A Theory of Public Saving Glut*

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Abstract

This paper proposes a new theoretical concept: the *public saving glut*. When the interest rate falls below the growth rate, public debt bubbles—a situation in which the government can borrow without a primary surplus to service its debt—can arise. I study a model with liquidity constraints and show that such bubbles emerge only in a dynamically efficient economy. Dynamic inefficiency arises when the government engages in excessive saving. Only in such a situation of the *public saving glut* does debt issuance help to alleviate dynamic inefficiency.

JEL Codes: E22, E44, E62, H63

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1 Introduction

Since Blanchard's (2019) presidential address at the American Economic Association, there has been a surge of research on fiscal policy under the condition where the interest rate on public bonds falls below the economic growth rate, i.e., $r < g$.[1] When $r < g$ holds, the government benefits from low-cost borrowing and can roll over debt without recording a primary surplus as a source of repayment.[2] Brunnermeier, Merkel, and Sannikov (2022) and Kocherlakota (2022, 2023a, 2023b, 2025) refer to such a situation as public debt bubbles.[3] Turning to the seminal works (Diamond, 1965; Tirole, 1985), $r < g$ corresponds to a condition of dynamic inefficiency, under which public debt bubbles are deemed desirable as a corrective mechanism.[4] In particular, $r < g$ implies that capital is overaccumulated beyond the "Golden Rule" level (Phelps, 1961), and crowding out of capital through public debt issuance can raise consumption at all points in time, thereby achieving Pareto improvements. This traditional view on dynamic efficiency implicitly assumes that the interest rate coincides with the return on capital.

However, as Abel, Mankiw, Summers, and Zeckhauser (1989) points out, in reality the interest rate has persistently remained below the return on capital, and the two do not coincide. Moreover, the return on capital typically exceeds the growth rate, so the economy is dynamically efficient. Recent studies likewise confirm that while the interest rate is below the growth rate, the return on capital remains above it, indicating dynamic efficiency (Barro, 2023; Ball and Mankiw, 2023). Reis (2021) attributes this divergence to the fact that public debt possesses greater liquidity and safety than capital.

This paper proposes a new theoretical concept: the *public saving glut*. I analyze a simple infinite-horizon model with liquidity constraints on equity finance and show that public debt bubbles can arise only in a dynamically efficient economy, thereby failing to serve as a remedy for dynamic inefficiency. Dynamic inefficiency, in contrast, occurs when the government saves excessively. Only under such circumstances of the *public saving glut* can debt issuance contribute to alleviating dynamic inefficiency. As Abel, Mankiw, Summers, and Zeckhauser (1989) emphasizes, in reality the return on capital exceeds the growth rate, and dynamic inefficiency is not observed. Yet by departing from the current reality of historically unprecedented public debt in major economies, and instead considering a hypothetical situation in which the government acts as a net saver, one can theoretically recover the possibility of dynamic inefficiency. The concept of the public saving glut paradoxically indicates that debt issuance does not mitigate dynamic inefficiency in real-world economies with mounting public debt.

Public bonds are more liquid than capital due to the liquidity constraints on equity fi-

[1] Cochrane (2021) emphasizes that "I believe Blanchard's fiscal policy address will be seen as important in our time as Milton Friedman's monetary policy address Friedman (1968) was in his." Reis (2022) is a recent survey on $r < g$ and its implications for fiscal policy.

[2] Since the rate of increase in interest payments is lower than the economic growth rate, perpetual rolling over of government bonds becomes possible under $r < g$ without causing an explosive increase in the debt-to-GDP ratio (Ihori, 1978), allowing for a certain level of primary deficits.

[3] Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2024) documents a large gap between the market and fundamental values of U.S. debt, implying that Treasuries are overpriced.

[4] The desirability of government debt issuance in the Diamond model is also discussed in Blanchard's recent book (Blanchard, 2023).

nance. This difference generates a liquidity premium that depresses the interest rate relative to the return on capital, giving rise to the condition $r < g$, under which public debt bubbles emerge. However, as the volume of government debt increases, the liquidity premium diminishes, and eventually the interest rate rises above the growth rate, making public debt bubbles unsustainable. I further consider the situation in which public debt expands into negative territory, that is, when the government engages in saving rather than borrowing. Although $r < g$ still holds, in this case the government cannot enjoy the benefits of low-cost borrowing and must instead raise tax revenues in excess of expenditures.

I show that public debt bubbles can exist only under a dynamically efficient economy, whereas dynamic inefficiency arises when the government is in a state of excessive saving. In other words, I demonstrate that the Golden Rule of capital accumulation—the threshold between dynamic efficiency and inefficiency—emerges only when the government is a net saver. The Golden Rule holds when the return on capital coincides with the growth rate. To identify where the Golden Rule lies, it is necessary to understand how public debt influences the return on capital, and hence capital formation.

Public bonds exert two opposing effects on capital accumulation. First, it generates a crowding-out effect, as debt issuance competes with capital and raises the interest rate. Second, it produces a crowding-in effect, as government bonds provide liquidity services that foster investment. The interplay between these two effects[5] produces a hump-shaped relationship between the return on capital and the level of public debt. At the point where the government is a net saver, the return on capital coincides with the growth rate, and the Golden Rule of capital accumulation is attained.

When public saving falls short of this Golden Rule level, or when the government is a net borrower, the economy remains dynamically efficient. Accordingly, public debt bubbles that finance government spending through borrowing can emerge only in a dynamically efficient economy. Conversely, once public saving expands excessively beyond the Golden Rule threshold, the government falls into a dynamically inefficient state that I term the *public saving glut*.

The rest of the paper proceeds as follows. After a review of the related literature, Section 2 presents the model. Section 3 analyzes public debt bubbles from the perspective of liquidity and explains government saving behavior under $r < g$. Section 4 introduces a new concept, the public saving glut, in comparison with public debt bubbles. Section 5 concludes.

Related Literature

Blanchard (2019) argued that $r < g$ has been the "historical norm" and that the fiscal and welfare costs of public debt issuance under $r < g$ are negligible, providing an important impetus to reconsider fiscal policy in a low interest rate environment.[6] In the standard

[5]Holmström and Tirole (1998), and recently Angeletos, Collard, and Dellas (2023) also discuss these two effects, but do not address dynamic efficiency.

[6]The $r < g$ condition has been examined from various perspectives (Bassetto and Cui, 2018; Brumn, Feng, Kotlikoff, and Kubler, 2022): liquidity—our focus—(Reis, 2021; Angeletos, Collard, and Dellas, 2023; Miao and Su, 2024), safety (Barro, 2023; Kocherlakota, 2023a, 2023b; Brumn, Feng, Kotlikoff, and Kubler, 2024; Aguiar, Amador, and Arellano, 2024; Brunnermeier, Merkel, and Sannikov, 2024), market power (Ball and Mankiw, 2023), monopolistic debt provison (Choi, Kirpalani, and Perez, 2024), and bond-in-utility (Mehrotra and Sergeyev, 2021, Mian, Straub, and Sufi, 2025).

deterministic overlapping generations (OLG) model (Diamond, 1965), which also features prominently in Blanchard (2019, 2023), there is no wedge between the interest rate on government bonds and the return on capital. Hence, $r < g$ not only implies that debt rollover is feasible, but also that the economy is dynamically inefficient.

