ON THE INTERACTION OF COMMITMENT AND COMMUNICATION CONSTRAINTS IN THE DYNAMIC BEHAVIOUR OF THE DISTRIBUTION OF WEALTH

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ABSTRACT. Within an economy, individual fortunes fluctuate unceasingly, and as a result each individual desires to moderate these fluctuations. This moderation is potentially achievable by averaging incomes in cooperation with others via a contract to share risk, but the contract must be incentive compatible, that is, it must induce individual participants to reveal the true state of their income, as they have the incentive to misrepresent their incomes as low. Green (1985) demonstrated that an incentive-compatible contract is in fact feasible.

What is not so obvious is that the moderation achieved via the contract can be equivalently accomplished by markets in which individuals borrow and save assets; I have established in previous publications under specialized preferences and technology that the equivalence is exact. Now we can ask, what interest rate achieves the best risk sharing? There is an answer, and if imposed, it results, surprisingly, in an ever-widening distribution of wealth.

To actually implement this interest rate you have to get everyone to stay committed to remaining in the equivalent contract. Because some individuals are inevitably luckier than others, and because the lucky ones will be asked to sacrifice some of their wealth to the unlucky, they will want to renege. Can we design the contract so that the desire of the lucky to renege never happens? Yes, by explicitly adding a constraint—a defection constraint—to the contract optimization problem, although the efficacy of the risk sharing the contract enables is impaired.

By adding the defection constraint to the contract, you have solved the so-called decentralization problem for asset markets: asset markets can function independently of government, and potentially effect the risk-sharing that was the desideratum.

In this paper I develop and simulate the contract model and roughly reproduce US consumption, interest rate and wealth distribution patterns. I demonstrate that the marginal value of information embodied in the Lagrange multiplier for the contract's incentive constraint, and the marginal value of commitment, embodied in the Lagrange multiplier for the contract's defection constraint, affect the interest rate, and the quantitative impact of these effects.

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1. Introduction

In this paper I develop and simulate a model that roughly reproduces US consumption, interest rate and wealth distribution patterns. Rather than building on standard macroeconomic underpinnings—utility maximization over consumption, labor and savings in the face of asset markets—I view the macroeconomy as the implementation of a dynamic contract in which there are incentives to hide information and to defect from the contract. I demonstrate that the marginal value of information embodied in the Lagrange multiplier for the contract's incentive constraint, and the marginal value of commitment, embodied in the Lagrange multiplier for the contract's defection constraint, affect the interest rate, and the quantitative impact of these effects.

It has become standard to model agents or firms in dynamic, optimisation-based models to impose a transversality condition on the agents in the model. In a planner problem this is a correct way to model the terminal condition. However in a heterogeneous-agent model in which individual agents each choose their own investment in the face of an equilibrium interest rate path or process, the individuals have the incentive to over-borrow unless there is an enforcement mechanism to ensure that they obey an appropriate terminal condition. It is straightforward to demonstrate that enforcing a terminal condition that enforces long run budget balance on an individual is equivalent to enforcing a transversality condition, so that in the aggregate the equilibrium is equivalent to a planner problem. (This is just an instance of the first welfare theorem.)

The supposition that there is some enforcement mechanism for budget feasibility, even if unstated and in the background, does not clash with fundamental modelling esthetics and principles in deterministic models. This is because these models automatically entail perfect foresight, and so the future time paths of all consumption plans are determined at each instant, and implicitly observable to all, including any background enforcement mechanism. Any deviation from budget balance can therefore be deterred.

In stochastic models this is a little less clear. Because income shocks are stochastic, consumption can also be stochastic. Budget balance can only be achieved on average, and in particular it makes sense for consumption to temporarily exceed income when income is low, for the sake of consumption-smoothing in the service of long-run expected utility maximization. Consumption itself becomes a *process*, determined by an optimal consumption *function* that takes as arguments an individual's income state, history, and aggregate states such as the interest

rate. This function must have the property that long run budget balance is achieved in expectation. This in turn can work in equilibrium only if there are counterparties who lend, and these counterparties must be sure that long-run budget balance is maintained by each individual, that is, the equilibrium must ensure that they will be repaid in expectation.

It is again reasonable to suppose that there is an enforcement mechanism in the background, as long as idiosyncratic income shocks can be publicly observed. In that case, the consumption function is public knowledge because it is optimal, and so it can be verified whether an individual is consuming in accord with this function.

If income is publicly unobservable this logic fails. Each individual has the incentive to report income that is below his true income, in order to justify excess borrowing; in practice one can envision this as individuals paying their credit card invoice using their credit card. One can then carry an arbitrary credit card debt, growing in size indefinitely due to the accruing interest charges. If all individuals pursue this strategy then all become permanent borrowers. This is resource-infeasible and cannot yield an equilibrium.

Imposing borrowing limits and constraints solves the problem: borrowing states are externally verifiable. But this raises the question: what is the socially optimal borrowing constraint? Is there an improvement over credit equilibria with borrowing constraints? Indeed, because borrowing constraints are not directly related to prices, it is not at all obvious that the welfare theorem will continue to apply.

The contract approach encompasses these questions: there is a planner in the role of principal in an agency contract, with individuals filling the role of agent. The agents' effort is presumed unobservable, but the principal takes the role of insurer against the agent's shocks. The optimal contract—optimal in the sense that the principal optimizes the contract subject to an incentive constraint—then mimics debt in its role of intertemporally substituting shocks, and there is an equilibrium interest rate that is determined by the contract.

Does the optimal contract mimic debt, and does it entail borrowing constraints? Yes: borrowing constraints are implied by an additional constraint, the requirement that the agent not defect from the contract. The desire to defect arises from the persistence of stochastic income or productivity shocks: a positive shock leads the agent to want to essentially renege on his obligation to fund unlucky individuals under the contract. This requirement is formally dual to borrowing constraints whose purpose is to prevent the unbounded borrowing that

would take place without borrowing constraints. Because the borrowing constraints are inherent in the contract, one can then assert that the borrowing constraints are optimal.

Because it is dual to a defection-free contract, one can view a credit equilibrium with an optimal borrowing constraint as *self-enforcing* and therefore *decentralizable*. [1]

Outline of the contract. The contract model has two basic features: there is an unobservable action by an agent simultaneously with the potential for the agent to defect from the contract. The agent seeks to participate in the contract to insure against stochastic fluctuations in productivity. The contract works so that the principal's continuation value behaves like an asset, and for this reason the contract can be construed as mimicking asset equilibria.

The model is an amalgamation of the continuous-time agency model of Sannikov [20] and the continuous-time model of defection constraints of Miao and Zhang [14]. These models cannot be easily or tractably combined however, so an additional dimension must be introduced, namely two-factor production, with labor and capital. This additional element then convexifies the agency aspect of the model.

More specifically, Miao and Zhang model consumption as a flow from the stock of a state variable Y_t; one could think of this flow as interest on a stock of capital that grows autonomously and stochastically, although they do not model it as such; their model does not encompass production, and they model the state as observable by both principal and agent. In contrast, Sannikov models the principal as receiving a flow from the agent's effort A_t, which generates disutility for the agent, but the current state X_t, which is controlled by the agent, does not yield utility directly, and consumption for the agent, which is a flow,

[1] It is necessary to bring up a nuance. In the dynamic contract that is immune to defection, defection would be punished by permanently excluding the defector from the contract. In the equivalent credit equilibrium, all participants in the equilibrium would have to agree to exclude any defectors by denying them the opportunity to re-enter the contract at a later date. In practice this would mean refusing to lend to a defector or to any coalition of defectors. In equilibrium, no-one defects, but the threat must nevertheless be real and committed to by all participants. At a deeper level the contract must be willing to punish any member who does not participate in punishing a defector. To capture this requirement in the agency contract we would allow the principal to defect as well as the agent; [14] explore this dimension of their contract with defection constraints. Going forward I ignore this additional layer of complexity.

is simply subtracted from his effort. Because of this, if the agent defects, he will not suffer any stochastic shocks, and there is therefore no natural way to model defection.

To combine agency with defection, I model consumption as the implicit flow from a production function entailing the state variable Y_t as in Miao and Zhang, but with that state interacting with labor; a geometric (Cobb-Douglas) utility function effects this.

Economic interpretation of the multipliers in the contract. Why bother combining agency and defection in a contract that is intended to mimic a macroeconomy? Given that monetary assets can be interpreted as the implementation of an agency contract with private information in the sense of Green [9], it is a short step to asking what price the contract is paying to induce revelation: inducing revelation is necessary so that budget balance can be maintained in the contract, that is, taxing the lucky to repay the unlucky in the contract. The implicit multiplier for the revelation constraint, which in the notation of this paper will be the marginal value of the principal's continuation value with respect to the continuation value of the agent J_W, is then the price of revelation.

Similarly, the defection constraint induces individuals to remain in the contract is expressed via the multiplier for the defection constraint, Z_t; one can view this multiplier as inducing commitment to the contract, and thus is also the marginal compensation for an agent to remain in the contract.

Continuing with this logic, the marginal values can be viewed as payments by the contract for information and for commitment. These payments can be expressed as reductions in the implicit interest rate, and those reductions in the interest rate can be compared to the underlying interest rate for the principal, who is a stand-in for the contract. I explore these issues in the numerical simulation of the model.

Outline of the paper. In the next section I set out the basic notation, preferences and technology surrounding the agent and principal. In the subsequent section I develop and solve the agent's autarky problem, post-defection, in order to characterize the defection value in the defection constraint. This section further fleshes out the technical elements of the model.

In Sections 4-7 I set out the details of the contract, starting from the agent's contract problem, then moving on to the principal's problem, with a separate section treating the details of the defection constraint, and ending with the solution of the principal's problem. In Sections 8-10 I describe the equilibrium, presenting two key aspects of the contract

that ensure that it does, in fact, mimic asset equilibria. Finally, in Sections 11-13 I calibrate and simulate the model and compare the simulated results to empirical data.

Following the conclusion in Section 14 there are several appendices. Appendices A and B provide detailed derivations of the autarky (post-defection) solution of the agent's problem for geometric utility and production functions. Appendix C presents a supplementary discussion of how the model might be re-framed in terms of a risk-neutral measure.

