A Theory of Portfolio Choice for Heterogeneous Investors*

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Abstract

This paper proposes a continuous-time heterogeneous agent model with common noise to study portfolio choice for heterogeneous investors. When bearing common financial risk, households pay an "asset-exposure premium" (AEP) in lifetime welfare based on their wealth. The AEP effect serves as a non-monotonic third term in optimal portfolio choice. Near the borrowing constraint, the AEP effect is large and negative, overwhelming other motives and explaining limited participation. As wealth increases, the AEP effect transitions from negative to positive, intensifies in the middle-wealth range, and then gradually weakens until it disappears at the highest wealth levels, generating a hump-shaped profile of risky portfolio share. The AEP mechanism enables households to optimally respond to the risk characteristics of their portfolios. In a complete model incorporating both idiosyncratic risk and systematic risk, this mechanism explains why low wealth households exhibit underdiversification and why middle-to-high wealth households hold diversified portfolios. Furthermore, this paper introduces a steady state that accommodates arbitrary shock histories. Finally, the presence of common noise solves the degeneration problem of the wealth distribution in the continuous-time ABH framework.

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1 Introduction

Many of the investor behaviors in "household finance" (Campbell, 2006) are hard to reconcile with rational optimizing models (Campbell, 2018). These behaviors demonstrate the following empirical phenomena: the vast majority of low-wealth households do not participate in risky asset markets (Vissing-Jørgensen, 2002; Campbell, 2006); and low-wealth households hold severely underdiversified portfolios with highly idiosyncratic risk, while wealthier households hold diversified portfolios (Calvet, Campbell, & Sodini, 2007, 2009; Goetzmann & Kumar, 2008). Over the life cycle, households' risk portfolio share exhibit a "hump-shaped" profile (Heaton & Lucas, 2000; Poterba & Samwick, 2001; Benzoni, Collin-Dufresne, & Goldstein, 2007; Ameriks & Zeldes, 2011; Fagereng, Gottlieb, & Guiso, 2017; Gomes & Smirnova, 2021; Parker et al., 2025). While these stylized phenomena span disparate domains, they all point to the need for a portfolio theory regarding heterogeneous investors. So, do we have such a theoretical framework?

The seminal work of Merton (1969, 1971, 1973) establishes a continuous-time model that has become one of the cornerstones of portfolio theory. The Merton framework is hard to uniformly explain empirical phenomena related to heterogeneous investor behaviors. Based on Hamilton-Jacobi-Bellman (HJB) equations, it treats financial risk as an undifferentiated aggregate, failing to distinguish between systematic and idiosyncratic components. In this framework, the optimal portfolio choice consists of the speculative (or myopic) demand, which reflects the risk-return properties of assets, as well as the wage-offsetting (or labor hedging) effect of labor income associated with risky assets. Aggregate income risk thus becomes the key channel for explaining investor behaviors. In practice, much of the risk associated with labor income is idiosyncratic and therefore unhedgeable (Campbell, 2006). Furthermore, the literature debates the correlation between income and assets, particularly regarding its stability (Guiso, Jappelli, & Terlizzese, 1996; Bertaut & Haliassos, 1997; Heaton & Lucas, 2000; Poterba, 2001; Ameriks & Zeldes, 2011) and range (Campbell et al., 2001; Angerer & Lam, 2009; Betermier et al., 2012; Huggett & Kaplan, 2016). These factors significantly weaken the explanatory power through the aggregate income risk channel.

The limitations manifest across these empirical phenomena. First, explaining underdiver-

sification requires rational optimizing models that can identify systematic and idiosyncratic risk in household portfolios. However, portfolio risk characteristics are captured only indirectly through income-asset correlations, without explicit modeling of these distinct risk components. Second, limited participation is hard to reconcile with "long-term investor behaviors" (Viceira, 2001; Campbell & Viceira, 2002). Given the fact that households hold stocks that are highly correlated with labor income (Campbell, 2018), the optimal response is to reduce risky asset demand when income and assets are positively correlated—a approach that can explain limited participation among low-wealth households. Yet this approach simultaneously implies that employed investors should hold lower risky portfolio share than retirees in the right tail over the life cycle, contradicting observed investor behavior. Third, the hump-shaped portfolio profile is particularly difficult to explain because it requires non-monotonic motives. Economists implicitly assume monotonicity to ensure analytic feasibility, requiring investors' speculative demand to decrease monotonically. This monotonicity assumption serves as the starting point in this paper, but it significantly hinders the ability to study hump-shaped portfolio share in life-cycle patterns because both motives are monotonic in the Merton framework.

This paper proposes a continuous-time heterogeneous agent (CTHA) model with common noise to study portfolio choice for heterogeneous investors. The model provides a unified theoretical framework for explaining the aforementioned empirical phenomena. This framework is based on Mean-Field-Games (MFG) with common noise, which couples backward stochastic HJB (BSHJB) equations (Peng, 1992) with stochastic Kolmogorov forward equations. The mathematical foundations of MFG without common noise are established in Lasry and Lions (2007). Cardaliaguet et al. (2019) first proposes the MFG with common noise and demonstrates its well-posedness, while Carmona, Delarue, et al. (2018) proves its convergence using probability theory. Despite the significant development of these rigorous mathematical tools, their application to specific economic or financial problems in MFG with common noise remains limited.

In this framework, households serve as market participants who make consumption-portfolio choice and view prices as partial equilibrium. This approach aligns with the Merton framework and Campbell (2018)'s perspective on heterogeneous investors. The core insight

of this paper is that aggregate market prices act as a carrier to transmit aggregate risk to households, forming their common risk at the micro level. Households hold assets priced at aggregate market prices, as well as labor, bearing common financial risk and common income risk. These common risks correspond to systematic risk and aggregate income risk, respectively, at the market level.

Common risk causes households to bear the welfare impact of aggregate price uncertainty and the corresponding premium, called the common risk premium. This paper's central theoretical contribution is identifying and analyzing the wealth-dependent component of the common risk premium: the "asset exposure premium" (AEP). It serves as the household's optimal response to systematic risk and distinguishes the risk characteristics of household portfolios. The AEP consists of a cost term and a compensation term, making it asymmetric. This asymmetric structure generates a non-monotonic, wealth-dependent AEP effect, which serves as a third component in the optimal portfolio choice.

The magnitude and sign of the AEP depends on households' positions in the wealth distribution and an effective relative risk aversion, denoted η_t. It measures the curvature of the value function with respect to wealth and captures households' intertemporal risk preference relative to wealth. For the low wealth near borrowing constraints, labor income almost dominates its consumption, resulting in extremely low sensitivity to changes in wealth. In this case, η_t converges to zero. Hence, the AEP effect becomes a overwhelmingly large negative value that dominates all the speculative demand and the wage-offsetting effect. The AEP reflects the high cost of common financial risk, while households' near-zero financial wealth prevents them from obtaining the corresponding compensation. They completely withdraw from the financial market, arising in limited participation.

As wealth increases, households' η_t gradually rise above one, making their AEP positive. Due to the net positive welfare effect of common financial risk, middle-to-high wealth households with $\eta_t > 1$ invest more in risky portfolio share than Merton (1969)'s result. For the highest wealth, its income become negligible, and its AEP eventually disappears by the scaling factor $1/a$. Hence, η_t converges to the CRRA coefficient γ, and thus, the highest wealth's sensitivity to wealth changes is entirely determined by their risk preference to consumption. This result yields an asymptotically linear ratio of consumption to wealth,

with the risky portfolio share eventually converging to Merton (1969)'s result. This process naturally generates a hump-shaped portfolio profile across the wealth distribution.

The AEP mechanism provides an explanation for underdiversification under homogeneous CRRA preferences. This paper shows that the risk characteristics of household portfolio, in terms of systematic risk and idiosyncratic risk, depend on the sign of the β exposure ratio $\Gamma(a_t)$. It consists of the AEP effect and the wage-offsetting effect that measures the impact of systematic risk on portfolio efficiency. Households in low wealth areas with $\Gamma_t < 0$ experience a negative welfare cost from systematic risk. These households rationally sacrifice some of their portfolio Sharpe ratio in order to hold a portfolio with high heterogeneous risk, resulting in underdiversification. Conversely, households with $\Gamma_t > 0$ are in middle-to-high wealth areas and benefit from systematic risk. This incentivizes these households to hold diversified portfolios or mutual funds that are fully correlated with the financial market. This risk-taking behavior in portfolio translates into wealth dynamics. The portfolio efficiency of households with $\Gamma_t < 0$ falls far from the efficient frontier, creating a self-reinforcing poverty traps. Meanwhile, the portfolio efficiency of households with $\Gamma_t > 0$ clusters on the efficient frontier, entering wealth acceleration channels.

In this framework, the household problem become an infinite-dimensional stochastic control problem. First, this paper proposes a "local steady state" (LSS) as the terminal condition. The LSS is defined as the moment at which each household expects the aggregate price to fluctuate around its expected value, with an independent increment of exactly zero ($\mathbb{E}_t[d\mathbf{W}_t] = 0$) and with common risk still remaining. The LSS expands analytical capabilities by incorporating arbitrary shock histories and generalizing the "stochastic steady state" (Futia, 1982; Duffie et al., 1994), which is employed in CTHA models without common noise (Fernández-Villaverde, Hurtado, & Nuno, 2023; Bilal, 2023). The stochastic steady state requires the entire cumulative shock process to be zero ($\mathbb{E}_t[\mathbf{W}_{t+1}] = 0$) to characterize long-run fixed points. The optimization problem involves estimating an infinite-dimensional object for a distribution measure. Second, this paper assumes that bounded rationality maps infinite-dimensional distribution to low-dimensional moment, based on the assumption that the expectation about the distribution will only lead to a numerical shift in household welfare without affecting the framework's theoretical results.

The presence of common risk in this framework solves the degenerate problem of the wealth distribution in CTHA models without common noise, which is a topic of interest in the macroeconomic literature. In these models, households do not save and the wealth distribution disappears. Based on the continuous-time Aiyagari (1994)-Bewley (1986)-Huggett (1993) (ABH) framework, these models have a boring economy where the wealth distribution degenerates into a Dirac mass point at the borrowing constraint because all households bear only idiosyncratic noise (Achdou et al., 2022). While these models maintain general equilibrium, analyzing policy responses is difficult. Blanchard (2025) argues that not everything is right in modern macroeconomics, with some models putting too much emphasis on general equilibrium implications from the start, rather than first on a partial equilibrium analysis of the phenomenon at hand. This point is verified in CTHA models.

This framework proposes a labor participation constraint, which is based on a economic intuition: households should only participate in the labor force when labor income brings positive welfare. However, CHTA models without common noise rely on idiosyncratic income risk to generate an artificial wealth distribution, which violates this rationality. To generate a non-degenerate wealth distribution near the borrowing constraint, these models require idiosyncratic income risk to be high enough to outweigh household impatience. This forces households near the borrowing constraint to have positive savings. For these households, this requirement means that the prospect of labor participation represents a source of welfare cost rather than a welfare-enhancing opportunity. If labor force participation itself constitutes a net loss in welfare compared to autarky, wouldn't rational households choose to exit the labor force entirely?

Applying partial differential equation (PDE) methods to the ABH framework is a significant methodological advance. Achdou et al. (2014) provide a comprehensive overview of this emerging field, framing CTHA models without common noise. The authors identify CTHA models with common noise as the most significant challenge because the entire distribution becomes an infinite-dimensional state variable in the stochastic control problem. This renders standard solution methods insufficient. This paper directly addresses this challenge. The continuous-time ABH framework is pioneered by Achdou et al. (2022), offers significant advantages over discrete-time formulations by avoiding integer problems (Stokey, 2008) and

generating sparse problems. Subsequent research by Fernández-Villaverde, Hurtado, and Nuno (2023) has extended these models to incorporate financial frictions.

Although both are based on CTHA models, the common noise (or shock) in this framework is different from the aggregate shock in CTHA models without common noise. Common noise acts on prices, and households, being price takers, bear the common risk. In these models (Bilal, 2023; Fernández-Villaverde, Hurtado, & Nuno, 2023), the aggregate shock acts on the capital stock, affecting the equilibrium price, typically the risk-free rate. Households respond to the risk-free rate by adjusting their savings behavior, which is then aggregated into the capital stock. Households' state space is actually free of any financial risk.

The presence of common noise renders numerical solutions for methods without common noise inapplicable. For these models, finite difference (FD) methods (Achdou et al., 2022) form the basis for solving the HJB equations. These models include state-space iteration (Ahn et al., 2018) and approximation of consistent beliefs via the neural network (Fernández-Villaverde, Hurtado, & Nuno, 2023). Recent innovations in solution techniques include deep learning (DL) methods (Han, Yang, et al., 2021), DL-based probabilistic approachs using the finite volume method (Huang, 2023), analytic perturbations (Bilal, 2023), physics-informed neural network methods (Gu et al., 2024), and sequence-space Jacobian methods (Bilal & Goyal, 2025). In this paper, the continuity of both wealth and income processes creates a high-dimensional state space. It makes standard grid-based techniques, such as FD methods, computationally infeasible. These methods without common noise address idiosyncratic income risk by using a simple, large, 2×2 sparse transition matrix to, which itself can be approximated by a two-state Markov process. In contrast, aggregate income risk requires the discretization of a far more complex state space. Moreover, this framework exhibits highly nonlinear and sharp dynamics near the borrowing constraint. FD methods become unstable when the non-convection term dominates and struggle to handle cross derivatives, which forces us to seek new methods.

The lack of homogeneity in the AEP makes solving the household problem under certain assumptions still difficult. To simplify the solution by obtaining analytical solutions, this paper employs the mathematical technique that assumes the value function has homogeneous forms, attempting to simplify partial derivatives of the value function with respect to income

to obtain analytical solutions. This technique is inspired by the log-linearization methods (Campbell, 1993; Campbell & Viceira, 1999, 2001; Viceira, 2001; Campbell, 2018) within the Merton framework, which aims to eliminate cross terms to obtain an approximate analytical solution. But the conclusion is that a necessary condition for the homogeneous form is the absence of the AEP mechanism.

Given the computational complexities involved, this paper first establishes the core theoretical properties of the underlying economic mechanism. This theoretical foundation is intended to guide future quantitative investigations. One promising computational strategy is the use of DL methods, given their strong generalization capabilities. Han, Jentzen, and E (2018) pioneers the DL framework that solves high-dimensional stochastic PDEs using the discretized Euler-Maruyama methods. Ji et al. (2022) solves stochastic HJB equations via the stochastic maximum principle with DL methods.

The remainder of this paper proceeds as follows. Section 2 develops the theoretical framework. Section 3 defines temporary equilibrium and steady state concepts. Section 4 derives the core theoretical results. Section 5 presents economic implications through graphical analysis of the theoretical mechanisms. Section 6 concludes.

2 Model Framework

We construct a continuous-time heterogeneous agent model with common noise. First, we consider the case in which households face only common financial risks to illustrate the core mechanism. In section 4.3, we discuss more complex issues that include idiosyncratic financial risks.

