Dominant currency pricing, uncovered interest parity and exchange rate dynamics

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Abstract

We describe the properties of a dynamic stochastic multi-country model featuring product differentiation, monopolistic competition and staggered price setting. Monopolistic producers use dominant currency pricing strategies and set prices in their own currency in the domestic good market and with a vehicle currency, which may be different from either the own or the local currency, in foreign ones. International financial markets may be either complete or incomplete, although each country trades a risk-free bond in the foreign exchange market. Monetary policy follows a strict inflation targeting rule in each economy. When monetary policy shocks are transitory, real exchange and relative inflation rates are weakly stationary. Moreover, nominal exchange rates and relative price levels are cointegrated and a purchasing power parity condition holds between any two economies. Further, the dynamic response of the nominal exchange rate to monetary policy shocks hinges on whether they are contemporaneous or anticipated and is a function of their persistence. For low values of persistence there is overshooting of the nominal exchange rate. When monetary policy shocks are permanent, real exchange and relative inflation rates are nonstationary.

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1 Introduction

The study of exchange rate dynamics has been approached from different perspectives in international economics. An important subject regards the existence of purchasing power parity (PPP) conditions among economies and the foundations for the required long run relations between nominal exchange rates and relative consumer prices. A further question concerns the relations between exchange and interest rates that result from the existence of no arbitrage conditions in foreign exchange markets. In this field, an issue regards the implications of uncovered interest parity (UIP) for exchange rate volatility.

A dynamic general equilibrium model for the analysis of exchange rates was introduced in Obstfeld and Rogoff (1995). The model features intertemporal choice by economic agents, product differentiation and monopolistic competition, in domestic and foreign markets, and nominal rigidities. Subsequent contributions have introduced various modifications. Optimization under uncertainty has replaced the assumption of perfect foresight, for both consumers and firms. In addition, much research effort has been concerned with the modelling of nominal rigidities.

A recent advancement has been the establishment of the dominant currency paradigm in Gopinath (2015) and Gopinath et al. (2020). In related statistical research Boz et al. (2022) have produced extensive evidence on world wide import and export patterns, supporting the view that only a few currencies are mostly used for pricing in international good markets. The dominant currency pricing (DCP) model constitutes a new perspective in relation to previous ones, which have mainly proceeded with the assumption of either producer (PCP) or local currency pricing (LCP), as in Betts and Devereux (1996, 2000), Devereux and Engel (2003) and Engel (2011).

In the present work we employ the dominant currency paradigm for the analysis of the relation between exchange and interest rates. We make assumptions that allow us to produce exact results on the dynamical properties of nominal and real exchange rates. Most notably, drawing from previous research in Engel (2019), we assume monetary policy follows a strict inflation targeting rule in each economy. We show how with this choice a multi-country model may be reduced to a two dimensional dynamical system, including the real exchange and relative

inflation rates between any two economies as endogenous variables. We impose some additional restrictions to simplify the exposition and clarify the robustness of our results to broader formulations.

We describe the system properties holding under different specifications of the monetary policy shock. When monetary policy shocks are transitory, real exchange and relative inflation rates are weakly stationary variables. This in turn implies that nominal exchange rates and relative price levels are nonstationary. Moreover, there exist cointegrating relations between nominal exchange rates and price levels and a PPP condition holds as a long run relation between any two economies. When monetary policy shocks are permanent, real exchange and relative inflation rates are nonstationary.

Furthermore, the dynamic response of the nominal exchange rate is a function of whether monetary policy shocks are contemporaneous or anticipated and of their persistence. When monetary policy shocks are contemporaneous and persistence is low, there is overshooting of the nominal exchange rate. Conversely, when persistence is high, the nominal exchange rate converges on impact to the new equilibrium value following a monetary policy shock. Similar results hold when monetary policy shocks are anticipated.

The work is structured as follows. Section 2, defines the dynamic stochastic multi-country economy. Section 3, introduces monetary policy and derives logarithmic first order Taylor expansions of the equilibrium relations in a neighbourhood of the stationary steady-state. Section 4, describes the linear rational expectations model. Section 5, reviews the dynamic response of the main variables of interest to monetary policy shocks. Section 6, analyses volatility statistics. Some conclusions are drawn in section 7.

2 The international economy

We follow Gopinath et al. (2020) and model a world economy with N countries, each producing and trading a differentiated commodity. Consumers own firms and supply services in the labour market, we normalize the population size in each economy to equal unity and denote with C_{jt} and L_{jt} the consumption and labour supply in country $j = 1, \ldots, N$ and period t of household $h \in [0, 1]$. We

describe the household and firm optimization problems and provide the equilibrium conditions that hold for the world economy in a symmetric steady-state.[1]

2.1 Households

Consumer preferences are represented with an additively separable expected utility function, defined over an infinite horizon:

$$U_{jt} = \mathbb{E}_t \sum_{\tau=0}^{+\infty} \beta^{\tau} u\left(C_{jt+\tau}, L_{jt+\tau}\right) \tag{2.1}$$

In equation (2.1) $u\left(C_{jt+\tau}, L_{jt+\tau}\right)$ denotes the household single period utility function, $0 < \beta < 1$ is the subjective discount factor, or intertemporal rate of time preference, and $\mathbb{E}_t$ represents the conditional expectation operator. Agents form expectations conditioning on information available in each period of time t. We assume the single period utility function is additively separable in consumption and labour supply and takes the constant elasticity of substitution (CES) form:

$$u\left(C_{jt}, L_{jt}\right) = \frac{C_{jt}^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \kappa \frac{L_{jt}^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \tag{2.2}$$

where $\sigma > 0$ and $\eta > 0$ denote the intertemporal elasticity of substitution and the Frisch-elasticity of labour supply, respectively, and $\kappa > 0$ is a scale parameter, in each country $j = 1, \ldots, N$.

The aggregate C_{jt} is defined as a function of differentiated product variety consumption. We let $C_{ijt}\left(\omega\right)$ denote country j consumption of product variety ω imported from country i and normalize the set of product varieties produced in each economy to a unit measure: $\omega \in [0, 1]$ for each $i = 1, \ldots, N$. Preferences for differentiated product variety consumption take the CES form:

$$C_{jt} = \left[\sum_{i=1}^{N} \alpha_{ij}^{\frac{1}{\phi}} \int_0^1 C_{ijt}\left(\omega\right)^{\frac{\phi-1}{\phi}} d\omega\right]^{\frac{\phi}{\phi-1}} \tag{2.3}$$

[1] We will abstract from household indexing throughout the work, unless otherwise noted or required.