However, as Mankiw (2022) emphasizes, Abel, Mankiw, Summers, and Zeckhauser (1989) shows that in reality the interest rate and the return on capital diverge, with the return on capital exceeding the growth rate, so that the economy is dynamically efficient. Taking this divergence and the dynamic efficiency of actual economies into account, a growing body of research—motivated by Blanchard (2019)—has re-examined the relationship between $r < g$ and fiscal policy (Reis, 2021; Ball and Mankiw, 2023; Abel and Panageas, 2025, among others). Reis (2021), within the framework of an infinite-horizon model, emphasizes the liquidity and safety of government bonds and derives an upper bound on ratio of government spending to the amount of debt outstanding—termed "fiscal capacity"—in situations where the growth rate exceeds the interest rate but also exceeds the return on capital, such that public debt bubbles arise in a dynamically efficient economy. Fiscal capacity depends on the wedge between the return on capital and the interest rate (the "liquidity premium" in this paper, which Reis (2021) calls the "bubble premium"), leading him to argue that fiscal space is finite. Ball and Mankiw (2023) constructs a model in which market power drives the return on capital above the interest rate. Based on a Solow model, they show that the economy is dynamically efficient, and further demonstrate in an OLG model that when the growth rate exceeds the interest rate but remains below the return on capital, the marginal welfare effects of debt issuance are ambiguous, warning against the perils of unrestrained borrowing in a low interest rate environment. Abel and Panageas (2025) introduces durability shocks to capital in an OLG model and show that the welfare-maximizing level of debt issuance coincides with the debt level at which $r = g$—that is, the upper limit of sustainable debt rollover—thus deepening Blanchard's (2019) insights. Although the analysis in this paper is most closely related to Reis's (2021) collateral-constraint model, to the best of my knowledge, none of the existing literature, including Reis (2021), has highlighted the central proposition of this paper: while public debt bubbles arise only in a dynamically efficient economy, dynamic inefficiency emerges from excessive government saving—a state I term the *public saving glut*.

In our model, the Golden Rule, which serves as the threshold between dynamic efficiency and inefficiency, emerges even though the framework is an infinite-horizon model rather than an OLG model. In OLG models, the finiteness of life limits intertemporal borrowing, which acts as a form of liquidity friction (Angeletos, Lian, and Wolf, 2024), thereby increasing the demand for government bonds as a saving vehicle. This mechanism leads to a situation where $r < g$ holds. By contrast, in standard infinite-horizon models with no frictions or uncertainty, the interest rate equals the return on capital and always exceeds the growth rate by the subjective discount rate, precluding the emergence of the Golden Rule. However, Michau, Ono, and Schlegl (2023) demonstrates that introducing a preference for wealth allows for the Golden Rule even in infinite-horizon settings with bubbles. Aguiar, Amador, and Arellano (2024) also discusses the Golden Rule, dynamic efficiency and its complementary concept in the incomplete market model. I show that the Golden Rule can arise by introducing liquidity constraints instead, when the government is a net saver.[7]

[7]In this paper, the Golden Rule is achieved when the government is a net saver, where steady-state consumption is maximized. Aiyagari and McGrattan (1998), the seminal paper on the optimum quantity of

This paper is also related to the literature on rational bubbles. Tirole (1985), using Diamond's (1965) OLG model, argues that when $r < g$ holds in a bubbleless equilibrium, the economy is dynamically inefficient, and this condition constitutes the basis for bubble formation. Farhi and Tirole (2012) and Martin and Ventura (2012) extend OLG models by introducing financial market frictions, showing that bubbles can also arise in dynamically efficient economies. The setup of this paper is closer to that of Hirano and Yanagawa (2017) and Guerron, Hirano, and Jinnai (2023), who incorporate financial frictions into infinite-horizon models. As an analogy to such bubbles, Brunnermeier, Merkel, and Sannikov (2022) introduce the concept of public debt bubbles. As explained by Kocherlakota (2022), public debt bubbles refer to equilibria where the interest rate perpetually falls below the growth rate, while the primary surplus is perpetually negative. Kocherlakota (2023a) analyzes public debt bubbles triggered by tail risk, and Kocherlakota (2025) decomposes the wedge between the return on capital and government bonds into risk and liquidity components, examining their implications for welfare. Building on this literature, I show that public debt bubbles arise only in dynamically efficient economies, consistent with the data, and then turn to discuss dynamic inefficiency that occurs under excessive government saving—referred to as the public saving glut.

2 Model

Our model consists of, households and firms, and fiscal authority. I extend a framework developed by Shi (2015), which is a simplified version of Kiyotaki and Moore (2019), to derive analytical implications.

2.1 Households and Firms

The economy is populated by a continuum of households with measure one. Each household has a unit measure of members. Some members become entrepreneurs with probability $\pi \in [0, 1]$, and others become workers. One period is divided into four stages: morning, day, evening, and night.

Morning

In the morning, all members of the household are together and share their assets which are n_{t-1} units of capital and b_{t-1} units of public bonds. Household members are identical at this stage.

The head of the household equally divides the assets among the members. She also gives each member the instructions on the choices in the period contingent on whether the member will be an entrepreneur or a worker in the day.

debt, calibrates an incomplete markets model to U.S. data and shows that steady-state welfare is maximized when the government engages in a moderate level of borrowing. In contrast, Nakajima and Takahashi (2017) applys the framework of Aiyagari and McGrattan (1998) to Japanese data and demonstrates that the optimal level of government debt is negative. Peterman and Sager (2022) extends Aiyagari and McGrattan (1998) by incorporating a life-cycle structure, showing that the optimal level of U.S. government debt is also negative—that is, steady-state welfare is maximized when the government is a net saver.

If the member becomes an entrepreneur, he or she spends i_t units of final goods to invest, and brings home the following items before night (when household members consume dinner): x_t^e units of final goods, n_t^e units of capital, and b_t^e units of public bonds. Likewise, if the member becomes a worker, he or she inelastically supplies l units of labor, and brings home the following items before night: x_t^w units of final goods, n_t^w units of capital, and b_t^w units of public bonds. After receiving these instructions, members go to the market and remain separated from each other until night.

Day

At the beginning of the day, members' roles are realized. Competitive firms produce final goods. Aggregate output at time t, y_t, is produced by competitive firms using capital, k_{t-1}, and effective labor, $(1+g)^t l^D$, where $1+g$ is an index of labor-augmenting productivity that grows at rate $g \geq 0$ and $l^D = (1-\pi) l$ is the aggregate labor services by workers. The production function is

$$y_t = A k_{t-1}^{\alpha} \left[(1+g)^t l^D \right]^{1-\alpha},$$

where α is the capital share, and A is constant technology level. The rental price of capital, m_t, is equated to the marginal product of capital as the result of profit maximization $m_t = \alpha \frac{y_t}{k_{t-1}}$. After production, workers receive wage income, $w_t = (1-\alpha) \frac{y_t}{l^D}$, equity holders receive compensation for capital, and a fraction δ of capital depreciates.

Evening

In the evening, entrepreneurs seek finance and undertake investment projects. An entrepreneur can transform i_t units of final goods into i_t units of new capital. In this stage, the asset market and the goods market are open. Individuals trade assets to finance new investments and to achieve the portfolio of asset holdings instructed earlier by the head of the household.

Night

At night, household members meet again. An entrepreneur consumes c_t^e units of final goods and a worker consumes c_t^w units of final goods. After consumption, members forget their identities. They start a new period as identical members.

The instructions given to the members of the household have to satisfy a number of constraints. First, the budget constraints have to be satisfied; for an entrepreneur, it is

$$x_t^e + \underbrace{i_t}_{\text{investment action}} + b_t^e = m_t n_{t-1} + q_t \left[\underbrace{i_t}_{\text{equity ownership}} + (1-\delta) n_{t-1} - n_t^e \right] + (1+r_{t-1}) b_{t-1}, \quad (1)$$

and for a worker, it is

$$x_t^w + b_t^w = m_t n_{t-1} + q_t \left[(1-\delta) n_{t-1} - n_t^w \right] + (1+r_{t-1}) b_{t-1} + w_t l, \quad (2)$$

where q_t is the equity price. r_{t-1} denotes net interest rate on public bond.

The members of the household share their assets at the end of the period:

$$n_t = \pi n_t^e + (1-\pi)\, n_t^w, \tag{3}$$

and

$$b_t = \pi b_t^e + (1-\pi)\, b_t^w. \tag{4}$$

In addition, the instructions must satisfy a feasibility constraint for consumption at night given by[8]

$$\pi x_t^e + (1-\pi)\, x_t^w + z_t = \pi c_t^e + (1-\pi)\, c_t^w \equiv c_t, \tag{5}$$

where z_t is a lump-sum subsidy/tax from the government.