Appendices D-F are concerned with the numerical simulation of the model. Appendix D relates the smooth-pasting boundary condition to a Neumann boundary condition that applies in the numerical simulation of partial differential equations, and Appendix E relates the Neumann condition to the Dirichlet boundary condition. Finally, Appendix F details the numerical solution algorithm, especially focusing on the treatment of the boundary conditions. The boundary is a so-called free boundary, that is, the boundary itself is endogenous to the model and must therefore be found recursively.

2. The basic setup of the model

The agent's objective is to maximize conditional expected utility of consumption C_t, less the disutility of effort A_t, conditional on the information available to the agent:

$$E\left[\int_t^\infty e^{-\rho(s-t)}\left(u(C_s)-v(A_s)\right)ds\Big|\mathcal{X}_t\right] \tag{1}$$

with $\mathcal{X}_t$ adapted to B_t. The agent has access to a production technology that generates a flow of output using effort and capital Y_t as inputs:

$$\xi A_t^\alpha Y_t^{1-\alpha} \tag{2}$$

In the absence of intervention by the contract capital accumulates after subtracting consumption, subject to stochastic shocks:

$$dY_t = (\xi A_t^\alpha Y_t^{1-\alpha} - C_t)Y_t dt + \sigma Y_t dB_t \tag{3}$$

where B_t is a standard Brownian process.

Under the contract, the agent receives payments from the principal, who takes over and observes the output process, but not the agent's effort process. The principal's objective is therefore the discounted excess of output less the contractual consumption the agent receives under the contract:

$$E\left[\int_t^\infty e^{-r(s-t)}\left(\xi A_s^\alpha Y_s^{1-\alpha} - C_s\right)dt\Big|\mathcal{Y}_t\right] \tag{4}$$

with $\mathcal{Y}_t \subset \mathcal{X}_t$. It is an important detail that the principal's discount rate, r, differs from the agent's discount rate ρ.

3. The agent's autarky problem under defection

In order to analyze the contract it is first necessary to analyze the consequence of defection for the agent. If he defects, he transitions to autarky, investing in capital Y_t via a choice of effort A_t and consumption C_t. The solution of this autarky optimization problem leads to a continuation value process W_t, and it is this continuation value that is then an input into the principal's optimal contract problem, entering the defection constraint as the defection value function U^d.

I assume that the post-defection autarky output process has the very conventional structure

$$dY_t = (\xi A_t^{\alpha} Y_t^{1-\alpha} - C_t)Y_t dt + \sigma Y_t dB_t \tag{5}$$

If the agent defects, he produces output $\xi A_t^{\alpha} Y_t^{1-\alpha}$ and accumulates capital Y_t via the law of motion in (5), where ξ is productivity. His HJB equation is then

$$\rho W = \max_{A,C} \left\{ u(C_t) - v(A) + (A^{\alpha} Y^{1-\alpha} - C_t) Y W_Y + \frac{1}{2}\sigma^2 Y^2 W_{YY} \right\} \tag{6}$$

that is, direct optimization of the utility of consumption, with accumulation of capital offset by consumption, as in the classical growth model. (To save notation, the productivity term ξ will be dropped in the subsequent exposition, but reintroduced in the numerical simulations of the model.)

The production function encompasses the Sannikov and Miao-Zhang formulations as extremes for $\alpha = 1$ and $\alpha = 0$ respectively. Also, in Miao and Zhang's model the state process is geometric Brownian motion, which guarantees that Y_t is positive, which in turn allows the use of geometric or logarithmic utility; by contrast, Sannikov's model is in levels.

With a geometric production function the optimality conditions with respect to A and C are

$$-v'(A) + \alpha Y A^{\alpha-1} Y^{1-\alpha} W_Y = 0 \tag{7}$$

and

$$u'(C) - Y W_Y = 0 \tag{8}$$

For geometric disutility of labor, $v(A) = A^{\gamma}$, the closed form solution for labor A is

$$\gamma A^{\gamma-1} = \alpha A^{\alpha-1} Y^{2-\alpha} W_Y \quad \rightarrow \quad A = \left(\frac{\alpha}{\gamma} Y^{2-\alpha} W_Y \right)^{\frac{1}{\gamma-\alpha}} \tag{9}$$

and for geometric (CRRA) utility, $u(C) = \frac{C^{1-\lambda}}{1-\lambda}$,

$$C^{-\lambda} = YW_Y \rightarrow C = (YW_Y)^{-\frac{1}{\lambda}} \tag{10}$$

The agent's value process. The agent's value process can now be derived. Using the state process

$$dY_t = (A_t^{\alpha} Y_t^{1-\alpha} - C_t)dt + \sigma Y_t dB_t, \tag{11}$$

apply Ito's lemma to the continuation value process W_t:

$$dW_t = ((A_t^{\alpha} Y_t^{1-\alpha} - C_t)W_Y + \frac{1}{2}\sigma^2 Y_t^2 W_{YY})dt + \sigma Y_t W_Y dB_t. \tag{12}$$

Now substitute from the HJB equation (6) and optimality conditions:

$$dW_t = (\rho W - u(C) + v(A))dt + \sigma \left(v'(A)\alpha^{-1} A^{1-\alpha} Y^{\alpha-1}\right) dB_t \tag{13}$$

or

$$dW_t = (\rho W - u(C) + v(A))dt + \sigma \left(u'(C)\right) dB_t \tag{14}$$

and substituting from (8) yields

$$dW_t = (\rho W - u(C) + v(A))dt + \sigma \left(YW_Y\right) dB_t \tag{15}$$

This is then a guide for the structure of the W_t process under a contract, that is, if defection occurs, then at the moment of defection the continuation value W_t must be the same under the contract and under autarky, that is, it must obey (15).

With the solution of W in hand we can then state that $W(Y_t) = U_d(Y_t)$ the defection utility. From then we can connect this formulation to the boundary condition analysis. Specifically, the boundary conditions from the contract model can be reformulated for the autarky problem. At the defection boundary $(\phi(W_t), W_T)$, if defection takes place, then the starting value of Y_t is $H(W_t)$.

Additional comments about feasibility. From the Y_t process in (5), if consumption equals output then the Y_t process becomes stationary and the overall process will be stationary. However it does not do violence to the model to assume that this condition is not met; there would then be growth, viewing Y_t as capital. With that interpretation however it would then be natural to assume that the risky interest rate would be described by the marginal product of capital $A^{\alpha}Y^{-\alpha}$, but with an adjustment for risk.

The agenda would then be to adjust the contract model so that it mimicked this behavior, that is, a capital market equilibrium, and that would in turn entail finding a discount rate for the principal, r, such

that r generated an equivalent rate of return (or the equivalent of the riskless rate of return).

4. The agent's problem under the contract

In this section I analyze the agent's problem under the contract. The contract will take the form of functions dictating consumption and effort with respect to a state process that is ultimately driven by the capital accumulation, represented by Y_t, that was developed in the previous section. However the agent does not see or directly control this state under the contract: the principal owns all output. In its stead, the principal presents the agent with a state variable, X_t, as an artificial construct, determines consumption based on this state, and creates incentives for the agent to undertake an optimal amount of effort by connecting that effort to the evolution of the state. This is done by setting down functions $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ that are respectively the drift and volatility of the artificial state process X_t. The artificial state process is then

$$dX_t = f(\tilde{A}(X_t, Z_t), X_t, Z_t)dt + \sigma g(X_t, Z_t)dB_t \tag{16}$$

where $\tilde{A}(\cdot)$ is the action chosen by the agent as a function of the state. As in Sannikov's model, I assume that the volatility is a function of the state alone.

In addition to the state process X_t there is a deterministic background state process Z_t,

$$dZ_t = -(\rho - r)Z_t dt \tag{17}$$

The motivation for this process will be developed later.

The agent's consumption is determined by a function $\hat{C}(\cdot,\cdot)$ is a function of the state under the contract. Thus the contract is then expressed via the collection of functions $\{f(\cdot,\cdot,\cdot), g(\cdot,\cdot), \hat{C}(\cdot,\cdot)\}$.

The agent solves

$$\sup_{\tilde{A}(\cdot)} E\left[\int_t^\infty e^{-\rho(s-t)}\left(u(\hat{C}(X_s, Z_s)) - v(\tilde{A}(X_s, Z_s))\right) ds \middle| \mathcal{X}_t\right] \tag{18}$$

subject to (16), and where $\mathcal{X}_t$ is the information available to the agent. The resulting HJB equation is

$$\begin{aligned}\rho W(x,z) = \max_a \Big\{ & u(\hat{C}(x,z)) - v(a) + f(a,x,z)W_X(x,z) \\ & - (\rho - r)zW_Z + \frac{1}{2}\sigma^2 g(x,z)^2 W_{XX}(x,z)\Big\}\end{aligned} \tag{19}$$

Equation (19) expresses the incentive constraint for the principal: the principal does not directly control the agent's action a, but rather

constructs a contract such that the agent chooses the action a optimally against the contract. The principal optimizes the contract, expressed as the functions f and g, subject to this optimization by the agent. Thus, more formally, the incentive constraint is

(20)
$$\tilde{A} \in \arg\max_a \left\{ u(\hat{C}(x,z)) - v(a) + f(a,x,z)W_X + \frac{1}{2}\sigma^2 g(x,z)^2 W_{XX} \right\}$$

in the principal's own optimization problem, which is set out later.

The agent's optimality condition with respect to A is then

$$-v'(a) + f_A(a,x,z)W_X = 0 \tag{21}$$

which implicitly determines the function $\tilde{A}(\cdot)$. In order for the contract to be consistent, the solution for a in (21) must be identical with the principal's "recommended" action $\hat{A}$.

The agent's continuation value process. Expanding W using Ito's lemma and equation (16) yields the agent's continuation value process,

$$dW_t = \Big(f(\tilde{A}_t, X_t, Z_t)W_X - (\rho - r)zW_Z + \frac{1}{2}g(X_t,Z_t)^2 W_{XX} \Big) dt + \sigma g(X_t,Z_t)W_X dB_t \tag{22}$$

Substituting from the HJB equation (19) yields

$$dW_t = \left(\rho W(X_t,Z_t) - u(C) + v(A)\right) dt + \sigma g(X_t,Z_t)W_X dB_t \tag{23}$$

In the subsequent development this process will be treated as the principal's state process.

Because the state process Z_t *(i)* is deterministic and also *(ii)* is not driven by the agent's action, it therefore does not directly affect the continuation process in (23).

