International Reserve Management under Rollover Crises*

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Abstract

This paper investigates how a government should manage international reserves when it faces the risk of a rollover crisis. We ask, should the government accumulate reserves or reduce debt to make itself less vulnerable? We show that the optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves as the government approaches a safe zone. Furthermore, we find that issuing additional debt to accumulate reserves can lead to a reduction in sovereign spreads. We provide evidence consistent with the theoretical predictions.

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1 Introduction

Governments are frequently exposed to episodes where investors suddenly lose confidence in the government's ability to repay the debt. This sudden loss of confidence can be self-fulfilling and result in situations where the government is unable to roll over the debt and defaults on its obligations. Given the significant costs associated with rollover crises, understanding how governments can reduce their vulnerability is crucial for economic policy.

In this paper, we investigate whether governments should accumulate international reserves as a safeguard against rollover crises. A common argument is that reserves provide the government with liquid resources when it cannot roll over its debt. However, a government can also lower its vulnerability by reducing sovereign debt. To the extent that reserves earn a lower interest than the one paid on the debt, it is unclear whether building up a stock of reserves is an effective way to reduce the government's vulnerability to a rollover crisis.

We provide a simple environment to study the optimal management of international reserves for a government subject to the risk of rollover crises. We show that if the government is highly indebted, accumulating reserves is not optimal. Rather, the government should lower its vulnerability by reducing the debt. We show that once the debt is reduced significantly, accumulating reserves becomes strictly optimal. In particular, the government should finance the reserve accumulation by increasing the amount of borrowing. Notably, issuing more debt in this case *reduces* sovereign spreads.

Our environment is a canonical model of rollover crises, following Cole and Kehoe (2000), augmented with reserve accumulation. The government starts with an initial stock of reserves and long-duration debt, receives a constant stream of income, and discounts the future at the same rate as external investors. To abstract from the insurance motive highlighted in Bianchi, Hatchondo and Martinez (2018), we assume that the only source of uncertainty is the possibility of a rollover crisis.[1] When the government is in good credit standing, it decides how much debt to issue, how many reserves to accumulate, and whether to repay the coupons due or default on its obligations. Upon default, the government faces a penalty and is permanently excluded from sovereign debt markets. However, it can keep its reserves and continue to accumulate reserves in the future.

Our analysis begins by characterizing how the economy can be in one of three zones depending on the initial portfolio of debt and reserves. In the safe zone, the government repays the debt regardless of whether investors continue to roll over the debt, and thus

[1] We also abstract from maturity management as an alternative way to reduce the vulnerability (see Aguiar and Amador, 2014, 2025).

it is not vulnerable to a run in equilibrium. In the Default Zone, the government finds it optimal to default regardless of whether investors are willing to lend. In the *crisis zone*, the government's default decision depends on investors' beliefs. If investors are willing to roll over the debt, it is optimal for the government to repay. On the other hand, if investors refuse to roll over the debt, the cost of repayment increases for the government, leading the government to default. In the crisis zone, the government is therefore vulnerable to self-fulfilling rollover crises, as in Cole and Kehoe (2000).

The key question we tackle is the following: Suppose a government is in the crisis zone. What is the best strategy to reach the safe zone? Should the government accumulate reserves or reduce its debt?

A key consideration for understanding our results is how gross positions affect the value of default and repayment for the government. An initial point to note is that higher reserves increase both the value of repayment and default. In contrast, higher debt reduces the value of repayment and does not affect the value of default, assuming full default. In an environment with one-period of debt, an increase in debt and reserves, both by one unit, leaves the value of repayment for the government unchanged. Because the value of default is increasing in reserves, an increase in debt and reserves makes the government more vulnerable to a rollover crisis by making default more attractive. In a scenario with one-period debt, accumulating reserves is therefore not optimal. The government should reduce the vulnerability to a rollover crisis by reducing debt.

When debt has long maturity, however, larger gross positions while holding net foreign assets constant allow the government to relax its budget constraint when it is unable to borrow. This is because when debt has a long maturity, only a fraction of the debt comes due every period. For a given net foreign asset position, an increase in debt and reserves therefore increases the resources available to the government and raises the value of repayment. As mentioned above, however, the value of default also increases with higher gross positions. Therefore, the overall effect of higher reserves and debt on the exposure to a rollover crisis is, in principle, ambiguous. Our analytical results elucidate when it is optimal to simply reduce the debt to reduce the vulnerability and when it is optimal to accumulate reserves.

Moreover, we show that the transition out of the crisis zone is in general *non-monotonic*. The optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves as the government exits the crisis zone. That is, when the government is highly indebted, it should raise the net foreign asset position exclusively through reductions in debt. The accumulation of reserves should be postponed until increasing reserves ensures that the government becomes safe from a rollover crisis. Interestingly, to exit the crisis zone,

the government issues more debt to accumulate reserves, and this operation lowers sovereign spreads.

Finally, we present two empirical patterns that are consistent with the rationale for reserves in our model. First, across 30 emerging economies, reserve accumulation financed by public external borrowing is associated with lower EMBI spreads. Second, during crisis episodes, reserves—once accumulated—are typically not drawn down. Together, these facts support the model's implication that reserves function as a liquidity backstop that deters rollover crises: by reducing investors' incentives to run, reserves lower default risk and sovereign spreads. This interpretation contrasts with the standard buffer-stock view of reserves, under which reserves are accumulated to be depleted during crises.

Related literature. Our paper belongs to the literature on international reserves. In particular, our paper is related to studies on the joint determination of reserves and defaultable sovereign debt that build on the workhorse model of sovereign default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; and Arellano, 2008).[2]

Alfaro and Kanczuk (2009) study reserve accumulation in the canonical sovereign default model with one-period debt and show that it is not optimal for the government to accumulate reserves. This is because reserves make default more attractive, thereby worsening debt sustainability. In a model with long-term debt, Bianchi, Hatchondo and Martinez (2018) show that accumulating reserves provides insurance against negative income shocks that raise borrowing costs. In their model, when the government issues debt to accumulate reserves, it effectively reallocates resources from states with high bond prices and low marginal utility to states with low bond prices and high marginal utility. On the other hand, higher gross positions lead to higher sovereign spreads. The optimal portfolio for the government trades off the insurance benefits against the costs of higher spreads.[3] In contrast to these studies, we consider the possibility of rollover crises and show that issuing debt to accumulate reserves can reduce the vulnerability to rollover crises and lead to a decrease in sovereign spreads. Hernandez (2018) provides numerical simulations in a model with fundamental risk and rollover risk. Our approach focuses exclusively on rollover risk and allows us to provide an

[2]See Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole and Stangebye (2016) for a review of the literature on sovereign debt and default.

[3]Bianchi and Sosa-Padilla (2023) show that under a fixed exchange rate regime, the hedging benefits lead to a macroeconomic stabilization motive for accumulating reserves.

analytical characterization of when issuing debt to accumulate reserves is optimal.[4]

Corsetti and Maeng (2024) study the joint accumulation of reserves and debt in a model of belief-driven sovereign risk crises proposed by Aguiar, Chatterjee, Cole and Stangebye (2022). They show that accumulating reserves and one-period debt is desirable because it rules out an inefficient equilibrium with depressed bond prices.[5] The distinct role of reserves in our model is that they provide the government with liquidity in the event that investors refuse to roll over the debt in the future.

Aguiar and Amador (2013, 2025) and Bocola and Dovis (2019) study optimal debt maturity structure in a model with rollover crises. In these studies, a sufficiently long maturity can prevent a rollover crisis, but it has the cost of exacerbating debt dilution.[6] Our results on the optimal deleveraging are connected in particular to those in Aguiar and Amador (2013), who show that it is optimal for the government to initially remain passive in long-term debt bonds, that is,actively reducing short-term debt, and to lengthen the maturity at the end of the process. Aguiar and Amador (2025) study maturity swaps and show how they can improve bond prices and welfare under rollover risk. Our paper considers an exogenous maturity structure and studies the optimal accumulation of international reserves. One difference in our portfolio problem is that reserves increase the value of the outside option (default), whereas the outside option of defaulting is exogenous in their model. The fact that reserves increase the value of default is important because it can potentially reduce the sustainable debt level, as in Bulow and Rogoff (1989).[7]

Conesa and Kehoe (2024) study a model of rollover crises where the government can commit to a tax rate in advance. They show that it is ex-ante optimal to set a high tax so that in the event of a rollover crisis, the government has sufficient tax revenue to repay the debt, thus deterring investors from running. However, this policy is suboptimal ex-post, as

[4]In his simulations, Hernandez (2018) finds negligible effects of changes in rollover risk on the accumulation of reserves, suggesting that the key motive for reserve accumulation in his model is related to fundamental risk instead of rollover risk. Our necessary condition for reserves to be optimal under a rollover crisis does not appear to be satisfied in his calibration.

[5]Aguiar et al. (2022) show how the equilibrium with depressed bond prices can also be avoided by issuing low or high amounts of debt, relative to the good equilibrium level. Central to the mechanism in this framework is that intra-period uncertainty about the relative payoff between default and repayment induces a price schedule that is a non-monotonic function of debt.

[6]The desirable properties of long-term debt are also highlighted in Cole and Kehoe (2000). See also Arellano and Ramanarayanan (2012) and Hatchondo, Martinez and Sosa-Padilla (2016) for studies of the maturity tradeoffs under fundamental risk. There is also an extensive literature on maturity tradeoffs in closed economies (Barro, 1999, 2003; Angeletos, 2002; Buera and Nicolini, 2004; Bhandari, Evans, Golosov and Sargent, 2017), in which the key aspects involve distortionary taxation and fluctuations in the endogenous risk-free rate instead of default risk.

[7]Bulow and Rogoff (1989) show that lending cannot be supported in equilibrium in the absence of direct punishments when the government has the ability to save abroad in reserves.

the government can borrow at the risk-free rate and would prefer to set a lower tax rate. They refer to this policy as "preemptive austerity."[8] In our model, reserves also serve a preemptive role, but using reserves to service the debt is ex-post optimal and does not require ex-ante commitment. Moreover, the accumulation of reserves allows the government to maintain a lower net foreign asset position and potentially higher consumption as it exits the crisis zone.

Outline. The remainder of the paper is organized as follows. Section 2 introduces the environment. Section 3 analyzes the optimal government portfolio. Section 4 concludes.

2 Environment

Consider a small open economy with time indexed by $t = 0, 1, 2, \ldots$. There is a single consumption good, which is freely tradable. The government in the small open economy receives a constant endowment of the consumption good, y, and trades bonds, b, and risk-free assets, a, with a continuum of external investors. Investors are risk-neutral and share the same discount factor, $\beta \in (0, 1)$, as the government.

2.1 Government Problem

The government's preferences are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) - \phi d_t] \,,$$

where c_t denotes consumption and $u(\cdot)$ is an increasing, twice-continuously differentiable, strictly concave function, and satisfies the Inada conditions $\lim_{c \to 0^+} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$. We denote by d_t the default decision that takes the value of 1 if the government is in default and 0 otherwise.

The government enters every period with an initial portfolio of assets and outstanding bonds (a, b). We refer to the assets as "reserves" and assume without loss of generality that

[8] See also a related literature on sovereign debt crises that followed the Eurozone crisis (e.g., Aguiar, Amador, Farhi and Gopinath, 2013, 2015; Bacchetta, Perazzi and Van Wincoop, 2018; Corsetti and Dedola, 2016; Corsetti and Maeng, 2023; Araujo, Leon and Santos, 2013; Lorenzoni and Werning, 2019; Ayres, Navarro, Nicolini and Teles, 2018, 2023; Camous and Cooper, 2019; Bassetto and Galli, 2019; Cole, Neuhann and Ordonez, 2024; Bianchi and Mondragon, 2022; Bianchi, Ottonello and Presno, 2023).

they have a one-period maturity.[9] The return on reserves is $1 + r = \beta^{-1}$.

The bonds the government issues are long-maturity bonds with geometrically decaying coupons, as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). A bond issued at any date t promises to pay $\left(\frac{\delta+r}{1+r}\right)[1, (1-\delta), (1-\delta)^2, ...]$ in periods $t+1, t+2, t+3....$ Notice that the coupon payments are normalized so that the price of the bond equals $1/(1+r)$ if there is no default risk. The Macaulay duration of the bond is parameterized by $1/\delta$. We highlight that we treat debt duration as given in order to focus on the management of international reserves.[10]

If the government has not defaulted in the past, we can write its budget constraint when it chooses to repay in period t as:

$$c_t = y + a_t - \left(\frac{\delta + r}{1 + r}\right) b_t - \frac{a_{t+1}}{1 + r} + q_t[b_{t+1} - (1 - \delta)b_t].$$

That is, the government collects its income and assets, consumes, pays the debt coupons, accumulates reserves, and issues bonds at a price schedule q_t. As we will see, in the Markov equilibrium, the bond price schedule will depend on the portfolio chosen by the government.

If the government defaults on its debt, it is excluded permanently from financial markets and faces a utility cost ϕ every period.[11] As occurs in practice, the government is able to keep its holdings of reserves and continue to adjust them over time.[12] The budget constraint in case of default is as follows: [13]

$$c_t = y + a_t - \frac{a_{t+1}}{1 + r}\,.$$

Timing. Following Cole and Kehoe (2000), we assume that the government chooses its portfolio—new debt issuance and reserve accumulation—at the beginning of the period, and makes the repayment/default decision at the end of the period. As highlighted in Aguiar and Amador (2014), this timing contrasts with the Eaton–Gersovitz setup, in which the government first decides whether to repay or default and, conditional on repayment, selects its portfolio within the period. We also assume that if the government chooses to default

[9] When reserves can be sold costlessly in a spot market, and there are no fluctuations in the risk-free interest rate, it is equivalent whether reserves are one-period or long-term assets. The equivalence would break down in the presence of shocks to the risk-free rate.

[10] Optimal maturity management is analyzed in Aguiar and Amador (2013, 2025).

[11] Our results about the optimal portfolio are qualitatively the same if we impose an income cost of defaulting.

[12] Under the Foreign Sovereign Immunities Act (FSIA), reserves cannot be legally seized by creditors.

[13] Following Aguiar and Amador (2023)'s implementation of the Cole-Kehoe rollover crises, we assume that in the case of default, the debt issuance revenue is lost for both the government and the lenders.

at the end of period, it can reoptimize the level of reserves. Like in the Cole and Kehoe model, the fact that investors are atomistic will open the door to coordination failures where a *good equilibrium* in which investors continue to roll over the bonds and the government repays coexists with a *bad equilibrium* where investors refuse to roll over the bonds and the government defaults.

We will use ζ to denote a sunspot variable that will determine the type of equilibrium. If $\zeta = 0$, investors will expect others to continue rolling over the bonds, while if $\zeta = 1$, investors will expect others to stop rolling over the bonds. The probability that $\zeta = 1$ is constant and denoted by λ. As we will see, whether a rollover crisis actually takes place will be determined endogenously and will depend on the initial portfolio of the government.

Recursive problem. If the government has not defaulted in the past, it chooses whether to repay or default, and its value function is given by

$$V(a, b, \zeta) = \max \{V_R(a, b, \zeta), V_D(a)\},$$

where V_D and V_R represent, respectively, the values of default and repayment. We assume without loss of generality that if the government is indifferent between repaying and defaulting, it repays.

The value of the government under default is given by

$$\begin{aligned} V_D(a) = \max_{c, a' \geq 0} \{u(c) - \phi + \beta V_D(a')\}, \\ \text{subject to} \\ c = y + a - \frac{a'}{1+r}. \end{aligned} \tag{1}$$

The value of repayment is given by

$$\begin{aligned} V_R(a, b, \zeta) = \max_{c, a' \geq 0, b'} \{u(c) + \beta \, \mathbb{E} V(a', b', \zeta')\}, \\ \text{subject to} \\ c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + q(a', b', s)\left(b' - (1-\delta)b\right), \end{aligned}$$

where we index the price schedule q by the initial state $s = (a, b, \zeta)$ to capture the possibility of multiple equilibria, as will become clear below. The expectation operator $\mathbb{E}$ is taken over the realization of the next-period sunspot ζ'. If the constraint set is empty, we set

$V_R(a, b, \zeta) = -\infty$.

2.2 Markov Equilibrium

Before defining the equilibrium, we present the no-arbitrage condition imposed by the presence of risk-neutral international investors. Investors' optimality implies that they equate the return on a risk-free asset to the expected return from government bonds. That is,

$$q(a', b', s) = \begin{cases} \frac{1}{1+r}\mathbb{E}\left[(1-d')\left(\frac{\delta+r}{1+r} + (1-\delta)q(a'', b'', s')\right)\right] & \text{if } d(s) = 0, \\ 0 & \text{if } d(s) = 1, \end{cases} \tag{2}$$

where $b'' = \hat{b}(a', b', \zeta')$, $a'' = \hat{a}(a', b', \zeta')$, and $d' = \hat{d}(a', b', \zeta')$ represent the policies the government is expected to follow in the next period. If the government is expected to repay in the current period, the bond price takes into account the coupon payments and the secondary market price of the bond in the next period (which depends on the future probabilities of default). If the government is expected to default at the end of the current period, the bond price is zero.

The Markov equilibrium in this economy is defined as follows:

Definition 1. A Markov equilibrium is defined by a set of policies for the government $\{c(\cdot), d(\cdot), a'(\cdot), b'(\cdot)\}$ and a bond price schedule $q(\cdot)$ such that

i) the policies solve the government problem given the bond price schedule;

ii) the price schedule satisfies (2) given the government policies.

2.3 Multiplicity of Equilibria

As in Cole and Kehoe (2000), the government can be subject to rollover crises where the government defaults on its debt because investors stop rolling over the bonds. To determine the states in which the government is vulnerable to a rollover crisis, we need to distinguish between two scenarios: one in which investors continue to roll over the bonds and one in which they do not.

Consider first a situation in which each individual investor expects others to continue

rolling over the bonds. In this case, the government solves the following problem:

$$V_R^+(a,b) = \max_{c,a'\geq 0,b'} \{u(c) + \beta\, \mathbb{E}V(a',b',\zeta')\}\,, \tag{3}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + \tilde{q}(a',b')\,(b' - (1-\delta)b)\,,$$

where $\tilde{q}$ denotes the "fundamental" bond price:

$$\tilde{q}(a',b') = \frac{1}{1+r}\mathbb{E}\left\{(1-d')\left[\left(\frac{\delta + r}{1+r}\right) + (1-\delta)\, q(a'',b'',s')\right]\right\}. \tag{4}$$

Consider now a situation in which the government would like to issue new debt, but investors are unwilling to lend. In this case, the value of repayment for the government is given by

$$V_R^-(a,b) = \max_{c,a'\geq 0} \{u(c) + \beta\, \mathbb{E}V(a',(1-\delta)b,\zeta')\}\,, \tag{5}$$
$$\text{subject to}$$
$$c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r},$$

An immediate implication is that $V_R^+(a,b) \geq V_R^-(a,b)$. That is, a government that can roll over its debt obtains at least the same value as a government that cannot.[14]

When $V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)$, we have multiple equilibria. If the government faces the fundamental price $\tilde{q}$, then the government repays. If the government faces a price $q = 0$, then the government defaults. We examine below for which initial portfolios self-fulfilling debt crises are possible.

The safe zone, the crisis zone, and the Default Zone. Given the value functions (1), (3), and (5), we can split the economy into three different zones depending on the initial

[14] One element implicit in the budget constraint in problem (5) is that if the government were to repurchase debt when investors are unwilling to lend, the price of bonds would rise to the fundamental price (see Aguiar and Amador, 2014).

portfolio (a, b):

$$\begin{aligned}
\mathbf{S} &= \left\{(a,b) : V_D(a) \leq V_R^-(a,b)\right\}, \\
\mathbf{D} &= \left\{(a,b) : V_D(a) > V_R^+(a,b)\right\}, \\
\mathbf{C} &= \left\{(a,b) : V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)\right\}.
\end{aligned}$$

S is the safe zone: the government is better off repaying, regardless of the sunspot realization. **D** is the Default Zone: the government is better off defaulting, irrespective of the sunspot realization. **C** is the crisis zone: the government finds it optimal to repay when lenders are willing to lend, whereas it finds it optimal to default if the lenders are unwilling to lend.

Debt thresholds. The value functions of repayment are strictly decreasing in debt and strictly increasing in reserves and continuous in both.[15]

We can then define thresholds $b^-(a)$ and $b^+(a)$ that determine respectively the boundary between **S** and **C** and the boundary between the **C** and **D**, for every a. That is, $b^-(a)$ is given by

$$V_R^-(a, b^-(a)) = V_D(a), \tag{6}$$

and $b^+(a)$ is given by

$$V_R^+(a, b^+(a)) = V_D(a). \tag{7}$$

Notice that these two thresholds are uniquely determined for any given a. Thus we have constructed two functions that map an initial level of reserves to a level of debt that makes the government indifferent between repaying and defaulting, depending on whether investors are willing to roll over the debt or not. A central aspect of our analysis will be how these thresholds vary with reserves and what this implies for the optimal portfolio.

2.4 Equilibrium Payoffs

Given the definition of $\mathbf{S}, \mathbf{C}$, and $\mathbf{D}$, we can write the payoff for the government as

$$V(a, b, \zeta) = \begin{cases} V_R^+(a,b) & \text{if } (a,b) \in \mathbf{S}, \\ V_R^+(a,b) & \text{if } (a,b) \in \mathbf{C} \;\&\; \zeta \in \{0\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{C} \;\&\; \zeta \in \{1\}, \\ V_D(a) & \text{if } (a,b) \in \mathbf{D}. \end{cases} \tag{8}$$

[15] Monotonicity is shown in Lemma B.1.

When $(a, b) \in \mathbf{C}$, the equilibrium outcome is undetermined, and the government's payoff depends on the sunspot. Notice that the equilibrium payoff for the government never takes the value V_R^-, as this is an off-equilibrium payoff. Nonetheless, V_R^- is essential for determining which zone the government is in and thus, the government's ultimate payoff.

We now analyze the values in each zone.

Safe zone. In Cole and Kehoe (2000) under $\beta(1+r) = 1$, when the government is in the safe zone, it stays in the safe zone with a constant level of debt and consumption. The logic is that once the government reaches the safe zone, it can achieve the level of consumption that would prevail in the absence of default risk. Crucial for this result is that if the government remains in the safe zone, it can borrow at the risk-free rate. If the government were to choose a portfolio outside the safe zone, the bond price would fall, reflecting the positive probability of default, and the government would incur the expected costs of defaulting.

In our model with reserves, maintaining a constant path of consumption requires that for any initial (a, b), consumption is given by

$$c = y + (1 - \beta)(a - b),$$

and any portfolio choice $(a', b') \in \mathbf{S}$ satisfying $a' - b' = a - b$ is consistent with this path. Because utility is concave, it follows directly that any other portfolio in the safe zone that induces non-stationary consumption yields strictly lower lifetime utility.