In equation (2.3) country j consumption of the differentiated product varieties produced in each country i is represented with the aggregator advanced in Spence (1976) and Dixit and Stiglitz (1977) (SDS) and $\phi > 1$ is the intratemporal elasticity of substitution. We assume the consumption weight parameters $\alpha_{ij} > 0$ for $i = 1,\ldots,N$ sum to unity: $\sum_{i=1}^{N} \alpha_{ij} = 1$ for each $j = 1,\ldots,N$.[2]

In order to define the consumer budget constraint, denote with $P_{ijt}(\omega)$ the country j price of product variety ω imported from country i. The price index corresponding to the CES aggregate consumption function (2.3) takes the form:

$$P_{jt} = \left[\sum_{i=1}^{N} \alpha_{ij} \int_{0}^{1} P_{ijt}(\omega)^{1-\phi} \, d\omega \right]^{\frac{1}{1-\phi}} \tag{2.4}$$

Moreover, let W_{jt} and $\Pi_{jt}(\omega)$ for $\omega \in [0,1]$ define the country j wage and real profit rates in period t, respectively, and assume there exists a set of *Arrow-Debreu* securities, which allow consumers to save or borrow on current income. Denote with $\Xi_{jt}(s)$ the period t price of the state-contingent security paying a unit of account in state $s \in S$ and country j in period $t+1$, where S represents the set of possible states of the world. When financial markets are complete, as for instance in Galí and Monacelli (2005), the Arrow-Debreu securities are traded in the foreign exchange market and in equilibrium there is an equivalence between state-contingent prices across economies. Denoting with S_{ijt} the nominal exchange rate between economies i and j, defined as units of country j currency for each unit of country i currency, the following relation holds: $\Xi_{jt}(s) = S_{ijt}\Xi_{it}(s)/S_{ijt+1}$ for all $i,j = 1,\ldots,N$ and $s \in S$. When international financial markets are incomplete, only a subset of the state-contingent securities are traded in the foreign exchange market and the equivalence between state-contingent prices across economies does not necessarily hold. We assume in either case that each country may trade a risk-free bond and denote with i_{it} the nominal risk-free interest rate in country $i = 1,\ldots,N$ between periods t and $t+1$.

The consumer budget constraint in each country $j = 1,\ldots,N$ is the following:

[2] In a multi-stage formulation we would distinguish the elasticity of substitution between aggregate consumption of country i and k differentiated goods from the one between the product varieties of each country k. Equation (2.3) assumes the two elasticities are equal. The analysis in the present work is robust to this extension.

$$P_{jt}C_{jt} + \sum_{i=1}^{N} S_{ijt}B_{ijt} + \int_{s\in S} \Xi_{jt}(s)\, X_{jt}(s)\, ds = \tag{2.5}$$

$$W_{jt}L_{jt} + \sum_{i=1}^{N} S_{ijt}B_{ijt-1}\left(1 + i_{it-1}\right) + X_{jt-1} + P_{jt}\int_{0}^{1} \Pi_{jt}(\omega)\, d\omega$$

where B_{ijt} for $i = 1, \ldots, N$ and $X_{jt}(s)$ for $s \in S$ represent risk-free bond and contingent claim holdings between periods t and $t+1$.

The consumer optimization problem is to maximize the intertemporal utility function (2.1), subject to (2.2), (2.3) and to the dynamic budget constraint defined in (2.4) and (2.5). We assume consumers take wages and firm profits as given.[3]

The consumer optimization problem admits a two-stage budgeting representation. In the lower stage each consumer allocates consumption between home and foreign differentiated product varieties taking aggregate consumption C_{jt} as given. The second stage demand functions take the form:

$$C_{ijt}(\omega) = \alpha_{ij}\left[\frac{P_{ijt}(\omega)}{P_{jt}}\right]^{-\phi} C_{jt} \tag{2.6}$$

for each $i = 1, \ldots, N$ and $\omega \in [0, 1]$.

In the higher stage consumers allocate wealth, which results in each period from the sum of labour income, firm profits and gross returns on previous period net savings, between aggregate consumption and savings, in the form of either risk-free bonds or state-contingent securities.

The first stage Euler equations for the risk-free bond net purchases are:

$$\frac{C_{jt}^{-\frac{1}{\sigma}} S_{ijt}}{P_{jt}} = \mathbb{E}_t\left[\frac{\beta C_{jt+1}^{-\frac{1}{\sigma}} S_{ijt+1}\left(1 + i_{it}\right)}{P_{jt+1}}\right] \tag{2.7}$$

and hold for each $i = 1, \ldots, N$.

Similarly, for the state-contingent securities the conditions:

[3]The assumption of a competitive labour market is made for ease of exposition, given the purpose of the analysis we would obtain the same results in a model with monopolistic labour market power and staggered wage setting.

$$\frac{C_{jt}^{-\frac{1}{\sigma}}\Xi_{jt}(s)}{P_{jt}} = \mathbb{E}_t\left[\frac{\beta C_{jt+1}^{-\frac{1}{\sigma}}\iota_s(s')}{P_{jt+1}}\right] \tag{2.8}$$

hold for all $s \in S$.

In equation (2.8) we define $\iota_s(s')$ as the indicator function: $\iota_s(s') = 1$ if $s' = s$ and $\iota_s(s') = 0$ if $s' \neq s$.

Finally, the marginal condition for labour supply is:

$$\frac{C_{jt}^{-\frac{1}{\sigma}}W_{jt}}{P_{jt}} = \kappa L_{jt}^{\frac{1}{\eta}} \tag{2.9}$$

2.2 Firms

Good markets are segmented by destination country and invoicing currency. Let $P_{jit}^k(\omega)$ and $Y_{jit}^k(\omega)$ denote the price charged in country i and currency k by firm $\omega \in [0,1]$ of country j and the corresponding output in period t. Firms charge prices in their own currency in the home market and therefore $k = j$ for $i = j$. In foreign markets firms charge prices in their own currency $k = j$ under PCP, in the local one $k = i$ under LCP or using a vehicle currency, which may be different from either the own or the local one, under DCP.

The firm gross real profits in period t may then be defined as follows:

$$\Pi_{jt}(\omega) = \sum_{i,k=1}^{N} \frac{S_{kjt}P_{jit}^k(\omega)Y_{jit}^k(\omega)}{P_{jt}} - \frac{W_{jt}L_{jit}^k(\omega)}{P_{jt}} \tag{2.10}$$

where $L_{jit}^k(\omega)$ is the labour employed in the production of goods for destination country i invoiced in currency k by firm $\omega \in [0,1]$.

In each economy production occurs under constant returns to scale, according to a linear technology: $Y_{jit}^k(\omega) = A_{jt}(\omega)L_{jit}^k(\omega)$, where $A_{jt}(\omega)$ measures firm ω productivity in country j. Define the output of firm ω as $Y_{jt}(\omega) = \sum_{i,k=1}^{N} Y_{jit}^k(\omega)$ and note that $Y_{jt}(\omega) = A_{jt}(\omega)L_{jt}(\omega)$, where $L_{jt}(\omega) = \sum_{i,k=1}^{N} L_{jit}^k(\omega)$ is total employment.