2.1 The Environment

There is a continuum of infinitely lived households indexed by $i \in [0, 1]$ with unit mass. Households have two basic endowments: financial wealth a_t and labor income y_t. Each household i begins with initial wealth $a_{i,0} > 0$ and initial income $y_{i,0} > 0$, giving rise to an initial joint distribution $g(a_0, y_0)$.

Households' endowments correspond to two priceable markets. These markets are subject to a risk source, which is a two-dimensional Wiener process:

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{W}_{1,t} & \mathbf{W}_{2,t} \end{bmatrix}^\top, \tag{1}$$

where $\mathbf{W}_{1,t}$ is the primary risk source and $\mathbf{W}_{2,t}$ is the orthogonal component satisfying $\mathbb{E}_t[d\mathbf{W}_{1,t} d\mathbf{W}_{2,t}] = 0$. Financial assets can be traded instantaneously in a centralized financial market. This market is fully driven by the primary risk source. Labor markets are decentralized and not instantaneous; they partially respond to the primary risk source. Let ρ_{sy} denote the magnitude of this partial response, where $\rho_{sy} \in [-1, 1]$. The primary risk source can be interpreted as the systematic component of a fundamental economic shock in the literature.[1]

Each household views the others as market participants. The aggregate price $\mathbf{P}_t$ of these markets acts as a carrier that transfers aggregate risks to households, forming their common risks. We use the term "noise" to refer to a risk source at the household level specifically. This includes common noise, denoted by $\mathbf{W}_t$, and idiosyncratic noise, denoted by $\mathbf{B}_{i,t}$. These are locally uncorrelated, i.e., $\mathbb{E}_t[d\mathbf{W}_t d\mathbf{B}_{i,t}] = 0$.

The financial market trades ownership of financial assets. A *systematic risk asset* represents ownership of the risky component of financial assets and is usually measured by a market index. Its price S_t acts as a carrier that transfers systematic risk to households:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s d\mathbf{W}_{s,t}, \tag{2}$$

where $\mu_s > 0$ is the expected return, $\sigma_s > 0$ is common financial risk representing systematic risk. We have that $\mathbf{W}_{s,t} = \mathbf{W}_{1,t}$.

A risk-free asset represents ownership of the risk-free component of financial assets, such as typically cash or short-term government bonds. Its price b_t does not respond to any risk source:

$$\frac{db_t}{b_t} = r_f dt, \tag{3}$$

[1] This extensive body of work, surveyed in Ramey (2016), is dedicated to identificating and propagating various structural impulses, such as monetary policy, fiscal actions, and technology innovations.

where $r_f > 0$ is the risk-free rate.

Labor supply $n_{i,t}$ of household i remains purely idiosyncratic:

$$\frac{dn_{i,t}}{n_{i,t}} = \tilde{\mu}_{n_i} dt + \tilde{\sigma}_{n_i} d\mathbf{B}_{n_i,t}, \tag{4}$$

where $\tilde{\mu}_{n_i}$ is the drift parameter and $\tilde{\sigma}_{n_i} > 0$ is idiosyncratic income risk.

Labor supply is priced through an *aggregate wage rate* w_t in these labor markets. This wage rate acts as a carrier that transfers aggregate income risk to households:

$$\frac{dw_t}{w_t} = \mu_y dt + \sigma_y d\mathbf{W}_{y,t}, \tag{5}$$

where μ_y is the drift parameter and $\sigma_y > 0$ is common income risk. We decompose the common income noise as

$$d\mathbf{W}_{y,t} = \rho_{sy} d\mathbf{W}_{1,t} + \sqrt{1 - \rho_{sy}^2} d\mathbf{W}_{2,t}. \tag{6}$$

Applying Itô's Lemma, the household i's labor income $y_{i,t} \equiv w_t n_{i,t}$ follows

$$\begin{aligned} \frac{dy_{i,t}}{y_{i,t}} &= \frac{dw_t}{w_t} + \frac{dn_{i,t}}{n_{i,t}} \\ &= (\mu_y + \tilde{\mu}_{n_i}) dt + \sigma_y d\mathbf{W}_{y,t} + \tilde{\sigma}_{n_i} d\mathbf{B}_{n_i,t}. \end{aligned} \tag{7}$$

Hence, the aggregate price in these markets is given by:

$$\mathbf{P}_t = \{S_t, w_t\}. \tag{8}$$

2.2 The Household's Problem

Households exhibit constant relative risk aversion (CRRA) over consumption, characterized by the instantaneous utility function:

$$U(c_t) := \frac{c_t^{1-\gamma} - 1}{1 - \gamma}, \tag{9}$$

where $\gamma > 1$ is the CRRA coefficient.

Household i chooses optimal consumption $c_{i,t}$ to maximize expected lifetime utility:

$$\max_{c_{i,t}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{i,t})\, dt \right], \tag{10}$$

where $\rho > 0$ is the discount rate.

Each household decides consumption-portfolio choice to maximize expected utility. A simple portfolio consists only of the systematic risk asset S_t and the risk-free asset b_t.

The household's wealth evolves as

$$da_t = \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\, dt + \omega_t a_t \sigma_s d\mathbf{W}_{s,t}, \tag{11}$$

where ω_t is the portfolio share of the asset S_t.

Households face a borrowing constraint $a_t \geq 0$ and a short-selling constraint:

$$\omega_t \geq 0. \tag{12}$$

It implies that households will not short sell any risky asset with a positive premium ($\mu_s - r_f > 0$). Near the borrowing constraint, consumption cannot exceed current labor income, i.e., $0 < c_t \leq y_t$.

The household's problem can be formulated as the BSHJB equation $v(a_t, y_t, g_t, \mathbf{W}_t)$:

$$dv_t = \{u(c_t) + \mathcal{L}[v_t] + \mathbb{E}_t \mathcal{A}[v_t] - \mathrm{div}(\mathcal{V}_t)\}\, dt + \mathcal{V}_t d\mathbf{W}_t, \tag{13}$$

where $d\mathbf{W}_t$ is the independent increment of the common noise, $\mathcal{A}[\cdot]$ is the functional derivative of the distribution measure, $\mathrm{div}(\cdot)$ is the divergence operator, $\mathcal{V}_t$ is the $\mathcal{F}$-adapted process, and $\mathcal{L}[\cdot]$ is the infinitesimal generator of the underlying state process:

$$\begin{aligned} \mathcal{L}[v_t] \equiv \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} \\ + \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y}. \end{aligned} \tag{14}$$

In MFG with only idiosyncratic noise, the law of large numbers ensures that the both of the value function and the distribution evolve deterministically, eliminating stochastic components at the population level (Cardaliaguet & Porretta, 2020). The expected evolution of the household's value function is fully described by a HJB equation consisting of the operator $\mathcal{L}[v_t]$ and $\mathcal{A}[v_t]$.

This deterministic framework breaks down in MFG with common noise. Common noise affects each household simultaneously and cannot be eliminated by the law of large numbers. The distribution now becomes a stochastic process, forcing households to base their optimization on the entire noise history, denoted by $\mathbf{W}_{\tau\{0\leq\tau\leq t\}}$. Therefore, the value function must evolve as a random field $v(a_t, y_t, g_t, \mathbf{W}_t)$, requiring a BSHJB equation. The BSHJB equation is first proposed by Peng (1992), and then used in Cardaliaguet et al. (2019) to develop the coupled system consisting of the BSHJB equation and the stochastic Kolmogorov forward equation in MFG with common noise.

The presence of common noise alters the random path of the household state variables. We define the impact of this random path on a household's lifetime welfare:

Definition 1. A *common risk impact* is the stochastic change in the household welfare caused by the exogenous realization of the common noise:

$$\Psi_t \equiv \mathcal{V}_t d\mathbf{W}_t = \begin{bmatrix} \mathcal{V}_{1,t} & \mathcal{V}_{2,t} \end{bmatrix} d\mathbf{W}_t, \tag{15}$$

where the $\mathcal{F}$-adapted process $\mathcal{V}_t$ follows:

$$\begin{aligned} \mathcal{V}_{1,t} =& \omega_t a_t \sigma_s \frac{\partial v}{\partial a} + y_t \sigma_y \rho_{sy} \frac{\partial v}{\partial y} \\ \mathcal{V}_{2,t} =& y_t \sigma_y \sqrt{1-\rho_{sy}^2} \frac{\partial v}{\partial y}. \end{aligned} \tag{16}$$

It reflects households' expectations of future aggregate price. When households expect that the aggregate price will fluctuate around expected returns, the common risk impact is zero, i.e., $\mathbb{E}_t[\Psi_t] = 0$. At this time, the household problem is unaffected by the common risk impact.

Definition 2. A *common risk premium* is the endogenous welfare cost or compensation that

households pay for bearing common risk:

$$\begin{aligned}\Phi_t \equiv & -\operatorname{div}(\mathcal{V}_t) = -\frac{\partial \mathcal{V}_{1,t}}{\partial a} - \frac{\partial \mathcal{V}_{2,t}}{\partial y} \\ = & -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1-\rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right).\end{aligned} \tag{17}$$

It represents an additional adjustment term for the stochastic optimization problem, even when the expected value of the common risk impact is zero. The common risk premium depends on the sensitivity process $\mathcal{V}_t$, which is determined by the household's state space.

Mathematically, the common risk impact and the common risk premium play two distinct but necessary roles. The common risk impact ensures that the value function is adapted to the filtration of the common noise. Economically, this means the household's welfare at time t depends only on the history of common noise up to that point. the common risk premium is an additional correction term required by the Itô-Kunita-Wentzell formula (Kunita, 1990), which is the chain rule applied to random fields. This correction is essential because the common noise simultaneously drives both the household's state variables and the value function itself (as a random field). When applying the chain rule to the household's value function, its randomness interacts with the the randomness in the state variables. This interaction, a quadratic covariation or "bracket term," generates an additional drift. The common risk premium is precisely the negative of this drift, and its inclusion serves to cancel out this mathematically-generated drift.

The common risk premium comprises a premium for risks actively chosen through financial investments and a premium for risks passively imposed through labor participation. We formalize the former:

Definition 3. An *asset-exposure premium* (AEP) is the component of the common risk premium that households decide their portfolio share ω_t to maximize their lifetime welfare under common risk:

$$AEP_t \equiv -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right). \tag{18}$$

The defining characteristic of the AEP is its inherent asymmetric structure. It consists of two asymmetric components: a negative exposure cost ($-\omega_t \sigma_s \frac{\partial v}{\partial a}$), driven by the marginal

utility of wealth, and a positive exposure compensation ($-\omega_t \sigma_s a_t \frac{\partial^2 v}{\partial a^2}$), driven by the curvature of the value function. Crucially, only the AEP compensation term is related to household wealth. This asymmetry gives the AEP its non-monotonicity, explaining hump-shaped profile. For low-wealth households, high marginal utility leads to a high AEP cost, while low wealth results in nearly zero AEP compensation. This net welfare cost is the root cause of limited participation. As wealth increases, however, this trade-off reverses. AEP compensation exceeds AEP costs, making wealthy households more inclined to bear common financial risk.

Definition 4. An *income-exposure premium* (IEP) is the component of the common risk premium arising from labor income that cannot be scaled by risky portfolio share under common risk:

$$IEP_t \equiv -y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1-\rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right). \tag{19}$$

It represents the endogenous welfare cost that households incur by participating in the labor market and bearing common income risk. The IEP has a structure similar to the AEP, as well as a cross-effect term of common risk. This structure may impose a higher welfare cost on low-wealth households. These households will comprehensively consider the benefits of income, background risk, and the IEP comprehensively before deciding whether to participate in the labor market. We propose a labor participation constraint to satisfy the economic intuition that households are averse to their welfare losses:

Assumption 1 (**Labor Participation Constraint**)**.** Any household that actively participates in the labor market satisfies the condition that the total marginal contribution of the income process to the value function must be positive:

$$y_t \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} + IEP_t \geq 0. \tag{20}$$

It implis that continued labor participation must result in a net improvement in household welfare. Just as a risk-averse investor would only hold a risky asset if it offers a positive welfare, a rational household would not remain in the labor force if doing so reduced their welfare compared to autarky. When the participation constraint is violated, households

would choose not to participate in the labor force. This may provide a theoretical explanation for a form of natural unemployment.

Households expect the welfare impact of the common noise and the impact on the evolution of the distribution. The value function is a McKean-Vlasov equation that depends on the distribution. The stochastic Kolmogorov forward (SKF) equation measures the evolution of the distribution:

$$dg_t = (\mathcal{L}^* g_t)dt - \text{div}(\mathcal{V}_t g_t)d\mathbf{W}_t, \tag{21}$$

where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}^*[g_t] \equiv &- \left\{ \frac{\partial}{\partial a}\Big(\left((\omega_t(\mu_s - r_f) + r_f)\, a_t + y_t - c_t \right) g_t \Big) + \frac{\partial}{\partial y}\Big((\mu_y + \tilde{\mu}_{n_i}) y_t g_t \Big) \right\} \\ &+ \left\{ \frac{\partial^2}{\partial a^2}\left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) \right\} \\ &+ \frac{\partial^2}{\partial a \partial y}\Big(\omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y g_t \Big),\end{aligned} \tag{22}$$

and the divergence term follows

$$\begin{aligned}\text{div}(\mathcal{V}_{1,t} g_t) =& \frac{\partial}{\partial a}\Big((\omega_t a_t \sigma_s + y_t \sigma_y \rho_{sy}) g_t \Big) \\ \text{div}(\mathcal{V}_{2,t} g_t) =& \frac{\partial}{\partial y}\Big(y_t \sigma_y \sqrt{1 - \rho_{sy}^2}\, g_t \Big).\end{aligned} \tag{23}$$

In the presence of common noise, the distribution g_t now becomes a stochastic process, so household's decisions must be contingent on the aggregate price. Therefore, each household recognizes that g_t evolves stochastically because the aggregate price affects each household's wealth and income simultaneously. In the absence of common noise, the distribution g_t evolves deterministically according to the law of large numbers, and each household takes it as a given drift when making their optimization decisions.

From a computational perspective, the BSHJB equation and the SKF equation together play different roles and present different challenges. The SKF equation is a forward-looking law of motion for the population density. Given the common noise, we can view the distribution g_t as a conditional measure and simulate the SKF equation, i.e., $g(\mathbf{X} + d\mathbf{W}_t)$.

This likes a task of population bookkeeping. In contrast, the BSHJB equation represents the backward-looking optimization for the household problem. The computational challenge is the curse of dimensionality. The household problem does not depend on the current infinite-dimensional distribution, but rather on the entire future stochastic path of the distribution, which itself is driven by future realizations of common noise. This dependence makes the household problem intractable.

