Portfolio Choice for Heterogeneous Investors*

Mingzhe Li†

Latest version
First version: June 17, 2025
This version: January 27, 2026

Abstract

This paper proposes a portfolio choice model that explains a wide variety of investor behaviors in household finance, including limited participation, underdiversification, and hump-shaped risky share over the life cycle. The model introduces an asset-exposure premium (AEP) as the marginal welfare valuation of systematic risk, driven by the intertemporal risk preference to wealth that increases monotonically from zero at the borrowing constraint to the CRRA coefficient at infinite wealth. By evolving from a prohibitive welfare cost near the borrowing constraint to a welfare benefit and eventually vanishing at high wealth, the AEP naturally rationalizes these phenomena. The model posits a fundamentally intuition: households optimize portfolios based on the wealth-dependent welfare valuation of systematic versus idiosyncratic risk of assets.

*I owe a special debt of gratitude to John Y. Campbell for his criticisms and insights on the earliest versions of this work. The author declares no conflicts of interest and received no external funding for this research.

†Shihezi University. Email: mingzheli@stu.shzu.edu.cn

1 Introduction

Many of the investor behaviors in "household finance" (Campbell, 2006) are hard to reconcile with rational optimizing models (Campbell, 2018). These behaviors demonstrate the following empirical phenomena: the vast majority of low-wealth households do not participate in risky asset markets (Vissing-Jørgensen, 2002; Campbell, 2006); and low-wealth households hold severely underdiversified portfolios with highly idiosyncratic risk, while wealthier households hold diversified portfolios (Calvet, Campbell, & Sodini, 2007, 2009; Goetzmann & Kumar, 2008). Over the life cycle, households' risky portfolio share tends to be "hump shaped" (Heaton & Lucas, 2000; Poterba & Samwick, 2001; Benzoni, Collin-Dufresne, & Goldstein, 2007; Ameriks & Zeldes, 2011; Fagereng, Gottlieb, & Guiso, 2017; Gomes & Smirnova, 2021; Parker et al., 2025).

The seminal work of Merton (1969, 1971, 1973) provides a benchmark model of portfolio choice, yet it encounters three limitations in providing an explaination for these phenomena. First, explaining underdiversification requires models that should distinguish between systematic and idiosyncratic risk in household portfolios. The speculative (or myopic) demand fail to do so. Instead, this distinction is driven by aggregate income risk through the wage-offsetting (or labor hedging) effect of income associated with systematic risk. The literature debates the magnitude and robustness of its correlation. In practice, much of the risk associated with labor income is idiosyncratic and therefore unhedgeable (Campbell, 2006). Second, limited participation is hard to reconcile with long-term investor behaviors that, with idiosyncratic income risk, the optimal portfolio share is unambiguously larger for employed investors than for retired investors (Viceira, 2001; Campbell & Viceira, 2002). This also implies that, investors accumulate wealth through human capital, so those with lower wealth should hold a higher risky portfolio share. Third, while generating a hump-shaped portfolio profile requires non-monotonic wealth-driven motives, the speculative demand is assumed to be strictly monotonic for tractability.

As yet, there is no unified economic explanation for these household investment patterns. We lack a framework that distinguishes the welfare implications of systematic versus idiosyncratic risk of risky assets at the household level.

Starting from an intuitive perspective, households act as price-takers who perceive the idiosyncratic risk in their portfolios as orthogonal to the pricing of systematic assets. Hence, fluctuations in the market price of systematic risk only impact the welfare derived from risk exposure, leaving the welfare implications of idiosyncratic risk isolated. This separation allows us to decompose the marginal welfare effects of different risk types. This avoids the need for Ito-Wentzell integrals, which arise when the value function itself evolves stochastically with market prices in which a mathematically complex extension currently under our development.

We model the household's marginal welfare valuation of systematic risk exposure as the asset-exposure premium (AEP), which is fundamentally driven by the intertemporal risk preference relative to wealth, denoted by η_t. We prove that η_t increases monotonically from zero at the borrowing constraint to the CRRA coefficient γ as wealth approaches infinity in which the monotonicity is derived from the sufficient conditions for precautionary savings. This evolution creates a mechanism: for $\eta_t < 1$, households exhibit an aversion to systematic risk, whereas for $\eta_t > 1$, they exhibit a preference for it, effectively seeking systematic exposure to enhance welfare.

Near borrowing constraints, the AEP manifests as a prohibitive welfare cost of systematic risk. Because labor income dominates consumption, households exhibit near-zero wealth sensitivity, driving the AEP to a large negative value that overwhelms both speculative and hedging demands. This forces a rational withdrawal from the market, generating limited participation.

As wealth accumulates, the intertemporal risk preference rises, reflecting a shift from risk-aversion to a net positive welfare effect from systematic exposure. This transition leads middle-to-high wealth households to invest more aggressively than the Merton benchmark. At the wealth apex, where labor income is negligible, the AEP vanishes and the risky share converges back to the Merton (1969)'s result as consumption becomes asymptotically linear to wealth. This non-monotonic transition naturally yields a hump-shaped portfolio profile.

The AEP mechanism, synthesized with the wage-offsetting effect, dictates the household's portfolio risk characteristics through the β exposure ratio, $\Gamma(a_t)$. This ratio aggregates the total impact of systematic risk on both income and assets. We show that the sign of Γ_t

acts as a watershed for portfolio efficiency: households with $\Gamma_t < 0$ (typically the poor) perceive systematic risk as a net welfare threat. To mitigate this, they rationally sacrifice portfolio efficiency, accepting higher idiosyncratic risk and lower Sharpe ratios, which results in underdiversification. In contrast, households with $\Gamma_t > 0$ leverage systematic risk to enhance welfare, incentivizing them to hold diversified portfolios aligned with the efficient frontier.

This model introduces significant non-homogeneity, rendering standard analytical approximations inadequate. We explore advanced Deep Learning methods to tackle the resulting HJB equations, but the inherent stiffness caused by borrowing constraints poses substantial convergence challenges. To illuminate the core economic mechanism, we adopt a parsimonious consumption rule that matches the theoretical properties.

2 Model

We consider the case in which households face only systematic financial risks to illustrate the core mechanism. In section 3.3, we discuss more complex issues that include idiosyncratic financial risks.

2.1 The Environment

There is a continuum of infinitely lived households indexed by $i \in [0, 1]$ with unit mass. Each households i's endowments consist of financial wealth $a_{i,t}$ and labor income $y_{i,t}$ at time t. Each household begins with initial wealth and initial income.

Households' endowments correspond to two priceable markets. These markets are subject to risk source, which is a two-dimensional Wiener process:

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{W}_{1,t} & \mathbf{W}_{2,t} \end{bmatrix}^\top, \tag{1}$$

where $\mathbf{W}_{1,t}$ is the primary risk source and $\mathbf{W}_{2,t}$ is the orthogonal component satisfying $\mathbb{E}_t[d\mathbf{W}_{1,t} d\mathbf{W}_{2,t}] = 0$. Financial assets can be traded instantaneously in a centralized financial market. This market is fully driven by the primary risk source. Labor markets are

decentralized and not instantaneous; they partially respond to the primary risk source. Let ρ_{sy} denote the magnitude of this partial response, where $\rho_{sy} \in [-1, 1]$.

Noise, as risk sources, is categorized into two components: systematic noise, denoted by $\mathbf{W}_t$, and idiosyncratic noise, denoted by $\mathbf{B}_{i,t}$. These are locally uncorrelated, i.e., $\mathbb{E}_t[d\mathbf{W}_t d\mathbf{B}_{i,t}] = 0$.

Households act as price takers in these markets. The market prices denoted by $\mathbf{P}_t$ transfers noise to households. The prices are $\mathcal{F}_t$-adapted, implying that households fully observe the realized price path at each time $t \in \mathbb{T} \equiv [0, \infty)$.

A risk-free asset represents ownership of the risk-free component of financial assets, such as typically cash or short-term government bonds. Its price denoted by b_t does not respond to noise:

$$\frac{db_t}{b_t} = r_f dt, \tag{2}$$

where $r_f > 0$ is the risk-free rate.

A *systematic risk asset* represents ownership of the risky component of financial assets, typically measured by a market index. Its price denoted by S_t responds full to noise:

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s d\mathbf{W}_{s,t}, \tag{3}$$

where $\mathbf{W}_{s,t} = \mathbf{W}_{1,t}$, $\mu_s > r_f$ is the expected return, and $\sigma_s > 0$ is systematic risk.

Household i's labor supply, $n_{i,t}$, remains purely idiosyncratic:

$$\frac{dn_{i,t}}{n_{i,t}} = \tilde{\mu}_{n_i} dt + \tilde{\sigma}_{n_i} d\mathbf{B}_{n_i,t}, \tag{4}$$

where $\tilde{\mu}_{n_i}$ is the drift parameter and $\tilde{\sigma}_{n_i} > 0$ is idiosyncratic income risk.

Labor supply is priced through a wage rate, denoted by w_t. This rate transfers noise to households:

$$\frac{dw_t}{w_t} = \mu_y dt + \sigma_y d\mathbf{W}_{y,t}, \tag{5}$$

where

$$d\mathbf{W}_{y,t} = \rho_{sy} d\mathbf{W}_{1,t} + \sqrt{1 - \rho_{sy}^2} d\mathbf{W}_{2,t}, \tag{6}$$

μ_y is the drift parameter, and $\sigma_y > 0$ is aggregate income risk.

By Itô's Lemma, the household i's labor income, $y_{i,t} \equiv w_t n_{i,t}$, follows

$$\begin{aligned} \frac{dy_{i,t}}{y_{i,t}} &= \frac{dw_t}{w_t} + \frac{dn_{i,t}}{n_{i,t}} \\ &= (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y d\mathbf{W}_{y,t} + \tilde{\sigma}_{n_i} d\mathbf{B}_{n_i,t}. \end{aligned} \tag{7}$$

Hence, we have the market prices:

$$\mathbf{P}_t = \{S_t, w_t\}. \tag{8}$$

2.2 The Household's Problem

Households exhibit constant relative risk aversion (CRRA) over consumption, characterized by the instantaneous utility function:

$$U(c_t) := \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \tag{9}$$

where $\gamma > 1$ is the CRRA coefficient.

Each household i chooses its optimal consumption, $c_{i,t}$, to maximize expected lifetime utility:

$$\max_{c_{i,t}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{i,t})\, dt \right], \tag{10}$$

where $\rho > 0$ is the discount rate.

Each household chooses its optimal portfolio consisting of the systematic risk asset S_t and the risk-free asset b_t. The household's wealth evolves as

$$da_t = \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\, dt + \omega_t a_t \sigma_s d\mathbf{W}_{s,t}, \tag{11}$$

where ω_t is the portfolio share of the asset S_t.

Households will not short sell any risky asset with a positive premium ($\mu_s - r_f > 0$), that is, there is a short-selling constraint:

$$\omega_t \geq 0. \tag{12}$$

Households face a borrowing constraint $a_t \geq 0$. Near the borrowing constraint, consumption cannot exceed income, i.e., $0 < c_t \leq y_t$, as $a \to 0$.

The infinitesimal generator of the underlying state process $\mathcal{L}[\cdot]$ follows

$$\begin{aligned}\mathcal{L}[v_t] \equiv &\{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} \\ &+ \omega_t a_t y_t \rho_{sy}\sigma_s\sigma_y \frac{\partial^2 v}{\partial a \partial y} + (\mu_y + \tilde{\mu}_{n_i})y_t \frac{\partial v}{\partial y} + \frac{1}{2}y_t^2(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)\frac{\partial^2 v}{\partial y^2}.\end{aligned} \tag{13}$$

We define the *asset-exposure premium* (AEP) as the component of the risk-exposure premium that households decide their portfolio share ω_t to maximize their lifetime welfare under systematic risk:

$$\begin{aligned}AEP_t \equiv &-\frac{\partial}{\partial a}\left(\omega\sigma_s a \frac{\partial v}{\partial a}\right) \\ = &-\omega_t\sigma_s\left(a_t\frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a}\right).\end{aligned} \tag{14}$$

It consists of two asymmetric components: a negative exposure cost ($-\omega_t\sigma_s\frac{\partial v}{\partial a}$), driven by the marginal utility of wealth, and a positive exposure compensation ($-\omega_t\sigma_s a_t\frac{\partial^2 v}{\partial a^2}$), driven by the curvature of the value function. The AEP compensation term is related to household wealth. This asymmetry gives the AEP its non-monotonicity, explaining hump-shaped profile. For low-wealth households, high marginal utility leads to a high AEP cost, while low wealth results in nearly zero AEP compensation. This net welfare cost is the root cause of limited participation. As wealth increases, this trade-off reverses. AEP compensation exceeds AEP costs, making wealthy households more inclined to bear systematic risk.

