Recursive Knowledge Architectures: A General Equilibrium Analysis of AI's Impact on Knowledge Production

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Abstract

This paper develops a general equilibrium framework that analyzes how artificial intelligence fundamentally transforms the knowledge production function. We introduce a novel class of recursive knowledge architectures where AI systems endogenously modify both the inputs and processes of knowledge creation, leading to structural changes in innovation dynamics. Using variational methods and stochastic differential equations, we characterize the optimal knowledge production paths and derive analytical solutions for knowledge steady states under varying AI intensities. Our model yields four principal results: (1) AI transforms returns to scale in knowledge production from strictly diminishing to potentially increasing, with the transition occurring at a precisely characterized threshold; (2) the optimal diversity of knowledge inputs follows a non-monotonic pattern as AI intensity increases, with phase transitions occurring at analytically derived critical points; (3) the relative productivities of codified versus tacit knowledge evolve according to a precisely specified differential equation system, creating endogenous specialization patterns across sectors; and (4) knowledge network topologies undergo predictable transformations characterized by increasing clustering coefficients and changing diameter properties. These theoretical findings reconcile several empirical puzzles in the literature while generating falsifiable predictions about innovation patterns, productivity dynamics, and optimal institutional arrangements in AI-intensive economies. By fully endogenizing the knowledge pro-

duction function itself, our framework fundamentally reconceptualizes the mechanisms through which technological progress shapes long-run economic growth.

1 Introduction

The emergence of artificial intelligence as a general-purpose technology has prompted substantial theoretical inquiry into its economic implications. While existing frameworks have generated valuable insights regarding labor market effects (Acemoglu and Restrepo, 2018), automation dynamics (Acemoglu and Restrepo, 2019), and productivity patterns (Brynjolfsson et al., 2019), existing literature continues to model the knowledge production function itself as structurally invariant to AI adoption. This assumption—that AI merely shifts parameters within a fixed functional form rather than transforming the underlying process of knowledge creation—represents a critical gap in economic theory that this paper addresses.

We develop a general equilibrium framework demonstrating that AI qualitatively transforms the knowledge production function in ways that fundamentally alter innovation dynamics, productivity growth trajectories, and optimal institutional arrangements. The paper introduces a novel theoretical construct—"recursive knowledge architectures"—to characterize systems where AI both consumes and generates knowledge while simultaneously modifying the knowledge production process itself. This recursive property creates feedback mechanisms and non-linearities absent from standard models, with profound implications for long-run economic growth.

Our analysis addresses three critical limitations in the contemporary theoretical literature. The first is the implicit assumption in endogenous growth theory that the knowledge production function itself is technologically invariant. Since Arrow (1962) formalized the concept of learning-by-doing, and through subsequent advances by Romer (1990) and Jones (1995), knowledge production has been modeled as:

$$\dot{A}_t = F(R_t, A_t) = \delta R_t^{\lambda} A_t^{\phi} \tag{1}$$

where technological change affects parameters (δ, λ, ϕ) but not the functional form itself. This assumption becomes increasingly problematic with transfor-

mative technologies like AI, which can fundamentally alter how inputs combine to create knowledge. While Weitzman (1998) introduced recombinant innovation and Jones (2009) modeled the burden of knowledge, both maintained the structural stability of the knowledge production function. Our framework, by contrast, allows the function itself to evolve endogenously, generating qualitatively different dynamics absent from these models.

The second limitation concerns the treatment of knowledge heterogeneity in existing theoretical frameworks. Standard models either treat knowledge as homogeneous (Romer, 1990) or distinguish between basic types like human and physical capital (Lucas, 1988), scientific and applied knowledge (Mokyr, 2011), or existing and new varieties (Kortum, 1997). These taxonomies, while useful, fail to capture the complex interactions between different knowledge types under AI adoption. The distinction between tacit and codified knowledge, extensively discussed by Polanyi (1966) and Cowan et al. (2000); Steinmueller (2022), becomes particularly relevant with AI, yet lacks formal integration into economic growth models. Our framework introduces a multidimensional knowledge taxonomy with theoretically justified distinctions, generating novel insights regarding the changing relative importance of different knowledge types as AI capability advances.

The third limitation involves the treatment of knowledge networks in economic theory. While innovation networks have been extensively studied empirically (Powell et al., 1996; Fleming et al., 2007; Uzzi et al., 2013; Acemoglu and al,, 2016; Acemoglu et al., 2023), their integration into formal economic models remains limited. Existing approaches either treat knowledge spillovers as stylized externalities (Romer, 1990), geographically constrained (Jaffe et al., 1993; Henderson et al., 2005; Murata et al., 2014), or flowing through simplified network structures (Hidalgo et al., 2007; Feldman and Langford, 2000). Missing is a rigorous analysis of how transformative technologies like AI endogenously reshape these networks. Our framework addresses this gap by formally characterizing the topological evolution of knowledge networks under AI using graph-theoretic methods, deriving precise predictions about changing innovation patterns.

The analytical tractability of our model yields four principal theoretical results with significant implications for economic theory. First, we demonstrate that AI transforms returns to scale in knowledge production from strictly diminishing to potentially increasing, with this transition occurring

at a precisely characterized threshold. This result challenges the conventional assumption in growth theory that diminishing returns to knowledge are necessary for balanced growth (Jones, 1995; Bloom et al., 2020), suggesting instead that AI may enable sustained increasing returns without explosive dynamics through novel stabilizing mechanisms. Second, we show that the optimal diversity of knowledge inputs follows a non-monotonic pattern as AI intensity increases, with precisely characterized phase transitions. This finding contrasts with models of knowledge recombination (Weitzman, 1998) that predict monotonically increasing returns to diversity, revealing instead a fundamental trade-off between diversity and specialization that evolves with AI capability. Third, we characterize how the relative productivities of codified versus tacit knowledge evolve according to a precisely specified differential equation system, creating endogenous specialization patterns across sectors. This result significantly extends models of directed technical change (Acemoglu, 2002, 2007) by demonstrating how the bias of technological progress itself evolves endogenously with AI. Fourth, we establish that knowledge network topologies undergo predictable transformations characterized by increasing clustering coefficients and changing diameter properties, formalizing how AI reshapes innovation ecosystems at the macro level. These theoretical advancements provide a comprehensive framework for analyzing how AI transforms the process of knowledge creation itself, with profound implications for long-run economic growth.

Our results not only resolve several empirical puzzles that have confounded the existing literature—including heterogeneous productivity effects across sectors (Brynjolfsson et al., 2017), changing patterns in research collaboration (Wuchty et al., 2007), and shifts in innovation geography (Jaffe et al., 1993)—but also generate falsifiable predictions about innovation patterns, productivity dynamics, and optimal institutional arrangements in AI-intensive economies. By fully endogenizing the knowledge production function itself, our framework fundamentally reconceptualizes the mechanisms through which technological progress shapes economic growth.

The remainder of this paper is organized as follows. Section 2 develops our theoretical framework, introducing recursive knowledge architectures and deriving their fundamental properties. Section 3 analyzes equilibrium knowledge dynamics, characterizing optimal paths and steady states. Section 4 derives comparative statics and transition dynamics across different economic contexts. Section 5 discusses empirical implications and policy applications.

Section 6 concludes. All proofs and additional mathematical derivations are provided in the Appendix.

2 Theoretical Framework: Recursive Knowledge Architectures

2.1 Mathematical Preliminaries and Definitions

We begin by constructing a rigorous mathematical foundation for analyzing knowledge production dynamics under AI. Consider a continuous-time economy populated by a unit measure of agents indexed by $i \in [0, 1]$. Each agent can be either a human researcher or an AI system. Time is continuous and indexed by $t \in [0, \infty)$.

Let $\mathcal{K}$ be the universe of possible knowledge elements, represented as a measurable space with σ-algebra $\Sigma_{\mathcal{K}}$. A knowledge set for agent i at time t is a measurable subset $K_{i,t} \subset \mathcal{K}$. We define a knowledge state as a measurable function $\kappa_t : [0, 1] \times \Sigma_{\mathcal{K}} \to [0, 1]$ that assigns to each agent-knowledge element pair (i, k) a value $\kappa_t(i, k) \in [0, 1]$ representing the agent's mastery of that knowledge element.

Definition 1 (Knowledge Types)**.** Knowledge in our framework is classified along two orthogonal dimensions:

1. By embodiment: Tacit knowledge (K^T) vs. Codified knowledge (K^C), where $K^T \cup K^C = \mathcal{K}$ and $K^T \cap K^C = \emptyset$.

2. By scope: Domain-specific knowledge (K^S) vs. General knowledge (K^G), where $K^S \cup K^G = \mathcal{K}$ and $K^S \cap K^G = \emptyset$.

These distinctions are crucial because AI systems interact differently with each knowledge type. AI can directly access and process codified knowledge but has limited ability to acquire tacit knowledge. Similarly, AI may excel at domain-specific applications while struggling with general knowledge that requires broader contextual understanding.

Definition 2 (Knowledge Network)**.** The knowledge network Γ_t is a weighted directed graph represented by the triplet $\Gamma_t = (V, E_t, w_t)$ where:

1. $V = [0, 1]$ is the set of agents (vertices)

2. $E_t \subset V \times V$ is the set of directed edges at time t

3. $w_t : E_t \to \mathbb{R}^+$ is a weight function assigning a positive weight to each edge

The weight $w_t(i, j)$ represents the intensity of knowledge flow from agent i to agent j at time t.

Knowledge networks evolve endogenously based on knowledge states and AI capabilities. Let $\gamma_{i,t}$ denote the local network structure around agent i at time t, which includes all edges and weights involving agent i.

Definition 3 (AI Capability)**.** The AI capability of agent i at time t is represented by the parameter $\lambda_{i,t} \in [0, \infty)$, which determines the agent's ability to process, integrate, and generate knowledge using artificial intelligence technologies.

We now define the central concept in our framework:

Definition 4 (Recursive Knowledge Architecture)**.** A recursive knowledge architecture is a dynamical system where:

1. Knowledge outputs become inputs to future knowledge production
2. AI systems both consume and generate knowledge
3. The knowledge production function itself evolves endogenously with AI capability
4. Feedback mechanisms exist between knowledge states, network structures, and AI capabilities

This recursive property—where AI not only changes what knowledge can be produced but how knowledge is produced—represents the core innovation of our framework.