In the absence of initial reserves, postponing default is never optimal. If the government were to leave the safe zone to default tomorrow, it would be strictly better off to default today; hence, the government could not have been in the safe zone today. However, when the government has access to initial reserves, it is no longer guaranteed that the optimal policy keeps the economy in the safe zone. The reason for this, perhaps surprising, result is that the government can use reserves to service current payments coupons, postpone the realization of default costs, and thereby deliver a smoother path of consumption.[16] We henceforth restrict attention to policy strategies in which the government never transitions from **C** or **S** to **D**, and once in the safe zone, it remains there. That the safe zone is an absorbing state implies that if for a given (a, b), keeping $c_t = y + (1 - \beta)(a - b)$ is feasible, then there are no other policies can deliver higher utility. In Appendix D, we provide a condition on δ under which

[16] This logic connects with the result in Bulow and Rogoff (1989) that access to reserves reduces debt sustainability.

this assumption is indeed consistent with optimality.[17]

We then have that the value of being in the safe zone is

$$V(a,b) = \frac{u(y+(1-\beta)(a-b))}{1-\beta} \equiv V_S(a-b), \tag{9}$$

and the policy functions satisfy

$$c(a,b) = y + (1-\beta)(a-b) \equiv c_S(a-b),$$
$$a'(a,b) - b'(a,b) = a - b.$$

with $(a'(a,b), b'(a,b)) \in \mathbf{S}$. Notice that the payoff for the government depends only on the net foreign asset position (NFA), defined as $a-b$, and not on the gross positions. Moreover, while the choice for consumption and the NFA are unique in the safe zone, there are generically multiple portfolios that are optimal. That is, as long as the new portfolio keeps the same NFA and the government remains in the safe zone next period, the portfolio is (weakly) optimal.

Default zone. Given our assumption that $\beta(1+r) = 1$, under default, the government keeps reserves constant and consumes its income plus the annuity value of the reserves. Thus, the value of default is given by

$$V_D(a) = \frac{u(y+(1-\beta)a) - \phi}{1-\beta}. \tag{10}$$

Crisis zone. Recall that the problem the government faces when investors are willing to roll over the bonds is given by (3). Using the results above, we can write the continuation value for the government as follows:

$$\mathbb{E}V(a',b',\zeta') = \begin{cases} V_S(a'-b') & \text{if } (a',b') \in \mathbf{S}, \\ (1-\lambda)V_R^+(a',b') + \lambda V_D(a') & \text{if } (a',b') \in \mathbf{C}, \\ V_D(a') & \text{if } (a',b') \in \mathbf{D}. \end{cases} \tag{11}$$

That is, if the government chooses a portfolio in the safe zone, the continuation value is given by $V_S(a'-b')$. If the government chooses a portfolio in the crisis zone, then the government

[17] As discussed in the appendix, for $a = 0$, a transition from $\mathbf{S}$ to $\mathbf{D}$ is never optimal under any configuration. For $a > 0$, we show that there is a threshold for δ above which this transition can be ruled out. Intuitively, a policy strategy where the government lowers reserves, repays today, and defaults tomorrow with probability one becomes less likely the shorter the maturity, since paying today's coupons and then defaulting tomorrow becomes less attractive.

will obtain $V_D(a')$ with probability λ and $V_R^+(a', b')$ with probability $1 - \lambda$. Finally, if the government chooses a portfolio in the Default Zone, its continuation value is given by $V_D(a')$.

Staying in the crisis zone is costly for the government. If the government faces a good sunspot tomorrow, it pays ex post a high interest rate on the debt. If the government faces a bad sunspot tomorrow, it does not pay the debt, but it faces the cost of default. While the government pays an actuarially fair interest rate to investors, it still bears the cost of defaulting. As shown by Cole and Kehoe (2000), the government has therefore incentives to deleverage until it reaches the safe zone.[18] Exiting the crisis zone, however, is also costly. To the extent that the utility function is strictly concave, the government may try to exit the crisis zone slowly. Depending on the realization of the sunspot, the government may eventually be able to reach the safe zone, or it may default along the way. How fast the government attempts to exit depends on the perceived probability of facing the bad sunspot. In particular, a higher λ induces a faster exit of the crisis zone.[19]

Under the operating assumption that the safe zone is an absorbing state, we can iterate on investors' break-even condition (4) to arrive at an expression for the bond price faced by the government. In particular, we can use that the bond price becomes $q = 1/(1 + r)$ the period in which the government chooses a portfolio that takes it to the safe zone in the next period. Starting from $t = 0$, suppose that the portfolio choice today is (a', b') and the policy function of the government is such that the government reaches the safe zone in period T so long as $\{\zeta_t\}_{t=0}^{T-1}$.[20] We have that for any $T > 0$ the bond price is given by

$$q(a', b') = \frac{\delta + r}{1 + r} \sum_{t=1}^{T-1} \left(\frac{1 - \lambda}{1 + r} \right)^t (1 - \delta)^{t-1} + \left[\frac{(1 - \lambda)(1 - \delta)}{1 + r} \right]^{T-1} \frac{1}{1 + r} \tag{12}$$

The first term captures the bond coupon payments investors expect to receive, and the second term reflects the risk-free price of the bond once the government exits the crisis zone.

In the Cole and Kehoe model with one-period debt, consumption is constant over time

[18] As explained in Aguiar and Amador (2013), the logic has to do with the fact that default is not a zero-sum game. If a government is indifferent between repaying and defaulting, and flips a coin to decide whether to repay or default, it generates a cost to bondholders but provides no benefits to the government since it is indifferent.

[19] Like in Cole and Kehoe (2000), as the probability λ gets close to zero, a government may decide to avoid any increase in savings necessary to escape the crisis zone, and hence it will eventually default once the bad sunspot is realized. In our model with reserves, there is another possibility: the government may draw down the reserve holdings and move to the default region. In Appendix D, we provide a condition for δ that rules this out in a Markov equilibrium. Under our baseline calibration, this requires debt maturities exceeding 200 years, well above the empirical range.

[20] As in Aguiar and Amador (2013), we assume that if the government is indifferent between policies that imply different exit periods, it chooses the one with lower exit period.

while the government tries to exit the crisis zone, and once the government reaches the safe zone, consumption increases and then stays constant thereafter. The idea is that keeping consumption low allows the government to reduce its debt and reach the safe zone—and once the government is safe, debt can be kept constant. Proposition 1 provides a similar result in our economy.

Proposition 1 (Monotonically increasing consumption path)**.** *Consider an initial portfolio* $(a_0, b_0) \in \mathbf{C}$ *such that the government exit time is* T*. Then, if* $\zeta_t = 0$ *for all* $t \leq T-1$*, we have* $c_{t+1} \geq c_t$ *for all* $t \leq T$*.*

Proof. In Appendix A.1. □

A subtle difference with the canonical model is that consumption may be strictly increasing as the government tries to exit the crisis zone. This result is due to the dilution effect from long-term debt. In particular, the bond price at which investors are willing to lend depends on the expected T at which the government will exit (see eq. 12). Therefore, if keeping current consumption low relative to next-period reduces the exit time T, the government can obtain a better price on the debt. As we will see in the numerical simulation, consumption will be constant between two periods for which a small change in the portfolio would not affect the exit time and increasing otherwise.[21]

2.5 Solving for the debt thresholds

There are two cases to consider when solving for the debt thresholds, $b^-(a)$ and $b^+(a)$. In the first case, the government repays the coupons in a run and remains vulnerable in the future. In the second case, the government repays the coupons but is no longer vulnerable going forward.

In the former case, the two thresholds $b^-(a)$ and $b^+(a)$ must be solved for jointly, since the value V_R^+ depends on the bond price schedule, which in turn depends on $b^-(a)$, and the value V_R^- also depends on V_R^+.

In the latter case, $b^-(a)$ can be obtained independently from V_R^+. Moreover, its value can be computed from

$$\max_{a' \geq 0} u\left(y + a - \left(\frac{\delta + r}{1+r}\right) b^-(a) - \frac{a'}{1+r}\right) + \beta V_S\Big(a' - (1-\delta)b^-(a)\Big) = V_D(a)\,,$$

[21] In a continuous time model with outside option shocks and long-term bonds, Aguiar and Amador (2023) find that consumption is *strictly* increasing over time as the government tries to reach the safe zone. This is because the exit time in their model is continuous and strictly increasing in debt within the crisis zone.

where the continuation value $V_S(\cdot)$ reflects that the government is safe tomorrow after facing a run today and repaying the coupons.

Optimal reserves in a run. The next lemma presents a result on the optimal reserve policy when the government faces a run and compares the two thresholds.

Lemma 1 (Optimal reserves in a run)**.** *Consider a government facing a run given an initial portfolio* (a, b)*. Then:*

(i) Suppose that $a < \delta b$*. If* $(0, (1-\delta)b) \in \mathbf{S}$*, the optimal policy is* $a' = 0$*.*

(ii) Suppose that $a \geq \delta b$*. If* $(a - \delta b, (1-\delta)b) \in \mathbf{S}$*, the optimal policy is* $a' = a - \delta b$*.*

Moreover, if $a \geq \delta b$ *and* $(a - \delta b, (1-\delta)b) \in \mathbf{S}$*, then* $b^-(a) = b^+(a)$*. Otherwise,* $b^-(a) < b^+(a)$*.*

Proof. In Appendix A.2. □

Result (i) says that when the government faces a run and is safe next period with zero reserves, it will choose a corner solution with zero reserves if $a < \delta b$. The intuition is that when the government cannot roll over the bonds and starts with low reserves, it chooses the minimum feasible reserves if it anticipates that it will be safe in the future. Note, however, that if the government remains vulnerable tomorrow after repaying during a run, it may choose to further reduce current consumption and keep positive reserves in order to reduce its vulnerability in the future.[22] In either case, we have $b^-(a) < b^+(a)$. That is, a government under a run, defaults for lower levels of debt.

Result (ii) says that if the government has a high level of reserves, it can achieve the unconstrained level of consumption even when investors refuse to roll over the debt. Notice that this result requires that the government is not vulnerable tomorrow after repaying the coupons due today. This is because the reduction in reserves needed to make the coupon payments today may make the government vulnerable tomorrow, thereby lowering the value from repaying today in the event of a run.

One interesting observation from (ii) is that reserves do not need to be large enough to pay for all coupons in order to be safe from a rollover crisis. That is, given $r > 0$, there exists

[22] If a government has an initial portfolio at $(0, b^-(0))$, faces a run and repays the coupons, it will be safe next period. This is because $V_R^-(0, (1-\delta)b^-(0)) > V_R^-(0, b^-(0)) = V_D(0)$. Similarly, as δ approaches one, there is little debt left after repaying today, so the government is safe in the next period. For our baseline calibration, the government is indeed safe in the period following the run. However, none of the theoretical results that follow depend on this particular case. See Appendix G for further discussion and for configurations under which a government remains vulnerable in the next period following a run.

$a \in [\delta b, \left(\frac{\delta+r}{1+r}\right) b]$ such that the government is safe. To be safe, the government needs just enough reserves to repay the fraction of the debt that would allow the government to keep the NFA constant. In effect, when the government has a negative NFA, it uses a fraction of the endowment to pay the interest on the debt.[23]

3 Reserve Management under Rollover Crises

In this section, we turn to the main question we investigate in this paper: *How should the government choose its portfolio to manage the risk of a rollover crisis?*

3.1 The Three Zones

We start the analysis by showing how the initial portfolio determines which zone the economy is in. Figure 1 presents the results.[24] The blue-solid line and the red-dashed line denote respectively $b^+(a)$ and $b^-(a)$. The two lines indicate the boundaries between the crisis zone and the Default Zone and between the crisis zone and the safe zone. Recall that a government that starts with a portfolio $(a, b^-(a))$ is indifferent between defaulting and repaying when it cannot roll over the bonds, while a government that starts with a portfolio $(a, b^+(a))$ is indifferent between defaulting and repaying when it can roll over the bonds. Thus, we must have $b^+(a) \geq b^-(a)$, as observed in the figure. We can also see that $b^+(a) = b^-(a)$ for large values of reserves, a result in line with part (ii) of Lemma 1. That is, for large a, the crisis zone is empty. Defaults only happen due to fundamentals.

Consider any pair (a, b) at the $b^-(a)$ boundary. Holding a fixed, we can see that a lower level of debt takes the government to the safe zone, and a higher level of debt takes it to the crisis zone. Moreover, we can see that $b^-(a)$ has a positive slope: holding b fixed, a higher level of reserves takes the government to the safe zone, while a lower level of reserves puts the government in the crisis zone. The figure shows that $b^+(a)$ also has a positive slope, but it is less steep than $b^-(a)$. To understand better this results, we totally differentiate (6) and (7) and obtain

[23] To see this more clearly, suppose the government starts with the portfolio $(\delta b, b)$ and thus the NFA is equal to $-(1-\delta)b$. If the government cannot roll over the debt, $b' = (1-\delta)b$, and if sets $a' = 0$, the government keeps the same NFA and achieves the ideal level of consumption, i.e., $c = y - (1-\beta)(1-\delta)b$.

[24] Throughout, we use calibrated parameter values listed in Table 1, to be described below.

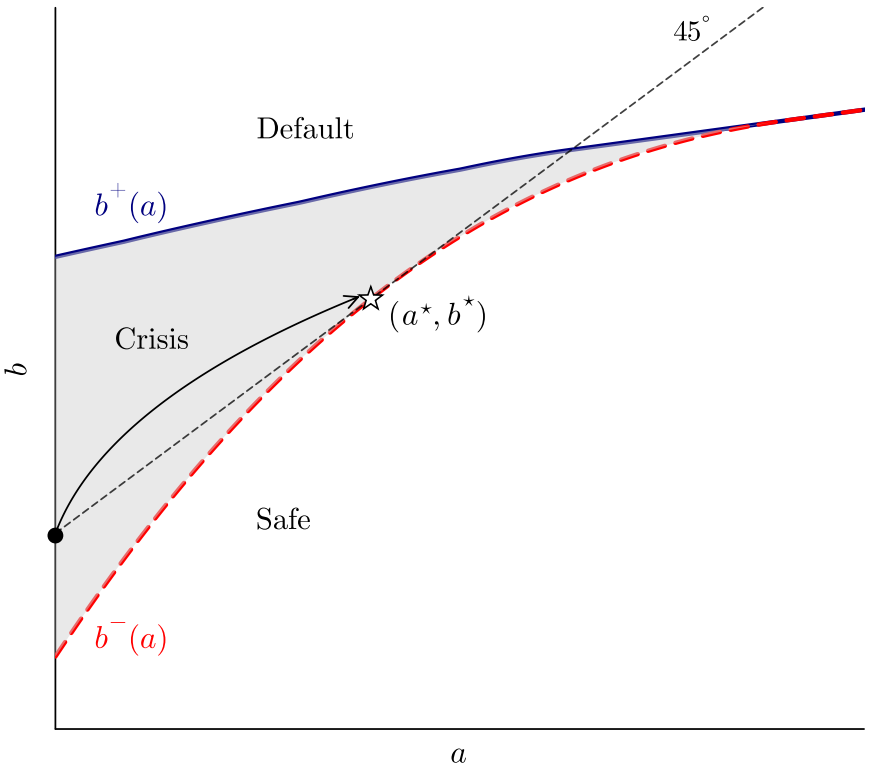


Figure 1: The three zones. The boundaries $b^-(a)$ and $b^+(a)$ are defined in (6) and in (7), respectively.

$$\frac{\partial b^-(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-\left(a,b^-(a)\right)}{\partial a}}{\frac{\partial V_R^-(a,b^-(a))}{\partial b}} \qquad (13.1) \qquad \frac{\partial b^+(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+\left(a,b^+(a)\right)}{\partial a}}{\frac{\partial V_R^+(a,b^+(a))}{\partial b}} \qquad (13.2)$$

That is, the slopes of b^- and b^+ are determined by how much the value functions of repayment and default change when we vary a and b at the indifference points.[25]

As we show formally below, both numerators in (13.1) and (13.2) are negative. For the case of b^-, the result follows because when debt cannot be rolled over, making the debt payment forces an increase in the government's marginal utility of consumption. For the case of b^+, the result follows because when the government can borrow, it reduces consumption to try to exit the crisis zone, as we showed in Proposition 1. Given that the denominators in (13.1) and (13.2) are negative, this implies that both slopes are positive.

[25]Note that while the bond price is discontinuous, the differentiability of V_R^+ can be established following the arguments in Clausen and Strub (2020) and Aguiar and Amador (2013). V_R^- may have points of non-differentiability, but it is differentiable in the neighborhood of zero reserves.

We summarize these results in the following proposition.

Proposition 2 (Monotonicity)**.** *In any Markov equilibrium, $b^-(\cdot)$ and $b^+(\cdot)$ are strictly increasing in a, for all a.*

Proof. In Appendix A.3. □

We have seen that reserves help expand the safe zone. Even though this suggests that reserves could be desirable, the government in principle, could also reduce its vulnerability by decreasing debt instead. The key issue is whether increasing both debt and reserves leads to an increase in $V_R^-(a,b)$ that offsets the increase in $V_D(a)$. More concretely, consider a government portfolio that lies slightly above the red-dashed curve in Figure 1. Does an increase in debt and reserves by one unit push the government into the crisis zone or the safe zone? We examine this question next.

3.2 A Joint Increase in Debt and Reserves

Each of the portfolios in the $b^-(a)$ boundary is associated with a different level of net foreign assets. It will be useful to examine the portfolio in the safe zone with the lowest NFA position. We refer to this portfolio as the "lowest-NFA safe portfolio", and we denote it by $(a^\star, b^\star)$.

Definition 2. The lowest-NFA safe portfolio $(a^\star, b^\star)$ is the portfolio in the safe zone with the lowest net foreign asset position.

Formally, we have that the lowest-NFA safe portfolio is given by

$$(a^\star, b^\star) = \underset{a\geq 0, b}{\operatorname{argmin}} \; a - b \qquad (14)$$
$$\text{s.t.} \quad (a,b) \in \mathbf{S}.$$

Using that $(a,b) \in \mathbf{S}$ if $b \leq b^-(a)$ and assuming a *strictly interior solution for* $a^\star$, we obtain that the lowest-NFA safe portfolio satisfies $\frac{\partial b^-(a^\star)}{\partial a} = 1$. Graphically, this portfolio corresponds to the point highlighted in Figure 1 where the 45° line is tangent to the red-dashed line. It is then immediate that if the government starts from a point on or below this tangent line, it can move to the safe zone by increasing reserves and debt by the same amount (thus keeping the same NFA). The arrow in the figure illustrates how a government at the tangent line with zero initial reserves can jump to the safe zone by choosing $(a^\star, b^\star)$.

The lowest-NFA safe portfolio will constitute a focal point. When a government is deep in the crisis zone (i.e., it is above the aforementioned tangent line), it needs to increase its NFA to reach the safe zone. The larger the required increase in the NFA, the higher the cost of exiting the crisis zone because of the concavity of the utility function. The portfolio $(a^\star, b^\star)$ makes the government safe and minimizes the need for consumption cuts.

The key question we tackle next is when do we have $a^\star > 0$?

A condition for $a^\star > 0$. The proposition below provides a sufficient condition guaranteeing that the solution to (14) is strictly interior.

Proposition 3 (Positive reserves)**.** *Suppose that the boundary of the crisis region at zero reserves $b^-(0)$ satisfies the following condition:*

$$\beta(1-\delta)\left[u'\left(y-\left(\frac{\delta+r}{1+r}\right)b^-(0)\right)-u'\left(y-(1-\beta)(1-\delta)b^-(0)\right)\right]>u'(y). \tag{15}$$

Then, the lowest-NFA safe portfolio features a strictly positive level of reserves. That is, $a^\star > 0$.

Proof. We first show that starting from zero reserves, an increase in reserves raises the b^- threshold by more than one unit:

$$\left.\frac{d\,b^-(a)}{da}\right|_{a=0} > 1. \tag{16}$$

where note that the derivative exists by the Implicit Function Theorem and the fact that both V_D and V_R^- are continuously differentiable in a neighborhood around $a = 0$.

To prove (16), we first note that taking the envelope condition in (5) and evaluating it at $(0, b^-(0))$ we obtain

$$\frac{\partial V_R^-\left(0,b^-(0)\right)}{\partial b}=-u'\left(y-\left(\frac{\delta+r}{1+r}\right)b^-(0)\right)\left(\frac{\delta+r}{1+r}\right)-\beta(1-\delta)u'\left(c_S(-(1-\delta)b^-(0))\right),$$

where the expression uses that $a' = 0$ and that the government is safe the period after the run, which both follow from Lemma 1 and $a = 0 < \delta b^-(0)$.

Replacing the previous equation and the value for $\partial V_D(a)/\partial a$ in (13.1), and using (15), we obtain the inequality in (16).

Toward a contradiction, suppose that the lowest-NFA safe portfolio $(a^\star, b^\star)$ is such that $a^\star = 0$. Consider an alternative portfolio $(\tilde{a}, b^-(\tilde{a}))$, for $\tilde{a} > 0$ approximately close to zero.

We have

$$b^-(\tilde{a}) - b^-(0) = \int_0^{\tilde{a}} \frac{\partial b^-(a)}{\partial a} da > \int_0^{\tilde{a}} da = \tilde{a}. \tag{17}$$

where the strict inequality follows from (16). Rearranging (17), we arrive at

$$\tilde{a} - b^-(\tilde{a}) < 0 - b^-(0),$$

That is, there is a portfolio $(\tilde{a}, b^-(\tilde{a})) \in \mathbf{S}$ with a lower net foreign asset position than the portfolio with zero reserves $(0, b^-(0))$. We thus find a contradiction that $a^\star = 0$. That is, the lowest-NFA safe portfolio features strictly positive reserves. □

The argument behind the proof is that starting from a point with zero reserves where the government is indifferent between repaying and defaulting, an increase in reserves of one unit implies that there exists an increase in debt of more than one unit that keeps the government indifferent. That is, for low a, the effect of one more unit of reserves on the value of repayment in a run exceeds the marginal cost of repaying the debt and the marginal value of reserves under default. That is, for low level of reserves $\frac{\partial V_R^-}{\partial a} > \frac{\partial V_R^-}{\partial a} + \frac{\partial V_D}{\partial a}$. As the initial level of reserves increase, however, the marginal value of reserves fall. At some point, we then have $\frac{\partial V_R^-}{\partial a} = \frac{\partial V_R^-}{\partial a} + \frac{\partial V_D}{\partial a}$. This is exactly the portfolio $(a^\star, b^\star)$. Once the portfolio is a the lowest-NFA safe portfolio, a marginal decrease or increase in reserves puts the government in the crisis zone.

As condition (15) indicates, a key determinant for the lowest-NFA safe portfolio featuring strictly positive reserves is the curvature of the utility function. Intuitively, when the government cannot roll over the bonds, this implies a reduction in consumption. When the utility function is highly concave, this means that there is a large drop in utility, and thus a large drop in V_R^-. In this situation, a higher level of reserves has a particularly high marginal utility and can offset the fact that higher reserves also increase the value of default.