Following Kiyotaki and Moore (2019),[9] I assume that an entrepreneur can issue new equity on, at most, a fraction θ of investment. In short, θ of equity ownership from investment is saleable

$$n_t^e \geq (1-\theta)\, i_t. \tag{6}$$

This liquidity constraint on equity finance makes capital illiquid, whereas government bonds remain fully liquid. To rule out unlimited liquidity creation through public bonds, I assume that entrepreneurs cannot engage in short-selling of public bonds

$$b_t^e \geq 0. \tag{7}$$

The head of the household chooses instructions to its members to maximize the utility

$$\sum_{t=0}^{\infty} \beta^t \left[\pi \log\left(c_t^e\right) + (1-\pi) \log\left(c_t^w\right)\right],$$

subject to (1), (2), (3), (4), (5), (6) and (7).

Rewriting the constraints is helpful to understand the households problem. First, multiplying π and $(1-\pi)$ by (1) and (2) respectively, adding them up, and then substituting (3), (4), and (5), we obtain:

$$\pi c_t^e + (1-\pi)\, c_t^w + q_t n_t + b_t = \left[m_t + q_t\,(1-\delta)\right] n_{t-1} + (1-\pi)\, w_t l + (1+r_{t-1})\, b_{t-1} + z_t + \pi\,(q_t - 1)\, i_t. \tag{8}$$

When q_t exceeds one, i_t relaxes budget constraint. The household makes money from investment. But there is a limit on this activity because funding is limited by the liquidity constraint (6). I assume θ is sufficiently small so that the liquidity constraint always binds.

Substituting (1) into (6), we can derive the following feasibility constraint for investment:[10]

$$\underbrace{(1-\theta q_t)\, i_t}_{\text{minimum cost to conduct } i_t} = \underbrace{m_t n_{t-1} + q_t\,(1-\delta)\, n_{t-1} + (1+r_{t-1})\, b_{t-1}}_{\text{maximum liquidity an investor can attain}}. \tag{9}$$

[8] x_t^e is a trick, introduced by Guerron-Quintana, Hirano, and Jinnai (2023), to exclude c_t^e and z_t from entrepreneurs' liquidity on hand and focus on the analyses of private and public assets (n_t and b_t).

[9] Similar framework appears in Ajello (2016), Cui (2016), Del Negro, Eggertsson, Ferrero, and Kiyotaki (2017), Cui and Radde (2020), Gutkowski (2021), and Guerron-Quintana and Jinnai (2022), among many others. They assume that existing capital is also illiquid, but I do not, in order to derive analytical solutions.

[10] As in Guerron-Quintana, Hirano, and Jinnai (2023), both b_t^e and x_t^e are zero in equilibrium to maximize the liquidity to use.

The left-hand side is the minimum cost investors face to self-finance in order to conduct investment i_t. This cost is smaller than i_t because investors can issue θi_t of equity against the new capital; $1-\theta q_t$ can be interpreted as the down payment for investment. The right-hand side is the maximum liquidity an investor can attain.

Combining (8) and (9), we obtain

$$\begin{gathered}\pi c_t^i + (1-\pi)\, c_t^s + q_t n_t + b_t \\ = m_t n_{t-1} + q_t\,(1-\delta)\, n_{t-1} + (1-\pi)\, w_t l + (1+r_{t-1})\, b_{t-1} + z_t \ , \\ +\pi\lambda_t \left\{ [m_t + q_t\,(1-\delta)]\, n_{t-1} + (1+r_{t-1})\, b_{t-1} \right\}\end{gathered}$$

where

$$\lambda_t \equiv \frac{q_t - 1}{1-\theta q_t}. \tag{10}$$

λ_t is called the liquidity service in Shi (2015), measuring how much profit investors can make using a unit of liquidity. Recall that they can convert a unit of liquidity to $\frac{1}{1-\theta q_t}$ units of capital by investment with leverage, each of which is worth q_t in the market but its marginal cost is one.

The first-order conditions are

$$c_t^e = c_t^w = c_t,$$

$$q_t = \beta \left(\frac{c_t}{c_{t+1}} \right) (1+\pi\lambda_{t+1}) \left[m_{t+1} + q_{t+1}\,(1-\delta) \right], \tag{11}$$

and

$$1 = \beta \left(\frac{c_t}{c_{t+1}} \right) (1+\pi\lambda_{t+1})\,(1+r_t)\,. \tag{12}$$

The first equation states that the marginal utility from consumption has to be equalized across members of the household. The second equation is the Euler equation for equity (capital), where λ_t appears because capital is not only a production factor but also a means of providing liquidity to investors. The third equation is the Euler equation for public bonds.

2.2 Fiscal Authority

Government policies are simple. In each period, the government provides subsidies z_t (levies tax if $z_t < 0$) and spends interest payments $(1+r_{t-1})\, b_{t-1}^S$. To balance the government budget, the fiscal authority issues public bonds b_t^S

$$b_t^S = z_t + (1+r_{t-1})\, b_{t-1}^S. \tag{13}$$

following the fiscal rule:

$$\tilde{b} = \frac{b_t^S}{k_t},$$

setting the debt-to-capital ratio constant. As in Reis (2021) and Abel and Panageas (2025), this paper uses $\tilde{b}$ as a measure of the size of government debt.

2.3 Market Clearing

Competitive equilibrium is defined in a standard way; all agents optimize given prices, and the market clearing conditions are satisfied for capital

$$n_t = k_t,$$

government bonds

$$b_t = b_t^S,$$

labor services

$$l^D = (1 - \pi)\, l,$$

and final goods

$$y_t = c_t + \pi i_t,$$

for all t.

The law of motion for capital is

$$k_t = \pi i_t + (1 - \delta)\, k_{t-1},$$

which automatically holds by Walrass law.

3 Public Debt as Bubbly Liquidity

I study the role of public debt as a source of liquidity and its bubble-like nature. Since public bonds are more liquid than capital, their interest rate is depressed by a liquidity premium. When the interest rate falls below the growth rate ($r < g$), public debt bubbles—where the government can borrow without a primary surplus to service its debt—can arise. As the volume of public debt increases, the per-unit value of bonds as a source of liquidity declines, causing the interest rate to rise. Once the interest rate exceeds the growth rate, public debt bubbles become unsustainable.

Conversely, I also consider the case in which public debt expands in the negative direction, that is, when the government saves rather than borrows. In such a situation, although the condition $r < g$ persists, the government cannot enjoy the benefits of low-interest borrowing and is unable to pursue a fiscal expansion. Instead, a contractionary fiscal policy is implemented, whereby the government records a primary surplus while accumulating public savings.

3.1 Liquidity Premium: $r < MPK - \delta$

From this section onward, I focus on balanced growth paths. For the notational convention, I use variables without time subscripts as constant values along balanced growth paths.

When the liquidity constraint (6) is binding, the saleability of capital is constrained. In contrast, public bonds are fully liquid, and a liquidity premium that the bond interest

rate falls below the return on capital arises. From equations (11) and (12), we obtain the relationship between the interest rate and the return on capital:

$$\underbrace{r}_{\text{interest rate}} = \frac{m}{q} - \delta < \underbrace{m - \delta}_{\text{return on capital}}. \tag{14}$$

To purchase one unit of capital goods, q units of consumption goods are required, and the marginal return on capital is m. Thus, subtracting the depreciation rate δ from $\frac{m}{q}$ yields the return from holding capital in consumption units. By the no-arbitrage condition, this return must equal the bond interest rate r in equilibrium. When the liquidity constraint (6) is binding, the equity price q exceeds one, and the bond interest rate r becomes lower than the return on capital $m-\delta$. The difference between the bond interest rate and the return on capital can be interpreted as a liquidity premium stemming from the difference in liquidity between bonds and capital.

The equity price q, which is the source of the liquidity premium, can be derived analytically along a balanced growth path as (see Appendix 1):

$$q = \frac{\beta(1-\pi) - \pi\frac{1+g}{\delta+g}\tilde{b}}{\left(\frac{1+g}{\delta+g} - \beta\right)\pi + \beta\theta}. \tag{15}$$

This expression implies that public debt issuance (higher $\tilde{b}$) reduces the capital price, as equity becomes a less attractive source of liquidity in the presence of abundant public liquidity. Conversely, when the level of public debt is low and liquidity is scarce, the equity price rises and the liquidity premium may increase.