This intractable problem is reflected in the functional derivative $\mathcal{A}[v_t]$. It captures how changes in the distribution affect the value function:

$$\begin{aligned}\mathcal{A}[v_t] &\equiv \left\langle \frac{\delta v}{\delta g}, dg_t \right\rangle + \frac{1}{2}\left\langle \frac{\delta^2 v}{\delta g^2}, d\langle g, g\rangle_t \right\rangle \\ &= \int_{\mathbb{R}^2} \mathcal{L}^* \frac{\delta v}{\delta g(x')} g(x') + \frac{1}{2}\int_{\mathbb{R}^{2\times 2}} \frac{\delta^2 v}{\delta g(x')\delta g(x'')}\Big(\operatorname{div}\big(\mathcal{V}g(x')\big)\cdot \operatorname{div}\big(\mathcal{V}g(x'')\big)\Big),\end{aligned} \tag{24}$$

where $x \equiv (a, y)$ is the state vector, $\langle \cdot, \cdot \rangle$ is the inner product, and $\frac{\delta v}{\delta g}$ is the Fréchet derivative, which quantifies the marginal impact of the change of the distribution.

Since the functional derivative $\mathcal{A}[v_t]$ is a function of the distribution g_t, households require a measure of g_t to forecast the change in the value function. We define the first moment of the distribution g_t as the macroeconomic variables:

$$\begin{aligned} K_t &= \iint_i \omega_{i,t} a_{i,t} g_t \\ B_t &= \iint_i (1-\omega_{i,t}) a_{i,t} g_t. \end{aligned} \tag{25}$$

The density satisfies the normalization, i.e., $\iint g_t = 1$. The initial distribution g_0 implies that the initial macroeconomic variables $\{K_0, B_0\}$ exist.

3 Equilibrium

The household problem requires each household to predict the stochastic evolution of the distribution, which is an infinite-dimensional object. Given the realistic constraints of "limited cognitive resources" (Enke, 2024) and "limited information processing capabilities" (Javed & Sutton, 2024), household expectations about g_t are an extreme manifestation of the

"curse of dimensionality" (Krusell & Smith, 1998; Dend Haan, 1996; Fernández-Villaverde, Nuño, & Perla, 2024). Rational expectations require all decision makers in the model to share information in order to achieve a Markov equilibrium (Sargent, 2025). This assumption make the household problem impractical for measuring the stochastic evolution of the object. Recent work argues that the assumption of rational expectations is unnecessarily complicates computations in CTHA models (Moll, 2025; Moll & Ryzhik, 2025).

To overcome these dimensionality challenges, households simplify their expectations about the distribution in realistic ways. We assume that households focus on one key variable that measures the distribution, i.e., $\mathbb{E}_t[g_t] \approx K_t$. The evolution of the capital is the *actual law of motion* (ALM):

$$dK_t = \mu_{k,t}dt + \Sigma_{k,t}d\mathbf{Z}_t, \tag{26}$$

where $\mu_{k,t}$ and $\Sigma_{k,t}$ are the drift and diffusion coefficients, respectively. Our idea is inspired by "approximate aggregation" in a general equilibrium model (Krusell & Smith, 1998).

For the coupled system to converge, households must form expectations about the ALM. We define a *perceived law of motion* (PLM) as the household's subjective belief about the ALM, $d\hat{K}_{i,t}$:

$$d\hat{K}_{i,t} \equiv \hat{\mu}_k(K_t, B_t)dt + \hat{\Sigma}_k(K_t, B_t)d\mathbf{W}_t + \hat{\sigma}_{K_i}(x_{i,t})d\mathbf{B}_{K_i,t}, \tag{27}$$

where $\hat{\mu}_k$ and $\hat{\Sigma}_k$ are the drift and diffusion coefficients of a consensus belief, $x_{i,t}$ is the household i's information, $d\mathbf{B}_{K_i,t}$ is heterogeneous beliefs (or observable noise), and $\hat{\sigma}_{K_i}$ is its diffusion coefficient.

We assume that households do not have heterogeneous beliefs if information is costless, i.e., $\hat{\sigma}_{K_i} = 0$. This concept is inspired by a continuous-time model (Grossman & Shiller, 1982), in which tradable assets and costless information lead to heterogeneous beliefs that do not affect the equilibrium price.[2] The concept of "consensus belief" arises from the Marcet and Sargent (1989)-G. W. Evans and Honkapohja (2001) approach, which forms a basis for the approximate aggregation.[3]

[2]The broader literature on heterogeneous beliefs includes studies by Shiller, Fischer, and Friedman (1984), De Long et al. (1990), Anderson, Ghysels, and Juergens (2005), Jouini and Napp (2007), Bhamra and Uppal (2014), Andrade et al. (2019), Borovička (2020), and Bigio, Silva, and Zilberman (2025).

[3]This approach has been advanced by G. W. Evans and Honkapohja (2012) and Ljungqvist and Sargent (2018).

We formalize these assumptions:

Assumption 2 (**Bounded Rationality**)**.** Assume that:

1. Households simplify their forecasting objective from a full measure of the distribution to moment estimation of the capital: $v(\mathbb{E}_t[g_t], \cdot) \mapsto v(\mathbb{E}_t[K_t], \cdot)$;

2. Households have a subjective belief about the ALM that defines the PLM (27);

3. Costless information precludes idiosyncratic beliefs: $(\hat{\sigma}_{k_i} = 0)$.

Then, we have the expectation of the functional derivative $\mathcal{A}[v_t]$ in the value function:

$$\begin{aligned}\mathbb{E}_t\mathcal{A}[v_t] \approx & \mathbb{E}_t\left\langle \frac{\partial v}{\partial K}, dK_t \right\rangle + \frac{1}{2}\mathbb{E}_t\left\langle \frac{\partial^2 v}{\partial K^2}, d\langle K, K\rangle_t \right\rangle \\ \approx & \mathbb{E}_t\left[\mu_{k,t} \mid K_t, B_t\right]\frac{\partial v}{\partial K} + \frac{1}{2}\mathbb{E}_t\left[\Sigma^2_{k,t} \mid K_t, B_t\right]\frac{\partial^2 v}{\partial K^2} \\ \approx & \hat{\mu}_{k,t}\frac{\partial v}{\partial K} + \frac{1}{2}\hat{\Sigma}^2_{k,t}\frac{\partial^2 v}{\partial K^2}.\end{aligned} \tag{28}$$

Although this bounded rationality assumption causes numerical drift in real values, it does not affect our theoretical mechanism. Even with this assumption, the solution still remains complex. According to Jensen's inequality, the household's problem is a nonlinear function, depending on the entire shape of the distribution and the common noise history, i.e., $dK = f(g_t, \mathbf{W}_\tau)_{\{0\leq\tau\leq t\}}$. This is not a sufficient statistic; households must implement a sophisticated nonlinear filter to track the evolution of the unobserved higher moments of the distribution based on $\mathbf{W}_\tau$.[4] In CTHA models without common noise, household expectations about the distribution dynamics are relatively simple. For example, households either track the total bond (Fernández-Villaverde, Hurtado, & Nuno, 2023) or do not track the dynamics (Ahn et al., 2018).

This bounded rationality assumption leads to the concept of temporary equilibrium:

Definition 5 (Equilibrium)**.** Under the assumptions 1, 2, and 4[5], a *temporary equilibrium*

[4]Before performing numerical simulations, we cannot determine $\hat{\Sigma}_{k,t}$. We suspect that it will not be close to zero, as in the Krusell and Smith (1998) model. This is because our framework is highly nonlinear, making it difficult for consensus belief to satisfy the certainty equivalence principle using least-squares learning.

[5]See Appendix A for a detailed discussion about the well-posedness of the coupled system.

consists of the tradable assets S_t (2) and B_t (3), the non-tradable labor income y_t (7), and the PLM (27) such that:

1. Each household i maximizes its value function (13) through consumption-portfolio choice $\{c_{i,t}, \omega_{i,t}\} \in (0,\infty) \times [0,\infty)$;

2. The distribution g_t evolves according to the SKF equation (21);

Temporary equilibrium describes the state of the coupled system at any given moment:

$$\begin{aligned} dv_t = &\max_{c,\omega} \left\{ u(c_t) + \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} + \hat{\mu}_k \frac{\partial v}{\partial K} \right\} dt \\ &+ \left\{ \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} + \frac{1}{2} \hat{\Sigma}_{k,t}^2 \frac{\partial^2 v}{\partial K^2} \right\} dt \\ &- \left\{ \omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) + y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sigma_y \sqrt{1 - \rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right) \right\} dt \\ &+ \begin{bmatrix} \omega_t a_t \sigma_s \frac{\partial v}{\partial a} + y_t \sigma_y \rho_{sy} \frac{\partial v}{\partial y} & y_t \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial v}{\partial y} \end{bmatrix} d\mathbf{W}_t \end{aligned} \tag{29}$$

$$\begin{aligned} dg_t = &- \left\{ \frac{\partial}{\partial a} \left(\left((\omega_t(\mu_s - r_f) + r_f) a_t + y_t - c_t \right) g_t \right) + \frac{\partial}{\partial y} \left((\mu_y + \tilde{\mu}_{n_i}) y_t g_t \right) \right\} dt \\ &+ \left\{ \frac{\partial^2}{\partial a^2} \left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) + \frac{\partial^2}{\partial a \partial y} \left(\omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y g_t \right) \right\} dt \\ &- \begin{bmatrix} \frac{\partial}{\partial a} (\omega_t a_t \sigma_s g_t) + \frac{\partial}{\partial a} (y_t \sigma_y \rho_{sy} g_t) & \frac{\partial}{\partial y} \left(y_t \sigma_y \sqrt{1 - \rho_{sy}^2} g_t \right) \end{bmatrix} d\mathbf{W}_t. \end{aligned} \tag{30}$$

It allows for discrepancies between the PLM and the ALM. The concept of temporary equilibrium is originally developed by Lindahl (1939) and Hicks (1946) and formalized by Grandmont (1977, 1989).[6]

Our analysis is based on the terminal condition of the household problem as a steady state, which is the convergence behavior of the coupled system.[7]

Definition 6 (Steady State)**.** In equilibrium 5, a *local steady state* is a terminal condition in which each household expects the aggregate price to fluctuate around the expected value.

[6]Recent applications of temporary equilibrium in modern macroeconomic models include Woodford (2013), Piazzesi and Schneider (2016), García-Schmidt and Woodford (2019), and Farhi and Werning (2019).

[7]Coeurdacier, Rey, and Winant (2011) proposes that a deterministic steady state (DSS) is a terminal condition in which households do not anticipate future common risk. In this case, the expectation of common risk vanishes ($\sigma(\mathbf{W}_t) = 0$) and the model degenerates to an ABH framework.

That is, the independent increment of the common noise is zero ($d\mathbf{W}_t = 0$) and common risk still remains ($\sigma(\mathbf{W}_t) > 0$). The LSS is characterized by the following system:

$$\begin{aligned} \rho v_t =& \max_{c,\omega} \{u(c) + \mathcal{L}[v_t] + \mathbb{E}_t \mathcal{A}[v_t] - \text{div}(\mathcal{V}_t)\}. \\ \frac{\partial g}{\partial t} =& \mathcal{L}^*[g_t]. \end{aligned} \tag{31}$$

The brilliance of the LSS lies in its conceptual separation of the impact of common risk from the premium. Defining $\mathbb{E}_t[d\mathbf{W}_t] = 0$ shuts off stochastic fluctuations in the next moment. The common risk premium remains fully activated. The LSS thus acts as a theoretically controlled experiment, allowing us to precisely measure the endogenous welfare cost that households pay for being exposed to common risk alone, without the numerical disturbances caused by future common noise. The LSS is a short-run fixed point at which the aggregate price continues moving along inertia after receiving a shock until the next moment when the new common noise changes. It also enables an coupled system with common risk to transmit toward long-run equilibrium. Formally, given the optimal consumption-portfolio choice $\mathbb{E}_t[\{dc_t, d\omega_t\}] = \arg v(\{c_t, \omega_t\}, \mathbb{E}_t[d\mathbf{W}_t])$ satisfies the convergence behavior, i.e., $\{0, 0\} = \arg v(\{c^*, \omega^*\}, 0)$. When the ALM equals the PLM, the LSS features a "self-confirming equilibrium" (Fudenberg & Levine, 1993; Kamada, 2010), which allows us to construct a tractable and economically meaningful convergence behavior of the coupled system.

The LSS generalizes "stochastic steady state" approaches (Futia, 1982; Duffie et al., 1994; Schenk-Hoppé & Schmalfuss, 2001; Coeurdacier, Rey, & Winant, 2011), which are used as an analytical benchmarks in CTHA models without common noise (Fernández-Villaverde, Hurtado, & Nuno, 2023; Bilal, 2023). Stochastic steady states are long-run fixed points that households expect the realization of aggregate shocks to be zero, i.e., ($\mathbb{E}_t[\mathbf{W}_{t+1}] = 0$). This assumption restricts the analysis with zero-probability path in which the entire history of the aggregate shocks must cancel out perfectly to achieve a deterministic long-run equilibrium and ergodic distributions.

4 Theoretical Results

Our analysis focuses on how common financial risk is transmitted to households through aggregate prices, triggering the AEP effect and shaping their risky portfolio share across the distribution. First, we derive the formula for the optimal consumption-portfolio choice from the household's problem (29) and examine household behavior at the extreme ends and in the middle of the wealth distribution. Next, we present a comprehensive model in which household portfolios incorporate two risk characteristics: systematic risk and idiosyncratic risk. Finally, we address broader theoretical issues.