Maturity also plays a critical role for condition (15) to hold. In particular, the condition can only be satisfied for intermediate values of maturity. In the case of $\delta = 0$, the bond becomes a perpetuity, in which case rollover risk becomes irrelevant as the government only pays interest and never pays any principal. Conversely, in the case of $\delta = 1$, condition (15) cannot be satisfied either. Intuitively, when debt is one-period, the value of repayment for the government depends only on $a - b$ (i.e., it is independent of the gross positions). Because V_D increases with a, it thus follows that a one-unit increase in a and b must lower $V_R^-(a, b) - V_D(a)$. This implies that if a government is indifferent between repaying while facing a run and defaulting, an increase in debt and reserves will always push the economy into the crisis zone. The proposition below formalizes that if $\delta = 0$ or $\delta = 1$, the lowest

NFA-safe portfolio must feature $a^\star = 0$.

Proposition 4. *Suppose that $\delta = 0$ or $\delta = 1$. Then, the lowest-NFA safe portfolio features zero reserves, $a^\star = 0$.*

Proof. In Appendix A.4. □

We next show that there exist an intermediate range of δ for which condition (15) holds, thus guaranteeing that reserves are strictly positive.

Proposition 5. *For any $\{\beta, y\}$, there exists a non-empty set of values for $\{\phi, \delta\}$ such that condition (15) is satisfied, and therefore $a^\star > 0$.*

Proof. In Appendix A.5 □

The proof of the proposition draws on the result that for any candidate value for the b^- boundary with zero reserves, there is a unique default cost ϕ that rationalizes such debt level and that in turn this value of ϕ does not affect condition (15). When the level of $b^-(0)$ is sufficiently large, an Inada condition guarantees that the marginal value of having one more unit of reserves and paying a fraction $\frac{\delta+r}{1+r}$ of the existing stock of debt becomes arbitrarily large. As a result, having one more unit of debt and reserves increases the value V_R^- more than V_D. This implies that $\frac{\partial b^-(a)}{\partial a}|_{a=0} > 1$ holds, which guarantees $a^* > 0$.

We highlight that the proposition is valid for any utility function satisfying the usual properties highlighted above. In Appendix C, we provide an example with quadratic utility, which allows for a closed-form expression for $b^-(0)$ and a complete characterization of the set of parameters that guarantee that $a^\star > 0$.

3.3 Exiting the crisis zone

We analyze now what the best strategy is for a government that is trying to exit the crisis zone: Should the government reduce its debt or increase reserves? If reserves are optimal, should the government slowly build up its stock of reserves?

Iso-T regions. We first examine how long it takes for the government to exit the crisis zone, depending on the initial portfolio. Following the terminology of Aguiar and Amador (2013), we construct "iso-T regions." Starting from $t = 0$, we compute in which period the government will reach the safe zone (so long as it does not get hit by a bad sunspot before then).

Figure 2 shows the exit time T as a function of the initial portfolios (a, b). As we can see, when the gross positions are close to $(a^\star, b^\star)$, the government exists in one period. That is, $T = 1$. As we move up and toward the left (that is, increasing debt and lowering reserves), it takes more periods to exit.

For this parameterization, we can see regions with $T = 2$, $T = 3$, and $T = 4$. When the government starts from debt levels above $b^+(a)$, the government strictly prefers to default in equilibrium.

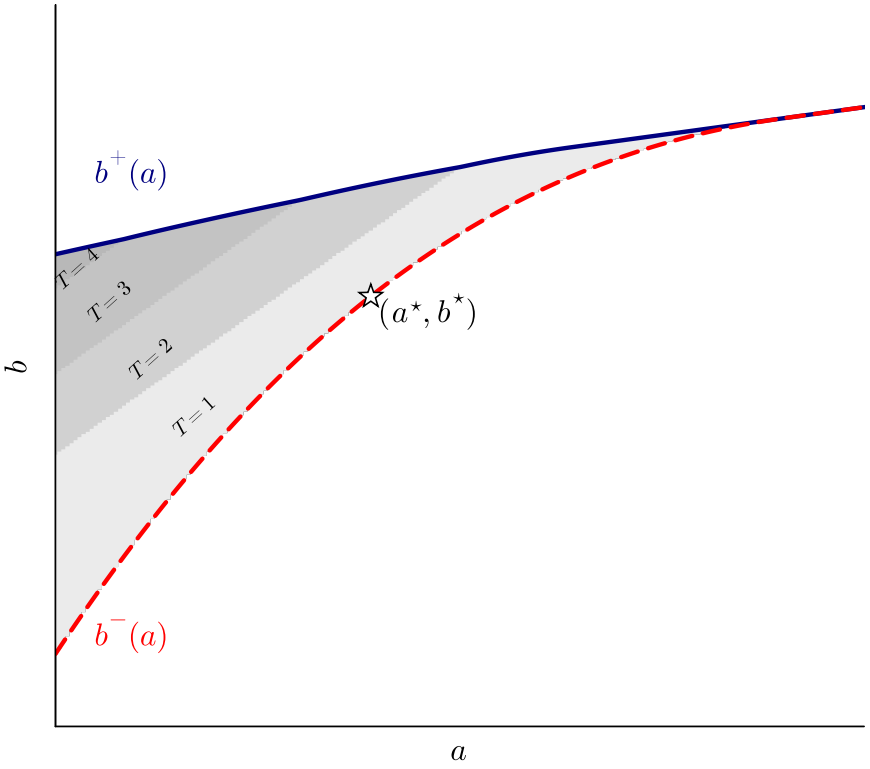


Figure 2: Iso-T Regions. The shaded areas describe how many periods it takes for the government to exit the crisis zone.

Optimal portfolio. We now analyze the optimal portfolio choice for the government that is today in the crisis zone and is able to borrow.

The proposition below characterizes the solution.

Proposition 6 (Optimal portfolio)**.** *Consider an initial* $(a, b) \in \mathbf{C}$*. The optimal portfolio choice* $\{a'(a, b), b'(a, b)\}$ *that solves* (3) *satisfies the following conditions:*

(i) Suppose that $a - b < a^\star - b^\star$*. Then, if* $(a'(a, b), b'(a, b)) \in \mathbf{S}$ *we have that* $a'(a, b) = a^\star$ *and* $b'(a, b) = b^\star$*.*

(ii) Suppose that $(a'(a,b), b'(a,b)) \in \mathbf{C}$*. Then, we have that* $a'(a,b) = 0$*.*

Proof. In Appendix A.6. □

Part (i) considers a government in the crisis zone that has an NFA that is lower than the lowest-NFA safe portfolio. The proposition shows that if the government chooses a portfolio that puts it in the safe zone, then the government chooses the lowest-NFA safe portfolio $(a^\star, b^\star)$. The idea is that choosing $(a^\star, b^\star)$ allows the government to exit the crisis zone minimizing the cut in consumption. Portfolios in the safe zone with lower amounts of reserves would imply the government needs to reduce debt by more than the reduction in reserves. Portfolios in the safe zone with higher amounts of reserves would allow the government to borrow more but the increase in borrowing would be smaller than the increase in reserves.

Part (ii) considers a situation where the government chooses a portfolio that keeps it in the crisis zone in the next period. The proposition shows that it is not optimal to accumulate reserves in this case. To establish the result, we show that any policy strategy with positive reserves can be improved by an alternative policy. In particular, we consider an alternative strategy where the government issues less debt and accumulates less reserves, in such a way that keeps the *continuation value under repayment unchanged*.[26] The exact changes in the portfolio that deliver this goal lead to an increase in current consumption, but, with lower future reserves, also lead to a decrease in consumption under an eventual default next period. Because the government is trying to deleverage toward the safe zone, consumption under repayment is strictly lower than the consumption under default; hence, the alternative strategy improves welfare over the initial policy with $a' > 0$.[27] Therefore, issuing debt to accumulate reserves in the crisis zone is strictly suboptimal as long as the government remains in the crisis zone.[28]

Putting these two parts together establishes that *the optimal exit strategy is to delay the accumulation of reserves until the government is ready to exit the crisis zone.* We summarize this in the following corollary.

Corollary 1 (Optimal exit strategy)**.** *Consider a portfolio* $(a,b) \in \mathbf{C}$ *such that* $a - b < a^\star - b^\star$ *and the government exits after* T *periods for* $T < \infty$*, provided that* $\{\zeta_t\}_{t=0}^{T-1} = 0$*. Then, we*

[26] This perturbation is in the spirit of the zero-cost trades of Aguiar and Amador (2013, 2025), but here we keep the continuation value under repayment rather than consumption constant. In their case, they show how a swap between short-term and long-term debt can help a government escape the crisis zone.

[27] A reader may wonder whether this result relies on the assumption that the cost of default is in terms of utility and not resources. We can show, however, that the same result holds under an income cost of defaulting.

[28] While in the crisis zone, another result proved in Lemma **??** is that if the government starts with $a - b < a^\star - b^\star$ and $(a', b') \in \mathbf{C}$, then $a' - b' < a^\star - b^\star$.

have $a_{t+1} = 0$ *for all* $t < T - 1$ *and* $a_T = a^\star$, $b_T = b^\star$.

Proof. In Appendix A.7. □

We turn next to the case when the government has an initial NFA higher than the one in the lowest-NFA safe portfolio. The proposition below shows that the government exits the crisis zone in one period.

Proposition 7 (Immediate exit)**.** *Consider an initial portfolio* $(a, b) \in \mathbf{C}$ *and suppose that* $a - b \geq a^\star - b^\star$. *Then, exiting in one period achieves the optimal allocation. Moreover, any portfolio* $(a', b') \in \mathbf{S}$ *such that* $a - b = a' - b'$ *is optimal.*

Proof. In Appendix A.8. □

The logic for why the government exits in one period is that it is feasible for the government to choose a portfolio that takes it to the safe zone without having to reduce consumption relative to the unconstrained optimal. That is, exiting the crisis zone is not costly in this situation. Notice that in this case, there are a range of portfolios that are optimal, including in particular, $(a^\star, b + a^\star - a)$.

Illustration. In Figure 3 we show three simulations, one where the government exits in one period (panel [a]), one where it exits in two periods (panel [b]), and one where it exits in three periods (panel [c]). As is consistent with Proposition 6, we can see that when the government exits in two periods, it chooses zero reserves in the first period, and when the government exits in three periods, it chooses zero reserves in periods 1 and 2.

Panel (d) of Figure 3 shows all the possible portfolios chosen throughout the transition for the entire range of possible initial portfolios with an NFA lower than the lowest-NFA safe portfolio. As the figure illustrates, the government never accumulates reserves unless doing so takes it to the safe zone. It is also interesting to note that there are "holes" in the figure in the sense that there are a range of b' values that are never chosen by the government in equilibrium. This is because, for those levels, the government is better off either borrowing slightly less and exiting in fewer periods (thus, obtaining a higher bond price) or borrowing slightly more and exiting in the same number of periods (thus, achieving a higher intertemporal smoothing of consumption).

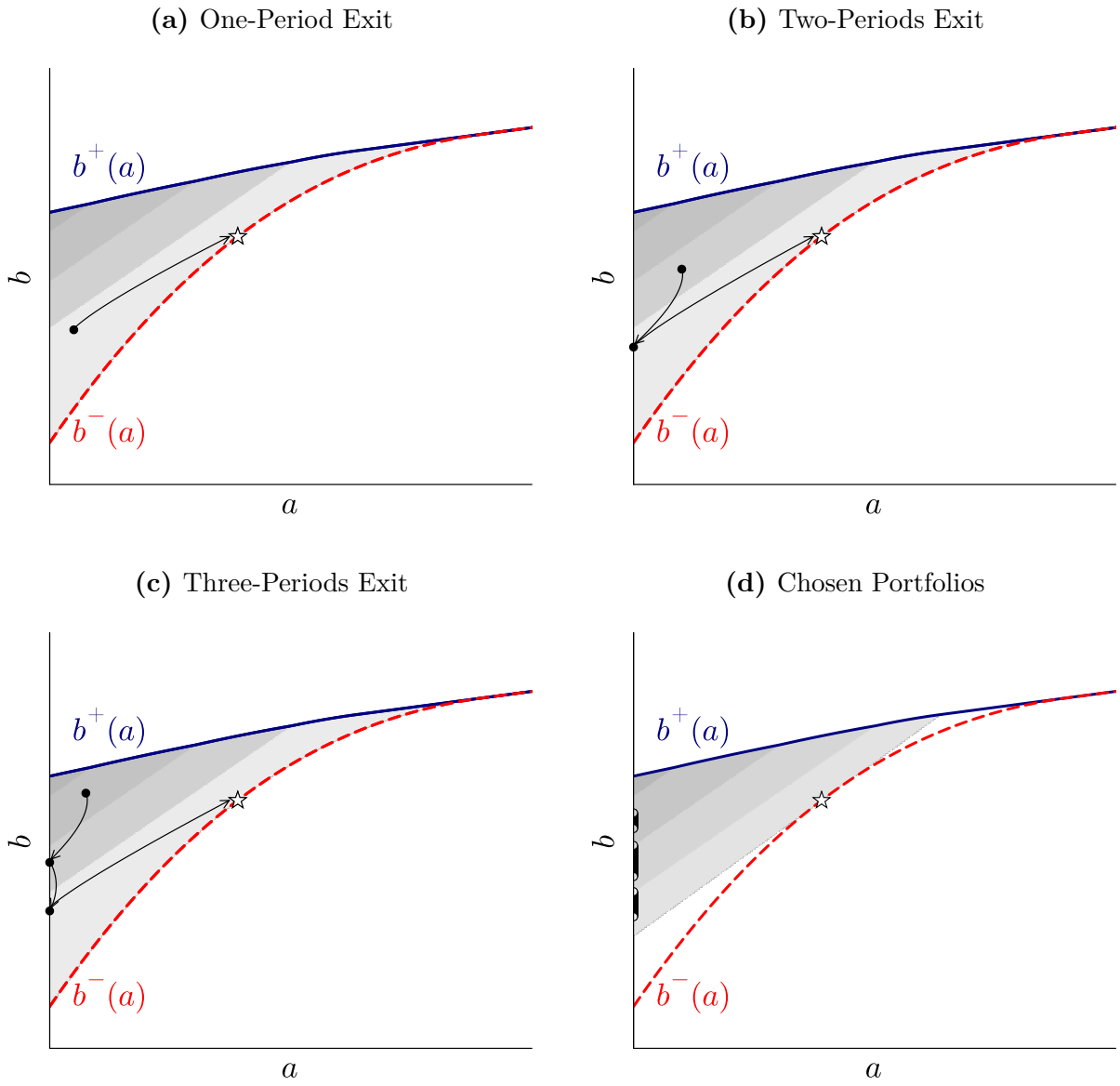


Figure 3: Periods to exit and portfolios. The figure presents examples where the government exits in one period (panel a), two periods (panel b), and three periods (panel c). Panel (d) shows all the possible portfolios the government chooses for any initial portfolio in the crisis zone such that $a - b < a^\star - b^\star$. The star point in all four panels indicate the portfolio $(a^\star, b^\star)$.

Policy correspondences. To further inspect the portfolio of the government, Figure 4 presents the policies for reserves, borrowing, NFA, and consumption in panels (a),(b),(c), and (d), respectively. These policies correspond to the solution of the government problem (3) (i.e., they are the policies conditional on repayment and having access to the bond market). The plots are presented for a range of initial values for b and for $a = 0$. The grey areas denote correspondences.

The figure highlights three vertical lines: the first vertical line denotes $b^-(0)$, the middle vertical line denotes $b^-(a^\star) - a^\star$, and the third vertical line denotes $b^+(0)$. We note that the middle vertical line corresponds to a portfolio with zero initial reserves and with the same NFA as the lowest-NFA safe portfolio. In terms of Figure 1, this corresponds to the solid dot where the arrow originates.

Let us start from the first vertical line, which indicates the boundary between the safe zone and crisis zone for zero reserves. For debt levels to the left of this line, the government is already in the safe zone and there is a range of portfolios that deliver the optimum solution, as illustrated in the shaded area in panels (a) and (b). Consumption and the NFA, however, are uniquely determined (panels [c] and [d]). In particular, consumption is given by $c = y - (1 - \beta)b$ and the NFA equals $-b$.

For debt levels between the first and the middle vertical line, the government is initially in the crisis zone. Because the government starts from an NFA position that is higher than the lowest-NFA safe portfolio, it can exit the crisis zone without cutting consumption, as implied by Proposition 7. As in the previous case, portfolios are correspondences and the NFA is kept constant. However, in this region, *the government must accumulate a strictly positive amount of reserves to exit the crisis zone*. Moreover, it finances the accumulation of reserves with new debt issuances.

For debt levels immediately to the right of the middle vertical line, the government still exits in one period, but it now needs to increase the net foreign asset position to reach the safe zone. Consistent with Proposition 6, the government finds it strictly optimal to choose $(a^\star, b^\star)$. One can also see that as the initial debt level increases further, there is a sharper drop in consumption (panel [d]). This is intuitive because while the initial debt level is increased, the government continues to choose the portfolio $(a^\star, b^\star)$. It is also interesting to note that the policy function for debt lies above the 45-degree line (panel [b]). That is, *the government increases the amount of debt to exit the crisis zone.*

As we move to the right (still between the middle and the third vertical lines), borrowing falls discretely and the government chooses zero reserves. At this point, the government now takes two periods to exit. We can also see that the policy for b' lies below the 45° line. That

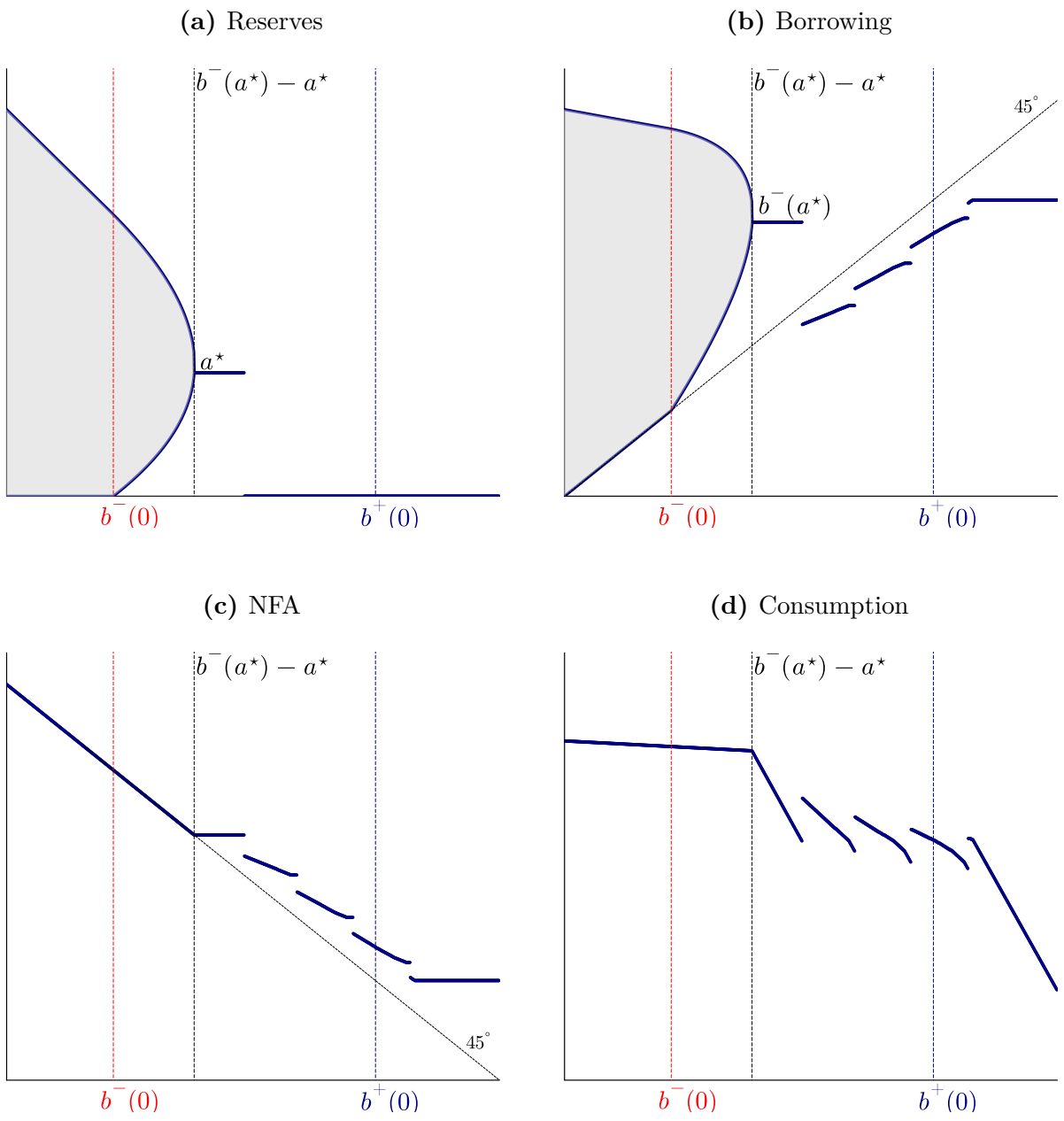


Figure 4: Policy correspondences. The plots are for an initial level of reserves $a = 0$ and a range of values for initial debt (in the horizontal axis). The top panels denote $a'(0, b)$ and $b'(0, b)$ and the bottom panels denote $a'(0, b) - b'(0, b)$, and $c(0, b)$. Light grey areas represent the portfolio indeterminacy in that region.

is, the government reduces its debt and delays the accumulation of reserves. As we move

further to the right, we see another point of discontinuity when the government postpones by one more period the planned exit of the crisis zone. Specifically, at the point of discontinuity, the government is indifferent between choosing a certain level of borrowing and a significantly higher one that delays the exit time by one period. When the government chooses to postpone by one period the exit, we can see that consumption jumps upward as we increase the initial debt (and then falls continuously with the initial debt level).[29] Once debt exceeds the third vertical line, denoting $b^+(0)$, the government chooses to default in equilibrium.[30]

To conclude, we have characterized the optimal strategy for a government that seeks to exit the crisis zone. The key takeaway is that when the government is deep in the crisis zone, the optimal policy is to reduce the debt and keep zero reserves. As the government approaches the safe zone, it is optimal to increase borrowing and accumulate reserves.