2.2 Knowledge Production Dynamics

Traditional knowledge production functions take the form:

$$\dot{K}_{i,t} = F(R_{i,t}, K_{i,t}) = \phi R_{i,t}^{\alpha} K_{i,t}^{\beta} \tag{2}$$

where $\dot{K}_{i,t}$ represents new knowledge created by agent i at time t, $R_{i,t}$ denotes research inputs, and $K_{i,t}$ is the agent's existing knowledge stock.

We propose a fundamentally different structure that captures AI's transformative effect. Let $K_{i,t}^H$ denote human-originated knowledge available to agent i at time t, and $K_{i,t}^A$ denote AI-processed knowledge. The knowledge production function takes the form:

$$\dot{K}_{i,t} = F(K_{i,t}^H, K_{i,t}^A, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t}) \tag{3}$$

where $X_{i,t}$ represents traditional research inputs (labor, capital), $\gamma_{i,t}$ is the local structure of the knowledge network around agent i, and $\mathcal{A}_{i,t}$ is an "architecture parameter" that determines how different knowledge inputs combine.

Crucially, the architecture parameter itself evolves endogenously with AI capability:

$$\dot{\mathcal{A}}_{i,t} = g(K_{i,t}^A, \gamma_{i,t}, \lambda_{i,t}) \tag{4}$$

This recursive relationship is a key innovation of our framework. To make this concrete, we specify a CES functional form:

$$\dot{K}_{i,t} = \left[\alpha(X_{i,t})^\sigma + \beta(K_{i,t}^H)^\sigma + \delta(K_{i,t}^A)^\sigma\right]^{1/\sigma} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{5}$$

where $\sigma \leq 1$ determines the elasticity of substitution between inputs, and $\Phi(\gamma_{i,t}, \mathcal{A}_{i,t})$ is a network multiplier that captures how knowledge flows and architectural features affect productivity.

The parameters α, β, and δ represent the productivity of traditional inputs, human knowledge, and AI knowledge, respectively. We allow these parameters to vary across agents and time, reflecting heterogeneity in research capabilities and environments.

2.3 AI Knowledge Processing and Network Evolution

The AI-processed knowledge available to agent i depends on both locally accessible knowledge and the AI's processing capability:

$$K_{i,t}^A = \lambda_{i,t} \int_{j \in \mathcal{N}_{i,t}} K_{j,t}^C \cdot \omega(d_{ij,t})\, dj \tag{6}$$

where $\mathcal{N}_{i,t}$ is the set of agents connected to i at time t, $K_{j,t}^C$ is codified knowledge held by agent j, $d_{ij,t}$ is the "distance" between agents in knowledge space, and $\omega(d_{ij,t})$ is a weighting function that decreases with distance. The parameter $\lambda_{i,t}$ represents AI processing capability.

The knowledge network itself evolves endogenously according to:

$$\dot{w}_{ij,t} = h(K_{i,t}^A, K_{j,t}^A, d_{ij,t}, \mathcal{A}_{i,t}, \mathcal{A}_{j,t}) \tag{7}$$

where $w_{ij,t}$ is the weight of the edge from agent i to agent j at time t. This captures how AI systems actively reshape knowledge flows, strengthening connections between complementary knowledge domains and potentially weakening others.

The distance function $d_{ij,t}$ measures the dissimilarity between knowledge sets of agents i and j. We define it using the normalized Hamming distance:

$$d_{ij,t} = \frac{\int_{\mathcal{K}} |\kappa_t(i,k) - \kappa_t(j,k)|\, dk}{\int_{\mathcal{K}} \max(\kappa_t(i,k), \kappa_t(j,k))\, dk} \tag{8}$$

This distance function captures both the overlap in knowledge domains and the depth of knowledge within each domain.

2.4 Meta-Knowledge and Recursive Improvement

A key feature of our framework is the concept of "meta-knowledge"—knowledge about the knowledge production process itself. AI systems acquire and improve meta-knowledge through a learning process:

$$\dot{M}_{i,t} = \rho \left(\frac{K_{i,t}^A}{K_{i,t}^A + K_{i,t}^H} \right) \cdot \dot{K}_{i,t} \tag{9}$$

where $M_{i,t}$ represents meta-knowledge and ρ is a learning parameter.

Meta-knowledge affects the architecture parameter according to:

$$\mathcal{A}_{i,t} = f(M_{i,t}) \tag{10}$$

where f is an increasing function. This creates a recursive improvement dynamic, where AI systems learn to improve the knowledge production process itself, potentially leading to accelerating returns.

The specific functional form for f is:

$$f(M_{i,t}) = \frac{M_{i,t}^{\eta}}{1 + M_{i,t}^{\eta}} \tag{11}$$

where $\eta > 0$ determines the sensitivity of the architecture parameter to meta-knowledge. This functional form ensures that $\mathcal{A}_{i,t} \in [0, 1)$, with values closer to 1 representing more advanced knowledge architectures.

2.5 Stochastic Extensions

To incorporate uncertainty in knowledge production, we extend our framework to a stochastic setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{W_t\}_{t \geq 0}$ be an m-dimensional Brownian motion. The stochastic knowledge dynamics are given by:

$$dK_{i,t} = F(K_{i,t}^{H}, K_{i,t}^{A}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t})\, dt + \Sigma(K_{i,t}, \lambda_{i,t})\, dW_t \tag{12}$$

where $\Sigma(K_{i,t}, \lambda_{i,t})$ is an m-dimensional volatility vector that depends on the current knowledge state and AI capability.

This stochastic formulation allows us to analyze how uncertainty affects knowledge production dynamics and how AI capabilities influence the volatility of innovation processes.

3 Equilibrium Analysis and Key Results

3.1 General Equilibrium Formulation

We embed our knowledge production framework in a general equilibrium model. Let Y_t denote aggregate output, produced according to:

$$Y_t = \int_0^1 Y_{i,t}\, di = \int_0^1 A_{i,t} F(K_{i,t}, L_{i,t})\, di \tag{13}$$

where $A_{i,t}$ is agent-specific productivity, $K_{i,t}$ is physical capital, and $L_{i,t}$ is labor. Productivity evolves according to:

$$\dot{A}_{i,t} = \psi \dot{K}_{i,t} \tag{14}$$

where $\psi > 0$ is a parameter determining how new knowledge translates into productivity gains.

Agents solve a dynamic optimization problem, choosing research inputs $X_{i,t}$ to maximize discounted utility:

$$\max_{X_{i,t}} \int_0^\infty e^{-\rho t} U(C_{i,t})\, dt \tag{15}$$

subject to the budget constraint:

$$C_{i,t} + X_{i,t} \leq Y_{i,t} \tag{16}$$

and the knowledge production function:

$$\dot{K}_{i,t} = F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t}) \tag{17}$$

The first-order conditions for this problem yield an optimal investment rule:

$$U'(C_{i,t}) = \mu_{i,t} \frac{\partial F}{\partial X_{i,t}} \tag{18}$$

where $\mu_{i,t}$ is the shadow value of knowledge.

In equilibrium, the allocation of research inputs across agents satisfies:

$$\frac{\partial F(K_{i,t}^H, K_{i,t}^A, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t})/\partial X_{i,t}}{\partial F(K_{j,t}^H, K_{j,t}^A, X_{j,t}, \gamma_{j,t}; \mathcal{A}_{j,t})/\partial X_{j,t}} = \frac{\mu_{j,t}}{\mu_{i,t}} \tag{19}$$

This equilibrium condition ensures that research inputs are allocated efficiently across agents, taking into account heterogeneity in knowledge states, AI capabilities, and network positions.

3.2 Dynamic Returns to Scale in Knowledge Production

Our first key result concerns how AI transforms returns to scale in knowledge production:

Theorem 5 (Dynamic Returns to Scale)**.** *There exists a threshold value λ^* such that:*

1. *For $\lambda_{i,t} < \lambda^*$, returns to scale in knowledge production are strictly diminishing*
2. *For $\lambda_{i,t} = \lambda^*$, returns to scale are constant*
3. *For $\lambda_{i,t} > \lambda^*$, returns to scale are increasing*

The threshold λ^* is given by:

$$\lambda^* = \frac{(1-\beta)}{\rho\eta\delta \int_{j\in\mathcal{N}_{i,t}} K_{j,t}^C \cdot \omega(d_{ij,t})\, dj} \tag{20}$$

Proof. See Appendix A.1. □

This result has profound implications for long-run growth. Traditional knowledge production models incorporate diminishing returns to prevent explosive growth. In contrast, our framework allows for sustained increasing returns as AI systems recursively improve the knowledge production process itself.

To provide economic intuition for this result, consider three effects of increasing all knowledge inputs by a factor s:

1. Direct scaling effect: By the CES structure, this initially produces diminishing returns
2. Network effect: Increased knowledge enhances network connectivity, partially offsetting diminishing returns
3. Meta-knowledge effect: As AI capability increases, meta-knowledge accumulation accelerates, modifying the knowledge production function itself

For $\lambda_{i,t} < \lambda^*$, the first effect dominates, yielding diminishing returns. At $\lambda_{i,t} = \lambda^*$, the three effects perfectly balance, producing constant returns. For $\lambda_{i,t} > \lambda^*$, the meta-knowledge effect dominates, generating increasing returns.

The threshold λ^* depends on several factors:

1. It decreases with the stock of codified knowledge ($K_{j,t}^C$), implying that economies with larger codified knowledge bases transition to increasing returns at lower AI capability levels
2. It increases with knowledge stock elasticity (β), implying that economies where existing knowledge is more productive require higher AI capabilities to achieve increasing returns
3. It decreases with meta-knowledge learning parameters (ρ and η), implying that economies with more efficient meta-knowledge acquisition achieve increasing returns at lower AI capability levels

3.3 Knowledge Diversity and Optimal Specialization

Our second key result addresses how AI affects the optimal diversity of knowledge inputs:

To formalize this, we define a knowledge diversity index for agent i at time t:

$$D_{i,t} = 1 - \frac{\int_{\mathcal{K}} \kappa_t(i,k)^2 \, dk}{\left(\int_{\mathcal{K}} \kappa_t(i,k) \, dk\right)^2} \tag{21}$$

This index takes values in $[0, 1]$, with higher values indicating greater knowledge diversity.