Comparison with Sannikov's dynamic agency model. The action side of the model is very similar to Sannikov's dynamic agency model [20], albeit with a more complicated production function. However the optimality condition (21) is essentially the same condition as Sannikov's optimal action in equation (4); this is then reflected in the volatility term in Sannikov's equation (5), similar to equation (23) here.

What is missing from Sannikov's model is the additional feedback built into the model because the principal's continuation value process ultimately becomes the agent's state process; this alters the principal's optimization process by leaving the agent's continuation value process in the form in (23), that is, without the indirect influence of the action

on the volatility term. If it were included, then the g function would become a function of the action A, and the agent would then have to react to it, leading to an infinite regress. Sannikov avoids this infinite regress by not making the agent's state process equivalent to the principal's value process; indeed the agent is treated as effectively myopic, in that the agent only solves a static optimization problem as reflected in equation (4), but not a dynamic control problem with a state as in the model of this paper.

In principle one could substitute the optimality condition in (21) for the W_X term in (23); in the sequel I demonstrate that due to feedback effects between the principal and agent, this substitution is vacuous: once the principal's value process is equated to the agent's state, the resulting feedback effect neutralizes the higher-order effects found by Sannikov.

5. The principal's problem

Formulating the principal's optimal contract problem. As I have outlined, the principal's optimal contract problem entails choosing the functions f and g; the principal's problem is a variational problem and as such is potentially very complicated. As will be demonstrated however, the principal's problem can be formulated as an optimal control problem, with the functions f and g falling out as natural expressions stemming from this control problem.

The principal receives output from a production function if the agent has not defected. To encompass the potential for defection I distinguish between the capital state (which I denote Y_t to be consistent with Miao and Zhang) and output, which is a function of capital and labor. If the production function is geometric, it takes the form

$$\xi A^{\alpha} Y^{1-\alpha} \tag{24}$$

where ξ is an arbitrary productivity parameter. The model also maps the agent's continuation value state into a value of capital,

$$Y_t = H(W_t) \tag{25}$$

Thus, when the agent defects, he receives the capital that the principal has created in the course of executing the contract. This is the capital equivalent to the starting point of the agent's autarky value process at the moment of defection. Thus, using Miao and Zhang's notation, the value of defection is

$$U^d(H(W_t)) \tag{26}$$

and before defection, the production function is

$$\xi A^{\alpha} H(W_t)^{1-\alpha} \tag{27}$$

Thus, before defection, the production function is fully consistent with the production function and initial value of capital that come into play at the moment of defection.

Structure of the H function. The U_d function is simply the value of defection given the state of Y_t. That value must be consistent with the agent's continuation value prior to defection, that is, there cannot be a jump in the value W_t at the moment of defection. For that reason we have the additional requirement that

$$W_t = U^d(H(W_t)) \tag{28}$$

We can therefore $H(\cdot)$ implicitly via the inverse of the U_d function:

$$H(W_t) \equiv U_d^{-1}(W_t) \tag{29}$$

The U_d function can be evaluated numerically by solving the agent's autarky optimal control problem, yielding the value function as a function of the capital state Y_t, and it is moderately easy to do this. The inverse can then be numerically calculated as well. Thus, it is straightforward to determine the H function both theoretically and numerically.

The principal's objective. The principal is assumed risk-neutral, with flow payoff equal to output less the consumption payment to the agent,

$$\xi A^{\alpha} H(W_t)^{1-\alpha} - C_t \tag{30}$$

and with state process W_t as expressed in equation (23); in addition the defection constraint (38) must be satisfied, so that the multiplier Z_t as determined in equation (37) becomes an additional state variable.

The state W_t is determined by the stochastic process dW_t as expressed in equation (23); this process can also be expressed in integrated form, similarly to the defection constraint. When this is done, an integrated multiplier, derived in a way similar to the derivation of the defection multiplier Z_t, can be derived. The result of this procedure, originally set out by Bismut, is the standard Hamilton-Jacobi-Bellman equation of stochastic control theory, but as a more explicit extension of classical deterministic control theory, with state and costate variables, and their relationship expressed via derivatives of a Hamiltonian function; with stochastic control Ito's lemma comes into play and also the quadratic covariation, which result in extra terms in the Hamiltonian formulation. The key insight is that the stochastic costate process is in fact equivalent to the first partial derivative of the value process of the control problem. The procedure for this argument is set out in Brock and Malliaris [4], p. 111 ff; also see Bismut [6]; see also Pham [18], Theorem 6.4.7, p. 151. The upshot of the argument is that the stochastic costate process is in fact equal to the first partial derivative of the value process with respect to the state. Thus, in the agency model here there are two costate variables, Z_t, and the costate associated with the agent's value process.

6. The defection constraint

The defector is assumed to take his stock of assets Y^i with him when he defects. The integral of the continuum of defection constraints term in the Lagrangian for the principal is then

(31)
$$E\left[\int_t^\infty e^{-r(s-t)}\lambda_s\left(\int_s^\infty e^{-\rho(\tau-s)}\left(u(C_\tau)-v(A_\tau)\right)d\tau - U^d(Y_s)\right)ds\Big|\mathcal{Y}_t\right]$$

where U^d is the defection value for the agent, and where $\mathcal{Y}_t$ represents the information available to the principal at time t. The agent's discounted continuation value under the contract, $e^{-\rho(s-t)}\left(u(C_s)-v^i(A_s)\right)ds$, must not be smaller than the defection value $U^d(Y_t)$ at the moment of defection; this must hold at all times t.

The defection constraint can be further refined. Because the agent's continuation value under the contract, W_t, is equivalent to the continuation value under autarky at the moment of defection, there is a function $H(\cdot)$ that maps the continuation value under the contract to the wealth Y_t under autarky, and therefore the constraint can be re-expressed as

(32)
$$E\left[\int_t^\infty e^{-r(s-t)}\lambda_s\left(\int_s^\infty e^{-\rho(\tau-s)}\left(u(C_\tau)-v(A_\tau)\right)d\tau - U^d(H(W_s))\right)ds\right]$$

Following the similar arguments in Miao and Zhang [14] and He and Pages [10], and stating the integral from time zero, an integration by parts derivation yields

$$\begin{aligned}(33)\quad & E\left[\int_0^\infty e^{-rt}\lambda_t\left(\int_t^\infty e^{-\rho(s-t)}\left(u(C_s)-v(A_s)\right)ds\right)dt\right] \\ & \qquad = E\left[\int_0^\infty\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)e^{-\rho t}\left(u(C_t)-v(A_s)\right)dt\right]\end{aligned}$$

Now define the costate process

$$(34)\qquad X_t \equiv \int_0^t e^{(\rho-r)s}\lambda_s ds + \phi \qquad dX_t \equiv e^{(\rho-r)t}\lambda_t dt \qquad X_0 = \phi$$

The constraint can then be reformulated as

(35)
$$-E\left[\int_0^\infty e^{-\rho t}U^d(H(W_t))dX_t\right]+E\left[\int_0^\infty e^{-rt}\left(e^{-(\rho-r)t}X_t\left(u(C_t)-v(A_s)\right)\right)dt\right]-X_0$$

Again as in Miao and Zhang and He and Pages, define the costate variable Z_t as

$$Z_t \equiv e^{-(\rho-r)t}X_t \qquad dZ_t = \frac{Z_t}{X_t}dX_t - (\rho - r)Z_t dt \qquad Z_0 = X_0 = z \tag{36}$$

A key point is that in the continuation region the defection constraint is not binding, and therefore the dX_t process has no stochastic element, that is, $dX_t = 0$; we can therefore drop the first term in the constraint, $-E\left[\int_0^\infty e^{-\rho t}U^d(H(W_t))dX_t\right]$. As a result, in the continuation region the Z_t process has the deterministic form,

$$dZ_t = -(\rho - r)Z_t dt \tag{37}$$

and the constraint can then be expressed in terms of Z_t:

$$E\left[\int_0^\infty e^{-rt}Z_t\left(u(C_t) - v(A_s)\right)dt\middle|\mathcal{Y}_t\right] - X_0 \tag{38}$$

in the continuation region, so that the discounting in the constraint now agrees with the principal's overall discount rate r. The constraint can therefore be brought in to the principal's objective, moment by moment, with multiplier Z_t, and in addition Z_t can now be treated as a state variable but with a deterministic law of motion, (37).

7. Solution of the principal's problem

As a result of this formulation we can state the principal's objective using the reformulated defection constraint from (38), as follows:

$$E\left[\int_0^\infty e^{-rt}\left(\xi A_t^\alpha H(W_t)^{1-\alpha} - C_t + Z_t\left(u(C_t) - v(A_t)\right)\right)dt \middle| \mathcal{Y}_0\right] \tag{39}$$

and subject to the state process in equation (71).

Because it is an agency model, the principal will determine both the desired actions of the agent, A_t, and the reward function for the agent, C_t.

The principal's HJB equation is therefore

$$\begin{aligned} rJ(w) = \max_{a,c}\Big\{ & \xi A^\alpha H(w)^{1-\alpha} - c \\ & + z\left(u(c) - v(a)\right) + (r-\rho) z J_Z \\ & + J_W\left(\rho w - u(c) + v(a)\right) + \frac{1}{2}\sigma^2\nu(w)^2 J_{WW}\Big\} \end{aligned} \tag{40}$$

with the partial derivative J_W having the interpretation as a costate process associated with the state process W_t. The HJB equation can be rearranged as

$$\begin{aligned} rJ = \max_{c,a}\Big\{ & \xi A^\alpha H(w)^{1-\alpha} - c \\ & + (z - J_W)\left(u(c) - v(a)\right) \\ & + \rho w\, J_W + \frac{1}{2}\sigma^2\nu(w)^2 J_{WW} \\ & + (r-\rho) z J_Z\Big\} \end{aligned} \tag{41}$$

It is evident from this formulation of the HJB equation that the terms involving the Lagrange multiplier for the defection constraint, Z_t, occur with the terms driven by incentive compatibility with the coefficient J_W, which thus has the same units as the multiplier Z_t, which converts units of utility to units of output. Thus, the effects of the two constraints are additive via the sum $Z_t - J_W$. The relative influence of the two constraints can be calculated; this will be done in the numerical analysis of the model.