3.2 Public Debt Bubbles: $r < g$

The liquidity premium exerts a downward pressure on the interest rate. When the condition of $r < g$ holds, the government can borrow while running a primary deficit. I turn to study this phenomenon referred to as public debt bubbles by Brunnermeier, Merkel, and Sannikov (2022) and Kocherlakota (2022, 2023a, 2023b, 2025).

The optimality condition of public bonds (12) and the government budget constraint (13) can be written as

$$1 = \beta\frac{1+r}{1+g}(1+\pi\lambda), \text{ where } \lambda = \frac{q-1}{1-\theta q}, \tag{16}$$

and

$$\underbrace{\tilde{z}}_{\text{primary deficit}} = \left(1 - \frac{1+r}{1+g}\right)\underbrace{\tilde{b}}_{\text{debt outstanding}}. \tag{17}$$

where $\tilde{z}_t \equiv \frac{z_t}{k_t}$. If the liquidity constraint (6) is slack, the equity price q becomes one. In this case, the inerest rate coincides with the return on capital (no liquidity premium) and always exceeds the growth rate ($1 + r > \beta(1+r) = 1+g$) from (16). From (17), servicing outstanding debt always requires a primary surplus ($\tilde{z} < 0$).

However, when the liquidity constraint (6) is binding, q exceeds one, and the interest rate is suppressed by the liquidity premium ($r < m - \delta$). The ineterest rate can be smaller than

the growth rate due to the existence of liquidity services λ. A primary deficit can coexist with outstanding positive debt, thereby public debt bubbles are feasible.

I now investigate in greater detail the conditions under which public debt bubbles can exist. From equations (15) and (16), the interest rate response to public debt issuance can be decomposed as:

$$\frac{dr}{d\tilde{b}} = \underbrace{\frac{dr}{d\lambda}}_{<0}\underbrace{\frac{d\lambda}{dq}}_{>0}\underbrace{\frac{dq}{d\tilde{b}}}_{<0} > 0. \tag{18}$$

This decomposition provides the following interpretation. Public debt issuance reduces the equity price as a source of liquidity ($\frac{dq}{d\tilde{b}} < 0$). A lower q diminishes the liquidity service ($\frac{d\lambda}{dq} > 0$), because capital becomes less attractive relative to its marginal cost; the net revenue from holding one unit of equity $(q-1)$ shrinks, and the required down payment for investment $1-\theta q$ increases. These effects jointly lower liquidity services. As λ falls, public bonds provide less liquidity to investors, requiring higher interest payments to compensate for the reduced liquidity benefit ($\frac{dr}{d\lambda} < 0$). Consequently, the interest rate rises with public debt issuance.[11]

From (18), debt issuance lowers the liquidity value per unit of debt and pushes up the interest rate. Since the interest rate may exceed the growth rate when the level of debt becomes sufficiently large, there must be an upper bound on the sustainable size of public debt bubbles. I define this size of public debt as $\tilde{b}^f$:[12]

$$\lim_{\tilde{b}\to\tilde{b}^f} r\left(\tilde{b}\right) = g. \tag{19}$$

From (16), I also define the equity price corresponding to the upper bound on the sustainable size of public debt bubbles $\tilde{b}^f$ as

$$q^f \equiv \lim_{r\to g} q\left(r\right) = \frac{1-\beta+\beta\pi}{\left(1-\beta\right)\theta+\beta\pi}.$$

Using equation (15), $\tilde{b}^f$ can be derived as

$$\tilde{b}^f = \frac{\left[\beta\left(1-\theta\right)+\beta\left(1-\pi\right)-1\right]\left(1+g\right)-\beta\left(1-\delta\right)\left(1-\theta\right)}{\left[\left(1-\beta\right)\theta+\beta\pi\right]\left(1+g\right)}.$$

$\tilde{b}^f$ is positive if and only if

$$1+g > \frac{\beta\left(1-\delta\right)\left(1-\theta\right)}{\beta\left(1-\theta\right)+\beta\left(1-\pi\right)-1}. \tag{20}$$

The growth rate must be sufficiently high to support public debt bubbles. I assume this condition is always satisfied.

Proposition 1. *Public debt bubbles exist if and only if the amount of public debt is sufficiently small and the economic growth rate is sufficiently high.*

[11] This mechanism offers a microfoundation for the empirical observation by Krishnamurthy and Vissing-Jorgensen (2012) and Greenwood, Hanson, and Stein (2015).

[12] $\tilde{b}^f$ can be interpreted as a pure bubble (Miao and Su, 2024), since a primary fiscal balance ($\tilde{z}=0$) is satisfied and a repayment sources to service $\tilde{b}^f$ is zero.

When the volume of debt exceeds $\tilde{b}^f$, public debt bubbles become unsustainable. The interest rate surpasses the growth rate, requiring a primary surplus ($\tilde{z} < 0$) for high interest payments according to equation (17). Unlimited expansion of a primary deficit by debt issuance is not feasible. This limitation on government spending is another rationale for fiscal capacity discussion by Reis (2021), and hump-shaped deficit-debt diagram (fiscal space) raised by Mian, Straub, and Sufi (2025).

3.3 Public Savings

Let me shift perspective and consider the case in which public debt is reduced into the negative range ($\tilde{b} < 0$). When public debt is negative, it can be interpreted as the government engaging in saving rather than borrowing. From equation (18), as public debt expands in the negative direction, the interest rate declines further; hence, even when the government is saving, the condition $r < g$ continues to hold. However, even under $r < g$, if the government chooses to save, it cannot benefit from borrowing at a low interest rate and thus cannot pursue fiscal expansion. From equation (17), when $\tilde{b}$ is negative under $r < g$, $\tilde{z}$ also takes on negative values. The government saves while running a primary surplus.

Proposition 2. *Even under $r < g$, if the government engages in saving, it cannot enjoy the benefits of low-interest borrowing and necessarily records a primary surplus.*

4 Public Saving Glut

The classical literature regards bubbles as a remedy for dynamic inefficiency (Diamond, 1965; Tirole, 1985, among others). These models, however, abstract from liquidity premia. By contrast, our framework generates liquidity premia, as outlined in Section 3.1. I demonstrate that public debt bubbles can emerge only in a dynamically efficient economy and therefore cannot serve to alleviate dynamic inefficiency in our model.

Dynamic inefficiency occurs when the government engages in excessive saving. I refer to this situation as the *public saving glut*. It is only under such a situation of the public saving glut that debt issuance can help mitigate dynamic inefficiency.

4.1 The Golden Rule with Liquidity Premium: $r < g = MPK - \delta$

Firstly, I show that in our model the Golden Rule of capital accumulation, which serves as the threshold between dynamic efficiency and inefficiency, is achieved only when the government is a net saver.

The Golden Rule holds when the growth rate equals the return on capital. Figure 1 illustrates the steady-state comparative statics of r, g, and $m - \delta$ with respect to $\tilde{b}$. In our model, the growth rate is constant and unaffected by public debt issuance. According to equation (18), the interest rate increases with debt issuance. In contrast, due to the liquidity premium, the return on capital exceeds the interest rate and follows an inverse U-shaped pattern rather than increasing monotonically (the details of this shape will be discussed shortly). As shown in Figure 1, the return on capital coincides with the economic growth rate when the government saves at the level of $\tilde{b}^g$.