3 Theoretical Results

The household's problem can be simply expressed as[1]

[1]Here, we used the Itô integral to compromise with the complexity of current mathematics in order to illustrate economic intuition. From the perspective of rigorous stochastic analysis, we should be using the Itô-Wentzell integral, which is something we are currently working on.

$$\begin{aligned}\rho v_t = &\max_{c,\omega} u(c_t) + \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i})y_t\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \sigma_s)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}y_t^2(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)\frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy}\sigma_s\sigma_y\frac{\partial^2 v}{\partial a \partial y} \\ &- \omega_t\sigma_s\left(a_t\frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a}\right).\end{aligned} \tag{15}$$

By the envelope condition, we have the optimal consumption:

$$c_t^* = (u')^{-1}\left(\frac{\partial v}{\partial a}\right). \tag{16}$$

Taking the first-order condition (FOC), we have the optimal portfolio choice:

$$\omega_t^* = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left\{\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)\frac{\partial v}{\partial a} + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)y_t\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_s}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right\}. \tag{17}$$

We define the *asset-exposure premium effect* as the component of households' optimal portfolio choice with respect to the systematic risk asset S, given by $\Omega^*(a_t)$, which directly arises from the asset-exposure premium (14) within their optimization problem when choosing among a set of feasible investments including the risk asset S and the risk-free asset B:

$$\Omega^*(a_t) \equiv \frac{1}{a_t\sigma_s}\left[1 + \frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\frac{\partial v}{\partial a}\right]. \tag{18}$$

The curvature-to-slope ratio of the value function with respect to wealth is defined as the *effective relative risk aversion* (ERRA):

$$\eta(a_t) \equiv -a_t\frac{v_{aa}}{v_a}, \tag{19}$$

where $v_a \equiv \partial v/\partial a$ and $v_{aa} \equiv \partial^2 v/\partial a^2$. The effective relative risk aversion η_t measures households' intertemporal risk preferences across wealth. In contrast, the Arrow-Pratt measure of relative risk aversion is defined as $R(c_t) \equiv -c_t u''/u'$, which measures instantaneous risk preferences. The mathematical form of η_t is proposed by Merton (1969, 1971). Given the

standard conditions ($a > 0$, $v_a > 0$, and $v_{aa} < 0$), we have

$$\eta_t > 0. \tag{20}$$

Substituting the equation (19) into the equations (14), (18), and (17) yields

$$AEP_t = \omega_t \sigma_s \big(\eta_t - 1\big) \frac{\partial v}{\partial a}, \tag{21}$$

$$\Omega^*(a_t) = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t}\right), \tag{22}$$

and

$$\omega_t^* = \frac{1}{\eta_t} \Big[\underbrace{\frac{\mu_s - r_f}{\sigma_s^2}}_{\text{myopic demand}} + \underbrace{\frac{\rho_{sy}\sigma_y}{\sigma_s} y_t \left(\frac{\partial v}{\partial a}\right)^{-1} \frac{\partial^2 v}{\partial a \partial y}}_{\text{wage-offsetting effect}} + \underbrace{\frac{1}{a_t \sigma_s}\big(\eta_t - 1\big)}_{\text{AEP effect}} \Big]. \tag{23}$$

It decomposes the portfolio choice into three motives scaled by η_t. We call the first component the speculative demand, which reflects the households' intertemporal decision regarding the returns on risky assets, accounting for its excess returns and variance.[2] To distinguish idiosyncratic risk in labor productivity $n_{i,t}$ from income, we call the second term the wage-offsetting effect. It reflects balancing or compensating in the optimal portfolio share due to the primary systematic noise through the aggregate wage rate w_t. The third term is the AEP effect.

Households have solely the speculative demand when their portfolios is subject purely to idiosyncratic risk. The presence of systematic risk additionally triggers both of the AEP effect and the wage-offsetting effect. The first and second terms were pioneered by Merton (1969, 1971). Merton (1992) and Campbell and Viceira (2002) provide in-depth analyses for portfolio choice and intertemporal portfolio choice, respectively.

3.1 ERRA and Portfolio Choice at Both Ends

To align the household problem with economic intuition, we link η_t to observable microe-

[2]The first term is often referred to as myopic demand in the literature. According to the definition by Campbell (2018), at a given point in time, investors with given preferences has the same optimal portfolio, regardless of their time horizon. Myopic portfolio choice is usually satisfied in an approximately linear framework. This nonlinear model emphasizes the wealth-dependent property of portfolio choice.

conomic behavior. Let the marginal propensity to consume out of wealth $\text{MPC}_a \equiv \partial c_t^*/\partial a$. We have[3]

$$\eta_t = \frac{a_t}{c_t^*} \cdot \text{MPC}_a \cdot \gamma. \tag{24}$$

It transforms the invariant risk preference γ into time-varying risk-taking behavior at different wealth levels through the elasticity of consumption with respect to wealth, $\epsilon_{c,a} \equiv (a_t/c_t^*) \cdot \text{MPC}_a$.

3.1.1 ERRA and Portfolio Choice at the Borrowing Constraint

Near the borrowing constraint $\underline{a}$, households' optimal consumption is nearly determined by labor income. This implies that a_t/c_t^* approaches zero as wealth approaches $\underline{a}$.

Proposition 1 (**ERRA at the Borrowing Constraint**). *For any household with finite CRRA ($\gamma < \infty$) and bounded marginal propensity to consume out of wealth ($MPC_a < \infty$), the effective relative risk aversion vanishes at the borrowing constraint:*

$$\lim_{a \to 0^+} \eta_t = 0. \tag{25}$$

Proof. See Appendix A.1. □

It implies that, near the borrowing constraint, households' risk aversion to wealth becomes relatively negligible compared to the marginal benefits of wealth. The economic intuition is that the low wealth haves a strong incentive to pursue excess returns on risky assets in order to escape poverty.

Corollary 1 (**Portfolio Choice at the Borrowing Constraint**). *At the borrowing constraint, households do not hold the systematic asset S_t and the asset-exposure premium effect is singular:*

$$\lim_{a \to 0^+} \Omega^*(a_t) = -\infty, \qquad \lim_{a \to 0^+} \omega_t^* = 0. \tag{26}$$

Proof. See Appendix A.2. □

[3]We exploit the envelope condition (16) and derive $v_{aa} = u''(c_t^*)\frac{\partial c^*}{\partial a}$. We use the standard CRRA specification $\gamma = -c_t^* u''/u'$ and substitute these equations into (19). Then, $\eta(a_t) = -a_t \cdot \frac{v_{aa}}{u'(c^*)} = -a_t \cdot \frac{\partial c^*}{\partial a} \cdot \frac{u''(c^*)}{u'(c^*)} = \frac{a_t}{c_t^*} \cdot \text{MPC}_a \cdot \gamma$.

It reveals the core mechanism of limited participation: near the borrowing constraint, the magnitude of the different motives diverges. On the one hand, the speculative demand tends to positive infinity when the wage-offsetting effect is smaller than the speculative demand. Both motives are of order $O(n)$. This reflects the strong incentive for the poor to ignore risk in pursuit of asset returns. The AEP effect completely overwhelms all of this. With the scaling factor $1/a$, the AEP effect rushes toward negative infinity much faster, at the order of $O(n^2)$. The negative force is an order of magnitude larger than the other two forces. As wealth approaches the borrowing constraint, households' optimal decisions are entirely governed by the AEP effect. They rationally price the substantial AEP cost of holding the systematic asset and choose to exit the market entirely.

3.1.2 ERRA and Portfolio Choice of the Wealthy

As wealth grows arbitrarily large, the limiting case becomes a benchmark for understanding the behavior of the wealthy. In this regime, the influence of labor income diminishes, and wealth dynamics become paramount. As in a long line of recent studies, consumption and wealth become proportional at the limit.

Let c_t^* be the optimal consumption rule derived from the solution to the equation (15).

Lemma 1 (**Consumption-Wealth Ratio**). *The consumption-to-wealth ratio converges asymptotically to the constant $\bar{k}$ as wealth approaches infinity:*

$$\lim_{a\to\infty} \mathbb{E}_t\left[\frac{c_t^*}{a_t}\right] = \bar{k}, \tag{27}$$

where

$$\bar{k} = \frac{1}{\gamma}\left(\rho + (\gamma-1)\left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right)\right). \tag{28}$$

Proof. See Appendix B.1. □

It is consistent with the result of Merton (1969).

Proposition 2 (**ERRA of the Wealthy**). *As wealth approaches infinity, the effective*

relative risk aversion converges to the finite CRRA coefficient:

$$\lim_{a\to\infty} \eta_t = \gamma. \tag{29}$$

Proof. See Appendix B.2. □

It implies that the wealthy's intertemporal risk preference to wealth behaves in the same way as their instantaneous risk preference to consumption. This is because their consumption depends entirely on asset returns.

Corollary 2 (**Portfolio Choice of the Wealthy**). *As wealth approaches infinity, the asset-exposure premium effect vanishes, and the portfolio choice converges to a constant:*

$$\lim_{a\to\infty} \Omega^*(a_t) = 0, \qquad \lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\gamma}. \tag{30}$$

Proof. See Appendix B.3. □

It is consistent with the classical Merton ratio. In the high-wealth limit, the asymptotically optimal behavior alters the relative magnitudes of the three economic forces in the portfolio choice. As wealth approaches infinity, the effective relative risk aversion converges asymptotically to γ, and the speculative demand for risky assets is determined by a time-invariant risk aversion parameter. The speculative demand converges to a non-zero constant of order $O(1)$ remaining robustly active. The wage-offsetting effect vanishes as labor income becomes negligible relative to wealth. The AEP effect also vanishes by the scaling factor $1/a$.

Households' optimal portfolio share converges to the speculative component, which is consistent with the standard benchmark. In other words, for a given γ, changes in asset risk-return characteristics are the only feedback to the portfolio choice because the level of wealth does not affect portfolio decisions and remains constant.[4] The dynamics of wealth inequality are therefore driven by fundamentally different forces across the distribution. The

[4]This behavior robustly converges back to a representative agent model, which is a continuous-time framework of the Phelps-Ramsey problem (Phelps, 1962). The static analysis can be found in Merton (1969).

most significant portfolio heterogeneity driven by the AEP mechanism is concentrated among low-to-moderate wealth households.

3.2 Limited Participation and Life-Cycle Pattern

We now apply the AEP mechanism to provide a unified explanation for limited participation and the hump-shaped profile of the risky portfolio share. These phenomena stem from the inherent dynamics of the effective relative risk aversion as it changes with wealth. We restrict households' intertemporal risk aversion so that it is well-defined:

Assumption 1 (**Well-defined Intertemporal Risk Aversion**)**.** The effective relative risk aversion η_t is a monotonic increasing function of wealth a:

$$\frac{\partial \eta}{\partial a} > 0, \qquad \forall a < \infty. \tag{31}$$

We make this assumption based on the economic intuition: as wealth increases, households' effective relative risk aversion to a risky portfolio also increases, because their consumption depends more on asset returns. This assumption is equivalent to the consumption behavior that the elasticity of consumption with respect to wealth is monotonic, i.e., $\frac{\partial \epsilon_{c,a}}{\partial a} > 0$.

Lemma 2. *Assumption 1 holds if the value function satisfies*[5]

$$\frac{v_{aaa} v_a}{v_{aa}^2} \geq \frac{u''' u'}{(u'')^2}, \tag{32}$$

Proof. See Appendix C.1. □

In the low-wealth region, the equation (21) shows that when $\eta_t < 1$, the AEP is negative, forcing $\Omega_t < 0$. This occurs when the exposure cost exceeds the exposure compensation. For households near the borrowing constraint, their effective relative risk aversion η_t approaches

[5]This condition also serves as a sufficient condition for the precautionary saving (Carroll & Kimball, 1996).

zero. The excess returns of any risky asset induce these households to have a strong the speculative demand by the factor $1/\eta$. It also amplifies the AEP effect, which is dominated by the scaling factor $1/a$. This results in a negative effect that is order of magnitude larger than the other two motivations in the low-wealth region. The AEP mechanism endogenizes limited participation through a zero risky portfolio share.[6] As wealth increases, the impact of the scaling factor on the AEP effect gradually weakens, while η_t also departing from zero. When η_t approaches the left side of a critical point $\eta_t = 1$, the negative impact of the AEP effect becomes smaller than the positive impact of the excess returns. Households thus gradually become willing to participate in the market. This is reflected by a rapid increase in the risky portfolio share held.

