Targeted Advertising Platforms: Data Sharing and Customer Poaching*

Klajdi Hoxha†

November 3, 2025

Abstract

E-commerce platforms are rolling out ambitious *targeted advertising* initiatives that rely on merchants *sharing customer data* with each other via the platform. Yet current platform designs fail to address participating merchants' concerns about *customer poaching*. This paper proposes a model of designing targeted advertising platforms that incentivizes merchants to voluntarily share customer data despite poaching concerns. I characterize the optimal mechanism that maximizes a weighted sum of platform's revenues, customer engagement and merchants' surplus. In sufficiently large platforms, the optimal mechanism can be implemented through the design of three markets: *i*) **selling market**, where merchants can sell all their data at a posted price p, *ii*) **exchange market**, where merchants share all their data in exchange for high click-through rate (CTR) ads, and *iii*) **buying market**, where high-value merchants buy high CTR ads at the full price. The model is broad in scope with applications in other market design settings like the *greenhouse gas credit markets* and *reallocating public resources*, and points toward new directions in designing *combinatorial exchange markets*.

Keywords: Exchange market, click-through rate (CTR), customer data, bundling, mechanism design, platform, poaching, scoring rule, targeted advertising.

*I am grateful to Michael Ostrovsky, Andrzej Skrzypacz and Weijie Zhong for their guidance and support. I am thankful to Bing Liu for numerous discussions during the early stages of the paper, as well as Ellen Muir, Spencer Pantoja, Ilya Segal, Eric Tang, Frank Yang and seminar participants at Stanford University for helpful comments and suggestions. I owe special thanks to Varanya Chaubey for exceptional editing advice.

The author declares that he has no relevant or material financial interests that relate to the research described in this paper.

†Graduate School of Business, Stanford University. Email: hklajdi@stanford.edu.

1 Introduction

E-commerce platforms are rolling out ambitious targeted advertising initiatives that rely on merchants sharing customer data with each other via the platform. These initiatives are in response to recent privacy policies—such as California's CCPA, European Union's GDPR and Apple's ATT—limiting third-party cookie tracking, a cornerstone of targeted online advertising.[1] With Safari, Firefox and Edge having phased out third-party cookies, Chrome is now the only major browser still relying on them as it seeks alternative solutions.[2] One promising solution involves e-commerce platforms pooling customer data from merchants' own datasets. Doing so helps the platform build a larger and more targeted dataset, thereby delivering targeted ads to customers from their preferred merchants.

Yet current platform designs fail to address participating merchants' concerns about customer poaching. These designs give participating merchants the opportunity to run advertising campaigns on the platform that target every customer in the shared dataset. To determine which merchants get to target which customers, such platforms commonly allocate targeted ads by running ad auctions based on bid amounts and ad quality scores, where the latter measures the probability of a click. If competitors with ads at least of similar quality bid more aggressively, merchants' ad campaigns may reach fewer customers than before joining the platform.

Our running example that illustrates this setting is Shopify, an e-commerce platform that powers the (online) stores of millions of merchants. In 2022, Shopify pioneered these initiatives by launching Shopify Audiences, a tool designed to pool customer data from participating merchants to improve their targeted advertising. However, a frequently asked question on Shopify's official page reveals a growing concern: can merchants "exclude competitors from using their data"?[3] Shopify's ambiguous response—"...the data you contribute will be used to benefit other participating merchants"—only validates the customer poaching concerns.

This paper proposes a model of designing targeted advertising platforms and solves for the optimal platform design. Specifically, a designer designs a platform where a finite number of merchants participate by contributing exclusive customer datasets and compete for targeted advertisements designed using the *aggregated dataset*. Each customer is represented by a profile of *click-through rates* (CTRs), one for each merchant's ad. Merchants

[1] 96% of iOS users have opted out of app tracking and 80% of advertisers rely on third-party cookies according to https://www.wordstream.com/blog/ws/2021/10/26/online-advertising-statistics and https://www.imore.com/96-iphone-users-have-opted-out-app-tracking-ios-145-launched

[2] https://privacysandbox.com/news/privacy-sandbox-update/

[3] https://help.shopify.com/en/manual/promoting-marketing/shopify-audiences/faq

earn a private profit-margin per click (type).[4] To elicit this information truthfully, the designer assigns a participation fee (transfer)—possibly negative—and credibly commits to some advertising mechanism such that all merchants are willing to participate. The designer maximizes a weighted sum of platform revenue and social value.

The novelty in the model stems from the heterogeneity in customers' CTRs for each merchant's product and the exclusivity of customer datasets initially held by individual merchants. This modeling choice is motivated by the existing inefficiencies of merchants' individual targeting efforts outside the platform using their exclusive customer datasets. Such decentralized efforts yield both smaller audience reach and reduced targeting precision due to data sparsity (Lambrecht and Tucker (2013), Bleier and Eisenbeiss (2015)). On the platform, however, participating merchants compete for the ability to target *any* customer within the shared dataset. Given a large aggregated dataset and heterogeneity in customers CTRs, the platform can potentially design targeted ads that deliver higher expected clicks for most—if not all—merchants.

Using a mechanism design framework, I solve for the optimal mechanism that improves upon individual targeting efforts while ensuring merchants' willingness to share their customer data. I show that the optimal advertising (direct) mechanism allocates targeted ads in bundles according to a non-decreasing *merchant-specific scoring rule* that is a function of each merchant's (type). Specifically, each customer in the dataset is targeted by the merchant with the highest *quality score* defined as the *weighted* merchant-specific score by the customer-specific CTR. In the case of ties, the designer constructs a customer-specific tie-breaking rule that ensures participation of every type. Due to the differences in CTRs and merchants' types, some poaching may still occur in the optimum. To alleviate poaching concerns, adversely affected merchants with negative *net* expected clicks receive positive monetary transfers to guarantee their participation and often give strictly surplus.

By allocating targeted ads through a single merchant-specific scoring rule, the designer *pools* merchants' misreporting incentives for different ads—hence reducing information rents. Consequently, a single bid determines each merchant's entire on-platform advertising campaigns: which of her own customers she continues to target and which new customers from other merchants she gains access to. On the platform, merchants anticipate retaining advertising access to only their high CTR customers, while selling the rest to the designer. Simultaneously, they expect to buy high CTR customers from competing merchants' datasets. This creates *countervailing incentives*: merchants have incentives to overestimate data value when selling and underestimate it when buying. The designer lever-

[4] I assume for simplicity that customers always convert (i.e. buy the product) if they click the ad.

ages these countervailing incentives by constructing a single scoring rule—based on each merchant's initial dataset—to optimally design menus of *data bundles* and *non-itemized transfers* that essentially describe on-platform advertising campaigns.

Designing data bundles often involves resolving for each customer tie-breaks among merchants with identical quality scores, making this a multidimensional mechanism design problem, hence computationally untractable. In the second part of the paper, I extend the model by allowing a continuum of CTR realizations, in which case ties occur with probability zero. This gains tractability and I provide an explicit system of non-linear equations which solution to characterizes merchant-specific scoring rules.

I then consider a stylized setting under additional i.i.d. and symmetry assumptions and solve for the optimal mechanism when the number of merchants grows large. In a large market, the optimal design simplifies nicely which can be ex-post implementable through the design of three markets: *i*) selling market, where merchants can sell all their data at a posted price p, *ii*) exchange market, where merchants share all their data in exchange for high click-through rate (CTR) ads, and *iii*) buying market, where high-value merchants buy high CTR at the full price. The posted price is set such that equating the marginal cost of procuring data from the selling and the exchange markets. Almost every merchant with types higher than posted price p participate in the exchange market. Intuitively, the designer finds it cheaper to exchange data with high types than to elicit their private information—provided there's a sufficiently heterogeneous data pool to choose from, which the large market assumption guarantees. Additionally, the large market ensures that any residual data can be sold at the highest price in the buying market.

The exchange market design is novel compared to existing mechanisms for combinatorial double auctions and, more generally, combinatorial exchange markets. Double auctions are used to mediate trades between multiple buyers and sellers, where in equilibrium prices emerge from the interaction of submitted bids and asks; as such, bid-ask price mechanisms aggregate dispersed information across participants and serve as micro foundation for price formation in large markets. Combinatorial exchange markets generalize double-auction environments to settings where agents can buy or sell, or do both, bundles of heterogeneous goods rather than individual items. A prominent example includes the FCC spectrum where carriers trade bundles of licenses across regions and bands. Mechanisms like the FCC Incentive Auction (IA) (Ausubel et al. (2012)) and the Combinatorial Clock Exchange (CCE) (Hoffman and Menon (2010)) are two-sided, multi-round spectrum market mechanisms that use iterative price adjustments and algorithmic clearing to align supply with demand. The insights from our model suggest that it may be worth analyzing new

mechanisms that introduce spectrum exchanges across regions and bands. I conclude the paper by describing the model's scope and its applications in other market design settings like greenhouse gas credit market and reallocation of public resources.

Related Literature. Theoretically, the model generalizes the models of bilateral trading (Myerson and Satterthwaite (1983)) and partnership dissolution (Cramton et al. (1987), Loertscher and Wasser (2019)) by trading multiple *heterogeneous* goods. It also relates to the bundling literature since the optimal design trades data in bundles. Yang (2023) studies nested bundling in a framework with one-dimensional heterogeneity where preferences are non-additive. In my model I have one-dimensional heterogeneity and preferences are non-additive. In canonical multidimensional mechanism design problems with one-dimensional heterogeneity and additive preferences bundling strategies do not improve seller's objective. However, since the model generates different countervailing incentives for each good, optimal bundling strategies arise that "pool" these incentives.

On the practical side, the paper contributes to the vast literature on digital platforms. Bergemann and Bonatti (2024) and Bergemann et al. (2024) study digital platforms where advertisers can reach consumers on and off the platform with a focus on optimal ad pricing and efficient product matches. Some studies consider information design and pricing when the database is controlled by the platform (see Bergemann and Bonatti (2015), Elliott et al. (2022)). See also Bergemann et al. (2022), Galperti et al. (2024) and Galperti and Perego (2022) for settings when consumers control how much data to provide to these intermediaries and its implications. Lastly, see Farboodi and Veldkamp (2023) for a comprehensive review on data and markets.

Roadmap. The remainder of the paper proceeds as follows. Section 2 presents the model. Section 3 presents some illustrative examples. Section 4 presents the optimal mechanism in the finite case, sketches the main proof and characterizes the mechanism in a stylized setting. Section 5 extends the model in the continuum case and presents the optimal design in large platforms. Section 6 revisits model assumptions and presents some applications. Section 7 concludes. Omitted proofs are given in the appendices.

2 Model

A designer ($i = 0$) designs a platform that targets a unit mass of customers in the market through ads using exclusive customer datasets provided by each merchant $i \in \mathcal{N} = \{1, \dots, N\}$ for some finite $N \geq 2$.

Each merchant $i \in \mathcal{N}$ can supply unlimited units of their product at a private profit

margin (type) $\theta_i \in \Theta_i = [0,1]$. Customers have unit-demand and discover products exclusively through ads either on the platform or outside the platform via merchants' independent advertising campaigns. A customer is represented by a vector profile $\omega = (\omega_i)_{i\in\mathcal{N}}$ where $\omega_i \in \Omega_i \subseteq [0,1]$ denotes the click-through-rate (CTR) on merchant i's ad. That is, merchant i's expected profits from targeting customer ω is given by $\theta_i\omega_i$.[5]

Assume for now that $\Omega := \times_{i\in\mathcal{N}}\Omega_i$ is a finite set and let the *discrete* distribution $\alpha \in \Delta(\Omega)$ denote the *aggregate dataset* (henceforth, dataset) in the market.[6] Initially, every customer belongs exclusively to some merchants i's *individual datasets* $\alpha_i = \{\alpha_i^\omega\}_{\omega\in\Omega}$ such that

$$\sum_{i\in\mathcal{N}} \alpha_i^\omega = \alpha(\omega), \quad \forall \omega \in \Omega.$$

That is, for every $i \in \mathcal{N}$ and $\omega \in \Omega$, α_i^ω denotes the mass of customers ω that merchant i can exclusively target outside the platform.

Distributional Assumptions. Assume types θ_i are drawn from independent Borel probability measures $dF_i \in \Delta(\Theta_i)$ with cdf. F_i that admits continuous and strictly positive densities f_i and satisfies *regularity*: the virtual value and virtual cost functions defined by $\phi_i^B(\theta_i) = \theta_i - (1 - F_i(\theta_i))/f_i(\theta_i)$ and $\phi_i^S(\theta_i) = \theta_i + F_i(\theta_i)/f_i(\theta_i)$, respectively, are strictly increasing.

Informational Assumptions. The primitives F_i and α are common knowledge. The latter implies that both the designer and merchant i agree on the CTR values ω_i of every customer ω belonging to their datasets α_i.[7] I discuss in Section 6 the substance of the later assumption by connecting it to the literature on mechanism design with limited commitment. For the first I point the reader to the vast literature on 'detail-free' mechanism design, also known as the Wilson doctrine.

2.1 Platform Design

I describe how the platform operates by specifying the policy on data sharing and the design of targeted advertising campaigns.

[5]For a micro-foundation of the merchant's profit function, consider each merchant having a private constant marginal cost of c_i of selling a unit of the product. Suppose customers can be separated into two groups who are either interested in buying the product or not. Now assume the interested group has willingness to pay $v_i \sim G_i$ for some distribution G_i. Then private profit margin θ_i is given by calculating profits $\theta_i(c_i) = (p(c_i) - c_i)G_i(p(c_i))$ from the optimal posted price $p(c_i)$.

[6]I extend the model to a continuum of CTR realizations in Section 5.

[7]For example, if merchant i gives the designer access to data about their customers' past shopping behavior, the assumption implies that both parties agree on (predicted) CTRs ω_i.

Data Sharing. The designer requires participating merchants to give up all targeting rights to their exclusive customers by sharing their individual datasets with the designer. Assuming the designer can always design targeted advertising campaigns that give back same access to every merchant at a negligible cost, it follows readily that it is without loss of generality to restrict to settings where every merchant participates in the platform.[8]

Targeted Advertising Campaigns. Let $(x,t) = (x_i, t_i)_{i\in\mathcal{N}}$ denote a *targeted advertising campaign* (henceforth, campaign) consisting of i) *targeted ads* $x_i = (x_i^\omega)_{\omega\in\Omega}$ where $x_i^\omega \in [0,1]$ is the mass of customers ω that discover merchant i's product via the platform such that

$$\sum_{i\in\mathcal{N}} x_i^\omega \leq \alpha(\omega), \quad \forall \omega \in \Omega, \tag{Fsb}$$

and ii) (monetary) *transfers* $t_i \in \mathbb{R}$ to participate in the platform. Note that (Fsb) restricts the designer to displaying only one (winning) ad to every customer. Since customers have unit-demand and prefer products with higher ω_i, (Fsb) is without loss of generality. Moreover, transfers t_i can be negative depending on net expected clicks from targeted ads x_i and outside option α_i.

It is common practice for eCommerce platforms to run advertising campaigns for participating merchants by letting them choose an advertising budget (t_i) that will determine their individual ad campaign performance (x_i). Here, the designer faces an additional layer of complexity: choosing advertising campaigns that guarantee each merchant weakly prefers platform participation to running independent targeted ads.

2.2 Designer's Problem.

The designer's problem is to design a campaign (x,t) that maximizes a weighted sum of platform revenue, customer engagement and merchants' surplus subject to participation constraints.

Platform Revenue. The platforms' revenue is the sum of transfers across all merchants:

$$R(t) := \sum_{i\in\mathcal{N}} t_i.$$

Designing such platforms can be costly for the designer so I consider optimal designs in which the designer cares about platform revenue. I treat the costs of building these platforms as sunk, hence I don't explicitly introduce them in the model.

Customer Engagement. Customer engagement is defined as the improved targeting precision, defined as the net increase in expected clicks across all participating merchants'

[8] I formally prove this claim in an online supplementary appendix.

ads. Specifically, customer engagement for given target ads x is defined by

$$W(x) := \sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}} \omega_i(x_i^\omega - \alpha_i^\omega).$$

I interpret $W(x)$ as a measure of customer (consumer) welfare in this setting. Since product prices are not part of the model, I interpret CTR ω_i as a measure of customer ω satisfaction with merchant i's product.

There are many practical reasons beyond the model's scope for maximizing customer engagement. Poor targeting leads to ad fatigue and user attrition, while precise targeting enhances user experience and sustained platform engagement. Furthermore, higher engagement improves advertiser relationships and platform reputation, and these metrics frequently determine the platform's success.