The household's problem yields the envelope condition for the optimal consumption:

$$c_t^* = (u')^{-1}\left(\frac{\partial v}{\partial a}\right). \tag{32}$$

Taking the first-order condition (FOC) with respect to the portfolio share ω_t yields the portfolio choice:

$$\omega_t^* = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left\{\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)\frac{\partial v}{\partial a} + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)y_t\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_s}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right\}. \tag{33}$$

Definition 7. The *asset-exposure premium effect* is the component of households' optimal portfolio choice with respect to the systematic risk asset S, given by $\Omega^*(a_t)$, which directly arises from the asset-exposure premium (18) within their optimization problem when choosing among a set of feasible investments including the risk asset S and the risk-free asset B:

$$\Omega^*(a_t) \equiv \frac{1}{a_t\sigma_s}\left[1 + \frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\frac{\partial v}{\partial a}\right]. \tag{34}$$

We define the curvature-to-slope ratio of the BSHJB equation with respect to wealth as an *effective relative risk aversion* (ERRA):

$$\eta(a_t) \equiv -a_t\frac{v_{aa}}{v_a}, \tag{35}$$

where $v_a \equiv \partial v/\partial a$ and $v_{aa} \equiv \partial^2 v/\partial a^2$. The effective relative risk aversion η_t measures households' intertemporal risk preferences across wealth. In contrast, the Arrow-Pratt measure

of relative risk aversion is defined as $R(c_t) \equiv -c_t u''/u'$, which measures instantaneous risk preferences. The mathematical form of η_t is proposed by Merton (1969, 1971). Given the standard conditions ($a > 0$, $v_a > 0$, and $v_{aa} < 0$), we have

$$\eta_t > 0. \tag{36}$$

Substituting the equation (35) into the AEP (18), the AEP effect (34), and the optimal portfolio choice (33) yields

$$AEP_t = \omega_t \sigma_s \big(\eta_t - 1\big) \frac{\partial v}{\partial a}, \tag{37}$$

$$\Omega^*(a_t) = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t}\right), \tag{38}$$

and

$$\omega_t^* = \frac{1}{\eta_t} \Big[\underbrace{\frac{\mu_s - r_f}{\sigma_s^2}}_{\text{speculative demand}} + \underbrace{\frac{\rho_{sy}\sigma_y}{\sigma_s} y_t \left(\frac{\partial v}{\partial a}\right)^{-1} \frac{\partial^2 v}{\partial a \partial y}}_{\text{wage-offsetting effect}} + \underbrace{\frac{1}{a_t \sigma_s}\big(\eta_t - 1\big)}_{\substack{\text{AEP effect} \\ \text{(New Term)}}} \Big]. \tag{39}$$

It decomposes the household portfolio choice into three motives scaled by η_t. We call the first component the speculative demand. It reflects the households' intertemporal decision regarding the returns on risky assets, accounting for its excess returns and variance. To distinguish idiosyncratic noise in labor productivity $n_{i,t}$ from income, we call the second term the wage-offsetting effect. It reflects balancing or compensating in the optimal portfolio share due to the primary common noise through the aggregate wage rate w_t. The third term is the AEP effect. It reflects the impact of the AEP on portfolio choice. For a portfolio with idiosyncratic noise, households only have the speculative demand. Under common financial risk, the systematic risk asset additionally triggers both of the AEP effect and the wage-offsetting effect. These effects characterize common financial risk. The first and second terms were pioneered by Merton (1969, 1971). Merton (1992) and Campbell and Viceira (2002) provide in-depth analyses for portfolio choice and intertemporal portfolio choice, respectively.[8]

[8]The first term is often referred to as myopic demand in the literature. According to the definition by Campbell (2018), at a given point in time, investors with given preferences has the same optimal portfolio, regardless of their time horizon. Myopic portfolio choice is usually satisfied in an approximately linear framework. Our framework is highly nonlinear and dynamic. Meanwhile, we emphasize the wealth-dependent

4.1 ERRA and Portfolio Choice at Both Ends

To align the household problem with economic intuition, we link η_t to observable microeconomic behavior. Let the marginal propensity to consume out of wealth $\text{MPC}_a \equiv \partial c_t^*/\partial a$. We have[9]

$$\eta_t = \frac{a_t}{c_t^*} \cdot \text{MPC}_a \cdot \gamma. \tag{40}$$

It transforms the invariant risk preference γ into time-varying risk-taking behavior at different wealth levels through the elasticity of consumption with respect to wealth, $\epsilon_{c,a} \equiv (a_t/c_t^*) \cdot \text{MPC}_a$.

4.1.1 ERRA and Portfolio Rule at the Borrowing Constraint

Near the borrowing constraint $\underline{a}$, households' optimal consumption is nearly determined by labor income. This implies that a_t/c_t^* approaches zero as wealth approaches $\underline{a}$.

Proposition 1 (**ERRA at the Borrowing Constraint**). *For any household with finite CRRA ($\gamma < \infty$) and bounded marginal propensity to consume out of wealth ($MPC_a < \infty$), the effective relative risk aversion vanishes at the borrowing constraint:*

$$\lim_{a \to 0^+} \eta_t = 0. \tag{41}$$

Proof. See Appendix B.1. □

It implies that, near the borrowing constraint, households' risk aversion to wealth becomes relatively negligible compared to the marginal benefits of wealth. The economic intuition is that the low wealth haves a strong incentive to pursue excess returns on risky assets in order to escape poverty.

Corollary 1 (**Portfolio Rule at the Borrowing Constraint**). *At the borrowing constraint, households do not hold the systematic asset S_t and the asset-exposure premium effect*

property of portfolio choice, rather than the term "hedging" effect.

[9] We exploit the envelope condition (32) and derive $v_{aa} = u''(c_t^*)\frac{\partial c^*}{\partial a}$. We use the standard CRRA specification $\gamma = -c_t^* u''/u'$ and substitute these equations into (35). Then, $\eta(a_t) = -a_t \cdot \frac{v_{aa}}{u'(c^*)} = -a_t \cdot \frac{\partial c^*}{\partial a} \cdot \frac{u''(c^*)}{u'(c^*)} = \frac{a_t}{c_t^*} \cdot \text{MPC}_a \cdot \gamma$.

is singular:

$$\lim_{a\to 0^+} \Omega^*(a_t) = -\infty, \qquad \lim_{a\to 0^+} \omega_t^* = 0. \tag{42}$$

Proof. See Appendix B.2. □

It reveals the core mechanism of limited participation: near the borrowing constraint, the magnitude of the different motives diverges. On the one hand, the speculative demand tends to positive infinity when the wage-offsetting effect is smaller than the speculative demand. Both motives are of order $O(n)$. This reflects the strong incentive for the poor to ignore risk in pursuit of asset returns. The AEP effect completely overwhelms all of this. With the scaling factor $1/a$, the AEP effect rushes toward negative infinity much faster, at the order of $O(n^2)$. The negative force is an order of magnitude larger than the other two forces. Thus, as wealth approaches the borrowing constraint, households' optimal decisions are entirely governed by the AEP effect. They rationally price the substantial AEP cost of holding the systematic asset and choose to exit the market entirely.

4.1.2 ERRA and Portfolio Choice of the Wealthy

As wealth grows arbitrarily large, the limiting case becomes a benchmark for understanding the behavior of the wealthy. In this regime, the influence of labor income diminishes, and wealth dynamics become paramount. As in a long line of recent studies, consumption and wealth become proportional at the limit.

Let $c^*(a, y, K, t)$ be the optimal consumption rule derived from the solution to the BSHJB equation (29).

Theorem 1 (Consumption-Wealth Ratio). *As wealth approaches infinity, the consumption-to-wealth ratio converges asymptotically to the deterministic constant $\bar{k}_L$ at the LSS:*

$$\lim_{a\to\infty} \mathbb{E}_t \left[\frac{c^*(a, y, K, t)}{a_t} \right] = \bar{k}_L, \tag{43}$$

where

$$\bar{k}_L = \frac{1}{\gamma} \left(\rho + (\gamma - 1) \left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2} \right) \right). \tag{44}$$

Proof. See Appendix C.1. □

It is consistent with the result of Merton (1969). Modern analyses show that the policy function becomes asymptotically linear for the highest level of wealth (Benhabib, Bisin, & Zhu, 2015; Achdou et al., 2022).

Proposition 2 (**ERRA of the Wealthy**). *As wealth approaches infinity, the effective relative risk aversion converges to the finite CRRA coefficient:*

$$\lim_{a\to\infty} \eta_t = \gamma. \tag{45}$$

Proof. See Appendix C.2. □

It implies that the wealthy's intertemporal risk preference to wealth behaves in the same way as their instantaneous risk preference to consumption. This is because their consumption depends entirely on asset returns.

Corollary 2 (**Portfolio Choice of the Wealthy**). *As wealth approaches infinity, the asset-exposure premium effect vanishes, and the portfolio choice converges to a constant:*

$$\lim_{a\to\infty} \Omega^*(a_t) = 0, \qquad \lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\gamma}. \tag{46}$$

Proof. See Appendix C.3. □

It is consistent with the classical Merton ratio. In the high-wealth limit, the asymptotically optimal behavior alters the relative magnitudes of the three economic forces in the portfolio choice. As wealth approaches infinity, the effective relative risk aversion converges asymptotically to γ, and the speculative demand for risky assets is determined by a time-invariant risk aversion parameter. The speculative demand converges to a non-zero constant of order $O(1)$ remaining robustly active. The wage-offsetting effect vanishes as labor income becomes negligible relative to wealth. The AEP effect also vanishes by the scaling factor $1/a$.

Households' optimal portfolio share converges to the speculative component, which is consistent with the standard benchmark. In other words, for a given γ, changes in asset risk-return characteristics are the only feedback to the portfolio choice because the level of

wealth does not affect portfolio decisions and remains constant.[10] The dynamics of wealth inequality are therefore driven by fundamentally different forces across the distribution. The most significant portfolio heterogeneity driven by the AEP mechanism is concentrated among low-to-moderate wealth households.

4.2 Limited Participation and Life-Cycle Pattern

We now apply the AEP mechanism to provide a unified explanation for limited participation and the hump-shaped profile of the risky portfolio share. These phenomena stem from the inherent dynamics of the effective relative risk aversion as it changes with wealth. We restrict households' intertemporal risk aversion so that it is well-defined:

Assumption 3 (**Well-defined Intertemporal Risk Aversion**)**.** The effective relative risk aversion η_t is a monotonic increasing function of wealth a:

$$\frac{\partial \eta}{\partial a} > 0, \qquad \forall a < \infty. \tag{47}$$

We make this assumption based on the economic intuition: as wealth increases, households' effective relative risk aversion to a risky portfolio also increases, because their consumption depends more on asset returns. This assumption is equivalent to the consumption behavior that the elasticity of consumption with respect to wealth is monotonic, i.e., $\frac{\partial \epsilon_{c,a}}{\partial a} > 0$. Furthermore, the sufficient condition for this assumption is that the value function satisfies $\frac{v_{aaa} v_a}{v_{aa}^2} \geq \frac{u''' u'}{(u'')^2}$, see Appendix D.1.[11] When this assumption does not hold, we cannot obtain clear analytical results because of ambiguous investment behavior. For example, the speculative demand may fluctuate as wealth increases.

In the low-wealth region, the equation (37) shows that when $\eta_t < 1$, the AEP is negative, forcing $\Omega_t < 0$. This occurs when the exposure cost exceeds the exposure compensation. For

[10] This behavior robustly converges back to a representative agent model, which is a continuous-time framework of the Phelps-Ramsey problem (Phelps, 1962). The static analysis can be found in Merton (1969).

[11] This condition is closely linked to the precautionary saving in the discrete framework (Carroll & Kimball, 1996).

households near the borrowing constraint, their effective relative risk aversion η_t approaches zero. The excess returns of any risky asset induce these households to have a strong the speculative demand by the factor $1/\eta$. It also amplifies the AEP effect, which is dominated by the scaling factor $1/a$. This results in a negative effect that is order of magnitude larger than the other two motivations in the low-wealth region. The AEP mechanism endogenizes limited participation through a zero risky portfolio share.[12] As wealth increases, the impact of the scaling factor on the AEP effect gradually weakens, while η_t also departing from zero. When η_t approaches the left side of a critical point $\eta_t = 1$, the negative impact of the AEP effect becomes smaller than the positive impact of the excess returns. Households thus gradually become willing to participate in the market. This is reflected by a rapid increase in the risky portfolio share held.

Lemma 1. *Given the CRRA coefficient $\gamma > 1$, there exists the critical point $a^* \in a$ such that*

$$\eta(a^*) = 1 \tag{48}$$

and

$$\eta(a) > 1, \qquad \forall a > a^*. \tag{49}$$

Proof. See Appendix D.2. □

At the critical point a^*, the AEP cost perfectly offsets the AEP compensation, causing the AEP effect vanishes completely, i.e., $\Omega(a^*) = 0$. It implies that the portfolio choice is determined by the speculative demand and the wage-offsetting effect.

In the medium-to-high wealth region, where $a > a^*$, the AEP translates into a net benefit, generating a positive AEP effect, $\Omega_t > 0$. At this point, the AEP compensation exceeds the AEP cost. As wealth increases beyond a^*, the sensitivity to marginal utility and the curvature of risk aversion remain substantial. The interaction between these factors pushes the positive AEP effect to its peak value, providing an additional incentive to hold

[12] The literature has proposed explanations, categorized into three main streams: fixed participation costs that deter small investors, non-standard preferences such as ambiguity aversion or loss aversion, and the effect of background risk. For a comprehensive review of these approaches, see Campbell (2018). Recent studies on limited participation include Athreya, Ionescu, and Neelakantan (2023), Bharath and Cho (2023), and Melcangi and Sterk (2025).

the systematic risk asset beyond standard portfolio motives. As wealth continues to grow, the scaling factor $1/a$ becomes dominant, causing the AEP effect to gradually decline and ultimately converge to zero from above.

As households accumulate wealth over their lifetimes with positive savings, they naturally traverse wealth zones in which the AEP effect first inhibits, then encourages, and ultimately normalizes risky asset holdings. This process culminates in the portfolio choice converging asymptotically to the classical Merton benchmark. The complete evolutionary path of the asset-driven, non-monotonic AEP mechanism generates the hump-shaped profile of portfolio choice over the life cycle. This profile aligns with the "hump-shaped profile" (Heaton & Lucas, 2000; Poterba & Samwick, 2001; Ameriks & Zeldes, 2011; Parker et al., 2025) and the "hump-shaped saving behavior" (Graaff, 1950; Ando & Modigliani, 1963).[13]

4.2.1 Labor Income Channel

With well-defined intertemporal risk aversion, canonical portfolio theory explains the hump-shaped profile through the labor income channel. This raises the question: How does this framework operate within the labor income channel?

We propose that households treat labor endowment as an implicit wealth, defined as the expected present value of future labor income. Although non-tradable labor income cannot be evaluated independently of its risk, the Assumption 1 ensures that this implicit wealth improves households' lifetime welfare, even in the presence of aggregate income risk. This alters their attitude toward tradable financial assets. Our perspective is similar to the point proposed by Heaton and Lucas (2000) that although income is risky, it limits bad outcomes relative to investment income, which reduces the effective relative risk aversion η_t.

Formally, we provide a proof of this logic. Following the "proper property" Pratt and Zeckhauser (1987) and Gollier and Pratt (1996), the absolute risk aversion $-u''/u'$ is monotonically decreasing in wealth. We measure the welfare impact of wealth through an *effective absolute risk aversion*, $A(a) \equiv -v_{aa}/v_a$. As established by Carroll and Kimball (1996), we have that $v_{aaa}v_a - v_{aa}^2 > 0$. This is a sufficient condition for $A(a)$ to be

[13]Recent studies of life-cycle portfolio choice include Hubener, Maurer, and Mitchell (2016), Addoum (2017), Fagereng, Gottlieb, and Guiso (2017), Michaelides and Zhang (2017), Chang, Hong, and Karabarbounis (2018), Gomes (2020), Catherine (2022), and Catherine, Sodini, and Zhang (2024).

monotonically decreasing in wealth, i.e., $\frac{\partial A(a)}{\partial a} < 0$. The implicit wealth causes households to behave as if they possess greater wealth. This leads to a lower $A(a)$ for any given wealth and a lower η_t.