Lemma 3. *Given the CRRA coefficient $\gamma > 1$, there exists the critical point $a^* \in a$ such that*

$$\eta(a^*) = 1 \tag{33}$$

and

$$\eta(a) > 1, \qquad \forall a > a^*. \tag{34}$$

Proof. See Appendix C.2. □

At the critical point a^*, the AEP cost perfectly offsets the AEP compensation, causing the AEP effect vanishes completely, i.e., $\Omega(a^*) = 0$. It implies that the portfolio choice is determined by the speculative demand and the wage-offsetting effect.

In the medium-to-high wealth region, where $a > a^*$, the AEP translates into a net benefit, generating a positive AEP effect, $\Omega_t > 0$. At this point, the AEP compensation exceeds the AEP cost. As wealth increases beyond a^*, the sensitivity to marginal utility and the curvature of risk aversion remain substantial. The interaction between these factors pushes the positive AEP effect to its peak value, providing an additional incentive to hold the systematic risk asset beyond standard portfolio motives. As wealth continues to grow,

[6] The literature has proposed explanations, categorized into three main streams: fixed participation costs that deter small investors, non-standard preferences such as ambiguity aversion or loss aversion, and the effect of background risk. For a comprehensive review of these approaches, see Campbell (2018). Recent studies on limited participation include Athreya, Ionescu, and Neelakantan (2023), Bharath and Cho (2023), and Melcangi and Sterk (2025).

the scaling factor $1/a$ becomes dominant, causing the AEP effect to gradually decline and ultimately converge to zero from above.

As households accumulate wealth over their lifetimes with positive savings, they naturally traverse wealth zones in which the AEP effect first inhibits, then encourages, and ultimately normalizes risky asset holdings. This process culminates in the portfolio choice converging asymptotically to the classical Merton benchmark. The complete evolutionary path of the asset-driven, non-monotonic AEP mechanism generates the hump-shaped profile of portfolio choice over the life cycle. This profile aligns with the "hump-shaped profile" (Heaton & Lucas, 2000; Poterba & Samwick, 2001; Ameriks & Zeldes, 2011; Parker et al., 2025) and the "hump-shaped saving behavior" (Graaff, 1950; Ando & Modigliani, 1963).[7]

3.2.1 Labor Income Channel

With well-defined intertemporal risk aversion, canonical portfolio theory explains the hump-shaped profile through the labor income channel. This raises the question: How does this model operate within the labor income channel?

We propose that households treat labor endowment as an implicit wealth, defined as the expected present value of future labor income. Although non-tradable labor income cannot be evaluated independently of its risk, even in the presence of aggregate income risk. This alters their attitude toward tradable financial assets. Our perspective is similar to the point proposed by Heaton and Lucas (2000) that although income is risky, it limits bad outcomes relative to investment income, which reduces the effective relative risk aversion η_t.

Formally, we provide a proof of this logic. Following the "proper property" Pratt and Zeckhauser (1987) and Gollier and Pratt (1996), the absolute risk aversion $-u''/u'$ is monotonically decreasing in wealth. We measure the welfare impact of wealth through an *effective absolute risk aversion*, $A(a) \equiv -v_{aa}/v_a$. As established by Carroll and Kimball (1996), we have that $v_{aaa}v_a - v_{aa}^2 > 0$. This is a sufficient condition for $A(a)$ to be monotonically decreasing in wealth, i.e., $\frac{\partial A(a)}{\partial a} < 0$. The implicit wealth causes households to behave as if they possess greater wealth. This leads to a lower $A(a)$ for any given wealth

[7]Recent studies of life-cycle portfolio choice include Hubener, Maurer, and Mitchell (2016), Addoum (2017), Fagereng, Gottlieb, and Guiso (2017), Michaelides and Zhang (2017), Chang, Hong, and Karabarbounis (2018), Gomes (2020), Catherine (2022), and Catherine, Sodini, and Zhang (2024).

and a lower η_t.

Proposition 3. *The effective relative risk aversion η_t is a monotonic decreasing function of the implicit wealth a':*

$$\frac{\partial \eta_t}{\partial a'} < 0. \tag{35}$$

Proof. $\eta_t' = a_t A(a_t + a') = -a_t \frac{v_{aa}(a_t+a')}{v_a(a_t+a')} < -a_t \frac{v_{aa}(a_t)}{v_a(a_t)} = \eta_t$, where $a' > 0$ is the implicit wealth. □

It implies that implicit wealth reduces households' intertemporal risk aversion to wealth, leading them to desire a larger share of a risky portfolio by the factor $1/\eta$. Over the life cycle, while the labor productivity $n_{i,t}$ may cause the income $y_{i,t}$ to be hump-shaped; implicit wealth is monotonically decreasing. This indicates that, at the same wealth level, young households are more willing to take intertemporal risks in pursuit of capital gains. The same logic applies to employed households compared to retirees.

Idiosyncratic income risk causes the wage-offsetting effect to disappear. The portfolio share of the systematic risk asset is clearly determined by two wealth-dependent motives. In the presence of aggregate income risk, the wage-offsetting effect depends on the uncertain empirical value of the correlation ρ_{sy}, so its net impact is indeterminate without further parameterization. A more detailed discussion of this topic is deferred to Section 3.3.

3.2.2 Labor income in Merton Framework

The Merton framework faces multiple theoretical challenges when addressing these puzzles. Non-tradeable labor income depends critically on two risk characteristics of income processes. However, both of these channels are theoretically problematic.

Idiosyncratic income risk exacerbates limited participation. Early studies by Bodie, Merton, and Samuelson (1992) and Heaton and Lucas (1997) examine the effect of the labor income in the presence of idiosyncratic noise on portfolio decisions. According to Proposition 3, income increases household welfare and results in greater speculative demand. This is similar to the economic intuition proposed by Viceira (2001) and Campbell and Viceira (2002) that low-wealth households relying on labor income have low effective relative risk aversion, and a high risky portfolio share.

Aggregate income risk cannot simultaneously explain limited participation and the hump-shaped profile of portfolio share. Using microdata, The literature calibrates the correlation between stock returns and labor income (Jagannathan, Kocherlakota, et al., 1996; Davis & Willen, 2000; Viceira, 2001; Haliassos & Michaelides, 2003; Cocco, Gomes, & Maenhout, 2005; Benzoni, Collin-Dufresne, & Goldstein, 2007). Typically, a large value of the correlation is required to make the effect significant for all households because the numerator $\frac{\partial^2 v}{\partial a \partial y}$ is relatively small than the denominator $\frac{\partial v}{\partial a}$. This effect is more sensitive to low-wealth households.

Given the fact that households hold stocks that are highly correlated with labor income (Campbell, 2018), the optimal response is to reduce risky asset demand when income and assets are positively correlated—a approach that can explain limited participation among low-wealth households. Yet this approach simultaneously implies that employed investors should hold lower risky portfolio share than retirees in the right tail over the life cycle, contradicting observed investor behavior. This result is inconsistent with life-cycle patterns. When the correlation is negative, the wage-offsetting effect increases the risky portfolio share for all households. Low-wealth households experience a more pronounced increase, which exacerbates limited participation further.

The fragility of the aggregate income risk is further underscored by the well-documented instability of the correlation (Guiso, Jappelli, & Terlizzese, 1996; Bertaut & Haliassos, 1997; Heaton & Lucas, 2000; Poterba, 2001; Ameriks & Zeldes, 2011), with empirical estimates varying across countries and periods. Some studies have found correlations close to zero or positive (Campbell et al., 2001; Angerer & Lam, 2009; Betermier et al., 2012; Huggett & Kaplan, 2016).

3.3 Portfolio Risk Characteristics: From Underdiversification to Diversification

To explain underdiversification, we move beyond exogenous frictions—such as transaction costs or portfolio size constraints—which lack consistent empirical support (Goetzmann & Kumar, 2008). Instead, following evidence that portfolio risk patterns are primarily

shaped by wealth distribution Calvet, Campbell, and Sodini (2007, 2009), Goetzmann and Kumar (2008), and Parker and Vissing-Jørgensen (2009), we develop a mechanism where risk characteristics arise endogenously from the household's optimization problem. While the Merton framework relies on aggregate income risk to indirectly reflect these patterns, our model identifies systematic risk as the internal driver. In this setup, the transition from underdiversification to diversification is an optimal response to the shifting marginal welfare valuation of risk as households accumulate wealth.

Let the dynamics of the risk component of the household portfolio be as follows:

$$\frac{dp_t}{p_t} = \mu_p + \beta_p \sigma_s d\mathbf{W}_{s,t} + \tilde{\sigma}_p d\mathbf{B}_{p,t}, \tag{36}$$

where μ_p is the expected return of the risky asset, $\beta_p \sigma_s$ is the systematic risk, $\tilde{\sigma}_p$ is the idiosyncratic risk, and $\mathbf{B}_{p,t}$ is the idiosyncratic noise.

Consider two portfolios F and N with completely different risk characteristics. The risk-free component of these portfolios includes only the risk-free asset b_t. The risk component of portfolio F contains only systematic risk, i.e., $\tilde{\sigma}_F = 0$ and $\beta_F = 1$. The risk component of portfolio N contains only idiosyncratic risk, i.e., $\beta_N = 0$. The portfolio F is the market portfolio, which serves as a benchmark.

We define a function that measures the household's preference for systematic risk in its portfolio:

Definition 1. A β *exposure ratio* in the household's portfolio p is defined as:

$$\Gamma_{p,t} \equiv \beta_p \left\{ \rho_{sy} \sigma_y y_t \left(\frac{\partial v}{\partial a} \right)^{-1} \frac{\partial^2 v}{\partial a \partial y} + \frac{1}{a_t} (\eta_t - 1) \right\}, \tag{37}$$

where ρ_{sy} is the structural correlation.

It is beyond the risk-return profile of the household portfolio. The first term reflects the offsetting effect of labor on systematic risk in portfolio choice. The remaining labor terms, such as the IEP, income risk, and income, only affect household welfare. In other words, households are unaffected by these terms when deciding on the risk composition of their portfolios.

Next, we use the sign of the β exposure ratio to classify households' rational behavior at different positions in the wealth distribution. First, we determine these households' preferences for their portfolios versus the market portfolio based on equivalent risk-return characteristics. Then, based on equivalent welfare levels, we determine the Sharpe ratio premium that households are willing to pay for taking on systematic risk. In other words, we analyze the portfolio efficiency of these households under the same consumption levels.

3.3.1 Underdiversification

Let Γ_t be the β exposure ratio of the market portfolio. The following proposition holds:

Proposition 4 (**Portfolio Risk Preference**). *When the β exposure ratio is negative ($\Gamma_t < 0$), then:*

(i) For equivalent risk-return characteristics, the optimal portfolio share satisfies:

$$\omega_F^* < \omega_N^*. \tag{38}$$

(ii) Under equivalent welfare levels, households demand a strictly higher Sharpe ratio from the market portfolio:

$$\mathcal{S}_F > \mathcal{S}_N \tag{39}$$

with the required Sharpe ratio premium given by:

$$\mathcal{S}_F - \mathcal{S}_N = -\Gamma_t > 0. \tag{40}$$

Proof. See Appendix D.1. □

When the β exposure ratio is negative ($\Gamma_t < 0$), low-wealth households bear the welfare cost of systematic risk. To minimize their welfare loss, they will seek portfolios that are low or even zero beta and that uncorrelated with financial market fluctuations. In real-world asset allocation, these households may invest in a few companies, local businesses, or specific industries. This behavior objectively constitutes underdiversification, which results in their portfolios being exposed to more idiosyncratic risk. In this process, households are willing

to accept portfolios with lower Sharpe ratios in exchange for better welfare. This rational behavior reduces exposure to systematic risk; however, the price they pay is the Sharpe ratio loss associated with holding more idiosyncratic portfolio. This explains the findings of Calvet, Campbell, and Sodini (2007) and Goetzmann and Kumar (2008) that households with less wealth are less likely to participate, and even if they do, they are more likely to invest inefficiently. For households with near-borrowing constraints, β exposure ratios are very high, almost completely excluding them from the market for systematic risk assets. This explains the empirical evidence from Blume and Friend (1975) and Kelly (1995) that households own severely underdiversified portfolios of individual stocks. Although Calvet, Campbell, and Sodini (2007) finds that a small number of Swedish households do suffer from significant underdiversification, we speculate that this is because Swedish households have higher wealth, resulting in less β exposure ratio loss.