Merchants' Surplus. For given targeted ads x_i and type θ_i, merchant i's surplus is the net expected profits given by

$$V_i(\theta_i, x_i) = \theta_i \sum_{\omega\in\Omega} \omega_i(x_i^\omega - \alpha_i^\omega).$$

That is, $V_i(\theta_i, x_i)$ are the net expected profits made from net expected clicks between ad campaigns on the platform and outside of it. The designer puts a nonnegative weight on maximizing total merchants' surplus $V(\theta, x)$ given by

$$V(\theta, x) := \sum_{i\in\mathcal{N}} V_i(\theta_i, x_i).$$

Satisfied merchants may participate more often, share new data in future interactions and increase their advertising expenditures—providing strong practical reasons to care about maximizing their surplus.

Designer's Objective and Participation Constraints. Now that the necessary terminology is in place, I formally state the designer's objective. For a given type profile $\theta \in \Theta$ and campaign (x, t), the designer's objective function is given by

$$\psi_\eta(\theta, x, t) := \eta_v V(\theta, x) + \eta_w W(x) + \eta_r R(t), \tag{Obj$_\theta$}$$

for some non-negative *welfare weight* $\eta = (\eta_v, \eta_w, \eta_r)$ such that $\eta_v + \eta_w + \eta_r = 1$ and $\eta_r > 0$. Solving for the case $\eta_r = 0$ would require additional restrictions on the choice of platform design, which is not the scope of this paper. It is also not generally true that the optimal mechanism converges as $\eta_r \to 0$; see Section 5 for an illustration when it is not the case.

Since the designer is restricted to feasible targeted ads x, transfers t must ensure that platform participation is *individually rational* (IR) for every merchant. Letting $U_i(\theta_i, x_i, t_i) := V_i(\theta_i, x_i) - t_i$ be the *expected payoff* for type θ_i receiving targeted ads x_i and transfers t_i,

we say that a campaign (x,t) is IR at type profile $\theta \in \Theta$ if

$$U_i(\theta_i, x_i, t_i) \geq 0, \quad \forall i \in \mathcal{N}. \tag{IR$_\theta$}$$

Therefore, ideally the designer would like to design at each type profile $\theta \in \Theta$ an IR campaign $(x^{FB}(\theta), t^{FB}(\theta))$ that maximizes objective function $\psi_\eta(\theta, x, t)$. However, we know this is generally not possible because the designer faces *incentive compatibility* (IC) constraints: merchants must truthfully report their private information (θ) to implement $(x^{FB}(\theta), t^{FB}(\theta))$. I therefore appeal to mechanism design theory to formally address these constraints and solve for the second-best platform design.

2.3 Mechanism Design

We invoke Revelation Principle and set up the designer's problem by restricting to the space of IR and IC *direct mechanisms*.

Direct Mechanisms. By standard arguments, the Revelation Principle implies that the designer can restrict, without loss of generality, to direct mechanisms (henceforth, *mechanisms*) which are measurable maps given by *targeted ads rule* (henceforth, ads rule) $x = (x_i)_{i\in\mathcal{N}}$ such that $x_i = (x_i^\omega)_{\omega\in\Omega}$ and $x_i^\omega : \Theta \to \mathbb{R}_+$, and transfer rule $t = (t_i)_{i\in\mathcal{N}}$ such that $t_i : \Theta \to \mathbb{R}$, that satisfy *interim* IC and *interim* IR constraints

$$U_i(\theta_i, X_i(\theta_i), T_i(\theta_i)) \geq U_i(\theta_i, X_i(\tilde{\theta}_i), T_i(\tilde{\theta}_i)), \qquad \forall \theta_i, \tilde{\theta}_i \in \Theta_i, \tag{IC}$$

$$U_i(\theta_i, X_i(\theta_i), T_i(\theta_i)) \geq 0, \qquad \forall \theta_i \in \Theta_i, \tag{IR}$$

and *ex-post* feasibility constraints

$$\sum_{i\in\mathcal{N}} x_i^\omega(\theta) \leq \alpha(\omega), \quad \forall \omega \in \Omega, \forall \theta \in \Theta, \tag{Fsb$_\theta$}$$

where $X_i = (X_i^\omega)_{\omega\in\Omega}$ is such that $X_i^\omega(\theta_i) := \mathbb{E}_{\theta_{-i}}[x_i^\omega(\theta_i, \theta_{-i})]$ and $T_i(\theta_i) := \mathbb{E}_{\theta_{-i}}[t(\theta_i, \theta_{-i})]$ for every $\theta_i \in \Theta_i$ and $i \in \mathcal{N}$. I adopt the Bayes-Nash solution for the analysis of IC constraints, which is less restrictive and allows for richer and more flexible mechanisms. In many real-world mechanisms that platforms run, like standard auction formats, agents act strategically conditional on beliefs about others' private information. The more restrictive assumption is requiring only the mechanism ti satisfy only interim IR constraints: merchants are guaranteed a nonnegative expected payoff, but not necessarily ex-post. Our main result for the optimal design in large markets (see Section 5) will turn out to be both ex-post IC and IR.

Let $\mathcal{M}^\alpha(\Theta, \times dF_i)$ denote the space of mechanisms. Note that $\mathcal{M}^\alpha(\Theta, \times dF_i) \neq \emptyset$ since the mechanism with $x_i^\omega \equiv \alpha_i^\omega$ for every $i \in \mathcal{N}$, $\omega \in \Omega$, and $t \equiv 0$ (henceforth, $(x,t) \equiv (\alpha, 0)$)

satisfies (IC), (IR) and (Fsb$_\theta$).

Designer's Problem. For given mechanism (x, t), letting $\mathcal{W}(x) := \mathbb{E}[W(x(\theta))]$, $\mathcal{R}(t) := \mathbb{E}[R(t(\theta))]$ and $\mathcal{V}(x) := \mathbb{E}[V(\theta, x(\theta))]$, the designer's objective is given by

$$\psi_\eta(x, t) := \eta_v \mathcal{V}(x) + \eta_w \mathcal{W}(x) + \eta_r \mathcal{R}(t). \tag{Obj}$$

The designer's problem (P) is therefore given by

$$\Pi(\alpha, \eta) := \sup_{(x,t)\in\mathcal{M}^\alpha(\Theta,\times dF_i)} \psi_\eta(x, t), \tag{P}$$

where I refer to $\Pi(\alpha, \eta)$ as the *value of platform design* with welfare weight η and dataset α. Note that the value $0 \leq \Pi(\alpha, \eta) < \infty$ since the elements of $\mathcal{M}^\alpha(\Theta, \times dF_i)$ are bounded measurable functions $(x, t \in \mathcal{L}^\infty(\Theta, \times dF_i))$ and $(\alpha, 0) \in \mathcal{M}^\alpha(\Theta, \times dF_i)$.

3 Illustrative Examples

I illustrate the optimal mechanism for two merchants and a revenue-maximizing (i.e. $\eta_r = 1$) platform in settings for which the optimal mechanism is known from the literature, such as *monopoly pricing*, *bilateral trading* and *partnership dissolution*. I conclude the section by presenting an novel example that departs from traditional models and build intuition for the general characterization of the optimal mechanism in Section 4.

Let $i = 1, 2$ and $\theta_i \overset{iid}{\sim} \text{Uniform}[0, 1]$, and welfare weight $\eta_r = 1$. Suppose there are only two types of customers: *a*) *exclusive customers* with CTR $\omega_i = 1$ for some merchant $i = 1, 2$ and have no intent in purchasing from the other merchant $j \neq i$, i.e. $\omega_j = 0$; *b*) *inclusive customers* who see both merchants as perfect substitutes and have $\omega = (1, 1)$.

Monopoly Pricing. Suppose that all customers are exclusive with CTR $\omega_1 = 1$ and all are initially part of merchant 2's dataset, i.e. $\alpha_2^{(1,0)} = 1$. Then clearly, the optimal mechanism is a classical monopoly pricing problem: the designer procures the dataset at no cost and sells a unit mass of targeted ads to merchant 1 according to a posted price $p^* = 1/2$—thus only the types $\theta_1 \geq 1/2$ take the offer and run campaigns on the platform.

Bilateral Trade and Partnership Dissolution. Suppose that all customers are inclusive and consider first the case in which merchant 2 owns all the inclusive dataset, i.e. $\alpha_2^{(1,1)} = 1$. Then the problem becomes analogous to the profit-maximizing bilateral trading model via a mediator where merchant 1 is the *buyer* (B) and merchant 2 is the *seller* (S) as in Myerson and Satterthwaite (1983). They show (Theorem 4) that the optimal mechanism assigns scores equal to the virtual value $\phi^B_{\eta,1}(\theta_1) = 2\theta_1 - 1$ and virtual cost $\phi^S_{\eta,2}(\theta_2) = 2\theta_2$ and data trading between merchants takes if and only if $\phi^B_\eta(\theta_1) \geq \phi^S(\theta_2)$—or $\theta_1 - \theta_2 \geq 1/2$.

Now consider the case in which merchant $i = 1, 2$ holds an ownership share of $r_i \in (0, 1)$ in the inclusive dataset, i.e. $r_i := \alpha_i^{(1,1)}$ such that $r_1 + r_2 = 1$. The problem becomes analogous to dissolving partnerships (Cramton et al. (1987)) at a profit via a mediator (Loertscher and Wasser (2019)). Similarly, the optimal mechanism assigns full ownership the the merchant with the highest score $g_i(\theta_i)$, where g_i is derived by ironing the *virtual type function* $\phi_i(\theta_i, \theta_i')$[9] at some optimal *critical worst-off* type θ_i'—specifically, there exists some $z = (z_1(\theta_1'), z_2(\theta_2'))$ such that $g_i(\theta_i) = \phi_i^S(\theta_i)$ if $\phi_i^S(\theta_i) < z_i$, $g_i(\theta_i) = \phi_i^B(\theta_i)$ if $\phi_i^B(\theta_i) > z_i$ and $g_i(\theta_i) = z_i$ otherwise. For instance, when $r_i = 1/2$ then $z_i = 1/2$ implying trade takes place whenever $\theta_i > \theta_j$ unless $\theta_i \in [1/4, 3/4]$ (i.e. $g_i(\theta_i) = 1/2$) for each $i = 1, 2$. In the latter case, ties are broken uniformly. Note that the trade volume (and revenues) increase compared to bilateral trade (i.e. $r_i = 1$). This is due to the presence of countervailing incentives that weaken incentive constraints: a high (low) type is more likely to buy (sell) and thus has an incentive to under(over)-report, whereas the types in the "middle" have the weakest incentives to misreport since they retain their shares in expectation by selling/buying "half" of the time.

Departing from Traditional Models. Suppose now that at least one merchant owns data about both types of customers. In particular, let $\alpha(1, 1) \in (0, 1)$ be equally distributed across both merchants as above, i.e. $\alpha_1^{(1,1)} = \alpha_2^{(1,1)} = \alpha(1, 1)/2$. Let the remaining $1 - \alpha(1, 1)$ mass of customers be exclusive towards merchant 2 but which are initially owned by merchant 1, i.e. $\alpha_1^{(0,1)} = 1 - \alpha(1, 1)$. What's new in this setting is that "goods" to be traded are heterogeneous across merchants. In particular, merchant 2 can be both a seller *and* a buyer: in the optimal mechanism merchant 2 sometimes sells their $\alpha(1, 1)/2$ mass of inclusive customer but is always a potential buyer for the $1 - \alpha(1, 1)$ mass of exclusive customers.

A naive strategy for the designer is to design separate mechanisms to assign targeted ads for inclusive and exclusive customers as in the preceding analysis; see Figure 1 where $z^E = 0$ and $z^I = 1/2$ identify the tie-breaking regions for optimal ad design of exclusive and inclusive datasets, respectively. Moreover, the optimal expected transfers T_i^E and T_i^I can be computed as following: $T_1^E \equiv 0$ and $T_2^E(\theta_2) = (1 - \alpha(1, 1)) \times (1/2)$, and for each

[9] The virtual type function $\phi_i(\theta_i, \theta_i')$ at *critical type* θ_i' is defined by $\phi_i(\theta_i, \theta_i') = \phi_i^S(\theta_i)$ for all $\theta_i < \theta_i'$ and $\phi_i(\theta_i, \theta_i') = \phi_i^B(\theta_i)$ for all $\theta_i \geq \theta_i'$.

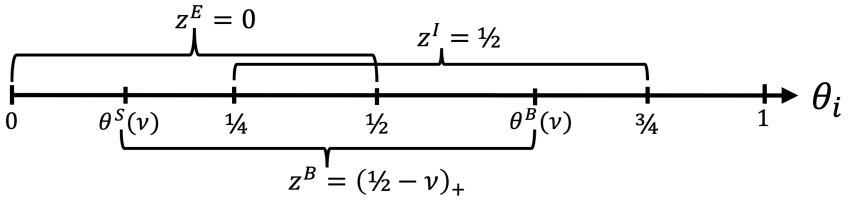


Figure 1: Optimal ad design separately (z^E and z^I) and in bundles (z^B).

$i = 1, 2$

$$T_i^I = \begin{cases} \frac{\alpha(1,1)}{2}\left(\theta_i^2 - \frac{1}{4}\left(1 - \frac{1}{4}\right)\right), & \text{if } \theta_i \in \left[0, \frac{1}{4}\right] \\ \frac{\alpha(1,1)}{2}\left(\theta_i^2 - \frac{3}{4}\left(1 - \frac{3}{4}\right)\right), & \text{if } \theta_i \in \left[\frac{3}{4}, 1\right] \\ 0, & \text{otherwise.} \end{cases}$$

Now, I show that the designer can in fact do *strictly* better by designing targeted ads in *bundles*. Specifically, the optimal mechanism assigns scores g_i that irons the virtual type $\phi_i(\theta_i, \theta_i')$ with ironing parameter $z^B = (1/2 - \nu)_+$ where $\nu := (1 - \alpha(1,1))/\alpha(1,1)$ and $z^B > 0$ if and only if $\alpha(1,1) > 2/3$. The corresponding tie-breaking region is thus given by

$$\left[\theta^S(\nu), \theta^B(\nu)\right] = \left[\left(\frac{1}{4} - \frac{\nu}{2}\right)_+, \max\left(\frac{1}{2}, \frac{3}{4} - \frac{\nu}{2}\right)\right].$$

The optimal design for targeted ad bundles is as follows: merchant 2 is always assigned the exclusive targeted ads whenever $g_2(\theta_i) > 0$—which holds a.e. if $z^B > 0$. On the other hand, the inclusive targeted ads are assigned to the merchant with the highest score $g_i(\theta_i)$ where ties are broken in favor of merchant 1 according to $p_1^{(1,1)} = \min(1, 1/2 + \nu)$. Moreover, the optimal *non-itemized* expected transfers from targeted ad bundles T_i^B can be written as a sum of the following *hypothetical* itemized expected transfers: $T_i^B \equiv \tilde{T}_i^E + \tilde{T}_i^I$ where $\tilde{T}_1^E \equiv 0$ and $\tilde{T}_2^E(\theta_2) = (1 - \alpha(1,1))\theta^S(\nu)$ for $\theta_2 < \theta^S(\nu)$, $\tilde{T}_2^E(\theta_2) = (1 - \alpha(1,1))\theta^B(\nu) > 0$ for $\theta_2 > \theta^B(\nu)$ and zero otherwise. On the other hand, for each $i = 1, 2$

$$\tilde{T}_i^I = \begin{cases} \frac{\alpha(1,1)}{2}\left(\theta_i^2 - (1 - \theta^S(\nu)(1 - \theta^S(\theta)))\right), & \text{if } \theta_i \in [0, \theta^S(\nu)] \\ \frac{\alpha(1,1)}{2}\left(\theta_i^2 - (1 - \theta^B(\nu)(1 - \theta^B(\theta)))\right), & \text{if } \theta_i \in [\theta^B(\nu), 1] \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $\theta_2 < \theta^S(\nu)$, although $\tilde{T}_2(\theta_2) < 0$ overall, the itemized transfers satisfy $T_2^I(\theta_2) < \tilde{T}_2^I(\theta_2) < 0$ and $\tilde{T}_2^E(\theta_2) = (1 - \alpha(1,1))\theta^S(\nu) > 0$—a strictly positive per-unit posted price $p^* = \theta^S(\nu)$ that is higher than merchant 2's profit margin θ_2. Intuitively, when merchant 2 is a low type, she expects to trade the inclusive customers and only target the exclusive ones. Since the incentives to over-report (from selling) dominates the incentives

to under-report (from buying), the designer exploits it by offering her a bundle of ads at a single non-itemized price. Thus, if $\theta_i < 1/4$ then $\tilde{T}_i(\theta_i) > T_i^E(\theta_i) + T_i^I(\theta_i) = T_i^I(\theta_i)$ since $\tilde{T}_i(\theta_i) = 0 > T_i^I(\theta_i)$ for any $\theta_i \in (\theta^S(\nu), 1/4)$. Hence, on $[0, 1/4]$ the designer has strictly higher revenues. One can do the same analysis for $\theta_i > 1/2$ and find that for merchant $i = 1$ the designer still assigns zero transfers in $[1/4, 1/2]$, does better in $[1/2, \theta^B(\nu)]$ by selling inclusive ads but worse in $[\theta^B(\nu), 3/4]$ by selling them at a lower price. On the other hand, for merchant $i = 2$ the designer does worse in region $[1/4, 1/2]$ by not selling any exclusive ads, but does better in $[1/2, 3/4]$ altogether by selling ads in bundles rather than separately. Overall, it can be shown that the designer achieves higher revenues from designing ads in bundles. I generalize these insights in Section 4.2 by explicitly characterizing the optimal mechanism for any arbitrary given symmetric type distributions, exclusive/inclusive dataset and welfare weight.

