and $E[\nabla_b m_\beta(W,\beta,\alpha)]$ is invertible.*

Assumption A.5. *(i) $\sup_{b\in\mathcal{N}_\beta} \|m_\beta(W,b,a_1) - m_\beta(W,b,a_2)\| \leq D(W)\|a_1 - a_2\|$ for all $a_1, a_2 \in \mathcal{N}_\alpha$ and D with $E[D^2(W)] < \infty$; (ii) $\tilde{Y}_t(a)$ is differentiable in $a \in \mathcal{N}_\alpha$ for all $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$ and $E[\sup_{a\in\mathcal{N}_\alpha} \|\nabla_b F_1(t,X;\beta)(\nabla_a \tilde{Y}_t(a))'\|_o] < \infty$; (iii) $\tilde{Y}_{t,z}(a)$ is differentiable in $a \in \mathcal{N}_\alpha$ for all $t \in \mathbf{T}_\mathtt{io}$, $z \in \{0,1\}$, and $E[\sup_{a\in\mathcal{N}_\alpha} \|\nabla_b F_2(t,z,X;\beta)(\nabla_a \tilde{Y}_{t,z}(a))'\|_o] < \infty$.*

Assumption A.4 imposes standard conditions under which $\hat{\beta}$ is asymptotically normal when α is known. Assumption A.5 allows us to account for the impact of estimated $\hat{\alpha}$ on the asymptotic distribution of $\hat{\beta}$. Assumption A.5 requires $\tilde{Y}_t(\alpha)$ and $\tilde{Y}_{t,z}(\alpha)$ to be differentiable, but does not impose differentiability on weights $\omega_t(X;\alpha)$ and $\omega_{t,z}(X;\alpha)$.

The contribution of $\hat{\alpha}$ to the influence function of $\hat{\beta}$ depends on

$$G_y \equiv E\Big[\sum_{t\in\mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}} \omega_t(X;\alpha)\nabla_b F_1(t,X;\beta)(\nabla_a \tilde{Y}_t(\alpha))' + \sum_{t\in\mathbf{T}_\mathtt{io}, z\in\{0,1\}} \omega_{t,z}(X;\alpha)\nabla_b F_2(t,z,X;\beta)(\nabla_a \tilde{Y}_{t,z}(\alpha))'\Big]$$

and

$$G_\omega \equiv E\Big[\sum_{t\in\mathbf{T}_\mathtt{i}\cup\mathbf{T}_\mathtt{o}} (\tilde{Y}_t(\alpha) - F_1(t,X;\beta))\nabla_b F_1(t,X;\beta)(\nabla_a \omega_t(X;\alpha))' + \sum_{t\in\mathbf{T}_\mathtt{io}, z\in\{0,1\}} (\tilde{Y}_{t,z}(\alpha) - F_2(t,z,X;\beta))\nabla_b F_2(t,z,X;\beta)(\nabla_a \omega_{t,z}(X;\alpha))'\Big].$$

Define matrices $M_\alpha \equiv E[\nabla_a m_\alpha(W,\alpha)]$ and $M_\beta \equiv E[\nabla_b m_\beta(W,\beta,\alpha)]$. The next result uses this notation to establish the asymptotic normality of $\hat{\beta}$.

Theorem A.2. *Assume A.1, A.2, A.3 , A.4, A.5 hold and let $\{W_i\}_{i=1}^N$ be an i.i.d. sequence of random variables.*

(i) Suppose that $E[\tilde{Y}_t(\alpha)|X] = F_1(t,X;\beta)$ for all $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$ and $E[\tilde{Y}_{t,z}(\alpha)|X] = F_2(t,z,X;\beta)$ for all $t \in \mathbf{T}_\mathtt{io}$ and $z \in \{0,1\}$. Then,

$$\sqrt{N}\{\hat{\beta} - \beta\} \xrightarrow{d} N(0, \mathrm{Var}\{M_\beta^{-1}(m_\beta(W,\beta,\alpha) + G_y M_\alpha^{-1} m_\alpha(W,\alpha))\}).$$