Proposition 3. *The effective relative risk aversion η_t is a monotonic decreasing function of the implicit wealth a':*[14]

$$\frac{\partial \eta_t}{\partial a'} < 0. \tag{50}$$

It implies that implicit wealth reduces households' intertemporal risk aversion to wealth, leading them to desire a larger share of a risky portfolio by the factor $1/\eta$. Over the life cycle, while the labor productivity $n_{i,t}$ may cause the income $y_{i,t}$ to be hump-shaped; implicit wealth is monotonically decreasing. This indicates that, at the same wealth level, young households are more willing to take intertemporal risks in pursuit of capital gains. The same logic applies to employed households compared to retirees.

Idiosyncratic income risk causes the wage-offsetting effect to disappear. The portfolio share of the systematic risk asset is clearly determined by two wealth-dependent motives. In the presence of aggregate income risk, the wage-offsetting effect depends on the uncertain empirical value of the correlation ρ_{sy}, so its net impact is indeterminate without further parameterization. A more detailed discussion of this topic is deferred to Section 4.3.

4.2.2 Labor income in Merton Framework

The Merton framework faces multiple theoretical challenges when addressing these puzzles. Non-tradeable labor income depends critically on two risk characteristics of income processes. However, both of these channels are theoretically problematic.

Idiosyncratic income risk exacerbates limited participation. Early studies by Bodie, Merton, and Samuelson (1992) and Heaton and Lucas (1997) examine the effect of the labor income in the presence of idiosyncratic noise on portfolio decisions. According to Proposition 3, income increases household welfare and results in greater speculative demand. This is similar to the economic intuition proposed by Viceira (2001) and Campbell and Viceira

[14] $\eta_t' = a_t A(a_t + a') = -a_t \frac{v_{aa}(a_t+a')}{v_a(a_t+a')} < -a_t \frac{v_{aa}(a_t)}{v_a(a_t)} = \eta_t$, where $a' > 0$ is the implicit wealth.

(2002) that low-wealth households relying on labor income have low effective relative risk aversion, and a high risky portfolio share.

Aggregate income risk cannot simultaneously explain limited participation and the hump-shaped profile of portfolio share. Using microdata, The literature calibrates the correlation between stock returns and labor income (Jagannathan, Kocherlakota, et al., 1996; Davis & Willen, 2000; Viceira, 2001; Haliassos & Michaelides, 2003; Cocco, Gomes, & Maenhout, 2005; Benzoni, Collin-Dufresne, & Goldstein, 2007). Typically, a large value of the correlation is required to make the effect significant for all households because the numerator $\frac{\partial^2 v}{\partial a \partial y}$ is relatively small than the denominator $\frac{\partial v}{\partial a}$. This effect is more sensitive to low-wealth households. When the correlation is negative, the wage-offsetting effect increases the risky portfolio share for all households. Low-wealth households experience a more pronounced increase, which exacerbates limited participation further. When the correlation is positive, the wage-offsetting effect affects employed households, but not retirees, who hold a larger risky portfolio share than employed households. This result is inconsistent with life-cycle patterns. The fragility of the aggregate income risk is further underscored by the well-documented instability of the correlation (Guiso, Jappelli, & Terlizzese, 1996; Bertaut & Haliassos, 1997; Heaton & Lucas, 2000; Poterba, 2001; Ameriks & Zeldes, 2011), with empirical estimates varying across countries and periods. Some studies have found correlations close to zero or positive (Campbell et al., 2001; Angerer & Lam, 2009; Betermier et al., 2012; Huggett & Kaplan, 2016).

In the next section, we add idiosyncratic risk to households' portfolio to characterize its risk characteristics, which provides a reasonable economic explanation for the logical contradiction in the Merton framework.

4.3 Portfolio Risk Characteristics: From Underdiversification to Diversification

We are required to use a theoretical framework to explain underdiversification. Some exogenous hypotheses that aim to explain underdiversification face empirical challenges. For example, there is no evidence in the U.S. market that portfolio size or transaction

costs constrain diversification (Goetzmann & Kumar, 2008). Calvet, Campbell, and Sodini (2007, 2009), Goetzmann and Kumar (2008), and Parker and Vissing-Jørgensen (2009) find that household portfolio risk preferences depend primarily on their position in the wealth distribution. These findings motivate us to consider the different risk characteristics of household portfolios and propose an endogenous, wealth-dependent mechanism that can explain both underdiversification and diversification across the wealth distribution. The Merton framework, based on the representative agent model, fails to identify the different risk characteristics of household portfolios. It can only indirectly reflect these characteristics through aggregate income risk, which is not an endogenous property of assets.

Let the dynamics of the risk component of the household portfolio be as follows:

$$\frac{dp_t}{p_t} = \mu_p + \beta_p \sigma_s d\mathbf{W}_{s,t} + \tilde{\sigma}_p d\mathbf{B}_{p,t}, \tag{51}$$

where μ_p is the expected return of the risky asset, $\beta_p \sigma_s$ is the systematic risk, $\tilde{\sigma}_p$ is the idiosyncratic risk, and $\mathbf{B}_{p,t}$ is the idiosyncratic noise.

Consider two portfolios F and N with completely different risk characteristics. The risk-free component of these portfolios includes only the risk-free asset b_t. The risk component of portfolio F contains only systematic risk, i.e., $\tilde{\sigma}_F = 0$ and $\beta_F = 1$. The risk component of portfolio N contains only idiosyncratic risk, i.e., $\beta_N = 0$. The portfolio F is the market portfolio, which serves as a benchmark.

We define a function that measures the household's preference for systematic risk in its portfolio:

Definition 8. A β *exposure ratio* in the household's portfolio p is defined as:

$$\Gamma_{p,t} \equiv \beta_p \left\{ \rho_{sy} \sigma_y y_t \left(\frac{\partial v}{\partial a} \right)^{-1} \frac{\partial^2 v}{\partial a \partial y} + \frac{1}{a_t} (\eta_t - 1) \right\}, \tag{52}$$

where ρ_{sy} is the structural correlation.

It is beyond the risk-return profile of the household portfolio. The first term reflects the offsetting effect of labor on systematic risk in portfolio choice. The remaining labor terms, such as the IEP, income risk, and income, only affect household welfare. In other words,

households are unaffected by these terms when deciding on the risk composition of their portfolios.

Next, we use the sign of the β exposure ratio to classify households' rational behavior at different positions in the wealth distribution. First, we determine these households' preferences for their portfolios versus the market portfolio based on equivalent risk-return characteristics. Then, based on equivalent welfare levels, we determine the Sharpe ratio premium that households are willing to pay for taking on systematic risk. In other words, we analyze the portfolio efficiency of these households under the same consumption levels.

4.3.1 Underdiversification

Let Γ_t be the β exposure ratio of the market portfolio. The following proposition holds:

Proposition 4 (**Portfolio Risk Preference**). *When the β exposure ratio is negative ($\Gamma_t < 0$), then:*

(i) For equivalent risk-return characteristics, the optimal portfolio share satisfies:

$$\omega_F^* < \omega_N^*. \tag{53}$$

(ii) Under equivalent welfare levels, households demand a strictly higher Sharpe ratio from the market portfolio:

$$\mathcal{S}_F > \mathcal{S}_N \tag{54}$$

with the required Sharpe ratio premium given by:

$$\mathcal{S}_F - \mathcal{S}_N = -\Gamma_t > 0. \tag{55}$$

Proof. See Appendix E.1. □

When the β exposure ratio is negative ($\Gamma_t < 0$), low-wealth households bear the welfare cost of systematic risk. To minimize their welfare loss, they will seek portfolios that are low or even zero beta and that uncorrelated with financial market fluctuations. In real-world asset allocation, these households may invest in a few companies, local businesses, or specific

industries. This behavior objectively constitutes underdiversification, which results in their portfolios being exposed to more idiosyncratic risk. In this process, households are willing to accept portfolios with lower Sharpe ratios in exchange for better welfare. This rational behavior reduces exposure to systematic risk; however, the price they pay is the Sharpe ratio loss associated with holding more idiosyncratic portfolio. This explains the findings of Calvet, Campbell, and Sodini (2007) and Goetzmann and Kumar (2008) that households with less wealth are less likely to participate, and even if they do, they are more likely to invest inefficiently. For households with near-borrowing constraints, β exposure ratios are very high, almost completely excluding them from the market for systematic risk assets. This explains the empirical evidence from Blume and Friend (1975) and Kelly (1995) that households own severely underdiversified portfolios of individual stocks. Although Calvet, Campbell, and Sodini (2007) finds that a small number of Swedish households do suffer from significant underdiversification, we speculate that this is because Swedish households have higher wealth, resulting in less β exposure ratio loss.

When evaluating the efficiency of a portfolio with idiosyncratic risk and systematic risk, we determine the welfare loss incurred by a household portfolio taking on systematic risk and how this response can lead to seemingly suboptimal choices. Let a portfolio P follows the euqation (51), then the following corollary holds:

Corollary 3 (Preference for Risk Assets). *When the β exposure ratio is negative ($\Gamma_t < 0$), then:*

(i) For equivalent risk-return characteristics, the optimal portfolio share satisfies:

$$\omega_P^* > \omega_F^*. \tag{56}$$

(ii) Under equivalent welfare levels, the required Sharpe ratio for portfolio P lies strictly between the individual asset requirements:

$$\mathcal{S}_N < \mathcal{S}_P \tag{57}$$

with the specific relationship:

$$\mathcal{S}_F - \mathcal{S}_P = (\beta_P - 1)\Gamma_t. \tag{58}$$

Proof. See Appendix E.2. □

We can make a quantitative assessment based on the relative Sharpe ratio loss (RSRL), a mean-variance measure proposed by Calvet, Campbell, and Sodini (2007). The RSRL of portfolio P is

$$RSRL_{P,t} = \frac{\beta_P - 1}{\mathcal{S}_F}\Gamma_t. \tag{59}$$

When the portfolio P has $0 < \beta_P < 1$, the household requires its Sharpe ratio to be less than that of the market portfolio. Conversely, if portfolio P has $\beta_P > 1$, the household requires that its Sharpe ratio be greater than that of the market portfolio. Because the β exposure ratio is an endogenously determined function of each household and the aggregate price is observable, households only need to compare the Sharpe ratio of the market portfolio to determine their portfolio's Sharpe ratio. This measure allows us to directly map our theoretical prediction of welfare loss to an empirically measurable quantity, providing a path for future quantitative validation.

The risk attitudes predicted by our framework resonate with several findings in behavioral finance.[15] Some studies have linked the tendency to overweight small probabilities to holding an underdiversified portfolio (Polkovnichenko, 2005; Dimmock et al., 2021).

4.3.2 Diversification

Households with positive β exposure ratio ($\Gamma_t > 0$) prefer to take on systematic risk, either by holding the market portfolio or through diversification. Systematic risk offers these households a net welfare benefit beyond financial returns. Thus, these households decide not

[15] This is similar to the risk-seeking behavior in the loss domain of prospect theory (Kai-Ineman, Tversky, et al., 1979; Fiegenbaum & Thomas, 1988; Thaler & Johnson, 1990; Tversky & Kahneman, 1992; McDermott, Fowler, & Smirnov, 2008; Kahneman & Tversky, 2013; N. C. Barberis, 2013). This preference for high-risk, low-return assets echoes research showing that investors view certain stocks as "lottery tickets" (Kumar, 2009; N. Barberis & Huang, 2008; Boyer & Vorkink, 2014; Bali, Cakici, & Whitelaw, 2011; Eraker & Ready, 2015).

based on how much relative Sharpe ratio loss they need to bear systematic risk, but rather on how willing they are to accept this loss in their portfolio's idiosyncratic risk to gain the added benefit of systematic risk.

If the market portfolio already offers the highest Sharpe ratio, then it is on the efficient frontier. In this case, it is difficult for households to find assets with higher Sharpe ratios. Furthermore, the subjective valuation of these households' portfolios will change, requiring the "adjusted" Sharpe ratio of these portfolios to satisfy at least $\mathcal{S}_F + (\beta - 1)\Gamma_t$. These households find the market portfolio more attractive than portfolios with $\beta < 1$. Therefore, they will exhibit a strong preference for diversification over portfolios with $\beta < 1$. Since the market portfolio is already on the efficient frontier, households will be reluctant to hold portfolios with $\beta < 1$. This explains the empirical phenomenon that most participating Swedish households' portfolios are actually close to the efficient frontier. (Calvet, Campbell, & Sodini, 2007).

When portfolios have $\beta > 1$, these households are willing to take on additional idiosyncratic risk. They allocate more to the systematic risk to balance the idiosyncratic risk. This is also confirmed by the findings of Calvet, Campbell, and Sodini (2007, 2009) and Goetzmann and Kumar (2008) that households with higher levels of wealth accumulation tend to invest more efficiently but also more aggressively. Overallocation is the driving force behind the hump in a hump-shaped portfolio. If their portfolios have $\beta > 1$, the systematic risk may transform the hump profile into a flat-topped profile across the wealth distribution.

As wealth grows further, the β exposure ratio decreases and converges to zero from above. This means that the welfare cost of bearing systematic risk is also almost zero. For the wealthiest households, the risk characteristics of their portfolio are irrelevant; only the risk-return characteristics matter. Economic intuition suggests that, in the market with a single, richest investor, it consider itself as a single investor. This is why convergence to the Merton framework occurs in this case. Campbell (2006)'s empirical findings support this logic, showing that the wealthiest households in the United States hold a large share of private businesses.

The rate at which households accumulate wealth changes based on the risk characteristics of their portfolio. Households with positive β exposure ratio follow fluctuations in the

financial market, accelerating their wealth accumulation. This optimization behavior acts as a "wealth accelerator," propelling these households upward in the distribution.

4.4 Analytical Solution under Homogeneity

This section explores when the value function in this framework degenerates into a homogeneous form, yielding an analytical solution under common risk. Although we adopt the Merton framework's assumption that households have a CRRA utility function, our analysis of the optimal portfolio choice reveals that the AEP effect breaks the homogeneity of the value function. The value function can only be homogeneous under certain conditions. We summarize this finding as follows:

Proposition 5 (**Necessary Condition for Homogeneity**). *A necessary condition for the value function to be a homogeneous function of income is that the AEP effect must be zero:*

$$\Omega^*(a_t) = 0. \tag{60}$$

Proof. See Appendix F.1. □

This implies that an analytical solution exists only for households at the critical point $\eta = 1$, or at the upper bound of wealth under the LSS. For these households unaffected by the AEP, the problem degenerates into the Merton framework for representative investors.

Furthermore, this necessary condition prevents us from using mathematical methods to handle partial derivatives with respect to income in order to obtain an analytical solution, increasing the difficulty of solving the household problem.