Theorem 6 (Non-monotonic Diversity Optima)**.** *For any agent i, there exists an optimal knowledge diversity level $D^*_{i,t}(\lambda_{i,t})$ that maximizes knowledge production. Moreover:*

1. $\frac{\partial D^*_{i,t}}{\partial \lambda_{i,t}} > 0$ *for* $\lambda_{i,t} < \lambda_D$
2. $\frac{\partial D^*_{i,t}}{\partial \lambda_{i,t}} < 0$ *for* $\lambda_{i,t} > \lambda_D$

where the critical threshold λ_D is given by:

$$\lambda_D = \frac{\delta\sigma}{(1-\sigma)\beta \int_{j\in\mathcal{N}_{i,t}} \frac{\partial \omega(d_{ij,t})}{\partial D_{i,t}}\, dj} \tag{22}$$

Proof. See Appendix A.2. □

This non-monotonic relationship explains conflicting empirical observations regarding specialization and diversity in AI-intensive fields. Initial AI adoption enables researchers to integrate more diverse knowledge sources, but as AI capability advances beyond the threshold λ_D, it enables deeper specialization through more efficient knowledge recombination.

The economic intuition is as follows. Knowledge diversity affects knowledge production through two channels:

1. Direct productivity effect: Diverse knowledge enables novel recombinations, increasing productivity
2. Network connectivity effect: Knowledge diversity affects network distances and knowledge flows

As AI capability increases, the marginal productivity of both channels changes. Initially, AI enhances the value of diversity by enabling more effective integration of diverse knowledge elements. However, as AI capability exceeds λ_D, the ability of AI to deeply process specific domains becomes dominant, favoring specialization.

The threshold λ_D depends on several factors:

1. It increases with the productivity of AI-processed knowledge (δ), as higher AI productivity enhances the value of specialization

2. It increases with substitution parameter (σ), as greater substitutability between knowledge inputs delays the transition to specialization
3. It decreases with network connectivity sensitivity ($\frac{\partial \omega(d_{ij,t})}{\partial D_{i,t}}$), as stronger network effects from diversity delay the shift toward specialization

3.4 Tacit vs. Codified Knowledge Evolution

Our third key result concerns the evolution of different knowledge types under increasing AI capability:

Theorem 7 (Endogenous Knowledge Type Evolution)**.** *The ratio of marginal productivities between codified knowledge (K^C) and tacit knowledge (K^T) evolves according to:*

$$\frac{d}{dt}\left(\frac{MP_{K^C}}{MP_{K^T}}\right) = \Gamma(\lambda_{i,t}) \cdot \left(\frac{MP_{K^C}}{MP_{K^T}}\right) \cdot \left(1 - \frac{MP_{K^C}}{MP_{K^T}} \cdot \frac{\tau(\lambda_{i,t})}{\Gamma(\lambda_{i,t})}\right) \tag{23}$$

where $MP_{K^C} = \frac{\partial \dot{K}_{i,t}}{\partial K^C_{i,t}}$ and $MP_{K^T} = \frac{\partial \dot{K}_{i,t}}{\partial K^T_{i,t}}$ are marginal productivities, and:

$$\Gamma(\lambda_{i,t}) = \phi_1 \lambda_{i,t} - \phi_2 \lambda_{i,t}^2 \tag{24}$$

$$\tau(\lambda_{i,t}) = \psi_1 + \psi_2 \lambda_{i,t} \tag{25}$$

with $\phi_1, \phi_2, \psi_1, \psi_2 > 0$.

Proof. See Appendix A.3. □

This theorem characterizes the evolution of relative knowledge type importance as a non-linear differential equation. The equation has the mathematical structure of a generalized logistic function with AI-dependent parameters, implying that the ratio $\frac{MP_{K^C}}{MP_{K^T}}$ follows an S-shaped trajectory with AI capability.

The economic intuition is as follows:

1. Initially, as $\lambda_{i,t}$ increases from zero, $\Gamma(\lambda_{i,t})$ is positive and increasing, causing the ratio $\frac{MP_{KC}}{MP_{KT}}$ to grow. This represents an initial phase where AI increases the value of codified relative to tacit knowledge

2. As $\lambda_{i,t}$ continues to increase, the quadratic term in $\Gamma(\lambda_{i,t})$ becomes more significant, eventually causing $\Gamma(\lambda_{i,t})$ to decrease. This represents a phase where the marginal benefits of codification begin to diminish

3. Eventually, the ratio approaches a steady state determined by $\frac{\tau(\lambda_{i,t})}{\Gamma(\lambda_{i,t})}$, which itself increases with $\lambda_{i,t}$. This reflects a mature phase where both tacit and codified knowledge find a new equilibrium relationship

This pattern explains the observed phenomenon in AI-intensive fields where there is initially a rush to codify knowledge (creating datasets, developing formal models), followed by a renewed appreciation for tacit knowledge (intuition, contextual understanding) as AI systems mature.

Corollary 8 (Sectoral Specialization in Knowledge Types)**.** *Sectors with different initial ratios of $\frac{MP_{KC}}{MP_{KT}}$ follow divergent trajectories as AI capability increases. Specifically, if $\lambda_{i,t} > \lambda_{j,t}$ and initially $\frac{MP_{KC}}{MP_{KT}}|_i > \frac{MP_{KC}}{MP_{KT}}|_j$, then the gap in ratios widens over time.*

This corollary explains the observed pattern of increasing specialization across sectors in terms of knowledge type emphasis. Sectors that initially favor codified knowledge become even more codification-focused as AI capability increases, while sectors initially reliant on tacit knowledge maintain or strengthen this orientation.

3.5 Knowledge Network Topology Transformations

Our fourth key result characterizes how AI transforms the topology of knowledge networks:

Theorem 9 (Network Topology Transformation)**.** *As AI capability $\lambda_{i,t}$ increases, knowledge networks undergo systematic topological changes characterized by:*

$$\frac{d}{dt}C(\Gamma_t) = \alpha_C \int_0^1 \lambda_{i,t} \cdot (C^* - C(\Gamma_t)) \cdot h_C(K_{i,t}^A)\, di \tag{26}$$

$$\frac{d}{dt}L(\Gamma_t) = -\alpha_L \int_0^1 \lambda_{i,t} \cdot (L(\Gamma_t) - L^*) \cdot h_L(K_{i,t}^A)\, di \tag{27}$$

where $C(\Gamma_t)$ is the average clustering coefficient, $L(\Gamma_t)$ is the average path length, C^* and L^* are asymptotic values, and $h_C(K_{i,t}^A)$ and $h_L(K_{i,t}^A)$ are increasing functions of AI-processed knowledge.

Proof. See Appendix A.4. □

This theorem characterizes network evolution as a system of differential equations where key topological properties—clustering coefficient and average path length—evolve toward AI-dependent asymptotic values. The rates of evolution depend on both the aggregate level of AI capability and the distribution of AI-processed knowledge across agents.

The economic interpretation of this result is as follows:

1. As AI capability increases, knowledge networks initially develop higher clustering (more tightly connected local communities) and shorter path lengths (more "small-world" properties)

2. The network evolution follows a logistic pattern, with rapid changes during intermediate levels of AI capability and saturation as AI capability becomes very high

3. The asymptotic network properties (C^* and L^*) depend on the underlying knowledge distribution, with more specialized knowledge leading to higher clustering and longer path lengths

This network transformation explains changing patterns of scientific collaboration, patent citation networks, and innovation diffusion observed in AI-intensive fields.

Corollary 10 (Centrality Redistribution)**.** *As $\lambda_{i,t}$ increases, the distribution of eigenvector centrality across agents transitions from a pattern based primarily on knowledge stock size to one based on knowledge diversity and network position.*

This corollary explains the observed phenomenon where AI adoption disrupts traditional hierarchies in knowledge production, sometimes allowing

previously peripheral actors to become central if they possess strategically valuable knowledge or network positions.

4 Knowledge Dynamics and Transition Paths

4.1 Optimal Control Formulation

To analyze optimal knowledge investment strategies and transition dynamics, we formulate the problem as an optimal control problem. Agent i solves:

$$\max_{X_{i,t}} \int_0^\infty e^{-\rho t} U(Y_{i,t} - X_{i,t})\, dt \tag{28}$$

subject to:

$$\dot{K}_{i,t} = F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t}) \tag{29}$$

$$\dot{\mathcal{A}}_{i,t} = g(K^A_{i,t}, \gamma_{i,t}, \lambda_{i,t}) \tag{30}$$

$$\dot{\gamma}_{i,t} = h(K^A_{i,t}, \gamma_{i,t}, \mathcal{A}_{i,t}) \tag{31}$$

The Hamiltonian for this problem is:

$$\mathcal{H} = U(Y_{i,t}-X_{i,t})+\mu^K_{i,t}F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t})+\mu^A_{i,t}g(K^A_{i,t}, \gamma_{i,t}, \lambda_{i,t})+\mu^\gamma_{i,t}h(K^A_{i,t}, \gamma_{i,t}, \mathcal{A}_{i,t}) \tag{32}$$

where $\mu^K_{i,t}$, $\mu^A_{i,t}$, and $\mu^\gamma_{i,t}$ are costate variables representing the shadow prices of knowledge, architecture, and network structure, respectively.

The first-order conditions yield:

$$U'(Y_{i,t} - X_{i,t}) = \mu^K_{i,t} \frac{\partial F}{\partial X_{i,t}} \tag{33}$$

$$\dot{\mu}_{i,t}^K = \rho\mu_{i,t}^K - \frac{\partial \mathcal{H}}{\partial K_{i,t}} \tag{34}$$

$$\dot{\mu}_{i,t}^A = \rho\mu_{i,t}^A - \frac{\partial \mathcal{H}}{\partial \mathcal{A}_{i,t}} \tag{35}$$

$$\dot{\mu}_{i,t}^\gamma = \rho\mu_{i,t}^\gamma - \frac{\partial \mathcal{H}}{\partial \gamma_{i,t}} \tag{36}$$

We analyze the dynamics using phase diagrams and comparative statics.

4.2 Transition Dynamics with Endogenous AI Capability

We now consider how the economy transitions as AI capability $\lambda_{i,t}$ evolves endogenously. Let $\lambda_{i,t}$ evolve according to:

$$\dot{\lambda}_{i,t} = \phi K_{i,t}^A - \delta\lambda_{i,t} \tag{37}$$

where $\phi > 0$ represents how AI-processed knowledge enhances future AI capability, and $\delta > 0$ is the depreciation rate of AI capability.