4 Optimal Mechanism

I show the existence of an optimal mechanism in which targeted ads are allocated according to some nonnegative and nondecreasing *scoring rules* $g_i : \Theta_i \to \mathbb{R}_+$ such that for given type profile θ, a customer $\omega \in \Omega$ is targeted by some merchant $i \in \mathcal{N}$ with the highest weighted score $\omega_i g_i(\theta_i)$. Ties are broken according to some *customer-specific* tie-breaking rules $p^\omega : \Theta \to \Delta(\mathcal{N}_0)$ such that participation constraints of *worst-off* types (obtaining the least payoff) binds.[10]

Definition 1 (*Scoring Mechanism*)**.** *A scoring mechanism* $(g, p) = ((g)_{i\in\mathcal{N}}, (p^\omega)_{\omega\in\Omega})$ *is a pair constituting of some nonnegative and non-decreasing scoring rules* $g_i : \Theta_i \to \mathbb{R}_+$ *for every* $i \in \mathcal{N}$ *and customer-specific tie-breaking rules* $p^\omega : \Theta \to \Delta(\mathcal{N}_0)$ *for every* $\omega \in \Omega$ *such that* $\forall\theta \in \Theta, \forall\omega \in \Omega, \forall i \in \mathcal{N}$*:*

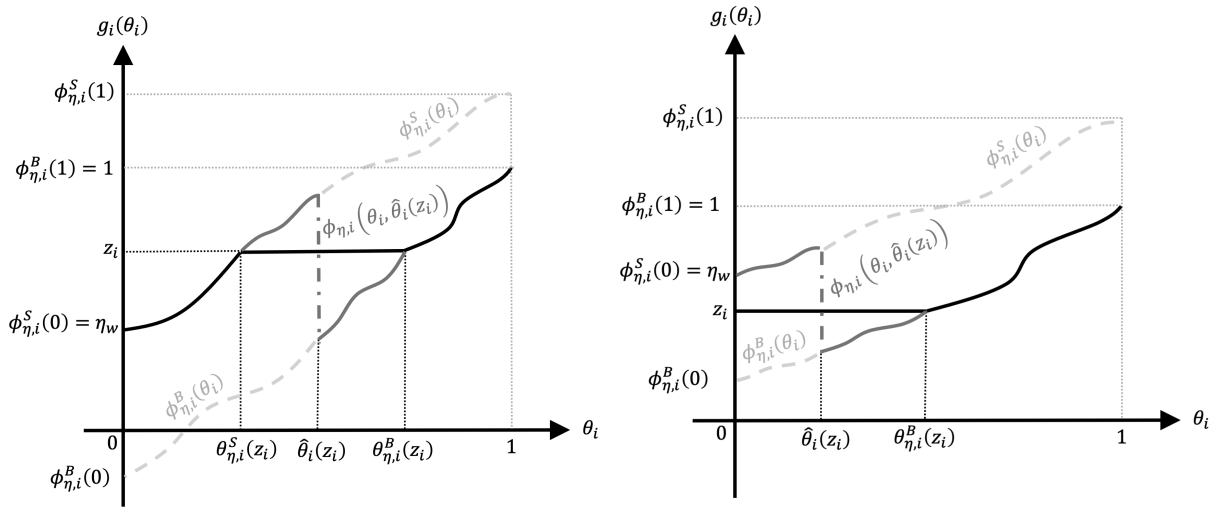
$$x_i^\omega(\theta) = p_i^\omega(\theta)\alpha(\omega) \quad \textit{and} \quad p_i^\omega(\theta) = 0 \textit{ if } i \notin \mathcal{N}^\omega(\theta) := \operatorname*{argmax}_{j\in\mathcal{N}_0}\{\omega_j g_j(\theta)\}, \tag{1}$$

where for simplicity of exposition let $\omega_0 g_0 \equiv 0$ *be the score assigned to the designer.*

To state the main theorem, I first introduce some definitions and a lemma.

Definition 2 (*Weighted Virtual Type Functions*)**.** *For each* $k \in \{B, S\}$ *let* $\phi_{\eta,i}^k(\theta_i)$ *denote the weighted virtual value* $(k = B)$ *and cost* $(k = S)$ *functions defined by* $\phi_{\eta,i}^k(\theta_i) :=$

[10] I let $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$ and $p_0^\omega(\theta)$ denote the mass of customers ω that are not targeted by any merchant. While one can show that there always exists a mechanism in which every customer is targeted by some merchant, i.e. $p^\omega(\theta) \in \Delta(\mathcal{N})$ instead, in some applications (see, for example, Section 4.2) I focus on a more practical class of ads rules for which some customers do not get targeted by any merchant.



(a) $\phi^B_{\eta,i}(0) < 0$. (b) $\phi^B_{\eta,i}(0) \geq 0$.

Figure 2: Graphical representation of (optimal) scoring rules and weighted virtual functions.

$\eta_\omega + \eta_v\theta_i + \eta_r\phi^k_i(\theta_i)$ *for every* $\theta_i \in \Theta_i$ *and* $i \in \mathcal{N}$. *Let* $\phi_{\eta,i}(\theta, \theta'_i)$ *denote the weighted virtual type function at some critical type* $\theta'_i \in \Theta_i$ *defined by* $\phi_{\eta,i}(\theta_i, \theta'_i) = \phi^S_{\eta,i}(\theta_i)$ *if* $\theta_i < \theta'_i$ *and* $\phi_{\eta,i}(\theta_i, \theta'_i) = \phi^B_{\eta,i}(\theta_i)$ *if* $\theta_i \geq \theta'_i$, *for every* $\theta_i \in \Theta_i$ *and* $i \in \mathcal{N}$. *Lastly, let* $\theta^k_{\eta,i}(y) := (\phi^k_{\eta,i})^{-1}(y)$ *for some* $y \in [\phi^k_{\eta,i}(0), \phi^k_{\eta,i}(1)]$.[11] *See Figure 2 for a visual description.*

Definition 3 (*Worst-Off Types*)**.** *For every* $i \in \mathcal{N}$, *let* $\hat{\Theta}_i(x,t)$ *denote the set of worst-off types* θ_i *induced by mechanism* (x,t), *i.e.*

$$\hat{\Theta}_i(x,t) := \operatorname*{argmin}_{\theta'_i \in \Theta_i} U_i(\theta'_i, X_i(\theta'_i), T_i(\theta'_i)).$$

Let $\hat{\Theta}(x,t) = \times_{i\in\mathcal{N}}\hat{\Theta}_i(x,t)$ *denote the set of worst-off types* $\theta \in \Theta$ *induced by mechanism* (x,t).

Lemma 1 (*Characterizing Worst-Off Types à la Cramton et al. (1987)*)**.** *Given* (IC) *and* (IR) *mechanism* (x,t), *the set of worst-off types can be characterized in terms of ads rules* x *only, i.e.* $\hat{\Theta}_i(x,t) = \hat{\Theta}_i(x)$ *where*

$$\hat{\Theta}_i(x) = \begin{cases} \{\theta_i \in \Theta_i : S_i(\theta_i) = a_i\}, & \text{if } \exists \theta_i : S_i(\theta_i) = a_i, \\ \{\theta_i \in \Theta_i : S_i(y) < a_i, \forall y < \theta_i \text{ and } S_i(y) > a_i, \forall y > \theta_i\}, & o/w, \end{cases} \tag{2}$$

where $S_i(\theta_i) := \sum_{\omega\in\Omega}\omega_i X^\omega_i(\theta_i)$ *and* $a_i := \sum_{\omega\in\Omega}\omega_i\alpha^\omega_i$ *denote the (interim) expected clicks on and outside the platform, respectively. Similarly, let* $\hat{\Theta}(x) = \times_{i\in\mathcal{N}}\hat{\Theta}_i(x)$.

[11] Note that $\theta^S_{\eta,i}(y) < \theta^B_{\eta,i}(y)$ for every $y \in [\phi^S_{\eta,i}(0), \phi^B_{\eta,i}(1)]$ given regular F_i and $\eta_r > 0$.

We are now ready to present the first main result on existence of an optimal scoring mechanism:

Theorem 1 (*Optimal Mechanism*). *For any arbitrarily given welfare weight η and dataset α, there exists a scoring mechanism (g, p) that attains the value $\Pi(\alpha, \eta)$ in (P). In particular, for every $i \in \mathcal{N}$ there exists $z_i \in [\underline{z}_i, \overline{z}_i] = \left[\max\{0, \phi^B_{\eta,i}(0)\}, \phi^S_{\eta,i}(1)\right]$ such that g_i is the ironed virtual function $\phi_{\eta,i}(\theta_i, \hat{\theta}_i(z_i))$ with critical type $\hat{\theta}_i(z_i)$ uniquely defined by $\hat{\theta}_i(z_i) = \mathbb{E}\left[\overline{\phi}_{\eta,i}(\theta_i, z_i) - \eta_w - \eta_v\theta_i\right]/\eta_r$, that is,*

$$g_i(\theta_i) := \overline{\phi}_{\eta,i}(\theta_i, z_i) = \begin{cases} \phi^S_{\eta,i}(\theta_i), & \text{if} \quad \theta_i < \theta^S_{\eta,i}(z_i), \\ z_i, & \text{if} \quad \theta^S_{\eta,i}(z_i) \le \theta_i \le \theta^B_{\eta,i}(z_i), \\ \phi^B_{\eta,i}(\theta_i), & \text{if} \quad \theta_i > \theta^B_{\eta,i}(z_i). \end{cases} \tag{3}$$

The tie breaking rule p is chosen such that the critical type $\hat{\theta}_i(z_i)$'s IR constraint binds, i.e. $\hat{\theta}_i(z_i) \in \hat{\Theta}_i(x)$ is a worst-off type. The interim expected transfers satisfy

$$T_i(\theta_i) = \theta_i(S_i(\theta_i) - a_i) - \int_{\hat{\theta}_i(z_i)}^{\theta_i} (S_i(y) - a_i)dy, \quad \forall \theta_i \in \Theta_i \text{ and } i \in \mathcal{N}. \tag{4}$$

By assigning a single merchant-specific scoring rule g_i, the incentives to over (under) report profit margins θ_i on customers ω with low (high) CTR ω_i are "pooled" in a single report $g_i(\theta_i)$ that is nondecreasing in θ_i. Then from (1) the scoring mechanism is such that customers ω are targeted by the merchant i with highest quality score $\omega_i g_i(\theta_i)$. Since tie-breaks $\omega_i z_i = \omega_j z_j$ for some $i \neq j \in \mathcal{N}$ can occur with positive probability, in the characterization of the optimal mechanism I show that the tie-breaking rule is essential only in guaranteeing that the critical type $\hat{\theta}_i(z_i) \in \hat{\Theta}_i(x)$.

If $z_i > 0$ then the worst-off types $\hat{\theta}_i \in [\theta^S_{\eta,i}, \theta^B_{\eta,i}]$ receive expected clicks $S_i(\hat{\theta}_i) = a_i$, where for some type profiles θ_{-i} the targeted customers $x^\omega_i(\hat{\theta}_i, \theta_{-i})$ may be different from customers ω from the initial dataset α_i. In a way, the designer is *exchanging* customers between merchants to guarantee their outside option. Moreover, the designer can mitigate customer poaching by increasing z_i for adversely affected merchants to boost their scores. And when there are ties, the designer can favor some merchants in the tie-breaks.

Next, I outline the main steps of the proof of Theorem 1 and leave any remaining details to Appendix A. As we will see, while (IC) depends only on the one-dimensional monotonicity of expected clicks S_i, (IR) depends on the multidimensional tie-breaking rule p.

4.1 Proof Sketch

The first steps consist of eliminating (IC) and (IR) constraints and the dependence on transfer rule t from (P). I then proceed to show the existence of an optimal scoring mechanism.

Eliminating (IC). Using standard Myersonian approach to characterize incentive compatibility, I reduce the designer's problem in (P) to optimizing over the space of ads rules $x \in \mathcal{X}(\Theta, \times dF_i)$ subject to suitable monotonicity constraints and transfers $t(\hat{\theta})$ at some worst-off type profile $\hat{\theta}$.

Specifically, by standard characterization of (interim) incentive compatibility the expected payoffs $U_i(\theta_i) = \theta_i(S_i(\theta_i) - a_i) - T_i(\theta_i)$ are attained by some (IC) mechanism (x, t) if and only if

$$U_i(\theta_i) = U_i(\theta_i') + \int_{\theta_i'}^{\theta_i} (S_i(y) - a_i)dy, \quad \forall \theta_i, \theta_i' \in \Theta_i, \tag{ICFOC}$$

$$S_i(\theta_i) \text{ is nondecreasing.} \tag{M}$$

Now, fix an arbitrary critical type $\theta_i' \in \Theta_i$ for every $i \in \mathcal{N}$. Using (ICFOC), the (interim) expected transfers $T_i(\theta_i)$ at every $\theta_i \in \Theta_i$ are pinned down in terms of expected clicks S_i and payoff $U_i(\theta_i')$:

$$T_i(\theta_i) = \theta_i(S_i(\theta_i) - a_i) - \int_{\theta_i'}^{\theta_i} (S_i(y) - a_i)dy - U_i(\theta_i'). \tag{5}$$

Note that $U_i(\theta_i')$ depends implicitly on transfers $T_i(\theta_i')$ assigned to the critical type θ_i'. Then fix $(x, t) \in \mathcal{M}^{\alpha}(\Theta, \times dF_i)$ that satisfies (IC)—hence (ICFOC) and (M). By standard integration by parts techniques and using (ICFOC) and (5), we can simplify (Obj) as follows:

$$\psi_\eta(x, t) = \psi_\eta(x, \theta') - \eta_r \sum_{i \in \mathcal{N}} U_i(\theta_i'), \tag{6}$$

where the *virtual objective function* $\psi_\eta(x, \theta')$ is given by

$$\psi_\eta(x, \theta') = \mathbb{E}\left[\sum_{i \in \mathcal{N}} \left(\sum_{\omega \in \Omega} \omega_i x_i^{\omega}(\theta) - a_i\right) \phi_{\eta,i}(\theta_i, \theta_i')\right]. \tag{7}$$

Note that θ' is chosen arbitrarily, so the identity in (6) holds for any $\theta' \in \Theta$.

In standard mechanism design problems, the worst-off type $\hat{\theta}$ is typically predetermined—often taken to be either $\hat{\theta} = \underline{\theta}$ or $\hat{\theta} = \overline{\theta}$—*independently* of the optimal ads rule x^*. By optimality, such types receive expected payoffs $U_i(\hat{\theta}_i) = 0$ for every $i \in \mathcal{N}$. In those cases, one sets $\theta' = \hat{\theta}$, allowing (5) (hence (6) too) to be written solely in terms of the ads rule x. However, in settings with countervailing incentives like ours the worst-off type(s)

and ads rule x are *interdependent*, which complicates the analysis. I next show how one can circumvent such complications.

Eliminating (IR) and Existence. I show that the optimization problem in (P) can be written as a *minimax* problem, which I then use to i) show existence of an optimal mechanism, and ii) characterize the optimal ads rule x^* *independently* of the term $U_i'(\theta')$ in (6).

By a similar argument as in Loertscher and Wasser (2019) and Loertscher and Muir (2024), the set of worst-off types admits the following characterization in terms of the virtual objective function:

$$\hat{\Theta}(x) = \operatorname*{argmin}_{\theta' \in \Theta} \psi_\eta(x, \theta'). \tag{8}$$

Then we can formulate the designer's problem as a minimax problem:

$$\Pi(\alpha, \eta) = \max_{x \in \mathcal{X}(\Theta, \times dF_i)} \min_{\theta' \in \Theta} \psi_\eta(x, \theta'), \tag{Max-Min}$$

where $\mathcal{X}(\Theta, \times dF_i)$ denotes the space of measurable ads rules x that satisfy (Fsb_θ) and (M). I then show that a solution to the (Max-Min) problem exists by a generalized version of von Neumann's minimax theorem (von Neumann (1928)). Moreover, the theorem implies that one can swap the order in (Max-Min) as

$$\Pi(\alpha, \eta) = \min_{\theta' \in \Theta} \max_{x \in \mathcal{X}(\Theta, \times dF_i)} \psi_\eta(x, \theta'), \tag{Min-Max}$$

and that a saddle point $(x^*, \hat{\theta})$ exists

$$\max_{x \in \mathcal{X}(\Theta, \times dF_i)} \psi_\eta(x, \hat{\theta}) = \psi_\eta(x^*, \hat{\theta}) = \min_{\theta' \in \Theta} \psi_\eta(x^*, \theta'), \tag{SP}$$

with $\Pi(\alpha, \eta) = \psi_\eta(x^*, \hat{\theta})$. As a result, one characterizes an optimal mechanism by finding saddle points $(x^*, \hat{\theta})$ that separately solve each of the optimization problems in (SP).