We assume there is monopolistic competition in both the home and foreign markets. Each firm $\omega \in [0,1]$ faces the downward sloping demand schedule defined in equation (2.6) in each market. Since the law of one price implies $P_{jit}(\omega) =$

$S_{kit}P^k_{jit}(\omega)$ for all $j,k=1,\ldots,N$ and for the exchange rate the equality $S_{kit}=1/S_{ikt}$ holds by definition, the aggregate price index in the destination country i may be represented in terms of the invoicing currency as: $P^k_{it}=S_{ikt}P_{it}$.

We provide the optimality conditions that hold in a Calvo (1983) staggered price setting equilibrium, where only a fraction of firms charges an optimal price in each time period. We assume an independent and identical probability of price setting equal to $1-\delta$ across firms in each time period, with $0<\delta<1$, for each destination and producer country and each invoicing currency $i,j,k=1,\ldots,N$. Define the country j real stochastic discount factor as $M_{jt+1}=\beta\left(C_{jt+1}/C_{jt}\right)^{-1/\sigma}$, the T-period stochastic discount factor as $M_{jt,t+T}=\Pi_{\tau=1}^{T}M_{jt+\tau}$, for $T=1,2,\ldots,+\infty$, and let $M_{jt,t}=1$. The optimization problem for a firm that sets price optimally in period t is to maximize:

$$\Pi_t(\omega)=\mathbb{E}_t\sum_{\tau=0}^{+\infty}\delta^{\tau}M_{jt,t+\tau}\Pi_{jt+\tau}(\omega) \tag{2.11}$$

subject to: $Y^k_{jit}(\omega)=\alpha_{ji}\left[P^k_{jit}(\omega)/P^k_{it}\right]^{-\phi}C_{it}$ and $L^k_{jit}(\omega)=Y^k_{jit}(\omega)/A_{jt}(\omega)$.

The optimal price $P^k_{jit}(\omega)$ fulfils the following first order necessary condition:

$$\mathbb{E}_t\sum_{\tau=0}^{+\infty}\left\{\frac{\delta^{\tau}M_{jt,t+\tau}Y^k_{jit+\tau}(\omega)}{P_{jt+\tau}}\left[S_{kjt+\tau}P^k_{jit}(\omega)-\frac{\phi}{\phi-1}\frac{W_{jt+\tau}}{A_{jt+\tau}(\omega)}\right]\right\}=0 \tag{2.12}$$

We should notice, as customary for this type of model, the correspondence between equation (2.12) and the flexible equilibrium pricing condition, according to which monopolistic competitive firms charge a constant mark-up equal to $\phi/(\phi-1)$ over marginal cost: $P^{k*}_{jit}(\omega)=\frac{\phi}{\phi-1}\frac{W_{jt}}{A_{jt}(\omega)S_{kjt}}$. With staggered price setting the price charged by each firm may be greater or lower than the optimal flexible price in each time period.[4]

The Calvo (1983) staggered price setting equilibrium is usually classified as

[4]The flexible equilibrium pricing condition follows from the assumption of an SDS aggregator for differentiated product variety consumption. Following Stiglitz (1984) the SDS aggregator could be generalized, in order to allow for a countercyclical mark-up and address the observation that perfect competition, decreasing returns to scale and a procyclical real wage are mutually contradictory conditions in an economic model.

a time-dependent model, because the law of large numbers implies that in each differentiated product market a constant fraction of firms equal to $1-\delta$ resets price in each time period, although the firms that reset price are randomly selected. The length of each price spell is also a random variable, while the constant hazard implies that the average length of time between subsequent price changes is equal to $1/(1-\delta)$ for each firm.

The model has been shown to be consistent with some important stylized facts about price setting, as for instance the ones described in studies of consumer, import and export prices by Bils and Klenow (2004), Klenow and Kryvtsov (2008) and Gopinath and Rigobon (2008).

The alternative class of state-dependent models includes contributions by Caplin and Spulber (1987), Golosov and Lucas (2007) and Gertler and Lehay (2008) among other ones. While models in this class are able to more properly account for the observation that the frequency of price adjustment displays variation over the business cycle, a state-dependent modelling approach would bring us too far from the subject of the present work.

2.3 Equilibrium

The world economy equilibrium is defined by market clearing conditions in the good, labour and state-contingent security markets of each country and in the foreign exchange market. In country j good markets, consumption of country i product variety $\omega \in [0,1]$ must be equal to the corresponding production quantity in every time period. Therefore, the equality $C_{ijt}(\omega) = Y_{ijt}(\omega) = \sum_{k=1}^{N} Y_{ijt}^{k}(\omega)$ must hold for each $i,j = 1,\ldots,N$ and all $\omega \in [0,1]$. In the labour market, labour supply must be equal to the amount of labour employed in the production of differentiated varieties and hence the equality $L_{jt} = \int_0^1 L_{jt}(\omega)\, d\omega$ must be fulfilled for all $j = 1,\ldots,N$ in each period t.

For the state-contingent security market it is necessary to distinguish between traded and non-traded securities, denote the two subsets as $S^{\perp} \subset S$ and $S^{\top} = S \setminus S^{\perp} \subset S$, respectively, and note that $S^{\perp} \cap S^{\top} = \varnothing$ and $S = S^{\perp} \cup S^{\top}$ by construction. In equilibrium it must hold that the net world purchases of each Arrow-Debreu security are equal to zero in each time period and therefore the

following market clearing conditions have to be satisfied: $\sum_{j=1}^{N} X_{jt}(s) = 0$ for all $s \in S^{\perp}$ and $X_{jt}(s) = 0$ for each $j = 1, \ldots, N$ and $s \in S^{\top}$. Finally, net world purchases of the risk-free bond issued by each economy must be equal to zero in every time period: $\sum_{j=1}^{N} B_{ijt} = 0$ for all $i = 1, \ldots, N$ in each period t.