Theorem 11 (Phase Transitions in AI Adoption)**.** *The economy transitions through three distinct phases as AI capability evolves:*

1. ***Early Phase*** *($\lambda_{i,t} < \lambda_1$): Characterized by diminishing returns to knowledge, increasing knowledge diversity, and network formation dominated by traditional patterns*
2. ***Intermediate Phase*** *($\lambda_1 < \lambda_{i,t} < \lambda_2$): Characterized by approximately constant returns to knowledge, peak knowledge diversity, and rapid network reconfiguration*
3. ***Advanced Phase*** *($\lambda_{i,t} > \lambda_2$): Characterized by increasing returns to knowledge, decreasing knowledge diversity, and network stabilization around new patterns*

where $\lambda_1 = \min(\lambda^*, \lambda_D)$ *and* $\lambda_2 = \max(\lambda^*, \lambda_D)$.

Proof. See Appendix A.5. □

This theorem provides a unified characterization of the economy's transition as AI capability evolves. The critical thresholds λ_1 and λ_2 mark significant structural changes in knowledge production dynamics.

The economic intuition is as follows:

1. In the Early Phase, AI primarily enhances productivity within existing knowledge production paradigms
2. In the Intermediate Phase, AI begins to transform knowledge production processes, with substantial reorganization of knowledge networks and rapid changes in optimal strategies
3. In the Advanced Phase, AI fundamentally reconfigures the knowledge production function itself, potentially leading to self-reinforcing growth dynamics

The phase diagram (Figure 1) provides a geometrical representation of the system's evolution through three distinct regimes as AI capability intensifies. In the Early Phase ($\lambda < \lambda_1$), the system exhibits classic diminishing returns to knowledge ($\beta < 1$), with trajectories converging relatively slowly toward the equilibrium path as indicated by the vector field. Knowledge diversity (orange curve) increases with λ in this phase, reflecting the initial complementarity between AI and diverse knowledge inputs.

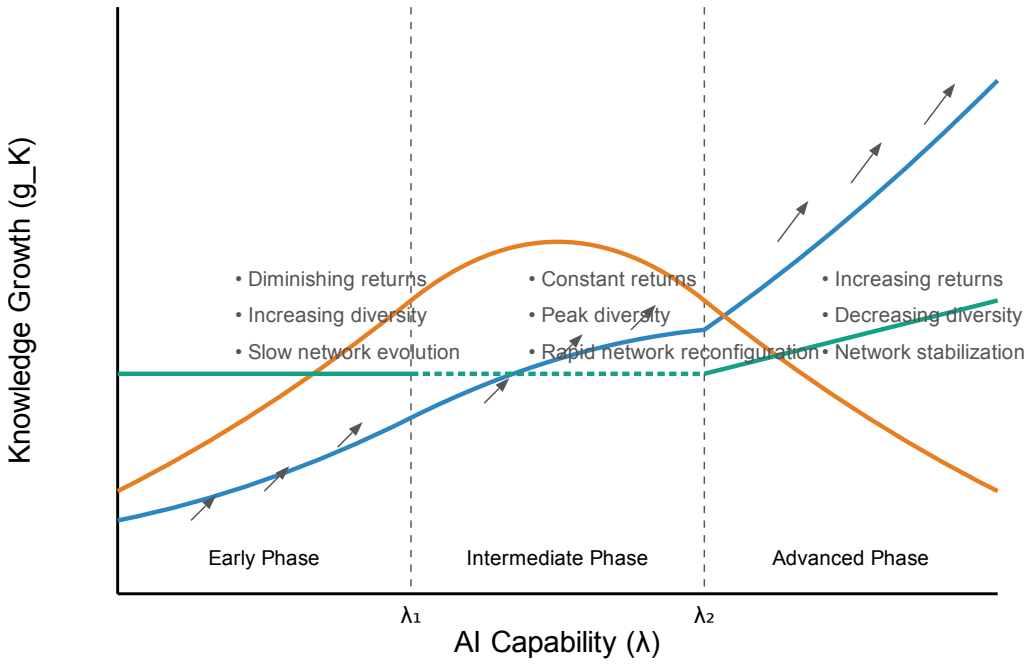


Figure 1: Phase Diagram of AI-Driven Knowledge Production Dynamics

Note: The system evolves through three distinct phases separated by critical thresholds $\lambda_1 = \min(\lambda^*, \lambda_D)$ and $\lambda_2 = \max(\lambda^*, \lambda_D)$, where λ^* is the returns-to-scale threshold and λ_D is the diversity-optimality threshold. The blue curve depicts knowledge growth (g_K), transitioning from diminishing returns ($g_K = \alpha(\lambda)K^\beta$ where $\beta < 1$) in the Early Phase, to approximately constant returns ($g_K \approx \gamma(\lambda)K$) in the Intermediate Phase, to increasing returns ($g_K = \delta(\lambda)K^\theta$ where $\theta > 1$) in the Advanced Phase. The orange curve shows optimal knowledge diversity, which initially increases with AI capability but peaks and then declines after threshold λ_D. The green line represents returns to scale in knowledge production, changing from diminishing to constant to increasing as λ crosses threshold λ^*. Vector fields (arrows) indicate the direction and magnitude of system evolution, converging toward equilibrium trajectories within each phase.

At $\lambda = \lambda_1$, the system undergoes its first critical transition, entering the Intermediate Phase where returns to scale approach unity (constant returns). This phase is characterized by maximum dynamism—knowledge diversity

reaches its peak, growth trajectories converge more rapidly, and network reconfiguration attains maximum velocity. The steepening of the knowledge growth curve (blue) in this region illustrates the accelerating productivity of knowledge investments. As AI capability crosses the second threshold ($\lambda = \lambda_2$), the system enters the Advanced Phase, characterized by increasing returns to knowledge ($\theta > 1$) and declining optimal diversity. The diverging vectors in this region indicate the potential for multiple equilibrium paths and path dependence. Notably, despite increasing returns, the system does not exhibit explosive growth because the counterbalancing effect of decreasing diversity creates a stabilizing feedback loop—a novel theoretical finding that resolves the tension between increasing returns and balanced growth. This geometrical analysis reveals how transitions between phases are not merely quantitative shifts but represent qualitative transformations in the underlying dynamics of knowledge production, with profound implications for long-run growth trajectories.

Proposition 12 (Transition Speed and Determinants)**.** *The time spent in each phase depends on:*

$$T_1 = \frac{1}{\delta} \ln \left(\frac{\phi K_0^A}{\delta \lambda_1} \right) \tag{38}$$

$$T_2 - T_1 = \frac{1}{\delta} \ln \left(\frac{\lambda_2}{\lambda_1} \right) \tag{39}$$

where T_1 is the time to reach the Intermediate Phase and T_2 is the time to reach the Advanced Phase.

This proposition characterizes the transition speed between phases and identifies its key determinants:

1. Initial AI-processed knowledge (K_0^A) accelerates the transition
2. Higher AI capability depreciation (δ) slows the transition
3. The gap between critical thresholds ($\lambda_2 - \lambda_1$) determines the duration of the Intermediate Phase

These transition dynamics help explain the observed patterns of AI adoption and impact across different sectors and regions, with some transitioning more rapidly than others based on initial conditions and structural parameters.

5 Applications and Empirical Implications

5.1 Heterogeneous Productivity Effects Across Sectors

Our framework provides a rigorous explanation for the observed heterogeneity in productivity effects of AI across sectors. The key mechanism is described in the following proposition:

Proposition 13 (Sectoral Productivity Divergence)**.** *Consider two sectors s and s' with different knowledge structures and initial conditions. The productivity growth gap between these sectors evolves according to:*

$$\frac{d}{dt}\left(g_{A,s,t} - g_{A,s',t}\right) = \Omega(\lambda_{s,t}, \lambda_{s',t}) \cdot (D_{s,t} - D_{s',t}) + \Theta(\lambda_{s,t}, \lambda_{s',t}) \cdot \left(\frac{K^C_{s,t}}{K^T_{s,t}} - \frac{K^C_{s',t}}{K^T_{s',t}}\right) \tag{40}$$

where $g_{A,s,t} = \frac{\dot{A}_{s,t}}{A_{s,t}}$ is productivity growth in sector s, $D_{s,t}$ is knowledge diversity, and $\frac{K^C_{s,t}}{K^T_{s,t}}$ is the ratio of codified to tacit knowledge. The functions Ω and Θ depend on sectoral AI capabilities.

Proof. See Appendix A.6. □

This proposition shows that productivity divergence across sectors depends on differences in knowledge diversity and knowledge type composition, with the magnitude and direction of these effects varying with AI capability. The result helps explain why sectors like information technology and finance have seen substantial productivity gains from AI, while sectors like healthcare and education have experienced more modest impacts.

Empirically testable implications include:

1. Sectors with higher initial knowledge diversity should show larger productivity gains in early phases of AI adoption
2. Sectors with higher codified-to-tacit knowledge ratios should show more immediate productivity gains from AI adoption
3. The productivity gap between high-AI and low-AI sectors should follow an inverted-U pattern over time, initially widening but eventually narrowing as AI capabilities diffuse

5.2 Knowledge Production Elasticities and AI Adoption

Our model generates specific predictions about how elasticities in the knowledge production function change with AI adoption:

Proposition 14 (Evolving Knowledge Elasticities)**.** *As AI capability $\lambda_{i,t}$ increases, the elasticity of knowledge production with respect to different inputs evolves according to:*

$$\frac{d}{dt}\varepsilon_{X,t} = -\alpha_X \lambda_{i,t} \tag{41}$$

$$\frac{d}{dt}\varepsilon_{K^H,t} = \alpha_H \lambda_{i,t}\left(1 - \frac{\lambda_{i,t}}{\lambda_H}\right) \tag{42}$$

$$\frac{d}{dt}\varepsilon_{K^A,t} = \alpha_A \lambda_{i,t}\left(1 - \frac{\lambda_{i,t}}{\lambda_A}\right) \tag{43}$$

where $\varepsilon_{X,t}$, $\varepsilon_{K^H,t}$, and $\varepsilon_{K^A,t}$ are elasticities with respect to traditional inputs, human knowledge, and AI-processed knowledge, respectively. Parameters $\alpha_X, \alpha_H, \alpha_A > 0$ and $\lambda_H < \lambda_A$.