Ironing. Let $(x^*, \hat{\theta})$ be a saddle point. We've already characterized the set of worst off types $\hat{\Theta}(x^*)$ in (2). To characterize the optimal ads rules x^* in terms of the worst-off type $\hat{\theta}$, we need to solve

$$\max_{x \in \mathcal{X}(\Theta, \times dF_i)} \psi_\eta(x, \hat{\theta}) = \max_{x \in \mathcal{X}(\Theta, \times dF_i)} \mathbb{E}\left[\sum_{i \in \mathcal{N}} \left(\sum_{\omega \in \Omega} \omega_i x_i^\omega(\theta) - a_i \right) \phi_{\eta,i}(\theta_i, \hat{\theta}_i) \right]. \tag{9}$$

Note that pointwise maximization of (9) does not necessarily result in non-decreasing $S_i^*(\theta_i)$ since $\phi_{\eta,i}(\theta_i, \hat{\theta}_i)$ is non-monotone at $\theta_i = \hat{\theta}_i$. To circumvent this, the solution requires ironing techniques from Myerson (1981): for each $i \in \mathcal{N}$ let $z_i(\hat{\theta}_i)$ be the unique ironing parameter

that satisfies

$$\mathbb{E}[\overline{\phi}_{\eta,i}(\theta_i, z_i)] = \mathbb{E}[\phi_{\eta,i}(\theta_i, \hat{\theta}_i)] = \eta_w + \eta_v \mathbb{E}[\theta_i] + \eta_r \hat{\theta}_i.$$

Then, by similar arguments as in Myerson (1981) optimizing (9) is equivalent to

$$\max_{\substack{x \text{ pointwise} \\ \text{s.t. } (\text{Fsb}_\theta)}} \overline{\psi}_\eta(x, \hat{\theta}) = \max_{\substack{x \text{ pointwise} \\ \text{s.t. } (\text{Fsb}_\theta)}} \mathbb{E}\left[\sum_{i\in\mathcal{N}} \left(\sum_{\omega\in\Omega} \omega_i x_i^\omega(\theta) - a_i\right) \overline{\phi}_{\eta,i}(\theta_i, z_i(\hat{\theta}_i))\right]. \tag{10}$$

That is, since scoring rule $\overline{\phi}_{\eta,i}(\theta_i, z_i(\hat{\theta}_i))$ is non-decreasing in θ_i, pointwise maximization of (10) yields non-decreasing $S_i^*(\theta_i)$—thus satisfying (M). Thus, let $x^{\text{pw}}(\hat{\theta})$ be a solution to (10) which by similar arguments as in Myerson (1981) it attains the optimal value, i.e.

$$\overline{\psi}_\eta(x^{\text{pw}}(\hat{\theta}), \hat{\theta}) = \psi_\eta(x^*, \hat{\theta}).$$

Scoring Mechanisms. I describe the pointwise maximizing ads rule x^{pw} in (10) and show that is a scoring mechanism.

Specifically, letting θ^* be a critical worst-off type, pointwise maximization wrt. ads rule $x^\omega(\theta)$ solves for every $\omega \in \Omega$ and $\theta \in \Theta$:

$$\max_{x^\omega(\theta)} \sum_{i\in\mathcal{N}} x_i^\omega(\theta)\omega_i \phi_{\eta,i}(\theta_i, z_i(\theta_i^*)), \quad \text{subject to } \sum_{i\in\mathcal{N}} x_i^\omega(\theta) \leq \alpha(\omega). \tag{x-pw}$$

If follows readily that the solution is given by the scoring ads rule given in (1). Lastly, the designer chooses customer-specific tie-breaking rule p such that $\theta^* \in \hat{\Theta}(x^{\text{pw}}(\theta^*))$. The expected transfers follow by evaluating (5) at $\theta' = \theta^*$.

4.2 A Stylized Setting: Exclusive & Inclusive Customers

I fully characterize the optimal scoring mechanism for $i = 1, 2$ and symmetric distributions $F_i \equiv F$ by restricting CTRs $\omega_i \in \{0, 1\}$. I perform comparative statics on the model parameters and interpret the results.

Specifically, a customer is said to be *exclusive* if they intend to purchase solely from some merchant $i = 1, 2$, and *inclusive* if they are indifferent among every merchant $i = 1, 2$. Formally, let $\Omega_1 = \Omega_2 = \{0, 1\}$ where customers $\omega = (1, 0)$ and $\omega = (0, 1)$ are exclusive to merchant 1 and 2 respectively, whereas customer $\omega = (1, 1)$ is inclusive to both merchants. For simplicity of exposition, let $\omega_1 := (1, 0)$ and $\omega_2 := (0, 1)$. Without loss of generality set $\alpha(0, 0) = 0$ and $\alpha_i^{\omega_i} = 0$ for each $i = 1, 2$.[12] Let $\mathcal{A}_{0,1} \subset \Delta(\{0, 1\}^2)$ denote the set of all feasible inclusive and exclusive datasets.

Note that for any scoring rule with $z_i(\hat{\theta}_i) > 0$ pointwise maximization in (x-pw) implies

[12] Customers $\omega = (0, 0)$ are irrelevant, and there's no gain from reallocating exclusive customer that are already part of their preferred merchants' dataset.

that in the optimal mechanism all exclusive customers ω_i are targeted from their preferred merchant i, and in particular, the worst-off type $\hat{\theta}_i$ receives $x^{*\omega_i}_i(\hat{\theta}_i, \theta_j) = \alpha^{\omega_i}_j$ for every $\theta_j \in \Theta_j$. However, if $z_i(\hat{\theta}_i) = 0$ then there can be multiplicity regarding the choice of the optimal tie-breaking rules between type $\hat{\theta}_i$, any type θ_j (since $\omega_j = 0$) and the designer ($g_0\omega_0 \equiv 0$). I define a class of scoring mechanisms that break ties for exclusive customers in favor of their preferred merchant and then construct an optimal mechanism within this class.

Definition 4 (*Exclusive-Priority (EP) Scoring Mechanism*). *Let $\alpha \in \mathcal{A}_{0,1}$ and scoring mechanism (g, p) with $g_i \equiv \phi_{\eta,i}(\cdot, z_i(\hat{\theta}_i))$ for some $\hat{\theta} \in \Theta$. The tie-breaking rule p is said to be exclusive-priority (EP) if whenever there exists $i \in \mathcal{N}$ such that $z_i(\hat{\theta}_i) = 0$, tie breaking rule $p^{\omega_i}_i$ satisfies $p^{\omega_i}_i(\hat{\theta}_i, \theta_j) = \min\{\alpha^{\omega_i}_j, \alpha^{(1,1)}_i\}/\alpha^{\omega_i}_j$ for every $\theta_j \in \Theta_j$.*

If at some type profile θ choosing between available inclusive customers $\omega = (1, 1)$ and exclusive customers ω_i affects neither the designer's objective function nor the merchants' payoffs, then EP scoring mechanism prioritizes targeting first exclusive customers. I show that for any arbitrarily given dataset $\alpha \in \mathcal{A}_{0,1}$, welfare weight η and (regular) distribution F there exists an (essentially unique) EP scoring mechanism (g, p) and provide conditions that characterize ironing parameter z and tie-breaking rule p explicitly.

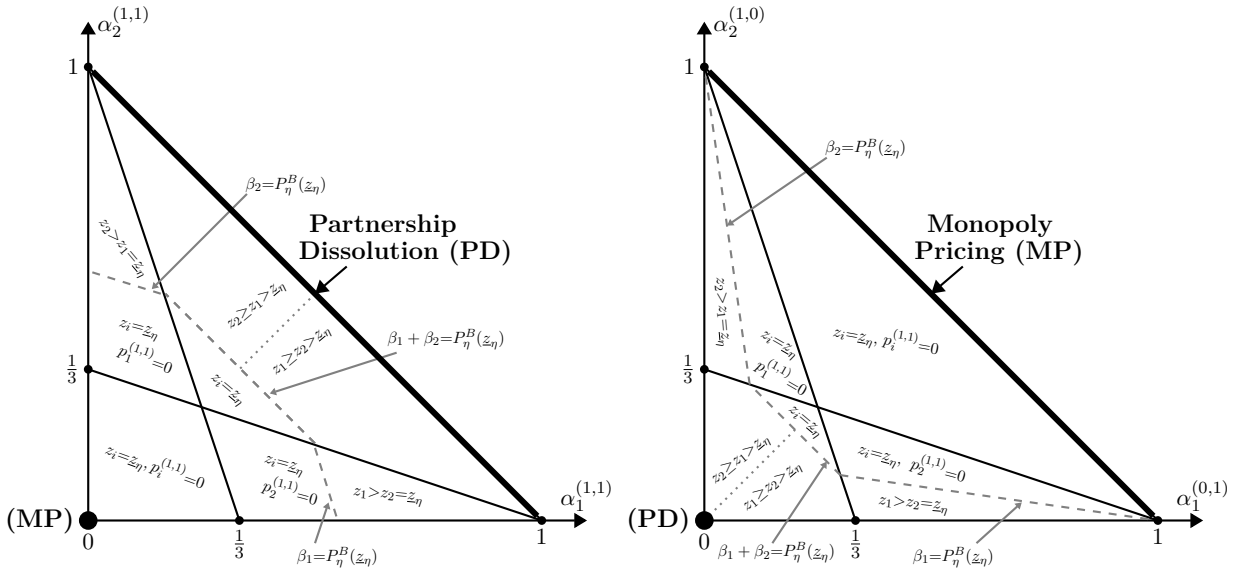
Before stating the result, I first introduce some additional terminologies. In what follows let $i \neq j \in \{1, 2\}$ and $\beta_i := \max\{0, (\alpha^{(1,1)}_i - \alpha^{\omega_i}_j)/\alpha(1, 1)\}$ denote merchant i's *adjusted* shares from $\alpha(1, 1)$ mass of inclusive customers. Note that $\alpha^{\omega_i}_j$ mass of exclusive customers are initially 'dormant' as they never purchase from merchant j; the designer may then costlessly offer merchant i ads that target these customers. Therefore, of the total $\alpha(1, 1)$ mass of inclusive customers, the designer retains residual ownership over a fraction $1-\beta_1-\beta_2$.[13] Next, define $P^S_\eta(z_i) := \mathbb{P}_F\left(\phi^S_{\eta,j}(\theta_j) \leq z_i\right)$ and $P^B_\eta(z_i) := \mathbb{P}_F\left(\phi^B_{\eta,j}(\theta_j) \leq z_i\right)$. If the optimal scoring rule is such that $z_i \leq z_j$ then the probability that the critical worst-off type $\hat{\theta}_i(z_i)$ ($\hat{\theta}_j(z_j)$) has a *strictly higher* score than merchant j (i) is given by $P^S_\eta(z_i)$ ($P^B_\eta(z_j)$). If $z_i = z_j = z$ then the probability that the critical worst-off type $\hat{\theta}_i(z_i)$ has equal score to merchant j is given by $P^B_\eta(z) - P^S_\eta(z)$. Finally, let $\underline{z}_\eta := \max\{0, \phi^B_{\eta,i}(0)\}$.

Because of symmetry I state the results for the case $\beta_1 \geq \beta_2$:

Proposition 1. *For any arbitrarily chosen $\alpha \in \mathcal{A}_{0,1}$ such that $\beta_1 \geq \beta_2$ and welfare weight η there exists (essentially unique) EP-scoring mechanism (g, p) such that:*

i) if $\beta_1, \beta_2 > 0$ and $\beta_1 + \beta_2 > P^B_\eta(\underline{z}_\eta)$, let $\tilde{z} > \underline{z}_\eta$ be uniquely defined by $P^B_\eta(\tilde{z}) + P^S_\eta(\tilde{z}) = \beta_1 + \beta_2$ and suppose first that $P^B_\eta(z) - P^S_\eta(z) \geq \beta_1 - \beta_2$. Then $z_1 = z_2 = \tilde{z}$ and the inclusive

[13] We have $\beta_1 + \beta_2 \leq \left(\alpha^{(1,1)}_1 + \alpha^{(1,1)}_2\right)/\alpha(1, 1) = 1$.



(a) $\alpha_1^{(0,1)}=\alpha_2^{(1,0)}$ (b) $\alpha_1^{(1,1)}=\alpha_2^{(1,1)}$

Figure 3: Exclusive & Inclusive Customers: Optimal EP Scoring Mechanisms

tie-breaking rule $p^{(1,1)}$ *is uniquely given by*

$$p_1^{(1,1)} = \frac{1}{2}\left(1 + \frac{\beta_1 - \beta_2}{P_\eta^B(z) - P_\eta^S(z)}\right).$$

On the other hand, if $P_\eta^B(z) - P_\eta^S(z) < \beta_1 - \beta_2$*, then there exists* $z_1 \geq \tilde{z} \geq z_2 (> \underline{z}_\eta)$*, with at least one inequality being strict, that uniquely solve*

$$P_\eta^B(z_1) = \beta_1 \quad \textit{and} \quad P_\eta^S(z_2) = \beta_2. \tag{11}$$

Lastly, if $\beta_1 + \beta_2 \leq P_\eta^B(\underline{z}_\eta)$ *then* $z_1 = z_2 = \underline{z}_\eta (= 0)$ *and the unique inclusive tie-breaking rule* $p^{(1,1)}$ *satisfies*

$$p_1^{(1,1)} = \frac{\beta_1}{P_\eta^B(0)} \quad \textit{and} \quad p_2^{(1,1)} = \frac{\beta_2}{P_\eta^B(0)}. \tag{12}$$

ii) if $\beta_1 > \beta_2 = 0$ *then an optimal scoring rule* g_2 *with* $z_2 = \underline{z}$ *exists. Moreover, there exists a unique scoring rule* g_1 *such that if* $\beta_1 > P_{F,\eta}^B(\underline{z}_\eta)$ *then* $z_1 = (P_{F,\eta}^B)^{-1}(\beta_1) > \underline{z}_\eta$*. On the other hand, if* $\beta_1 \leq P_{F,\eta}^B(\underline{z}_\eta)$ *then* $z_1 = \underline{z}_\eta (= 0)$ *and the unique inclusive tie-breaking rule* $p^{(1,1)}$ *satisfies*

$$p_1^{(1,1)} = \frac{\beta_1}{P_{F,\eta}^B(\underline{z}_\eta)} \quad \textit{and} \quad p_2^{(1,1)} = 0.$$

iii) if $\beta_1 = \beta_2 = 0$ *then* $z_1 = z_2 = \underline{z}_\eta$*. Moreover, ties occur with positive probability if* $P_{F,\eta}^B(\underline{z}_\eta) < 0$*, which in that case* $p^{(1,1)} \equiv 0$*.*

See Figure 3 for an illustration of the optimal ironing parameters $z = (z_1, z_2)$ and inclusive tie-breaking rule $p^{(1,1)}$ for arbitrary welfare weight η and individual datasets α_i which have a symmetric mass of either exclusive customers (a) or inclusive customers (b).

Consider first Figure 3, (a) with $\alpha_1^{(0,1)} = \alpha_2^{(1,0)}$. Notice first that the diagonal corresponds to datasets that consists of inclusive customers only, in which case we are back to the case of partnership dissolution (PD). As we move away from the diagonal, we notice that the optimal scoring rules have lower ironing parameters z_i. As the share of exclusive customer grows compared to the inclusive ones, it becomes cheaper for the designer to satisfy merchants' outside options by assigning more ads that target exclusive customers—which in the first place the designer procured at no cost. Thus, the further we move away from the diagonal, the more merchants are treated as buyers (hence lower z_i) of any residue of the inclusive customers. Moreover, when merchant i contributes a larger share of inclusive customers, to mitigate customer poaching the designer assigns either a strictly larger score ($z_i > z_j$) or favors them with with tie-breaks $p_i^{(1,1)} > p_j^{(1,1)}$. Lastly, we can immediately see that the corner case with no inclusive customers is simply the monopoly pricing (MP) problem.

The results in Figure 3, (b) are symmetric to (a). One takeaway from this exercise is that characterizing the optimal mechanism beyond this stylized setting becomes highly untractable: the multidimensional nature of having multiple instruments kicks in when choosing the optimal tie-breaking rules for various $\omega \in \Omega$. I therefore enrich the model by introducing continuum realizations in Ω, which nicely overcomes the issue of multidimensionality and makes the analysis more tractable.