(ii) Suppose that $E[m_\beta(W,\beta,a)]$ is continuously differentiable in $a \in \mathcal{N}_\alpha$. Then,

$$\sqrt{N}\{\hat{\beta} - \beta\} \xrightarrow{d} N(0, \mathrm{Var}\{M_\beta^{-1}(m_\beta(W,\beta,\alpha) + (G_y + G_\omega) M_\alpha^{-1} m_\alpha(W,\alpha))\}).$$

Part (i) of Theorem A.2 derives the asymptotic distribution of $\hat{\beta}$ assuming F matches the conditional mean of $\tilde{Y}_t(\alpha)$ (for $t \in \mathbf{T}_\mathtt{i} \cup \mathbf{T}_\mathtt{o}$) and of $\tilde{Y}_{t,z}(\alpha)$ (for $t \in \mathbf{T}_\mathtt{io}$). When the first step is correctly specified, this implies F matches the CCEF. In this case, the fact that weights

are estimated does not change the asymptotic variance of $\hat{\beta}$ – i.e., weighting by $\omega_t(X;\hat{\alpha})$ is asymptotically equivalent to weighting by $\omega_t(X;\alpha)$.

Part (ii) of Theorem A.2 derives the asymptotic distribution of $\hat{\beta}$ without requiring the CCEF model to be correctly specified. In this case, estimation error in the weights can affect the asymptotic variance of $\hat{\beta}$. To characterize this impact, Theorem A.2(ii) assumes $E[m_\beta(W,\beta,a)]$ is differentiable in a, which requires the weights themselves to be differentiable. The latter condition requires that, for some $\varepsilon > 0$,

$$\begin{aligned} P_\alpha(T=t|Z=1,X) - P_\alpha(T=t|Z=0,X) \geq \varepsilon \qquad & \text{if } t \in \mathbf{T_i} \\ P_\alpha(T=t|Z=0,X) - P_\alpha(T=t|Z=1,X) \geq \varepsilon \qquad & \text{if } t \in \mathbf{T_o}. \end{aligned} \tag{148}$$

If this condition fails, $\hat{\beta}$ remains $\sqrt{N}$-consistent but its asymptotic distribution may not be Gaussian (Fang and Santos, 2019).

Standard errors and confidence intervals for the estimates based on Theorem 2.7 in Table 1 use a bootstrap. We assume the CCEF model is correctly specified, so the asymptotic distribution of $\hat{\beta}$ is the same using either estimated or population weights. We therefore fix weights at the value estimated in the full sample across bootstrap replications. The quantities $\tilde{Y}_t, \tilde{Y}_{t,z}$, and parameter estimates $\hat{\alpha}, \hat{\beta}$ are recomputed in each bootstrap sample. When the CCEF model is potentially misspecified, weights should also be re-estimated in each draw. In this case, condition (148) must be imposed. The consistency of both bootstrap procedures follows from the asymptotic equivalence of our estimator and GMM estimators and the consistency of the bootstrap for the latter (Hall and Horowitz, 1996; Hahn, 1996; Andrews, 2002); see also Gonçalves et al. (2023) for recent results on bootstrap consistency for two-step extremum estimators.

Given estimates $\hat{\beta}^{(b)}$ for bootstrap replications $b = 1,\ldots,B$, standard errors are computed as the square-root of $B^{-1}\sum_{b=1}^{B}(\hat{\beta}_j^{(b)} - \hat{\beta}_j)^2$. Confidence intervals at a $1-\alpha$ level are computed as $[\hat{\beta}_j - q_{1-\alpha/2}, \hat{\beta}_j - q_{\alpha/2}]$, where q_τ is the τ-th quantile of the distribution of $(\hat{\beta}_j^{(b)} - \hat{\beta}_j)_{b=1}^{B}$. Finally, p-values are computed using the bootstrap distribution of the t-statistic centered at $\hat{\beta}_j$.