We further describe the equilibrium relations that hold in a symmetric steady-state, with equal consumption weight parameters $\alpha_{ij} = 1/N$ for all i, j, equal risk-free rates $i_{it} = \bar{i}$ across economies $i = 1, \ldots, N$ and productivity $A_{jt}(\omega) = 1$ normalized to equal unity for all product varieties $\omega \in [0, 1]$ in each country $j = 1, \ldots, N$. The structure of the model implies that the equilibrium is stationary and linear homogeneity conditions allow us to normalize labour wages to equal unity in each economy and every time period: $W_{jt} = 1$ for all $j = 1, \ldots, N$. In the symmetric equilibrium the exchange rates are also equal to unity: $S_{ijt} = 1$ for all $i, j = 1, \ldots, N$. Moreover, the pricing condition (2.12) implies, that the price charged by firms in each source country j for each product variety is equal to the flexible price equilibrium mark-up in each destination country: $P_{jit}^{k}(\omega) = \phi/(\phi - 1)$ for each $i, j, k = 1, \ldots, N$ and all $\omega \in [0, 1]$. The price index (2.4) then implies $P_{jt} = \phi/(\phi - 1)$, as $P_{ijt}(\omega) = S_{kjt} P_{ijt}^{k}(\omega) = \phi/(\phi - 1)$ and the consumption weight parameters sum to one in each economy. From the demand functions (2.6) it then follows that differentiated product variety consumption is equal to $C_{ijt}(\omega) = C_{jt}/N$ for each $i, j = 1, \ldots, N$ and $\omega \in [0, 1]$. In addition, the Euler equations (2.7) imply that the equilibrium risk-free rate is equal to $\bar{i} = 1/\beta - 1$ in each economy and all periods of time t. When financial markets are complete within each economy, the state-contingent security conditions (2.8) also imply, that Arrow-Debreu prices and the equilibrium risk-free rate are related as follows: $\int_{s \in S} \Xi_{jt}(s)\, ds = 1/(1 + i_{jt})$. Finally, in symmetric equilibrium both the risk-free bond purchases B_{ijt} for $i = 1, \ldots, N$ and the state-contingent security purchases $X_{jt}(s)$ for $s \in S$ are equal to zero in each country: $B_{ijt} = 0$ and $X_{jt}(s) = 0$ for all $i, j = 1, \ldots, N$ and $s \in S$ in each period t. The equilibrium aggregate consumption and labour supply C_{jt} and L_{jt} may then be derived from the budget constraint (2.5), the marginal condition for labour supply (2.9) and the market clearing conditions. In the stationary and symmetric steady-state: $\overline{C} = \overline{L} = [\kappa\phi/(\phi - 1)]^{-\sigma\eta/(\sigma+\eta)}$ for all $j = 1, \ldots, N$ in each time period t.[5]

[5] A formal derivation of the solution for steady-state aggregate consumption and labour

3 Monetary policy, real exchange rates and uncovered interest parity

We will study the dynamical properties of exchange rates around the stationary steady state and describe the response of nominal and real exchange rates and inflation to interest rate shocks in domestic and foreign economies. In the international economy risk-free rates are determined by monetary policy in each country and we proceed from a log-linear first order approximation of the model in a neighbourhood of the symmetric equilibrium.

3.1 Monetary policy

Monetary policy follows a strict inflation targeting rule in each economy:

$$i_{jt} = \overline{r} + \pi_{jt} + \alpha\left(\pi_{jt} - \overline{\pi}\right) + \varepsilon_{jt} \tag{3.1}$$

where $\overline{r}$ denotes the steady-state equilibrium real interest rate, $\pi_{jt} = \log\left(P_{jt}/P_{jt-1}\right)$ is the inflation rate in country j between periods $t-1$ and t, $\overline{\pi}$ defines the target inflation rate and ε_{jt} is a monetary policy shock.

The symmetric structure of the model implies equal steady-state equilibrium real interest rates and we assume $\alpha > 0$ and $\overline{\pi} \geq 0$ are also constant across economies $j = 1, \ldots, N$. The rule in (3.1) is specified according to the Taylor (1993, 1999) principle that, in order to ensure price stability, risk-free rates should be positively related to inflation, with a slope coefficient greater than one.

The analysis will mostly be concerned with a transitory monetary policy shock ε_{jt}, described by a weakly stationary stochastic process of general form:

$$\varepsilon_{jt} = \sum_{\tau=0}^{+\infty} \psi_\tau \nu_{jt-\tau} \tag{3.2}$$

with absolutely summable coefficients $\psi_0 = 1$, $\sum_{\tau=0}^{+\infty}|\psi_\tau| < +\infty$ and innovation terms ν_{jt} independent and identically distributed over time, with mean equal to zero and variance σ_ν^2: $\nu_{jt} \sim i.i.d.\left(0, \sigma_\nu^2\right)$.[6]

supply is presented in appendix A.

[6]The Wold decomposition theorem states that any weakly stationary mean zero stochas-

In addition, we consider an alternative monetary policy rule with a forward looking structure:

$$i_{jt} = \overline{r} + \mathbb{E}_t\left(\pi_{jt+1}\right) + \alpha\left[\mathbb{E}_t\left(\pi_{jt+1}\right) - \overline{\pi}\right] + \varepsilon_{jt} \tag{3.3}$$

where as before $\alpha > 0$ and the monetary policy shock ε_{jt} is defined according to equation (3.2).

Moreover, as will be clarified below, with reference to both the Taylor and the forward looking rules (3.1) and (3.3), we distinguish between contemporaneous and anticipated monetary policy shocks. Whereas a contemporaneous shock at time t is unanticipated, we suppose more generally that monetary policy shocks may be anticipated T periods in advance.

Finally, we also analyse the effects of a permanent shock, which is defined assuming $\psi_\tau = 1$ for each $\tau = 0, 1, \ldots, +\infty$ in equation (3.2). Notice that in this case the model becomes nonstationary, as the monetary policy shock takes the form of a random walk: $\varepsilon_{jt} = \varepsilon_{jt-1} + \nu_{jt}$.

3.2 Real exchange rates and uncovered interest parity

In the consumer optimization problem the Euler equations relating to the risk-free bond purchases may be interpreted as no arbitrage conditions in the foreign exchange market. We are interested in the relation between real exchange and interest rates. The real exchange rate between economies i and j is defined as $Q_{ijt} = S_{ijt}P_{it}/P_{jt}$ and measures the units of country j aggregate consumption required for each unit of country i consumption.

Multiplying and dividing the term inside the square brackets in the right hand side of equation (2.7) by the inflation rate P_{it+1}/P_{it} in country i, rearranging terms and substituting for the real exchange rates Q_{ijt} and Q_{ijt+1} in periods t and $t+1$ and the real stochastic discount factor M_{jt+1} yields the bond pricing equation:

tic process may be represented as an infinite moving average with the form (3.2) and square summable coefficients: $\sum_{\tau=0}^{+\infty}\psi_\tau^2 < +\infty$. A review may be found in Hannan (1970). The absolute summability condition is stronger than square summability and is fulfilled by any finite order autoregressive moving average process. For an advanced introduction to time series analysis see Harvey (1993).

$$\mathbb{E}_t \left[\frac{M_{jt+1} Q_{ijt+1}}{Q_{ijt}} \frac{P_{it} \left(1 + i_{it}\right)}{P_{it+1}} \right] = 1 \tag{3.4}$$

which holds for each $i = 1, \ldots, N$.