3.4 Quantitative Results

In this section, we present a calibration of the model to assess the quantitative role of reserves. We use data from Italy to calibrate the model. Parameter values are presented in Table 1.[31]

Table 1: Parameter values

Parameter	Value	Description	Source
y	1	Endowment	Normalization
σ	2	Risk-aversion	Standard
r	3%	Risk-free rate	Standard
$1/\delta$	6	Maturity of debt	Italian Debt
β	0.97	Discount factor	$\beta(1+r)=1$
$\underline{c}$	0.68	Consumption floor	Bocola and Dovis (2019)
λ	0.5%	Sunspot probability	Baseline
ϕ	0.33	Default Cost	Debt-to-income =100%

A model period is one year and income is normalized to one. As in Conesa and Kehoe

[29] The saw-tooth pattern for NFA and consumption in panels (c) and (d) respectively, share some common features with Cole and Kehoe (2000)'s model with one-period debt and no assets. One subtle difference with long-term debt is that the policy function for NFA has several flat regions. This is related to the debt dilution effect of long-term debt which implies that the government is deterred from a slight increase in borrowing despite the higher initial stock of debt. This is because higher borrowing would increase the exit period and discretely reduce the bond price. Notice that at these flat portions, consumption experiences a sharper drop.

[30] As mentioned above, the figure presents the policy conditional on repayment. Notice also that in the figure borrowing never exceeds $b^+(0)$ (because the government would face a zero bond price in that case).

[31] The numerical algorithm is described in Appendix H.

(2017) and Bocola and Dovis (2019), we assume the utility function takes the form

$$u(c) = \frac{(c - \underline{c})^{1-\sigma}}{1-\sigma},$$

where $\underline{c}$ stands for the level of consumption that cannot be changed in the short run. Following Bocola and Dovis (2019), we set $\underline{c} = 0.68$, based on the measure of non-discretionary spending for the Italian government. We set $\sigma = 2$ and $r = 0.03$, which are common values in the literature.

We calibrate the cost of default ϕ so that the midpoint between $b^-(0)$ and $b^+(0)$ is roughly 100% of GDP, the level of Italy's debt in the run-up to the sovereign debt crisis in 2012.[32] The maturity of the debt is set to 6 years, which implies $\delta = 1/6$. Finally, we use $\lambda = 0.5\%$ as a baseline value. This value is important for the speed at which the economy exits the crisis zone but does not affect the lowest-NFA safe portfolio $(a^\star, b^\star)$ for this parametrization.[33]

Simulation results. We obtain that the lowest-NFA safe portfolio is $a^\star = 0.08, b^\star = 1.02$. Recall that we normalize income to one, so the values can be interpreted as fractions of GDP. Figure 5 shows the time series simulation for a government that starts in the crisis zone. For the initial portfolio at $t = 0$ considered, the government reaches the safe zone in four periods in both cases. For $t \geq 4$, all variables therefore remain constant.

As can be observed in panels (a) and (e), the NFA increases monotonically while the debt level fall initially and then increase at $t = 3$ upon exiting the crisis zone. In addition, consumption (panel [c]) is weakly increasing over time, as shown in Proposition 1. In particular, consumption is constant for $t = 0, 1, 2, 3$ and increases at $t = 4$ once the economy reaches the safe zone. Finally, panel (f) illustrates how the bond price increases monotonically over time. As highlighted in the recursion (12), as the government approaches the safe zone, the probability of a future default falls and the bond price increases.

[32] The calibrated default penalty $\phi = 0.33$ implies a lifetime utility loss equivalent to a permanent 0.11% reduction in consumption. This figure is obtained by comparing the consumption equivalents of V_S and V_D, evaluated at the lowest-NFA safe portfolio $(a^\star, b^\star)$.

[33] The result that $(a^\star, b^\star)$ is invariant to λ holds for this parametrization, and many others, but is not general. See more results for alternative calibrations of λ in Appendix F.

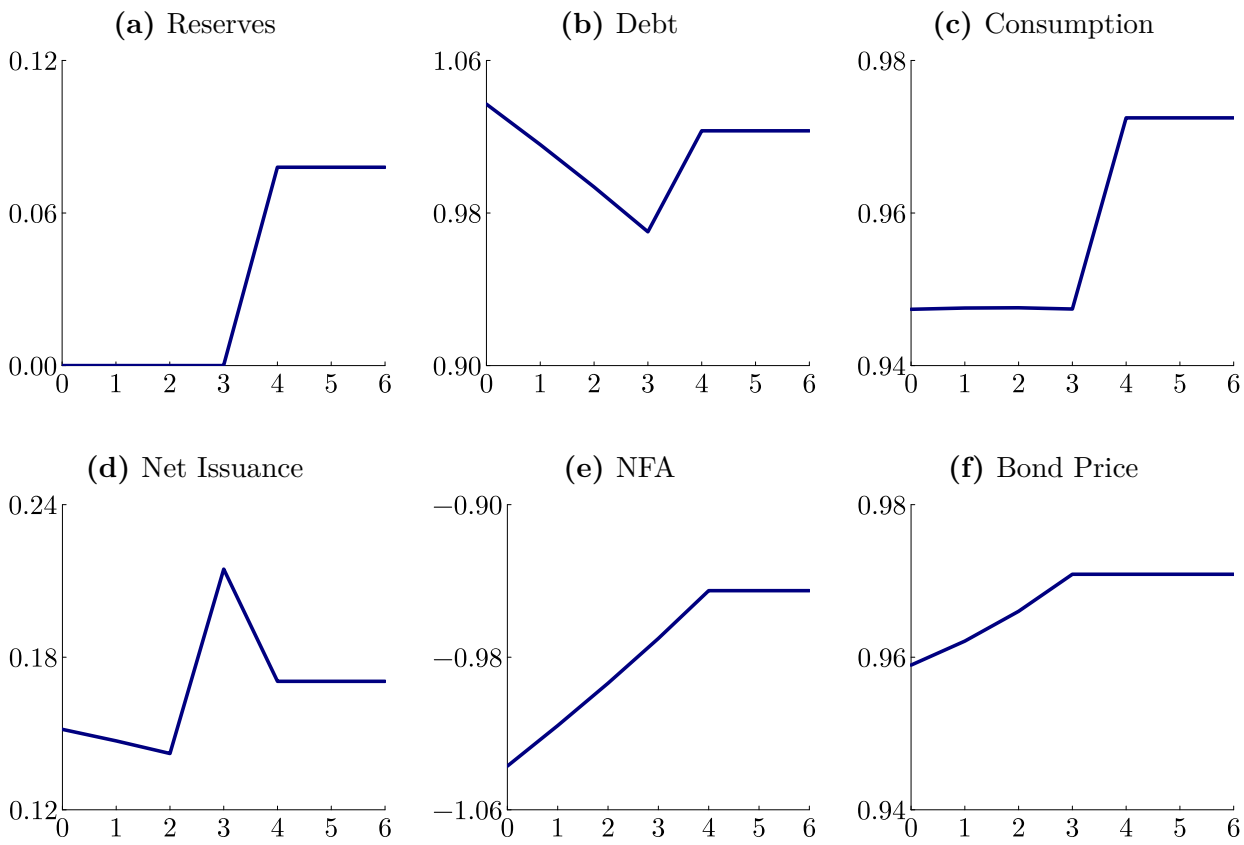


Figure 5: Deleveraging dynamics. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves. Panels (a), (b), and (e) plot beginning-of-period levels of reserves, debt, and NFA, respectively.

Sensitivity. In Figure 6, we examine how changes in the parameters for the debt duration and risk aversion alter the optimal portfolio for the government. Specifically, we vary δ and σ and recalibrate ϕ to match the same $b^-(0)$ as in the baseline calibration. The figure presents the value of $a^\star$.

Panel (a) shows that when the debt maturity becomes shorter, the government accumulates a larger amount of reserves. A shorter maturity implies that a larger fraction of the debt becomes due each period.[34] Therefore, the government needs a higher amount of reserves to be safe from a rollover crisis. For a maturity of 4 years, the stock of reserves can reach close to 30% of GDP. Panel (b) shows that for higher risk aversion, the government accumulates a larger amount of reserves. Higher risk aversion implies greater curvature in the utility function. Thus, reserves provide a higher marginal value when the government faces a run.

[34]Recall, however, that in the limit when the maturity is one period there is no scope for reserve accumulation, as established in Proposition 4.

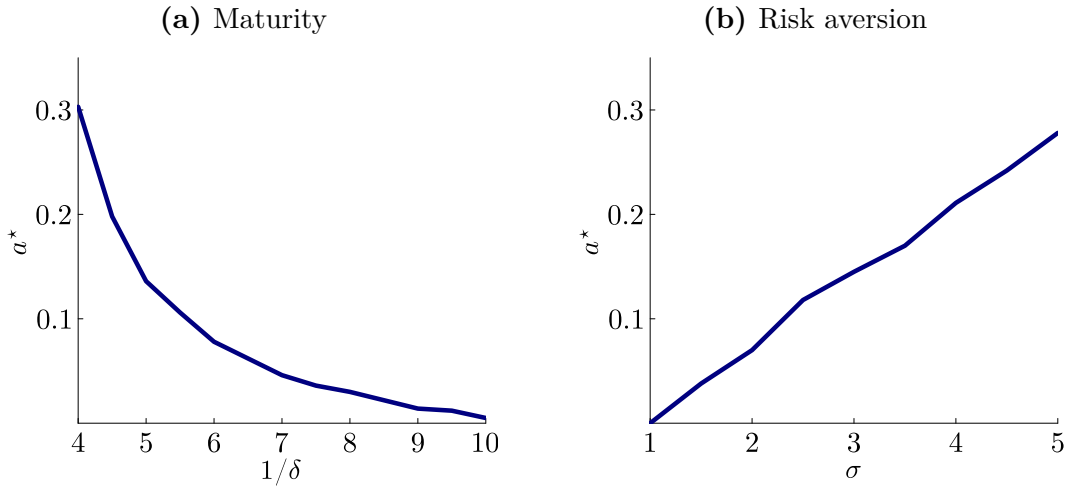


Figure 6: Sensitivity analysis. The panels show the level of reserves $a^\star$ for different parameter values for the (inverse of the) maturity δ and the risk aversion σ. In the simulations, we recalibrate the value of ϕ to match the same debt level $b^-(0)$ as in the baseline calibration.

Counterfactual without reserves. We next compare our benchmark economy with an economy where the government never accumulates reserves under repayment. Namely, the government starts with portfolio (a, b) at some t, and from $t+1$ onward, it is forced to set $a' = 0$.[35]

Figure 7 compares the bond price schedules with and without reserves. One can see that the bond price with reserves is always above the economy without reserves, strictly so for a range of values of debt. In the absence of reserves, exiting the crisis zone becomes more costly, as now the government's terminal point is $(0, b^-(0))$ instead of $(a^\star, b^\star)$. That is, the government must either cut consumption more through the deleveraging process or take more time to reach the safe zone. Because exiting is more costly, it follows that the debt threshold b^+ is reduced. The lack of access to reserves makes debt less sustainable.[36]

[35] We impose the restriction to never accumulate reserves only under repayment, so this means that V_D remains the same as in the baseline.

[36] Appendix E presents more details. Figure E.1 presents the policy function for NFA and consumption without reserves with the same baseline parameters. Figure E.2 presents a simulation where the government without reserves takes more time to exit the crisis zone and chooses lower consumption during this transition as it deleverages.

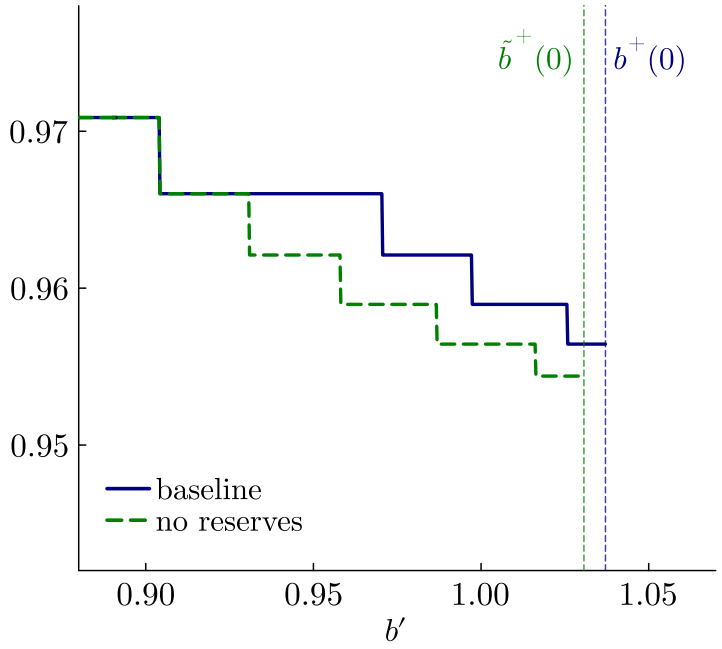


Figure 7: Bond prices schedules with and without reserves. The blue-solid line denotes the bond price schedule in the baseline model at zero reserves, $q(0, b')$. The green-dashed line denotes the bond price schedule in the economy without reserves, $\tilde{q}(b')$. The green vertical line $\tilde{b}^+(0)$ is given by $\tilde{V}_R^+(\tilde{b}^+(0)) = V_D(0)$, where $\tilde{V}_R^+$ corresponds to the value function when the government cannot accumulate reserves under repayment in the future.

3.5 Empirical Analysis

In this section, we provide empirical evidence consistent with the role for reserves in our theoretical framework. The theory generates two predictions. First, issuing debt and accumulating reserves can reduce sovereign spreads. Second, once accumulated, reserves are typically not drawn down. As discussed, the first prediction emerges in our model because reserves provide a liquidity backstop that reduces the probability of default, and the second feature emerges because the role of reserves is not to be used as insurance, but as a liquidity backstop that deters runs on the equilibrium path.

These predictions of our model contrast with those that point out to an insurance role of reserves, in particular Bianchi et al. (2018), where higher gross positions lead to higher spreads and the government depletes reserves in times of fiscal stress. In that framework, higher gross positions provide insurance against increases in borrowing costs. This is because higher reserves allow the government to smooth consumption when a persistent adverse

income shock leads to an increase in the probability of future default and sovereign spreads.[37] On the other hand, higher gross positions make the government more likely to default in the future, and thus sovereign spreads rise. Hence, the government trades off the insurance benefits from accumulating higher gross positions with the costs of higher spreads.

To examine whether the model predictions in our paper are borne out in the data, we collect data on international reserve holdings, sovereign spreads, external debt, and other macroeconomic variables for a group of 30 emerging economies. See Appendix I for full details of the dataset. We examine each of the two predictions in turn.

(i) Issuing debt to accumulate reserves reduces spreads. We test this prediction by estimating the following spreads regression for our panel of countries:

$$\text{Spread}_{it} = \beta_{\text{Res}} \, \text{Reserves}_{it} + \beta_{\text{Debt}} \, \text{Sov.Debt}_{it} + \beta_{\text{Int}} \, \text{Reserves}_{it} \times \text{Sov.Debt}_{it} + \beta'_{\text{M}} \, \text{M}_{it} + \epsilon_{it}$$

where β_{Res}, β_{Debt}, and β_{Int} are the coefficients of main interest, M_{it} is a vector of macroeconomic control variables and ϵ_{it} is a random error.[38]

Columns 1 and 2 in Table 2 confirm this theoretical prediction in our panel data: ceteris paribus, increasing debt and reserves in the same amount results in a statistically significant reduction in spreads. In particular, the discrete marginal effect of increasing debt and reserves simultaneously by one percentage point of GDP is a (statistically significant) reduction in spreads of roughly 10 basis points.[39] Moreover, our theory predicts that this portfolio adjustment, that is, simultaneously increasing debt and reserves, is particularly effective at lowering spreads when debt is relatively low (i.e., the economy is close to the safe zone). Columns 3 and 4 in Table 2 shows evidence consistent with this feature. In particular, the average marginal effect is approximately -15 bps (and significant) for the low-debt subsample

[37] In particular, states with high spreads are associated with states high marginal utility of consumption, and therefore by issuing debt and reserves, the government transfers resources effectively from periods of low marginal utility to high marginal utility.

[38] Following Sosa-Padilla and Sturzenegger (2023), the vector M_{it} includes measures for global risk aversion, the world interest rate, the country rating, and private external debt to GDP. We also include country fixed effects and year dummies. Table I.1 in Appendix I reports the full set of coefficients.

[39] We estimate the discrete marginal effect as:

$$\mathbb{E}\left[\text{Spread} \,|a_0+1, b_0+1\right] - \mathbb{E}\left[\text{Spread} \,|a_0, b_0\right] = \hat{\beta}_{\text{Res}} + \hat{\beta}_{\text{Debt}} + \hat{\beta}_{\text{Int}} \left[(a_0+1)\,(b_0+1) - a_0 b_0\right]$$

where a_0 and b_0 denote the initial (reference) values for the reserves and debt ratios (expressed as percentage points of GDP), respectively. Following our theory, we report the marginal effect for $a_0 = 0$ (as the typical exit path from the crisis zone involves carrying zero reserves until the last step) and setting b_0 to the mean debt ratio in our dataset. Results are similar if we evaluate the effect using the mean reserves ratio, and even stronger if we compute the effect at low debt levels.

versus an insignificant -11 bps for the high-debt subsample.

(ii) Reserves are 'not used.' Our model predicts that once a country accumulates reserves and exits the crisis zone, it does not draw down these reserves. The logic is that having reserves deters investors from running on the government bonds. That is, reserves are not used as insurance, but to rule out runs on the equilibrium path. We test this prediction by asking whether, in crisis windows, higher initial reserve buffers systematically forecast subsequent drawdowns of reserves.

Figure 8 plots the change in reserves during crisis episodes against the level of reserves held at the onset of the crisis, using data from the 2008–09 Global Financial Crisis and the 2020–21 Covid pandemic. Each point represents a country-crisis pair. The figure shows that for the majority (78%) of the country-crisis pairs feature an increase in reserves by the end of the crisis. In addition, the estimated relationship is effectively flat, and the associated p-value of 0.41 indicates no statistically significant correlation between initial reserve levels and reserve usage during these periods of global financial stress. These feature contrast with existing models of reserves which emphasize an insurance role of reserves (see e.g., Bianchi et al., 2018). In models of that class, higher initial reserve stocks should be drawn down in crises and larger initial buffers would allow for a larger drawdown to smooth out the crisis. Instead, the data align with our model's prediction: once accumulated, reserves are not used.[40]

Taken together, the evidence suggests that increases in debt and reserves are associated with reductions in spreads and that there is a lack systematic reserve drawdowns in times of fiscal stress. This empirical pattern is consistent with our model and contrasts with the insurance role of reserves in buffer-stock.

[40] Formally, we estimate

$$\Delta\text{Reserves}_i = \text{intercept} + \beta_{\text{Res}}\,\text{Reserves}_{i,0} + \epsilon_i,$$

where $\Delta\text{Reserves}_i$ is the change in reserves over the crisis window for the country-crisis pair i, $\text{Reserves}_{i,0}$ is the reserve ratio for that country at the onset of the crisis, and ϵ_i is a random error. Buffer-stock models predict $\beta_{\text{Res}} < 0$ and $\mathbb{E}[\Delta\text{Reserves}_i] < 0$; we find $\hat{\beta}_{\text{Res}} \approx 0$ (insignificant) and a positive mass of $\Delta\text{Reserves}_i$, consistent with no on-path use of reserves.

Table 2: Panel regressions

Dep. Variable:	Spread (in bps)			
	Full Sample		Low Debt	High Debt
	(1)	(2)	(3)	(4)
Reserves	-12.01***	-13.18***	-7.96***	-27.39***
	(0.71)	(1.13)	(1.55)	(2.74)
Sov.Debt	3.05***	2.32***	-1.04	2.35
	(0.64)	(0.84)	(1.76)	(1.57)
Reserves × Sov.Debt		0.05	-0.29***	0.42***
		(0.04)	(0.06)	(0.07)
Marg. Effect (in bps)	-8.96***	-9.61**	-14.58***	-11.10
[p-value]	[0.001]	[0.028]	[0.002]	[0.128]
Num.Obs.	4,497	4,497	2,464	2,033
R2 Adj.	0.412	0.412	0.552	0.420

Note: All specifications include year dummies, country FEs, and additional macroeconomic controls. Reserves and Sov.Debt are measured in percent of GDP. Robust standard errors are in parentheses. *p<0.1; **p<0.05; ***p<0.01. The 'Marg. Effect' row reports the discrete marginal effect (on spreads) of increasing reserves and debt by 1% of GDP (see footnote 39).

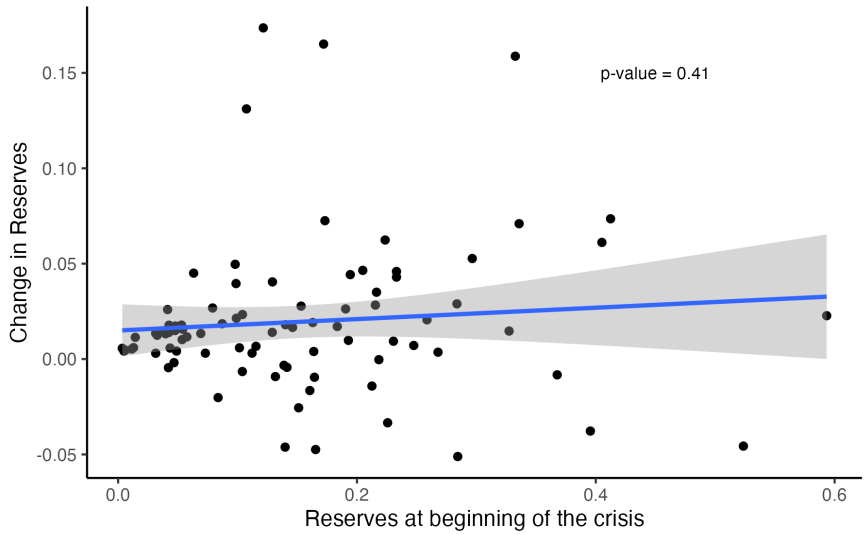


Figure 8: Reserves 'not used' during crises. Each dot represents a country-crisis observation. The figure also shows the line of best fit together with the 95% confidence interval.

4 Conclusions

This paper considers a government subject to the risk of a rollover crisis and asks, should the government accumulate reserves or reduce debt to make itself less vulnerable? The theory shows that the optimal policy entails initially reducing debt, followed by a subsequent increase in both debt and reserves. That is, a highly indebted government should first raise its net foreign asset position by lowering debt and abstaining from holding reserves. Once the government approaches the safe zone, it becomes optimal to issue more debt to accumulate reserves. Furthermore, the theory reveals how issuing debt to accumulate reserves can lower spreads by reducing the vulnerability to a rollover crisis.

Our findings speak to central bank policy discussions on the appropriate level of international reserves (e.g., IMF, 2016, 2023). Following a debt crisis, the IMF often prescribes that countries accumulate reserves to improve their liquidity position. Our results show, however, that holding reserves is not optimal at the beginning of a deleveraging process. Rather, a highly indebted government should first reduce debt and postpone the accumulation of reserves until the increase in reserves makes the government safe from a rollover crisis.