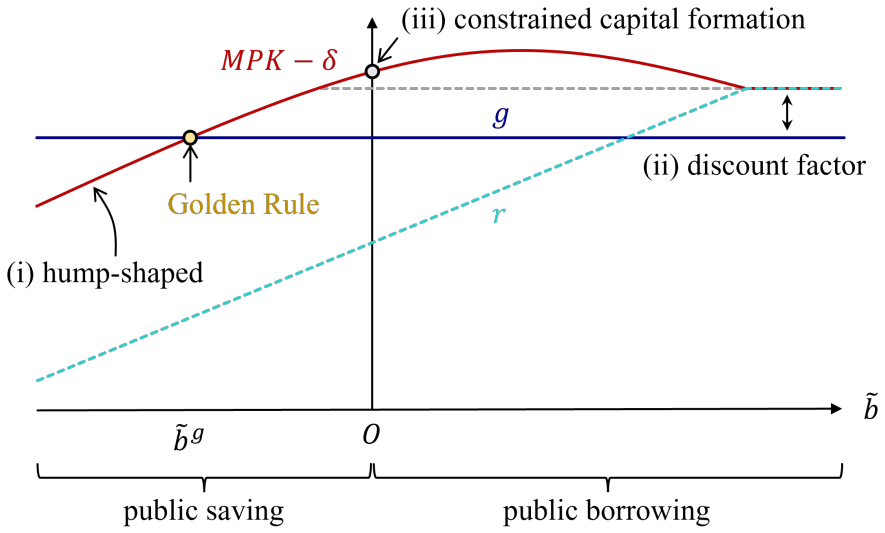


Figure 1: The Golden Rule with Liquidity Premium: $r < g = MPK - \delta$

Proposition 3. *The Golden Rule of capital accumulation can be achieved only when the government is a net saver, such that the growth rate equals the return on capital, while exceeding the interest rate* $(r < g = m - \delta)$.

Why does the Golden Rule occur when the government saves, not borrows? I answer this question based on three key mechanisms as described in Figure 1.

(i) Hump-shaped $MPK - \delta$. Crowding-out and crowding-in effects compete with each other, causing the return on capital $m-\delta$ to be hump-shaped. Equation (14) can be rewritten as:

$$m = q\,(r + \delta)\,.$$

The marginal product of capital m is affected by both the capital price q and the bond interest rate r. From (18), as the level of government debt increases, the bond interest rate r rises ($\frac{dr}{d\tilde{b}} > 0$) to compensate for the decline in liquidity services. This increase in r exerts upward pressure on m, discouraging capital accumulation — **the crowding-out effect**. On the other hand, public debt also serves as a source of liquidity for investment — **the crowding-in effect**. This effect is reflected in the capital price q: as public debt increases, aggregate market liquidity rises, q declines ($\frac{dq}{d\tilde{b}} < 0$) from (15), stimulating capital accumulation, thereby reducing m. These two opposing effects create a hump-shaped profile for $m - \delta$. As government bond issuance increases, the interest rate rises, and the total repayment on public bonds, $(1 + r)\,\tilde{b}$, increases sharply. So does net worth available for investment — the right-hand side of equation (9). Thus, in our model, the crowding-in effect may dominate the crowding-out effect at sufficiently high debt levels, resulting in an inverse-U-shaped $m - \delta$ curve. See Appendix 2 for details.

(ii) Discount factor. The $m-\delta$ curve flattens when liquidity is so abundant that constraint (6) becomes slack. In this neutral region, public debt becomes Ricardian and has no effect on any prices in the economy. In this case, capital is no longer valuable as a source of liquidity ($q = 1$). The liquidity premium (λ) vanishes, and the Euler equation implies that $r > g$

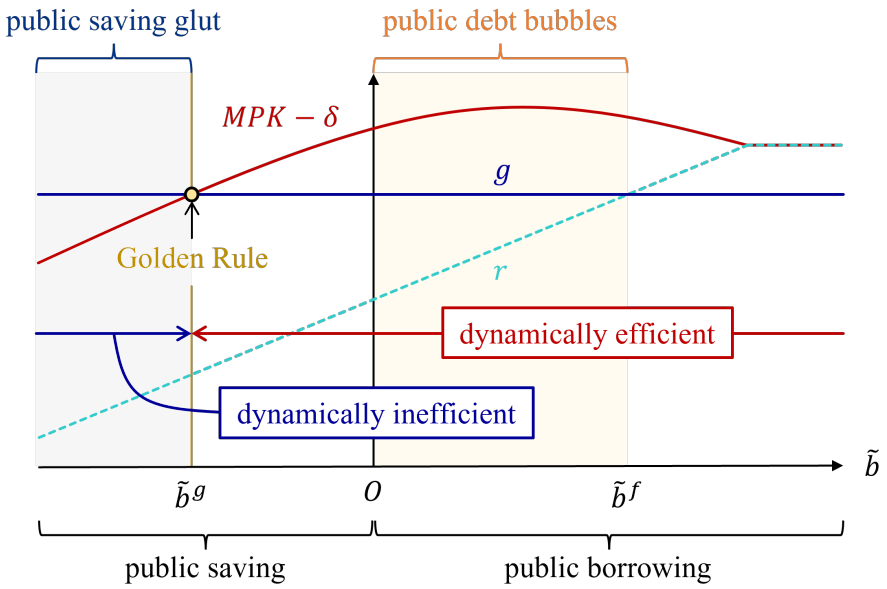


Figure 2: Public Debt Bubbles and Public Saving Glut

holds because $1+g=\beta(1+r)$. The slackness of the liquidity constraint implies no liquidity premium ($r=m-\delta$), hence $g<m-\delta$ is satisfied. See Appendix 3 for details.

(iii) Constrained capital formation. $g<m-\delta$ also holds when no government debt is issued ($\tilde{b}=0$). If capital formation is constrained by financial frictions and no external liquidity is injected into the economy, the capital stock is lower—and its marginal product higher—than in a frictionless setting. As noted earlier, $g<m-\delta$ holds in the liquidity-unconstrained region. Therefore, $g<m-\delta$ is also satisfied in a liquidity-constrained economy without any public liquidity provision.

From these three considerations, I conclude that the Golden Rule is satisfied only when the government is a net saver, that is, when $r<g=m-\delta$. Appendix 4 provides a complete mathematical proof.

4.2 Dynamically Efficient Public Debt Bubbles: $r<g<MPK-\delta$

The Golden Rule serves as the threshold between dynamic efficiency and inefficiency. When the return on capital ($m-\delta$) is less than the growth rate (g), it is possible to increase consumption at all points in time—i.e., achieve a Pareto improvement—by reducing the capital stock. In this sense, capital is overaccumulated, and the economy is dynamically inefficient when $g>m-\delta$. Conversely, when $g<m-\delta$, the economy is dynamically efficient.

Public debt bubbles refer to a situation in which the government borrows without a primary surplus as a repayment source, and they arise under the condition $r<g$. In Figure 2, the region $0<\tilde{b}<\tilde{b}^f$ represents public debt bubbles.

Proposition 4. *Public debt bubbles emerge only in a dynamically efficient economy.*

As is evident from the figure, in the region $0<\tilde{b}<\tilde{b}^f$, the growth rate exceeds the interest

rate while remaining below the return on capital. Thus, public debt bubbles ($r < g$) arise only under a dynamically efficient economy ($g < m - \delta$).

Traditionally, within the framework of the overlapping generations (OLG) model, public debt bubbles have been considered to arise in dynamically inefficient economies and to serve as a remedy for such inefficiency (Diamond, 1965; Tirole, 1985). This view implicitly assumes that the interest rate coincides with the return on capital. In other words, the dynamic efficiency test that compares g and $m - \delta$ is directly linked to the comparison between r and g, which determines the feasibility of public debt bubbles. In the standard OLG model, only crowding-out effects ($\frac{dr}{d\bar{b}} > 0$) occur, and there is no liquidity premium ($r = m - \delta$). Accordingly, the return on capital increases monotonically with debt issuance, just like the interest rate. Public debt bubbles can exist and alleviate dynamic inefficiency only up to the Golden Rule threshold, where the growth rate coincides with both the return on capital and the interest rate.

Empirically, however, the interest rate remains below the return on capital (Abel, Mankiw, Summers, and Zeckhauser (1989), Reis, 2021; Ball and Mankiw, 2023; Barro, 2023, among others). Even if $r < g$ holds and debt rollover is feasible, it would be misleading to conclude that public debt bubbles are desirable from the perspective of alleviating dynamic inefficiency, since r is merely the interest rate and does not necessarily reflect the return on capital.