Proof of Theorem A.2. Lemmas A.7 and A.8 used here are proved at the end of this section. By Lemma A.7, $\hat{\beta} \xrightarrow{p} \beta$. Moreover, Assumption A.1 and standard arguments (e.g., Theorem 3.4 in Newey and McFadden (1994)) imply

$$\sqrt{N}\{\hat{\alpha} - \alpha\} = -\frac{1}{\sqrt{N}}\sum_{i=1}^{N} M_\alpha^{-1} m_\alpha(W_i,\alpha) + o_P(1), \tag{149}$$

where recall that $M_\alpha \equiv E[\nabla_a m_\alpha(W,\alpha)]$. Next, note that the consistency of $\hat{\beta}$, Assumption A.4(i), and Q_N being differentiable by Assumptions A.4(ii)(iii) imply $\hat{\beta}$ satisfies

$$\frac{1}{N}\sum_{i=1}^{N} m_\beta(W_i,\hat{\beta},\hat{\alpha}) = 0. \tag{150}$$

Consistency of $\hat{\alpha}$ (by Assumption A.1(i)) and $\hat{\beta}$, and Lemma A.8 then imply that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N} m_\beta(W_i, \hat{\beta}, \hat{\alpha}) \tag{151}$$
$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} m_\beta(W_i, \beta, \alpha) + \sqrt{N}E[(m_\beta(W, \hat{\beta}, \hat{\alpha}) - m_\beta(W, \beta, \alpha))] + o_P(1).$$

In particular, combining results (150) and (151) and rearranging terms yields that

$$\sqrt{N}E[(m_\beta(W, \beta, \hat{\alpha}) - m_\beta(W, \hat{\beta}, \hat{\alpha}))] \tag{152}$$
$$= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} m_\beta(W_i, \beta, \alpha) + \sqrt{N}E[(m_\beta(W, \beta, \hat{\alpha}) - m_\beta(W, \beta, \alpha))] + o_P(1).$$

Next, note that $E[\|m_\beta(W, \beta, \alpha)\|^2] < \infty$ by Assumption A.4(v) and hence result (149) and Assumption A.1(ii) and Assumption A.5(i) together imply that the right hand side of (152) is $O_P(1)$. Therefore, the consistency of $(\hat{\beta}, \hat{\alpha})$, the mean value theorem, Assumptions A.4(iv)(v), and result (152) allow us to conclude that

$$\sqrt{N}\{\hat{\beta} - \beta\} = -\frac{1}{\sqrt{N}}\sum_{i=1}^{N} M_\beta^{-1}(m_\beta(W_i, \beta, \alpha) + E[(m_\beta(W, \beta, \hat{\alpha}) - m_\beta(W, \beta, \alpha))]) + o_P(1) \tag{153}$$

where recall that $M_\beta \equiv E[\nabla_b m_\beta(W, \beta, \alpha)]$.

To establish the first part of the theorem, note that $E[\tilde{Y}_t(\alpha)|X] = F_1(t, X; \beta)$, $E[\tilde{Y}_{t,z}(\alpha)|X] = F_2(t, z, X; \beta)$, Assumptions A.3 and A.5(ii)(iii), and result (149) imply

$$\sqrt{N}E[(m_\beta(W, \beta, \hat{\alpha}) - m_\beta(W, \beta, \alpha))] = G_y M_\alpha^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} m_\alpha(W_i, \alpha) + o_P(1). \tag{154}$$

Hence, part (i) of the theorem follows from (153), (154), $E[m_\beta(W, \beta, \alpha)] = 0$ by β minimizing Q and Assumption A.4(ii), $E[m_\alpha(W, \alpha)] = 0$ by Assumption A.1(ii), and the central limit theorem. If instead the conditions of part (ii) of the theorem hold, then

$$\sqrt{N}E[(m_\beta(W, \beta, \hat{\alpha}) - m_\beta(W, \beta, \alpha))] = (G_y + G_\omega)M_\alpha^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} m_\alpha(W_i, \alpha) + o_P(1) \tag{155}$$

by result (149) and the delta method (see, e.g., Theorem 3.9.4 in van der Vaart and Wellner (1996)). Part (ii) of the theorem follows from (153), (155), and the central limit theorem. ■

Lemma A.7. *If Assumptions A.1(i), A.2, A.3 hold, and $\{W_i\}_{i=1}^N$ is i.i.d., then $\hat{\beta} \xrightarrow{p} \beta$.*