Boundary conditions. Reasoning as in [14], the principal's boundary conditions arising from the defection constraint are value matching and smooth pasting, implicitly defining the boundary function ϕ:

$$J_z(w, \phi(w)) = U^d(H(\phi(w)), w) \qquad \text{and} \qquad J_{zz}(w, \phi(w)) = 0 \tag{42}$$

In practical terms numerically one would solve for an iteration of J using the previous iteration's value of ϕ, and then use that solution to find the next value of ϕ.

Optimal consumption and inverse Euler equation. The solution for the optimality condition for consumption implicitly defines the contract consumption function:

$$C = u'^{-1}\left(\frac{1}{Z_t - J_W}\right) \tag{43}$$

Equation (43), is an Euler condition. In a corresponding credit model in which the agent optimizes consumption in the face of a dynamic budget constraint intermediated by debt, we would have the optimality condition

$$u'(C) = \lambda \tag{44}$$

where λ is the multiplier associated with the budget constraint, rather than (43); in (43) the multipliers in the Euler condition by contrast are inverted. Thus, the contract optimal consumption entails an "inverse Euler equation" just as in standard dynamic agency models such as in Rogerson [19]; an additional argument will therefore be needed to justify the interpretation of the contract model as equivalent to an asset model.

Optimal effort. The optimality condition for the principal with respect to effort is

$$\xi\alpha A_t^{\alpha-1} H(W_t)^{1-\alpha} - (Z_t - J_W)\, v'(A_t) = 0 \tag{45}$$

with the solution implicitly defining the principal's recommended action $\hat{A}(\upsilon^{-1}(W_t))$. The next step is to reconcile the recommended and actual effort, and to characterise the contract functions f and g.

Solving the principal's continuation value process. The drift and volatility of the principal's continuation value process are the expression of the contract, as the agent treats the principal's continuation value process as its state process. To solve for the continuation value, I first solve the optimality conditions of the principal's problem, then combine these conditions with the principal's HJB equation and Ito's lemma. I will then reinterpret the principal's continuation value process J_t as equivalent to the abstract contract state process X_t in (16).

8. The equilibrium contract

The temptation is to designate the agent's optimality condition (21) as a first-order incentive constraint for the principal, with an associated multiplier. However this would entail deeming the agent's action A as subsequently under the control of the principal. Instead, it is proper to regard the agent's action A as not under the control of the principal; rather, the principal determines a recommended action via a separate optimality condition, and then we can check that the resulting recommended action is followed in equilibrium. This equality does in fact hold if state-variable duality is imposed, that is, if the agent treats the principal's continuation value as its state variable and, vice versa, that the principal treat the agent's continuation value W as its state variable. The following lemma expresses the key feature of this duality:

Lemma 1. *Let $X \equiv J$. Then*

$$W_X = W_J = \frac{1}{J_W} \tag{46}$$

Proof. If W and J are smooth functions then the result is immediate. □

This result makes it possible to re-express the agent's value process in terms of J itself, that is, we can exchange the volatility term W_X in (23) for $\frac{1}{J_W}$ using Lemma 1. This reduces the principal's problem to conventional form, and will allow us to establish the equality of the agent's action and the principal's recommended action.

Proposition 2. *The optimal contract functions are as follows:*

$$\begin{aligned} g(x,z) &= 1 \\ f(a,w,z) &\equiv \left(rJ - \xi a^{\alpha} H(w)^{1-\alpha} + c + z(v(a) - u(c))\right) \end{aligned} \tag{47}$$

and therefore the principal's continuation value process is

$$dJ = \left(rJ - \xi A^{\alpha} H(W)^{1-\alpha} - Z_t(u(C_t) - v(A_t)) + C_t\right) dt + \sigma dB_t \tag{48}$$

Proof. Posit that $g(x) = 1$. Using the agent's state process from (23) and Ito's lemma, the principal's value process is then

(49)

$$dJ = \left((\rho W - u(C_t) + v(A_t))J_W + \frac{1}{2}\sigma^2 \left(W_X(\psi(W_t), Z_t)\right)^2 J_{WW} + (r-\rho) Z_t J_Z \right) dt + \sigma W_X(\psi(W_t), Z_t) J_W dB_t$$

Substituting from the principal's HJB equation (41) yields

(50)
$$dJ_t = \left(rJ_t - \xi\hat{A}(W_t)^{\alpha}H(Z,W)^{1-\alpha} - Z_t(u(\hat{C}(W_t)) - v(\hat{A}(W_t)) + \hat{C}(W_t)\right)dt + \sigma W_X(\psi(W_t), Z_t)J_W dB_t$$

using the principal's optimality conditions (43) and (45). Because the drift of the J process in (48) is the contract function f, we have

$$f(a,w,z) \equiv \left(rJ - \xi a^{\alpha}H(w)^{1-\alpha} + c + z(v(at) - u(c))\right) \tag{51}$$

and

$$f_A = -\xi\alpha a^{\alpha-1}H(w)^{1-\alpha} + zv'(a) \tag{52}$$

Now posit that the artificial state, X_t, that the principal presents to the agent as his state under the contract, is equivalent to the principal's continuation value, J_t, so that $X_t \equiv J_t(W_t) = \psi(W_t)$. The volatility coefficent in (50) is then

$$g(w,z) \equiv W_X(\psi(W_t), Z_t)J_W = \frac{1}{J_W}J_W = 1 \tag{53}$$

using the duality relation $J_W = \frac{1}{W_J}$ from Lemma 1.

□

A consistency result. The further consequence of the duality between the principal's state and the agent's state is that the principal's optimisation over the action A, which implicitly defines the structure of the contract by defining the drift of the J_t process $f(A,W,Z)$, is consistent with the agent's optimisation over A.

Proposition 3. *(Incentive compatibility) Let the principal's continuation value process J_t be the agent's state process, and let the agent's continuation value process W_t be the principal's state process. Then the principal's and the agent's optimality conditions for A are equivalent, that is,*

$$\hat{A}(w) = \tilde{A}(v^{-1}(w)) \tag{54}$$

Proof. To see this, first recall from the proof of Proposition 2 that $J_W = \frac{1}{W_J}$. The principal's sub-objective with respect to A is

$$\xi A_t^{\alpha}H(W_t)^{1-\alpha} - (Z_t - J_W)\,v(A_t) + \frac{1}{2}\sigma^2\left(\frac{v'(A_t)}{f_A}\right)^2 J_{WW} \tag{55}$$

The third term,

$$\frac{v'(A_t)}{f_A} = W_J = \frac{1}{J_W}$$

where the first equality follows from the *agent's* optimality condition, and the second from the mutual state process construction; we can then note that $\frac{1}{J_W}$ is not a function of A. The objective then becomes

$$\xi A_t^\alpha H(W_t)^{1-\alpha} - (Z_t - J_W)\, v(A_t) + \frac{1}{2}\sigma^2 \left(\frac{1}{J_W}\right)^2 J_{WW} \tag{56}$$

or simply

$$\xi A_t^\alpha H(W_t)^{1-\alpha} - (Z_t - J_W)\, v(A_t) \tag{57}$$

as the third term is not a function of A. The optimality condition for the principal is then

$$\xi\alpha A_t^{\alpha-1} H(W_t)^{1-\alpha} - (Z_t - J_W)\, v'(A_t) = 0 \tag{58}$$

which can be written as

$$-\xi\alpha A_t^{\alpha-1} H(W_t)^{1-\alpha} + Z_t v'(A_t) = J_W v'(A_t) = f_A \tag{59}$$

which is again the optimality condition of the agent in equation (21). Thus, the two optimisation problems are equivalent. □

Substituting the the consistency result in Proposition 3 into the HJB equation (41), the principal's HJB equation simplifies to

$$\begin{aligned} rJ = \max_{C,A} \Big\{ & \xi A^\alpha H(W)^{1-\alpha} + \rho W\, J_W \\ & + (Z_t - J_W)\, u(C_t) - C_t \\ & - (Z_t - J_W)\, v(A_t) + \frac{1}{2}\sigma^2 \left(\frac{1}{J_W}\right)^2 J_{WW} \\ & + (r-\rho) Z_t J_Z \Big\} \end{aligned} \tag{60}$$

This is the equation that will be used in numerical simulations.

Equilibrium contract. Although the principal has been presumed unable to observe the agent's true underlying state Y_t and the incremental shock process B_t, the principal can observe the output process resulting from the observation of the agent's output and determine a desired degree of action for the agent, A_t, and reward the agent for following that recommendation via the consumption reward C_t, which the agent does not control and takes as given via the contract process. The principal therefore faces an optimisation problem with two control variables: consumption C_t and also the recommended action A_t. This leads to the following definition.

Definition 4. *An agency equilibrium consists of functions* $f(\cdot,\cdot,\cdot)$, $g(\cdot,\cdot)$, $A(\cdot,\cdot)$, $C(\cdot,\cdot)$, $\phi(\cdot)$ *and processes* J_t *and* W_t *such that*

(i) $\tilde{A}(\cdot,\cdot)$ *solves the agent's problem* (20)
(ii) $\hat{A}(\cdot,\cdot)$ *and* $\hat{C}(\cdot,\cdot)$ *solve the principal's problem* (60)
(iii) *The defection constraint* (38) *holds*
(iv) $\hat{A}(\cdot,\cdot) = \tilde{A}(\cdot,\cdot)$, *that is, the contract is incentive-compatible*

The key aspect of this definition is item (iv), the requirement that the effort function $A(\cdot)$ must satisfy the optimization problem of *both* the agent and the principal, that is, that the agent's choice of optimal effort is identical to the effort recommended by the principal.

The agreement between the principal and the agent is achieved by equating the artificial state process that the principal uses as the expression of the contract, X_t, with his own value process, J_t, with the consequence that the contract functions $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ do not need to be found directly via the solution of a variational problem, but rather arise naturally as the drift and volatility of the J_t process itself. Using Lemma 1, Proposition 2 and Proposition 3, the equilibrium is achieved.

9. Asset interpretation of the J function

The principal's continuation value process is taken as the state process by the agent inside the contract. The drift of the J process then behaves in such a way that J can be interpreted as an asset process. The asset J has a negative sign and so is interpreted as debt; J is a function of the underlying state W_t, and because by the reasoning so far W_t is in turn a function of the state J_t, then J_t is then recursive. There is however a proviso: J is also a function of the additional state Z_t. Any discussion of the economic intuition underlying the behaviour of the asset process must therefore take account of this additional state. More fundamentally, empirical test of asset models in this context must identify the Z_t process, not just the J_t process.