5 Continuum of CTRs

When customers ω have a positive measure then resolving tie-breaks between merchants with equal quality scores $\omega_i z_i$ becomes computationally intractable. In this section, I enrich the model by considering a continuum of realizations ω. I show the existence of optimal scoring mechanism analogous to the finite case and derive a system of (nonlinear) equations to explicitly compute the scoring rules g_i (i.e. ironing parameters z_i). In a stylized setting with i.i.d. symmetric datasets, I perform comparative statics and derive the optimal large-market mechanism, which is ex post implementable via a selling, a buying, and an exchange market.

Specifically, let $\Omega = [0,1]^N$ and the probability space $(\Omega, \mathcal{B}(\Omega), \alpha)$ be endowed with the Borel σ-algebra $\mathcal{B}(\Omega)$. For each $i \in \mathcal{N}$ let the individual dataset α_i be defined by the tuple

$\alpha_i := (\lambda(i), \mu_i)$ where $\lambda \in \Delta(\mathcal{N})$ is a strictly positive *discrete* measure that represents the *mass distribution* across merchants' datasets, and $\mu_i \in \Delta(\Omega)$ is a *continuous* $\mathcal{B}(\Omega)$-measure that represents the distribution of customers ω in merchant i's dataset. Thus, the aggregate dataset α is the $\mathcal{B}(\Omega)$-measure defined by

$$\alpha(E) := \sum_{i \in \mathcal{N}} \lambda(i)\mu_i(E), \quad \forall E \in \mathcal{B}(\Omega). \tag{13}$$

5.1 Mechanism Design

I define the space of direct mechanisms that accommodate a *continuum* of instruments (x_i^ω) and then define the designer's problem analogously to the finite case.

Direct Mechanism. Let $\mathcal{X}$ be the space of $\mathcal{B}(\Omega)$-measurable functions $x : \Omega \to \mathbb{R}_+$. A (direct) revelation mechanism $(x, t) = ((x_i)_{i \in \mathcal{N}_0}, (t_i)_{i \in \mathcal{N}_0})$ is given by measurable ads rules $x_i : \Theta \to \mathcal{X}$—that is, $\forall \theta \in \theta$, $x_i(\theta) : \Omega \to \mathbb{R}_+$ is a $\mathcal{B}(\Omega)$-measurable function—and measurable transfer rules $t_i : \Theta \to \mathbb{R}$. Since the designer allocates target a unit mass of customers drawn from the aggregate dataset α, a direct mechanism is subject to the following feasibility constraints:

Definition 5 (*Feasibility*). *A mechanism (x, t) is feasible if for every $\theta \in \Theta$:*

i) $x(\theta)$ distributes a unit mass of customers:

$$\mathbb{E}_\alpha[x(\theta)] := \mathbb{E}_\alpha\left[\sum_{i \in \mathcal{N}_0} x_i^\omega(\theta)\right] = 1.^{14} \tag{Unit}$$

ii) the distribution of customers induced by $x(\theta)$ agrees with the true distribution:

$$\mathbb{E}_\alpha\left[\sum_{i \in \mathcal{N}_0} x_i^\omega(\theta) \cdot \mathbb{1}_E(\omega)\right] = \alpha(E), \quad \forall E \in \mathcal{B}(\Omega). \tag{Marg}$$

Observe that (Marg) implies (Unit) by taking $E = \Omega$.

Payoffs. The expected clicks outside the platform are given by

$$a_i := \lambda(i)\mathbb{E}_{\mu_i}[\omega_i], \quad \forall i \in \mathcal{N}.$$

For given ads rule x, define analogously to the finite case the expected clicks S_i on the platform by

$$S_i(\theta_i) := \mathbb{E}_{(\alpha, F_{-i})}[\omega_i x_i^\omega(\theta_i, \theta_{-i})], \quad \forall \theta_i \in \Theta_i, i \in \mathcal{N}.$$

[14] Without loss of generality the designer $i = 0$ serves as the 'sink' for all customers (if any) that are not getting targeted by any merchant $i \in \mathcal{N}$.

Thus, given mechanism (x,t) type θ_i gets expected payoffs $U_i(\theta_i, S_i(\theta_i), T_i(\theta_i)) = \theta_i(S_i(\theta_i) - a_i) - T_i(\theta_i)$. The definitions of (IC) and (IR) constraints thus follow as in the finite case. Let $\mathcal{M}(\Theta, \times dF_i)$ denote the space of feasible, (IC) and (IR) mechanisms (x,t).

The following example illustrates a mechanism that guarantees every merchant's outside option, hence showing that $\mathcal{M}(\Theta, \times dF_i) \neq \emptyset$ and that the participation of every merchant in the platform is without loss of generality.

Example 1 (*Outside-Option Guarantee Mechanism*)**.** *Consider the ads rule x^{RN} by letting $x_0 \equiv 0$ and for every $i \in \mathcal{N}$ and $\theta \in \Theta$,*

$$x_i^{\omega}(\theta) := \lambda(i) h_i(\omega), \quad \forall \omega \in \Omega,$$

where the measurable function $h_i : \Omega \to \mathbb{R}_+$ is given by

$$h_i \equiv \frac{d\mu_i}{d\alpha},$$

i.e. h is the Radon-Nikodym derivative of the merchant-specific measure μ_i with respect to the aggregate measure α. By definition (13), μ_i is absolutely continuous with respect to α, thus the RN derivative exists (and is unique a.e.). Now, x^{RN} is feasible since it satisfies (Marg):

$$\begin{aligned}\mathbb{E}_{\alpha}\left[\sum_{i\in\mathcal{N}_0} x_i^{\omega}\cdot \mathbb{1}_E(\omega)\right] &= \sum_{i\in\mathcal{N}} \lambda(i)\mathbb{E}_{\alpha}[h_i(\omega)\cdot\mathbb{1}_E(\omega)] \\ = \sum_{i\in\mathcal{N}} \lambda(i)\mathbb{E}_{\mu_i}[\mathbb{1}_E(\omega)] &= \sum_{i\in\mathcal{N}} \lambda(i)\mu_i(E) = \alpha(E).\end{aligned}$$

Moreover, letting $t \equiv 0$, the mechanism $\left(x^{RN}, 0\right)$ guarantees every merchant's outside option:

$$\mathbb{E}_{\alpha}[\omega_i x_i^{\omega}] = \lambda(i)\mathbb{E}_{\alpha}[\omega_i h_i(\omega)] = \lambda(i)\mathbb{E}_{\mu_i}[\omega_i], \quad \forall i \in \mathcal{N}.$$

Designer's Problem. The designer's problem is defined analogously as in (P).

5.2 Optimal Mechanism

I now define a class of mechanisms to be used in characterizing the optimal mechanism.

Let partition P of the measurable space $(\Omega, \mathcal{B}(\Omega))$ be a collection of *countable* disjoint subsets $P = \{E_j\}_{j\in J}$ for some index set J such that $E_j \in \mathcal{B}(\Omega)$ for all $j \in J$ and $\cup_{j\in J} E_J = \Omega$. Let $\mathscr{P}$ be the set of all partitions P of Ω.

Definition 6 (*Targeted Ads Rules*)**.** *An ads rule x is partitional if for every $\theta \in \Theta$ there*

exists a measurable partition $P(\theta) = \{E_i(\theta)\}_{i\in\mathcal{N}\cup\{0\}}$ *of* $(\Omega, \mathcal{B}(\Omega))$ *such that*

$$x_i^\omega(\theta) = 1_{E_i(\theta)}(\omega), \quad \forall \omega \in \Omega.$$

Moreover, a partitional ads rule x *is targeted if there exists a critical worst-off type* $\hat{\theta}_i \in \hat{\Theta}(x)$ *such that for every* $\theta \in \Theta$ *the partition* $P^z(\theta) = \{E_i^z(\theta)\}_{i\in\mathcal{N}\cup\{0\}}$ *is such that for every* $i \in \mathcal{N}$

$$E_i^z(\theta) := \Big\{\omega \in \Omega : \omega_i \overline{\phi}_{\eta,i}(\theta_i, z_i(\hat{\theta}_i)) > \omega_j \overline{\phi}_{\eta,j}(\theta_j, z_j(\hat{\theta}_j)), \forall j \neq i, j \in \mathcal{N}\Big\},$$

and $E_0^z(\theta) = \Omega \setminus \cup_{i\in\mathcal{N}} E_i^z(\theta)$.

Theorem 2 (*Optimal Mechanism*)**.** *For any arbitrarily chosen welfare weight* η *and dataset* α*, there exists an optimal targeted ads rule* x *for some critical worst-off type* $\hat{\theta} \in \hat{\Theta}(x)$*. Moreover, if* $z_i(\hat{\theta}) = z_i > 0$ *for at least some* $i \in \mathcal{N}$*, then either* $z_i = \phi_{\eta,i}^S(1)$ *or it satisfies*

$$\mathbb{E}_\alpha\left[\omega_i\left(\prod_{j\neq i, j\in\mathcal{N}} \mathbb{P}_{F_j}\left(\omega_i z_i > \omega_j \overline{\phi}_{\eta,j}(\theta_j, z_j)\right)\right)\right] = a_i. \tag{Opt-z}$$

Lastly, the expected (interim) transfers follow similarly from (4)*.*

Proof Sketch. The first observation is that the main steps in the proof of Theorem 1 go through, thus allowing us to reduce to the following optimization problem in finding the optimal ads rule x:

$$\max_{\substack{x \text{ pointwise a.e.} \\ \text{s.t. (Marg)}}} \mathbb{E}_{(\alpha, \times dF_i)}\left[\sum_{i\in\mathcal{N}} \omega_i x_i^\omega(\theta) \overline{\phi}_{\eta,i}(\theta_i, z_i(\hat{\theta}_i))\right]. \tag{14}$$

I show in Appendix B that the optimal ads rule takes the form $x_i^\omega(\theta) = \mathbb{1}_{E_i^z(\theta)}(\omega)$ a.e. in Ω. But then, this allows us to compute explicitly the interim expected clicks at $\hat{\theta}(z)$. Specifically, for each $i \in \mathcal{N}$,

$$\begin{aligned} S_i(\hat{\theta}_i(z_i)) &= \mathbb{E}_{(\alpha, F_{-i})}\left[\omega_i \mathbb{1}_{E_i^z\left(\hat{\theta}_i(z_i), \theta_{-i}\right)}(\omega)\right] = \mathbb{E}_\alpha\left[\omega_i \mathbb{E}_{F_{-i}}\left[\mathbb{1}_{E_i^z\left(\hat{\theta}_i(z_i), \theta_{-i}\right)}(\omega)\middle|\omega\right]\right] \\ &= \mathbb{E}_\alpha\left[\omega_i \mathbb{P}_{F_{-i}}\left(E_i^z\left(\hat{\theta}_i(z_i), \theta_{-i}\right)\middle|\omega\right)\right] = \mathbb{E}_\alpha\left[\omega_i\left(\prod_{j\neq i, j\in\mathcal{N}} \mathbb{P}_{F_j}\left(\omega_i z_i > \omega_j \overline{\phi}_{\eta,j}(\theta_j, z_j)\right)\right)\right] \end{aligned} \tag{15}$$

where the first equality is derived using the law of iterated expectations and the independence of the distributions F and α, and the last equality uses the independence of F_i's. Moreover, since $a_i > 0$, if $z_i > 0$ for some $i \in \mathcal{N}$ the from the last term in (15) we can see that it must also be the case that $z_i > 0$ for all $i \in \mathcal{N}$. Therefore, from definition of $\hat{\Theta}(x)$ it follows that if no z_i that makes (15) equal to a_i exists, then $z_i = \phi_{\eta,i}^S(1)$ and $S_i(\hat{\theta}_i(z_i)) < a_i$. □

I now provide some comparative statics of the optimal mechanism in an example with $N = 2$ and uniform distributions:

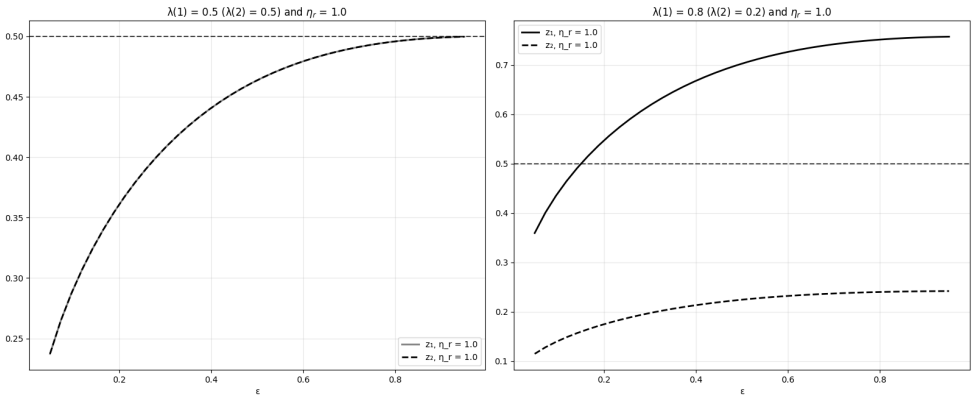


Figure 4: Optimal $(z_1(\varepsilon), z_2(\varepsilon))$ for $\eta_r = 1$, $\lambda(1) = 0.5$ (left) and $\lambda(1) = 0.8$ (right).

Example 2 (Optimal Scoring Rules)**.** *Consider $N = 2$ with $\theta_i \sim$ Uniform$[0,1]$, $\omega_i \sim$ Uniform$[\varepsilon, 1]$ for some $\varepsilon \geq 0$, $\eta_w = 0$ and $\lambda(1) > \lambda(2)$.*[15] *In Figure 4 I plot the optimal solutions for $\eta_r = 1$ and mass distributions $\lambda_1 = 0.5$ and $\lambda_2 = 0.8$ while varying data quality ε. Note that as $\varepsilon \to 1$ for $\eta_r = 1$ and $\lambda(1) = 0.5$, the optimal $z_i \to 1/2$, thus agreeing with the profit maximizing mechanism in the partnership dissolution model as illustrated in Section 3. Moreover, lower data quality implies that merchants expect to benefit more from the platform and be net buyers (thus z_i small). Lastly, merchants with larger contributions to the platform datasets get assigned scoring rules g_i with higher ironing parameter z_i.*

Next, I study a stylized setting under additional i.i.d. and symmetry assumptions, and solve for the optimal design as the platform grows large ($N \to \infty$).

5.3 A Stylized Setting: I.I.D. and Symmetric Datasets

For all $i \in \mathcal{N}$ consider symmetric $F_i \equiv F$ and $\lambda(i) = 1/N$, and let the distribution μ_i be a product of i.i.d. draws ω_i, i.e. $\mu_i := \prod_{i \in \mathcal{N}} \alpha_i$ where $\alpha_i \in \Delta(\Omega)$.[16] It then follows that the aggregate dataset is also distributed according to $\alpha \equiv \prod_{i \in \mathcal{N}} \alpha_i$, and so we can write (Opt-z) as

$$\mathbb{E}_{\alpha_i}\left[\omega_i \prod_{j \neq i, j \in \mathcal{N}} \mathbb{E}_{\alpha_j}\left[\mathbb{P}_F\Big(\omega_i z_i > \omega_j \overline{\phi}_{\eta,j}(\theta_j, z_j)\Big)\Big|\omega_i\right]\right] = \frac{\mathbb{E}_{\alpha_i}[\omega_i]}{N}, \quad \forall i \in \mathcal{N}. \tag{16}$$

Moreover, by symmetry there exists $z_N \geq 0$ such that $z_i = z_N$ every $i \in \mathcal{N}$, and (16) is

[15] I interpret the parameter ε as data quality and heterogeneity where higher ε makes the data more homogeneous and likely have higher CTRs.

[16] I abuse notation by denoting α_i the individual datasets and the distribution of CTRs ω_i.