Our analysis suggests several directions for future research. For example, introducing a time-varying probability of rollover crises and fundamental shocks would be valuable to quantitatively examine the dynamics of international reserves, government debt, and spreads. It would also be interesting to study the joint management of international reserves and debt maturity structure.

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A Proofs

In this Appendix, we prove the propositions laid out in the main text. In Appendix B we state a number of auxiliary results that are used in the proofs below.

A.1 Proof of Proposition 1

Proof. Assume, by way of contradiction, that the optimal sequence is such that $c_t > c_{t+1}$ for some t. We will show that there is an alternative policy that delivers a strict welfare improvement. In particular, we consider a deviation where the government consumes less in period t and from period $t+1$ onward, the government follows the Markov equilibrium policies given the lower borrowing in period t.

Denote by $\mathcal{C}, \mathcal{B}, \mathcal{A}$ the policies in the Markov equilibrium. In particular, consider an alternative policy such that $\tilde{c}_s = \mathcal{C}(a_s, b_s), \tilde{b}_{s+1} = \mathcal{B}(a_s, b_s), \tilde{a}_{s+1} = \mathcal{A}(a_s, b_s)$ for all $s = 0, 1, \ldots, t-1$ and

$$\begin{aligned} \tilde{c}_t &= c_t - \varepsilon \\ \tilde{a}_{t+1} &= a_{t+1} \\ \tilde{b}_{t+1} &= b_{t+1} - \frac{\varepsilon}{q(a_{t+1}, b_{t+1})} \end{aligned}$$

and for $s = t+1, t+2, \ldots, T$, $\tilde{c}_s = \mathcal{C}(\tilde{a}_s, \tilde{b}_s), \tilde{b}_{s+1} = \mathcal{B}(\tilde{a}_s, \tilde{b}_s), \tilde{a}_{s+1} = \mathcal{A}(\tilde{a}_s, \tilde{b}_s)$.

Notice that the alternative policy satisfies the budget constraint in period t and thus is feasible. This follows from Lemma B.2, which shows that starting from an optimal policy, a small reduction in debt does not change the bond price.

The deviation yields expected lifetime utility

$$\tilde{U} = u(\tilde{c}_t) + \beta(1-\lambda)V_R^+(a_{t+1}, \tilde{b}_{t+1}) + \beta\lambda V_D(a_{t+1}),$$

where we used that $\tilde{a}_{t+1} = a_{t+1}$. The original plan gives

$$U = u(c_t) + \beta(1-\lambda)V_R^+(a_{t+1}, b_{t+1}) + \beta\lambda V_D(a_{t+1}).$$

Hence, the change in lifetime utility is, when divided by $\varepsilon > 0$

$$\begin{aligned}\frac{\Delta U}{\varepsilon} &= \frac{\tilde{U} - U}{\varepsilon} \\ &= \frac{[u(c_t - \varepsilon) - u(c_t)]}{\varepsilon} + \beta(1-\lambda)\frac{\left[V_R^+\left(a_{t+1}, b_{t+1} - \frac{\varepsilon}{q(a_{t+1},b_{t+1})}\right) - V_R^+(a_{t+1}, b_{t+1})\right]}{\varepsilon}.\end{aligned}$$

We have

$$\lim_{\varepsilon \downarrow 0} \frac{u(c_t - \varepsilon) - u(c_t)}{\varepsilon} = -u'(c_t)$$

and with the envelope condition

$$\lim_{\varepsilon \downarrow 0} \frac{V_R^+\left(a_{t+1}, b_{t+1} - \frac{\varepsilon}{q(a_{t+1},b_{t+1})}\right) - V_R^+(a_{t+1}, b_{t+1})}{\varepsilon} = \frac{u'(c_{t+1})}{q(a_{t+1}, b_{t+1})}\left[\left(\frac{\delta + r}{1+r}\right) + (1-\delta)q(a_{t+2}, b_{t+2})\right]$$

From the recursive definition of the fundamental price, we have

$$q(a_{t+1}, b_{t+1}) = \frac{(1-\lambda)}{1+r}\left[\left(\frac{\delta + r}{1+r}\right) + (1-\delta)q(a_{t+2}, b_{t+2})\right].$$

It is important to notice that the bond price does not change under the alternative policy. So, using $\beta(1+r) = 1$, we can substitute the recursive price formulation and simplify to obtain:

$$\lim_{\varepsilon \downarrow 0} \frac{\Delta U}{\varepsilon} = u'(c_{t+1}) - u'(c_t).$$

If, as assumed at the beginning of the proof, $c_t > c_{t+1}$, then by strict concavity of the utility function, we have that $u'(c_{t+1}) > u'(c_t)$, implying $\Delta U > 0$.

Therefore, a small reduction in c_t induced by slightly reduced borrowing raises lifetime utility, contradicting the optimality of the original plan, and we must have $c_t \leq c_{t+1}$ for all t. □

A.2 Proof of Lemma 1

Proof. We prove each statement in turn.

Part (i). Assume $(0, (1-\delta)b) \in \mathbf{S}$. Repaying coupons while facing a run implies that the government will be safe tomorrow, and the run problem reduces to choosing reserves $a' \geq 0$ with a continuation value in the safe zone. The objective function is strictly concave in a',

and the optimality condition implies

$$u'\Big(y+a-\Big(\frac{\delta+r}{1+r}\Big)\,b-\frac{a'}{1+r}\Big)\;\geq\;\beta(1+r)\,u'\Big(y+(1-\beta)\Big(a'-(1-\delta)b\Big)\Big)\;,$$

with equality if $a'>0$. Using $\beta(1+r)=1$, the interior condition implies equality of the arguments,

$$y+a-\Big(\frac{\delta+r}{1+r}\Big)\,b-\frac{a'}{1+r}=y+(1-\beta)\Big(a'-(1-\delta)b\Big)\,,$$

which solves to $a'=a-\delta b$, if feasible. Under the maintained condition of part (i), $a<\delta b$, the interior solution is not feasible, and the optimum is the corner $a'=0$.

Part (ii). If $a\geq\delta b$, then the unconstrained solution $a'=a-\delta b\geq 0$ is feasible. Substituting into the budget constraint and using $\beta(1+r)=1$, the induced consumption is constant at

$$c_S(a-b)=y+(1-\beta)(a-b)\,,$$

so this policy delivers a perfectly smooth consumption path. Hence, the optimal policy is $a'=a-\delta b$.

Comparison of $b^-(a)$ and $b^+(a)$. Recall that for any (a,b), we have $V_R^+(a,b)\geq V_R^-(a,b)$ because in the no-run scenario, the government can always replicate the feasible choices available under a run.

Suppose first that either $a<\delta b$ or $a\geq\delta b$ and $(a-\delta b,(1-\delta)b)\in\mathbf{C}$. In this case, the problem under a run is strictly constrained. If $a<\delta b$, the unconstrained choice $a'=a-\delta b$ is not feasible and the solution is at the corner $a'=0$. If instead $a\geq\delta b$ but $(a-\delta b,(1-\delta)b)\in\mathbf{C}$, then repaying today's coupons leaves the economy in the crisis zone, so under a run the government is restricted to continuation portfolios that remain vulnerable to rollover risk. By contrast, when the government does not face a run, it can issue bonds at the fundamental price and choose portfolios that relax the constraints faced under a run. Hence, the feasible set is strictly larger without a run, implying $V_R^+(a,b)>V_R^-(a,b)$.

Evaluating at $b=b^-(a)$ yields

$$V_R^+(a,b^-(a))>V_R^-(a,b^-(a))=V_D(a)=V_R^+(a,b^+(a))\,.$$

Since $V_R^+(a,b)$ is strictly decreasing in b, we conclude that $b^-(a)<b^+(a)$.

Finally, suppose that $a \geq \delta b$ and $(a - \delta b, (1-\delta)b) \in \mathbf{S}$. Then the policy

$$a' = a - \delta b, \qquad b' = (1-\delta)b$$

is feasible both with and without a run. By construction, following this policy yields constant consumption $c_S(a-b)$ and a safe continuation, and it therefore achieves the same value in the run and no-run problems: $V_R^-(a,b) = V_R^+(a,b)$. From this, and using the definitions of $b^-(a)$ and $b^+(a)$ from equations (7) and (6) we obtain $b^-(a) = b^+(a)$. □

A.3 Proof of Proposition 2

Proof. We have that $V_D(a) = V_R^-(a, b^-(a))$. Totally differentiating with respect to a, we obtain

$$\frac{\partial V_D(a)}{\partial a} = \frac{\partial V_R^-(a, b^-(a))}{\partial a} + \frac{\partial V_R^-(a, b^-(a))}{\partial b}\frac{\partial b^-}{\partial a}.$$

Rearranging, we get

$$\frac{\partial b^-}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-\left(a, b^-(a)\right)}{\partial a}}{\frac{\partial V_R^-(a, b^-(a))}{\partial b}}. \tag{A.1}$$

Proceeding analogously for $b^+(a)$, we have that

$$\frac{\partial b^+}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+\left(a, b^+(a)\right)}{\partial a}}{\frac{\partial V_R^+(a, b^+(a))}{\partial b}}. \tag{A.2}$$

Applying the envelope condition in (3), (5), and (10), we obtain

$$\begin{aligned}
\frac{\partial V_R^-(a,b)}{\partial a} &= u'\left(c_R^-(a,b)\right), \\
\frac{\partial V_D(a)}{\partial a} &= u'\left(c_D(a)\right), \\
\frac{\partial V_R^+(a,b)}{\partial a} &= u'\left(c_R^+(a,b)\right).
\end{aligned}$$

The denominators in (A.1) and (A.2) are negative by Lemma B.1. In the auxiliary Lemmas B.5 and B.6, we establish that $c_R^-(a,b) < c_D(a)$ and that $c_R^+(a,b) < c_D(a)$, respectively. Given these inequalities, the result of this proof that $b^-(\cdot)$ and $b^+(\cdot)$ are strictly increasing in a follows from the strict concavity of $u(\cdot)$. □

A.4 Proof of Proposition 4

Proof. With $\delta = 1$, the value of repayment in a run becomes

$$V_R^-(a,b) = \max_{a' \geq 0} u\left(y + a - b - \frac{a'}{1+r}\right) + \beta V_S(a'),$$

where we have used that the government is safe tomorrow after repaying all the debt. The solution for a' is given by $a' = \max\{0, a-b\}$. Under the assumption that $a < b$, and thus a negative net foreign asset position, which we will verify below, we have $a' = 0$. Replacing the values of V_S and V_D from (9) and (10), we have that the government is indifferent between repaying in a run and defaulting if the following holds:

$$u(y + a - b) + \frac{\beta}{1-\beta} u(y) = \frac{u(y + a(1-\beta)) - \phi}{1-\beta}\,.$$

The lowest-NFA safe portfolio solves

$$\begin{aligned} &\min_{a \geq 0, b} \; a - b, \\ &\text{subject to} \\ &u(y + a - b) + \frac{\beta}{1-\beta} u(y) \geq \frac{u(y + a(1-\beta)) - \phi}{1-\beta}\,. \end{aligned}$$

Since $u(\cdot)$ is strictly increasing and continuous, the constraint must bind. Then the original problem is equivalent to the following:

$$\begin{aligned} &\min_{a \geq 0, b} \; a - b, \\ &\text{subject to} \\ &\qquad a - b = \mathcal{L}(a) \equiv u^{-1}\left[\frac{u(y + (1-\beta)a) - \phi - \beta u(y)}{1-\beta}\right] - y\,. \end{aligned}$$

Solving the problem above is equivalent to minimizing $\mathcal{L}(a)$. Since $u(\cdot)$ is strictly increasing and continuous, so is $u^{-1}(\cdot)$. Thus, $\mathcal{L}(\cdot)$ is strictly increasing in a. Therefore, the minimum of $\mathcal{L}(a)$ is attained at the lowest possible value for a, namely $a^\star = 0$. Moreover, notice that plugging this into the constraint gives us

$$u(y - b^\star) = u(y) - \frac{\phi}{1-\beta}\,.$$

Since $u(\cdot)$ is strictly increasing and continuous and $\phi > 0$, we must have $b^\star > 0$. Therefore, we have obtained $0 = a^\star < b^\star$, which verifies the assumption above.

With $\delta = 0$, the value of repaying in a run becomes $V_R^-(a,b) = \frac{u(y+(a-b)(1-\beta))}{1-\beta}$ and the lowest-NFA safe portfolio solves

$$\min_{a\geq 0, b} \; a - b,$$
$$\text{subject to}$$
$$\frac{u(y+(a-b)(1-\beta))}{1-\beta} \geq \frac{u(y+a(1-\beta))-\phi}{1-\beta}.$$

Once again, the constraint must bind. Therefore, following the same logic from above, the problem becomes

$$\min_{a\geq 0} \tilde{\mathcal{L}}(a) = \frac{u^{-1}\left(u(y+(1-\beta)a)-\phi\right)-y}{1-\beta}.$$

Since $u(\cdot)$ and its inverse are strictly increasing and continuous, we have that $\tilde{\mathcal{L}}(a)$ is strictly increasing in a. Therefore, the minimum is attained at zero, namely $a^\star = 0$. □

A.5 Proof of Proposition 5

Proof. The proof is by construction. Fix income y and the discount factor β, with $\beta(1+r) = 1$.

We first show that, for appropriate choices of the debt threshold $b^-(0)$ and maturity δ, a default cost ϕ rationalizes $b^-(0)$ given δ. Consider a debt level x and a maturity parameter δ such that

$$\delta < \frac{y(1+r)}{x} - r. \tag{A.3}$$

There exists a unique value of the default cost ϕ such that $b^-(0) = x$. To see this, define $\phi(x)$ implicitly by the indifference condition

$$V_R^-(0,x) = V_D(0),$$

which yields

$$\phi(x) = u(y) - \left[(1-\beta)u\Big(y - \Big(\tfrac{\delta+r}{1+r}\Big)x\Big) + \beta u(y-(1-\beta)(1-\delta)x)\right].$$

This function is well defined for $x < \frac{y(1+r)}{\delta+r}$. Moreover, ϕ is continuous in x, satisfies

$\phi(0) = 0$, is strictly increasing ($\phi'(x) > 0$), and diverges to infinity as $x \to \frac{y(1+r)}{\delta+r}$ from below. Hence, there exists a unique $\bar{\phi} = \phi(b^-(0))$. This implies that, in the relevant region of the parameter space, $b^-(0)$ can be treated as being externally pinned down by ϕ when varying δ as long as (A.3) is satisfied.

Next, choose x and δ such that (A.3) holds and

$$y < x < \frac{y(1+r)}{r}\,.$$

These inequalities ensure that the right-hand side of (A.3) lies in the interval $(0, 1)$. By the argument above, we can select ϕ so that $b^-(0) = x$. Define

$$\Delta \equiv \frac{y(1+r)}{x} - r$$

and

$$F(\delta, x) \equiv \beta(1-\delta)\Big[u'\Big(y - \Big(\tfrac{\delta+r}{1+r}\Big)\,x\Big) - u'(y - (1-\beta)(1-\delta)x)\Big] - u'(y)\,.$$

Using $\beta(1+r) = 1$, we have $F(0, b^-(0)) = -u'(y) < 0$. Moreover, by the Inada condition,

$$\lim_{\delta \uparrow \Delta} F(\delta, b^-(0)) = \infty\,,$$

where ϕ is adjusted so as to keep $b^-(0) = x$, as established above. Since F is continuous in δ, there exists $\varepsilon > 0$ such that

$$F(\delta, b^-(0)) > 0 \quad \text{for all } \delta \in (\Delta - \varepsilon, \Delta)\,.$$

This establishes that condition (15) is satisfied. □

A.6 Proof of Proposition 6

The proof has two parts:

Proof. **Part (i).** Suppose the government chooses $(\tilde{a}, \tilde{b}) \in \mathbf{S}$ such that $(\tilde{a}, \tilde{b}) \neq (a^\star, b^\star)$. We

can show that the utility is then lower. To see this, we need to show that

$$u\left(y+a-b+\left(\frac{\tilde{b}-\tilde{a}}{1+r}\right)\right)+\frac{\beta}{1-\beta}u\left(y+(1-\beta)(\tilde{a}-\tilde{b})\right) \\ < u\left(y+a-b+\left(\frac{b^\star-a^\star}{1+r}\right)\right)+\frac{\beta}{1-\beta}u\left(y+(1-\beta)(a^\star-b^\star)\right). \quad \text{(A.4)}$$

Define the following objects:

$$\Delta \equiv \frac{b^\star-a^\star}{1+r}-\frac{\tilde{b}-\tilde{a}}{1+r}>0,$$
$$\hat{c} \equiv y+a-b+\left(\frac{b^\star-a^\star}{1+r}\right) < y+(1-\beta)(a^\star-b^\star) \equiv c^\star.$$

where the first and second inequality follow respectively from the definition of $(a^\star, b^\star)$ and $a-b<a^\star-b^\star$. Using these expressions, we can rewrite (A.4) as

$$u(\hat{c}-\Delta)+\frac{\beta}{1-\beta}u\left(c^\star+\left(\frac{1-\beta}{\beta}\right)\Delta\right) < u(\hat{c})+\frac{\beta}{1-\beta}u\left(c^\star\right).$$

Rearranging, we need to show that the following holds:

$$u(\hat{c})-u(\hat{c}-\Delta) > \frac{\beta}{1-\beta}\left[u\left(c^\star+\left(\frac{1-\beta}{\beta}\right)\Delta\right)-u(c^\star)\right]. \quad \text{(A.5)}$$

The result follows from an application of the mean-value theorem, the strict concavity of $u(\cdot)$, and the fact that $\hat{c}<c^\star$. Namely, there exists $x\in(\hat{c}-\Delta,\hat{c})$, such that

$$u'(x)\Delta = u(\hat{c})-u(\hat{c}-\Delta). \quad \text{(A.6)}$$

Similarly, there exists $z\in\left(c^\star, c^\star+\left(\frac{1-\beta}{\beta}\right)\Delta\right)$, such that

$$u'(z)\Delta = \frac{u\left(c^\star+\left(\frac{1-\beta}{\beta}\right)\Delta\right)-u(c^\star)}{(1-\beta)/\beta}. \quad \text{(A.7)}$$

Since $z>c^\star>\hat{c}>x, \Delta>0$ and $u(\cdot)$ is strictly concave, we have that $u'(x)>u'(z)$. Using this strict inequality and rearranging (A.6) and (A.7), we obtain (A.5), as we wanted to show.

□

Proof. **Part (ii).** Suppose, toward a contradiction, that $a'>0$. Consider an alternative

portfolio given by

$$(\tilde{a}, \tilde{b}) = \left(a' - \varepsilon, b' - \frac{\varepsilon(1-\lambda)}{q(a', b')(1+r)} \right),$$

where $\varepsilon > 0$.

We first show that for small ε, the bond price remains the same. There are two cases to consider. If the government is in the interior of an iso-T region, then by continuity, it is straightforward to show that it can reduce reserves and debt and remain in the same iso-T region. If the government is at the boundary between two iso-T regions, Auxiliary Lemma B.7 shows that this alternative portfolio with lower reserves and lower debt moves the government into the interior of the same iso-T region, keeping the bond price unchanged.

With this alternative policy $(\tilde{a}, \tilde{b})$, consumption in period t is therefore given by

$$\begin{aligned} \tilde{c}_R^+(a, b) &= c_R^+(a, b) + \frac{\varepsilon}{1+r} - q(a', b') \frac{\varepsilon(1-\lambda)}{q(a', b')(1+r)} \\ &= c_R^+(a, b) + \frac{\lambda \varepsilon}{1+r}. \end{aligned}$$

Notice that there is a drop in consumption under default in $t+1$, since the alternative portfolio has lower reserves ($a' > a' - \varepsilon$). The change in lifetime welfare induced by this alternative portfolio, which we denote by ΔU, is given by

$$\begin{aligned} \Delta U = &\left[u(\tilde{c}_R^+(a, b)) - u(c_R^+(a, b)) \right] + \beta\lambda \left[V_D(a' - \varepsilon) - V_D(a') \right] \\ &+ \beta(1-\lambda) \left[V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a', b') \right]. \end{aligned} \tag{A.8}$$

Equivalently, with $\varepsilon > 0$

$$\begin{aligned} \frac{\Delta U}{\varepsilon} = &\frac{\left[u(\tilde{c}_R^+(a, b)) - u(c_R^+(a, b)) \right]}{\varepsilon} + \beta\lambda \frac{\left[V_D(a' - \varepsilon) - V_D(a') \right]}{\varepsilon} \\ &+ \beta(1-\lambda) \frac{\left[V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a', b') \right]}{\varepsilon}. \end{aligned}$$

We want to establish that $\lim_{\varepsilon \downarrow 0} \frac{\Delta U}{\varepsilon} > 0$. We have that

$$\lim_{\varepsilon \downarrow 0} \frac{\left[u(c_R^+(a, b) + \frac{\lambda \varepsilon}{1+r}) - u(c_R^+(a, b)) \right]}{\varepsilon} = u'(c_R^+(a, b)) \frac{\lambda}{1+r}, \tag{A.9}$$

and, from the envelope condition:

$$\lim_{\varepsilon \downarrow 0} \frac{V_D(a' - \varepsilon) - V_D(a')}{\varepsilon} = -u'(c_D(a'))\,. \tag{A.10}$$

As established in Lemma B.7, we have

$$\lim_{\varepsilon \downarrow 0} \frac{\left[V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a', b')\right]}{\varepsilon} = 0\,. \tag{A.11}$$

Plugging (A.9), (A.10) and (A.11) into (A.8), and using $\beta(1+r) = 1$, we arrive at:

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \frac{\Delta U}{\varepsilon} &= \frac{\lambda}{1+r}\left[u'(c_R^+(a,b)) - u'(c_D(a'))\right] \\
&\geq \frac{\lambda}{1+r}\left[u'(c_R^+(a',b')) - u'(c_D(a'))\right] \\
&> 0\,.
\end{aligned}$$

The first inequality follows from the strict concavity of the $u(\cdot)$ and from Proposition 1, which establishes $c_R^+(a,b) \leq c_R^+(a',b')$. The last inequality uses again the strict concavity of $u(\cdot)$ and the fact that $c_R^+(a',b') < c_D(a')$, a result established in Auxiliary Lemma B.5.

This deviation is profitable, contradicting the optimality of (a', b'). Therefore, we conclude that $a' = 0$. □

A.7 Proof of Corollary 1

Proof. Part (ii) of Proposition 6 indicates that the government chooses $a_{t+1} = 0$ for all $t < T$ (i.e., as long as it remains in the crisis zone). Part (i) of Proposition 6 indicates that the government exits the crisis zone by choosing $(a^\star, b^\star)$. Once the government reaches the safe zone, it stays in the safe zone. □

A.8 Proof of Proposition 7

Proof. If the government picks $(a^\star, b + a^\star - a)$, then the government is safe next period. This follows from the fact that $b + a^\star - a \leq b^\star$ and that $b^-(a)$ is increasing in a, as shown in Proposition 2. In addition, notice that $c = y + (1-\beta)(a-b)$ and the value of choosing this portfolio is given by $V_S(a-b)$. Given that $V_R^+(a,b) \leq V_S(a-b)$, from Lemma B.3, it thus follows that the portfolio $(a^\star, b + a^\star - a)$, achieves the optimal solution. □

Online Appendix to "International Reserve Management under Rollover Crises"

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B Auxiliary Lemmas

Here we state and prove a number of auxiliary lemmas that we use in the proofs of main propositions in Appendix A.