Examining (48), the J_t process behaves as a standard asset in that it incurs interest rate charges at rate r; any income from labor and capital, $A_t^{\alpha} H(W_t)^{1-\alpha}$, is added to the asset, reducing the debt. Consumption C_t increases the debt. This leaves the terms $Z_t(v(A_t) - u(C_t))$ to account for. It is clear that the defection multiplier Z_t converts utility units to consumption units. Any increase in consumption increases utility and this is then converted into an increment to the debt. However, because the principal wants to induce the agent to refrain from defecting, he *reduces* the debt from consumption via the term $Z_t u(C_t)$. Thus, effort and consumption still have the expected direct effects on debt, but there is a mitigation of the effect via the multiplier Z_t.[2]

Because Z_t increases as the J state moves *away* from the defection boundary, the mitigation effect is reduced close to the defection boundary where Z is small; far from the boundary there is mitigation, which rewards the agent for being far away from the boundary.

Interest rate implicit in the drift of J. Having interpreted the J_t process as identical to the state process for the agent X_t, expressing the J process in (48) in rate of return form yields

$$\frac{dJ}{J} = \left(r - \frac{A^{\alpha} H(W)^{1-\alpha} + Z_t(v(A_t) - u(C_t)) - C_t}{J} \right) dt + \frac{\sigma}{J} dB_t \tag{61}$$

Interpreting $A^{\alpha} H(W)^{1-\alpha}$ as income and also subtracting consumption, we are left with the implicit interest rate process as

$$r - \frac{Z_t(u(C_t) - v(A_t))}{J_t} \tag{62}$$

where C_t and A_t are evaluated at their optimized values. Thus, the interest rate is affected by the defection constraint; specifically, it is

[2]This effect is also present in Kehoe and Perri [16].

effectively reduced by the $Z_t(u(C_t) - v(A_t)$ and this reduction effect can be calculated in the numerical simulations.

Borrowing constraints. It is now straightforward to interpret the contract as imposing borrowing constraints. Because the state vector (z, w) is constrained to lie in the continuation region, $J(z, w)$ is constrained by that region. Because $J_Z(z, w) > J_Z(\phi(w), w)$ for (z, w) in the continuation region, it is immediate that the inequality $J(z, w) > J(\phi(w), w)$ holds for (z, w) in the continuation region. Interpreting J as the asset state for the agent (which can conceptually be negative as well as positive), this is identical to a borrowing constraint. However, unlike an arbitrary and *ad hoc* fixed borrowing constraint, the borrowing constraint depends on the W_t state in a highly nonlinear way; one must know the ϕ function as well as the W_t state in order to identify the borrowing constraint. Estimating the $\phi(\cdot)$ function in turn requires identifying the principal's value function J.

10. The marginal value of revelation

As discussed in the introduction, the reduction of the interest rate by the defection constraint multiplier Z_t can be viewed as a kind of payment to the agent in return for his commitment to the permanence of the contract, and this payment effect shows up as an adjustment of the implicit rate of return in the drift of the dJ_t process. In contrast however, the multiplier for the revelation constraint is only implicit—it is the J_W function, and this does not appear in the continuation process dJ_t that I am proposing as the equivalent of an asset process for the agent.

It is of interest to articulate the effect of the incentive (revelation) constraint because it expresses the need for each participant to signal his state so that insurance payments can be effected via the contract. Each individual then contributes to the pooling that diversifies the risk faced by all agents.

To get at the the effect of the revelation-constraint multiplier, one can approach it a different way: in order to be a stand-in for the aggregate equilibrium, the principal must not supply or subtract resources from the agent in expectation.[3] In earlier pure exchange models—see [26] and [25]—this was modelled by presuming that the principal's drift is zero, thus yielding aggregate feasibility for the contract. This zero-drift condition was satisfied by the choice of the principal's discount rate (r in this setting). Thus, we can also calculate the impact on the implicit return, requiring that the agent consume all output.

This reasoning can be generalized. In autarky, an individual might invest in the capital state Y_t, yielding growth. Along with this growth there will be an implicit risky rate of return. The contract model can mimic this in principle but must still not add or subtract resources to the economy, thus the implicit rate of return from the contract must be equivalent to the growth equilibrium interest rate.

Further remarks on the marginal value of revelation. We can undertake the following thought experiment: find the value of the principal's discount rate r such that the principal's expected profit is zero. In a model with labor only this would translate to zero drift for the agent's continuation value (and also the principal's), but in a model with capital growth such as this one zero drift is not key. Rather, the zero profit condition for the principal ensures that we can map the

[3]It is sufficient to require that require resources must balance in expectation, because in the analogous planner model one aggregates over the continuum of agents in the economy using the probability measure of their equilibrium state, and this integral will be a constant under the resource balance constraint.

agency model to an equilibrium model in which no resources enter or leave the economy.

Because the model is so nonlinear it is hopeless to characterize the zero expected profit condition for the principal analytically. Rather it must be found numerically by looking for ergodic realizations of the state processes (W_t and Z_t). Once this is found the *average* impact of the defection constraint on the interest can be numerically calculated from the ergodic behavior of the sample paths of W_t and Z_t. Once this has been calculated and subtracted from the difference $\rho - r$, the residual $\rho - r - Z(u - v)$ is the marginal effect of the revelation constraint.

The challenge is then to calculate the critical value of r, which must be done by constructing the ergodic sample paths of W_t and Z_t and searching for the value of r that yields zero expected profit for the principal.

Risk and incentive effects of the marginal value of revelation. The diffusion term in the principal's HJB equation (60) is the second-order term $\frac{\sigma^2}{2} J_{WW}$; if the PDE had only this term and the J_Z term (along with the complicated "source" terms like ρJ) then it would be a diffusion equation. The solution would be a probability density. The incentive effect of the contract is expressed via this term; the $(1/J_W^2)$ thus *suppresses* the risk term in the PDE.

Keeping in mind that the J_W term is the shadow multiplier for the incentive constraint, one can interpret J_W as the marginal value of revelation; as this marginal value increases, the impact of the risk term $\frac{\sigma^2}{2} J_{WW}$ shrinks; thus the term expresses the insurance effect of the contract. Because the coefficient $(1/J_W^2)$ stems from the feedback effect of the contract—see Proposition 1—we see the diversification of risk via asset equilibria expressed here.

11. Numerical simulation: Preliminary discussion

A numerical challenge is to find a way to make the boundary function $\phi(\cdot)$ and J jointly recursive. The fundamental boundary condition $J_z(\phi(w), w) = U^d(H(w)) = w$ unfortunately buries ϕ inside J, so the boundary is only implicitly determined.

There is an indirect approach to the joint recursion. The slope of the boundary function, at the boundary, can be expressed as

$$\phi'(w) = \frac{J_w(\phi(w), w)}{J_z(\phi(w), w)} \tag{63}$$

Because of the fundamental boundary condition $J_z = U^d(H(w)) = w$, this becomes (assuming $U^d(H(w)) = w$)

$$\phi'(w) = \frac{J_w(\phi(w), w)}{w} \tag{64}$$

Integrating,

$$\phi(w) = \int_{\underline{w}}^{w} \frac{J_w(\phi(w), w)}{w} dw + \phi(\underline{w}) \tag{65}$$

which is a recursion in ϕ, and which depends on J. [4]

Formulation as a differential equation. The solution in (65) presumes that an initial condition is known, and builds on the previous iteration of J and ϕ. Another way to approach the problem is to view (64) as a (nonlinear) differential equation in ϕ, with J known (from the previous iteration). One could then solve numerically, imposing a boundary condition just as $J_w(\phi(\underline{w}), \underline{w}) = 0$.[5] As a practical approach to determining the initial value of ϕ in (65) I manually adjust the initial value so that the formal boundary conditions (42) are satisfied by the solution of the PDE for J. See Appendix F for further details of the algorithm.

[4] Moreover, there is some potential to express the derivative J_w via a Green's function:

$$J_w = \int \partial_w G(z, z', w, w') f(z, z', w, w') dw' dz' \tag{66}$$

writing abstractly. Thus, we have a recursion set up with Green functions involving J and also ϕ, with the right hand side involving integration so that the recursion is a functional equation. The potential therefore exists to demonstrate that the recursion is a contraction; see for example Stuart [23].

[5] This approach is similar to the approach in Peskir and Shiryaev [17], p. 205, equation (13.2.22)

12. Numerical simulation results

Because the model is to a large extent a combination of the Miao-Zhang and Sannikov models, the numerical examples for those models provide benchmarks for the numerical simulations in the combined model. The key feature of the Miao-Zhang model is the ϕ function. That function is an upward sloping function of the state variable y, with the active ("no-jump") region above the boundary. In the model here ϕ is a function of w, but because w is a positive function of the output process y via $H(w)$, it must inherit these same properties.

The Sannikov numerical example has three basic properties: first, the action function A is a decreasing function of the agent's continuation value state w, however there is a local "dip" in the function for low values of w due to the proximity of the boundary. Second, the consumption function $C(w)$ is flat for low values of w, and is increasing for high values of w. Third, the drift of the J process (F in Sannikov's paper) is positive for low values of the state and negative for high values of the state, and close to a linear function of w for intermediate values of the state w. The interior region is thus stable, and so the stable region is near linear. This last property means that J can appropriately be interpreted as a quasi-asset from the perspective of the agent, in that resource balance can be achieved. Both Miao and Zhang and Sannikov assume that the discount rates of the principal and agent are equal, however by allowing the principal's discount rate r to differ from the agent's discount rate ρ, this resource balance can be achieved.

There are some other properties that need to hold for the solution to make sense. The partial derivative J_Z needs to be negative or small in relation to the potential values of Z, but $J_Z - w$ needs to be increasing in Z so as to properly define the boundary and active region. Second, the function J_{ZZ} must be zero or close to zero, as this is implied by smooth pasting. Third, the partial derivative J_W needs to be negative, as $Z - J_W$ appears in a number of places in the model.

Example 1. In the first example I choose parameters as stated in Table 1.

To determine $H(w)$ I numerically solve the agent's autarky problem to calculate the U^d function, then invert this to obtain H. For this example,

$$U^d(y) = -1.823y^{-1.12} \qquad H(w) = 1.72(-w)^{-.9} \tag{67}$$

The plot of H is presented in Figure 1.