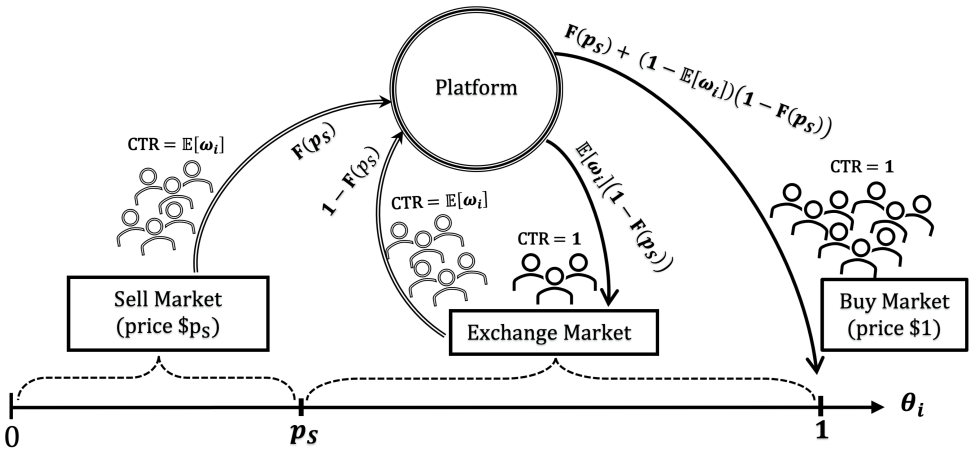


Figure 5: Optimal Large Platform Design ($N \to \infty$).

simplified to a single (non-linear) equation:

$$\mathbb{E}_{\alpha_1}\left[\omega_1\left(\mathbb{E}_{\alpha_2}\left[\mathbb{P}_F\left(\omega_1 z_N > \omega_2 \overline{\phi}_{\eta,2}(\theta_2, z_N)\right)\middle|\omega_1\right]\right)^{N-1}\right] = \frac{\mathbb{E}_{\alpha_1}[\omega_1]}{N}. \tag{17}$$

Lemma 2. *There exists a unique optimal ironing parameter $0 < z_N < 1$ that solves (17) converges to $z_\infty = 1$ as $N \to \infty$.*

A consequence of Lemma 2 is the following (ex-post) implementation of the optimal mechanism:

Proposition 2 (*Optimal Large Platform Design*)**.** *The optimal mechanism purchases data and designs ads using three markets (see also Figure 5):*

i) A selling market for which merchants can sell all their data at a posted price $p_S = \left(\phi^S_{\eta,i}\right)^{-1}(1)$ per unit of CTR.

ii) An exchange market in which merchants sell all their data and get in return ads with $CTR = 1$ that generate a_i (outside option) expected clicks.

iii) A buying market in which high types $\theta_i = 1$ buy ads with $CTR = 1$ at the full price $p_B = 1$.

Surprisingly, for sufficiently large platforms the optimal mechanism can be approximated by a simple ex-post mechanism that jointly designs a buying, a selling, and an exchange market. I discuss next the intuition of the results and conclude the section by discussing asymptotic efficiency of the optimal mechanism.

Intuition for the Optimal Mechanism. Consider the alternative approach where the designer solve for the optimal *profit-maximizing* mechanism in the limiting case with no aggregate uncertainty. At a posted price p per unit of CTR, the designer faces a supply curve of $F(p)$ units of expected CTR equal to $\mathbb{E}_{\mu_i}[\omega_i]$. Now, consider initially the optimal design of a selling and a buying market only where the seller decides on posted prices p_B and p_S (*bid-ask* prices). Because of the large market assumption, every consumer ω available to the designer can be matched efficiently to some merchant $i = 1$ such that $\theta_i \omega_i \approx 1$ and sold at the price $p_B = 1$. Therefore, $p_B = 1$, and at a posted price p the profits from the selling market and buying markets are given by

$$\pi_F^{B,S}(p) = (1 - p\mathbb{E}_{\mu_i}[\omega_i])F(p).$$

The optimal posted price p_S is then calculated by equating marginal revenue in the buying market, $\mathrm{MR}_B(1) = 1$, with the marginal cost in the selling market, $\mathrm{MC}_S(p) = \phi_i^S(p)\mathbb{E}_{\mu_i}[\omega_i]$,[17] resulting in the optimal posted price

$$\tilde{p} = \left(\phi_i^S\right)^{-1}(1/\mathbb{E}_{\mu_i}[\omega_i]).$$

Next suppose the designer introduces an exchange market. The profits from selling the residual units of the exchange market in the buying market are given by

$$\pi_F^{B,E}(p) = (1 - \mathbb{E}_{\mu_i}[\omega_i])(1 - F(p)).$$

In the exchange market the designer exchanges a unit mass of customers of expected CTR equal to $\mathbb{E}_{\mu_i}[\omega_i]$, with $\mathbb{E}_{\mu_i}[\omega_i]$ unit mass of customers of CTR equal to 1. Thus, the profits from operating a selling market at a posted price p alongside the exchange market are given by

$$\pi_F^{B,ES}(p) = \pi_F^{B,S}(p) + \pi_F^{B,E}(p).$$

Since the marginal revenues of each unit procured in the exchange and selling market is equal to 1, the optimal posted price is calculated by equating the marginal costs in each market, i.e. $\mathrm{MC}_S(p)$ with $\mathrm{MC}_E(p) = \mathbb{E}_{\mu_i}[\omega_i]$, yielding optimal posted price

$$p_S = \left(\phi_i^S\right)^{-1}(1) \ (\leq \tilde{p}),$$

where p_S is equal to the posted price in Proposition 2 obtained in the limit.

Note that the designer always does weakly better by introducing an exchange market,

[17] See Bulow and Roberts (1989) for an interpretation of the marginal revenue and cost curves in mechanism design.

i.e.

$$\pi_F^{B,ES}(p_S) \geq \pi_F^{B,S}(\tilde{p}),$$

since the optimal mechanism excludes a positive measure of types from participation when there's a strictly positive bid-ask spread $1 - \tilde{p} > 0$. In particular, the revenues are strictly higher if $\mathbb{E}_{\mu_i}[\omega_i] < 1$. One can generalize these insights and conclude that the value of platform design $\Pi(\alpha, \eta)$ is increasing in the inefficiency of merchants' initial data allocation (low expected CTR $\mathbb{E}_{\mu_i}[\omega_i]$), for any arbitrarily chosen welfare weight η.

Asymptotic Efficiency. Now I discuss the asymptotic relative efficiency (ARE) of the platform for given dataset α and welfare weight η, denoted by $\text{ARE}(\alpha, \eta)$. Formally, $\text{ARE}(\alpha, \eta)$ is measured by the ratio of the value of design $\Pi^\infty(\alpha, \eta)$ to the total surplus $\text{TS}^\infty(\alpha, \zeta)$ for some nonnegative surplus weight $\zeta = (\zeta_w, z_v)$ where

$$\text{ARE}(\alpha, (\eta, \zeta)) = \frac{\Pi^\infty(\alpha, \eta)}{\text{TS}^\infty(\alpha, \zeta)} = \frac{\eta_w \mathcal{W}^\infty(\alpha, \eta) + \eta_v \mathcal{V}^\infty(\alpha, \eta) + \eta_r \mathcal{R}^\infty(\alpha, \eta)}{\zeta_w \mathcal{W}^{\max}(\alpha) + \zeta_v \mathcal{V}^{\max}(\alpha)}. \tag{ARE}$$

From Proposition 2 we know that in the optimal mechanism there's efficient targeting, i.e. every customer ω is targeted by some merchants i with $\omega_i = 1$, thus yielding

$$\mathcal{W}^\infty(\alpha, \eta) = \mathcal{W}^{\max}(\alpha) = 1 - \mathbb{E}_{\mu_i}[\omega_i].$$

As a result, I analyze (ARE) by setting $\eta_w = \zeta_w = 0$. Next, total merchants' surplus that can be achieved from efficient targeted ad design absent any information asymmetries is given by

$$\mathcal{V}^{\max}(\alpha) = \lim_{N \to \infty} \left(\mathbb{E}_{(\alpha, F)} \left[\max_{i \in \mathcal{N}} \omega_i \theta_i \right] - \sum_{i \in \mathcal{N}} \lambda(i) \mathbb{E}_{(\mu_i, F_i)}[\omega_i \theta_i] \right) = 1 - \mathbb{E}_{\mu_i}[\omega_i] \mathbb{E}_{F_i}[\theta_i].$$

Moreover, from our preceding calculations we have $\mathcal{R}^\infty(\alpha, \eta) = \pi_F^{B,ES}(p_S)$. Lastly, only the merchants that participate in the selling market enjoy positive surplus, hence

$$\mathcal{V}^\infty(\alpha, \eta) = \mathbb{E}_{\mu_i}[\omega_i] \mathbb{E}_{F_i}\left[(p_S - \theta_i)_+ \right].$$

Therefore,

$$\text{ARE}(\alpha, \eta_r) = \frac{(1 - \eta_r) \mathbb{E}_{\mu_i}[\omega_i] \mathbb{E}_{F_i}\left[(p_S - \theta_i)_+ \right] + \eta_r \pi_F^{B,ES}(p_S)}{1 - \mathbb{E}_{\mu_i}[\omega_i] \mathbb{E}_{F_i}[\theta_i]}. \tag{18}$$

I now solve an example and perform comparative statics on (18) by varying $\mathbb{E}_{\mu_i}[\omega_i]$:

Example 3. *Let $\theta_i \sim$ Uniform$[0, 1]$ and $\mathbb{E}_{\mu_i}[\omega_i] = \mu \in [0, 1]$. Letting $\eta_w = 0$, the optimal posted price is given by $p_S = 1/(1 + \eta_r)$, thus (18) simplifies to*

$$ARE(\mu, \eta_r) = \Big((1 - \eta_r) \Big(\mu (p_S)^2 / 2 \Big) + \eta_r ((1 - p_S \mu) p_S + (1 - \mu)(1 - p_S)) \Big) \Big/ (1 - \mu/2).$$

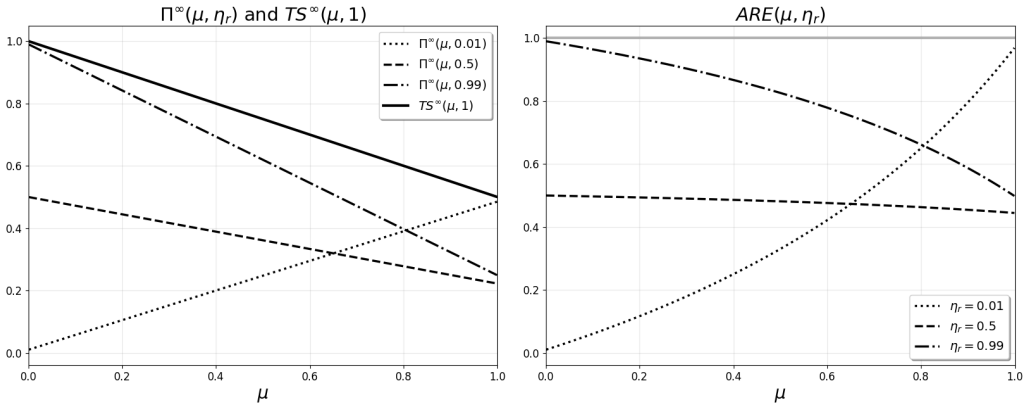


Figure 6: $\Pi^\infty(\varepsilon, \eta_r)$ and $\mathrm{TS}^\infty(\varepsilon, 1)$ (left) and $\mathrm{ARE}(\varepsilon, \eta_r)$ (right) for $\eta_r \in \{0.01, 0.5, 0.99\}$.

In Figure 6 I plot $ARE(\mu, \eta_r)$ for $\eta_r \in \{0.01, 0.5, 0.99\}$. When expected CTRs are low ($\mu \approx 0$), most of the surplus is enjoyed by the designer since the cost of operating the exchange and selling markets are sufficiently small. Thus for sufficiently high $\eta_r \approx 1$ we achieve asymptotic efficiency. If expected CTRs are high ($\mu \approx 1$), the designer's profits from operating the exchange market are close to zero. The only way to get efficiency is by paying merchants the highest price $p_S \approx 1$ —which is optimal when $\eta_r \approx 0$—thus giving all the surplus to the merchants.

These observations generalize to arbitrary distributions F. In particular, $\mathrm{ARE}(\mu, \eta_r) < 1$ unless $\mathbb{E}_{\mu_i}[\omega_i] = 0$ *and* $\eta_r = 1$, or $\mathbb{E}_{\mu_i}[\omega_i] = 1$ *and* $\eta_r = 0$. The main source of inefficiency comes from the observation that merchants are compensated at a price p_S per unit of clicks on their *individual ads*, i.e. $\mathbb{E}_{\mu_i}[p_S \omega_i] = p_S \mathbb{E}_{\mu_i}[\omega_i]$, instead of at a price p_S per unit of expected clicks on *efficient ads*, i.e. $\mathbb{E}_{\mu_i}[p_S \max_j(\omega_j)] = p_S \mathbb{E}_{\mu_i}[\max_j(\omega_j)]$. That is, for a customer ω initially in merchant i's dataset, the contract between the two parties does not internalize the customer's market value $\max_j(\omega_j)$; instead, it is written on the merchant-specific CTR ω_i.

6 Revisiting Model Assumptions and Applications

6.1 Revisiting Model Assumptions

Limits of Data Sharing. Another frequently asked question—"Can I exclude specific customers from my audiences?"[18]—raises the question of whether Shopify should make the

[18] https://help.shopify.com/en/manual/promoting-marketing/shopify-audiences/faq#can-i-exclude-specific-customers-from-my-audiences

choice to choose which data to share part of the design. Notably, Shopify does not give the flexibility to their merchants: "...you can't add or exclude customers to a specific audience." In our setting, doing so is without loss of generality. Our modeling choice of letting CTRs be observable is crucial in this aspect. One can think of settings in which merchants know more about their customers than the designer does, hence they may be selective in what audience they share if they had the flexibility to do so. Since merchants might know more about their customers than the designer, merchants can potentially take advantage of the flexibility by not sharing data about their most valuable customers. This in turn would be rationalized by the designer and analyzing the optimal design requires a different approach. Formally, this speaks to the literature on mechanism design with limited commitment pioneered by Bester and Strausz (2001) (and more recently Doval and Skreta (2022)). Generally, analyzing such problems can be challenging and is left for future research.

Data Complementarities. Recall that customers have unit-demand and merchants don't compete on prices; hence, without loss of generality, the designer assigns a single "winner" to target each customer. In practice, the advantages of designing such platforms can be significant when leveraging additional data insights from complementary products—"the Audiences algorithms might provide insights based on people who have purchased complementary products in the past."[19] For example, if you sell swimsuits, then the designer can provide you with targeted ads about customers who have purchased complementary products like sandals, sunglasses, sunscreen and sunhat. Extending the analysis to platforms that exploit complementarities and serve multi-unit-demand customer is left for future research.

6.2 Applications

The model introduced in Section 2 is broad in scope and is applicable to other economic settings that involve trading of goods among a set of agents.

Greenhouse gas (GHG) credit market. A GHG credit (e.g., carbon credit) is "a tradable instrument that conveys a claim to avoided GHG emissions or to the enhanced removal of GHG from the atmosphere."[20] The GHG credit market is a trading system in which firms buy and sell rights to emit specific GHGs. Each firm $i \in \mathcal{N}$ holds a portfolio of credits $\alpha_i = \{\alpha_i^\omega\}$ across GHG $\omega \in \Omega$. Let ω_i represent the intensity of using GHG ω at the industry where i operates. Firms need to comply with emissions targets set by regulatory

[19] https://www.shopify.com/partners/blog/shopify-audiences
[20] https://en.wikipedia.org/wiki/Carbon_offsets_and_credits

agencies and have private compliance cost of θ_i—for example, outdated, emissions-intensive tech and fossil-heavy energy have a higher compliance cost θ_i than modern and cleaner tech, thus willing to pay more for a unit of GHG credit.

Article 6 of the 2015 Paris Agreement on climate change enables parties to cooperate in reducing their greenhouse gas emissions—this means that emission reductions can be transferred between countries and counted towards nationally determined contributions. The Paris Agreement offers ways in which parties may cooperate on a voluntary basis including market mechanisms like the carbon credit market as well as non-market-based approaches. This paper offers a market mechanism to trade all types of GHG to help meet the desired emission levels.

Reallocating public resources. Public resources like water/land rights and fishing quotas are often distributed based on outdated criteria or arbitrarily. For example, California's Water Allocation during the 2012–2016 Drought distributed water rights based on a seniority system dating back to the 19th century.

The paper proposes a market mechanism in which locals ($\mathcal{N}$) trade resources ω (like water/land rights and fishing quotas) where ω_i represents how accessible/ease of use the resource ω is to local i, and θ_i represents personal costs (e.g., time, money, effort) to capitalize on gains from any resource.

Combinatorial Exchange Markets. Combinatorial exchanges enable trading when agents have non-additive valuations over bundles of goods—a setting where determining market-clearing prices and efficient allocations is computationally challenging (often NP-hard). The literature proposes clock-based mechanisms that reveal preferences through dynamic price adjustments (Rothkopf et al. (1998), Cramton (1998), Ausubel et al. (2012)). Likewise, bundle trading models in financial markets (Srinivasan (1999), Abrache et al. (2005), Wang et al. (2021)) allow traders to submit consolidated bids to buy or sell packages (bundles) of assets simultaneously, rather than trading individual assets separately. They consider price-based mechanisms to reflect traders' private valuations for bundles, ensuring market clearing. Due to computational difficulty of finding the optimal design, (traditional) options markets trade contracts individually.

Our finding point toward new directions in combinatorial market exchange designs, such exploring novel mechanisms that design exchange markets alongside the price-based mechanisms. The possibility of *pure* bundle exchange without bid-ask quotes can potentially help in mitigating information asymmetries and improve upon current designs.

7 Conclusions

This paper proposes proposes a model of targeted advertising platforms that host merchants who share their data and compete with one another for targeted ads. I then solve for an optimal design that mitigates poaching concerns by compensating merchants with either higher CTR ads or transfers. In a large market, the optimal design simplifies to the design of three markets: a selling, a buying and an exchange market. I discuss model extension on data complementarities and data sharing policies. Lastly, I note the model's broader applicability in market design settings such as the GHG credit markets and reallocation of public resources, and highlight promising directions in designing novel mechanisms for combinatorial market exchanges.