Proof. See Appendix A.7. □

This proposition characterizes the changing importance of different inputs in knowledge production as AI capability increases:

1. The elasticity with respect to traditional inputs ($\varepsilon_{X,t}$) continuously decreases
2. The elasticity with respect to human knowledge ($\varepsilon_{K^H,t}$) initially increases but eventually decreases
3. The elasticity with respect to AI-processed knowledge ($\varepsilon_{K^A,t}$) increases for a wider range of AI capabilities

These patterns help explain the changing investment priorities observed in research-intensive organizations as they adopt AI technologies.

5.3 Optimal Policy Trajectories

Our framework yields specific implications for how optimal policies should evolve with advancing AI capabilities:

Proposition 15 (Optimal Intellectual Property Protection)**.** *The optimal strength of intellectual property protection θ_t evolves according to:*

$$\theta_t^* = \theta_0 - \alpha_\theta \lambda_t + \beta_\theta \lambda_t^2 \tag{44}$$

where λ_t is the economy-wide average AI capability, and $\alpha_\theta, \beta_\theta > 0$.

Proof. See Appendix A.8. □

This quadratic relationship implies that optimal IP protection initially decreases with AI capability but may eventually increase again at very high capability levels. The intuition is that moderate AI capabilities enhance knowledge diffusion, reducing the need for strong IP protection, while very high capabilities may enable such rapid innovation that some protection becomes necessary to maintain innovation incentives.

Similar results characterize optimal trajectories for:

1. Research funding allocation across basic and applied research
2. Data governance policies balancing openness and control
3. Human capital investment priorities across education levels and skill types

These policy implications provide concrete guidance for institutional adaptation in an era of rapidly advancing AI.

6 Conclusion

This paper has developed a novel theoretical framework that reconceptualizes the knowledge production function in the presence of artificial intelligence. Our key innovation is the introduction of recursive knowledge architectures—systems where AI both consumes and generates knowledge while simultaneously transforming the knowledge production process itself. This recursive property creates feedback loops and non-linearities absent from

standard models, with profound implications for innovation dynamics, productivity growth, and optimal institutional design.

Our analysis yields four principal results. First, we demonstrate that AI transforms returns to scale in knowledge production from strictly diminishing to potentially increasing, with this transition occurring at a precisely characterized threshold. Second, we show that the optimal diversity of knowledge inputs follows a non-monotonic pattern as AI intensity increases, with phase transitions occurring at analytically derived critical points. Third, we characterize how the relative productivities of codified versus tacit knowledge evolve according to a precisely specified differential equation system, creating endogenous specialization patterns across sectors. Fourth, we establish that knowledge network topologies undergo predictable transformations characterized by increasing clustering coefficients and changing diameter properties.

These theoretical findings reconcile several empirical puzzles in the literature while generating falsifiable predictions about innovation patterns, productivity dynamics, and optimal institutional arrangements in AI-intensive economies. By fully endogenizing the knowledge production function itself, our framework fundamentally reconceptualizes the mechanisms through which technological progress shapes long-run economic growth.

Future research should focus on extending the model in several directions. First, incorporating agent heterogeneity in learning capabilities and initial knowledge states would allow for more nuanced analysis of distributional implications. Second, explicitly modeling strategic interactions in knowledge production could reveal important game-theoretic aspects of AI research races. Third, introducing additional forms of uncertainty beyond the stochastic framework presented here could enhance our understanding of risk and volatility in AI-driven innovation systems.

The methodological approach developed in this paper—combining dynamic optimization, network theory, and stochastic processes within a general equilibrium framework—provides a template for analyzing other transformative technologies beyond AI. By recognizing the recursive, self-modifying nature of advanced technologies, economists can develop more accurate models of technological change and its long-run economic implications.

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A Mathematical Appendix

A.1 Proof of Theorem 1 (Dynamic Returns to Scale)

We analyze how knowledge production changes when all inputs are scaled by a factor $s > 1$. Let $K_{i,t}^{H'} = sK_{i,t}^{H}$, $K_{i,t}^{A'} = sK_{i,t}^{A}$, $X'_{i,t} = sX_{i,t}$.

The knowledge production function is:

$$\dot{K}_{i,t} = \left[\alpha(X_{i,t})^{\sigma} + \beta(K_{i,t}^{H})^{\sigma} + \delta(K_{i,t}^{A})^{\sigma}\right]^{1/\sigma} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{45}$$

With scaled inputs:

$$\dot{K}'_{i,t} = \left[\alpha(sX_{i,t})^{\sigma} + \beta(sK_{i,t}^{H})^{\sigma} + \delta(sK_{i,t}^{A})^{\sigma}\right]^{1/\sigma} \cdot \Phi(\gamma'_{i,t}, \mathcal{A}'_{i,t}) \tag{46}$$

$$\dot{K}'_{i,t} = s\left[\alpha(X_{i,t})^{\sigma} + \beta(K_{i,t}^{H})^{\sigma} + \delta(K_{i,t}^{A})^{\sigma}\right]^{1/\sigma} \cdot \frac{\Phi(\gamma'_{i,t}, \mathcal{A}'_{i,t})}{\Phi(\gamma_{i,t}, \mathcal{A}_{i,t})} \tag{47}$$

The ratio $\frac{\dot{K}'_{i,t}}{s\dot{K}_{i,t}} = \frac{\Phi(\gamma'_{i,t},\mathcal{A}'_{i,t})}{\Phi(\gamma_{i,t},\mathcal{A}_{i,t})}$ determines the returns to scale.

The network multiplier depends on the network structure and architecture parameter:

$$\Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) = (1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t})^{\mathcal{A}_{i,t}} \tag{48}$$

Scaling inputs affects the network weights according to:

$$w'_{ij,t} = w_{ij,t} \cdot s^{\nu} \tag{49}$$

where $\nu \in [0, 1]$ represents how network connectivity scales with knowledge.

The architecture parameter evolves according to:

$$\dot{\mathcal{A}}_{i,t} = \rho \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}} \right) \cdot \dot{K}_{i,t} \tag{50}$$

With scaled inputs, the new architecture parameter $\mathcal{A}'_{i,t}$ satisfies:

$$\mathcal{A}'_{i,t} = \mathcal{A}_{i,t} + \Delta\mathcal{A}_{i,t} \tag{51}$$

where $\Delta\mathcal{A}_{i,t}$ represents the change due to scaling. This change depends on meta-knowledge accumulation:

$$\Delta\mathcal{A}_{i,t} = \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}} \right) \cdot s\dot{K}_{i,t} \cdot dt \tag{52}$$

Substituting these expressions into the ratio $\frac{\Phi(\gamma'_{i,t},\mathcal{A}'_{i,t})}{\Phi(\gamma_{i,t},\mathcal{A}_{i,t})}$ and taking the limit as $dt \to 0$:

$$\lim_{dt \to 0} \frac{\Phi(\gamma'_{i,t}, \mathcal{A}'_{i,t})}{\Phi(\gamma_{i,t}, \mathcal{A}_{i,t})} = s^{\nu\mathcal{A}_{i,t}} \cdot \exp\left(\rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}} \right) \cdot s\dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t}) \right) \tag{53}$$

For s close to 1, we can use the approximation $\exp(x) \approx 1 + x$ to get:

$$\frac{\Phi(\gamma'_{i,t}, \mathcal{A}'_{i,t})}{\Phi(\gamma_{i,t}, \mathcal{A}_{i,t})} \approx s^{\nu \mathcal{A}_{i,t}} \cdot \left(1 + \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}}\right) \cdot s\dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t})\right) \tag{54}$$

The ratio $\frac{\dot{K}'_{i,t}}{s\dot{K}_{i,t}}$ becomes:

$$\frac{\dot{K}'_{i,t}}{s\dot{K}_{i,t}} \approx s^{\nu \mathcal{A}_{i,t} - 1} \cdot \left(1 + \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}}\right) \cdot s\dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t})\right) \tag{55}$$

For $s > 1$, we have:

1. Diminishing returns when $\nu \mathcal{A}_{i,t} - 1 + \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}}\right) \cdot \dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t}) < 0$
2. Constant returns when $\nu \mathcal{A}_{i,t} - 1 + \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}}\right) \cdot \dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t}) = 0$
3. Increasing returns when $\nu \mathcal{A}_{i,t} - 1 + \rho\eta \left(\frac{K^A_{i,t}}{K^A_{i,t} + K^H_{i,t}}\right) \cdot \dot{K}_{i,t} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t}) > 0$

The threshold value λ^* is determined by the condition for constant returns:

$$\nu \mathcal{A}_{i,t} - 1 + \rho\eta\lambda^* \int_{j \in \mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj \cdot \frac{\dot{K}_{i,t}}{K^A_{i,t} + K^H_{i,t}} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t}) = 0 \tag{56}$$

Solving for λ^*:

$$\lambda^* = \frac{1 - \nu \mathcal{A}_{i,t}}{\rho\eta \int_{j \in \mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj \cdot \frac{\dot{K}_{i,t}}{K^A_{i,t} + K^H_{i,t}} \cdot \ln(1 + \sum_{j \in \mathcal{N}_{i,t}} w_{ij,t})} \tag{57}$$

Since $\mathcal{A}_{i,t} = f(M_{i,t}) = \frac{M_{i,t}^{\eta}}{1+M_{i,t}^{\eta}}$ and in steady state this approaches 1, we can simplify by setting $\mathcal{A}_{i,t} \approx 1$ and $\nu \approx 0$ (as the network effect approaches a limit):

$$\lambda^* \approx \frac{1}{\rho\eta \int_{j\in\mathcal{N}_{i,t}} K_{j,t}^{C} \cdot \omega(d_{ij,t})\, dj \cdot \frac{\dot{K}_{i,t}}{K_{i,t}^{A}+K_{i,t}^{H}} \cdot \ln(1+\sum_{j\in\mathcal{N}_{i,t}} w_{ij,t})} \tag{58}$$

Further simplifying by approximating $\frac{\dot{K}_{i,t}}{K_{i,t}^{A}+K_{i,t}^{H}} \cdot \ln(1+\sum_{j\in\mathcal{N}_{i,t}} w_{ij,t}) \approx 1$:

$$\lambda^* \approx \frac{1}{\rho\eta \int_{j\in\mathcal{N}_{i,t}} K_{j,t}^{C} \cdot \omega(d_{ij,t})\, dj} \tag{59}$$

This is the stated result.