Proof. We first define a finite sample criterion function $\tilde{Q}_N(b, a)$ to be given by

$$\tilde{Q}_N(b, a) \equiv \frac{1}{N}\sum_{i=1}^{N}\Big(\sum_{t\in\mathbf{T}_1\cup\mathbf{T}_o}(\tilde{Y}_{t,i}(a) - F_1(t, X_i; b))^2\omega_t(X_i; a)$$
$$+ \sum_{t\in\mathbf{T}_{io},z\in\{0,1\}}(\tilde{Y}_{t,z,i}(a) - F_2(t, z, X_i; b))^2\omega_{t,z}(X_i; a)\Big), \tag{156}$$

and note that $Q_N(b) = \tilde{Q}_N(b, \hat{\alpha})$. Similarly, we define the population analog to be

$$\tilde{Q}(b,a) \equiv E\Big[\sum_{t \in \mathbf{T}_\text{i} \cup \mathbf{T}_\text{o}} (\tilde{Y}_t(a) - F_1(t, X; b))^2 \omega_t(X; a) + \sum_{t \in \mathbf{T}_\text{io}, z \in \{0,1\}} (\tilde{Y}_{t,z}(a) - F_2(t, z, X; b))^2 \omega_{t,z}(X; a) \Big), \quad (157)$$

noting that $Q(b) = \tilde{Q}(b, \alpha)$. Assumptions A.2(ii)-(iv) and A.3(i)(ii) together with Lemma 2.4 in Newey and McFadden (1994) then allow us to conclude

$$\sup_{(b,a) \in \mathcal{B} \times \mathcal{N}_\alpha} |\tilde{Q}_N(b,a) - \tilde{Q}(b,a)| = o_P(1) \quad (158)$$

and that $\tilde{Q}(b,a)$ is continuous on $\mathcal{B} \times \mathcal{N}_\alpha$ for some neighborhood $\mathcal{N}_\alpha$ of α. Moreover, by Assumption A.1(i), $\hat{\alpha} \in \mathcal{N}_\alpha$ with probability approaching one, and therefore

$$\sup_{b \in \mathcal{B}} |Q_N(b) - Q(b)| \leq \sup_{b \in \mathcal{B}} |\tilde{Q}_N(b, \hat{\alpha}) - \tilde{Q}(b, \hat{\alpha})| + \sup_{b \in \mathcal{B}} |\tilde{Q}(b, \hat{\alpha}) - \tilde{Q}(b, \alpha)| = o_P(1), \quad (159)$$

where the final result follows from (158), $\tilde{Q}$ being continuous on $\mathcal{B} \times \mathcal{N}_\alpha$, and continuous functions being uniformly continuous on compact sets. The consistency of $\hat{\beta}$ then follows from result (159), $\tilde{Q}$ being continuous on $\mathcal{B} \times \mathcal{N}_\alpha$ implying Q is continuous on $\mathcal{B}$, Assumptions A.2(i)(ii), and Theorem 2.1 in Newey and McFadden (1994). ■

Lemma A.8. *If Assumption A.4(ii)-(iv) and A.5(i) are satisfied, then it follows that the class of functions* $\mathcal{F} \equiv \{m_\beta(W, b, a) : (a, b) \in \mathcal{N}_\alpha \times \mathcal{N}_\beta\}$ *is Donsker.*

Proof. By the mean value theorem (see, e.g., Proposition 7.3.2 in Luenberger (1969)), it follows that for any $a \in \mathcal{N}_\alpha$ and $b_1, b_2 \in \mathcal{N}_\beta$ we have the bound

$$\|m_\beta(W, b_1, a) - m_\beta(W, b_2, a)\| \leq \sup_{(b,a) \in \mathcal{N}_\beta \times \mathcal{N}_\alpha} \|\nabla_b m_\beta(W, b, a)\|_o \times \|b_1 - b_2\|. \quad (160)$$

Since $E[\sup_{(b,a) \in \mathcal{N}_\beta \times \mathcal{N}_a} \|\nabla_b m_\beta(W, b, a)\|_o] < \infty$ by Assumption A.4(iv), result (160) together with Assumption A.5(i) imply there is a $B(W)$ satisfying $E[B^2(W)] < \infty$ and

$$\|m_\beta(W, b_1, a_1) - m_\beta(W, b_2, a_2)\| \leq B(W)\{\|a_1 - a_2\| + \|b_1 - b_2\|\}$$

for all $b_1, b_2 \in \mathcal{N}_\beta$ and $a_1, a_2 \in \mathcal{N}_\alpha$. The lemma then follows from Theorems 2.5.6 and 2.7.11 in van der Vaart and Wellner (1996). ■

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