When evaluating the efficiency of a portfolio with idiosyncratic risk and systematic risk, we determine the welfare loss incurred by a household portfolio taking on systematic risk and how this response can lead to seemingly suboptimal choices. Let a portfolio P follows the euqation (36), then the following corollary holds:

Corollary 3 (**Preference for Risk Assets**). *When the β exposure ratio is negative ($\Gamma_t < 0$), then:*

(i) For equivalent risk-return characteristics, the optimal portfolio share satisfies:

$$\omega_P^* > \omega_F^*. \tag{41}$$

(ii) Under equivalent welfare levels, the required Sharpe ratio for portfolio P lies strictly between the individual asset requirements:

$$\mathcal{S}_N < \mathcal{S}_P \tag{42}$$

with the specific relationship:

$$\mathcal{S}_F - \mathcal{S}_P = (\beta_P - 1)\Gamma_t. \tag{43}$$

Proof. See Appendix D.2. □

We can make a quantitative assessment based on the relative Sharpe ratio loss (RSRL), a mean-variance measure proposed by Calvet, Campbell, and Sodini (2007). The RSRL of portfolio P is

$$RSRL_{P,t} = \frac{\beta_P - 1}{\mathcal{S}_F}\Gamma_t. \tag{44}$$

When the portfolio P has $0 < \beta_P < 1$, the household requires its Sharpe ratio to be less than that of the market portfolio. Conversely, if portfolio P has $\beta_P > 1$, the household requires that its Sharpe ratio be greater than that of the market portfolio. Because the β exposure ratio is an endogenously determined function of each household and the aggregate price is observable, households only need to compare the Sharpe ratio of the market portfolio to determine their portfolio's Sharpe ratio. This measure allows us to directly map our theoretical prediction of welfare loss to an empirically measurable quantity, providing a path for future quantitative validation.

The risk attitudes predicted by this model resonate with several findings in behavioral finance.[8] Some studies have linked the tendency to overweight small probabilities to holding an underdiversified portfolio (Polkovnichenko, 2005; Dimmock et al., 2021).

3.3.2 Diversification

Households with positive β exposure ratio ($\Gamma_t > 0$) prefer to take on systematic risk, either by holding the market portfolio or through diversification. Systematic risk offers these households a net welfare benefit beyond financial returns. Thus, these households decide not based on how much relative Sharpe ratio loss they need to bear systematic risk, but rather on how willing they are to accept this loss in their portfolio's idiosyncratic risk to gain the added benefit of systematic risk.

If the market portfolio already offers the highest Sharpe ratio, then it is on the efficient

[8]This is similar to the risk-seeking behavior in the loss domain of prospect theory (Kai-Ineman, Tversky, et al., 1979; Fiegenbaum & Thomas, 1988; Thaler & Johnson, 1990; Tversky & Kahneman, 1992; McDermott, Fowler, & Smirnov, 2008; Kahneman & Tversky, 2013; N. C. Barberis, 2013). This preference for high-risk, low-return assets echoes research showing that investors view certain stocks as "lottery tickets" (Kumar, 2009; N. Barberis & Huang, 2008; Boyer & Vorkink, 2014; Bali, Cakici, & Whitelaw, 2011; Eraker & Ready, 2015).

frontier. In this case, it is difficult for households to find assets with higher Sharpe ratios. Furthermore, the subjective valuation of these households' portfolios will change, requiring the "adjusted" Sharpe ratio of these portfolios to satisfy at least $\mathcal{S}_F + (\beta - 1)\Gamma_t$. These households find the market portfolio more attractive than portfolios with $\beta < 1$. Therefore, they will exhibit a strong preference for diversification over portfolios with $\beta < 1$. Since the market portfolio is already on the efficient frontier, households will be reluctant to hold portfolios with $\beta < 1$. This explains the empirical phenomenon that most participating Swedish households' portfolios are actually close to the efficient frontier. (Calvet, Campbell, & Sodini, 2007).

When portfolios have $\beta > 1$, these households are willing to take on additional idiosyncratic risk. They allocate more to the systematic risk to balance the idiosyncratic risk. This is also confirmed by the findings of Calvet, Campbell, and Sodini (2007, 2009) and Goetzmann and Kumar (2008) that households with higher levels of wealth accumulation tend to invest more efficiently but also more aggressively. Overallocation is the driving force behind the hump in a hump-shaped portfolio. If their portfolios have $\beta > 1$, the systematic risk may transform the hump profile into a flat-topped profile across the wealth distribution.

As wealth grows further, the β exposure ratio decreases and converges to zero from above. This means that the welfare cost of bearing systematic risk is also almost zero. For the wealthiest households, the risk characteristics of their portfolio are irrelevant; only the risk-return characteristics matter. Economic intuition suggests that, in the market with a single, richest investor, it consider itself as a single investor. This is why convergence to the Merton framework occurs in this case. Campbell (2006)'s empirical findings support this logic, showing that the wealthiest households in the United States hold a large share of private businesses.

The rate at which households accumulate wealth changes based on the risk characteristics of their portfolio. Households with positive β exposure ratio follow fluctuations in the financial market, accelerating their wealth accumulation. This optimization behavior acts as a "wealth accelerator," propelling these households upward in the distribution.

3.4 Analytical Solution under Homogeneity

This section explores when the value function in this model degenerates into a homogeneous form, yielding an analytical solution under systematic risk. Although we adopt the Merton framework's assumption that households have a CRRA utility function, our analysis of the optimal portfolio choice reveals that the AEP effect breaks the homogeneity of the value function. The value function can only be homogeneous under certain conditions. We summarize this finding as follows:

Proposition 5 (**Necessary Condition for Homogeneity**). *A necessary condition for the value function to be a homogeneous function of income is that the AEP effect must be zero:*

$$\Omega^*(a_t) = 0. \tag{45}$$

Proof. See Appendix E.1. □

This implies that an analytical solution exists only for households at the critical point $\eta = 1$, or at the upper bound of wealth. For these households unaffected by the AEP, the problem degenerates into the Merton framework for representative investors.

Furthermore, this necessary condition prevents us from using mathematical methods to handle partial derivatives with respect to income in order to obtain an analytical solution, increasing the difficulty of solving the household problem.

4 Visualization of the Theoretical Results

Solving the Merton framework typically models income with idiosyncratic risk, without the wage-offsetting effect. If modeling this effect, the current approach usually involves the log-linearization methods (Campbell, 1993; Campbell & Viceira, 1999, 2001; Viceira, 2001; Campbell, 2018), which aims to eliminate cross terms to obtain an approximate analytical solution. However, this method is suitable for approximately linear models in which assets and income be perfectly correlated.

The lack of homogeneity in the AEP makes solving the household problem under certain assumptions difficult. To simplify the solution, this paper employs the mathematical

technique that assumes the value function has homogeneous forms, attempting to simplify partial derivatives of the value function with respect to income. This technique is inspired by the log-linearization methods. But a necessary condition for the homogeneous form is the absence of the AEP mechanism by the Proposition 5.

Finding a solution in the household's problem remains an unresolved problem. Finite Difference methods are not ideal when modeling continuous labor while considering second-order effects. Physics-Informed Neural Networks (PINNs) offer a feasible approach, but they lack generalization capabilities. On the one hand, PINNs become very unstable after adjusting some economic parameters. On the other hand, the difference in magnitude between first- and second-order effects, the span of the state space, and the time dimension in the partial differential equations (PDEs), all combine to make convergence difficult. One promising computational strategy is the use of Deep Learning (DL) methods, given their strong generalization capabilities. Han, Jentzen, and E (2018) pioneers the DL framework that solves high-dimensional stochastic PDEs using the discretized Euler-Maruyama methods. Ji et al. (2022) solves stochastic HJB equations via the stochastic maximum principle with DL methods.

To visualize the core economic mechanisms, we adopt a parsimonious functional form for the household consumption rule $c(a)$, that is consistent with the theoretical results in Section 3. The function is defined as follows:

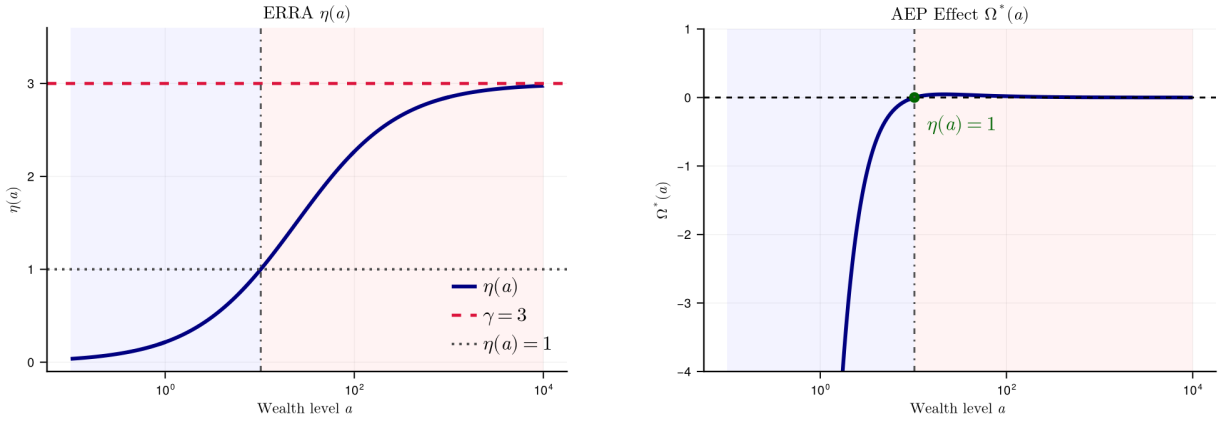
$$c(a) = (c_0^p + (\bar{k} \cdot a)^p)^{1/p}. \tag{46}$$

It ensures that consumption is sustained by a base level c_0 at low wealth and becomes asymptotically proportional to wealth, $c(a) \approx \bar{k} \cdot a$, at infinite wealth. This is consistent with Lemma 1. This formulation satisfies Assumption 1, which describes a monotonic elasticity of consumption with respect to wealth $\epsilon_{c,a}(a)$. As a result, the effective relative risk aversion, $\eta(a) = \gamma \cdot \epsilon_{c,a}(a)$, adheres to the boundary conditions established in Propositions 1 and 2. We isolate income risk to clearly present the main theoretical results. Since the function does not include aggregate income risk, we omit the underdiversification results from Section 3.3.

The model parameters are set as follows: the decay rate p is 0.8; the CRRA coefficient

γ is 3; the discount rate ρ is 0.05; the expected return for risky assets μ_s is 8%; the risk-free rate r_f is 3%. The mean income y_0 is normalized to 1.0. We set asset volatilities $\sigma_s = 0.30$ for risky assets and $\sigma_y = 0$ for income. For households at the borrowing constraint, their consumption c_0 equals their income.