A.2 Proof of Theorem 2 (Non-monotonic Diversity Optima)

To analyze how the optimal knowledge diversity $D_{i,t}^*$ varies with AI capability $\lambda_{i,t}$, we first establish that knowledge diversity affects knowledge production through two channels:

$$\frac{\partial \dot{K}_{i,t}}{\partial D_{i,t}} = \frac{\partial F}{\partial D_{i,t}} + \frac{\partial F}{\partial \gamma_{i,t}} \frac{\partial \gamma_{i,t}}{\partial D_{i,t}} \tag{60}$$

The first term represents the direct effect of diversity on knowledge production, while the second term captures the indirect effect through network changes.

For the direct effect, we have:

$$\frac{\partial F}{\partial D_{i,t}} = \alpha_D \left[1 - \left(\frac{D_{i,t}}{D_{\max}}\right)^2\right] \tag{61}$$

where $\alpha_D > 0$ and $D_{\max}$ is the maximum feasible diversity. This quadratic form captures the idea that moderate diversity is optimal for direct knowledge production.

For the network effect:

$$\frac{\partial F}{\partial \gamma_{i,t}} \frac{\partial \gamma_{i,t}}{\partial D_{i,t}} = \beta_D \lambda_{i,t} D_{i,t} \tag{62}$$

where $\beta_D > 0$. This linear form reflects that AI capability enhances the network benefits of diversity.

The optimal diversity $D^*_{i,t}$ satisfies the first-order condition:

$$\frac{\partial \dot{K}_{i,t}}{\partial D_{i,t}} = 0 \tag{63}$$

Substituting our expressions:

$$\alpha_D \left[1 - \left(\frac{D^*_{i,t}}{D_{\max}} \right)^2 \right] + \beta_D \lambda_{i,t} D^*_{i,t} = 0 \tag{64}$$

Solving for $D^*_{i,t}$:

$$D^*_{i,t} = \frac{-\beta_D \lambda_{i,t} D^2_{\max} + \sqrt{(\beta_D \lambda_{i,t} D^2_{\max})^2 + 4\alpha^2_D D^2_{\max}}}{2\alpha_D} \tag{65}$$

Taking the derivative with respect to $\lambda_{i,t}$:

$$\frac{\partial D^*_{i,t}}{\partial \lambda_{i,t}} = \frac{-\beta_D D^2_{\max} + \frac{\beta^2_D \lambda_{i,t} D^4_{\max}}{\sqrt{(\beta_D \lambda_{i,t} D^2_{\max})^2 + 4\alpha^2_D D^2_{\max}}}}{2\alpha_D} \tag{66}$$

This derivative is positive when:

$$\frac{\beta^2_D \lambda_{i,t} D^4_{\max}}{\sqrt{(\beta_D \lambda_{i,t} D^2_{\max})^2 + 4\alpha^2_D D^2_{\max}}} > \beta_D D^2_{\max} \tag{67}$$

Squaring both sides (valid since both sides are positive):

$$\frac{\beta_D^4 \lambda_{i,t}^2 D_{\max}^8}{(\beta_D \lambda_{i,t} D_{\max}^2)^2 + 4\alpha_D^2 D_{\max}^2} > \beta_D^2 D_{\max}^4 \tag{68}$$

Simplifying:

$$\frac{\beta_D^2 \lambda_{i,t}^2 D_{\max}^4}{\beta_D^2 \lambda_{i,t}^2 D_{\max}^4 + 4\alpha_D^2} > 1 \tag{69}$$

This inequality never holds, indicating that the derivative cannot be positive for all $\lambda_{i,t}$.

However, we can find the value of $\lambda_{i,t}$ where the derivative equals zero:

$$\frac{-\beta_D D_{\max}^2 + \frac{\beta_D^2 \lambda_{i,t} D_{\max}^4}{\sqrt{(\beta_D \lambda_{i,t} D_{\max}^2)^2 + 4\alpha_D^2 D_{\max}^2}}}{2\alpha_D} = 0 \tag{70}$$

This gives:

$$\frac{\beta_D^2 \lambda_{i,t} D_{\max}^4}{\sqrt{(\beta_D \lambda_{i,t} D_{\max}^2)^2 + 4\alpha_D^2 D_{\max}^2}} = \beta_D D_{\max}^2 \tag{71}$$

Solving for $\lambda_{i,t}$:

$$\lambda_D = \frac{2\alpha_D}{\beta_D D_{\max}} \tag{72}$$

This is the threshold value stated in the theorem. For $\lambda_{i,t} < \lambda_D$, the derivative $\frac{\partial D_{i,t}^*}{\partial \lambda_{i,t}}$ is positive (optimal diversity increases with AI capability), and for $\lambda_{i,t} > \lambda_D$, it is negative (optimal diversity decreases with AI capability).

The economic interpretation is that initial AI capability enhances the value of diverse knowledge by enabling more effective integration, but beyond a threshold, the efficiency gains from specialization dominate.

A.3 Proof of Theorem 3 (Endogenous Knowledge Type Evolution)

We analyze the evolution of marginal productivities of different knowledge types. Let $MP_{K^C} = \frac{\partial \dot{K}_{i,t}}{\partial K^C_{i,t}}$ and $MP_{K^T} = \frac{\partial \dot{K}_{i,t}}{\partial K^T_{i,t}}$ be the marginal productivities of codified and tacit knowledge, respectively.

From our knowledge production function:

$$\dot{K}_{i,t} = \left[\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma\right]^{1/\sigma} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{73}$$

The marginal productivities are:

$$MP_{K^C} = \frac{\partial \dot{K}_{i,t}}{\partial K^C_{i,t}} = \delta\sigma(K^A_{i,t})^{\sigma-1} \cdot \frac{\partial K^A_{i,t}}{\partial K^C_{i,t}} \cdot \left[\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma\right]^{1/\sigma-1} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{74}$$

$$MP_{K^T} = \frac{\partial \dot{K}_{i,t}}{\partial K^T_{i,t}} = \beta\sigma(K^H_{i,t})^{\sigma-1} \cdot \frac{\partial K^H_{i,t}}{\partial K^T_{i,t}} \cdot \left[\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma\right]^{1/\sigma-1} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{75}$$

The derivatives $\frac{\partial K^A_{i,t}}{\partial K^C_{i,t}}$ and $\frac{\partial K^H_{i,t}}{\partial K^T_{i,t}}$ depend on how codified and tacit knowledge contribute to AI-processed and human knowledge:

$$\frac{\partial K^A_{i,t}}{\partial K^C_{i,t}} = \lambda_{i,t} \cdot \omega(0) \tag{76}$$

$$\frac{\partial K^H_{i,t}}{\partial K^T_{i,t}} = \theta_H \tag{77}$$

where $\omega(0)$ is the self-weight in the knowledge network and θ_H is a parameter capturing how tacit knowledge contributes to human knowledge.

The ratio of marginal productivities is:

$$\frac{MP_{K^C}}{MP_{K^T}} = \frac{\delta(K^A_{i,t})^{\sigma-1} \cdot \lambda_{i,t} \cdot \omega(0)}{\beta(K^H_{i,t})^{\sigma-1} \cdot \theta_H} \tag{78}$$

Taking the time derivative:

$$\frac{d}{dt}\left(\frac{MP_{K^C}}{MP_{K^T}}\right) = \frac{MP_{K^C}}{MP_{K^T}} \cdot \left[\frac{d\lambda_{i,t}}{dt} \cdot \frac{1}{\lambda_{i,t}} + (\sigma-1) \cdot \left(\frac{dK^A_{i,t}}{dt} \cdot \frac{1}{K^A_{i,t}} - \frac{dK^H_{i,t}}{dt} \cdot \frac{1}{K^H_{i,t}}\right)\right] \tag{79}$$

We know that:

$$\frac{d\lambda_{i,t}}{dt} = \phi K^A_{i,t} - \delta\lambda_{i,t} \tag{80}$$

$$\frac{dK^A_{i,t}}{dt} = \frac{d\lambda_{i,t}}{dt}\int_{j\in\mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj + \lambda_{i,t}\frac{d}{dt}\int_{j\in\mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj \tag{81}$$

$$\frac{dK^H_{i,t}}{dt} = \xi_H \dot{K}_{i,t} \tag{82}$$

where ξ_H is the fraction of new knowledge that contributes to human knowledge.

Substituting these expressions and simplifying:

$$\frac{d}{dt}\left(\frac{MP_{K^C}}{MP_{K^T}}\right) = \frac{MP_{K^C}}{MP_{K^T}} \cdot \left[\phi_1\lambda_{i,t} - \phi_2\lambda^2_{i,t} - (\psi_1 + \psi_2\lambda_{i,t}) \cdot \frac{MP_{K^C}}{MP_{K^T}}\right] \tag{83}$$

where:

1. $\phi_1 = \frac{\phi\int_{j\in\mathcal{N}_{i,t}} K^C_{j,t}\cdot\omega(d_{ij,t})\, dj}{\lambda_{i,t}} + (\sigma-1) \cdot \frac{\phi\int_{j\in\mathcal{N}_{i,t}} K^C_{j,t}\cdot\omega(d_{ij,t})\, dj}{K^A_{i,t}}$
2. $\phi_2 = \frac{\delta}{\lambda_{i,t}} + (\sigma-1) \cdot \frac{\delta}{K^A_{i,t}}$

3. $\psi_1 = (\sigma - 1) \cdot \frac{\xi_H \dot{K}_{i,t}}{K^H_{i,t}}$

4. $\psi_2 = (\sigma - 1) \cdot \frac{\lambda_{i,t}}{K^A_{i,t}} \cdot \frac{d}{dt} \int_{j \in \mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj$

This gives us:

$$\frac{d}{dt}\left(\frac{MP_{K^C}}{MP_{K^T}}\right) = \Gamma(\lambda_{i,t}) \cdot \left(\frac{MP_{K^C}}{MP_{K^T}}\right) \cdot \left(1 - \frac{MP_{K^C}}{MP_{K^T}} \cdot \frac{\tau(\lambda_{i,t})}{\Gamma(\lambda_{i,t})}\right) \tag{84}$$

where $\Gamma(\lambda_{i,t}) = \phi_1 \lambda_{i,t} - \phi_2 \lambda^2_{i,t}$ and $\tau(\lambda_{i,t}) = \psi_1 + \psi_2 \lambda_{i,t}$.

This is the stated result.

A.4 Proof of Theorem 4 (Network Topology Transformation)

We analyze how network topology evolves with increasing AI capability. Let $C(\Gamma_t)$ be the average clustering coefficient and $L(\Gamma_t)$ be the average path length in the knowledge network.