I first present the solution for J of the PDE after convergence (six iterations) in Figure 2. The function J has a distinct saddle shape, as

TABLE 1. Example 1 Parametrization

Principal discount rate r	Agent discount rate ρ	Volatility σ	Agent CRRA coefficient λ	Effort disutility coefficient γ	Labor product-ivity α	Product-ivity ξ
.05	.05	3.0	3.0	3.0	0.3	5.0

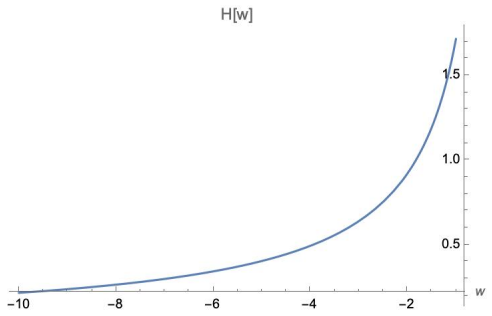


FIGURE 1. H function (ansatz)

is to be expected because its arguments are the direct state variable W_t and the Lagrange multiplier Z_t; it is apparent that J is concave in w and convex in z. It is clear that the function is decreasing in z, as

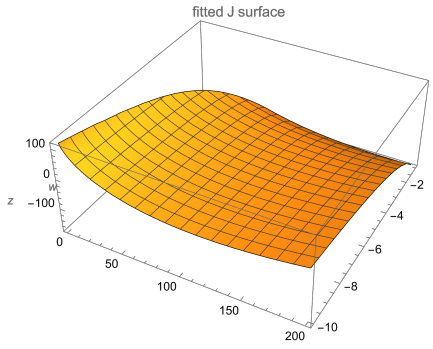


FIGURE 2. Converged J function

in Miao and Zhang Figure 2.

The function $J_z - U_d(H(w))$ is displayed in Figure 3 The figure clearly shows the intersection defining the function $\phi(w)$, with the inequality $J_z > U_d$ satisfied for z larger than $\phi(w)$, that is, the "no-jump" or active region. Figure 4 displays the resulting ϕ function.

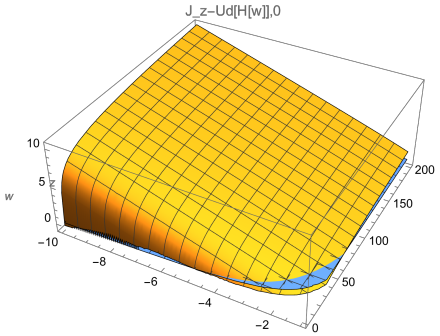


FIGURE 3. The $J_z - U_d$ manifold illustrating the implied $\phi(W)$ defection boundary

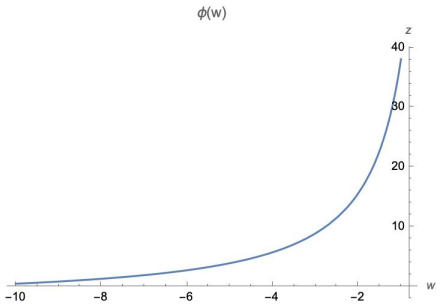


FIGURE 4. $\phi(w)$ function

Figure 5 displays the converged policy function for A. As with Sannikov's model, the effort function is mainly decreasing in w. It is also decreasing in z, that is, labor effort *increases* as the state approaches the defection boundary from above.

Figure 6 displays the converged policy function for consumption C. As with Sannikov's model, the consumption function is initially flat but is is increasing in w for larger values of w. It is also increasing in z, that is, consumption *decreases* as the state approaches the defection boundary from above. This jibes with the intuition that approaching defection is intuitively similar to approaching a borrowing constraint, the intuitive response to which would be to increase effort and thus output to stave off hitting the constraint, and also to reduce consumption, again reducing the probability of hitting the constraint.

Marginal value of revelation effects. Figure 7 shows that the absolute value of the marginal value of revelation multiplier J_W increases in absolute magnitude as the ϕ boundary is approached; this has the

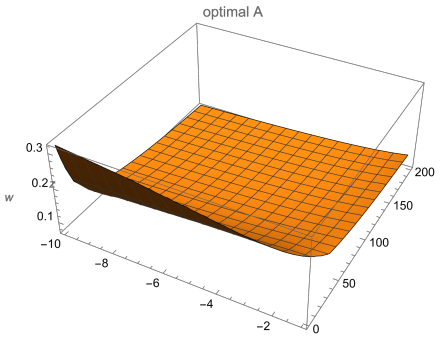


FIGURE 5. Optimal A function

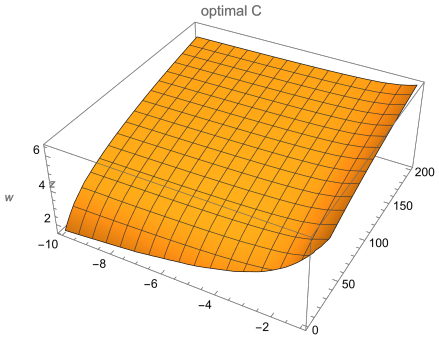


FIGURE 6. Optimal C function

effect of suppressing the volatility term in the principal's HJB equation (60). Thus, close to the boundary the principal is in effect offsetting more risk for the agent.

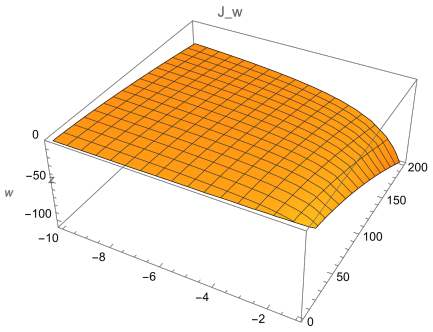


FIGURE 7. J_W

Relative influence of defection versus incentive multipliers. Finally, I examine the relative influence of the defection constraint versus the incentive constraint by comparing the ratio of the defection constraint multiplier Z_t to the implicit incentive constraint multiplier J_W by examining the ratio

$$\frac{Z}{Z - J_W} \tag{68}$$

which is displayed in Figure 8. It is evident that the relative importance of the defection multiplier decreases as the ϕ function is approached, as shown in the same context in Figure 3. Conversely, away from the defection boundary the defection multiplier Z dominates the incentive

constraint. As argued in the discussion of the asset nature of the J function, the Z multiplier mitigates the impact of the defection constraint away from the constraint, but the mitigation is removed as the constraint is approached; this in turn magnifies the impact of consumption on the J_t process, providing the incentive to avoid the defection constraint.

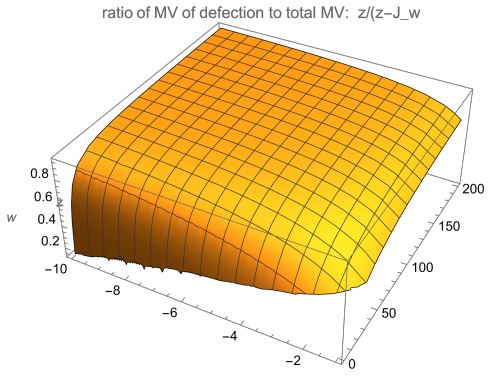


FIGURE 8. Defection fraction of total multiplier intensity $\frac{Z}{Z-J_W}$

Sample paths. To further characterize the behavior of the model it is helpful to generate and characterize sample paths. To do this to find the appropriate running maximum process. Recall that the integrated multiplier X_t grows only when the boundary $\phi(W_t)$ is hit.

From earlier work, first of all we have the definition of Z_t from X_t, where X_t is the cumulated Lagrange multiplier for the defection constraint:

$$Z_t \equiv e^{-(\rho-r)t}X_t \quad \rightarrow \quad dZ_t = \frac{Z_t}{X_t}dX_t - (\rho - r)Z_t dt \tag{69}$$

(See (36) or Miao and Zhang p. 14 equation (24).) Because X_t only increases, and because it is constant inside the continuation region, we have the running maximum relation

$$X_t = \max_{s\in[0,t]} e^{(\rho-r)s}\phi(W_s) \tag{70}$$

ϕ is the boundary function for Z_t, that is, $Z_t = \phi(W_t)$ at the boundary. (See Miao and Zhang p. 16.)

To implement sample paths we just need to generate sample paths by integrating the dW_t paths using (23).

$$dW_t = (\rho W(X,Z) - u(C) + v(A))\,dt + \sigma\frac{1}{J_W}dB_t \tag{71}$$

with X defined as J_t from the equivalence of the principal's state with the asset, so the sample paths of W_t can be generated numerically from

(72)
$$W_s = \int_0^s \left(\rho W_\tau - u(C(W_\tau, e^{-(\rho-r)\tau}X_\tau) + v(A(W_\tau, e^{-(\rho-r)\tau}X_\tau), e^{-(\rho-r)\tau}X_\tau)\right) d\tau + \sigma \int_0^t \frac{1}{J_W(W_\tau, e^{-(\rho-r)\tau}X_\tau)} dB_\tau$$

where the Z_t have been replaced with the definition in (69).

Once the sample paths of W_t have been generated, the corresponding paths of Z_t can be generated. These paths can then be put together to generate paths for the incremental payoff process for the J process, and an artificial sample of these paths can be generated. From these paths the drift process for the dJ_t process can be determined and the average drift can be generated. The average contribution of the defection constraint term $\frac{Z_t(u-v)}{J}$ to the reduction of the interest rate r can then be determined. Due to the necessarily finite domain of the numerical simulations, the sample paths eventually run off the left or upper edges of the (z, w) domain, so these measurements are estimates.

12.1. **Characterizing the incentive effect on the interest rate via sample path averages.** Following up on the analysis above, we can characterize the impact of the incentive effect relative to the defection effect on the effective or implied interest rate. Using the parameterization from Table 1, one can generate sample paths as described above.

We can examine the W_t and Z_t paths separately in Figures 9 and 10. It is apparent that Z_t has an upward trend, and the continuation value process W_t has a downward trend in the later part of the sample path; this is the anticipated result, as noted by Green [9] and Thomas and Worrall [27]. Examining the path of consumption in Figures 11 and 12, we see that consumption and output are roughly stable.

Sample path averages. Table 2 displays average consumption and average output. It is evident that average consumption far exceeds average output, suggesting that the interest rate structure cannot sustain a feasible contract. The table also displays the average of the $\frac{Z(u(C)-v(A))}{J}$ term, which is subtracted from the principal's discount rate r to determine the effective interest rate; we see that in this case the effective interest rate is slightly raised relative to r.