4.1 AEP and its Implications



(a) (b)

Figure 1: Effective Relative Risk Aversion and the AEP Effect

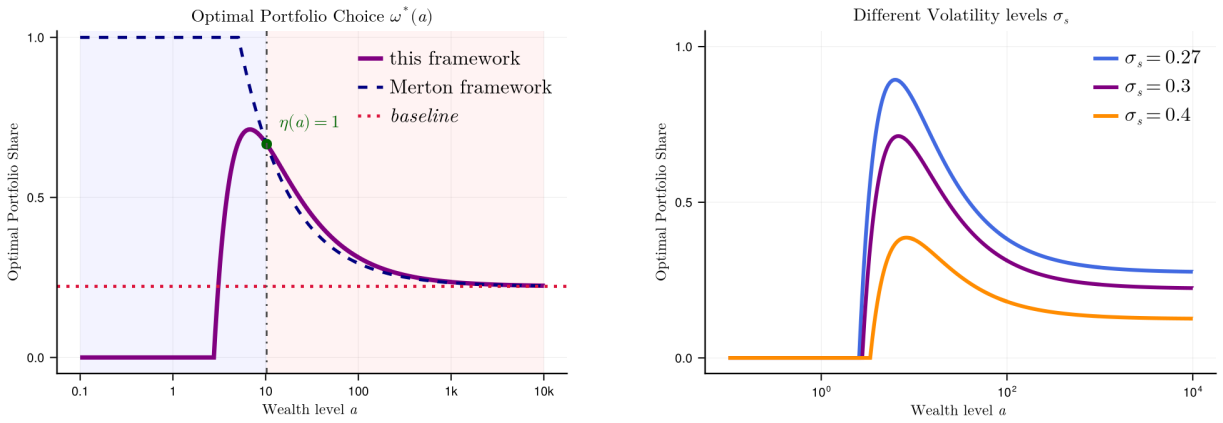
Figure 1a reveals the basis of the AEP mechanism: the functional relationship between the effective relative risk aversion $\eta(a_t)$ and household wealth. As Proposition 1 establishes, as households approach the borrowing constraint, η_t converges to zero. This reflects the economic intuition that, when consumption is nearly entirely determined by income, households' marginal utility out of wealth is nearly insensitive to changes in financial wealth. Conversely, Proposition 2 shows that as wealth approaches infinity, η_t converges to the CRRA coefficient γ. The smooth transition of η_t across the critical value of one is the engine of this model. It determines the sign, magnitude, and non-monotonicity of the AEP effect. This transition fosters heterogeneous preferences for systematic risk at different wealth levels: low-wealth households (with $\eta_t < 1$) are averse to this risk, while middle- and high-wealth households (with $\eta_t > 1$) receive net welfare compensation from taking on the risk.

Figure 1b visualizes the core meaning of this mechanism—the AEP effect. As shown, Ω_t^* drops sharply to negative infinity as wealth approaches zero. This corresponds to the

"overwhelming negative force" we identified in the theoretical section, providing a strong endogenous explanation for the "limited participation puzzle." As wealth grows, the AEP effect turns positive at $\eta_t > 1$, peaks, and then gradually converges back to zero from above, driven by the scaling factor $1/a$. This complete and non-monotonous path is the key to a unifying the explanation of the next set of empirical phenomena.

4.2 Optimal portfolio choice and Limited Participation

Figure 2a illustrates the ultimate impact of the AEP mechanism on the optimal portfolio share ω^* and compares it with the Merton framework in the absence of aggregate income risk. At low wealth levels, the strong negative AEP effect we observed in Figure 1b completely overwhelms speculative demand entirely, forcing the optimal portfolio share ω^* back to the short-selling constraint. This provides an endogenous theoretical basis for limited participation, which the Merton model struggles to explain. Once wealth exceeds the critical point $\eta_t = 1$, the AEP effect turns positive, providing an additional, modest incentive to hold the systematic risk asset. This force drives the portfolio share ω^* up, even exceeding the Merton framework and perfectly replicating the "hump-shaped" lifecycle profile observed in the empirical literature. Ultimately, as predicted in our Corollary 2, the optimal portfolio share converges back to the classic Merton ratio as the AEP effect disappears at the highest wealth level.



(a) (b)

Figure 2: Optimal portfolio choice

Figure 2b further explores the interaction between our mechanism and the financial market volatility σ_s. As expected, higher volatility reduces the demand for the systematic asset across all wealth levels, which is a standard outcome. The figure reveals a deeper prediction for this model: higher volatility exacerbates limited participation driven by the AEP. The wealth threshold required for market entry increases with volatility. The underlying economic logic is that higher volatility reduces the Sharpe ratio of the systematic asset, making the negative AEP costs in Figure 1b more unaffordable for low-wealth households.

5 Conclusion

This paper identifies that a household's endogenous valuation of systematic risk is structurally determined by its position in the wealth distribution. This AEP mechanism provides a unified framework for several empirical phenomena. The prohibitive welfare cost of systematic risk at low wealth levels generates limited participation and underdiversification. As wealth accumulates, the shifting AEP aligns household incentives with the efficient frontier, rationalizing the transition toward diversified portfolios.

While our analysis isolates the static determinants of optimal portfolio choice, it abstracts from inertia or return extrapolation to focus on the core interaction between welfare and risk. In this setup, households utilize parsimonious welfare rules to navigate a market environment, a tractable approach that highlights the fundamental economic mechanism. This framework offers a channel to assess policy interventions and provides a foundation to re-evaluate how market frictions interact with the endogenous willingness of heterogeneous households to bear systematic risk.

References

Addoum, J. M. (2017). Household portfolio choice and retirement. *Review of Economics and Statistics*, *99*(5), 870–883.

Ameriks, J., & Zeldes, S. P. (2011). *How do household portfolio shares vary with age?* SSRN.

Ando, A., & Modigliani, F. (1963). The" life cycle" hypothesis of saving: Aggregate implications and tests. *The American Economic Review*, *53*(1), 55–84.

Angerer, X., & Lam, P.-S. (2009). Income risk and portfolio choice: An empirical study. *The Journal of Finance*, *64*(2), 1037–1055.

Athreya, K., Ionescu, F., & Neelakantan, U. (2023). Stock market participation: The role of human capital. *Review of Economic Dynamics*, *47*, 1–18.

Bali, T. G., Cakici, N., & Whitelaw, R. F. (2011). Maxing out: Stocks as lotteries and the cross-section of expected returns. *Journal of Financial Economics*, *99*(2), 427–446.

Barberis, N., & Huang, M. (2008). Stocks as lotteries: The implications of probability weighting for security prices. *American Economic Review*, *98*(5), 2066–2100.

Barberis, N. C. (2013). Thirty years of prospect theory in economics: A review and assessment. *Journal of Economic Perspectives*, *27*(1), 173–196.

Benzoni, L., Collin-Dufresne, P., & Goldstein, R. S. (2007). Portfolio choice over the life-cycle when the stock and labor markets are cointegrated. *The Journal of Finance*, *62*(5), 2123–2167.

Bertaut, C. C., & Haliassos, M. (1997). Precautionary portfolio behavior from a life-cycle perspective. *Journal of Economic Dynamics and Control*, *21*(8-9), 1511–1542.

Betermier, S., Jansson, T., Parlour, C., & Walden, J. (2012). Hedging labor income risk. *Journal of Financial Economics*, *105*(3), 622–639.

Bharath, S. T., & Cho, D. (2023). Do natural disaster experiences limit stock market participation? *Journal of Financial and Quantitative Analysis*, *58*(1), 29–70.

Blume, M. E., & Friend, I. (1975). The asset structure of individual portfolios and some implications for utility functions. *The Journal of Finance*, *30*(2), 585–603.

Bodie, Z., Merton, R. C., & Samuelson, W. F. (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control*, *16*(3-4), 427–449.

Boyer, B. H., & Vorkink, K. (2014). Stock options as lotteries. *The Journal of Finance*, *69*(4), 1485–1527.

Calvet, L. E., Campbell, J. Y., & Sodini, P. (2007). Down or out: Assessing the welfare costs of household investment mistakes. *Journal of Political Economy*, *115*(5), 707–747.

Calvet, L. E., Campbell, J. Y., & Sodini, P. (2009). Measuring the financial sophistication of households. *American Economic Review*, *99*(2), 393–398.

Campbell, J. Y. (1993). Intertemporal asset pricing without consumption data. *The American Economic Review*, *83*(3), 487–512. http://www.jstor.org/stable/2117530

Campbell, J. Y. (2006). Household finance. *The Journal of Finance*, *61*(4), 1553–1604.

Campbell, J. Y. (2018). *Financial decisions and markets: A course in asset pricing*. Princeton University Press.

Campbell, J. Y., Cocco, J. F., Gomes, F. J., & Maenhout, P. J. (2001). Investing retirement wealth: A life-cycle model. In *Risk aspects of investment-based social security reform* (pp. 439–482). University of Chicago Press.

Campbell, J. Y., & Viceira, L. M. (1999). Consumption and portfolio decisions when expected returns are time varying. *The Quarterly Journal of Economics*, *114*(2), 433–495.

Campbell, J. Y., & Viceira, L. M. (2001). Who should buy long-term bonds? *American Economic Review*, *91*(1), 99–127.

Campbell, J. Y., & Viceira, L. M. (2002). *Strategic asset allocation: Portfolio choice for long-term investors*. Clarendon Lectures in Economic.

Carroll, C. D., & Kimball, M. S. (1996). On the concavity of the consumption function. *Econometrica: Journal of the Econometric Society*, 981–992.

Catherine, S. (2022). Countercyclical labor income risk and portfolio choices over the life cycle. *The Review of Financial Studies*, *35*(9), 4016–4054.

Catherine, S., Sodini, P., & Zhang, Y. (2024). Countercyclical income risk and portfolio choices: Evidence from sweden. *The Journal of Finance*, *79*(3), 1755–1788.

Chang, Y., Hong, J. H., & Karabarbounis, M. (2018). Labor market uncertainty and portfolio choice puzzles. *American Economic Journal: Macroeconomics*, *10*(2), 222–262.

Cocco, J. F., Gomes, F. J., & Maenhout, P. J. (2005). Consumption and portfolio choice over the life cycle. *The Review of Financial Studies*, *18*(2), 491–533.

Davis, S. J., & Willen, P. S. (2000). Occupation-level income shocks and asset returns: Their covariance and implications for portfolio choice.

Dimmock, S. G., Kouwenberg, R., Mitchell, O. S., & Peijnenburg, K. (2021). Household portfolio underdiversification and probability weighting: Evidence from the field. *The Review of Financial Studies*, *34*(9), 4524–4563.

Eraker, B., & Ready, M. (2015). Do investors overpay for stocks with lottery-like payoffs? an examination of the returns of otc stocks. *Journal of Financial Economics*, *115*(3), 486–504.

Evans, L. C. (2022). *Partial differential equations* (Vol. 19). American Mathematical Society.

Fagereng, A., Gottlieb, C., & Guiso, L. (2017). Asset market participation and portfolio choice over the life-cycle. *The Journal of Finance*, *72*(2), 705–750.

Fiegenbaum, A., & Thomas, H. (1988). Attitudes toward risk and the risk–return paradox: Prospect theory explanations. *Academy of Management Journal*, *31*(1), 85–106.

Goetzmann, W. N., & Kumar, A. (2008). Equity portfolio diversification. *Review of Finance*, *12*(3), 433–463.

Gollier, C., & Pratt, J. W. (1996). Risk vulnerability and the tempering effect of background risk. *Econometrica: Journal of the Econometric Society*, 1109–1123.

Gomes, F. (2020). Portfolio choice over the life cycle: A survey. *Annual Review of Financial Economics*, *12*(1), 277–304.

Gomes, F., & Smirnova, O. (2021). Stock market participation and portfolio shares over the life-cycle. *Available at SSRN 3808350*.

Graaff, J. d. V. (1950). Mr. harrod on hump saving. *Economica*, 81–90.

Guiso, L., Jappelli, T., & Terlizzese, D. (1996). Income risk, borrowing constraints, and portfolio choice. *The American Economic Review*, 158–172.

Haliassos, M., & Michaelides, A. (2003). Portfolio choice and liquidity constraints. *International Economic Review*, *44*(1), 143–177.

Han, J., Jentzen, A., & E, W. (2018). Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences*, *115*(34), 8505–8510.

Heaton, J., & Lucas, D. (1997). Market frictions, savings behavior, and portfolio choice. *Macroeconomic Dynamics*, *1*(1), 76–101.

Heaton, J., & Lucas, D. (2000). Portfolio choice and asset prices: The importance of entrepreneurial risk. *The Journal of Finance*, *55*(3), 1163–1198.

Hubener, A., Maurer, R., & Mitchell, O. S. (2016). How family status and social security claiming options shape optimal life cycle portfolios. *The Review of Financial Studies*, *29*(4), 937–978.

Huggett, M., & Kaplan, G. (2016). How large is the stock component of human capital? *Review of Economic Dynamics*, *22*, 21–51.

Jagannathan, R., Kocherlakota, N. R., et al. (1996). Why should older people invest less in stocks than younger people. *Federal Reserve Bank of Minneapolis Quarterly Review*, *20*, 11–20.

Ji, S., Peng, S., Peng, Y., & Zhang, X. (2022). Solving stochastic optimal control problem via stochastic maximum principle with deep learning method. *Journal of Scientific Computing*, *93*(1), 30.