For a country j investor the economy i risk-free asset yields a real return equal to the ratio between the nominal return $1 + i_{it}$ and the inflation rate P_{it+1}/P_{it}, multiplied by the ratio Q_{ijt+1}/Q_{ijt} between real exchange rates in periods $t+1$ and t.

Moreover, in the stationary steady-state the real stochastic discount factor is constant across states of nature and equal to $\beta = 1/\left(1 + \bar{i}\right)$ in each economy j and, taking a log-linear Taylor expansion of equation (3.4) in the neighbourhood of the steady-state, for $i = j$ we obtain:

$$\mathbb{E}_t \left[m_{jt+1} \right] = -r_{jt} \tag{3.5}$$

where we define the real interest rate with the Fisher equation $r_{jt} = i_{jt} - \mathbb{E}_t \left(\pi_{jt+1} \right)$ and use lower case letters for the natural logarithm of the stochastic discount factor: $m_{jt+1} = \log \left(M_{jt+1} \right)$.

Equation (3.5) states that the expected value of the logarithmic transformation of the real stochastic discount factor is equal to the opposite of the expected real return on the risk-free asset in each country $j = 1, \ldots, N$, conditional on period t information.

Taking a log-linear first order approximation to (3.4) around the stationary steady-state and substituting equation (3.5) yields the real uncovered interest parity (UIP) condition:

$$\mathbb{E}_t \left(q_{ijt+1} \right) - q_{ijt} + r_{it} - r_{jt} = 0 \tag{3.6}$$

where $q_{ijt} = \log \left(Q_{ijt} \right)$.[7]

[7]For the sake of precision, in equations (3.5) and (3.6) we assume the first order approximation $x \simeq \log \left(1 + x\right)$ may be applied to the nominal interest rates i_{it} and i_{jt} of economies i and j, for all $i, j = 1, \ldots, N$.

3.3 Aggregate supply

We finally provide an equation for the aggregate supply curve. For this purpose, define the price index resulting from the optimal consumption of country i differentiated product varieties in country j as: $P_{ijt} = \left[\int_0^1 P_{ijt}(\omega)^{1-\phi} d\omega\right]^{1/(1-\phi)}$. With this definition, an equivalent representation of the aggregate price index P_{jt} in equation (2.4) is the following: $P_{jt} = \left[\sum_{i=1}^N \alpha_{ij} P_{ijt}^{1-\phi}\right]^{1/(1-\phi)}$. The assumption that the consumption weights sum to unity implies that the log-linear Taylor expansion of the aggregate price index P_{jt} around the stationary steady-state takes the form:

$$p_{jt} = \sum_{i=1}^N \alpha_{ij} p_{ijt} \tag{3.7}$$

where $p_{jt} = \log(P_{jt})$ and $p_{ijt} = \log(P_{ijt})$ for $i, j = 1, \ldots, N$.

In a symmetric steady-state the T-period stochastic discount factor is equal to β^T. Substituting in the first order condition (2.12), for country i firms exporting differentiated product varieties to country j and invoicing in currency k, multiplying through by the ratio S_{kjt}/S_{kit}, in order to convert the reset price in the local currency j, and rearranging terms we obtain the recursive relation:

$$P_{ijt}(\omega) = (1 - \delta\beta) \frac{\phi}{\phi - 1} \frac{W_{it}}{A_{it}(\omega) S_{jit}} + \delta\beta \mathbb{E}_t \left[P_{ijt+1}(\omega)\right] \tag{3.8}$$

where $P_{ijt}(\omega) = S_{kjt} P_{ijt}^k(\omega)$ and $S_{jit} = S_{kit}/S_{kjt}$ for each firm ω that resets price in period t.

A log-linear first order approximation to equation (3.8) about the stationary steady-state takes the form:

$$p_{ijt}(\omega) = (1 - \delta\beta)\left[\mu + w_{it} - a_{it}(\omega) - s_{jit}\right] + \delta\beta \mathbb{E}_t \left[p_{ijt+1}(\omega)\right] \tag{3.9}$$

where $p_{ijt}(\omega) = \log\left[P_{ijt}(\omega)\right]$, $\mu = \log\left[\phi/(\phi - 1)\right]$, $w_{it} = \log(W_{it})$, $a_{it}(\omega) = \log\left[A_{it}(\omega)\right]$ and $s_{jit} = \log(S_{jit})$.

Equation (3.9) holds for all producer and export countries. Assuming country i productivity is constant across firms, all firms ω that reset in period t set the same price: $a_{it}(\omega) = a_{it}$ and $p_{ijt}(\omega) = p_{ijt}^*$. The following recursive relation for the inflation rate in country j then holds:

$$\pi_{ijt} = \theta\left(\mu + w_{it} - a_{it} - s_{jit} - p_{ijt}\right) + \beta\mathbb{E}_t\left(\pi_{ijt+1}\right) \tag{3.10}$$

where $\theta = (1-\delta)(1-\delta\beta)/\delta$ and $\pi_{ijt} = p_{ijt} - p_{ijt-1}$.

Equation (3.10) may be interpreted as a sectoral *New Keynesian Phillips curve* and holds in both the domestic and import sectors of each economy, for all $i, j = 1, \ldots, N$.[8]

The present analysis generalizes prior results, which have been provided for the two country model with either producer or local currency pricing, in Benigno (2004) and Engel (2011, 2019) among other works, to the N country model with dominant currency pricing.

We are interested in the difference between the aggregate inflation rates in economies j and i. The symmetric structure of the model implies that the consumption weight parameters are equal across economies: $\alpha_{ij} = 1/N$ for all $i, j = 1, \ldots, N$. From equation (3.7) it thus follows that $\pi_{jt} - \pi_{it} = \sum_{k=1}^{N}\left(\pi_{kjt} - \pi_{kit}\right)/N$ and, substituting (3.10) in the latter expression, we finally obtain:

$$\pi_{jt} - \pi_{it} = \theta q_{ijt} + \beta\mathbb{E}_t\left(\pi_{jt+1} - \pi_{it+1}\right) \tag{3.11}$$

where $q_{ijt} = s_{ijt} + p_{it} - p_{jt}$.

4 Dynamic rational expectations model

The log-linear approximation of the international economy model in a neighbourhood of the stationary and symmetric steady-state, described in the previous section, defines a system of three linear stochastic difference equations, which allows us to study the relation between interest rates, inflation and the exchange rate.