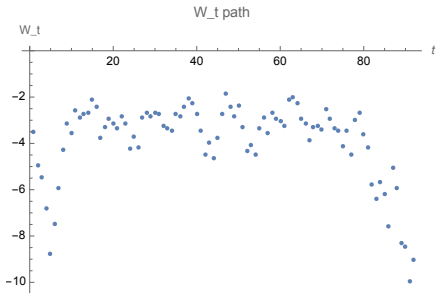


FIGURE 9. W path

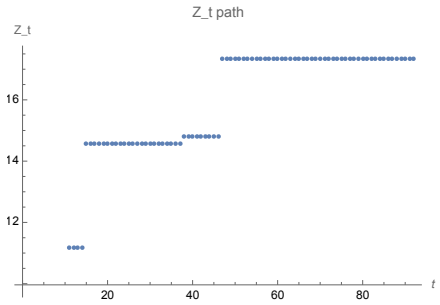


FIGURE 10. Z path

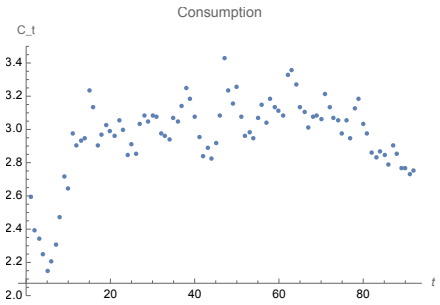


FIGURE 11. Consumption path

TABLE 2. Sample Path Averages for $\rho = r$

Mean Consumption	Mean output	Mean interest adjustment
2.96	2.08	-.046

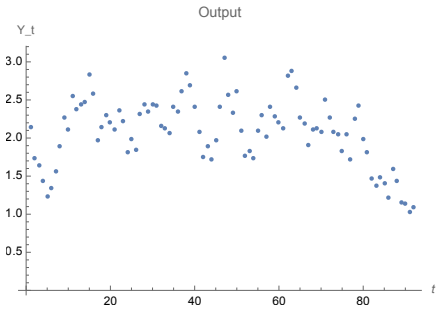


FIGURE 12. Output path

The distribution of wealth. One can interpret wealth as the inverse of the agent's utility function of consumption, applied to the continuation value process W_t; Figure 13. Viewing the simulated path of wealth as

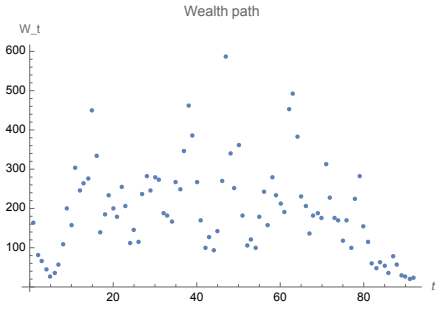


FIGURE 13. Wealth path

reflecting the ergodic distribution of wealth in the economy, one can generate the histogram of the distribution of wealth and its corresponding skewness, presented in Figure 14: The skewness is positive and relatively close the the empirical skewness for the US distribution of wealth; the histogram for the US distribution of wealth is presented in Figure 15; for comparison, the skewness for the US wealth histogram is 1.06. The skewness is roughly similar to the simulation value of .82.

Reduced principal's discount rate r. Reducing the principal's discount rate r from .05 to .01, keeping all other parameters fixed as in Table 3 that is, much less than the agent's discount rate ρ of .05 now yields a slightly upward-trending continuation value process W_t in Figure 16. The Z_t process in Figure 17 shrinks between jumps, as the dynamics are driven by the negative term $-(\rho - r)$ (see equation (37)).

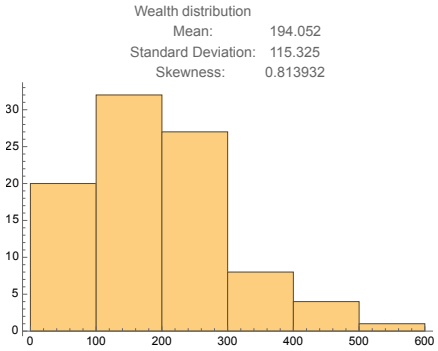


FIGURE 14. Wealth histogram

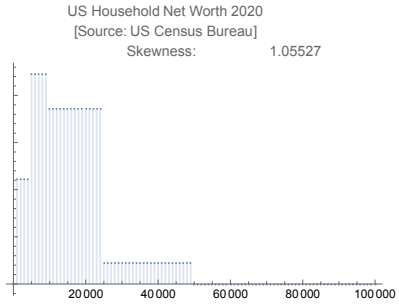


FIGURE 15. Wealth histogram

TABLE 3. Example 2 Parametrization

Principal discount rate r	Agent discount rate ρ	Volatility σ	Agent CRRA coefficient λ	Effort disutility coefficient γ	Labor productivity α	Productivity ξ
.01	.05	3.0	3.0	3.0	0.3	5.0

Mapping the Z_t and W_t processes together yields Figure 18, illustrating how the (W_t, Z_t) state drifts toward the boundary $\phi(W_t)$ between jumps.

Consumption and wealth also show a slight upward trend as in Figures 19 and 20. Thus, we can conclude that the revelation constraint is associated with the reduction of the principal's discount factor from $\rho = .05$ to $.01$, moving consumption and output closer together, thus mimicking a budget-balanced equilibrium with interest rate r, but with further adjustment downward due to the defection constraint.

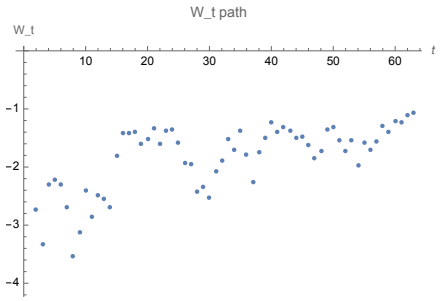


FIGURE 16. W path

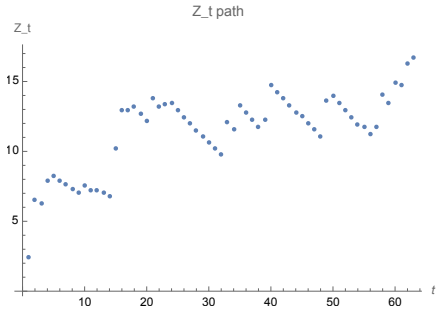


FIGURE 17. Z path

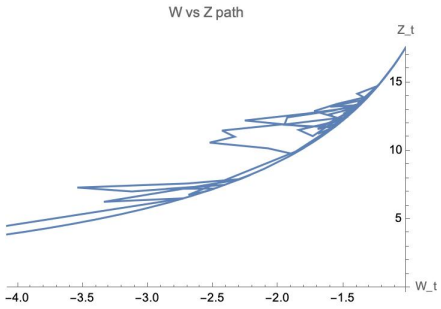


FIGURE 18. W vs Z path

Viewing the simulated path of wealth as reflecting the ergodic distribution of wealth in the economy, we can generate the histogram of the distribution of wealth and its corresponding skewness, presented in Figure 21: The skewness is again positive; again see Figure 15.

By averaging over the sample path we can calculate average consumption and output; Table 4 shows that average output now exceeds average consumption. Additionally, one can calculate the sample path

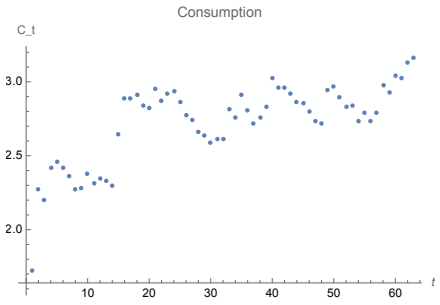


FIGURE 19. Consumption path

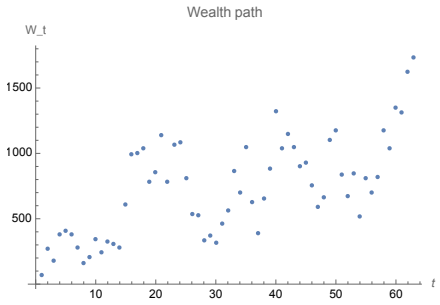


FIGURE 20. Wealth path

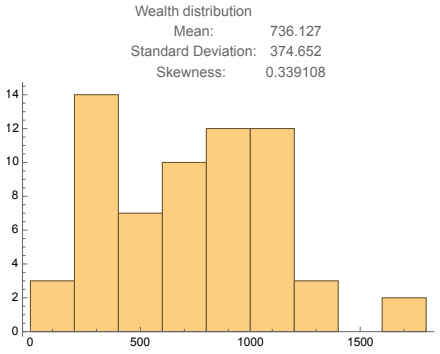


FIGURE 21. Wealth histogram

average of the interest rate mitigation term $\frac{Z_t(u(C_t)-v(A_t))}{J_t}$ (see equation (62)); this shows that the mitigation factor reduces the effective interest rate below its nominal value of .01 to .007. Thus, there are two effects: the reduction of the principal's discount rate from the agent's discount rate ρ to $r < \rho$ associated with the incentive compatibility

TABLE 4. Sample Path Averages for $\rho = .05$, $r = .01$

Mean Consumption	Mean output	Mean interest adjustment
2.72	3.47	.003

requirement, and the additional effect of the defection constraint via the mitigation factor.

Thus we can conclude that by raising the discount rates ρ and r further, and also increasing the difference $\rho - r$, we can achieve budget balance for the principal and then effectively have a credit equilibrium.

One can view the interest rate r as the risk-free rate in a standard growth model; under autarky such a model would hold. The contract model illustrates how this rate is reduced relative to the subjective rate of time discount via incentive effects arising from the constraints.

Comments concerning calibration. The model parameters have been chosen to enable tractable, illustrative and intuitively reasonable simulations. However it is possible to go a bit further in justifying the parameters. For the US, average real income per capita, 1960-2021 is \$39,790, and annual volatility is \$12,822, for a ratio of volatility to income of about .32. In the computed example with $\rho = .05$, $r = .01$, the output process resulting from the main example has an approximate average output on the order of 3.47; the volatility of output is .345, so the ratio of volatility to output is approximately .10; see Table 5.

The volatility of the log of GDP per capita for the US data is .322, whilst for the simulation with $\rho = .05$, $r = .01$, the volatility of the log of output is .18, which is roughly the same order of magnitude. One can also examine the ratio of the volatility of the logarithm of consumption to the volatility of the logarithm of output; in the data this is about 1.1, whilst in the simulation it is .47; again, a similar order of magnitude.