Kahneman, D., & Tversky, A. (2013). Prospect theory: An analysis of decision under risk. In *Handbook of the fundamentals of financial decision making: Part i* (pp. 99–127). World Scientific.

Kai-Ineman, D., Tversky, A., et al. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, *47*(2), 363–391.

Kelly, M. (1995). All their eggs in one basket: Portfolio diversification of us households. *Journal of Economic Behavior & Organization*, *27*(1), 87–96.

Kumar, A. (2009). Who gambles in the stock market? *The Journal of Finance*, *64*(4), 1889–1933.

McDermott, R., Fowler, J. H., & Smirnov, O. (2008). On the evolutionary origin of prospect theory preferences. *The Journal of Politics*, *70*(2), 335–350.

Melcangi, D., & Sterk, V. (2025). Stock market participation, inequality, and monetary policy. *Review of Economic Studies*, *92*(4), 2656–2690.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, *3*(4), 373–413.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica: Journal of the Econometric Society*, 867–887.

Merton, R. C. (1992). Continuous-time finance.

Michaelides, A., & Zhang, Y. (2017). Stock market mean reversion and portfolio choice over the life cycle. *Journal of Financial and Quantitative Analysis*, *52*(3), 1183–1209.

Parker, J. A., Schoar, A., Cole, A., & Simester, D. (2025). Household portfolios and retirement saving over the life cycle. *The Journal of Finance*.

Parker, J. A., & Vissing-Jørgensen, A. (2009). Who bears aggregate fluctuations and how? *American Economic Review*, *99*(2), 399–405.

Phelps, E. S. (1962). The accumulation of risky capital: A sequential utility analysis. *Econometrica: Journal of the Econometric Society*, 729–743.

Polkovnichenko, V. (2005). Household portfolio diversification: A case for rank-dependent preferences. *The Review of Financial Studies*, *18*(4), 1467–1502.

Poterba, J. M. (2001). Demographic structure and asset returns. *Review of Economics and Statistics*, *83*(4), 565–584.

Poterba, J. M., & Samwick, A. (2001). Household portfolio allocation over the life cycle. In *Aging issues in the united states and japan* (pp. 65–104). University of Chicago Press.

Pratt, J. W., & Zeckhauser, R. J. (1987). Proper risk aversion. *Econometrica: Journal of the Econometric Society*, 143–154.

Thaler, R. H., & Johnson, E. J. (1990). Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. *Management Science*, *36*(6), 643–660.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and uncertainty*, *5*(4), 297–323.

Viceira, L. M. (2001). Optimal portfolio choice for long-horizon investors with nontradable labor income. *The Journal of Finance*, *56*(2), 433–470.

Vissing-Jørgensen, A. (2002). Limited asset market participation and the elasticity of intertemporal substitution. *Journal of Political Economy*, *110*(4), 825–853.

Appendix

A

A.1 Proof of Proposition 1

Proof. We have the standard conditions $v_a > 0, v_{aa} < 0$ and $\eta_t > 0$. Let $I \equiv (0, \infty]$ be an interval containing the point 0. Let's define a function $\bar{\eta}_t \equiv \frac{a_t}{c_t^*} \cdot M \cdot \gamma$. Stated in the proposition, the assumptions implies that $\gamma < \infty$, and $\frac{\partial c}{\partial a} \equiv \frac{\partial c_t^*}{\partial a_t}$, is bounded, i.e., there exists some $M < \infty$ such that $\frac{\partial c}{\partial a} \leq M$ as $a_t \to 0^+$. Consider a household with a positive income, $y_t > 0$. As $a_t \to 0^+$, households' consumption does not fall to zero. It is sustained by income. The optimal consumption c_t^* converges to a strictly positive value, $c^*(0, y_t) > 0$. The limit of this upper-bound function is:

$$\lim_{a_t \to 0^+} \bar{\eta}_t = \left(\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} \right) \cdot M \cdot \gamma = \frac{0}{c^*(0, y_t)} \cdot M \cdot \gamma = 0. \tag{47}$$

By the Squeeze Theorem, $\forall a \in I$, we have

$$0 \leq \eta_t \leq \bar{\eta}_t \qquad \text{and} \qquad \lim_{a_t \to 0^+} 0 = \lim_{a_t \to 0^+} \bar{\eta}_t = 0. \tag{48}$$

Then

$$\lim_{a_t \to 0^+} \eta_t = 0. \tag{49}$$

□

A.2 Proof of Corollary 1

Proof. (i) The AEP effect is given by the equation (18):

$$\Omega_t^*(a_t) = \frac{1}{a_t \sigma_s} \frac{\eta_t - 1}{\eta_t} = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t} \right). \tag{50}$$

From Proposition 1, we have that as $a_t \to 0^+$, $\eta_t \to 0^+$.

Then

$$\lim_{a_t \to 0^+} \Omega^*(a_t) = \left(\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s} \right) \cdot \left(\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta_t} \right) \right) = (+\infty) \cdot (-\infty) = -\infty. \tag{51}$$

(ii) The optimal portfolio choice is given by the equation (17):

$$\omega_t^* = \frac{1}{\eta_t \sigma_s}\Big[\underbrace{\frac{\mu_s - r_f}{\sigma_s}}_{\text{speculative demand}} + \underbrace{\rho_{sy}\sigma_y y_t \left(\frac{\partial v}{\partial a}\right)^{-1} \frac{\partial^2 v}{\partial a \partial y}}_{\text{wage-offsetting effect}} + \underbrace{\frac{1}{a_t}(\eta_t - 1)}_{\substack{\text{AEP Effect} \\ \text{(New Term)}}} \Big] \tag{52}$$

We examine the order of magnitude of each motive as $a_t \to 0^+$. From Proposition 1, we have $\eta_t = (a_t/c_t^*) \cdot (\partial c_t^*/\partial a_t) \cdot \gamma$. Near the borrowing constraint, c_t^* and $\partial c_t^*/\partial a_t$ are both bounded, meaning η_t is on the order of a_t. Let $n = 1/a_t$. We denote $1/\eta \sim O(n)$ as $a_t \to 0^+$.

All effects are scaled by $1/(\eta_t \sigma_s)$, so we only need to consider the order in the brackets. The bracketed term of the speculative demand is a constant. Similarly, the bracketed term of the wage offsetting effect is also a constant. This is because it is unrealistic for household income to be infinite near the borrowing constraint. Since $1/\eta \sim O(n)$, these two motives diverge at a rate of $O(n)$. The bracketed term of the AEP effect diverges at a rate of $O(n)$. Hence, the AEP effect diverges at a rate of $O(n^2)$.

The limit of the unconstrained optimal portfolio share, $\omega_{t,\text{unc}}^*$, is dirven by this dominant term:

$$\lim_{a_t \to 0^+} \omega_{t,\text{unc}}^* = \lim_{a_t \to 0^+} \left[O(n) - O(n^2)\right] = -\infty. \tag{53}$$

If households face short-selling restrictions, which can be modeled as $\omega_t^* = \max(0, \omega_{t,\text{unc}}^*)$. Then

$$\lim_{a_t \to 0^+} \omega_t^* = \max(0, \omega_{t,\text{unc}}^*) = 0. \tag{54}$$

□

B

B.1 Proof of Lemma 1

Proof. The value function is given by

$$\rho v(a, y, t) = \sup_{c,\omega} \{u(c) + \mathcal{L}[v_t] + \mathbb{E}_t \mathcal{A}[v_t] + REP_t\} \tag{55}$$

It allows us to determine the asymptotic behavior of the optiaml rule as household wealth abecomes arbitrarily large. To analyze the behavior at infinity, we employ a scaling technique. This method allows us to investigate how the solution th steady-state HJB equation behaves asymptotically.

We introduce a scaling parameter $\xi > 0$ and define a value function v_ξ which normalizes

the original value function v by the scale. The transformation applies only to the individual-specific state variables (a, y).

Definition 2. For any $\xi > 0$, the scaled value function $v_\xi : [0, \infty) \times (0, \infty) \times \mathbb{R} \times [0, T] \to \mathbb{R}$ is defined as:

$$v_\xi(a, y, t) := \frac{1}{\xi^{1-\gamma}} v(\xi a, y, t). \tag{56}$$

Lemma 4. *The scaled value function $v_\xi(a, y, t)$ satisfies a BSHJB equation where the coefficients of the labor income process is scaled down by powers of ξ.*

Proof of Lemma 4. For brevity, we denote partial derivatives by subscripts (e.g., $v_a \equiv \frac{\partial v}{\partial a}$). We derive the dynamics of v_ξ by applying the chain rule to its definition. The partial derivatives of v_ξ relate to those of v as follows:

$$\begin{aligned}
\frac{\partial v_\xi}{\partial a} &= \xi^\gamma v_a(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial a^2} &= \xi^{\gamma+1} v_{aa}(\xi a, \dots), \\
\frac{\partial v_\xi}{\partial y} &= \xi^{\gamma-1} v_y(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial y^2} &= \xi^{\gamma-1} v_{yy}(\xi a, \dots), \\
\frac{\partial^2 v_\xi}{\partial a \partial y} &= \xi^\gamma v_{ay}(\xi a, \dots).
\end{aligned} \tag{57}$$

When the household's wealth a becomes infinite, its decision problem should converge to a control problem involving only wealth a. At infinity, the marginal effects of labor income and macroeconomic aggregation become negligible. As $\xi \to \infty$, the scaled value function $v_\xi(a, y, t)$ converges to a limiting function $v_\infty(a, t)$. This convergence is a standard result in such problems and can often be rigorously proven by the stability and consistency of viscosity solutions (Evans, 2022). If the value functions themselves converge, then their partial derivatives with respect to variables that become irrelevant in the limit should also converge to zero. Then, we have

$$\begin{aligned}
\lim_{\xi \to \infty} (v_\xi)_y(a, y, t) &= \frac{\partial v_\infty(a, t)}{\partial y} = 0, \\
\lim_{\xi \to \infty} (v_\xi)_{ay}(a, y, t) &= \frac{\partial^2 v_\infty(a, t)}{\partial a \partial y} = 0, \\
\lim_{\xi \to \infty} (v_\xi)_{yy}(a, y, t) &= \frac{\partial^2 v_\infty(a, t)}{\partial y^2} = 0.
\end{aligned} \tag{58}$$

Let $c_\xi = c/\xi$ and $\omega_\xi = \omega$ be the scaled controls. Substituting these into the original BSHJB equation, evaluating at $(\xi a, y, t)$, and multiplying the entire equation by $\xi^{\gamma-1}$ yields

the BSHJB equation for $v_\xi(a, y, t)$. The instantaneous utility transforms as:

$$\xi^{\gamma-1}u(c) = u(c/\xi) = u(c_\xi). \tag{59}$$

And other terms become:

$$\begin{aligned} &\xi^{\gamma-1}\Big[[\omega_t(\mu_s - r_f) + r_f](\xi a) + (y) - (\xi c_\xi) - \omega\sigma_s\Big]v_a(\xi a, \dots) \\ =&\Big[[\omega_t(\mu_s - r_f) + r_f]a + y/\xi - c - \omega\sigma_s\Big]v_{\xi,a}(a, \dots) \end{aligned} \tag{60}$$

and

$$\xi^{\gamma-1}\left[\frac{1}{2}(\omega(\xi a)\sigma_s)^2 - \omega(\xi a)\sigma_s\right]v_{aa}(\xi a, \dots) = \left[\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s\right]v_{\xi,aa}(a, \dots). \tag{61}$$

□

As $\xi \to \infty$,

$$\rho v_L = \sup_{c,\omega}\left\{u(c) + \Big[(\omega(\mu_s - r_f) + r_f)a - c - \omega\sigma_s\Big]v_L' + \left[\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s\right]v_L''\right\}. \tag{62}$$

The solution to this HJB equation is $v_L = B\frac{a^{1-\gamma}}{1-\gamma}$, where B is a constant. To derive $\bar{k}$, we substitute $v_L = B\frac{a^{1-\gamma}}{1-\gamma}$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ into the HJB equation (62).

The first and second derivatives of v_L are:

$$\begin{aligned} v_L' &= Ba^{-\gamma} \\ v_L'' &= -\gamma Ba^{-\gamma-1}. \end{aligned} \tag{63}$$

The envelope condition with respect to c is $u'(c) - v_L' = 0$, which gives:

$$c^{-\gamma} = Ba^{-\gamma} \implies c^* = B^{-1/\gamma}a. \tag{64}$$

Let $\bar{k} = B^{-1/\gamma}$, so $c^* = \bar{k}_L a$.