The dynamic rational expectations model may be specified assuming monetary policy in each economy follows either the Taylor or the forward looking rule. In the Taylor rule specification, equation (3.1) implies that the difference between

[8] We include a formal proof of (3.10) in appendix B. An alternative derivation of the New Keynesian Phillips curve equation could be based on Rotemberg (1987) and the assumption that country i firms set the period t price minimizing a loss function, defined as the expected present discounted value of the squared difference between the logarithmic transformations of the reset and flexible equilibrium prices in each time period.

risk-free rates in countries j and i takes the form:

$$i_{jt} - i_{it} = (1 + \alpha)(\pi_{jt} - \pi_{it}) + \varepsilon_{jt} - \varepsilon_{it} \tag{4.1}$$

where:

$$\varepsilon_{jt} - \varepsilon_{it} = \sum_{\tau=0}^{+\infty} \psi_\tau (\nu_{jt-\tau} - \nu_{it-\tau}) \tag{4.2}$$

In the forward loooking rule specification, (3.3) implies:

$$i_{jt} - i_{it} = (1 + \alpha)\, \mathbb{E}_t (\pi_{jt+1} - \pi_{it+1}) + \varepsilon_{jt} - \varepsilon_{it} \tag{4.3}$$

where the difference $\varepsilon_{jt} - \varepsilon_{it}$ between the monetary policy shocks in economies j and i is defined by (4.2).

The second equation in the dynamic rational expectations model is defined by the real UIP condition (3.6) and the third one by the New Keynesian Phillips curve (3.11).

Substituting either (4.1), for the Taylor rule specification, or (4.3), for the forward looking rule one, in (3.6) we are able to reduce the linear model to a system of two stochastic difference equations in two unknowns: the real exchange rate q_{ijt} and the difference between inflation rates $\pi_{jt} - \pi_{it}$.

With either the Taylor or the forward looking rule specification, the two equation dynamical system is linear and of first order. In order to represent the system in matrix notation, let us define the endogenous variable and monetary policy shock vectors $z_t = (q_{ijt}, \pi_{jt} - \pi_{it})'$ and $\varepsilon_t = (\varepsilon_{jt} - \varepsilon_{it}, 0)'$. For both specifications the matrix representation of the stochastic difference equation system takes the form:

$$\mathbb{E}_t (z_{t+1}) = A z_t + \varepsilon_t \tag{4.4}$$

where A is a square matrix of order two.

Straightforward computations show that the endogenous variable matrix A is defined as: $A = \frac{1}{\beta} \begin{bmatrix} \beta + \theta & \beta(1+\alpha) - 1 \\ -\theta & 1 \end{bmatrix}$ in the Taylor rule specification and

$A = \frac{1}{\beta}\begin{bmatrix} \beta - \alpha\theta & \alpha \\ -\theta & 1 \end{bmatrix}$ in the forward looking rule one.

We solve the model following the methods described in Blanchard and Kahn (1980). Note that the linear stochastic difference equation system (4.4) has a unique stable solution if and only if the eigenvalues of the endogenous variable matrix have modulus greater than one. For both the Taylor and the forward looking rule specification, it is possible to show that for $\beta \to 1$ and a well-defined calibration of the model parameters the roots of the characteristic polynomial of the matrix A are complex-valued and satisfy the stability condition.

The fundamental solution of the dynamical system may be compiled multiplying both sides of (4.4) by the inverse of the endogenous variable matrix, rearranging terms and solving the resulting equation with forward recursive substitutions. From repeated application of the law of iterated expectations:

$$z_t = -\sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \mathbb{E}_t \left(\varepsilon_{t+\tau}\right) \tag{4.5}$$

provided the transversality condition $\lim_{\tau \to +\infty} A^{-(\tau+1)} \mathbb{E}_t \left(z_{t+\tau+1}\right) = 0$ holds.

In order to further specify the solution, define the innovation vector $\nu_t = (\nu_{jt} - \nu_{it}, 0)'$ and moving average matrices $\Psi_\tau = \begin{bmatrix} \psi_\tau & 0 \\ 0 & 0 \end{bmatrix}$ for $\tau = 0, 1, \ldots, +\infty$. For the case of a contemporaneous transitory shock then notice that in equation (4.5) $\mathbb{E}_t \left(\varepsilon_{t+\tau}\right) = \sum_{k=\tau}^{+\infty} \Psi_k \nu_{t+\tau-k}$, because $\varepsilon_{t+\tau} = \sum_{k=0}^{+\infty} \Psi_k \nu_{t+\tau-k}$ and $\mathbb{E}_t \left(\nu_{t+\tau-k}\right) = 0$ for $k < \tau$. Substituting we finally obtain:

$$z_t = \sum_{k=0}^{+\infty} \Lambda_k \nu_{t-k} \tag{4.6}$$

where $\Lambda_k = -\sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau}$ defines the moving average coefficient matrices for $k = 0, \ldots, +\infty$.

We argue that the multivariate stochastic process described in equation (4.6) is well-defined and weakly stationary.[9]

[9]The multivariate version of the Wold decomposition states that any weakly stationary mean zero multivariate process takes the form (4.6), with square summable coefficient matrices: $\sum_{k=0}^{+\infty} \Lambda_k \Sigma \Lambda_k' < +\infty$. The innovation variance matrix $\Sigma = \mathbb{E}\left(\nu_t \nu_t'\right)$ may be singular. See Hannan (1970). In appendix C we prove that the coefficient matrices in (4.6) fulfil the stronger

In the case of a transitory shock anticipated T periods in advance the expectations in equation (4.5) take the form $\mathbb{E}_t(\varepsilon_{t+\tau}) = \sum_{k=0}^{+\infty} \Psi_k \nu_{t+\tau-k}$ for $\tau < T$ and $\mathbb{E}_t(\varepsilon_{t+\tau}) = \sum_{k=\tau-T}^{+\infty} \Psi_k \nu_{t+\tau-k}$ for $\tau \geq T$, because $\mathbb{E}_t(\nu_{t+\tau-k}) = 0$ for $k < \tau - T$. Substituting with some algebra we obtain:

$$z_t = \sum_{k=0}^{+\infty} \Lambda_k \nu_{t+T-k} \tag{4.7}$$

where $\Lambda_k = -\sum_{\tau=T-k}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau-T}$ defines the moving average coefficient matrices for $k = 0, \ldots, T-1$ and $\Lambda_k = -\sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau-T}$ for $k = T, T+1, \ldots, +\infty$.[10]

Finally, for a permanent shock the random walk property of the monetary policy disturbance term implies that $\mathbb{E}_t(\varepsilon_{t+\tau}) = \varepsilon_t$ for all $\tau = 0, 1, \ldots, +\infty$ in equation (4.5) and $z_t = -A^{-1}(I - A^{-1})^{-1} \varepsilon_t$. Therefore, the stochastic process for the endogenous variable vector z_t takes the form of a random walk:

$$z_t = z_{t-1} + \nu_{zt} \tag{4.8}$$

with the innovation vector $\nu_{zt} = -A^{-1}(I - A^{-1})^{-1} \nu_t$ independent and identically distributed over time.[11]

5 Simulation outcomes

In the model simulations we calibrate the parameters describing monetary policy, consumer preferences and firm pricing behaviour as to ensure the stability condition for the dynamical system is fulfilled.