In our liquidity-constrained economy, the direct linkage between dynamic efficiency and public debt bubbles is broken due to the liquidity premium ($r < m - \delta$).[13] Because of the liquidity premium, the growth rate exceeds the interest rate while remaining below the return on capital, so public debt bubbles arise only in a dynamically efficient economy. Hence, unlike the traditional argument, when a liquidity premium exists, public debt bubbles cannot serve as a remedy for dynamic inefficiency.

4.3 Public Saving Glut: $g > MPK - \delta$

What, then, are the circumstances under which dynamic inefficiency arises? As shown in Proposition 3, in our model the Golden Rule is achieved only when the government is a net saver. While the return on capital is hump-shaped with respect to public debt issuance, the growth rate remains flat. Once the government saves beyond the Golden Rule ($\tilde{b} < \tilde{b}^g$), the return on capital falls below the growth rate ($g > m - \delta$), as shown in Figure 2. In other words, the economy becomes dynamically inefficient when the government engages in excessive saving. I refer to this state of excessive government saving that renders the economy dynamically inefficient as the *public saving glut*.

Proposition 5. *Public debt issuance can mitigate dynamic inefficiency only under the condition of the public saving glut.*

Traditionally, public debt bubbles—financing primary deficits through borrowing—have been regarded as a remedy for dynamic inefficiency. However, once the liquidity premium is

[13] With financial frictions, Cass's (1972) dynamic efficiency test—comparing the interest rate and the growth rate—fails because of a wedge between the interest rate and the return on capital (Woodford, 1990; Farhi and Tirole, 2012; Martin and Ventura, 2012). It has been pointed out in the literature that when financial constraints are introduced, bubbles ($\tilde{b}^f$) arise in a dynamically efficient economy. However, to the best of my knowledge, no study has highlighted that the same applies to public debt bubbles ($0 < \tilde{b} < \tilde{b}^f$).

taken into account, dynamic inefficiency arises only under the public saving glut, and debt issuance becomes effective as a remedy only in this case. As established in Proposition 2, when the government saves under the condition of $r < g$, it cannot enjoy the benefit of low-interest borrowing and must instead run a primary surplus. Under the public saving glut, the government not only oversaves but also collects taxes exceeding its expenditures. It is only under such excessively austere fiscal policy that public debt issuance can be a remedy for dynamic inefficiency.

As pointed out by Abel, Mankiw, Summers, and Zeckhauser (1989), the real-world economy is dynamically efficient. Our model is consistent with this empirical finding, since the economy remains dynamically efficient when the government is a net borrower—that is, under the fiscal circumstances currently faced by most major economies. In light of the present reality, where large public debt stocks have become a burden for major economies, the theoretical result of this paper—that dynamic inefficiency arises only when the government is excessively saving—casts doubt on the effectiveness of public debt issuance in actual economies.

5 Conclusions

Since Blanchard (2019), there has been a surge of research on fiscal policy under the condition of $r < g$. When the interest rate falls below the growth rate, the government can borrow without recording a primary surplus that would otherwise serve as a resource for repayment, thereby enjoying the benefit of low-cost borrowing. Such public debt bubbles have inspired arguments for expansionary fiscal policy in the modern context (Furman and Summers, 2020), and have traditionally been regarded as a remedy for dynamic inefficiency (Diamond, 1965; Tirole, 1985). This view on dynamic efficiency, however, implicitly assumes that the interest rate coincides with the return on capital, whereas empirically the interest rate generally remains below the return on capital due to the presence of a liquidity premium.

Using a model with liquidity constraints, I have shown that public debt bubbles arise only in a dynamically efficient economy and therefore cannot serve to alleviate dynamic inefficiency. Dynamic inefficiency instead emerges when the government engages in excessive saving. Only under such circumstances—what I refer to as the *public saving glut*—can government debt issuance help to mitigate dynamic inefficiency. Most major economies today are characterized by rising public debt stocks and what may more appropriately be called a public borrowing glut. The concept of the public saving glut, paradoxically, underscores that debt issuance is not necessarily beneficial in real-world economies with mounting debt.

Appendix 1: Derivation of q

Define $\tilde{i}_t \equiv \frac{i_t}{k_t}$. From (9) and (14)

$$(1-\theta q)(1+g)\tilde{i} = m + q(1-\delta) + (1+r)\tilde{b} = (1+r)\left(q+\tilde{b}\right),$$

and

$$\tilde{i} = \frac{1}{\pi}\frac{\delta+g}{1+g}.$$

We obtain

$$1-\theta q = \pi\frac{1+g}{\delta+g}\frac{1+r}{1+g}\left(q+\tilde{b}\right).$$

From (10) and (12)

$$1-\theta q = \beta\frac{1+r}{1+g}\left[1-\theta q+\pi(q-1)\right].$$

We now obtain q by comparing two equations above

$$q = \frac{\beta(1-\pi) - \pi\frac{1+g}{\delta+g}\tilde{b}}{\left(\frac{1+g}{\delta+g}-\beta\right)\pi+\beta\theta}.$$

Hence, public debt issuance lowers capital value as a source of liquidity.

Appendix 2: Hump-Shaped $MPK - \delta$

Define bond price p as

$$p \equiv \frac{1}{1+r} = \beta(1+\pi\lambda).$$

The derivative of p with respect to $\tilde{b}$ is

$$\frac{dp}{d\tilde{b}} = \underbrace{\frac{dp}{d\lambda}}_{>0}\underbrace{\frac{d\lambda}{dq}}_{>0}\underbrace{\frac{dq}{d\tilde{b}}}_{<0} < 0.$$

$\frac{dp}{d\lambda} = \beta\pi$ and $\frac{dq}{d\tilde{b}} = -\frac{\pi\frac{1+g}{\delta+g}}{\left(\frac{1+g}{\delta+g}-\beta\right)\pi+\beta\theta}$ are constant. Setting $\frac{dq}{d\tilde{b}}$ as

$$\frac{dq}{d\tilde{b}} \equiv -\xi < 0.$$

Second derivative is

$$\frac{d^2p}{d\tilde{b}^2} = -\beta\pi\xi\frac{d^2\lambda}{dq^2}\frac{dq}{d\tilde{b}} = \beta\pi\xi^2\frac{d^2\lambda}{dq^2}.$$

Since $\frac{d^2\lambda}{dq^2} = \frac{2\theta(1-\theta)}{(1-\theta q)^3} > 0$,

$$\frac{d^2p}{d\tilde{b}^2} > 0.$$

Now we move on to $m = q(r+\delta) = q\left(\frac{1}{p} - 1 + \delta\right)$. Take a derivative with respect to $\tilde{b}$

$$\frac{dm}{d\tilde{b}} = -\xi\left(\frac{1}{p} - 1 + \delta\right) - q\frac{1}{p^2}\underbrace{\frac{dp}{d\tilde{b}}}_{<0}.$$

Second derivative is

$$\frac{d^2m}{d\tilde{b}^2} = 2\xi\frac{1}{p^2}\underbrace{\frac{dp}{d\tilde{b}}}_{<0}\left(1 + \frac{q}{p}\right) - \frac{q}{p^2}\underbrace{\frac{d^2p}{d\tilde{b}^2}}_{>0} < 0.$$

Hence, $MPK - \delta$ is concave with respect to $\tilde{b}$.

Appendix 3: Neutrality of Public Bonds: $r = MPK - \delta$

I consider the threshold beyond which public bonds cease to affect the real economy. From equation (15), q declines with $\tilde{b}$, but must remain above 1 when liquidity constraints are binding. Thus we can naturtally guess that there is a threshold which makes the liquidity constraint slack, defining this level of debt as $\tilde{b}^n$. From (15), $\tilde{b}^n$ satisfies

$$\tilde{b}^n = \lim_{q\to 1}\tilde{b}(q) = \frac{\beta(1-\theta)(\delta+g)}{\pi(1+g)} - 1.$$

When $\tilde{b} > \tilde{b}^n$, $q = 1$ always holds, and the liquidity premium disappears (i.e., $r = m - \delta$).