The FOC with respect to ω is:

$$[(\mu_s - r_f)a - \sigma_s]v_L' + (\omega\sigma_s^2a^2 - \sigma_s a)v_L'' = 0. \tag{65}$$

Substitute the derivatives v_L' and v_L'':

$$[(\mu_s - r_f)a - \sigma_s](Ba^{-\gamma}) + (\omega\sigma_s^2a^2 - \sigma_s a)(-\gamma Ba^{-\gamma-1}) = 0. \tag{66}$$

Dividing by $Ba^{1-\gamma}$ (since $B \neq 0$ and $a \neq 0$):

$$\omega^*(a) = \frac{\mu_s - r_f}{\gamma \sigma_s^2} + \frac{\gamma - 1}{\gamma a \sigma_s}. \tag{67}$$

We substitute the expressions for c^*, ω^*, v_L, v_L', and v_L'' into the HJB equation (62).

$$\begin{aligned} \rho B \frac{a^{1-\gamma}}{1-\gamma} =& \frac{(\bar{k}_L a)^{1-\gamma}}{1-\gamma} + \left((r_f + \omega^*(\mu_s - r_f))a - \bar{k}_L a - \omega^* \sigma_s\right) B a^{-\gamma} \\ &+ \left(\frac{1}{2}(\omega^* \sigma_s a)^2 - \omega^* a \sigma_s\right)(-\gamma B a^{-\gamma-1}). \end{aligned} \tag{68}$$

Divide the entire equation by $Ba^{1-\gamma}$:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k}^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma + \frac{\omega^* \sigma_s}{a}(\gamma - 1) \\ =& \frac{\bar{k}^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma, \qquad \text{as } a \to \infty. \end{aligned} \tag{69}$$

Since $\bar{k} = B^{-1/\gamma}$, we have $B = \bar{k}^{-\gamma}$. Substitute this into the first term:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k}^{1-\gamma}}{\bar{k}^{-\gamma}(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma \\ =& \frac{\bar{k}}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \bar{k} - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma. \end{aligned} \tag{70}$$

Combine terms involving $\bar{k}$:

$$\begin{aligned} \frac{\rho}{1-\gamma} =& \frac{\bar{k} - \bar{k}(1-\gamma)}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma \\ =& \frac{\gamma \bar{k}}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma. \end{aligned} \tag{71}$$

Multiply by $(1-\gamma)$:

$$\rho = \gamma \bar{k} + (1-\gamma)\left[r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^* \sigma_s)^2 \gamma\right]. \tag{72}$$

Now, substitute $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2}$ (as $a \to \infty$) into the equation:

$$\begin{aligned}\rho =& \gamma\bar{k} + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\left(\frac{\mu_s - r_f}{\gamma\sigma_s^2}\right)^2 \sigma_s^2\gamma\right] \\ =& \gamma\bar{k} + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2}\right] \\ =& \gamma\bar{k} + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right].\end{aligned} \tag{73}$$

Then

$$\bar{k} = \frac{1}{\gamma}\left(\rho - (1-\gamma)\left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right)\right). \tag{74}$$

□

B.2 Proof of Proposition 2

Proof. From Lemma 1, the marginal propensity to consume converges to the average propensity to consume by the L'Hôpital's rule:

$$\lim_{a\to\infty} \frac{c^*(a)}{a} = \lim_{a\to\infty} \frac{\partial c^*(a)}{\partial a} = \bar{k}. \tag{75}$$

The elasticity of consumption with respect to wealth converges to unity:

$$\lim_{a\to\infty} \frac{a}{c^*(a)} \frac{\partial c^*(a)}{\partial a} = (\bar{k})^{-1}\bar{k} = 1. \tag{76}$$

Then

$$\lim_{a\to\infty} \eta_t = \lim_{a\to\infty} \frac{a}{c^*(a)} \frac{\partial c^*(a)}{\partial a} \cdot \gamma = \gamma. \tag{77}$$

□

B.3 Proof of Corollary 2

Proof. (i) By definition, we have

$$\lim_{a\to\infty} \Omega^*(a_t) = \lim_{a\to\infty} \frac{1}{a_t\sigma_s}\left(1 - \frac{1}{\eta_t}\right) = 0. \tag{78}$$

(ii) From Lemma 1 and Proposition 2, we know that both of the wage-offsetting effect and the AEP effect vanish under the scaling $\xi \to \infty$.

Then

$$\lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2 \gamma}. \tag{79}$$

□

C

C.1 Proof of Lemma 2

Proof. From the definition of $\eta(a) = -a\frac{v_{aa}}{v_a}$ and its derivative with respect to wealth a, the condition for $\frac{d\eta}{da} > 0$ is that the numerator must be positive:

$$\frac{d\eta}{da} = \frac{a v_{aa}^2 - v_a v_{aa} - a v_a v_{aaa}}{(v_a)^2} > 0. \tag{80}$$

We introduce the sufficient condition:

$$v_a v_{aaa} \geq k v_{aa}^2, \tag{81}$$

where $k \equiv u'''u'/u''^2$.

Since $a > 0$, multiplying the condition by $-a$ reverses the inequality sign:

$$-a v_a v_{aaa} \leq -ak v_{aa}^2. \tag{82}$$

Substituting this into the condition (80), we obtain an upper bound for the derivative $\frac{d\eta}{da}$:

$$\frac{a v_{aa}^2 - v_a v_{aa} - ak v_{aa}^2}{(v_a)^2} \geq \frac{a v_{aa}^2 - v_a v_{aa} - a v_a v_{aaa}}{(v_a)^2} > 0. \tag{83}$$

This upper bound must be positive:

$$a v_{aa}^2 - v_a v_{aa} - ak v_{aa}^2 > 0. \tag{84}$$

We factor the expression above by first grouping terms with a and then factoring out v_{aa}:

$$v_{aa}\left(a v_{aa} - v_a - ak v_{aa}\right) > 0. \tag{85}$$

$$v_{aa}\left(a(1-k) v_{aa} - v_a\right) > 0. \tag{86}$$

Now, we factor out v_a and use the definition $\eta = -a\frac{v_{aa}}{v_a}$:

$$v_{aa}v_a\left(\frac{a(1-k)v_{aa}}{v_a} - 1\right) > 0 \tag{87}$$

$$v_{aa}v_a\left((k-1)\eta - 1\right) > 0. \tag{88}$$

The remaining term in the parentheses must be negative:

$$(k-1)\eta - 1 < 0 \tag{89}$$

$$(k-1)\eta < 1 \tag{90}$$

$$\eta < \frac{1}{k-1}. \tag{91}$$

We have

$$\begin{aligned} k =& \frac{\gamma(\gamma+1)c^{-2\gamma-2}}{(-\gamma c^{-\gamma-1})^2} \\ =& 1 + \frac{1}{\gamma}. \end{aligned} \tag{92}$$

From Proposition 2, then

$$\eta < \gamma, \qquad \forall a < \infty. \tag{93}$$

□

C.2 Proof of Lemma 3

Proof. By the Bolzano's intermediate value theorem, there exists a wealth interval $a \in (0, \infty)$ and $\eta : a \to \mathbb{R}$ is continuous. We have that $\eta(a)$ is an interval. With Proposition 1 and 2, $\eta(a)$ is the interval $[0, \gamma]$. Hence, $1 \in \eta(a)$ for $0 < 1 < \gamma$. Let $\eta(a^*) = 1$, where $a^* \in a$. According to Assumption 1, $\eta(a)$ is monotonic. Then, $\eta(a) > 1$ for $a \in (a^*, \infty)$. □

D

D.1 Proof of Proposition 4

Proof. (i) We regard the transition from portfolio N to portfolio F as adding a "perturbation term" $\mathcal{D}(v)$ to the HJB equations of portfolio N by the perturbation method. For equivalent risk-return characteristics, we have that $\sigma_N = \sigma_F = \sigma_s$ and $\mu_N = \mu_F = \mu_s$.

The household invests in the portfolio N, then its HJB equation is:

$$\begin{aligned}\rho v^N = &\max_{c,\omega}\Big\{u(c_t) + \{[\omega_t(\mu_N - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + y_t(\mu_y + \tilde{\mu}_{n_i})\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \tilde{\sigma}_N)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)y_t^2\frac{\partial^2 v}{\partial y^2}\Big\}.\end{aligned} \tag{94}$$

The household invests in the portfolio F, then its HJB equation is:

$$\begin{aligned}\rho v^F = &\max_{c,\omega}\Big\{u(c_t) + \{[\omega_t(\mu_F - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i})y_t\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \sigma_s)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)y_t^2\frac{\partial^2 v}{\partial y^2} + \omega_t a_t \sigma_s \Gamma_t v_a\Big\}.\end{aligned} \tag{95}$$

Subtracting the two equations gives the perturbation term:

$$\mathcal{D}(v) = \omega_t a_t \sigma_s \Gamma_t v_a \tag{96}$$

The HJB equation for portfolio F can be written as:

$$\rho v^F = \text{HJB}_N(v^F) + \mathcal{D}(v^F), \tag{97}$$

where HJB means replacing v^N in the HJB equation of portfolio N with v^F.

The FOC of portfolio F can be viewed as the FOC of portfolio N plus the partial derivative of the perturbation term with respect to ω:

$$\text{FOC}_F = \text{FOC}_N(v^F) + \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = 0, \tag{98}$$

where

$$\text{FOC}_N(v^F) = (\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F. \tag{99}$$

The partial derivative of the perturbation term with respect to ω is:

$$\frac{\partial}{\partial \omega}\mathcal{D}(v^F) = a_t \sigma_s \Gamma_t v_a \tag{100}$$

Now, the FOC of portfolio F can be rewritten as:

$$(\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F + \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = 0 \tag{101}$$

With $\eta_t = -a_t \frac{v_a}{v_{aa}}$, the FOC for portfolio N and F with respect to ω are:

$$\omega_N^* = \frac{1}{\eta_t^N \sigma_s} \frac{(\mu_s - r_f)}{\sigma_s} \tag{102}$$

$$\omega_F^* = \frac{1}{\eta_t^F \sigma_s} \left[\frac{(\mu_s - r_f)}{\sigma_s} + \Gamma_t \right]. \tag{103}$$

Let $f(\omega, v) := (\mu_s - r_f) a_t v_a + \omega (a_t \sigma_s)^2 v_{aa}$. The optimal policies satisfy:

$$f(\omega_F^*, v^F) \approx -\frac{\Gamma_t}{\eta_t \sigma_s} > 0 \approx f(\omega_N^*, v^N). \tag{104}$$

The household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Then

$$\omega_F^* < \omega_N^*. \tag{105}$$

(ii) We want to find the condition under which their welfare is equivalent, i.e., $v^F(a_t, y_t) = v^N(a_t, y_t)$, which implies $v_a^F = v_a^N = v_a$, $v_{aa}^F = v_{aa}^N = v_{aa}$, and $v_{ay}^F = v_{ay}^N = v_{ay}$. For welfare equivalence, we need the complete HJB equations to be equal. This requires:

$$\max_{\omega} \mathcal{H}_F(\omega) - \max_{\omega} \mathcal{H}_N(\omega) = 0, \tag{106}$$

where the Hamiltonians are defined as:

$$\mathcal{H}_F \equiv \max_{\omega} \left\{ \omega_t (\mu_F - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2} (\omega_t a_t \sigma_F)^2 \frac{\partial^2 v}{\partial a^2} + \omega_t a_t \sigma_s \Gamma_t v_a \right\} \tag{107}$$

and

$$\mathcal{H}_N \equiv \max_{\omega} \left\{ \omega_t (\mu_N - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2} (\omega_t a_t \sigma_N)^2 \frac{\partial^2 v}{\partial a^2} \right\}. \tag{108}$$

Taking the optimal portfolio shares ω_N^* and ω_F^* into these Hamiltonians, we have:

$$\sup_{\omega} \mathcal{H}_N = \frac{1}{2} \mathcal{S}_N^2 \frac{v_a^2}{-v_{aa}} \tag{109}$$

$$\sup_{\omega} \mathcal{H}_F = \frac{1}{2} \left(\mathcal{S}_F + \Gamma_t \right)^2 \frac{v_a^2}{-v_{aa}}, \tag{110}$$

where $\mathcal{S}$ is portfolio Sharpe ratio $\mathcal{S} = \frac{\mu - r_f}{\sigma}$.