Lemma B.1 (Monotonicity of value functions)**.** *The value functions $V_R^+(a,b)$ and $V_R^-(a,b)$ are strictly decreasing in debt. Moreover, $V_R^+(a,b)$, $V_R^-(a,b)$ and $V_D(a)$ are strictly increasing in reserves.*

Proof. Fix a and let $b_1 < b_2$. Let (a_2', b_2') be optimal at (a, b_2), and let c_2 be implied by the budget constraint. Consider state (a, b_1) and choose

$$(a_1', b_1') = (a_2', b_2')\,.$$

The budget constraint implies

$$c_1 = c_2 + \left(\frac{\delta + r}{1+r} + q(a_2', b_2')(1-\delta)\right)(b_2 - b_1) > c_2\,,$$

since $b_2 - b_1 > 0$, $q(a_2', b_2') \geq 0$. Because u is strictly increasing and the continuation state is the same, we have

$$V_R^+(a, b_1) > V_R^+(a, b_2)\,.$$

The other monotonicity results follow analogously. □

Lemma B.2. *Let $(a,b) \in \mathbf{C}$ and let (a', b') denote the optimal policies, such that $(a', b') \in \mathbf{C}$. Then, for sufficiently small $\varepsilon > 0$, we have $q(a', b' - \varepsilon) = q(a', b')$.*

Proof. Assume, by way of contradiction, that for any $\varepsilon > 0$, $\Delta_q(\varepsilon) \equiv q(a', b' - \varepsilon) - q(a', b') > 0$. The consumption under the original policies (a', b', c) is:

$$c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + q(a', b')(b' - (1-\delta)b)\,,$$

Notice that if $(b' - (1-\delta)b) \leq 0$, then $V_R^+(a,b) = V_R^-(a,b)$ and $(a,b) \notin \mathbf{C}$. Therefore, if $(a,b) \in \mathbf{C}$, we must have $b' - (1-\delta)b > 0$. Consider an alternative policy to be such that the chosen portfolio is $(a', b' - \varepsilon)$. Under this alternative policy, consumption is:

$$\begin{aligned} c_\varepsilon &= y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r} + q(a', b' - \varepsilon)(b' - \varepsilon - (1-\delta)b) \\ &= c - \varepsilon q(a', b' - \varepsilon) + \Delta_q(\varepsilon)(b' - (1-\delta)b). \end{aligned}$$

Observe that $q(a', b')$ is a step function, so that there exists a small enough ε_m such that for $\varepsilon \leq \varepsilon_m$, $\Delta_q(\varepsilon)$ is invariant in ε, that is $\Delta_q(\varepsilon) = \Delta_q$. Therefore, we can find a threshold level for the perturbation:

$$\bar{\varepsilon} \equiv \frac{\Delta_q(b' - (1-\delta)b)}{(\Delta_q + q(a', b'))}$$

such that any $\varepsilon \in (0, \min\{\bar{\varepsilon}, \varepsilon_m\})$ delivers a consumption level higher than the original plan, $c_\varepsilon > c$. Moreover, because the future debt is lower under this alternative policy and because $V_R^+(a', b')$ is strictly decreasing in b', the alternative policy is welfare increasing, contradicting the optimality of (a', b', c). □

Lemma B.3. *Let $(a,b) \in \mathbf{C}$. If there exists $(a', b') \in \mathbf{S}$, such that $a - b = a' - b'$, then*

$$c_R^+(a,b) = y + (1-\beta)(a-b) \equiv c_S(a-b).$$

Proof. For any alternative portfolio choice $(\tilde{a}, \tilde{b}) \in \mathbf{S}$, write its NFA as

$$\tilde{a} - \tilde{b} = (a-b) - \varepsilon\,.$$

The value ε measures the deviation from the original NFA. The current consumption and the continuation value are, respectively

$$\begin{aligned} c(\varepsilon) &= c_S(a-b) + \beta\varepsilon\,, \\ \mathbb{E}V(\varepsilon) &= V_S((a-b) - \varepsilon)\,. \end{aligned}$$

Let's define $\mathcal{E}$ as the set of all possible ε deviations from the original NFA that keep the government in $\mathbf{S}$,

$$\mathcal{E} = \left\{\varepsilon : \exists(\tilde{a}, \tilde{b}) \in \mathbf{S} \quad \text{such that} \quad \tilde{a} - \tilde{b} = (a-b) - \varepsilon\right\}.$$

The best alternative portfolio is therefore the one that solves the following univariate problem

$$\max_{\varepsilon \in \mathcal{E}} u(c_S(a-b)+\beta\varepsilon) + \beta \frac{u(c_S(a-b)-(1-\beta)\varepsilon)}{1-\beta}\,.$$

The first-order condition gives

$$u'(c_S(a-b)+\beta\varepsilon)\beta = \beta u'(c_S(a-b)-(1-\beta)\varepsilon)\,.$$

By the strict concavity of $u(\cdot)$, the unique solution is $\varepsilon = 0$, and the optimal portfolio must satisfy $a' - b' = a - b$. Thus, $c_R^+(a,b) = c_S(a-b)$. □

Lemma B.4. *The lowest-safe NFA portfolio* $(a^\star, b^\star) = \arg\min\{a - b : (a,b) \in \mathbf{S}\}$ *is such that* $a^\star < b^\star$.

Proof. By definition, $\forall (a,b) \in \mathbf{S}$, we must have $a^\star - b^\star \leq a - b$. Since $(0, b^-(0)) \in \mathbf{S}$, we have

$$a^\star - b^\star \leq 0 - b^-(0) < 0$$

and we conclude that $a^\star < b^\star$. □

Lemma B.5. *For any* $(a,b) \in \mathbf{C}$*, we must have* $c_R^+(a,b) < c_D(a)$.

Proof. Fix $(a_0, b_0) \in \mathbf{C}$ and let $\{c_t, a_{t+1}, b_{t+1}\}_{t\geq 0}$ denote the equilibrium allocations conditional on repayment and a good sunspot (rollover). Define $c_R^+(a_0,b_0) \equiv c_0$. Let $T \geq 0$ be the first date such that the beginning-of-period portfolio is in the safe zone, $(a_T, b_T) \in \mathbf{S}$.

Along repayment, the budget constraint implies the following:

$$\begin{aligned}
c_t &= y + a_t - \left(\frac{\delta + r}{1+r}\right) b_t - \frac{a_{t+1}}{1+r} + q_t(a_{t+1}, b_{t+1})(b_{t+1} - (1-\delta)b_t) \\
&\leq y + a_t - \left(\frac{\delta + r}{1+r}\right) b_t - \frac{a_{t+1}}{1+r} + \frac{1}{1+r}(b_{t+1} - (1-\delta)b_t) \\
&\leq y + a_t - b_t - \frac{1}{1+r}(a_{t+1} - b_{t+1})
\end{aligned}$$

where we used that $q_t \leq \frac{1}{1+r}$ and, as argued in Lemma B.2, $b_{t+1} > (1-\delta)b_t$ that whenever in the crisis zone and the government can rollover.

Multiplying the constraint above by β^t, summing from $t = 0$ to $t = T-1$, using

$\beta(1+r)=1$, we write

$$\sum_{t=0}^{T-1}\beta^t c_t \leq \sum_{t=0}^{T-1}\beta^t y + \Big(a_0 - b_0\Big) - \beta^T(a_T - b_T)\,.$$

Since $(a_T, b_T) \in \mathbf{S}$, consumption from T onward is constant at $c_S = y + (1-\beta)(a_T - b_T)$, and so we can write

$$\sum_{t=T}^{\infty}\beta^t c_t = \beta^T \frac{c_S}{1-\beta} = \beta^T\left(\frac{y}{1-\beta} + a_T - b_T\right).$$

Adding the last two expressions gives

$$\sum_{t=0}^{\infty}\beta^t c_t \leq \frac{y}{1-\beta} + (a_0 - b_0) = \frac{y + (1-\beta)(a_0 - b_0)}{1-\beta}\,.$$

Using $c_D(a_0) = y + (1-\beta)a_0$ and recalling that Proposition 1 implies $c_0 \leq c_t$ for all t:

$$\frac{c_0}{1-\beta} = \sum_{t=0}^{\infty}\beta^t c_0 \leq \sum_{t=0}^{\infty}\beta^t c_t \leq \frac{y + (1-\beta)(a_0 - b_0)}{1-\beta} < \frac{y + (1-\beta)(a_0)}{1-\beta} = \frac{c_D(a_0)}{1-\beta}\,.$$

That is, we have $c_R^+(a,b) < c_D(a)$, as we wanted to show. □

Lemma B.6. *For any $(a,b) \in \mathbf{C}$, we must have $c_R^-(a,b) < c_D(a)$.*

Proof. It suffices to show that $c_R^-(a,b) \leq c_S(a-b)$ since $c_S(a-b) \equiv y + (1-\beta)(a-b) < y + (1-\beta)a \equiv c_D(a)$. Notice that, using $\beta(1+r)=1$, we have:

$$\begin{aligned} c_R^-(a,b) &\equiv y + a - \left(\frac{\delta + r}{1+r}\right)b - \frac{a'}{1+r} \\ &= c_S(a-b) + \beta(a - \delta b - a')\,. \end{aligned}$$

If $a < \delta b$, as $a' \geq 0$ requires $a - \delta b - a' < 0$ and $c_R^-(a,b) < c_S(a-b)$, and the result is shown. If $a > \delta b$, notice that $a' = a - \delta b > 0$ gives a stationary consumption $c_S(a-b)$ if $(a - \delta b, (1-\delta)b) \in \mathbf{S}$. Such policy is feasible, and by Lemma B.3 optimal, under $V_R^+(a,b)$. Thus $V_R^-(a,b) = V_R^+(a,b)$, thereby $(a,b) \notin \mathbf{C}$.
Finally, recalling that $a > \delta b$, consider the first-order condition, together with the envelope

condition, when $(a', (1-\delta)b) \in \mathbf{C}$:

$$\begin{aligned} u'(c_R^-(a,b)) &\geq (1-\lambda)u'(c_R^+(a',(1-\delta)b)) + \lambda u'(c_D(a')) \\ &> (1-\lambda)u'(c_D(a')) + \lambda u'(c_D(a')) \\ &= u'(c_D(a')) \end{aligned}$$

where the second inequality uses $c_R^+(a',(1-\delta)b) < c_D(a')$ by Lemma B.5 and strict concavity of $u(\cdot)$. Next, assume by way of contradiction, that $c_R^-(a,b) \geq c_D(a)$. By the strict concavity of $u(\cdot)$, we have

$$u'(c_D(a)) \geq u'(c_R^-(a,b)) > u'(c_D(a'))$$

This requires $c_D(a) < c_D(a')$, and so $a' > a$. As a result, $0 > a - a' > a - \delta b - a'$. Therefore, $c_R^-(a,b) < c_S(a-b)$, which leads to $c_D(a) \leq c_R^-(a,b) < c_D(a)$, a contradiction. □

Lemma B.7. *Consider a government with initial portfolio $(a,b) \in \mathbf{C}$ that lies on the boundary of the iso-T region for $T = J$. That is, starting from (a,b), the sovereign is indifferent between exiting the crisis zone after J periods and after $J+1$ periods. Then there exists a small $\varepsilon > 0$ such that the alternative portfolio*

$$(\tilde{a}, \tilde{b}) \equiv \left(a - \varepsilon,\ b - \frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)} \right)$$

lies on the iso-T region for $T = J$. Moreover,

$$\lim_{\varepsilon \downarrow 0} \frac{V_R^+(\tilde{a},\tilde{b}) - V_R^+(a,b)}{\varepsilon} = 0\,.$$

Proof. Let $(a,b) \in \mathbf{C}$ be such that the government is indifferent between exiting the crisis zone in J or $J+1$ periods: $V_R^+(a,b) = W^J(a,b) = W^{J+1}(a,b)$, where $W^T(a,b)$ denotes the sovereign's value under a plan that exit the crisis zone in T periods. Define

$$\Delta W = W^J\left(a - \varepsilon, b - \frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)} \right) - W^{J+1}\left(a - \varepsilon, b - \frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)} \right)$$

We want to argue that $\frac{\Delta W}{\varepsilon} > 0$ when ε is sufficiently small. In that case, at $(\tilde{a},\tilde{b})$ the government strictly prefers to exit in J periods rather than $J+1$, so $(\tilde{a},\tilde{b})$ is in the interior of iso-T J, and shares the same price as the portfolio (a,b).

Using that $W^J(a,b) = W^{J+1}(a,b)$, we can write the above as

$$\frac{\Delta W}{\varepsilon} = \frac{\left[W^J\left(a-\varepsilon, b-\frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^J(a,b)\right]}{\varepsilon} - \frac{\left[W^{J+1}\left(a-\varepsilon, b-\frac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^{J+1}(a,b)\right]}{\varepsilon} \tag{B.1}$$

Now, we have that

$$\lim_{\varepsilon\downarrow 0} \frac{W^J\left(a-\varepsilon,\, b-\dfrac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^J(a,b)}{\varepsilon} = -\frac{\partial W^J(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^J(a,b)}{\partial b}.$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{W^{J+1}\left(a-\varepsilon,\, b-\dfrac{\varepsilon(1-\lambda)}{q(a,b)(1+r)}\right) - W^{J+1}(a,b)}{\varepsilon} = - \frac{\partial W^{J+1}(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^{J+1}(a,b)}{\partial b}.$$

Replacing these last two equations and taking the limit in (B.1)

$$\lim_{\varepsilon\downarrow 0}\frac{\Delta W}{\varepsilon} = \left[-\frac{\partial W^J(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^J(a,b)}{\partial b}\right] - \left[-\frac{\partial W^{J+1}(a,b)}{\partial a} - \frac{1-\lambda}{q(a,b)(1+r)}\frac{\partial W^{J+1}(a,b)}{\partial b}\right]$$

From the envelope conditions, we have

$$\begin{aligned}
\frac{\partial W^J(a,b)}{\partial a} &= u'(c^J)\,, \\
\frac{\partial W^J(a,b)}{\partial b} &= -u'(c^J)\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^J\right]\,, \\
\frac{\partial W^{J+1}(a,b)}{\partial a} &= u'(c^{J+1})\,, \\
\frac{\partial W^{J+1}(a,b)}{\partial b} &= -u'(c^{J+1})\left[\left(\frac{\delta+r}{1+r}\right) + (1-\delta)q^{J+1}\right]\,.
\end{aligned}$$

where c^T is the current consumption under repayment when exiting in T periods, q^T the bond price when exiting in T periods, for $T \in \{J, J+1\}$. Replacing these envelopes and

rearranging

$$\lim_{\varepsilon \downarrow 0} \frac{\Delta W}{\varepsilon} = u'(c^J)\left[-1 + \frac{1-\lambda}{q(a,b)(1+r)} \left(\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^J \right) \right] + u'(c^{J+1}) \left[1 - \frac{1-\lambda}{q(a,b)(1+r)} \left(\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^{J+1} \right) \right]. \tag{B.2}$$

The term in the first bracket is zero. To see why, recall that the price recursion implies that

$$q^{J+1} = \frac{1-\lambda}{1+r} \left[\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^J \right].$$

Recall that if the government starts with $(a, b) \in \mathbf{C}$ and is indifferent between exiting in J or $J+1$ periods, then, as in Aguiar and Amador (2013), equilibrium prices assume the government exits the crisis zone in J periods. Therefore, when that portfolio (a, b) was chosen, in the previous period, the government was exiting the crisis zone in $J+1$ periods, which means $q(a, b) = q^{J+1}$. As a result,

$$\frac{1-\lambda}{q(a,b)(1+r)} \equiv \frac{1-\lambda}{q^{J+1}(1+r)} = \left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^J . \tag{B.3}$$

As a result, we can write (B.2) as

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\Delta W}{\varepsilon} &= u'(c^{J+1}) \left[1 - \frac{\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^{J+1}}{\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^J} \right] \\ &= u'(c^{J+1}) \left[\frac{(1-\delta)(q^J - q^{J+1})}{\left(\frac{\delta + r}{1+r} \right) + (1-\delta) q^J} \right] > 0 . \end{aligned}$$

where the inequality follows from $q^J > q^{J+1}$. Therefore, we established that at $(\tilde{a}, \tilde{b})$ the government strictly prefers to exit the crisis zone in J rather than $J+1$ periods. As a result, $(\tilde{a}, \tilde{b})$ lies in the interior of the iso-T with $T = J$, and this portfolio shares the same bond price as (a, b), that is $q(\tilde{a}, \tilde{b}) = q(a, b)$.

Finally, we argue that $\lim_{\varepsilon \downarrow 0} \dfrac{V_R^+(\tilde{a}, \tilde{b}) - V_R^+(a, b)}{\varepsilon} = 0$. Using the envelope conditions, we

arrive at

$$\frac{\partial V_R^+(a,b)}{\partial a} = u'(c_R^+(a,b))$$
$$\frac{\partial V_R^+(a,b)}{\partial b} = -u'(c_R^+(a,b))\left[\left(\frac{\delta+r}{1+r}\right)+(1-\delta)q^J\right].$$

Therefore, we have

$$\lim_{\varepsilon\downarrow 0}\frac{\left[V_R^+(\tilde{a},\tilde{b})-V_R^+(a,b)\right]}{\varepsilon} = u'(c_R^+(a,b))\left[-1+\frac{1-\lambda}{q(a,b)(1+r)}\left(\left(\frac{\delta+r}{1+r}\right)+(1-\delta)q^J\right)\right].$$

Using (B.3), the term in brackets is zero. Therefore, we conclude that

$$\lim_{\varepsilon\downarrow 0}\frac{\left[V_R^+(\tilde{a},\tilde{b})-V_R^+(a,b)\right]}{\varepsilon} = 0.$$

□

C A special case for Proposition 3

In the main text, we provided a proof on the sufficient conditions for $a^\star$. In this appendix, we show a parametric condition for $a^\star > 0$ under quadratic utility. There are two parts. First, we derive the cutoff $b^-(0)$ as a function of the parameters $(y, r, \delta, \phi, \bar{c})$. Then, we substitute $b^-(0)$ into inequality (15). In what follows, we assume $\delta \in (0,1)$, $r > 0$ and $\phi > 0$.
Let us assume that the utility function is quadratic, defined for $c \leq \bar{c}$:

$$u(c) = -\frac{1}{2}(c - \bar{c})^2$$

with $y < \bar{c}$. Here is a summary of the results. Next we present the detailed derivations. The cutoff:

$$b^-(0) = \frac{(1+r)\left[(y-\bar{c})r + \sqrt{(y-\bar{c})^2 r^2 + 2r(r+\delta^2)\phi}\right]}{r(r+\delta^2)} > 0$$

The necessary condition:

$$\phi \; > \; \frac{(y-\bar{c})^2\, r\,[\,r + 2\delta - \delta^2\,]}{2\,(1-\delta)^2\,\delta^2} \equiv \underline{\phi} > 0$$

C.1 Deriving $b^-(0)$.

First, observe that the $b^-(0)$ equation is

$$u\left(y - \left(\frac{\delta + r}{1+r}\right) b^-(0)\right) + \beta \frac{u\left(y - (1-\beta)(1-\delta) b^-(0)\right)}{1-\beta} = \frac{u(y) - \phi}{1-\beta}$$

Let $x \equiv b^-(0)$ to simplify on notation and substitute the utility function

$$-\frac{1}{2}\left(y - \frac{\delta + r}{1+r}x - \bar{c}\right)^2 - \frac{1}{2}\frac{\beta}{(1-\beta)}\left(y - (1-\beta)(1-\delta)x - \bar{c}\right)^2 = \frac{-\frac{1}{2}(y-\bar{c})^2 - \phi}{1-\beta}$$

Multiplying by $-2(1-\beta)$ and simplifying:

$$(1-\beta)\left(y - \bar{c} - \frac{\delta + r}{1+r}x\right)^2 + \beta\left(y - \bar{c} - (1-\beta)(1-\delta)x\right)^2 = (y-\bar{c})^2 + 2\phi$$

Letting $\hat{y} \equiv y - \bar{c}$ and using $\beta(1+r) = 1$, we find

$$\frac{r}{1+r}\left(\hat{y} - \frac{\delta + r}{1+r}x\right)^2 + \frac{1}{1+r}\left(\hat{y} - \frac{r}{1+r}(1-\delta)x\right)^2 = \hat{y}^2 + 2\phi$$

After some algebra, we find

$$r(r+\delta^2)x^2 - 2\hat{y}r(1+r)x - 2\phi(1+r)^2 = 0$$

Hence, $b^-(0)$ solves the quadratic equation, with the unique positive solution, using $\hat{y} = y - c$:

$$b^-(0) = \frac{(1+r)\left[(y-\bar{c})r + \sqrt{(y-\bar{c})^2 r^2 + 2r(r+\delta^2)\phi}\right]}{r(r+\delta^2)}. \tag{C.1}$$

C.2 Parametric Condition for $a^\star > 0$

Next, take the $b^-(0)$ from (C.1) and plug into equation (15):

$$\beta(1-\delta)\left[u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right) - u'\left(y - (1-\beta)(1-\delta)b^-(0)\right)\right] > u'(y). \tag{C.2}$$

Observe that $u'(c) = -c + \bar{c} > 0$. Using $(\delta + r)/(1+r) = \delta\beta + 1 - \beta$:

$$u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right) = -y + (\delta\beta + (1-\beta))b^-(0) + \bar{c}$$

and

$$u'\left(y - (1-\beta)(1-\delta)b^-(0)\right) = -y + ((1-\beta) - \delta(1-\beta))b^-(0) + \bar{c}$$

After some algebra, condition (C.2) requires

$$b^-(0) > \frac{-y + \bar{c}}{\beta(1-\delta)\delta} \equiv \frac{(-y+\bar{c})(1+r)}{(1-\delta)\delta}$$

Using the formula we found at (C.1), and after some algebra, we arrive at

$$\phi \;>\; \frac{(y-\bar{c})^2\, r\,[\,r + 2\delta - \delta^2\,]}{2\,(1-\delta)^2\,\delta^2} \equiv \underline{\phi}$$

Notice that when $r > 0$, $\delta \in (0,1)$, $y < \bar{c}$, we must have $\underline{\phi} > 0$.