References

Abrache, J., T. G. Crainic, and M. Gendreau (2005): "Models for bundle trading in financial markets," *European Journal of Operational Research*, 160, 88–105.

Ausubel, L., J. Levin, and I. Segal (2012): "Incentive auction rules option and discussion," *Appendix to the FCCs*.

Bergemann, D. and A. Bonatti (2015): "Selling cookies," *American Economic Journal: Microeconomics*, 7, 259–294.

——— (2024): "Data, competition, and digital platforms," *American Economic Review*, 114, 2553–2595.

Bergemann, D., A. Bonatti, and T. Gan (2022): "The economics of social data," *The RAND Journal of Economics*, 53, 263–296.

Bergemann, D., A. Bonatti, and N. Wu (2024): "How Do Digital Advertising Auctions Impact Product Prices?" *Review of Economic Studies*, rdae087.

Bester, H. and R. Strausz (2001): "Contracting with imperfect commitment and the revelation principle: the single agent case," *Econometrica*, 69, 1077–1098.

Bleier, A. and M. Eisenbeiss (2015): "Personalized online advertising effectiveness: The interplay of what, when, and where," *Marketing Science*, 34, 669–688.

Bulow, J. and J. Roberts (1989): "The simple economics of optimal auctions," *Journal of political economy*, 97, 1060–1090.

Cramton, P. (1998): "The efficiency of the FCC spectrum auctions," *The Journal of Law and Economics*, 41, 727–736.

Cramton, P., R. Gibbons, and P. Klemperer (1987): "Dissolving a partnership efficiently," *Econometrica: Journal of the Econometric Society*, 615–632.

Doval, L. and V. Skreta (2022): "Mechanism design with limited commitment," *Econometrica*, 90, 1463–1500.

Elliott, M., A. Galeotti, A. Koh, and W. Li (2022): "Matching and information design in marketplaces," *Available at SSRN 4283968*.

Farboodi, M. and L. Veldkamp (2023): "Data and markets," *Annual Review of Economics*, 15, 23–40.

Galperti, S., A. Levkun, and J. Perego (2024): "The value of data records," *Review of Economic Studies*, 91, 1007–1038.

Galperti, S. and J. Perego (2022): "Competitive markets for personal data," *Available at SSRN 4309966*.

Hoffman, K. and D. Menon (2010): "A practical combinatorial clock exchange for spectrum licenses," *Decision Analysis*, 7, 58–77.

Lambrecht, A. and C. Tucker (2013): "When does retargeting work? Information specificity in online advertising," *Journal of Marketing research*, 50, 561–576.

Loertscher, S. and E. V. Muir (2024): "Optimal Hotelling auctions," *Mimeo*.

Loertscher, S. and C. Wasser (2019): "Optimal structure and dissolution of partnerships," *Theoretical Economics*, 14, 1063–1114.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of operations research*, 6, 58–73.

Myerson, R. B. and M. A. Satterthwaite (1983): "Efficient mechanisms for bilateral trading," *Journal of economic theory*, 29, 265–281.

Rothkopf, M. H., A. Pekeč, and R. M. Harstad (1998): "Computationally manageable combinational auctions," *Management science*, 44, 1131–1147.

Srinivasan, S. (1999): "Portfolio trading and electronic networks," *Available at SSRN 191368*.

von Neumann, J. (1928): "Zur theorie der gesellschaftsspiele," *Mathematische annalen*, 100, 295–320.

Wang, X., D. M. Pennock, N. R. Devanur, D. M. Rothschild, B. Tao, and M. P. Wellman (2021): "Designing a combinatorial financial options market," in *Proceedings of the 22nd ACM Conference on Economics and Computation*, 864–883.

Yang, F. (2023): "Nested bundling," Tech. rep., Working Paper.

Appendix

A Proofs for Finite Case

A.1 Proof of Lemma 1

By envelope theorem, $U_i'(\theta_i) = S_i^*(\theta_i) - a_i$ almost everywhere, implying that $U_i(\theta_i)$ is convex in θ_i by (M) and absolute continuity of U_i.[21] Then, similar to Cramton et al. (1987), the set $\hat{\Theta}_i(x,t)$ can be characterized in terms of ads rule x^* alone as in (2).

A.2 Proof of Theorem 1

The steps follow closely Loertscher and Wasser (2019) and Loertscher and Muir (2024).

Eliminating (IR). Fix an IC mechanism (x,t). By definition, it is sufficient that (IR) holds at every $\hat{\theta} \in \hat{\Theta}(x,t)$, which from optimality has $U_i(\hat{\theta}_i) = 0$. Now note that the RHS in (6) is constant for all $\theta' \in \Theta$. By definition, for any two arbitrary $\hat{\theta} \in \hat{\Theta}(x,t)$ and $\theta' \in \Theta \setminus \hat{\Theta}(x,t)$ we have $U_i(\theta') \geq U_i(\hat{\theta}_i)$ for all $i \in \mathcal{N}$, with at least one inequality being strict. It then follows that

$$\psi_\eta(x,\theta') > \psi_\eta(x,\hat{\theta}). \tag{19}$$

Since Θ is compact, the worst-off types admit the following characterization as a function of the ads rule x alone:

$$\hat{\Theta}(x) = \operatorname*{argmin}_{\theta' \in \Theta} \psi_\eta(x,\theta'). \tag{20}$$

It then follows that for any $\theta' = \hat{\theta}(x) \in \hat{\Theta}(x)$, using (6) we get

$$\Psi(x,t) = \psi_\eta(x,\hat{\theta}(x)) = \min_{\theta' \in \Theta} \psi_\eta(x,\theta'). \tag{21}$$

Therefore, the designer's problem is equivalently given by

$$\Pi(\alpha,\eta) = \sup_{x \in \mathcal{X}(\Theta, \times dF_i)} \min_{\theta' \in \Theta} \psi_\eta(x,\theta'). \tag{Sup-Min}$$

Lastly, note that $\mathcal{X}(\Theta, \times dF_i) \subseteq \mathcal{L}(\Theta, \times dF_i)$ where ads rules $x \equiv x'$ dF-a.e. are identified and $\mathcal{L}(\Theta, \times dF_i)$ is endowed with the weak*-topology. Since constraints on x are linear constraints, $\mathcal{X}(\Theta, \times dF_i)$ is a convex set and by Alaoglu's theorem $\mathcal{X}(\Theta, \times dF_i)$ is a compact set. Therefore, one can replace the sup in (Sup-Min) with max.

[21] If a function h is absolutely continuous and h' is increasing *almost everywhere* then h is convex.

Existence. Observe first that $\psi_\eta(\cdot,\theta') : \mathcal{X} \to \mathbb{R}$ is a linear functional for every $\theta' \in \Theta$, hence continuous. On the other hand, $\psi_\eta(x,\cdot) : \Theta \to \mathbb{R}$ is continuous at every $x \in \mathcal{X}(\Theta, \times dF_i)$ since we can write $\Psi(x,\theta')$ as

$$\Psi(x,\theta') = \sum_{i\in\mathcal{N}} \left(\int_0^{\theta_i'} (S_i(\theta_i) - a_i)\phi_{\eta,i}^S(\theta_i) dF_i(\theta_i) + \int_{\theta_i'}^1 (S_i(\theta_i) - a_i)\phi_{\eta,i}^B(\theta_i) dF_i(\theta_i) \right).$$

Since S_i, $\phi_{\eta,i}^S$ and $\phi_{\eta,i}^B$ are measurable functions, it follows that (see Royden, 4th Ed., Chapter 5, Theorem 14) $\Psi(x,\cdot)$ is absolutely continuous and

$$\frac{\partial \Psi(x,\theta')}{\partial \theta_i'} = S_i(\theta_i') - a_i \quad \text{and} \quad \frac{\partial^2 \Psi(x,\theta')}{\partial \theta_j' \partial \theta_i'} = 0 \quad dF\text{-a.e.}$$

Then, since $\partial\Psi(x,\cdot)/\partial\theta'$ increasing dF-a.e. by (M), $\Psi(x,\cdot)$ is convex at every $x \in \mathcal{X}(\Theta, \times dF_i)$. Lastly, $\Psi(\cdot,\theta')$ is continuous at each $\theta' \in \Theta$. Hence, applying the generalized version of von Neumann's minimax theorem we get the existence of a saddle point (i.e. an optimal mechanism) and that we can swap the order between max and min.

A.3 Proof of Proposition 1

I construct EP scoring mechanisms by finding (essentially unique) ironing parameter z and inclusive tie-breaking rule $p^{(1,1)}$ such that the solution x^* to (x-pw) satisfies $\hat{\theta}_i(z_i) \in \hat{\Theta}_i(x^*)$ for every $i = 1, 2$.[22]

Observe first that symmetry on F implies $\phi_{\eta,i}(s,z_i) \geq \phi_j(s,z_j)$ for every $s \in [\underline{\theta}, \overline{\theta}]$ if and only if $z_i \geq z_j$. Then, the expected clicks from inclusive customers satisfy $S_i^{(1,1)}(\hat{\theta}_i(z_i)) < S_j^{(1,1)}(\hat{\theta}_j(z_j))$ whenever $(\underline{z}_\eta \leq) z_i < z_j$ and $\alpha(1,1) > 0$, regardless of the inclusive tie-breaking rule $p^{(1,1)}$. Therefore, if merchant i has a higher adjusted share, i.e. $\beta_i \geq \beta_j$, then the optimal ironing parameter is such that $z_i \geq z_i$ to ensure (in expectation) the critical worst-off type $\hat{\theta}_i(z_i)$ their outside option.

i) $\alpha \in \mathcal{R}_{+,+}$ and $\beta_1 \geq \beta_2$. Consider first the case $\phi_{\eta,i}^B(0) < 0$.[23] I show that there exists $z_1 \geq z_2$ and $p^{(1,1)}$ such that $S_i(\hat{\theta}_i(z_i)) = a_i$.

Since for every $i = 1, 2$ the adjusted share $\beta_i > 0$ then $\alpha_i^{(1,1)} > \alpha_j^{\omega_i}$. By definition of EP scoring mechanism and optimality, the critical worst-off type $\hat{\theta}_i(z_i)$ is assigned all of the exclusive targeted ads $\alpha_j^{\omega_i}$, regardless of the value of z_i.

Then, suppose first that there exists $\tilde{z} \geq 0$ such that $z_1 = z_2 = \tilde{z}$. From (x-pw) the condition $S_i(\hat{\theta}_i(\tilde{z})) = a_i$ for each $i = 1, 2$ reduces to solving the following system of

[22] Note that $p^{(1,1)}(z)$ matters only when $z_1 = z_2$ and $\overline{\phi}_{\eta,i}(\theta_i, z_i) = z_i$ occurs with strictly positive probability.

[23] It follows that $\underline{z}_i := \max\{0, \phi_{\eta,i}^B(0)\} = 0$, $P_i^B(0) > 0$ and $P_i^S(z_i) = 0$ for every $z_i \in [0, \phi_{\eta,i}^S(0)]$, where $\phi_{\eta,i}^S(0) = \eta_w \geq 0$ for every $i = 1, 2$.

(nonlinear) equations:

$$\begin{cases} \alpha(1,1)\left(P_2^S(\tilde{z}) + p_1^{(1,1)}\left(P_2^B(\tilde{z}) - P_2^S(\tilde{z})\right)\right) + \alpha_2^{(1,0)} = \alpha_1^{(1,1)} \\ \alpha(1,1)\left(P_1^S(\tilde{z}) + p_2^{(1,1)}\left(P_1^B(\tilde{z}) - P_1^S(\tilde{z})\right)\right) + \alpha_1^{(0,1)} = \alpha_2^{(1,1)} \end{cases} \tag{22}$$

for some inclusive tie-breaking rule $p^{(1,1)}$ such that $p_1^{(1,1)} + p_2^{(1,1)} = 1$ whenever $\tilde{z} > 0$. For every $k = B, S$, since $P_1^k \equiv P_2^k$ by symmetry on F, define $P_\eta^k := P_i^k$ for every $i = 1, 2$ and rearrange (22) to get

$$p_i^{(1,1)} P_\eta^B(\tilde{z}) + \left(1 - p_i^{(1,1)}\right) P_\eta^S(\tilde{z}) = \beta_i, \quad \forall i = 1, 2. \tag{23}$$

Now, adding both sides of the equations in (23) gives

$$\left(p_1^{(1,1)} + p_2^{(1,1)}\right) P_\eta^B(\tilde{z}) + \left(2 - p_1^{(1,1)} - p_2^{(1,1)}\right) P_\eta^S(\tilde{z}) = \beta_1 + \beta_2. \tag{24}$$

Then, since P_η^S is non-decreasing and P_η^B is strictly increasing on $[0, 1]$, $\beta_1 + \beta_2 \leq 1$ and $p_1^{(1,1)} + p_2^{(1,1)} = 1$ whenever $\tilde{z} > 0$, from (24) it follows readily that $\tilde{z} = \underline{z}_i (= 0)$ if and only if $\beta_1 + \beta_2 \leq P_\eta^B(\underline{z}_i)$. In that case, using (23) it follows that there exists unique inclusive tie-breaking rules $p_i^{(1,1)} > 0$ such that $p_i^{(1,1)} = \beta_i / P_\eta^B(\underline{z}_i)$ for every $i = 1, 2$.

So, suppose instead that $\beta_1 + \beta_2 > P_\eta^B(\underline{z}_i)$. Then, $\tilde{z} > 0$ and $p_1^{(1,1)} + p_2^{(1,1)} = 1$, and the identity in (24) reduces to

$$P_\eta^B(\tilde{z}) + P_\eta^S(\tilde{z}) = \beta_1 + \beta_2. \tag{25}$$

Since $P_\eta^B(1) = 1$, it follows readily that there exists a unique $\tilde{z} \in [0, 1]$ satisfying (25). Then, using (23), solving for $p_1^{(1,1)}$ yields

$$p_1^{(1,1)} = \frac{1}{2}\left(1 + \frac{\beta_1 - \beta_2}{P_\eta^B(\tilde{z}) - P_\eta^S(\tilde{z})}\right), \tag{26}$$

where $P_\eta^B(\tilde{z}) - P_\eta^S(\tilde{z}) > 0$ since F is regular. Hence, a unique inclusive tie-breaking rule $p^{(1,1)}$ exists with $z_1 = z_2 = \tilde{z} > 0$ if and only if (26) is well-defined, i.e. $p_1^{(1,1)} \leq 1$ or $P_\eta^B(\tilde{z}) - P_\eta^S(\tilde{z}) \geq \beta_1 - \beta_2$.

Thus, suppose now that $\beta_1 + \beta_2 > P_\eta^B(\underline{z}_i)$ and $P_\eta^B(\tilde{z}) - P_\eta^S(\tilde{z}) < \beta_1 - \beta_2$, where $\tilde{z}$ is the solution to (25). I show that there exists unique $z_1 \geq \tilde{z} \geq z_2 > \eta_w$, with at least one of the inequalities being strict.

To see this, note that the analog of the nonlinear system in (22) that the designer solves when $z_1 > z_2$ is given by

$$\begin{cases} \alpha(1,1) P_\eta^B(z_1) + \alpha_2^{(1,0)} = \alpha_1^{(1,1)} \\ \alpha(1,1) P_\eta^S(z_2) + \alpha_1^{(0,1)} = \alpha_2^{(1,1)} \end{cases} \tag{27}$$

Rearranging (27) yields

$$P_\eta^B(z_1) = \beta_1 \text{ and } P_\eta^S(z_2) = \beta_2. \tag{28}$$

Then, since $\beta_2 > 0$ and $P_\eta^S(z_2) = 0$ for all $z_2 \leq \eta_w$, the ironing parameter z_2 must satisfy $z_2 > \phi_\eta^S(0)$. Moreover, since $P_\eta^B(\tilde{z}) + P_\eta^S(\tilde{z}) = \beta_1 + \beta_2$ and $P_\eta^B(\tilde{z}) - P_\eta^S(\tilde{z}) < \beta_1 - \beta_2$, it follows readily that $\beta_2 \leq P_\eta^S(\tilde{z}) < P_\eta^B(\tilde{z}) \leq \beta_1$. Then, since P_η^S is strictly increasing on $[\eta_w, \tilde{z}]$ and P_η^B is strictly increasing on $[\tilde{z}, 1]$ with $P_\eta^S(\eta_w) = 0$ and $P_\eta^B(1) = 1$, there exists unique $\tilde{z} \downarrow z_2$ and $\tilde{z} \uparrow z_1$ that solve (28).