5 Visualization of the Theoretical Results

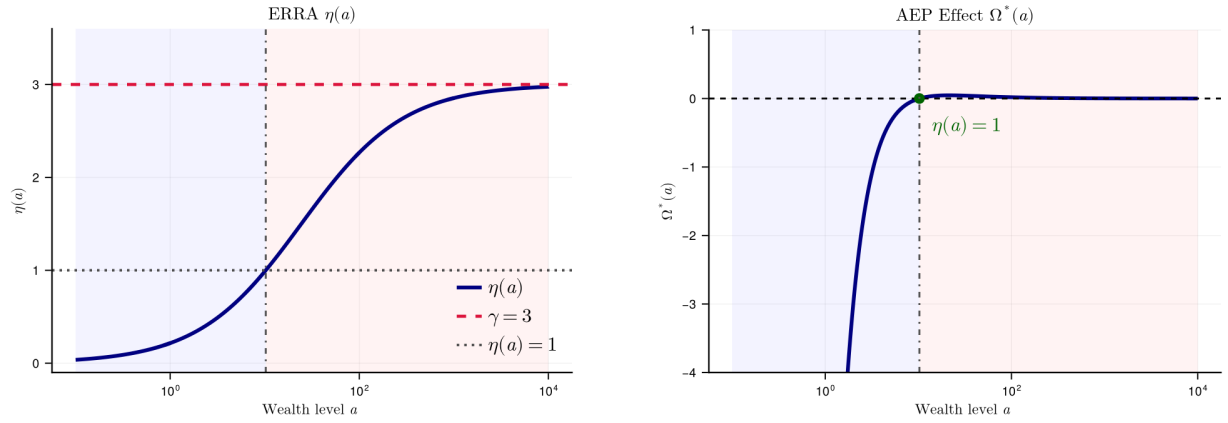
To visualize the core economic mechanisms, we adopt a parsimonious functional form for the household consumption rule $c(a)$, that is consistent with the theoretical results in Section 4. The function is defined as follows:

$$c(a) = (c_0^p + (\bar{k}_L \cdot a)^p)^{1/p}. \tag{61}$$

It ensures that consumption is sustained by a base level c_0 at low wealth and becomes asymptotically proportional to wealth, $c(a) \approx \bar{k}_L \cdot a$, at infinite wealth. This is consistent with Theorem 1. This formulation satisfies Assumption 3, which describes a monotonic elasticity of consumption with respect to wealth $\epsilon_{c,a}(a)$. As a result, the effective relative risk aversion, $\eta(a) = \gamma \cdot \epsilon_{c,a}(a)$, adheres to the boundary conditions established in Propositions 1 and 2. We isolate income risk to clearly present the main theoretical results. Since the function does not include aggregate income risk, we omit the underdiversification results from Section 4.3.

The model parameters are set as follows: the decay rate p is 0.8; the CRRA coefficient γ is 3; the discount rate ρ is 0.05; the expected return for risky assets μ_s is 8%; the risk-free rate r_f is 3%. The mean income y_0 is normalized to 1.0. We set asset volatilities $\sigma_s = 0.30$ for risky assets and $\sigma_y = 0$ for income. For households at the borrowing constraint, their consumption c_0 equals their income.

5.1 AEP and its Implications



(a) (b)

Figure 1: Effective Relative Risk Aversion and the AEP Effect

Figure 1a reveals the basis of the AEP mechanism: the functional relationship between the effective relative risk aversion $\eta(a_t)$ and household wealth. As Proposition 1 establishes, as households approach the borrowing constraint, η_t converges to zero. This reflects the economic intuition that, when consumption is nearly entirely determined by income, house-

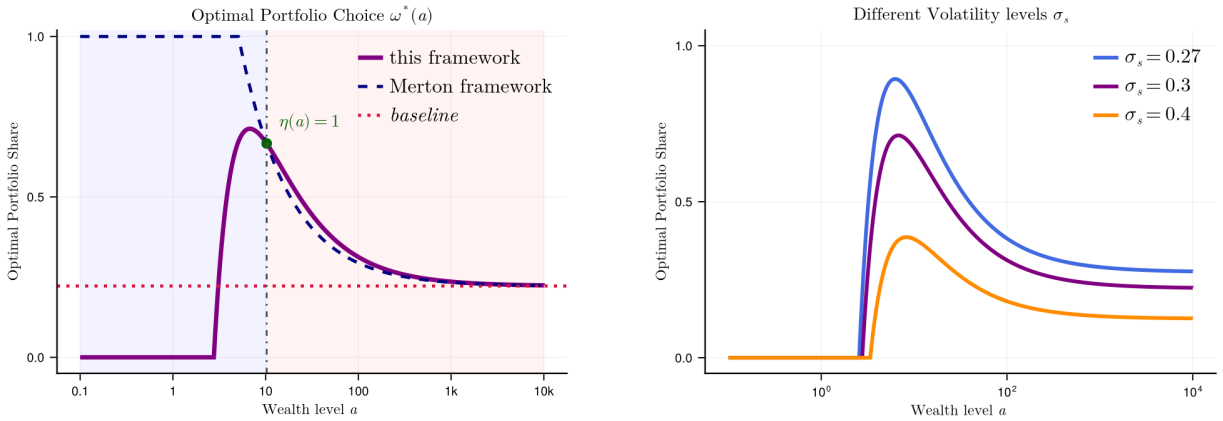
holds' marginal utility out of wealth is nearly insensitive to changes in financial wealth. Conversely, Proposition 2 shows that as wealth approaches infinity, η_t converges to the CRRA coefficient γ. The smooth transition of η_t across the critical value of one is the engine of our entire framework. It determines the sign, magnitude, and non-monotonicity of the AEP effect. This transition fosters heterogeneous preferences for common financial risk at different wealth levels: low-wealth households (with $\eta_t < 1$) are averse to this risk, while middle- and high-wealth households (with $\eta_t > 1$) receive net welfare compensation from taking on the risk.

Figure 1b visualizes the core meaning of this mechanism—the AEP effect. As shown, Ω_t^* drops sharply to negative infinity as wealth approaches zero. This corresponds to the "overwhelming negative force" we identified in the theoretical section, providing a strong endogenous explanation for the "limited participation puzzle." As wealth grows, the AEP effect turns positive at $\eta_t > 1$, peaks, and then gradually converges back to zero from above, driven by the scaling factor $1/a$. This complete and non-monotonous path is the key to a unifying the explanation of the next set of empirical phenomena.

5.2 Optimal portfolio choice and Limited Participation

Figure 2a illustrates the ultimate impact of the AEP mechanism on the optimal portfolio share ω^* and compares it with the Merton framework in the absence of aggregate income risk. At low wealth levels, the strong negative AEP effect we observed in Figure 1b completely overwhelms speculative demand entirely, forcing the optimal portfolio share ω^* back to the short-selling constraint. This provides an endogenous theoretical basis for limited participation, which the Merton model struggles to explain. Once wealth exceeds the critical point $\eta_t = 1$, the AEP effect turns positive, providing an additional, modest incentive to hold the systematic risk asset. This force drives the portfolio share ω^* up, even exceeding the Merton framework and perfectly replicating the "hump-shaped" lifecycle profile observed in the empirical literature. Ultimately, as predicted in our Corollary 2, the optimal portfolio share converges back to the classic Merton ratio as the AEP effect disappears at the highest wealth level.

Figure 2b further explores the interaction between our mechanism and the financial mar-



(a) (b)

Figure 2: Optimal portfolio choice

ket volatility σ_s. As expected, higher volatility reduces the demand for the systematic asset across all wealth levels, which is a standard outcome. The figure reveals a deeper prediction for our framework: higher volatility exacerbates limited participation driven by the AEP. The wealth threshold required for market entry increases with volatility. The underlying economic logic is that higher volatility reduces the Sharpe ratio of the systematic asset, making the negative AEP costs in Figure 1b more unaffordable for low-wealth households.

6 Conclusion

This paper finds that households' endogenous cost for bearing common risk is determined by its position in the wealth distribution. This AEP mechanism provides a unified explanation for several empirical phenomena. Common financial risk, transmitted through the AEP, squeezes low-wealth households out of systemic risk markets, endogenously explaining limited participation and underdiversification. Simultaneously, it pushes the portfolios of wealthier households toward the efficient frontier.

Our analysis focuses on the static determinants of optimal portfolio choice. The framework does not model portfolio inertia (the tendency to let allocations drift with returns) or return extrapolation. The theoretical deductions are based on a partial equilibrium analysis, in which households employ simplified forecasting rules to cope with a complex dynamic

environment.

This framework provides a foundation for future research agenda centered on heterogeneous investors. The AEP mechanism now offers a tractable channel to assess policy responses. Furthermore, this mechanism can now be integrated with other key themes in heterogeneous investors, providing a foundation to re-evaluate how private information, heterogeneous beliefs, asymmetric information, or market liquidity interact with households' endogenous willingness to bear common risk.

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Appendix

A Regularity and Well-posedness

To ensure the existence, uniqueness, and regularity of a classical solution to the resulting master equation, we adopt the standard assumptions from the MFG literature (Lasry & Lions, 2007; Cardaliaguet et al., 2019).

Assumption 4 (Regularity and Well-posedness of the MFG System). We assume:

1. Coercivity Condition: The coefficients of volatility is bounded constant ($\sigma > 0$) such that the Hamiltonian satisfies the coercivity condition.

2. Smoothness and Continuity: Hamiltonians are sufficiently smooth and globally Lipschitz continuous.

3. Monotonicity and Continuity: The PLM and the terminal condition is globally Lipschitz continuous and monotone.

4. Regularity Condition (Cardaliaguet et al., 2019): The coupled system satisfies the first-order regularity condition and the second-order regularity condition.

5. Transversality Condition (Merton, 1969): $\lim_{T\to\infty} \mathbb{E}_t \left[e^{-\rho(T-t)} v_t\right] = 0$.

The foundational theory is developed on a compact state space like the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ to avoid technicalities related to boundary conditions or conditions at infinity. We posit that the core economic mechanisms identified in this paper are robust to the specification of compact state space. For boundary conditions, both of the borrowing constraint $a \geq 0$ and the consumption constraint $y \geq c > 0$ implies that the state space have a compact support. For conditions at infinity, the standard transversality condition is satisfied. By the Carathéodory extension theorem (Amann & Escher, 2009), the distribution is an equivalence class of Borel probability measures. Then, we have the value function $v(a, y, K, t) : [0, \infty) \times (0, \infty) \times \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and that state space is compact. It implies that the short-selling constraint holds, i.e., $\omega \in [0, \infty)$. A rigorous mathematical treatment of boundary conditions and conditions at infinity is beyond the scope of this paper, which focuses on the characterization of the model's internal economic dynamics.

B

B.1 Proof of Proposition 1

Proof. Following Carroll and Kimball (1996), we assume the consumer is maximizing the time-additive present discounted value of utility from consumption and have the standard conditions $v_a > 0, v_{aa} < 0$ and $\eta_t > 0$.

Let $I \equiv (0, \infty]$ be an interval containing the point 0. Let's define a function $\bar{\eta}_t \equiv \frac{a_t}{c_t^*} \cdot M \cdot \gamma$. Stated in the proposition, the assumptions implies that $\gamma < \infty$, and $\frac{\partial c}{\partial a} \equiv \frac{\partial c_t^*}{\partial a_t}$, is bounded, i.e., there exists some $M < \infty$ such that $\frac{\partial c}{\partial a} \leq M$ as $a_t \to 0^+$. Consider a household with a positive income, $y_t > 0$. As $a_t \to 0^+$, households' consumption does not fall to zero. It is sustained by income. The optimal consumption c_t^* converges to a strictly positive value, $c^*(0, y_t) > 0$. The limit of this upper-bound function is:

$$\lim_{a_t \to 0^+} \bar{\eta}_t = \left(\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} \right) \cdot M \cdot \gamma = \frac{0}{c^*(0, y_t)} \cdot M \cdot \gamma = 0. \tag{62}$$

By the Squeeze Theorem, $\forall a \in I$, we have

$$0 \leq \eta_t \leq \bar{\eta}_t \qquad \text{and} \qquad \lim_{a_t \to 0^+} 0 = \lim_{a_t \to 0^+} \bar{\eta}_t = 0. \tag{63}$$

Then

$$\lim_{a_t \to 0^+} \eta_t = 0. \tag{64}$$

□

B.2 Proof of Corollary 1

Proof. (i) The AEP effect is given by the equation (34):

$$\Omega_t^*(a_t) = \frac{1}{a_t \sigma_s} \frac{\eta_t - 1}{\eta_t} = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t} \right). \tag{65}$$

From Proposition 1, we have that as $a_t \to 0^+$, $\eta_t \to 0^+$.

Then

$$\lim_{a_t \to 0^+} \Omega^*(a_t) = \left(\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s} \right) \cdot \left(\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta_t} \right) \right) = (+\infty) \cdot (-\infty) = -\infty. \tag{66}$$

(ii) The optimal portfolio choice is given by the equation (33):

$$\omega_t^* = \frac{1}{\eta_t \sigma_s}\Big[\underbrace{\frac{\mu_s - r_f}{\sigma_s}}_{\text{speculative demand}} + \underbrace{\rho_{sy}\sigma_y y_t \left(\frac{\partial v}{\partial a}\right)^{-1} \frac{\partial^2 v}{\partial a \partial y}}_{\text{wage-offsetting effect}} + \underbrace{\frac{1}{a_t}(\eta_t - 1)}_{\substack{\text{AEP Effect} \\ \text{(New Term)}}}\Big] \tag{67}$$

We examine the order of magnitude of each motive as $a_t \to 0^+$. From Proposition 1, we have $\eta_t = (a_t/c_t^*) \cdot (\partial c_t^*/\partial a_t) \cdot \gamma$. Near the borrowing constraint, c_t^* and $\partial c_t^*/\partial a_t$ are both bounded, meaning η_t is on the order of a_t. Let $n = 1/a_t$. We denote $1/\eta \sim O(n)$ as $a_t \to 0^+$.

All effects are scaled by $1/(\eta_t \sigma_s)$, so we only need to consider the order in the brackets. The bracketed term of the speculative demand is a constant. Similarly, the bracketed term of the wage offsetting effect is also a constant. This is because it is unrealistic for household income to be infinite near the borrowing constraint. Since $1/\eta \sim O(n)$, these two motives diverge at a rate of $O(n)$. The bracketed term of the AEP effect diverges at a rate of $O(n)$. Hence, the AEP effect diverges at a rate of $O(n^2)$.

The limit of the unconstrained optimal portfolio share, $\omega_{t,\text{unc}}^*$, is dirven by this dominant term:

$$\lim_{a_t \to 0^+} \omega_{t,\text{unc}}^* = \lim_{a_t \to 0^+} \left[O(n) - O(n^2)\right] = -\infty. \tag{68}$$

If households face short-selling restrictions, which can be modeled as $\omega_t^* = \max(0, \omega_{t,\text{unc}}^*)$. Then

$$\lim_{a_t \to 0^+} \omega_t^* = \max(0, \omega_{t,\text{unc}}^*) = 0. \tag{69}$$

□

C

C.1 Proof of Theorem 1

Proof. At the LSS, the economy converges allowing the optimal policy function to be invariant. The value function is given by

$$\rho v(a, y, K, t) = \sup_{c,\omega} \{u(c) + \mathcal{L}[v_t] + \mathbb{E}_t \mathcal{A}[v_t] + CRP_t\} \tag{70}$$

It allows us to determine the asymptotic behavior of the optiaml rule as household wealth abecomes arbitrarily large. To analyze the behavior at infinity, we employ a scaling technique. This method allows us to investigate how the solution th steady-state HJB equation behaves asymptotically.