Eliminate identical terms of both sides, we have:

$$\mathcal{S}_N = \mathcal{S}_F + \Gamma_t. \tag{111}$$

Then

$$\mathcal{S}_F - \mathcal{S}_N = -\Gamma_t > 0. \tag{112}$$

□

D.2 Proof of Corollary 3

Proof. (i) The portfolio P's return characteristics are expected excess return μ_P and volatility:

$$\sigma_P = \sqrt{\beta_P^2 \sigma_s^2 + \tilde{\sigma}_P^2}. \tag{113}$$

Since $\sigma_P = \sigma_s$ and $\tilde{\sigma}_P > 0$, we have $\beta_P < 1$.

The household invests in the portfolio P, then its HJB equation is:

$$\begin{aligned} \rho v^P = \max_{c,\omega_P} \Big\{ & u(c) + \left[(\omega_P(\mu_P - r_f) + r_f)\, a_t + y_t - c_t \right] \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} \\ & + \frac{1}{2}(\omega_P a_t \sigma_P)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_P a_t \sigma_s \beta_P \Gamma_t v_a \Big\}. \end{aligned} \tag{114}$$

Following the proof D.1, we can similarly obtain:

$$\omega_P^* < \omega_N^*. \tag{115}$$

For portfolio P, the complete perturbation term is:

$$\mathcal{D}_P(v) = (1 - \beta_P)\omega_P a_t \sigma_s \Gamma_t v_a. \tag{116}$$

The partial derivative of the perturbation term with respect to ω_P is:

$$\frac{\partial \mathcal{D}_P(v)}{\partial \omega_P} = (1 - \beta_P) a_t \sigma_s \Gamma_t v_a. \tag{117}$$

The FOC for portfolio P with respect to ω_P is:

$$(\mu_P - r_f) a_t v_a^P + \omega_P^* a_t^2 \sigma_P^2 v_{aa}^P + (1 - \beta_P) a_t \sigma_s \Gamma_t v_a = 0. \tag{118}$$

With $\eta_P = -a_t \frac{v_a}{v_{aa}}$, the FOC for portfolio P with respect to ω is:

$$\omega_P^* = \frac{1}{\eta_P \sigma_s} \left[\frac{(\mu_P - r_f)}{\sigma_s} + (1 - \beta_P)\Gamma_t \right]. \tag{119}$$

Let $f(\omega, v) := (\mu_s - r_f)a_t v_a + \omega(a_t\sigma_s)^2 v_{aa} + a_t\sigma_s\Gamma_t v_a$. The optimal policies satisfy:

$$f(\omega_P^*, v^P) \approx -\frac{\beta_P\Gamma_t}{\eta_P\sigma_s}(a_t\sigma_s)^2 v_{aa} < 0 \approx f(\omega_F^*, v^F). \tag{120}$$

The household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Then

$$\omega_N^* > \omega_P^* > \omega_F^*. \tag{121}$$

(ii) We want to find the condition under which the welfare of the household is equivalent, i.e., $v^P(a_t, y_t) = v^F(a_t, y_t)$, which implies $v_a^P = v_a^F = v_a$, $v_{aa}^P = v_{aa}^F = v_{aa}$, and $v_{ay}^P = v_{ay}^F = v_{ay}$. For welfare equivalence, we need the complete HJB equations to be equal. This requires:

$$\max_\omega \mathcal{H}_P(\omega, v^P) = \max_\omega \mathcal{H}_F(\omega, v^F), \tag{122}$$

where $\mathcal{H}_P$ and $\mathcal{H}_F$ are the Hamiltonians of portfolio P and F, respectively:

$$\mathcal{H}_P \equiv \max_\omega \left\{\omega_t(\mu_P - r_f)a_t\frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t\sigma_P)^2\frac{\partial^2 v}{\partial a^2} + \omega_t a_t\sigma_s\beta_P\Gamma_t v_a\right\} \tag{123}$$

and

$$\mathcal{H}_F \equiv \max_\omega \left\{\omega_t(\mu_F - r_f)a_t\frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t\sigma_F)^2\frac{\partial^2 v}{\partial a^2} + \omega_t a_t\sigma_s\Gamma_t v_a\right\}. \tag{124}$$

Taking the optimal portfolio shares ω_P^* and ω_F^* into these Hamiltonians, we have:

$$\sup_\omega \mathcal{H}_P = \frac{1}{2}(\mathcal{S}_P + \beta_P\Gamma_t)^2\frac{v_a^2}{-v_{aa}} \tag{125}$$

$$\sup_\omega \mathcal{H}_F = \frac{1}{2}(\mathcal{S}_F + \Gamma_t)^2\frac{v_a^2}{-v_{aa}} \tag{126}$$

Eliminate identical terms of both sides, we have:

$$\mathcal{S}_P + \beta_P\Gamma_t = \mathcal{S}_F + \Gamma_t. \tag{127}$$

Then

$$\mathcal{S}_F - \mathcal{S}_P = (\beta_P - 1)\Gamma_t. \tag{128}$$

□

E Proof of Analysis under Homogeneity

E.1 Proof of Proposition 5

Proof. We let the value function $v(a, y)$ be homogeneous with respect to the household's wealth and income:

$$v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x), \tag{129}$$

where $h(x)$ is an unknown function that is only related to the wealth-income ratio $x_t \equiv a_t/y_t$.

Lemma 5. *If the value function is homogeneous, the effective relative risk aversion $\eta(a_t)$ is equal to a distorted risk aversion function $\tilde{\eta}(x_t)$ determined only by the wealth-income ratio x:*

$$\eta(a_t) = \tilde{\eta}(x_t) \equiv -x_t \frac{h''(x)}{h'(x)}. \tag{130}$$

Proof. Our goal is to find an expression for $\tilde{\eta}(x)$ that is directly related to economic decision variables (such as consumption) under the assumption that the value function is homogeneous $v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x)$.

We start from the FOC for optimal household consumption, that is, marginal utility equals marginal value of wealth:

$$u'(c^*) = \frac{\partial v}{\partial a}. \tag{131}$$

For the CRRA utility function, we have $u'(c) = c^{-\gamma}$ and

$$\frac{\partial v}{\partial a} = \frac{\partial}{\partial a}\left(\frac{y^{1-\gamma}}{1-\gamma} h(x)\right) = \frac{y^{1-\gamma}}{1-\gamma} h'(x) \frac{\partial x}{\partial a} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{132}$$

Substituting both into FOC:

$$(c^*)^{-\gamma} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{133}$$

Now, we can solve the expression for $h'(x)$ from it:

$$h'(x) = (1-\gamma)\frac{(c^*)^{-\gamma}}{y^{-\gamma}} = (1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}. \tag{134}$$

The economic meaning of this expression is that the first-order derivative of the function $h(x)$ (reflecting the sensitivity of the value function to changes in the wealth-income ratio) is determined by the optimal consumption-income ratio c^*/y.

To find $h''(x)$, we take the derivative of $h'(x)$. The chain rule is needed here because c^*

itself is a function of a, which is also a function of x and y. To simplify the expression, we define the consumption-income ratio function $C(x) \equiv c^*(x)/y$.

$$h''(x) = \frac{d}{dx}\left[(1-\gamma)C(x)^{-\gamma}\right]. \tag{135}$$

Applying the chain rule:

$$\begin{aligned} h''(x) =& (1-\gamma)\cdot(-\gamma)\cdot C(x)^{-\gamma-1}\cdot\frac{dC(x)}{dx} \\ =& -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1} C'(x), \end{aligned} \tag{136}$$

where $C'(x) = \frac{d(c^*/y)}{dx}$.

We further analyze $C'(x)$:

$$C'(x) = \frac{d(c^*/y)}{d(a/y)} = \frac{d(c^*/y)}{da}\cdot\frac{da}{d(a/y)} = \left(\frac{1}{y}\frac{\partial c^*}{\partial a}\right)\cdot y = \frac{\partial c^*}{\partial a}. \tag{137}$$

It can be seen that $C'(x)$ is the marginal propensity to consume of the household.

Substituting into the above equation, we get the final expression for $h''(x)$:

$$h''(x) = -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}. \tag{138}$$

Now that we have expressions for $h'(x)$ and $h''(x)$, we can substitute them into the definition of $\tilde{\eta}(x)$:

$$\tilde{\eta}(x) \equiv -x\frac{h''(x)}{h'(x)}. \tag{139}$$

Substituting into:

$$\tilde{\eta}(x) = -x\frac{-\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}}{(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}}. \tag{140}$$

After simplification, we get:

$$\begin{aligned} \tilde{\eta}(x) =& x\cdot\gamma\cdot\left(\frac{y}{c^*}\right)\cdot\frac{\partial c^*}{\partial a} \\ =& \frac{a}{y}\cdot\gamma\cdot\frac{y}{c^*}\cdot\frac{\partial c^*}{\partial a}. \\ =& \gamma\cdot\frac{a}{c^*}\cdot\frac{\partial c^*}{\partial a}. \end{aligned} \tag{141}$$

Finally, we get:

$$\tilde{\eta}(x) = \eta(a). \tag{142}$$

□

The optimal portfolio choice (17) can be decomposed into three difference economic motives:

$$\omega_t^* = -\frac{1}{a_t}\frac{v_a}{v_{aa}}\frac{\mu_s - r_f}{\sigma_s^2} + \left(-\frac{v_{ay}}{a_t v_{aa}}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y + \frac{1}{a_t\sigma_s}\left(1 + \frac{v_a}{a_t v_{aa}}\right). \tag{143}$$

By Lemma 5, we have:

$$\text{Speculative Demand} = \frac{1}{\eta(a_t)}\frac{\mu_s - r_f}{\sigma_s^2} = \frac{1}{\tilde{\eta}(x_t)}\frac{\mu_s - r_f}{\sigma_s^2}. \tag{144}$$

We derive the expressions for v_{ay} and v_{aa}:

$$\begin{aligned} v_{ay} \equiv & \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial}{\partial y}\left(\frac{y^{-\gamma}}{1-\gamma}h'(x)\right) \\ = & \frac{-\gamma y^{-(\gamma+1)}}{1-\gamma}h'(x) + \frac{y^{-\gamma}}{1-\gamma}h''(x)\left(-\frac{a}{y^2}\right) \\ = & \frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right) \end{aligned} \tag{145}$$

and

$$v_{aa} \equiv \frac{\partial^2 v}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{y^{-\gamma}}{1-\gamma}h'(x)\right) = \frac{y^{-\gamma}}{1-\gamma}h''(x)\frac{\partial x}{\partial a} = \frac{y^{-(\gamma+1)}}{1-\gamma}h''(x). \tag{146}$$

Substituting these into the wage-offsetting effect expression:

$$\begin{aligned} \text{Wage-offsetting Effect} = & \left(-\frac{\frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right)}{a_t \frac{y^{-(\gamma+1)}}{1-\gamma}h''(x)}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \frac{\gamma h'(x) + x h''(x)}{a_t h''(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \left(\frac{\gamma}{y_t}\frac{h'(x)}{x_t h''(x)} + \frac{1}{y_t}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \left(-\frac{\gamma}{y_t\tilde{\eta}(x)} + \frac{1}{y_t}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}y \\ = & \frac{\tilde{\eta}(x) - \gamma}{\tilde{\eta}(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s}. \end{aligned} \tag{147}$$

We can simply use the result from Part (i) of this lemma.

$$\begin{aligned}\text{AEP Effect} =& \frac{1}{a_t\sigma_s}\left(1-\frac{1}{\eta(a_t)}\right)\\ =& \frac{1}{x_t y_t \sigma_s}\frac{\tilde{\eta}(x)-1}{\tilde{\eta}(x)}.\end{aligned} \tag{148}$$

We have

$$\tilde{\omega}^* = \frac{1}{\tilde{\eta}(x)}\frac{\mu_s - r_f}{\sigma_s^2} + \frac{1}{\tilde{\eta}(x)}\frac{\rho_{sy}\sigma_y}{\sigma_s y_t}(\tilde{\eta}(x)-\gamma) + \frac{1}{\tilde{\eta}(x)}\frac{1}{x_t y_t \sigma_s}(\tilde{\eta}(x)-1). \tag{149}$$

A necessary condition for homogeneity is that the optimal control is only related to x. Then, we require

$$\text{AEP Effect} = 0. \tag{150}$$

□