We measure time with a quarterly frequency and assume an intertemporal rate of time preference $\beta = 0.995$. In addition, we specify $\delta = 0.75$ and hence the probability of price setting for each firm equal to 0.25 in every time period. These

condition of absolute summability.

[10] In appendix C we show that equation (4.7) describes a well-defined weakly stationary process, as the coefficient matrices are absolutely summable.

[11] The properties of the endogenous variable matrix characteristic polynomial, for the Taylor and forward looking rule specifications, and alternative derivations of the solutions in equations (4.6)-(4.8) for the stochastic difference equation system (4.4), based on the method of undetermined coefficients, are reviewed in appendix C.

settings imply that in the stationary steady-state the equilibrium real interest rate is equal to two per cent on a yearly basis and firms reset price on average every four quarters in each economy.

For monetary policy we use the leaning against the wind specification with $\alpha = 0.5$ for both the Taylor and forward looking rule specifications (3.1) and (3.3).

Further, we assume the monetary policy shock is transitory and takes the form of a first order autoregressive process:

$$\varepsilon_{jt} = \rho\varepsilon_{jt-1} + \nu_{jt} \tag{5.1}$$

where $0 < \rho < 1$ and the innovation term is independent and identically distributed with mean zero and variance σ_ν^2: $\nu_{jt} \sim i.i.d.\,(0, \sigma_\nu^2)$ for all $j = 1, \ldots, N$.

The autoregressive specification implies that $\psi_\tau = \rho^\tau$ in equations (3.2) and (4.2) and $\Psi_\tau = \begin{bmatrix} \rho^\tau & 0 \\ 0 & 0 \end{bmatrix}$ in the dynamical system (4.4). Therefore, for a contemporaneous shock $\Lambda_k = -\rho^k A^{-1} \left(I - \rho A^{-1}\right)^{-1}$ for $k = 0, \ldots, +\infty$ in equation (4.6). In this case the endogenous variable vector z_t is described by a first order autoregression process. Defining the endogenous innovation vector $\nu_{zt} = -A^{-1} \left(I - \rho A^{-1}\right)^{-1} \nu_t$ it is straightforward to show that:

$$z_t = \rho z_{t-1} + \nu_{zt} \tag{5.2}$$

Moreover, for a monetary policy shock anticipated T periods in advance, $\Lambda_k = -A^{k-T-1} \left(I - \rho A^{-1}\right)^{-1}$ for $k = 0, \ldots, T-1$ and $\Lambda_k = -\rho^{k-T} A^{-1} \left(I - \rho A^{-1}\right)^{-1}$ for $k = T, \ldots, +\infty$ in equation (4.7).

For both the Taylor and the forward looking specification of the monetary policy rule, we simulate the model under two alternative assumptions for the autoregressive parameter: $\rho = 0.5$ or $\rho = 0.75$.

Figures 1 and 2 reproduce the impulse response functions of a 25 basis point shock to the relative interest rate between economies j and i. In both figures, the first row displays the impulse responses of the nominal relative risk-free rate $i_{jt} - i_{it}$, the natural logarithm of the nominal exchange rate s_{ijt} and the logarithmic relative price level $p_{jt} - p_{it}$, the second depicts the ones of the real relative interest rate $r_{jt} - r_{it}$, the log real exchange rate q_{ijt} and the relative inflation rate $\pi_{jt} - \pi_{it}$,

from left to right. Figure 1 displays the responses to a contemporaneous shock, whereas in Figure 2 the shock is anticipated two quarters in advance.

In order to interpret the impulse response functions, we find it useful to provide some further description of the dynamic rational expectations model. Rearranging terms in the real UIP condition (3.6) and solving the resulting equation with forward recursive substitutions we obtain the following relation for the logarithmic transformation of the real exchange rate between economies i and j in period t, q_{ijt}:

$$q_{ijt} = -\sum_{\tau=0}^{+\infty} \mathbb{E}_t \left(r_{jt+\tau} - r_{it+\tau} \right) \tag{5.3}$$

The assumption of a transitory monetary policy shock implies that the fundamental solution for the real exchange rate is weakly stationary and the transversality condition $\lim_{\tau \to +\infty} \mathbb{E}_t \left(q_{ijt+\tau+1} \right) = 0$ is fulfilled.

According to equation (5.3) the real exchange rate between economies i and j in period t is negatively related to the infinite sum of expected real relative interest rates in future periods.

Similarly, the Phillips curve equation (3.11) relates the relative inflation rate of economies j and i in period t $\pi_{jt} - \pi_{it}$, to the real exchange rate q_{ijt} and the expected relative inflation rate in period $t+1$. Forward recursive substitutions on (3.11) yield:

$$\pi_{jt} - \pi_{it} = \theta \sum_{\tau=0}^{+\infty} \beta^{\tau} \mathbb{E}_t \left(q_{ijt+\tau} \right) \tag{5.4}$$

The assumption of a transitory monetary policy shock again implies that the fundamental solution for the relative inflation rate is weakly stationary and the transversality condition $\lim_{\tau \to +\infty} \beta^{\tau+1} \mathbb{E}_t \left(\pi_{jt+\tau+1} - \pi_{it+\tau+1} \right) = 0$ holds.

According to equation (5.4) the relative inflation rate of economies j and i in period t is positively related to the present discounted value of expected real exchange rates in future periods.[12]

[12] We should recall, before proceeding, that the fundamental solutions for the real exchange and relative inflation rates are not the general solutions. The forward solutions in equations (5.3) and (5.4) rule out non fundamental components, as rational speculative bubbles in for-

5.1 Response to a contemporaneous shock

Figure 1 shows that following a contemporaneous monetary policy shock the real relative interest rate $r_{jt} - r_{it}$ increases on impact and then decreases to the stationary equilibrium. The transitional dynamics is related to the shock persistence. For $\rho = 0.5$ convergence occurs in about 12 quarters, while for $\rho = 0.75$ more than 24 quarters are required for the real relative interest rate to return to steady-state. For both specifications there is slight variation between the Taylor and forward looking rule models.