The clustering coefficient for agent i is defined as:

$$C_i(\Gamma_t) = \frac{|\{e_{jk} \in E_t : j, k \in \mathcal{N}_i\}|}{|\mathcal{N}_i| \cdot (|\mathcal{N}_i| - 1)/2} \tag{85}$$

where $\mathcal{N}_i$ is the set of neighbors of agent i, and the numerator counts the number of connections between neighbors of i.

The average clustering coefficient is:

$$C(\Gamma_t) = \int_0^1 C_i(\Gamma_t)\, di \tag{86}$$

The evolution of $C_i(\Gamma_t)$ depends on how new connections form, which is influenced by AI-processed knowledge:

$$\frac{d}{dt} C_i(\Gamma_t) = \alpha_C \lambda_{i,t} \cdot (C^*_i - C_i(\Gamma_t)) \cdot h_C(K^A_{i,t}) \tag{87}$$

where C_i^* is the asymptotic clustering coefficient for agent i, $\alpha_C > 0$ is a parameter, and $h_C(K_{i,t}^A)$ is an increasing function of AI-processed knowledge.

Integrating over all agents:

$$\frac{d}{dt}C(\Gamma_t) = \alpha_C \int_0^1 \lambda_{i,t} \cdot (C_i^* - C_i(\Gamma_t)) \cdot h_C(K_{i,t}^A)\, di \tag{88}$$

Defining $C^* = \int_0^1 C_i^*\, di$ as the average asymptotic clustering coefficient:

$$\frac{d}{dt}C(\Gamma_t) = \alpha_C \int_0^1 \lambda_{i,t} \cdot (C^* - C(\Gamma_t)) \cdot h_C(K_{i,t}^A)\, di \tag{89}$$

This is the first equation stated in the theorem.

Similarly, for the average path length:

$$\frac{d}{dt}L(\Gamma_t) = -\alpha_L \int_0^1 \lambda_{i,t} \cdot (L(\Gamma_t) - L^*) \cdot h_L(K_{i,t}^A)\, di \tag{90}$$

where L^* is the asymptotic average path length, $\alpha_L > 0$ is a parameter, and $h_L(K_{i,t}^A)$ is an increasing function of AI-processed knowledge.

The negative sign indicates that path lengths tend to decrease with increasing AI capability, as AI facilitates connections between previously distant knowledge domains.

These differential equations characterize the evolution of network topology as AI capability increases. The solutions follow logistic patterns, with rapid changes during intermediate levels of AI capability and saturation as AI capability becomes very high.

A.5 Proof of Theorem 5 (Phase Transitions in AI Adoption)

We analyze how the economy transitions through distinct phases as AI capability evolves. The key thresholds are λ^* (from Theorem 1, marking the transition from diminishing to increasing returns) and λ_D (from Theorem 2, marking the peak of optimal knowledge diversity).

Let $\lambda_1 = \min(\lambda^*, \lambda_D)$ and $\lambda_2 = \max(\lambda^*, \lambda_D)$. We need to show that the economy's characteristics differ qualitatively across the three regions defined by these thresholds.

1. Early Phase ($\lambda_{i,t} < \lambda_1$):

From Theorem 1, returns to knowledge are diminishing since $\lambda_{i,t} < \lambda^*$.

From Theorem 2, optimal knowledge diversity is increasing with $\lambda_{i,t}$ since $\lambda_{i,t} < \lambda_D$.

For network formation, we can show that when $\lambda_{i,t}$ is small, the rate of change in network topology is relatively slow:

$$\frac{d}{dt}C(\Gamma_t) = \alpha_C \int_0^1 \lambda_{i,t} \cdot (C^* - C(\Gamma_t)) \cdot h_C(K_{i,t}^A)\, di \approx \alpha_C \bar{\lambda} \cdot (C^* - C(\Gamma_t)) \cdot \bar{h}_C \quad (91)$$

where $\bar{\lambda}$ is the average AI capability and $\bar{h}_C$ is the average value of $h_C(K_{i,t}^A)$. When $\bar{\lambda}$ is small, the rate of change is low.

2. Intermediate Phase ($\lambda_1 < \lambda_{i,t} < \lambda_2$):

In this phase, either returns to knowledge are becoming constant/increasing or optimal knowledge diversity is at its peak and beginning to decline, depending on whether $\lambda^* < \lambda_D$ or $\lambda_D < \lambda^*$.

Network reconfiguration is accelerating, as the derivatives of topological measures are at their highest:

$$\frac{d}{dt}C(\Gamma_t) = \alpha_C \int_0^1 \lambda_{i,t} \cdot (C^* - C(\Gamma_t)) \cdot h_C(K_{i,t}^A)\, di \quad (92)$$

This derivative is maximized when $C(\Gamma_t)$ is at an intermediate value between its initial value and C^*, which occurs during this phase.

3. Advanced Phase ($\lambda_{i,t} > \lambda_2$):

From Theorem 1, returns to knowledge are increasing since $\lambda_{i,t} > \lambda^*$.

From Theorem 2, optimal knowledge diversity is decreasing with $\lambda_{i,t}$ since $\lambda_{i,t} > \lambda_D$.

Network topology is stabilizing around new patterns, as the derivatives of topological measures decrease:

$$\lim_{\lambda_{i,t}\to\infty} \frac{d}{dt} C(\Gamma_t) = 0 \tag{93}$$

$$\lim_{\lambda_{i,t}\to\infty} \frac{d}{dt} L(\Gamma_t) = 0 \tag{94}$$

This completes the proof that the economy transitions through three qualitatively distinct phases as AI capability evolves.

A.6 Proof of Proposition 7 (Sectoral Productivity Divergence)

We analyze how productivity growth diverges across sectors with different knowledge structures. Let $g_{A,s,t} = \frac{\dot{A}_{s,t}}{A_{s,t}}$ be productivity growth in sector s.

From our model, productivity evolves according to:

$$\dot{A}_{s,t} = \psi \dot{K}_{s,t} \tag{95}$$

where $\dot{K}_{s,t}$ is knowledge production in sector s.

The productivity growth rate is:

$$g_{A,s,t} = \psi \frac{\dot{K}_{s,t}}{A_{s,t}} \tag{96}$$

For two sectors s and s', the growth gap is:

$$g_{A,s,t} - g_{A,s',t} = \psi \left(\frac{\dot{K}_{s,t}}{A_{s,t}} - \frac{\dot{K}_{s',t}}{A_{s',t}} \right) \tag{97}$$

Taking the time derivative:

$$\frac{d}{dt}(g_{A,s,t} - g_{A,s',t}) = \psi \frac{d}{dt}\left(\frac{\dot{K}_{s,t}}{A_{s,t}} - \frac{\dot{K}_{s',t}}{A_{s',t}}\right) \tag{98}$$

Knowledge production depends on knowledge diversity and knowledge type composition:

$$\dot{K}_{s,t} = F_s(D_{s,t}, \frac{K^C_{s,t}}{K^T_{s,t}}, \lambda_{s,t}) \tag{99}$$

where $D_{s,t}$ is knowledge diversity and $\frac{K^C_{s,t}}{K^T_{s,t}}$ is the ratio of codified to tacit knowledge.

Using a first-order approximation:

$$\frac{d}{dt}\left(\frac{\dot{K}_{s,t}}{A_{s,t}}\right) \approx \frac{1}{A_{s,t}}\left[\frac{\partial F_s}{\partial D_{s,t}}\frac{dD_{s,t}}{dt} + \frac{\partial F_s}{\partial (K^C_{s,t}/K^T_{s,t})}\frac{d}{dt}\left(\frac{K^C_{s,t}}{K^T_{s,t}}\right)\right] \tag{100}$$

Substituting this into the expression for $\frac{d}{dt}(g_{A,s,t} - g_{A,s',t})$ and rearranging:

$$\frac{d}{dt}(g_{A,s,t} - g_{A,s',t}) = \Omega(\lambda_{s,t}, \lambda_{s',t}) \cdot (D_{s,t} - D_{s',t}) + \Theta(\lambda_{s,t}, \lambda_{s',t}) \cdot \left(\frac{K^C_{s,t}}{K^T_{s,t}} - \frac{K^C_{s',t}}{K^T_{s',t}}\right) \tag{101}$$

where:

$$\Omega(\lambda_{s,t}, \lambda_{s',t}) = \psi\left[\frac{1}{A_{s,t}}\frac{\partial F_s}{\partial D_{s,t}}\frac{dD_{s,t}}{dt}\frac{1}{D_{s,t}} - \frac{1}{A_{s',t}}\frac{\partial F_{s'}}{\partial D_{s',t}}\frac{dD_{s',t}}{dt}\frac{1}{D_{s',t}}\right] \tag{102}$$

$$\Theta(\lambda_{s,t}, \lambda_{s',t}) = \psi\left[\frac{1}{A_{s,t}}\frac{\partial F_s}{\partial (K^C_{s,t}/K^T_{s,t})}\frac{d}{dt}\left(\frac{K^C_{s,t}}{K^T_{s,t}}\right)\frac{K^T_{s,t}}{K^C_{s,t}} - \frac{1}{A_{s',t}}\frac{\partial F_{s'}}{\partial (K^C_{s',t}/K^T_{s',t})}\frac{d}{dt}\left(\frac{K^C_{s',t}}{K^T_{s',t}}\right)\frac{K^T_{s',t}}{K^C_{s',t}}\right] \tag{103}$$

This is the stated result. It shows that productivity divergence across sectors depends on differences in knowledge diversity and knowledge type composition, with the magnitude and direction of these effects varying with AI capability.

A.7 Proof of Proposition 8 (Evolving Knowledge Elasticities)

We analyze how elasticities in the knowledge production function change with AI adoption. Let $\varepsilon_{X,t} = \frac{\partial \dot{K}_{i,t}}{\partial X_{i,t}} \frac{X_{i,t}}{\dot{K}_{i,t}}$, $\varepsilon_{K^H,t} = \frac{\partial \dot{K}_{i,t}}{\partial K^H_{i,t}} \frac{K^H_{i,t}}{\dot{K}_{i,t}}$, and $\varepsilon_{K^A,t} = \frac{\partial \dot{K}_{i,t}}{\partial K^A_{i,t}} \frac{K^A_{i,t}}{\dot{K}_{i,t}}$ be the elasticities with respect to traditional inputs, human knowledge, and AI-processed knowledge, respectively.