We introduce a scaling parameter $\xi > 0$ and define a value function v_ξ which normalizes the original value function v by the scale. The transformation applies only to the individual-specific state variables (a, y), not the aggregate state K.

Definition 9. For any $\xi > 0$, the scaled value function $v_\xi : [0, \infty) \times (0, \infty) \times \mathbb{R} \times [0, T] \to \mathbb{R}$ is defined as:

$$v_\xi(a, y, K, t) := \frac{1}{\xi^{1-\gamma}} v(\xi a, y, K, t). \tag{71}$$

Lemma 2. *The scaled value function $v_\xi(a, y, K, t)$ satisfies a BSHJB equation where the coefficients of the labor income process is scaled down by powers of ξ.*

Proof of Lemma 2. For brevity, we denote partial derivatives by subscripts (e.g., $v_a \equiv \frac{\partial v}{\partial a}$). We derive the dynamics of v_ξ by applying the chain rule to its definition. The partial derivatives of v_ξ relate to those of v as follows:

$$\begin{aligned}
\frac{\partial v_\xi}{\partial a} &= \xi^\gamma v_a(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial a^2} &= \xi^{\gamma+1} v_{aa}(\xi a, \dots), \\
\frac{\partial v_\xi}{\partial y} &= \xi^{\gamma-1} v_y(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial y^2} &= \xi^{\gamma-1} v_{yy}(\xi a, \dots), \\
\frac{\partial^2 v_\xi}{\partial a \partial y} &= \xi^\gamma v_{ay}(\xi a, \dots), & \frac{\partial v_\xi}{\partial K} &= \xi^{\gamma-1} v_K(\xi a, \dots).
\end{aligned} \tag{72}$$

When the household's wealth a becomes infinite, its decision problem should converge to a control problem involving only wealth a. At infinity, the marginal effects of labor income and macroeconomic aggregation become negligible. As $\xi \to \infty$, the scaled value function $v_\xi(a, y, K, t)$ converges to a limiting function $v_\infty(a, t)$. It depends only on wealth and time, as it describes lifetime welfare that is no longer affected by income process or the PLM. This convergence is a standard result in such problems and can often be rigorously proven by the stability and consistency of viscosity solutions (L. C. Evans, 2022). If the value functions themselves converge, then their partial derivatives with respect to variables that become irrelevant in the limit should also converge to zero. Then, we have

$$\begin{aligned}
\lim_{\xi\to\infty} (v_\xi)_y(a, y, K, t) &= \frac{\partial v_\infty(a, t)}{\partial y} = 0, & \lim_{\xi\to\infty} (v_\xi)_K(a, y, K, t) &= \frac{\partial v_\infty(a, t)}{\partial K} = 0, \\
\lim_{\xi\to\infty} (v_\xi)_{ay}(a, y, K, t) &= \frac{\partial^2 v_\infty(a, t)}{\partial a \partial y} = 0, & \lim_{\xi\to\infty} (v_\xi)_{yy}(a, y, K, t) &= \frac{\partial^2 v_\infty(a, t)}{\partial y^2} = 0.
\end{aligned} \tag{73}$$

Let $c_\xi = c/\xi$ and $\omega_\xi = \omega$ be the scaled controls. Substituting these into the original BSHJB equation, evaluating at $(\xi a, y, K, t)$, and multiplying the entire equation by $\xi^{\gamma-1}$

yields the BSHJB equation for $v_\xi(a, y, K, t)$. The instantaneous utility transforms as:

$$\xi^{\gamma-1}u(c) = u(c/\xi) = u(c_\xi). \tag{74}$$

And other terms become:

$$\begin{aligned}&\xi^{\gamma-1}\Big[[\omega_t(\mu_s - r_f) + r_f](\xi a) + (y) - (\xi c_\xi) - \omega\sigma_s\Big]v_a(\xi a, \dots)\\ =&\Big[[\omega_t(\mu_s - r_f) + r_f]a + y/\xi - c - \omega\sigma_s\Big]v_{\xi,a}(a, \dots)\end{aligned} \tag{75}$$

and

$$\xi^{\gamma-1}\left[\frac{1}{2}(\omega(\xi a)\sigma_s)^2 - \omega(\xi a)\sigma_s\right]v_{aa}(\xi a, \dots) = \left[\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s\right]v_{\xi,aa}(a, \dots). \tag{76}$$

□

As $\xi \to \infty$, the BSHJB equation for v_ξ converges to the HJB equation at the LSS:

$$\rho v_L = \sup_{c,\omega}\left\{u(c) + \Big[(\omega(\mu_s - r_f) + r_f)a - c - \omega\sigma_s\Big]v_L' + \left[\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s\right]v_L''\right\}. \tag{77}$$

The solution to this HJB equation is $v_L = B\frac{a^{1-\gamma}}{1-\gamma}$, where B is a constant. To derive $\bar{k}_L$, we substitute $v_L = B\frac{a^{1-\gamma}}{1-\gamma}$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ into the HJB equation (77).

The first and second derivatives of v_L are:

$$\begin{aligned}v_L' &= Ba^{-\gamma}\\ v_L'' &= -\gamma Ba^{-\gamma-1}.\end{aligned} \tag{78}$$

The envelope condition with respect to c is $u'(c) - v_L' = 0$, which gives:

$$c^{-\gamma} = Ba^{-\gamma} \implies c^* = B^{-1/\gamma}a. \tag{79}$$

Let $\bar{k}_L = B^{-1/\gamma}$, so $c^* = \bar{k}_L a$.

The FOC with respect to ω is:

$$[(\mu_s - r_f)a - \sigma_s]v_L' + (\omega\sigma_s^2a^2 - \sigma_s a)v_L'' = 0. \tag{80}$$

Substitute the derivatives v_L' and v_L'':

$$[(\mu_s - r_f)a - \sigma_s](Ba^{-\gamma}) + (\omega\sigma_s^2a^2 - \sigma_s a)(-\gamma Ba^{-\gamma-1}) = 0. \tag{81}$$

Dividing by $Ba^{1-\gamma}$ (since $B \neq 0$ and $a \neq 0$):

$$\omega^*(a) = \frac{\mu_s - r_f}{\gamma \sigma_s^2} + \frac{\gamma - 1}{\gamma a \sigma_s}. \tag{82}$$

We substitute the expressions for c^*, ω^*, v_L, v_L', and v_L'' into the HJB equation (77).

$$\begin{aligned} \rho B \frac{a^{1-\gamma}}{1-\gamma} =& \frac{(\bar{k}_L a)^{1-\gamma}}{1-\gamma} + \left((r_f + \omega^*(\mu_s - r_f))a - \bar{k}_L a - \omega^* \sigma_s\right) B a^{-\gamma} \\ &+ \left(\frac{1}{2}(\omega^* \sigma_s a)^2 - \omega^* a \sigma_s\right)(-\gamma B a^{-\gamma-1}). \end{aligned} \tag{83}$$

Divide the entire equation by $Ba^{1-\gamma}$:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma + \frac{\omega^* \sigma_s}{a}(\gamma - 1) \\ =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma, \qquad \text{as } a \to \infty. \end{aligned} \tag{84}$$

Since $\bar{k}_L = B^{-1/\gamma}$, we have $B = \bar{k}_L^{-\gamma}$. Substitute this into the first term:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{\bar{k}_L^{-\gamma}(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma \\ =& \frac{\bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \bar{k}_L - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma. \end{aligned} \tag{85}$$

Combine terms involving $\bar{k}_L$:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k}_L - \bar{k}_L(1-\gamma)}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma \\ =& \frac{\gamma \bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma. \end{aligned} \tag{86}$$

Multiply by $(1-\gamma)$:

$$\rho = \gamma \bar{k}_L + (1-\gamma)\left[r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma\right]. \tag{87}$$

Now, substitute $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2}$ (as $a \to \infty$) into the equation:

$$\begin{aligned}\rho =& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\left(\frac{\mu_s - r_f}{\gamma\sigma_s^2}\right)^2 \sigma_s^2\gamma\right] \\ =& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2}\right] \\ =& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right].\end{aligned} \tag{88}$$

Then

$$\bar{k}_L = \frac{1}{\gamma}\left(\rho - (1-\gamma)\left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right)\right). \tag{89}$$

□

C.2 Proof of Proposition 2

Proof. From Theorem 1, the marginal propensity to consume converges to the average propensity to consume by the L'Hôpital's rule:

$$\lim_{a\to\infty} \frac{c^*(a)}{a} = \lim_{a\to\infty} \frac{\partial c^*(a)}{\partial a} = \bar{k}_L. \tag{90}$$

The elasticity of consumption with respect to wealth converges to unity:

$$\lim_{a\to\infty} \frac{a}{c^*(a)}\frac{\partial c^*(a)}{\partial a} = (\bar{k}_L)^{-1}\bar{k}_L = 1. \tag{91}$$

Then

$$\lim_{a\to\infty} \eta_t = \lim_{a\to\infty} \frac{a}{c^*(a)}\frac{\partial c^*(a)}{\partial a}\cdot\gamma = \gamma. \tag{92}$$

□

C.3 Proof of Corollary 2

Proof. (i) By definition, we have

$$\lim_{a\to\infty} \Omega^*(a_t) = \lim_{a\to\infty} \frac{1}{a_t\sigma_s}\left(1 - \frac{1}{\eta_t}\right) = 0. \tag{93}$$

(ii) From Theorem 1 and Proposition 2, we know that both of the wage-offsetting effect and the AEP effect vanish under the scaling $\xi \to \infty$.

Then

$$\lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2 \gamma}. \tag{94}$$

□

D

D.1 Proof of Necessary and Sufficient Condition

Proof. From the definition of $\eta(a) = -a\frac{v_{aa}}{v_a}$ and its derivative with respect to wealth a, the condition for $\frac{d\eta}{da} > 0$ is that the numerator must be positive:

$$\frac{d\eta}{da} = \frac{a v_{aa}^2 - v_a v_{aa} - a v_a v_{aaa}}{(v_a)^2} > 0. \tag{95}$$

We introduce the sufficient condition:

$$v_a v_{aaa} \geq k v_{aa}^2, \tag{96}$$

where $k \equiv u'''u'/u''^2$.

Since $a > 0$, multiplying the condition by $-a$ reverses the inequality sign:

$$-a v_a v_{aaa} \leq -ak v_{aa}^2. \tag{97}$$

Substituting this into the condition (95), we obtain an upper bound for the derivative $\frac{d\eta}{da}$:

$$\frac{a v_{aa}^2 - v_a v_{aa} - ak v_{aa}^2}{(v_a)^2} \geq \frac{a v_{aa}^2 - v_a v_{aa} - a v_a v_{aaa}}{(v_a)^2} > 0. \tag{98}$$

This upper bound must be positive:

$$a v_{aa}^2 - v_a v_{aa} - ak v_{aa}^2 > 0. \tag{99}$$

We factor the expression above by first grouping terms with a and then factoring out v_{aa}:

$$v_{aa}\left(a v_{aa} - v_a - ak v_{aa}\right) > 0. \tag{100}$$

$$v_{aa}\left(a(1-k) v_{aa} - v_a\right) > 0. \tag{101}$$

Now, we factor out v_a and use the definition $\eta = -a\frac{v_{aa}}{v_a}$:

$$v_{aa}v_a\left(\frac{a(1-k)v_{aa}}{v_a} - 1\right) > 0 \tag{102}$$

$$v_{aa}v_a\left((k-1)\eta - 1\right) > 0. \tag{103}$$

The remaining term in the parentheses must be negative:

$$(k-1)\eta - 1 < 0 \tag{104}$$

$$(k-1)\eta < 1 \tag{105}$$

$$\eta < \frac{1}{k-1}. \tag{106}$$

We have

$$\begin{aligned} k =& \frac{\gamma(\gamma+1)c^{-2\gamma-2}}{(-\gamma c^{-\gamma-1})^2} \\ =& 1 + \frac{1}{\gamma}. \end{aligned} \tag{107}$$

From Proposition 2, then

$$\eta < \gamma, \qquad \forall a < \infty. \tag{108}$$

□

D.2 Proof of Lemma 1

Proof. By the Bolzano's intermediate value theorem, there exists a wealth interval $a \in (0, \infty)$ and $\eta : a \to \mathbb{R}$ is continuous. We have that $\eta(a)$ is an interval. With Proposition 1 and 2, $\eta(a)$ is the interval $[0, \gamma]$. Hence, $1 \in \eta(a)$ for $0 < 1 < \gamma$. Let $\eta(a^*) = 1$, where $a^* \in a$. According to Assumption 3, $\eta(a)$ is monotonic. Then, $\eta(a) > 1$ for $a \in (a^*, \infty)$. □

E

E.1 Proof of Proposition 4

Proof. (i) We regard the transition from portfolio N to portfolio F as adding a "perturbation term" $\mathcal{D}(v)$ to the HJB equations of portfolio N by the perturbation method. For equivalent risk-return characteristics, we have that $\sigma_N = \sigma_F = \sigma_s$ and $\mu_N = \mu_F = \mu_s$.

The household invests in the portfolio N, then its HJB equation is:

$$\begin{aligned}\rho v^N = &\max_{c,\omega}\Big\{u(c_t) + \{[\omega_t(\mu_N - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + y_t(\mu_y + \tilde{\mu}_{n_i})\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \tilde{\sigma}_N)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)y_t^2\frac{\partial^2 v}{\partial y^2} + IEP_t\Big\}.\end{aligned} \tag{109}$$

The household invests in the portfolio F, then its HJB equation is:

$$\begin{aligned}\rho v^F = &\max_{c,\omega}\Big\{u(c_t) + \{[\omega_t(\mu_F - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i})y_t\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \sigma_s)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)y_t^2\frac{\partial^2 v}{\partial y^2} + \omega_t a_t \sigma_s \Gamma_t v_a + IEP_t\Big\}.\end{aligned} \tag{110}$$

Subtracting the two equations gives the perturbation term:

$$\mathcal{D}(v) = \omega_t a_t \sigma_s \Gamma_t v_a \tag{111}$$

The HJB equation for portfolio F can be written as:

$$\rho v^F = \text{HJB}_N(v^F) + \mathcal{D}(v^F), \tag{112}$$

where HJB means replacing v^N in the HJB equation of portfolio N with v^F.