D A Condition for Staying in the safe zone

In this appendix, we analyze the conditions under which policy strategies where the government transitions from the safe zone or the crisis zone to the Default Zone are not part of a Markov equilibrium. That is, we will rule out strategies where the government finds it optimal to default with probability one in the next period.

We can focus on the case where the economy is initially at a portfolio (a, b) where it obtains $V_S(a-b)$ and show the conditions under which repaying today and defaulting tomorrow is strictly worse than remaining at $V_S(a-b)$. That is, we will provide a sufficient condition (but not a necessary one) for staying in the safe zone.[41]

Let the initial portfolio (a, b) be such that the government is indifferent between staying safe and defaulting today. That is, we have that the initial portfolio satisfies

$$u(y + (1-\beta)(a-b)) = u(y + (1-\beta)a) - \phi\,. \tag{D.1}$$

It is straightforward to show that there is a unique $\hat{b}(a) > 0$ that satisfies (D.1), and that it is increasing in a and independent of δ.

The value of staying in the safe zone, $V_S(a-b)$, is given by (9). The alternative is for the government to repay in the current period and choose to default in the following period. Denoting the value of this alternative by $\tilde{V}(a, b)$, we have

$$\begin{aligned}\tilde{V}(a,b) = &\max_{a' \geq 0} u(c) + \beta V_D(a'),\\ &\text{subject to}\\ &c = y + a - \left(\frac{\delta + r}{1+r}\right) b - \frac{a'}{1+r}\,,\end{aligned}$$

where we already imposed that the bond price in the current period is zero. Assuming an interior solution for a',[42] the optimality condition is

$$a' = a - \left(\frac{\delta + r}{1+r}\right) b\,.$$

[41] To see why it is a sufficient condition, consider a case where the government is at the b^- boundary. For that case, we have $V_S(a-b) > V_R^-(a,b) = V_D(a)$. Thus, a policy of defaulting tomorrow could be preferable to defaulting today, but still lower than $V^S(a-b)$.

[42] The interior solution is guaranteed if $a \geq \left(\frac{\delta+r}{1+r}\right) b$. We present the case when $a < \left(\frac{\delta+r}{1+r}\right) b$, and therefore $a' = 0$, separately below.

This policy yields a perfectly smooth consumption path and implies

$$\tilde{V}(a,b) = \frac{u\left(y + (1-\beta)\left(a - \left(\frac{\delta + r}{1+r}\right) b\right)\right) - \beta\phi}{1-\beta}.$$

Whether this alternative policy is preferable depends on the comparison between $V_S(a-b)$ and $\tilde{V}(a,b)$. Let's define $\tilde{F}(\delta; a, b) \equiv \tilde{V}(a,b) - V_S(a-b)$ as the welfare gain attained by following this alternative policy. Recalling that we are considering an initial portfolio satisfying $V_S(a-b) = V_D(a)$, and using (D.1) to substitute out ϕ, we have that:

$$\begin{aligned} F(\delta; a, \hat{b}(a)) &\equiv (1-\beta)(\tilde{V}(a,b) - V_D(a)) \\ &= u\left(y + (1-\beta)\left[a - (1-\beta)\hat{b}(a) - \delta\beta\hat{b}(a)\right]\right) \\ &\qquad - \beta u(y + (1-\beta)a) - (1-\beta)u\left(y + (1-\beta)\left[a - \hat{b}(a)\right]\right), \end{aligned} \tag{D.2}$$

where we used $\left(\frac{\delta + r}{1+r}\right) = \delta\beta + (1-\beta)$. Having $F(\delta; a, \hat{b}(a)) > 0$ would imply that this deviation is profitable relative to staying in the safe zone.

Proposition D.1. *For any initial $(a, \hat{b}(a))$ with $a > 0$, there is a unique $\tilde{\delta}$ such that if $\delta \geq \tilde{\delta}$, the government never transitions into the default zone.*

Proof. Fix $(a, \hat{b}(a))$ such that $a \geq \left(\frac{\delta+r}{1+r}\right)\hat{b}(a)$ and let $F(\delta; a, \hat{b}(a))$ be as defined in (D.2). Observe that F is continuous and differentiable in δ. The derivative of F with respect to δ is

$$\frac{\partial F(\delta; a, \hat{b}(a))}{\partial \delta} = u'\left(y + (1-\beta)\left[a - (1-\beta)\hat{b}(a) - \delta\beta\hat{b}(a)\right]\right)(1-\beta)(-\beta\hat{b}(a)) < 0,$$

so F is strictly decreasing in δ. Next, we show that $F(0; a, \hat{b}(a)) > 0$:

$$\begin{aligned} F(0; a, \hat{b}(a)) = u\left(y + (1-\beta)(a - (1-\beta)\hat{b}(a))\right) \\ - \beta u(y + (1-\beta)a) - (1-\beta)u\left(y + (1-\beta)(a - \hat{b}(a))\right) \end{aligned}$$

Observing that we can write

$$y + (1-\beta)(a - (1-\beta)\hat{b}(a)) = \beta(y + (1-\beta)a) + (1-\beta)(y + (1-\beta)(a - \hat{b}(a))),$$

the claim that $F(0; a, \hat{b}(a)) > 0$ follows from the strict concavity of $u(\cdot)$:

$$u(y + (1-\beta)(a - (1-\beta)\hat{b}(a))) > \beta u(y + (1-\beta)a) + (1-\beta)u(y + (1-\beta)(a - \hat{b}(a)))\,.$$

Next, we show that $F(1; a, \hat{b}(a)) < 0$. Evaluation gives:

$$F(1; a, \hat{b}(a)) = -\beta \left[u\left(y + (1-\beta)(a - \hat{b}(a))\right) - u(y + (1-\beta)a)\right] = -\beta\phi < 0$$

Knowing that $F(\delta; a, \hat{b}(a))$ is continuous and strictly decreasing in δ, and that $F(0; a, \hat{b}(a)) > 0$ and $F(1; a, \hat{b}(a)) < 0$; it then follows from the Intermediate Value Theorem that there exists a unique $\tilde{\delta}$ satisfying $F(\tilde{\delta}; a, \hat{b}(a)) = 0$. □

An analytical expression for $\tilde{\delta}$. We now derive a closed-form expression for the threshold $\tilde{\delta}$. Define the auxiliary terms

$$\Phi \equiv y + (1-\beta)\Big(a - (1-\beta)\hat{b}(a)\Big), \qquad \Psi \equiv \beta u\Big(y + (1-\beta)a\Big) + (1-\beta)u\Big(y + (1-\beta)(a - \hat{b}(a))\Big)\,.$$

The indifference condition can be written as

$$u\left(\Phi - (1-\beta)\beta\delta\hat{b}(a)\right) = \Psi\,.$$

Inverting the utility function implies

$$\tilde{\delta} = \frac{\Phi - u^{-1}(\Psi)}{(1-\beta)\beta\hat{b}(a)}\,.$$

Using the definition of consumption in the safe zone $c_S(a - \hat{b}(a))$ and the fact that the initial portfolio satisfies $\phi = u(c_D(a)) - u(c_S(a - \hat{b}(a)))$, the threshold can be expressed as

$$\tilde{\delta} = \frac{c_S(a - \hat{b}(a)) + \beta\hat{b}(a) - u^{-1}\left\{u(c_S(a - \hat{b}(a)) + \beta\phi\right\}}{(1-\beta)\beta\hat{b}(a)}\,.$$

Accounting for $a' = 0$. We have assumed above that $a \geq \left(\frac{\delta+r}{1+r}\right)b$, so $a' \geq 0$ and the government achieves lifetime utility $\tilde{V}(a, b)$ (given by equation D) if it follows the alternative policy of repaying today and defaulting tomorrow. Whenever $a < \left(\frac{\delta+r}{1+r}\right)b$, we have $a' = 0$

and the lifetime utility attained by the alternative policy is instead

$$\hat{V}(a,b) = \frac{(1-\beta)\,u\left(y + a - \left(\frac{\delta+r}{1+r}\right)b\right) + \beta\,u(y) - \beta\phi}{1-\beta}\,.$$

Noticing that

$$y + (1-\beta)\left(a - \left(\frac{\delta+r}{1+r}\right)b\right) = (1-\beta)\left[y + a - \left(\frac{\delta+r}{1+r}\right)b\right] + \beta\,y\,,$$

we can appeal to the strict concavity of $u(\cdot)$ and use Jensen's inequality to establish that $\tilde{V}(a,b) > \hat{V}(a,b)$.

Since $\hat{V}(a,b)$ is strictly decreasing in δ, we have that the value $\hat{\delta}$ that solves $\hat{V}(a,b) = V_S(a-b)$ is necessarily lower than the value $\tilde{\delta}$ that solves $\tilde{V}(a,b) = V_S(a-b)$, for every a. After some algebra, we find that the cutoff value $\hat{\delta}$ is given by

$$\hat{\delta} \equiv \frac{c_S(a-\hat{b}(a)) + \beta a - u^{-1}\left\{\frac{u(c_S(a-\hat{b}(a))-\beta u(y)+\beta\phi}{1-\beta}\right\}}{\beta\hat{b}(a)}\,.$$

The special case of zero initial reserves, $a = 0$. Notice that when $a = 0$, we have $a' = 0$. As a result, if we evaluate the lifetime utility under the alternative policy at the threshold $\hat{\delta}$ we have:

$$\begin{aligned}
\hat{V}(0,\hat{b}(0)) &= u\left(y - \left(\frac{\hat{\delta}+r}{1+r}\right)\hat{b}(0)\right) + \beta V_D(0) \\
&= V_D(0) \\
&= u(c_D(0)) - \phi + \beta V_D(0) \\
&= u(c_S(0-\hat{b}(0))) + \beta V_D(0)\,,
\end{aligned}$$

where the second line uses $\hat{V}(0,\hat{b}(0)) = V_D(0)$ when $\delta = \hat{\delta}$, the third line follows from the definition of $V_D(0)$, and the fourth line uses equation (D.1). Therefore, we have

$$u\left(y - \left(\frac{\hat{\delta}+r}{1+r}\right)\hat{b}(0)\right) = u(y - (1-\beta)\hat{b}(0))\,.$$

Noticing that $\frac{\hat{\delta}+r}{1+r} = \hat{\delta}\beta + (1-\beta)$, we conclude that $\hat{\delta} = 0$ when $a = 0$.

Numerical results. Finally, we numerically compute the threshold value for δ over which the government optimally stays in the safe zone. Define $G(\delta; a, \hat{b}(a))$, the utility gain of following the alternative policy, as follows

$$G(\delta; a, \hat{b}(a)) = u\left(y + a - \left(\frac{\delta + r}{1 + r}\right)\hat{b}(a) - \frac{\mathcal{A}(a, \hat{b}(a))}{1 + r}\right) + \beta \frac{u(y + (1 - \beta)\mathcal{A}(a, \hat{b}(a))) - \phi}{1 - \beta}$$
$$- \frac{u(y + (1 - \beta)a) - \phi}{1 - \beta}$$

where $\mathcal{A}(a, \hat{b}(a)) = \max\left\{0, a - \left(\frac{\delta+r}{1+r}\right)\hat{b}(a)\right\}$ is the policy function for a'.

We then define the threshold $\bar{\delta}$ as the one satisfying $G(\bar{\delta}; a, \hat{b}(a)) = 0$. Since $G(\cdot)$ is continuous, strictly decreasing in δ, there is a single cutoff value $\bar{\delta} \in [0, 1]$ that zeros $G(\cdot)$.

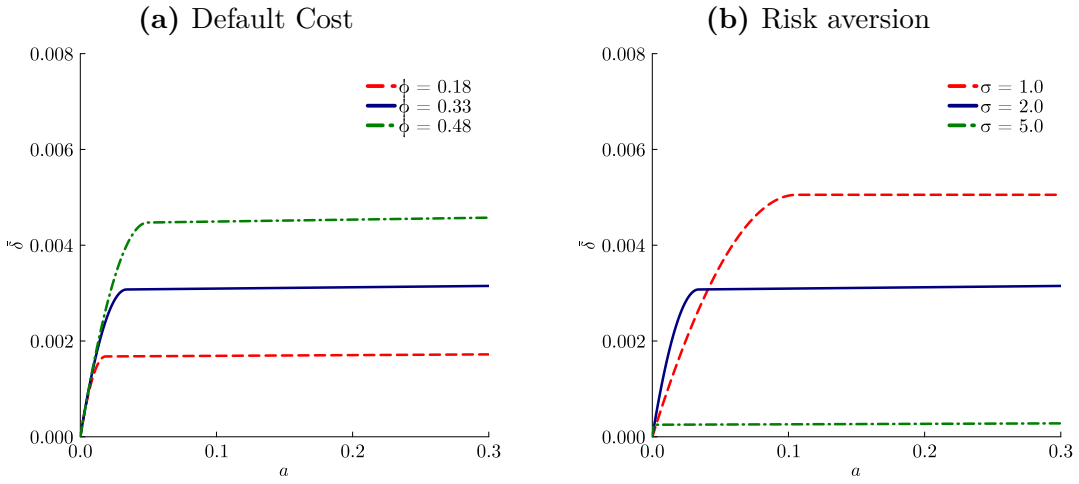


Figure D.1: Threshold $\bar{\delta}$ as a function of initial reserves. The figure plots the threshold $\bar{\delta}$ as a function of initial reserves a. Panel (a) varies the default cost parameter ϕ, and panel (b) varies the coefficient of risk aversion σ. In each panel, the solid blue line corresponds to the benchmark calibration. All remaining parameters are set as in Table 1.

Figure D.1 plots the threshold value $\bar{\delta}$ as a function of initial reserves a, computed as the solution to $G(\bar{\delta}; a, \hat{b}(a)) = 0$ under the benchmark calibration in Table 1. Panel (a) reports comparative statics with respect to the default cost parameter ϕ, while panel (b) reports the corresponding exercise for the coefficient of risk aversion σ.

Higher levels of initial reserves are associated with (weakly) higher values of $\bar{\delta}$, independent of the values of ϕ and σ. However, over the empirically relevant range of initial reserves, the threshold $\bar{\delta}$ remains small and well below the benchmark maturity parameter $\delta = 1/6$ used in the main quantitative analysis.

We highlight that the thresholds in Figure D.1 are constructed under the assumption that the government is initially at a point where it is indifferent between consuming $c_t = y + (1-\beta)(a-b)$ for all t and defaulting. As discussed above, if the government does not want to deviate and default tomorrow at that starting point, it will also not find it optimal to deviate starting from the b^- threshold. Finding a more relaxed condition in this case is more involved, but we have experimented numerically and found that for our baseline parameter values for $\{y, \phi, \beta, r, \sigma, \underline{c}\}$, there are no values of δ for which the government would want to transition from the safe zone into the default zone.

E Additional Figures: Economy without Reserves

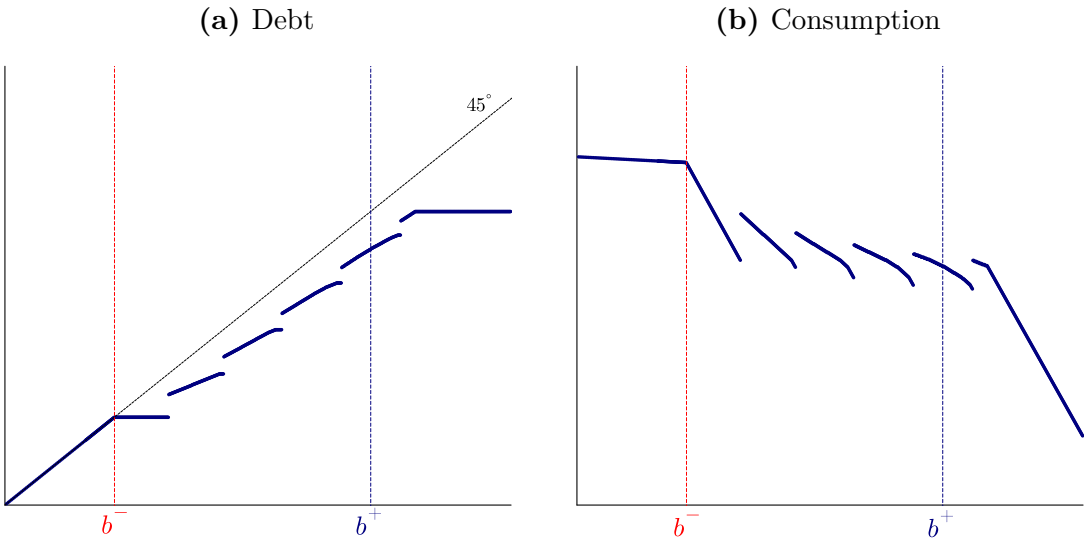


Figure E.1: Policy without reserves. The figure presents the policy functions for the economy when the government is restricted from accumulating reserves. Panel (a) shows the policy for debt, $b'(b)$. Panel (b) exhibits the policy for consumption, $c(b)$.

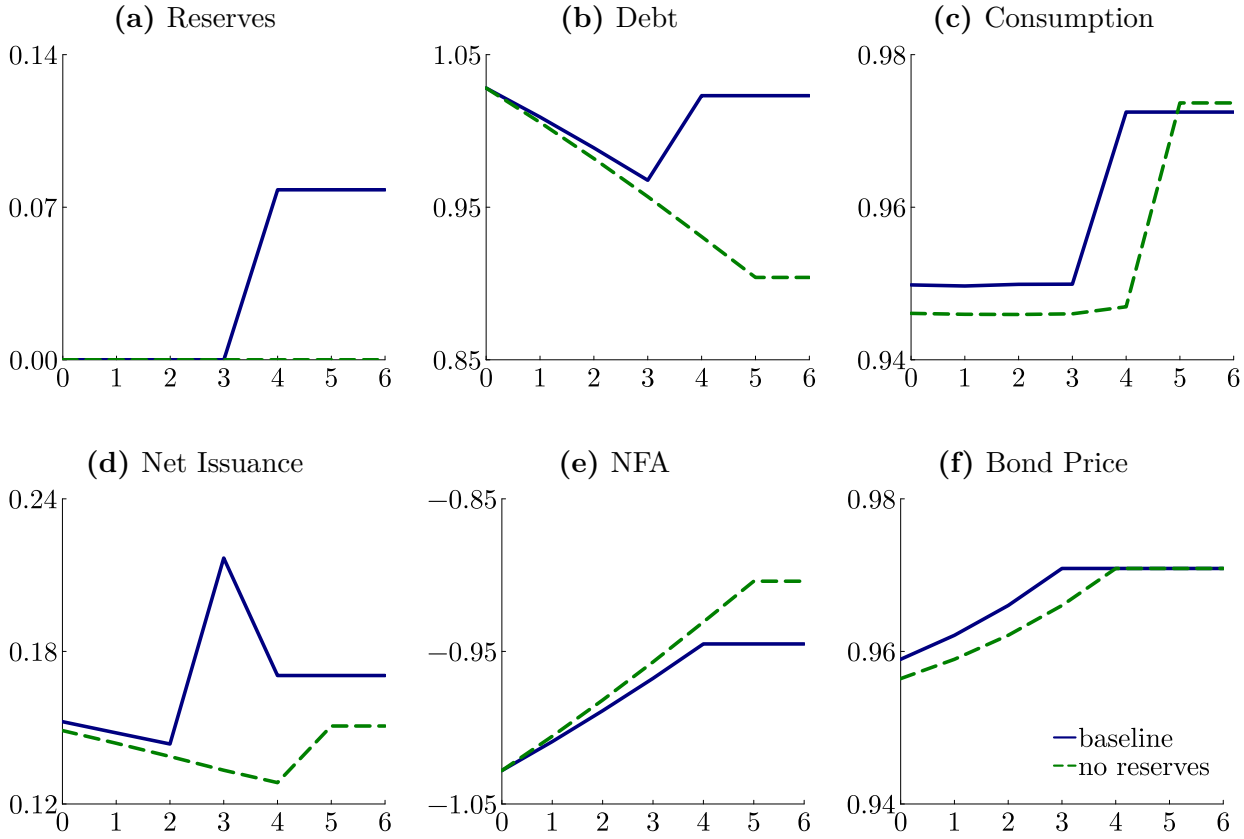


Figure E.2: Lower consumption and longer to exit without reserves. The government is assumed to start in the crisis zone with $b = 1.04$ and zero initial reserves.

F Varying the Probability of the Bad Sunspot, λ

In this Appendix, we present a sensitivity analysis for λ, the probability that the economy is hit by the bad sunspot.

Figure F.1 reports how four endogenous objects of interest vary with λ. Namely, we show the upper boundary of the crisis zone at zero reserves, $b^+(0)$, the number of periods required to converge to the lowest-NFA safe portfolio when starting from $(0, b^+(0))$, $T(0, b^+(0))$, and the lowest-NFA safe portfolio itself, $a^\star$ and $b^\star$, for different values of λ both below and above our benchmark calibration ($\lambda = 0.5\%$). All other parameters remain fixed at their benchmark values (see Table 1).

Panel (a) of Figure F.1 shows that a lower probability of the bad sunspot is associated with a higher $b^+(0)$. In the crisis zone, default is an undesirable outcome. When λ decreases, default becomes less likely, allowing the government to sustain a larger level of debt. Panel (b) of the same figure shows that the government finds it optimal to take longer to exit the crisis zone as λ decreases. This follows the same intuition: if the undesirable default state is less likely, the government sustains higher consumption by deleveraging more gradually. Given that λ also determines the default risk premium, a lower λ implies lower spreads for any given expected exit time, making it ex-post less costly to remain in the crisis zone while repaying.

Panels (c) and (d) of Figure F.1 show that $a^\star$ and $b^\star$ are invariant to λ under this parametrization. This arises because, with $(y, \sigma, \delta, r, c, \phi)$ fixed, we have both $(1-\delta)b^\star < b^-(0)$ and $a^\star - \delta b^\star < 0$, so a run at $(a^\star, b^\star)$ leads to the safe-zone continuation $(0, (1-\delta)b^\star)$. Along the entire segment of the $b^-(a)$ boundary up to $(a^\star, b^\star)$, facing a run will also lead to safe-zone continuations, so V_R^- depends only on V_S and is thereby locally independent of λ. We emphasize that this result holds for our benchmark calibration (and many others), but it is not a general result.

Figure F.2 shows deleveraging dynamics for alternative values of λ. For lower values of this probability, the government prefers to exit the crisis zone more slowly (i.e., taking more steps), and this comes with a more gradual deleveraging, higher consumption, and a worse NFA during the transition. When it comes to bond prices, we see that (i) conditional on being in the crisis zone in $t+1$, equilibrium bond prices in period t are lower for higher values of λ, and (ii) since higher values of λ induce the government to exit faster, the bond prices reach the risk-free level sooner.