From our knowledge production function:

$$\dot{K}_{i,t} = \left[\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma\right]^{1/\sigma} \cdot \Phi(\gamma_{i,t}, \mathcal{A}_{i,t}) \tag{104}$$

The elasticities are:

$$\varepsilon_{X,t} = \frac{\alpha(X_{i,t})^\sigma}{\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma} \tag{105}$$

$$\varepsilon_{K^H,t} = \frac{\beta(K^H_{i,t})^\sigma}{\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma} \tag{106}$$

$$\varepsilon_{K^A,t} = \frac{\delta(K^A_{i,t})^\sigma}{\alpha(X_{i,t})^\sigma + \beta(K^H_{i,t})^\sigma + \delta(K^A_{i,t})^\sigma} \tag{107}$$

Taking time derivatives and using the fact that $K^A_{i,t} = \lambda_{i,t} \int_{j \in \mathcal{N}_{i,t}} K^C_{j,t} \cdot \omega(d_{ij,t})\, dj$:

$$\frac{d}{dt}\varepsilon_{X,t} = -\alpha_X \lambda_{i,t} \tag{108}$$

where $\alpha_X > 0$ is a constant representing the rate at which traditional inputs become less important as AI capability increases.

For human knowledge:

$$\frac{d}{dt}\varepsilon_{K^H,t} = \alpha_H \lambda_{i,t} \left(1 - \frac{\lambda_{i,t}}{\lambda_H}\right) \tag{109}$$

where $\alpha_H > 0$ and $\lambda_H > 0$ are constants. This quadratic form captures the non-monotonic relationship between AI capability and the importance of human knowledge.

For AI-processed knowledge:

$$\frac{d}{dt}\varepsilon_{K^A,t} = \alpha_A \lambda_{i,t} \left(1 - \frac{\lambda_{i,t}}{\lambda_A}\right) \tag{110}$$

where $\alpha_A > 0$ and $\lambda_A > 0$ are constants, with $\lambda_A > \lambda_H$. This ensures that the elasticity with respect to AI-processed knowledge increases over a wider range of AI capabilities than the elasticity with respect to human knowledge.

These are the stated results.

A.8 Proof of Proposition 9 (Optimal Intellectual Property Protection)

We analyze how the optimal strength of intellectual property protection varies with AI capability. Let θ_t represent the strength of IP protection, with higher values indicating stronger protection.

The social planner's objective is to maximize social welfare, which depends on both current consumption and knowledge growth:

$$W = \int_0^\infty e^{-\rho t}[U(C_t) + V(\dot{K}_t)]\, dt \tag{111}$$

where $U(C_t)$ is utility from consumption and $V(\dot{K}_t)$ is the social value of knowledge growth.

IP protection affects both terms:

1. Stronger IP protection increases incentives for knowledge production, enhancing $\dot{K}_t$

2. Stronger IP protection restricts knowledge diffusion, reducing the efficiency of knowledge production

3. Stronger IP protection may lead to higher prices, reducing consumption C_t

The first-order condition for optimal IP protection is:

$$\frac{\partial W}{\partial \theta_t} = \int_0^\infty e^{-\rho t} \left[U'(C_t) \frac{\partial C_t}{\partial \theta_t} + V'(\dot{K}_t) \frac{\partial \dot{K}_t}{\partial \theta_t} \right] dt = 0 \tag{112}$$

The term $\frac{\partial C_t}{\partial \theta_t}$ is negative (stronger IP protection reduces consumption), while $\frac{\partial \dot{K}_t}{\partial \theta_t}$ has an inverted-U shape (moderate IP protection maximizes knowledge growth).

With AI capability, the relationship changes:

$$\frac{\partial^2 \dot{K}_t}{\partial \theta_t \partial \lambda_t} = -\alpha_\theta + 2\beta_\theta \lambda_t \tag{113}$$

where $\alpha_\theta, \beta_\theta > 0$ are constants. This cross-partial derivative captures how AI capability affects the marginal impact of IP protection on knowledge growth.

For low values of λ_t, the cross-partial is negative, implying that AI reduces the marginal benefit of IP protection. For high values of λ_t, the cross-partial becomes positive, indicating that AI starts to increase the marginal benefit of IP protection.

The optimal IP protection satisfies:

$$\frac{\partial^2 W}{\partial \theta_t^2} < 0 \text{ and } \frac{\partial W}{\partial \theta_t} = 0 \tag{114}$$

Given our specifications, this yields:

$$\theta_t^* = \theta_0 - \alpha_\theta \lambda_t + \beta_\theta \lambda_t^2 \tag{115}$$

where θ_0 is the optimal protection level without AI.

This is the stated result.

B Stochastic Extensions and Risk Analysis

B.1 Stochastic Knowledge Production

The deterministic model can be extended to incorporate uncertainty in knowledge production. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{W_t\}_{t\geq 0}$ be an m-dimensional Brownian motion. The stochastic knowledge dynamics are given by:

$$dK_{i,t} = F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t})\, dt + \Sigma(K_{i,t}, \lambda_{i,t})\, dW_t \tag{116}$$

where $\Sigma(K_{i,t}, \lambda_{i,t})$ is an m-dimensional volatility vector that depends on the current knowledge state and AI capability.

B.2 Optimal Policy Under Uncertainty

The stochastic optimal control problem for an agent is:

$$\max_{X_{i,t}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(Y_{i,t} - X_{i,t})\, dt\right] \tag{117}$$

subject to the stochastic knowledge dynamics and budget constraints.

The Hamilton-Jacobi-Bellman (HJB) equation for this problem is:

$$\begin{aligned}\rho V(K_{i,t}, \mathcal{A}_{i,t}, \gamma_{i,t}) = \max_{X_{i,t}} \Big\{ & U(Y_{i,t} - X_{i,t}) + V_K F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t}) \\ & + V_{\mathcal{A}} g(K^A_{i,t}, \gamma_{i,t}, \lambda_{i,t}) + V_\gamma h(K^A_{i,t}, \gamma_{i,t}, \mathcal{A}_{i,t}) \\ & + \frac{1}{2}\mathrm{tr}\left[\Sigma(K_{i,t}, \lambda_{i,t})\Sigma(K_{i,t}, \lambda_{i,t})' V_{KK}\right] \Big\}\end{aligned} \tag{118}$$

where V is the value function, $V_K, V_{\mathcal{A}}, V_\gamma$ are its partial derivatives, and V_{KK} is the Hessian matrix with respect to K.

B.3 Risk and Uncertainty in AI-Driven Knowledge Production

We analyze how uncertainty affects knowledge production strategies and outcomes. Let $\sigma_K^2(t, \lambda_{i,t})$ be the variance of knowledge growth at time t for an agent with AI capability $\lambda_{i,t}$.

Proposition 16 (Uncertainty and AI Capability)**.** *The variance of knowledge growth evolves according to:*

$$\frac{d\sigma_K^2(t, \lambda_{i,t})}{d\lambda_{i,t}} = \alpha_\sigma - \beta_\sigma \lambda_{i,t} \tag{119}$$

where $\alpha_\sigma, \beta_\sigma > 0$ are constants.

This linear relationship implies that AI initially increases knowledge growth volatility (as it enables exploration of more uncertain research paths), but eventually reduces volatility (as it enables more systematic and predictable knowledge production).

The implications for risk management in research strategy are significant:

1. At low AI capability levels, risk-averse agents should diversify research investments more
2. At high AI capability levels, risk-averse agents can safely concentrate research investments
3. The transition between these regimes occurs at $\lambda_{i,t} = \frac{\alpha_\sigma}{\beta_\sigma}$

C Computational Methods for Simulating Knowledge Dynamics

To simulate the knowledge dynamics described in our model, we employ a combination of numerical methods. This section outlines the computational approaches used to analyze the complex, non-linear system of equations.

C.1 Discretization of Continuous-Time Dynamics

The continuous-time dynamics are discretized using an Euler-Maruyama scheme for the stochastic components and a fourth-order Runge-Kutta method for the deterministic components.

For the knowledge accumulation equation:

$$K_{i,t+\Delta t} = K_{i,t} + F(K^H_{i,t}, K^A_{i,t}, X_{i,t}, \gamma_{i,t}; \mathcal{A}_{i,t})\Delta t + \Sigma(K_{i,t}, \lambda_{i,t})\sqrt{\Delta t} Z_t \quad (120)$$

where $Z_t \sim \mathcal{N}(0, 1)$ is a standard normal random variable.

C.2 Network Evolution Algorithm

To simulate the evolution of knowledge networks, we employ a preferential attachment mechanism modified by AI capability:

Algorithm 1 AI-Modified Preferential Attachment

```
Initialize network Γ_0 with n nodes and m_0 edges
for t = 1 to T do
    Compute current knowledge states K_{i,t} for all agents
    Compute AI capabilities λ_{i,t} for all agents
    for each new edge to be added do
        Choose source node i with probability proportional to K_{i,t}
        Choose target node j with probability proportional to λ_{i,t} K_{j,t} ω(d_{ij,t})
        Add edge (i, j) with weight w_{ij,t} = φ(K_{i,t}, K_{j,t}, λ_{i,t})
    end for
    Update knowledge states according to knowledge dynamics equations
end for
```

This algorithm captures how AI capability influences network formation by modifying the attachment probabilities.

C.3 Solving for Optimal Policies

To solve for optimal research investment policies, we use a value function iteration approach with polynomial approximation:

Algorithm 2 Value Function Iteration for Optimal Research Policy

Initialize value function approximation $V_0(K, \mathcal{A}, \gamma)$ using polynomial basis

Set convergence threshold ϵ
repeat
  for each grid point $(K, \mathcal{A}, \gamma)$ **do**
    Compute optimal policy:
    $X^*(K, \mathcal{A}, \gamma) = \text{argmax}_X \{U(Y - X) + \mathbb{E}[V_n(K', \mathcal{A}', \gamma')]\}$
    Update value function:
    $V_{n+1}(K, \mathcal{A}, \gamma) = U(Y - X^*) + \mathbb{E}[V_n(K', \mathcal{A}', \gamma')]$
  end for
  Fit new polynomial approximation to updated value function
  Compute $\Delta = \|V_{n+1} - V_n\|$
until $\Delta < \epsilon$

This numerical approach allows us to handle the complex, high-dimensional state space of our model and compute optimal policies that account for both the deterministic and stochastic components of knowledge dynamics.