The FOC of portfolio F can be viewed as the FOC of portfolio N plus the partial derivative of the perturbation term with respect to ω:

$$\text{FOC}_F = \text{FOC}_N(v^F) + \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = 0, \tag{113}$$

where

$$\text{FOC}_N(v^F) = (\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F. \tag{114}$$

The partial derivative of the perturbation term with respect to ω is:

$$\frac{\partial}{\partial \omega}\mathcal{D}(v^F) = a_t \sigma_s \Gamma_t v_a \tag{115}$$

Now, the FOC of portfolio F can be rewritten as:

$$(\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F + \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = 0 \tag{116}$$

With $\eta_t = -a_t \frac{v_a}{v_{aa}}$, the FOC for portfolio N and F with respect to ω are:

$$\omega_N^* = \frac{1}{\eta_t^N \sigma_s} \frac{(\mu_s - r_f)}{\sigma_s} \tag{117}$$

$$\omega_F^* = \frac{1}{\eta_t^F \sigma_s} \left[\frac{(\mu_s - r_f)}{\sigma_s} + \Gamma_t \right]. \tag{118}$$

Let $f(\omega, v) := (\mu_s - r_f) a_t v_a + \omega (a_t \sigma_s)^2 v_{aa}$. The optimal policies satisfy:

$$f(\omega_F^*, v^F) \approx -\frac{\Gamma_t}{\eta_t \sigma_s} > 0 \approx f(\omega_N^*, v^N). \tag{119}$$

The household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Then

$$\omega_F^* < \omega_N^*. \tag{120}$$

(ii) We want to find the condition under which their welfare is equivalent, i.e., $v^F(a_t, y_t) = v^N(a_t, y_t)$, which implies $v_a^F = v_a^N = v_a$, $v_{aa}^F = v_{aa}^N = v_{aa}$, and $v_{ay}^F = v_{ay}^N = v_{ay}$. For welfare equivalence, we need the complete HJB equations to be equal. This requires:

$$\max_{\omega} \mathcal{H}_F(\omega) - \max_{\omega} \mathcal{H}_N(\omega) = 0, \tag{121}$$

where the Hamiltonians are defined as:

$$\mathcal{H}_F \equiv \max_{\omega} \left\{ \omega_t (\mu_F - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2} (\omega_t a_t \sigma_F)^2 \frac{\partial^2 v}{\partial a^2} + \omega_t a_t \sigma_s \Gamma_t v_a \right\} \tag{122}$$

and

$$\mathcal{H}_N \equiv \max_{\omega} \left\{ \omega_t (\mu_N - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2} (\omega_t a_t \sigma_N)^2 \frac{\partial^2 v}{\partial a^2} \right\}. \tag{123}$$

Taking the optimal portfolio shares ω_N^* and ω_F^* into these Hamiltonians, we have:

$$\sup_{\omega} \mathcal{H}_N = \frac{1}{2} \mathcal{S}_N^2 \frac{v_a^2}{-v_{aa}} \tag{124}$$

$$\sup_{\omega} \mathcal{H}_F = \frac{1}{2} \left(\mathcal{S}_F + \Gamma_t \right)^2 \frac{v_a^2}{-v_{aa}}, \tag{125}$$

where $\mathcal{S}$ is portfolio Sharpe ratio $\mathcal{S} = \frac{\mu - r_f}{\sigma}$.

Eliminate identical terms of both sides, we have:

$$\mathcal{S}_N = \mathcal{S}_F + \Gamma_t. \tag{126}$$

Then

$$\mathcal{S}_F - \mathcal{S}_N = -\Gamma_t > 0. \tag{127}$$

□

E.2 Proof of Corollary 3

Proof. (i) The portfolio P's return characteristics are expected excess return μ_P and volatility:

$$\sigma_P = \sqrt{\beta_P^2 \sigma_s^2 + \tilde{\sigma}_P^2}. \tag{128}$$

Since $\sigma_P = \sigma_s$ and $\tilde{\sigma}_P > 0$, we have $\beta_P < 1$.

The household invests in the portfolio P, then its HJB equation is:

$$\begin{aligned} \rho v^P = \max_{c,\omega_P} \Big\{ & u(c) + \left[(\omega_P(\mu_P - r_f) + r_f)\, a_t + y_t - c_t \right] \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} \\ & + \frac{1}{2}(\omega_P a_t \sigma_P)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_P a_t \sigma_s \beta_P \Gamma_t v_a + IEP_t \Big\}. \end{aligned} \tag{129}$$

Following the proof E.1, we can similarly obtain:

$$\omega_P^* < \omega_N^*. \tag{130}$$

For portfolio P, the complete perturbation term is:

$$\mathcal{D}_P(v) = (1 - \beta_P)\omega_P a_t \sigma_s \Gamma_t v_a. \tag{131}$$

The partial derivative of the perturbation term with respect to ω_P is:

$$\frac{\partial \mathcal{D}_P(v)}{\partial \omega_P} = (1 - \beta_P) a_t \sigma_s \Gamma_t v_a. \tag{132}$$

The FOC for portfolio P with respect to ω_P is:

$$(\mu_P - r_f) a_t v_a^P + \omega_P^* a_t^2 \sigma_P^2 v_{aa}^P + (1 - \beta_P) a_t \sigma_s \Gamma_t v_a = 0. \tag{133}$$

With $\eta_P = -a_t \frac{v_a}{v_{aa}}$, the FOC for portfolio P with respect to ω is:

$$\omega_P^* = \frac{1}{\eta_P \sigma_s} \left[\frac{(\mu_P - r_f)}{\sigma_s} + (1 - \beta_P)\Gamma_t \right]. \tag{134}$$

Let $f(\omega, v) := (\mu_s - r_f)a_t v_a + \omega(a_t\sigma_s)^2 v_{aa} + a_t\sigma_s\Gamma_t v_a$. The optimal policies satisfy:

$$f(\omega_P^*, v^P) \approx -\frac{\beta_P\Gamma_t}{\eta_P\sigma_s}(a_t\sigma_s)^2 v_{aa} < 0 \approx f(\omega_F^*, v^F). \tag{135}$$

The household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Then

$$\omega_N^* > \omega_P^* > \omega_F^*. \tag{136}$$

(ii) We want to find the condition under which the welfare of the household is equivalent, i.e., $v^P(a_t, y_t) = v^F(a_t, y_t)$, which implies $v_a^P = v_a^F = v_a$, $v_{aa}^P = v_{aa}^F = v_{aa}$, and $v_{ay}^P = v_{ay}^F = v_{ay}$. For welfare equivalence, we need the complete HJB equations to be equal. This requires:

$$\max_\omega \mathcal{H}_P(\omega, v^P) = \max_\omega \mathcal{H}_F(\omega, v^F), \tag{137}$$

where $\mathcal{H}_P$ and $\mathcal{H}_F$ are the Hamiltonians of portfolio P and F, respectively:

$$\mathcal{H}_P \equiv \max_\omega \left\{\omega_t(\mu_P - r_f)a_t\frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t\sigma_P)^2\frac{\partial^2 v}{\partial a^2} + \omega_t a_t\sigma_s\beta_P\Gamma_t v_a\right\} \tag{138}$$

and

$$\mathcal{H}_F \equiv \max_\omega \left\{\omega_t(\mu_F - r_f)a_t\frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t\sigma_F)^2\frac{\partial^2 v}{\partial a^2} + \omega_t a_t\sigma_s\Gamma_t v_a\right\}. \tag{139}$$

Taking the optimal portfolio shares ω_P^* and ω_F^* into these Hamiltonians, we have:

$$\sup_\omega \mathcal{H}_P = \frac{1}{2}(\mathcal{S}_P + \beta_P\Gamma_t)^2\frac{v_a^2}{-v_{aa}} \tag{140}$$

$$\sup_\omega \mathcal{H}_F = \frac{1}{2}(\mathcal{S}_F + \Gamma_t)^2\frac{v_a^2}{-v_{aa}} \tag{141}$$

Eliminate identical terms of both sides, we have:

$$\mathcal{S}_P + \beta_P\Gamma_t = \mathcal{S}_F + \Gamma_t. \tag{142}$$

Then

$$\mathcal{S}_F - \mathcal{S}_P = (\beta_P - 1)\Gamma_t. \tag{143}$$

□

F Proof of Analysis under Homogeneity

F.1 Proof of Proposition 5

Proof. We let the value function $v(a, y)$ be homogeneous with respect to the household's wealth and income at the LSS:

$$v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x), \tag{144}$$

where $h(x)$ is an unknown function that is only related to the wealth-income ratio $x_t \equiv a_t/y_t$.

Lemma 3. *If the value function is homogeneous, the effective relative risk aversion $\eta(a_t)$ is equal to a distorted risk aversion function $\tilde{\eta}(x_t)$ determined only by the wealth-income ratio x:*

$$\eta(a_t) = \tilde{\eta}(x_t) \equiv -x_t \frac{h''(x)}{h'(x)}. \tag{145}$$

Proof. Our goal is to find an expression for $\tilde{\eta}(x)$ that is directly related to economic decision variables (such as consumption) under the assumption that the value function is homogeneous $v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x)$.

We start from the FOC for optimal household consumption, that is, marginal utility equals marginal value of wealth:

$$u'(c^*) = \frac{\partial v}{\partial a}. \tag{146}$$

For the CRRA utility function, we have $u'(c) = c^{-\gamma}$ and

$$\frac{\partial v}{\partial a} = \frac{\partial}{\partial a}\left(\frac{y^{1-\gamma}}{1-\gamma} h(x)\right) = \frac{y^{1-\gamma}}{1-\gamma} h'(x) \frac{\partial x}{\partial a} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{147}$$

Substituting both into FOC:

$$(c^*)^{-\gamma} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{148}$$

Now, we can solve the expression for $h'(x)$ from it:

$$h'(x) = (1-\gamma)\frac{(c^*)^{-\gamma}}{y^{-\gamma}} = (1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}. \tag{149}$$

The economic meaning of this expression is that the first-order derivative of the function $h(x)$ (reflecting the sensitivity of the value function to changes in the wealth-income ratio) is determined by the optimal consumption-income ratio c^*/y.

To find $h''(x)$, we take the derivative of $h'(x)$. The chain rule is needed here because c^*

itself is a function of a, which is also a function of x and y. To simplify the expression, we define the consumption-income ratio function $C(x) \equiv c^*(x)/y$.

$$h''(x) = \frac{d}{dx}\left[(1-\gamma)C(x)^{-\gamma}\right]. \tag{150}$$

Applying the chain rule:

$$\begin{aligned} h''(x) =& (1-\gamma)\cdot(-\gamma)\cdot C(x)^{-\gamma-1}\cdot\frac{dC(x)}{dx} \\ =& -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1} C'(x), \end{aligned} \tag{151}$$

where $C'(x) = \frac{d(c^*/y)}{dx}$.

We further analyze $C'(x)$:

$$C'(x) = \frac{d(c^*/y)}{d(a/y)} = \frac{d(c^*/y)}{da}\cdot\frac{da}{d(a/y)} = \left(\frac{1}{y}\frac{\partial c^*}{\partial a}\right)\cdot y = \frac{\partial c^*}{\partial a}. \tag{152}$$

It can be seen that $C'(x)$ is the marginal propensity to consume of the household.

Substituting into the above equation, we get the final expression for $h''(x)$:

$$h''(x) = -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}. \tag{153}$$

Now that we have expressions for $h'(x)$ and $h''(x)$, we can substitute them into the definition of $\tilde{\eta}(x)$:

$$\tilde{\eta}(x) \equiv -x\frac{h''(x)}{h'(x)}. \tag{154}$$

Substituting into:

$$\tilde{\eta}(x) = -x\frac{-\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}}{(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}}. \tag{155}$$

After simplification, we get:

$$\begin{aligned} \tilde{\eta}(x) =& x\cdot\gamma\cdot\left(\frac{y}{c^*}\right)\cdot\frac{\partial c^*}{\partial a} \\ =& \frac{a}{y}\cdot\gamma\cdot\frac{y}{c^*}\cdot\frac{\partial c^*}{\partial a}. \\ =& \gamma\cdot\frac{a}{c^*}\cdot\frac{\partial c^*}{\partial a}. \end{aligned} \tag{156}$$

Finally, we get:

$$\tilde{\eta}(x) = \eta(a). \tag{157}$$

□

The optimal portfolio choice (33) can be decomposed into three difference economic motives:

$$\omega_t^* = -\frac{1}{a_t}\frac{v_a}{v_{aa}}\frac{\mu_s - r_f}{\sigma_s^2} + \left(-\frac{v_{ay}}{a_t v_{aa}}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y + \frac{1}{a_t\sigma_s}\left(1 + \frac{v_a}{a_t v_{aa}}\right). \tag{158}$$

By Lemma 3, we have:

$$\text{Speculative Demand} = \frac{1}{\eta(a_t)}\frac{\mu_s - r_f}{\sigma_s^2} = \frac{1}{\tilde{\eta}(x_t)}\frac{\mu_s - r_f}{\sigma_s^2}. \tag{159}$$

We derive the expressions for v_{ay} and v_{aa}:

$$\begin{aligned} v_{ay} \equiv & \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial}{\partial y}\left(\frac{y^{-\gamma}}{1-\gamma}h'(x)\right) \\ = & \frac{-\gamma y^{-(\gamma+1)}}{1-\gamma}h'(x) + \frac{y^{-\gamma}}{1-\gamma}h''(x)\left(-\frac{a}{y^2}\right) \\ = & \frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right) \end{aligned} \tag{160}$$

and

$$v_{aa} \equiv \frac{\partial^2 v}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{y^{-\gamma}}{1-\gamma}h'(x)\right) = \frac{y^{-\gamma}}{1-\gamma}h''(x)\frac{\partial x}{\partial a} = \frac{y^{-(\gamma+1)}}{1-\gamma}h''(x). \tag{161}$$

Substituting these into the wage-offsetting effect expression:

$$\begin{aligned} \text{Wage-offsetting Effect} = & \left(-\frac{\frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right)}{a_t \frac{y^{-(\gamma+1)}}{1-\gamma}h''(x)}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \frac{\gamma h'(x) + x h''(x)}{a_t h''(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \left(\frac{\gamma}{y_t}\frac{h'(x)}{x_t h''(x)} + \frac{1}{y_t}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \left(-\frac{\gamma}{y_t\tilde{\eta}(x)} + \frac{1}{y_t}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \frac{\tilde{\eta}(x) - \gamma}{\tilde{\eta}(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s}. \end{aligned} \tag{162}$$

We can simply use the result from Part (i) of this lemma.

$$\begin{aligned}\text{AEP Effect} =& \frac{1}{a_t\sigma_s}\left(1-\frac{1}{\eta(a_t)}\right)\\ =& \frac{1}{x_t y_t \sigma_s}\frac{\tilde{\eta}(x)-1}{\tilde{\eta}(x)}.\end{aligned} \tag{163}$$

We have

$$\tilde{\omega}^* = \frac{1}{\tilde{\eta}(x)}\frac{\mu_s - r_f}{\sigma_s^2} + \frac{1}{\tilde{\eta}(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s y_t}(\tilde{\eta}(x)-\gamma) + \frac{1}{\tilde{\eta}(x)}\frac{1}{x_t y_t \sigma_s}(\tilde{\eta}(x)-1). \tag{164}$$

A necessary condition for homogeneity is that the optimal control is only related to x. Then, we require

$$\text{AEP Effect} = 0. \tag{165}$$

□