Finally, one can measure the volatility of the increments to the log of income and consumption; in a pure exchange model this would correspond to the volatility viewing the income process as a geometric brownian motion process. In the data the volatility for the log of the increments of GDP per capita is .023, and the consumption figure is .019; in the simulation with $\rho = .05$, $r = .01$ the figures are .112 and .051 respectively. Thus, the numerical model has much more volatility than the corresponding aggregate US data, however it is important to keep in mind that the model is intended as a model of individual income and consumption in the face of the implicit planner contract; thus we would expect that the variance of idiosyncratic fluctuations in income and consumption would significantly dominate the aggregate volatilities, as the idiosyncratic shocks would be averaged out in the aggregate. There are many nuances determining income volatility at the individual level such as age, sex and education. For example, using CPS data Ziliak, Hardy and Bollinger (2010) [29] find the volatilities for the percentage increments to individual income (which include the systematic component of the volatility) on the order of .5-1.0, much higher than the model's estimate.

TABLE 5. Calibrations

Quantity	Data	Simulation
Average real income/Income volatility	.32	.1
ln(GDP per capita) volatility	.322	.18
ln(consumption) volatility/ln (GDP) volatility	1.1	.47
Δ ln (GDP per capita) volatility	.023	.112
Δ ln(consumption per capita) volatility	.019	.051

13. Empirical implications

The theoretical model and simulations have demonstrated that the multiplier processes Z_t and J_W move around the (w, z) plane expressed in the simulations. Thus, reinterpreting the model as the implementation of a planner problem with the planner in the role of the principal, the path of consumption, labor and output applies to a single individual facing idiosyncratic risk that is insured via the incentive contract with the planner, and with each individual acting as an agent. It would be of interest to estimate the probability distribution of the multiplier processes as these express the influence of the constraints.

The estimation of the sum of the multipliers can be done via the observation of consumption: reasoning from equation (43), we have

$$C_t = u'^{-1}\left(\frac{1}{Z_t - J_W}\right) \rightarrow Z_t - J_W = \frac{1}{u'(C_t)}. \tag{73}$$

In a conventional growth-based model, the state price, which is equivalent to the Lagrange multiplier on the budget constraint in the underlying consumer optimization problem, would be directly proportional to the marginal utility of consumption; here, the Lagrange multipliers are inversely related to the marginal utility of consumption; this expresses the duality of the consumer optimization problem and the contract problem. Thus, the sum of the multipliers in (73) can be estimated directly as the inverse of the estimated state price process.

Examining Figure 8 of Barone-Adesi et al [5], it is apparent that the state price—what the authors label as the option-adjusted empirical pricing kernel, AKA the state price, is evidently a decreasing function of the overall value of the systematic part of the economy's wealth, as indexed by the value state of the discounted S&P500 index, as theory predicts.[6]

Taking this estimated state price as a starting point, as noted above, the inverse of this index is proportional to the marginal values of the defection and incentive constraints in the contract model, and therefore the prediction is that the inverse of the state price estimate will be an increasing function of the growth state of the economy. This is borne out in Figure 7; viewing the wealth state as correlated with the agent's continuation value state W_t, it is clear that $-J_W$ is increasing in W_t. Furthermore, as W_t increases, the defection constraint $\phi(W_t)$ binds,

[6]The logic is as follows: the state price is proportional to the marginal utility of consumption; this in turn is equal to the marginal value of the value function with respect to the wealth state by the envelope condition; by the concavity of the value function, the marginal value is inversely related to the wealth state.

forcing the Z_t state upward, so Z_t is also positively correlated with W_t in this sense.

These ideas can be explored further in the context of the Monte Carlo sample path examples. Using the artificial data generated in the second example (see Figure 18), we can examine the analogue of Barone-Adesi et al's Figure 8: the horizontal axis shows the deviations of the implicit asset for the agent J, normalized to its mean, $\frac{J(z,w)}{\overline{J}}$, analogous to the S&P500 state S_T/S_t in Barone-Adesi et al, and the normalized marginal value $\frac{J_W - Z_t}{J}$ is on the vertical axis. As predicted by the

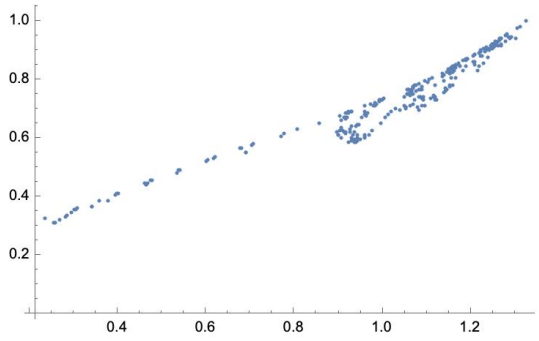


FIGURE 22. Normalized marginal value: $\frac{Z_t - J_W}{J}$ vs $\frac{J(z,w)}{\overline{J}}$

inverse relationship between the marginal value $Z - J_W$ and the state price in the conventional asset model, the slope should be the opposite of the negative slope predicted in Barone-Adesi *et al*'s formulation, and the positive slope in Figure 22 bears out this prediction.

It is also of interest to examine the impact of the defection constraint on the interest rate as discussed in equation (62); the defection multiplier reduces the effective interest rate. Using the data from the second example, Figure 23 shows the ratio $\frac{\rho Z_t U_d(H(W_t))}{J}$ versus the deviations of the effective asset value from its mean $\overline{J}$, $\frac{J(z,w)}{\overline{J}}$: (The utility value is approximated by the defection continuation value, converted to utility flow units $u(C) - v(A)$ as in equation (62) by the agent's subjective discount rate ρ.) It is evident that the interest rate adjustment is quite small; it is less than the .01 value of the principal's discount rate in the example. Thus, the implicit interest rate is very small but still positive.

Finally, we can explore the impact of the implicit incentive constraint via the shadow multiplier J_W; similar to the Z_t effect, one can interpret the effect on the principal's implicit interest rate: see equation (49). This is shown in Figure 24. Starting from the agent's subjective discount rate of $\rho = .05$ in the example, it is evident that there is a

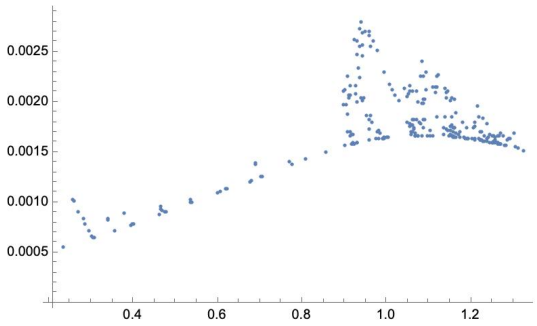


FIGURE 23. Interest rate adjustment $\frac{\rho Z_t U_d(H(W_t))}{J}$ versus the deviations of the asset value, $\frac{J(z,w)}{\overline{J}}$.

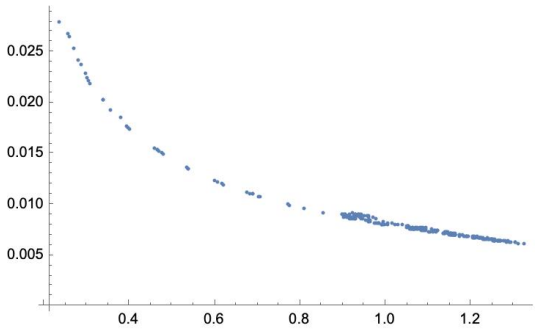


FIGURE 24. Interest rate adjustment $\frac{-\rho J_W U_d(H(W_t))}{J}$ versus the deviations of the asset value, $\frac{J(z,w)}{\overline{J}}$.

significant adjustment of the interest rate downward from that value as suggested by theory.

I end with two comments concerning extensions of the model. An additional topic of interest is to assess the separate magnitudes, and thus the relative magnitudes, of the marginal valuations implicit in Z_t and J_W. Unlike the reasoning above concerning the state price which yields an estimate of the sum $Z_t - J_W$, there is no obvious direct way to estimate Z_t and J_W separately. We can however estimate Z_t indirectly. To do this, note first that if we interpret J_t as the asset process in (61), then J_t is observable; one could proxy it via the S&P500. This process has a drift, and the components of the drift, namely output $A^\alpha H(W)^{1-\alpha}$, consumption C_t, labour A_t, and, if we are willing to admit functional forms for $u(\cdot)$ and $v(\cdot)$, the difference $u(C_t) - v(A_t)$. Whilst we cannot view the underlying principal's discount rate r and Z_t, we can in principle estimate the stochastic process of the drift function.

This stochastic process will itself have a nonstochastic trend, r, that can be estimated. With r determined, Z_t can then be estimated.

This estimation of the drift process is known to be difficult in general, as it depends on the underlying probability measure.

A final remark concerns the relationship between the interest rates in the model. In conventional models there is a distinction between the risk-free rate and the market return; idiosyncratic risk is averaged out; the subjective rate of time preference ρ is distinct from these interest rates. In the contract model the market rate would be equivalent to the market return, with the principal's underlying discount rate r equivalent to the risk-free rate. These equivalences can then be used to convert to a risk-neutral measure. These issues are analyzed further in Appendix C.

14. Conclusion

I have presented a dynamic contract model which combines a defection constraint with incentive compatible consumption and effort. The principal's continuation value process behaves like credit or debt from the perspective of the agent; the impact of the constraints is to reduce the rate of return on this asset. The defection constraint is dual to a borrowing constraint, but the borrowing constraint is strictly determined by the optimal contract that is dual to it.

The multipliers for the constraints are additive, and have the interpretation as the marginal value of relaxing these constraints, that is, the marginal value of inducing ongoing participation in the constraint, and the marginal value of inducing revelation about the underlying shock state. This latter multiplier can equally be considered as the marginal value of revealing the true income state of the agent and thus the marginal value of diversifying the income shocks of agents sharing an insurance contract. The contract imposes this price on the provision of insurance to the agent, but the price is raised even further to prevent defection, and hence reducing the overall efficiency of the contract.

The contract can therefore be interpreted as a decentralized asset equilibrium that enables individuals to share risk; conversely, asset equilibria can be interpreted as the implementation of an incentive-compatible dynamic contract.

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