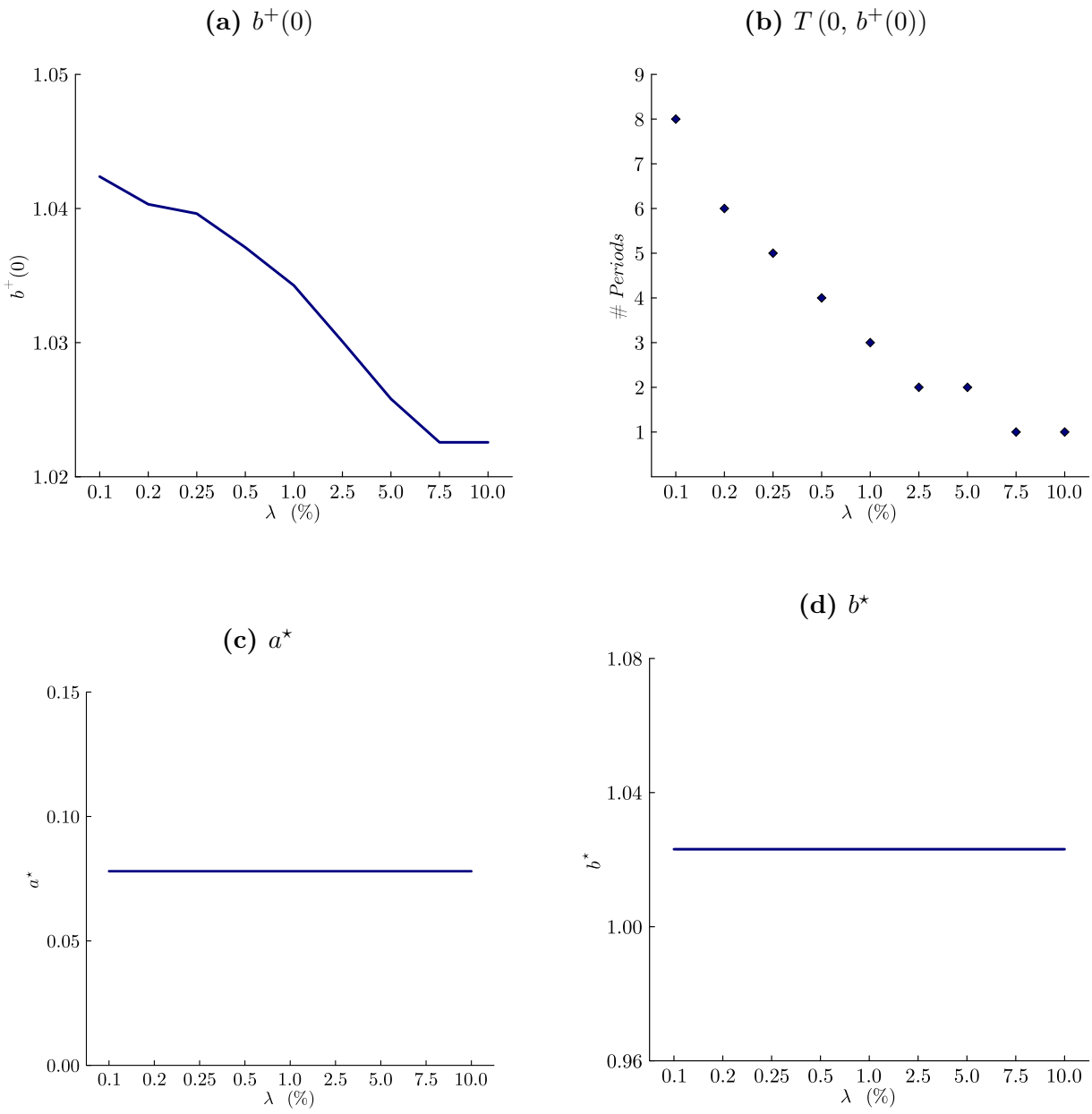


Figure F.1: Sensitivity to λ. Panel (a) presents $b^+(0)$, panel (b) the exit time $T(0, b^+(0))$, and panels (c) and (d) the lowest-NFA safe portfolio $a^\star$ and $b^\star$, for varying values of λ. All other parameters remain at their benchmark calibration.

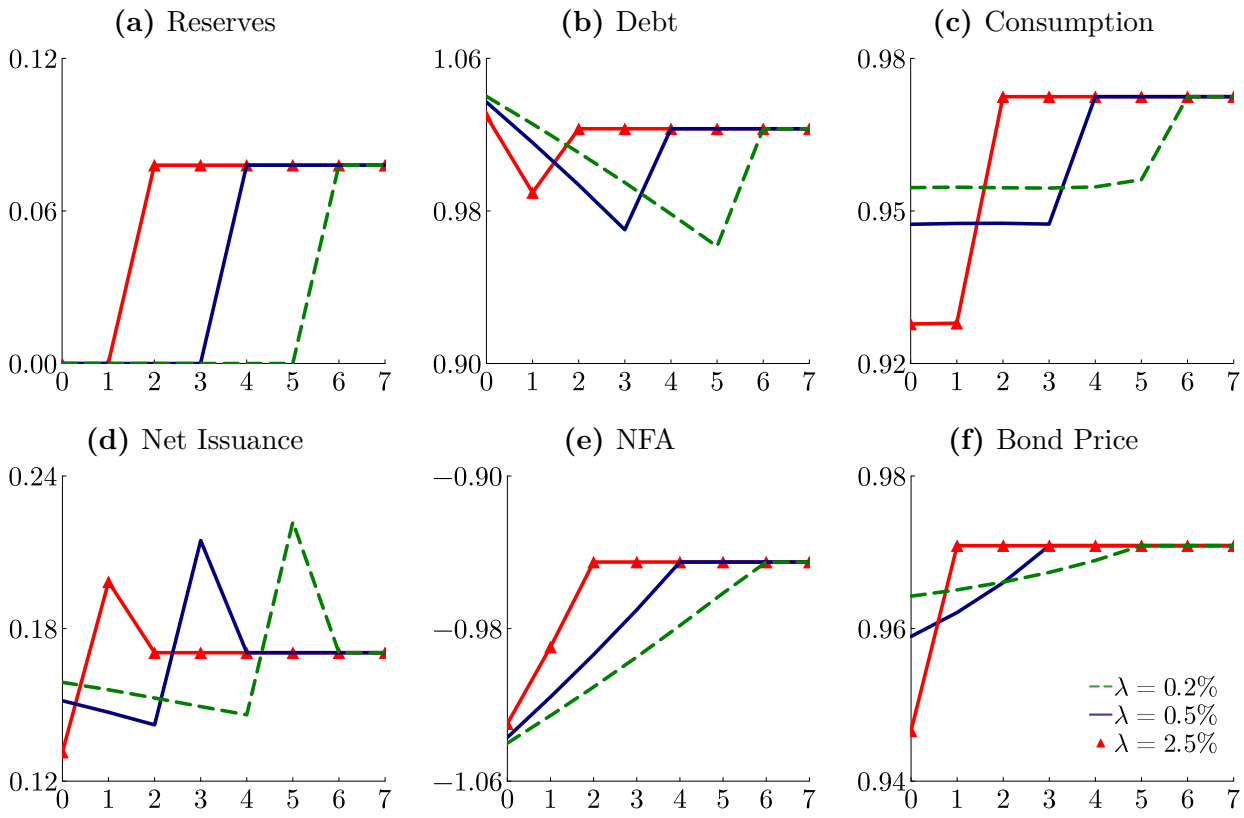


Figure F.2: Deleveraging dynamics for alternative λ. The government is assumed to start in the crisis zone with an initial portfolio $(0, b^+(0))$. Panels (a), (b), and (e) plot beginning-of-period levels of reserves, debt, and NFA, respectively. Notice that $b^+(0)$ is a function of λ. The solid blue line is for the baseline calibration, $\lambda = 0.5\%$. The dashed green line represents $\lambda = 0.2\%$, and the solid red line with triangles corresponds to $\lambda = 2.5\%$. All other parameters remain at their benchmark calibration.

G Vulnerability next-period after a run

In this appendix, we present a configuration under which the government remains vulnerable after repaying the coupons due during a run. The logic is that even though the government starts from lower debt tomorrow, the necessary reduction in reserves to repay may end up leaving the government still vulnerable.

Figure G.1 illustrates this situation. Suppose the government starts at point A, with initial portfolio $(\hat{a}, \hat{b}) \in \mathbf{C}$. Point B represents the portfolio tomorrow if the government pays the coupons, $b' = (1-\delta)\hat{b}$ and keeps zero reserves. The key point is that the portfolio $(0, (1-\delta)\hat{b})$ is not inside the safe zone. Point C shows the minimum level of reserves, denoted by $\bar{a}$, that makes the government safe tomorrow for $b' = (1-\delta)\hat{b}$. In this situation, the government effectively has to choose between points B or C. The tradeoff is that point B allows for higher consumption during the run, but point C implies that the government is now safe. Formally, the government chooses to save $a' = \bar{a}$ if

$$u\left(y + \hat{a} - \left(\frac{\delta + r}{1+r}\right)\hat{b} - \frac{\bar{a}}{1+r}\right) + \beta V_S(\bar{a} - b^-(\bar{a})) \geq$$

$$u\left(y + \hat{a} - \left(\frac{\delta + r}{1+r}\right)\hat{b}\right) + \beta\left[(1-\lambda)V_R^+(0, (1-\delta)\hat{b}) + \lambda V_D(0)\right]$$

and $a' = 0$ otherwise.

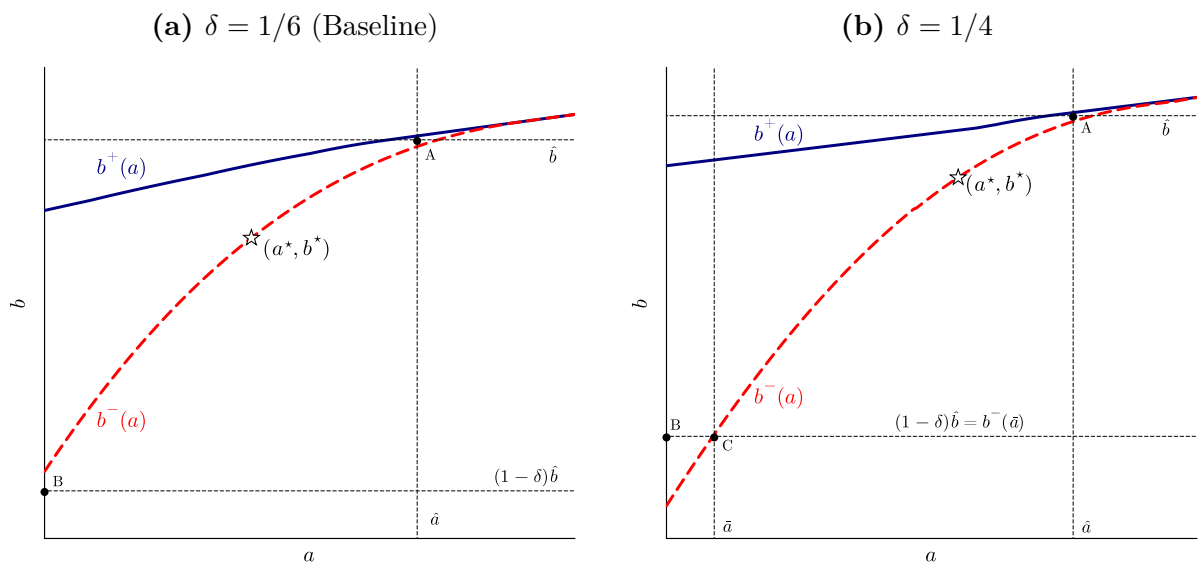


Figure G.1: Panel (a) shows the baseline parameters. Panel (b) shows a configuration where the government remains vulnerable after repaying the coupons in a run, with $\delta = 1/4$. All parameters are the same as in Table 1. Point A is $(\hat{a}, \hat{b}) \in \mathbf{C}$. In panel (a), point B is $(0, (1-\delta)\hat{b}) \in \mathbf{S}$, while in panel (b) B is $(0, (1-\delta)\hat{b}) \in \mathbf{C}$. Point C, only in panel (b), is $(\bar{a}, b^-(\bar{a})) \in \mathbf{S}$. Notice that $(1-\delta)\hat{b} = b^-(\bar{a})$.

H Numerical Algorithm

1. Set parameters $\{y, \sigma, \phi, \underline{c}, r, \delta, \lambda\}$. Set $\beta = (1+r)^{-1}$. Using bisection, compute $b^-(0)$ as the zero of the following function

$$F(x) = u(y) - \phi - (1-\beta)u\left(y - \left(\frac{\delta + r}{1+r}\right)x\right) - \beta u\left(y - (1-\beta)(1-\delta)x\right) .$$

   Notice that $F(b^-(0)) = 0$, with $F'(x) > 0$, $F(0) = -\phi < 0$ and $\lim_{x \uparrow \frac{y(1+r)}{(\delta+r)}} F(x) = \infty$.

2. Set a linearly spaced grid for reserves, $\mathcal{A}$, with $a_{\min} = 0.0$ and $a_{\max} = 0.35$, da $\equiv \mathcal{A}[2] - \mathcal{A}[1] = 0.001$ and $N_a = 351$.

3. Find the bounds for the $\mathcal{B}$ grid. Define $b_S(a)$ as the level of debt that makes the value of being safe equal to the value of default, given a, that is $V_S(a - b_S(a)) = V_D(a)$. Noticing that $b^+(a) \leq b_S(a)$, set $b_{\max}$ slightly above $b_S(a_{\max})$. Conversely, $b_{\min}$ is set slightly below $b^-(0)$. Set a linearly spaced grid for $\mathcal{B}$, with $b_{\min} = round(b^-(0); digits = 2) - 0.01$, $b_{\max} = ceil(b_S(a_{\max})) + 0.01$ and $N_b = \min\left\{ceil(\frac{b_{\max} - b_{\min}}{4\text{da}}; digits = 2) + 1, 3(N_a - 1) + 1\right\}$.[43]

4. For each $a \in \mathcal{A}$ compute $V_D(a) = \frac{u(y + (1-\beta)a) - \phi}{1-\beta}$.

[43] For the baseline calibration, $b_{\min} = 0.89$, $b_{\max} = 1.12$, and $N_b = 922$.

5. Set an iterator counter j to 1. For each value in the grid $(a, b) \in \mathcal{A} \times \mathcal{B}$, initialize the following guesses: $\tilde{q}_1(a, b) = \frac{1}{1+r}$, $V^+_{R,1}(a, b) = V^-_{R,1}(a, b) = \frac{u(y-(1-\beta)b_{\max})}{1-\beta}$, and $b^-_1(a) = b^+_1(a) = \infty$.

6. Given the price schedule $\tilde{q}_j$, solve for the value and policy functions under repayment. That is, for each $(a, b) \in \mathcal{A} \times \mathcal{B}$, go over all possible pairs $(\tilde{a}, \tilde{b}) \in \mathcal{A} \times \mathcal{B}$, searching for the optimal value as follows:

   (a) If the government can issue new debt, V^+_R

      i. If (a, b) satisfies $b \leq b^-_j(a)$, then government is in the safe zone; i.e., $(a, b) \in \mathbf{S}$. Set $(a'_j, b'_j) = (a, b)$ and $V^+_{R,\,j+1}(a, b) = V_S(a - b)$.

      ii. If (a, b) satisfies $b > b^-_j(a)$, then evaluate the welfare of all candidate pairs $(\tilde{a}, \tilde{b}) \in \mathcal{A} \times \mathcal{B}$ and store the maximizing policies (a'_j, b'_j). These policies imply a value under repayment given by $V^+_{R,j+1}(a, b)$.

   (b) If the government cannot issue new debt, V^-_R

      i. If (a, b) satisfies $b' \equiv (1 - \delta)b \leq b^-(0)$, the continuation value is safe for any $a' \geq 0$. From Lemma 1, we know that the optimal reserve policy is given by $a'_- = \max\{0, a - \delta b\}$. This implies:

$$V^-_{R,\,j+1}(a, b) = \begin{cases} V_S(a - b) & \text{if } a \geq \delta b \\ u\left(y + a - \left(\frac{\delta+r}{1+r}\right) b\right) + \beta V_S(-(1 - \delta)b) & \text{if } a < \delta b \end{cases}$$

      ii. If (a, b) satisfies $b' \equiv (1-\delta)b > b^-(0)$, then evaluate the welfare of all candidate future reserves $\tilde{a} \in \mathcal{A}$ and store the maximum:

$$V^-_{R,j+1}(a, b) = \max_{\tilde{a} \in \mathcal{A}} \left\{ u\left(y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{\tilde{a}}{1 + r}\right) + \beta \mathbb{E} V_j(\tilde{a}, (1 - \delta)b) \right\}$$

      whenever $b' \equiv (1 - \delta)b \notin \mathcal{B}$, interpolate $V^+_{R,j}(a, \cdot)$ and $V^-_{R,j}(a, \cdot)$ linearly.

7. Given $V_D(a)$, $V^+_{R,j+1}(a, b)$ and $V^-_{R,j+1}(a, b)$, first compute if

$$\varepsilon_v \equiv \|V_{j+1} - V_j\|_\infty, \quad \varepsilon_a \equiv \|a'_{j+1} - a'_j\|_\infty, \quad \varepsilon_b \equiv \|b'_{j+1} - b'_j\|_\infty$$

   and then update $V_j(a, b)$ to $V_{j+1}(a, b)$ and $\mathbb{E}V_j(a, b)$ to $\mathbb{E}V_{j+1}(a, b)$ following equations (8) and (11).

8. Given $V_D(a)$, $V^+_{R,j+1}(a, b)$, and $V^-_{R,j+1}(a, b)$, update $\{b^-_j(a), b^+_j(a)\}$ to $\{b^-_{j+1}(a), b^+_{j+1}(a)\}$ for each $a \in \mathcal{A}$, linearly interpolating $V^-_{R,j}(a, \cdot)$ and $V^+_{R,j}(a, \cdot)$ as needed. The three zones

are now given by $\mathbf{S}_{j+1}, \mathbf{C}_{j+1}, \mathbf{D}_{j+1}$.

9. Given $\mathbf{S}_{j+1}$, $\mathbf{C}_{j+1}$, $\mathbf{D}_{j+1}$, update the bond price schedule for each $(a', b') \in \mathcal{A} \times \mathcal{B}$ as follows:

$$\tilde{q}_{j+1}(a', b') = \begin{cases} \frac{1}{1+r} & \text{if } (a', b') \in \mathbf{S_{j+1}}, \\ 0 & \text{if } (a', b') \in \mathbf{D_{j+1}}, \\ \frac{1-\lambda}{1+r}\left(\frac{\delta+r}{1+r} + (1-\delta)\tilde{q}_j(a'', b'')\right) & \text{if } (a', b') \in \mathbf{C_{j+1}}. \end{cases}$$

where (a'', b'') is the optimal portfolio under state (a', b'); i.e., $a'' \equiv a'_{j+1}(a', b')$, and $b'' \equiv b'_{j+1}(a', b')$.

10. Check for convergence. Compute $\varepsilon_q \equiv \|\tilde{q}_{j+1} - \tilde{q}_j\|_\infty$. If $\max\{\varepsilon_q, \varepsilon_v, \varepsilon_a, \varepsilon_b\} < 10^{-8}$, stop. Otherwise, set

$$\tilde{q}_{j+1}(a, b) = \xi\tilde{q}_{j+1}(a, b) + (1 - \xi)\tilde{q}_j(a, b),$$

update the iterator counter to $j + 1$ and go back to step (6). We set the damping parameter as $\xi = 0.25$.

11. After convergence, compute $(a^\star, b^\star) = \arg\min\{a - b : (a, b) \in \mathbf{S} \cap (\mathcal{A} \times \mathcal{B})\}$.

I Empirical Appendix

I.1 Database

We use the same database as in Sosa-Padilla and Sturzenegger (2023). Here we provide a brief description of the time and country coverage as well as the main variables and their sources.[44]

Time and country coverage. We use monthly data from March of 1997 until September of 2019. We have 30 countries in the final panel: Algeria, Belarus, Belize, Brazil, Bulgaria, Colombia, Dominican Republic, Egypt, Georgia, Ghana, Indonesia, Ivory Coast, Jamaica, Jordan, Kazakhstan, Mexico, Morocco, Nigeria, Pakistan, Peru, Philippines, Russian Federation, Senegal, South Africa, Sri Lanka, Thailand, Tunisia, Turkey, Ukraine, and Venezuela.

Variables and sources. The main variables and their sources are:

1. Sovereign spreads: JP Morgan EMBI global index blended spread.
2. Reserves: Claims on Non-Residents. Taken from International Monetary Fund's International Financial Statistics.
3. Sovereign Debt: Public and publicly guaranteed debt from private creditors. Taken from the World Bank's International Debt Statistics (IDS).
4. Private Debt: External private (non-guaranteed) debt stocks. Taken from the World Bank's International Debt Statistics (IDS).
5. World Rate: US Treasury notes, 10-year constant maturity yield.
6. Risk aversion: Merrill Lynch ICE BofAML Option-Adjusted Spreads.
7. Country credit rating: An index based on Standard & Poor's rating. The index starts at 1 for Not Rated (NR) and goes up to the top at 29 for "AAA."

[44] We refer to the empirical appendix in Sosa-Padilla and Sturzenegger (2023) for further details.

I.2 Additional Regression Results

In Table I.1 we repeat the results from Table 2 in the main text, reporting all coefficients.

Table I.1: Panel regressions (with all control variables reported)

Dep. Variable:	Spread (in bps)			
	Full Sample		Low Debt	High Debt
	(1)	(2)	(3)	(4)
Reserves	-12.01***	-13.18***	-7.96***	-27.39***
	(0.71)	(1.13)	(1.55)	(2.74)
Sov.Debt	3.05***	2.32***	-1.04	2.35
	(0.64)	(0.84)	(1.76)	(1.57)
Reserves × Sov.Debt		0.05	-0.29***	0.42***
		(0.04)	(0.06)	(0.07)
Private.Debt	6.49***	6.65***	8.08***	6.36***
	(0.48)	(0.49)	(0.95)	(0.92)
log(Rating)	-154.77***	-155.13***	-575.00***	-126.20***
	(9.56)	(9.56)	(31.19)	(12.06)
log(Risk.Aversion)	466.89***	467.13***	525.12***	340.71***
	(19.16)	(19.16)	(19.34)	(32.63)
log(World.Rate)	19.28	19.31	101.09***	-55.45
	(29.98)	(29.97)	(30.72)	(48.42)
Marg. Effect (in bps)	-8.96***	-9.61**	-14.58***	-11.10
[p-value]	[0.001]	[0.028]	[0.002]	[0.128]
Num.Obs.	4,497	4,497	2,464	2,033
R2 Adj.	0.412	0.412	0.552	0.420

Note: All specifications include year dummies and country FEs. Reserves and Sov.Debt are measured in percent of GDP. Robust standard errors are in parentheses. *p<0.1; **p<0.05; ***p<0.01. The 'Marg. Effect' row reports the discrete marginal effect (on spreads) of increasing reserves and debt by 1% of GDP (see footnote 39).