Since $q^f > 1$ and $\frac{dq}{d\tilde{b}} < 0$, it follows that $\tilde{b}^n > \tilde{b}^f$ is always satisfied. In this region, a primary surplus is always needed to support the high debt level, and public bonds become Ricardian: they are fully tax-financed and have no impact on prices in the private economy.

Appendix 4: Region for Dynamic (In)Efficiency

When the Golden Rule is satisfied ($m - \delta = g$) , from (11),

$$g + \delta = q\left[\frac{1+g}{\beta(1+\pi\lambda)} - 1 + \delta\right]$$

is satisfied. Substituting $\lambda = \frac{q-1}{1-\theta q}$ and rearrenging this equation, we can obtain

$$\{(1+g)\theta - \beta(1-\delta)(\theta-\pi)\}q^2 - \{(1+g) + (g+\delta)(\theta-\pi) - \beta(1-\delta)(1-\pi)\}q + \beta(g+\delta)(1-\pi) = 0.$$

There are two solutions of q at most in this equation. I prove below that two solutions (q^* and q^{**}, $q^* > q^{**}$) exist, and also $\tilde{b}(q = q^*) = \tilde{b}^g < 0$ and $\tilde{b}(q = q^{**}) > \tilde{b}^n$ are satisfied.

When $m - \delta$ is equl to r, public bonds are neutral. From the Euler equation of public bond, $1 = \beta\frac{1+r}{1+g}$ is satisfied. Hence, the marginal product of capital, m, can be rewritten as $m = r + \delta = \frac{1+g}{\beta} - (1-\delta)$. From (11), we obtain

$$(q-1)\{[(1+g)\theta - \beta(1-\delta)(\theta-\pi)]q - [(1-\pi)(1+g) - \beta(1-\delta)(1-\pi)]\} = 0.$$

Thus, there are two solutions. I name the other one of the solution $q = 1 \left(\Leftrightarrow \tilde{b} = \tilde{b}^n \right)$ as q^N:

$$q^N = \frac{1-\pi}{\theta + \frac{\beta(1-\delta)}{(1+g)-\beta(1-\delta)}\pi}.$$

When $\tilde{b} = 0$, I name the corresponding q as q^0 which satisfies

$$q^0 = \frac{1-\pi}{\theta + \frac{(1+g)(1-\beta)+\beta(1-\delta)}{\beta[(1+g)-(1-\delta)]}\pi}.$$

Since

$$\frac{(1+g)(1-\beta)+\beta(1-\delta)}{\beta[(1+g)-(1-\delta)]} > \frac{\beta(1-\delta)}{\beta[(1+g)-(1-\delta)]} > \frac{\beta(1-\delta)}{(1+g)-\beta(1-\delta)},$$

we find that $q^N > q^0$ is always satisfied. I name the level of debt-to-capital ratio achieving q^N as $\tilde{b}^N$. We know that $\frac{dq}{d\tilde{b}} < 0$, and thus $\tilde{b}^N < 0$ is satisfied.

Now I check the sign of $\left.\frac{dm}{dq}\right|_{q=1}$. The derivative of m with respect to q is

$$\frac{dm}{dq} = \frac{(\theta-\pi)\left[(1+g)-\beta(1-\delta)(\theta-\pi)\right]\left[q^2 - 2\frac{1-\pi}{\theta-\pi}q + \frac{1-\pi}{\theta-\pi}\frac{(1+g)-\beta(1-\delta)(1-\pi)}{(1+g)-\beta(1-\delta)(\theta-\pi)}\right]}{\beta\left[(1-\pi)-(\theta-\pi)q\right]^2}.$$

Therefore, we can obtain

$$\left.\frac{dm}{dq}\right|_{q=1} = \frac{(1+g)(1-\theta-\pi)-\beta(1-\delta)(1-\theta)}{\beta(1-\theta)}.$$

From (20),

$$1+g > \frac{\beta(1-\delta)(1-\theta)}{\beta(1-\theta-\pi)-(1-\beta)} > \frac{\beta(1-\delta)(1-\theta)}{1-\theta-\pi}.$$

Hence, we find

$$\left.\frac{dm}{dq}\right|_{q=1} > 0.$$

So, we find that $m > \frac{1+g}{\beta} - (1-\delta) > g + \delta \Leftrightarrow m - \delta > g$ when $\tilde{b}^N < 0 \leq \tilde{b} < \tilde{b}^n$. We know that $m = \frac{1+g}{\beta} - (1-\delta) > g + \delta$ when public bond is neutral to the economy $\tilde{b} \geq \tilde{b}^n$. Thus, the economy is always dynamically efficient when the government is a net borrower ($\tilde{b} \geq 0$).

Also, we can get the values of m when $q = \frac{1}{\theta} > q^N$ and $q = 0 < 1$:

$$m\left(q = \frac{1}{\theta}\right) = -\frac{1-\delta}{\theta} < m\left(q = q^*\right) = g + \delta < \frac{1+g}{\beta} - (1-\delta) = m\left(q = q^N\right),$$

$$m\left(q = 0\right) = 0 < m\left(q = q^{**}\right) = g + \delta < \frac{1+g}{\beta} - (1-\delta) = m\left(q = 1\right).$$

From the intermediate value theorem, $\frac{1}{\theta} > q^* > q^N\,(>1)$ and $0 < q^{**} < 1$ are satisfied. Since $\frac{dq}{d\tilde{b}} < 0$, we find that $\tilde{b}\left(q = \frac{1}{\theta}\right) < \tilde{b}\left(q = q^*\right) = \tilde{b}^g < \tilde{b}^N < 0$ and $0 < \tilde{b}^n < \tilde{b}\left(q = q^{**}\right) < \tilde{b}\left(q = 0\right)$.

Thus, only $\tilde{b}^g < 0$ can achieve the Golden Rule of capital accumulation ($g = m - \delta$). Also, because

$$\tilde{b}\left(q = \frac{1}{\theta}\right) < \tilde{b}^g < \tilde{b}^N$$

and

$$m\left(q = \frac{1}{\theta}\right) < m\left(q = q^*\right) < m\left(q = q^N\right) \Leftrightarrow m\left[\tilde{b} = \tilde{b}\left(q = \frac{1}{\theta}\right)\right] < m\left(\tilde{b} = \tilde{b}^g\right) < m\left(\tilde{b} = \tilde{b}^N\right),$$

we can conclude that

$$\left.\frac{dm}{d\tilde{b}}\right|_{\tilde{b}=\tilde{b}^g} > 0.$$

To sum up, when $\tilde{b} < \tilde{b}^g$, the economy is dynamically inefficient ($g > m - \delta$). On the other hand, when $\tilde{b} \geq \tilde{b}^g$, the economy is dynamically efficient ($g \leq m - \delta$).

References

[1] **Abel, Andrew B., N. Gregory Mankiw, Lawrence H. Summers, and Richard J. Zeckhauser.** 1989. "Assessing Dynamic Efficiency: Theory and Evidence." *The Review of Economic Studies*, 56(1): 1-19.

[2] **Abel, Andrew B. and Stavros Panageas.** 2025. "Running Primary Decits Forever in a Dynamically Efficient Economy: Feasibility and Optimality." *Econometrica*, 93(5): 1601-1633.

[3] **Aguiar, Mark, Manuel Amador, and Cristina Arellano.** 2024. "Micro Risks and (Robust) Pareto-Improving Policies." *American Economic Review*, 114(11): 3669-3713.

[4] **Aiyagari, S. Rao and Ellen R. McGrattan.** 1998. "The optimum quantity of debt." *Journal of Monetary Economics*, 42(3): 447-469.

[5] **Ajello, Andrea**. 2016. "Financial Intermediation, Investment Dynamics and Business Cycle Fluctuations." *American Economic Review*, 106(8): 2256-2303.

[6] **Angeletos, George-Marios, Fabrice Collard, and Harris Dellas.** 2023. "Public Debt as Private Liquidity: Optimal Policy." *Journal of Political Economy*, 131(11): 3233-3264.

[7] **Angeletos, George-Marios, Chen Lian, and Christian K. Wolf.** 2024. "Can Deficits Finance Themselves?." *Econometrica*, 92(5): 1351-1390.