Lastly, consider the case $\phi_{\eta,i}^B(0) \geq 0$.[24] Here, we always have $\beta_1 + \beta_2 > P_\eta^B(\underline{z}_i) = 0$. The rest is identical to the preceding steps. □

ii) $\alpha \in \mathcal{R}_{+,0}$**.** Consider first the case $\phi_{\eta,i}^B(0) < 0$. I show that there exists $z_1 \geq z_2$ and $p^{(1,1)}$ such that $S_i(\hat{\theta}_i(z_i)) = a_i$.

Fix some optimal z_1. Since $\beta_2 = 0$, suppose first that $\alpha_2^{(1,1)} < \alpha_1^{(0,1)}$. Then, to have $S_2(\hat{\theta}_2(z_2)) = \alpha_2^{(1,1)}$ it must be that $z_2 = 0$, as otherwise by optimality merchant 2 would target all $\alpha_1^{(0,1)}$ mass of customers if $z_2 > 0$. Therefore, by definition of EP-scoring mechanism the designer sets $p_2^{(0,1)} = \alpha_2^{(1,1)}/\alpha_1^{(0,1)}$. On the other hand, if $\alpha_2^{(1,1)} = \alpha_1^{(0,1)}$, then any $0 \leq z_2 \leq \min\{z_1, \eta_w\}$ can be optimal. Suppose to the contrary that $(z_1 \geq) z_2 > \eta$. But then merchant 2 wins the right to target inclusive customers with probability $P_\eta^S(z_2) > 0$, yielding in total $S_2(\hat{\theta}_2(z_2)) > \alpha_2^{(1,1)}$. For simplicity of exposition, I let $z_2 = \underline{z}_\eta = 0$.

To solve for z_1, suppose first that $\beta_1 \leq P_\eta^B(\underline{z}_\eta)$. Then setting $z_1 = \underline{z}_\eta$ and inclusive tie-breaking rule $p^{(1,1)}$ such that $p_1^{(1,1)} = \beta_1/P_\eta^B(0)$ and $p_2^{(1,1)} = 0$ yields $S_i(\hat{\theta}_i(z_i)) = \alpha_i^{(1,1)}$ for every $i = 1, 2$. On the other hand, if $\beta_1 > P_\eta^B(0)$ then similar to part i) it follows readily that there exists unique $z_1 = (P_\eta^B)^{-1}(\beta_1) > 0$. If $z_2 = z_1$, then break ties in favor of merchant 1, i.e. $p_1^{(1,1)} = 1$.

Next, consider the case $\phi_{\eta,i}^B(0) \geq 0$. I show that there exists $z_1 \geq z_2$ and $p^{(1,1)}$ such that $S_i(\hat{\theta}_i(z_i)) = a_i$, unless $\phi_{\eta,i}^B(0) > 0$ *and* $\alpha_2^{(1,1)} < \alpha_1^{(0,1)}$, in which case $S_i(\hat{\theta}_i(z_i)) > a_i$.

Thus, suppose first $\phi_{\eta,i}^B(0) = 0$ or $\alpha_2^{(1,1)} = \alpha_1^{(0,1)}$. Solving for z_2 is similar to the preceding steps: any $\underline{z}_\eta \leq z_2 \leq \min\{z_2, \eta_w\}$ is optimal. On the other hand, since $\beta_1 > 0 = P_\eta^B(\underline{z}_\eta)$, it follows readily that $z_1 = (P_\eta^B)^{-1}(\beta_1) > \underline{z}_\eta$. Lastly, if $z_2 = z_1$, then similarly break ties in favor of merchant 1, i.e. $p_1^{(1,1)} = 1$.

Now suppose that $\phi_{\eta,i}^B(0) > 0$ and $\alpha_2^{(1,1)} < \alpha_1^{(0,1)}$. Since $\underline{z}_\eta = \phi_{\eta,i}^B(0) > 0$, then by optimality $p_2^{(0,1)} = 1$. Thus, any $\underline{z}_\eta \leq z_2 \leq \min\{z_2, \eta_w\}$ is optimal but gives $S_2(\hat{\theta}_2(z_2)) > a_2$. Solving for z_1 we have again $z_1 = (P_\eta^B)^{-1}(\beta_1) > \underline{z}_\eta$, where again ties are broken in favor of merchant 1. □

iii) $\alpha \in \mathcal{R}_{0,0}$**.** Clearly, $z_i = \underline{z}_i$ for each $i = 1, 2$. Now, if $\phi_{\eta,i}^B(\underline{z}_\eta) \leq 0$ then by definition of EP scoring mechanism the designer assigns $p_i^{\omega_i} = \alpha_i^{(1,1)}/\alpha_j^{\omega_i}$ and $p_i^{(1,1)} = 0$, yielding

[24] Now $\underline{z}_i := \phi_{\eta,i}^B(0) = \eta_w - \eta_r/f(0) \geq 0$, $P_i^B(\underline{z}_i) = 0$ and $P_i^S(z_i) = 0$ for every $z_i \in [\underline{z}_i, \eta_w]$ and $i = 1, 2$.

$S_i(\hat{\theta}_i(\underline{z}_i)) = a_i$ for each $i = 1, 2$.[25] Lastly, consider $\phi^B_{\eta,i}(\underline{z}_\eta) > 0$. Then by optimality the designer assigns $p_i^{\omega_i} = 1$, yielding $S_i(\hat{\theta}_i(\underline{z}_i)) = a_i$ for each $i = 1, 2$.[26] □

B Proofs for Continuum Case

B.1 Proof of Theorem 2

Fix some $\theta \in \Theta$. Suppose first that $\phi_{\eta,i}\Big(\theta_i, z_i(\hat{\theta}_i)\Big) > 0$ for some $i \in \mathcal{N}$. It then follows that for every $\omega \in \Omega$, there exists a unique $i^\omega \in \mathcal{N}_0$ such that $\omega \in E^z_{i_\omega}(\theta)$ (since it's a partition of the state space Ω). Then, x maximizing (14) pointwise implies that for every $i \in \mathcal{N}_0$ and $\omega \in \Omega$, $x_i^\omega(\theta) = 0$ for every $i \in \mathcal{N}$ such that $i \neq i_\omega$.

Thus, consider $i \in \mathcal{N}$ such that $E^z_i(\theta)$ has strictly positive measure. Then, to determine the (pointwise) optimal ads function $x_i^\omega(\theta)$ on the set of CTRs $E^z_i(\theta)$, using the fact that x must satisfy (Marg) and the preceding analysis, one has

$$\mathbb{E}_\alpha[x_i^\omega(\theta) \cdot \mathbb{1}_E(\omega)] = \alpha(E), \quad \forall E \subseteq E^z_i(\theta) \text{ and } E \in \mathcal{B}(\Omega).$$

Thus, since the function

$$\mu_{\theta,i}(E) := \mathbb{E}_\alpha[x_i^\omega(\theta) \cdot \mathbb{1}_E(\omega)], \quad \forall E \in \mathcal{B}(E^z_i(\theta))$$

defines a σ-finite measure on $\mathcal{B}(E^z_i(\theta))$, and $\mu_{\theta,i} \equiv \alpha|_{E^z_i(\theta)}$ then by Radon-Nikodym theorem the ads function $x_i(\theta)$ is the Radon-Nikodym derivative of the two measures, i.e.

$$x_i(\theta) = \frac{d\mu_{\theta,i}}{d\alpha|_{E^z_i(\theta)}} \equiv 1 \text{ a.e. on } E^z_i(\theta).$$

Thus, one concludes that

$$x_i^\omega(\theta) = \mathbb{1}_{E^z_i(\theta)}(\omega), \quad \forall \omega \in \Omega.$$

B.2 Proof of Lemma 2

I first show that there exists a unique $z_N \in (0, 1)$ which is strictly monotone in (sufficiently large) N. Define

$$x(z_k, \omega_1) := \mathbb{E}_{\alpha_2}\left[\mathbb{P}_F\Big(\omega_1 z_k > \omega_2 \overline{\phi}_{\eta,2}(\theta_2, z_k)\Big)\Big|\omega_1\right]$$

[25] Note the multiplicity in optimal tie-breaking rules and how the restriction to EP scoring mechanisms makes the exposition simpler.

[26] For example, in this case ties ($z_1 = z_2$) occur with probability 0, so any $p^{(1,1)}$ works.

where z_k solves

$$\mathbb{E}_{\alpha_1}\Big[\omega_1(x(z_k,\omega_1))^{k-1}\Big] = \frac{\mathbb{E}_{\alpha_1}[\omega_1]}{k}, \quad \forall k. \tag{29}$$

Fixing k, note that the function

$$h(z) := \mathbb{E}_{\alpha_1}\Big[\omega_1(x(z,\omega_1))^{k-1}\Big]$$

is strictly increasing in z with $h(0) = 0$ and $h(1) > \mathbb{E}_{\alpha_1}[\omega_1]/k$. When these conditions hold, then by intermediate value theorem we conclude that there exists a unique $0 < z_k < 1$ that solves (29).

To see that $h(0) = 0$ and $h(z)$ strictly increasing, observe that the probability

$$\mathbb{P}_F\Big(\omega_1 z > \omega_2\overline{\phi}_{\eta,2}(\theta_2, z)\Big) = \mathbb{P}\Big(\phi^B_{\eta,2}(\theta_2) < w_1 z/w_2\Big)\mathbb{1}_{\{w_1>w_2\}} + \mathbb{P}\Big(\phi^S_{\eta,2}(\theta_2) < w_1 z/w_2\Big)\mathbb{1}_{\{w_2>w_1\}}$$

is 0 for $z = 0$ and strictly increasing in z. Moreover, to show $h(1) > \mathbb{E}_{\alpha_1}[\omega_1]/k$, note that merchant i with type $\hat{\theta}_i(1)$ has score $g_i(\hat{\theta}_i(1)) = 1$. But then,

$$\mathbb{P}_F\Big(\omega_1 > \omega_2\overline{\phi}_{\eta,2}(\theta_2, 1)\Big) = \mathbb{1}_{\{w_1>w_2\}} + \mathbb{P}\Big(\phi^S_{\eta,2}(\theta_2) < w_1 z/w_2\Big)\mathbb{1}_{\{w_2>w_1\}},$$

implying that merchant type $\hat{\theta}_i(1)$ is assigned at least all the ads that target consumers with $\omega_i > \omega_j$, for all $j \neq i$. By symmetry, there are in total $1/N$ mass of such customers. But then, since μ_i is continuous with support on $[0,1]$, for any $k \geq 2$ the expected CTR of the $1/N$ mass of customers is strictly higher than the initial $\mathbb{E}_{\mu_i}[\omega_i]$, implying $h(1) > \mathbb{E}_{\alpha_1}[\omega_1]/k$.

Next, I show that z_N is strictly monotone. Since $z_N \in (0,1)$, we claim that $x(z_k,\omega_1) < 1$, for all ω_1. To see this, note that

$$\mathbb{E}_{\alpha_2}\Big[\mathbb{P}_F\Big(\omega_1 z_k > \omega_2\overline{\phi}_{\eta,2}(\theta_2, z_k)\Big)\Big|\omega_1\Big] < 1, \quad \forall \omega_1,$$

since the set

$$\{(\omega_2,\theta) : \omega_2\overline{\phi}_{\eta,2}(\theta_2, z_k) > \omega_1 z_k\}$$

has a positive measure ($\phi^B_{\eta,2}(\theta_2, z_k) > z_k$ for a positive measure of types θ_2). Next, since $x(z_k,\omega_1) \in (0,1)$, for $k = N$ sufficiently large we have

$$(x(z_N,\omega_1))^{N-1} - \frac{1}{N} < (x(z_N,\omega_1))^N - \frac{1}{N+1}, \quad \forall \omega_1.$$

It then follows that the solution to (29) for $k = N+1$ must be such that $z_{N+1} > z_N$. Thus, since z_N is strictly increasing and bounded, hence it converges (uniquely) to some $z_\infty \leq 1$. We claim that $z_\infty = 1$. Suppose to the contrary that $z_\infty < 1$. Then, by similar arguments $x(z_\infty,\omega) < 1$ for all ω_1. Since $x(z_k,\omega_1) \leq x(z_\infty,\omega_1) < 1$ for all k, we have

$$\lim_{N\to\infty} N(x(z_N,\omega_1))^{N-1} \leq \lim_{N\to\infty} N(x(z_\infty,\omega_1))^{N-1} \to 0 < \mathbb{E}_{\alpha_1}[\omega_1]. \tag{30}$$

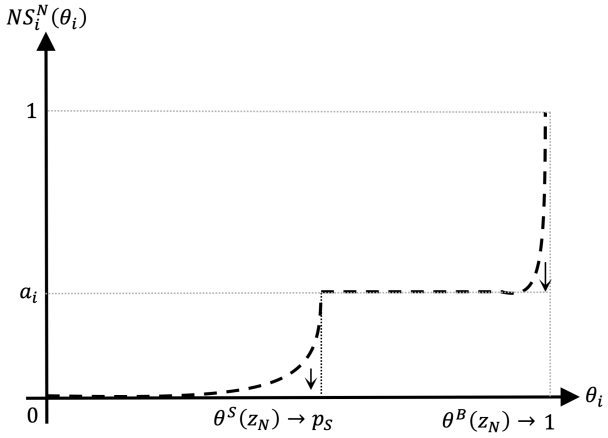


Figure 7: Expected clicks as $N \to \infty$.

Then, an immediate application of dominated convergence theorem implies that

$$\lim_{N\to\infty} \mathbb{E}_{\alpha_1}\Big[\omega_1\Big(N(x(z_N,\omega_1))^{N-1} - \mathbb{E}_{\alpha_1}[\omega_1]\Big)\Big] = \mathbb{E}_{\alpha_1}\Big[\omega_1\Big(\lim_{N\to\infty} N(x(z_N,\omega_1))^{N-1} - \mathbb{E}_{\alpha_1}[\omega_1]\Big)\Big] < 0,$$

contradicting with (29).

B.3 Proof of Proposition 2

By similar arguments as in (30) (see also Figure 7) we have

$$\lim_{N\to\infty} NS_i^N(\theta_i) = \begin{cases} 0, & \text{if } \theta_i < p_S, \\ \mathbb{E}_{\mu_i}[\omega_i], & \text{if } p_S \le \theta_i < 1. \end{cases} \tag{31}$$

where $p_S = \left(\phi^S_{\eta,i}\right)^{-1}(1)$. Given i.d.d. assumptions, in the limit types $\theta_i \in [p_S, 1)$ receive ads with $\omega_i \to 1$, so there's a residue of

$$F(p_S) + (1 - \mathbb{E}_{\mu_i}[\omega_i])(1 - F(p_S))$$

customers which the designer can sell at the (zero measure) buying market to some highest types $\theta_i \to 1$ having $\omega_i \to 1$. For $\theta_i < 1$, using equation (4) for transfers, the revenues from the buying and exchange market are given by

$$\begin{aligned} \mathbb{E}_F\Big[\lim_{N\to\infty} \sum_{i\in\mathcal{N}} T_i^N(\theta_i)\mathbb{1}_{\{\theta_i<1\}}\Big] &= \mathbb{E}_F\Big[\lim_{N\to\infty} \frac{1}{N}\sum_{i\in\mathcal{N}} NT_i^N(\theta_i)\mathbb{1}_{\{\theta_i<1\}}\Big] \\ &= \mathbb{E}_F\Big[\lim_{N\to\infty} \frac{1}{N}\sum_{i\in\mathcal{N}}\Big(\theta_i h_i^N(\theta_i) - \int_{\hat{\theta}_i(z_N)}^{\theta_i} h_i^N(s)ds\Big)\mathbb{1}_{\{\theta_i<1\}}\Big] \end{aligned}$$

where $h_i^N(\theta_i) := NS_i^N(\theta_i) - \mathbb{E}_{\mu_i}[\omega_i]$. Using (31) and dominated convergence theorem, taking the limit $N \to \infty$ we get pointwise convergence

$$\theta_i h_i^N(\theta_i) - \int_{\hat{\theta}_i(z_N)}^{\theta_i} h_i^N(s)ds \to \begin{cases} -p_S \mathbb{E}_{\mu_i}[\omega_i], & \text{if } \theta_i < p_S \\ 0, & \text{if } p_S \leq \theta_i < 1. \end{cases}$$

Thus, by a generalized version of the strong law of large numbers we get

$$\begin{aligned} &\mathbb{E}_F\left[\lim_{N\to\infty} \frac{1}{N} \sum_{i\in\mathcal{N}} \left(\theta_i h_i^N(\theta_i) - \int_{\hat{\theta}_i(z_N)}^{\theta_i} h_i^N(s)ds\right) \mathbb{1}_{\{\theta_i<1\}}\right] \\ &= \mathbb{E}_F\left[-p_S \mathbb{E}_{\mu_i}[\omega_i] \mathbb{1}_{\{\theta_i<p_S\}}\right] = -p_S \mathbb{E}_{\mu_i}[\omega_i] F(p_S), \end{aligned}$$

which equals to the total costs in the selling market from procuring $F(p_S)$ mass of customers at a price p_S per unit of expected CTR.