[8] **Ball, Laurence M, and N. Gregory Mankiw.** 2023. "Market Power in Neoclassical Growth Models." *Review of Economic Studies*, 90(2): 572-596.

[9] **Barro, Robert J.** 2023. "r Minus g." *Review of Economic Dynamics*, 48: 1-17.

[10] **Bassetto, Marco, and Wei Cui.** 2018. "The fiscal theory of the price level in a world of low interest rates." *Journal of Economic Dynamics and Control*, 89: 5-22.

[11] **Blanchard, Olivier.** 2019. "Public Debt and Low Interest Rates." *American Economic Review*, 109(4): 1197-1229.

[12] **Blanchard, Olivier.** 2023. *Fiscal Policy under Low Interest Rates*. The MIT Press.

[13] **Brumn, Johannes, Xiangyu Feng, Laurence Kotlikoff, and Felix Kubler.** 2022. "Are deficits free?" *Journal of Public Economics*, 208, 104627.

[14] **Brumn, Johannes, Xiangyu Feng, Laurence Kotlikoff, and Felix Kubler.** 2024. "When Interest Rates Go Low, Should Public Debt Go High?" *American Economic Journal: Macroeconomics*, 16(4): 432-469.

[15] **Brunnermier, Markus K., Sebastian Merkel, and Yuily Sannikov.** 2022. "The Fiscal Theory of Price Level with a Bubble." *NBER Working Paper 27116*.

[16] **Brunnermier, Markus K., Sebastian Merkel, and Yuily Sannikov.** 2024. "Safe Assets." *Journal of Political Economy*, 132: 200-223.

[17] **Cass, David.** 1972. "On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization." *Journal of Economic Theory*, 4(2): 200-223.

[18] **Choi, Jason, Rishabn Kirpalani, and Diego J. Perez.** 2024. "US Public Debt and Safe Asset Market Power." *Journal of Political Economy*, conditionally accepted.

[19] **Cochrane, John H.** 2021. "r<g." Stanford University manuscript.

[20] **Cui, Wei.** 2016. "Monetary–fiscal interactions with endogenous liquidity frictions." *European Economic Review*, 87: 1-25.

[21] **Cui, Wei, and Sören Radde.** 2020. "Search-based Endogenous Asset Liquidity and the Macroeconomy." *Journal of the European Economic Association*, 18(5): 2221-2269.

[22] **Del Negro, Marco, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki.** 2017. "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities." *American Economic Review*, 107(3): 824-857.

[23] **Diamond, Peter A.** 1965. "National Debt in a Neo-Classical Growth Model." *The American Economic Review*, 55(5): 1126-1150.

[24] **Farhi, Emmanuel, and Jean Tirole.** 2012. "Bubbly Liquidity." *The Review of Economic Studies*, 79(2): 678-706.

[25] **Friedman, Milton.** 1968. "The Role of Monetary Policy." *The American Economic Review* 58:1–17.

[26] **Furman, Jason, and Lawrence H. Summers.** 2020. "A Reconsideration of Fiscal Policy in the Era of Low Interest Rates." Harvard Univeristy manuscript.

[27] **Greenwood, Robin, Samuel G. Hanson, and Jeremy C. Stein.** 2015. "A Comparative-Advantage Approach to Government Debt Maturity." *The Journal of Finance*, 70(4): 1683-1722.

[28] **Guerron-Quintana, Pablo, and Ryo Jinnai.** 2022. "On Liquidity Shocks and Asset Prices." *Journal of Money, Credit and Banking*, 54(8): 2519-2546.

[29] **Guerron-Quintana, Pablo, Tomohiro Hirano, and Ryo Jinnai.** 2023. "Bubbles, Crashes, and Economic Growth: Theory and Evidence." *American Economic Journal: Macroeconomics*, 15(2): 333-371.

[30] **Gutkowski, Violeta A.** 2021. "Sovereign illiquidity and recessions." *Journal of Economic Dynamics and Control*, 122: 104029.

[31] **Hirano, Tomohiro, and Noriyuki Yanagawa.** 2017. "Asset Bubbles, Endogenous Growth, and Financial Frictions." *Review of Economic Studies*, 84(1): 406-443.

[32] **Holmström, Bengt, and Jean Tirole.** 1998. "Private and Public Supply of Liquidity." *Journal of Political Economy*, 106(1): 1-40.

[33] **Ihori, Toshihiro.** 1978. "The Golden Rule and the Role of Government in a Life Cycle Growth Model." *American Economic Review*, 68(3): 389-396.

[34] **Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh and Mindy Z. Xiaolan.** 2024. "The U.S. Public Debt Valuation Puzzle." *Econometrica*, 92(4): 1309-1347.

[35] **Kiyotaki, Nobuhiro, and John Moore.** 2019. "Liquidity, Business Cycles, and Monetary Policy." *Journal of Political Economy*, 127(6): 2926-2966.

[36] **Kocherlakota, Narayana.** 2022. "Stabilization with fiscal policy." *Journal of Monetary Economics*, 131: 1-14.

[37] **Kocherlakota, Narayana.** 2023a. "Public Debt Bubbles in Heterogeneous Agent Models with Tail Risk." *International Economic Review*, 64(2): 491-509.

[38] **Kocherlakota, Narayana.** 2023b. "Infinite Debt Rollover in Stochastic Economies." *Econometrica*, 91(5): 1629-1658.

[39] **Kocherlakota, Narayana.** 2025. "Public Debt Bubbles, Liquidity, and Risk: Policy Assessments Based on the Zero-Beta Interest." *NBER Working Paper 33897.*

[40] **Krishnamurthy, Arvind and Annette Vissing-Jorgensen.** 2012. "The Aggregate Demand for Treasury Debt." *Journal of Political Economy*, 120(2): 233-267.

[41] **Mankiw, N. Gregory.** 2022. "Government Debt and Capital Accumulation in an Era of Low Interest Rates." *NBER Working Paper 30024.*

[42] **Martin, Alberto, and Jaume Ventura.** 2012. "Economic Growth with Bubbles." *American Economic Review*, 102(6): 3033-3058.

[43] **Mehrotra, Neil, and Dmitriy Sergeyev.** 2021. "Debt sustainability in a low interest rate world." *Journal of Monetary Economics*,124S: S1-S18.

[44] **Miao, Jianjun, and Dongling Su.** 2024. "Fiscal and Monetary Policy Interactions in a Model with Low Interest Rates." *American Economic Journal: Macroeconomics*, 16(4): 35-76.

[45] **Mian, Atif, Ludwig Straub, and Amir Sufi.** 2025. "A Goldilocks Theory of Fiscal Deficits." Harvard University manuscript.

[46] **Michau, Jean-Baptiste, Yoshiyasu Ono, and Matthias Schlegl.** 2023. "Wealth preference and rational bubbles." *European Economic Review*, 156.

[47] **Nakajima, Tomoyuki, and Shuhei Takahashi.** 2017. "The optimum quantity of debt for Japan." *Journal of the Japanese and International Economies*, 46: 17-26.

[48] **Peterman, William B. and Erick Sager.** 2022. "Optimal Public Debt with Life Cycle Motives." *American Economic Journal: Macroeconomics*, 14(4): 404-437.

[49] **Phelps, Edmund S.** 1961. "The Golden Rule of accumulation: A fable for growthmen." *The American Economic Review*, 51(4): 638-643.

[50] **Reis, Ricardo.** 2021. "The constraint on public debt when $r < g$ but $g < m$." LSE manuscript.

[51] **Reis, Ricardo.** 2022. "Debt Revenue and the Sustainability of Public Debt." *Journal of Economic Perspectives*, 36(4): 103-124.

[52] **Shi, Shouyong.** 2015. "Liquidity, assets and business cycles." *Journal of Monetary Economics*, 70: 116-132.

[53] **Tirole, Jean.** 1985. "Asset Bubbles and Overlapping Generations." *Econometrica*, 53(6): 1499-1528.

[54] **Woodford, Michael.** 1990. "Public Debt as Private Liquidity." *The American Economic Review*, 80(2): 382-388.