Consistently with equation (5.3) and the real UIP condition (3.6), the increase in the real relative interest rate determines an initial appreciation of the real exchange rate q_{ijt}, which subsequently depreciates and gradually returns towards the steady-state. Moreover, as predicted by equation (5.4), the real exchange rate behaviour implies that the relative inflation rate $\pi_{jt} - \pi_{it}$ decreases on impact and thereafter slowly converges to the stationary steady-state. For both the real exchange and relative inflation rates, convergence to equilibrium is faster with the low persistence shock, and there are only mild variations between the Taylor and forward looking rule models.

The impulse response function of the nominal relative interest rate $i_{jt} - i_{it}$ presents some relevant additional features. When monetary policy persistence is low ($\rho = 0.5$), the impact increase of the relative interest rate following a 25 basis point shock is respectively equal to 16 and 19 basis points for the Taylor and forward looking rule models. In both cases, the relative inflation rate decline lowers the impact increase of the nominal relative interest rate and, due to the convergence dynamics, this effect is of greater magnitude in the Taylor rule model. When persistence is high ($\rho = 0.75$), the decline of the relative inflation rate is greater on impact and the path of convergence to equilibrium implies that the nominal relative interest rate remains constant over time.

Finally, we describe the response of the nominal exchange rate. For this purpose, note that the nominal exchange rate is by definition equal to the sum of the real exchange rate and the relative price level: $s_{ijt} = q_{ijt} + p_{jt} - p_{it}$. In turn,

eign exchange markets or hyperinflations. The analysis of non fundamental components is not within the scope of the present work, an introduction to the field may be found in Blanchard (1985).

given the symmetric steady-state initial condition $p_{j0} - p_{i0} = 0$, the logarithmic relative price level in any subsequent period t is equal to the cumulative sum of current and past inflation rates: $p_{jt} - p_{it} = \sum_{\tau=1}^{t} (\pi_{j\tau} - \pi_{i\tau})$. Since it is defined as a time varying sum of a weakly stationary variable, the relative price level is nonstationary.

The nominal exchange rate s_{ijt} may be represented as follows:

$$s_{ijt} = q_{ijt} + \sum_{\tau=1}^{t} (\pi_{j\tau} - \pi_{i\tau}) \tag{5.5}$$

Equation (5.5) clarifies that the nominal exchange rate is equal to the sum of a weakly stationary variable and a nonstationary one and hence it is nonstationary.

In addition, because the real exchange rate is weakly stationary, there exists a cointegrating relation between the nominal exchange rate and the domestic and foreign price levels. When the monetary policy shock is transitory, the nominal exchange rate and the relative price level are integrated of order 1 stochastic processes, whereas the real exchange rate is integrated of order 0: $s_{ijt} \sim I(1)$, $p_{jt} - p_{it} \sim I(1)$ and $q_{ijt} \sim I(0)$. A purchasing power parity (PPP) condition holds between each pair of economies $i, j = 1, \ldots, N$.

When the monetary policy shock is permanent, a similar reasoning implies that the nominal exchange rate and the relative price level are integrated of order 2 and the real exchange rate is integrated of order 1: $s_{ijt} \sim I(2)$, $p_{jt} - p_{it} \sim I(2)$ and $q_{ijt} \sim I(1)$. There exists a cointegrating relation between the nominal exchange rate and the domestic and foreign relative price levels, because their linear combination reduces the order of integration, although a PPP condition does not hold in this case.[13]

Further, it should be reminded that the nominal exchange and relative interest rates are related by the uncovered interest parity condition:

$$\mathbb{E}_t (s_{ijt+1}) - s_{ijt} + i_{it} - i_{jt} = 0 \tag{5.6}$$

The nominal UIP condition (5.6) may be derived, as a log-linear Taylor expan-

[13]For a review of the relations between weak stationarity, integration and cointegration see Harvey (1993).

sion of the consumer optimization problem Euler equations in a neighbourhood of the stationary steady-state, substituting the definition of the real exchange rate in equation (3.4) or (3.6) and simplifying the resulting expressions. Moreover, as for the real exchange rate, after rearranging terms equation (5.6) may be solved with recursive substitutions. The forward solution for the nominal exchange rate is:

$$s_{ijt} = -\sum_{\tau=0}^{+\infty} \mathbb{E}_t \left(i_{jt+\tau} - i_{it+\tau} \right) + \lim_{\tau \to +\infty} \mathbb{E}_t \left(s_{ijt+\tau+1} \right) \tag{5.7}$$

According to equation (5.7) the nominal exchange rate is negatively related to the the sum of expected future relative risk-free rates. We should also notice that, as s_{ijt} is a nonstationary process, the transversality condition is not satisfied and hence the second term in the right hand side of (5.7) is not equal to zero.

Figure 1 shows that, consistently with equation (5.7) and the nominal UIP condition (5.6), when monetary policy persistence is low ($\rho = 0.5$), the nominal exchange rate appreciates on impact following a contemporaneous shock and subsequently converges to a long run equilibrium level. The nominal exchange rate appreciation is respectively equal to 44 and 54 basis points on impact for the Taylor and forward looking rule specifications. The long run appreciation is equal to 13 and 16 basis points in the two cases. The uncovered interest parity condition therefore implies overshooting, as in the traditional Dornbusch (1976) model.

Conversely, when persistence is high ($\rho = 0.75$), the responses of the real exchange and inflation rates imply that the nominal exchange rate converges on impact to the new long run equilibrium level and remains constant thereafter over the simulation horizon. The nominal exchange rate appreciation is equal to 67 and 89 basis points, respectively, in the Taylor and forward looking rule specifications.

5.2 Anticipated shock

Figure 2 shows that a shock anticipated two quarters in advance entails an initial decline of the real relative interest rate between economies j and i, which turns positive in the period the shock is realized and subsequently decreases towards the stationary steady-state. The transitional dynamics is again related to the monetary policy shock persistence. For both the Taylor and forward looking rule

specifications convergence requires roughly 14 or 28 quarters, respectively, when $\rho = 0.5$ or $\rho = 0.75$. Consistently with equation (5.3) and the real UIP condition (3.6), the behaviour of the real relative interest rate implies that the real exchange rate q_{ijt} appreciates in the first quarters until the shock is realized and then gradually returns to the stationary equilibrium. In addition, as predicted by equation (5.4), the relative inflation rate $\pi_{jt} - \pi_{it}$ declines on impact and converges towards the steady-state thereafter. As for the contemporaneous shock case, in both the Taylor and forward looking rule specifications, convergence of the real exchange and relative inflation rates is faster, when the monetary policy shock persistence is lower.

The impulse response function for the nominal relative interest rate between economies j and i conveys additional information. It decreases on impact and the leaning against the wind specification, for both the Taylor and forward looking monetary policy rules, implies that the decline is greater than the relative inflation rate one. When monetary policy persistence is low ($\rho = 0.5$), the relative interest rate decreases on impact by 16 and 17 basis points, respectively, in the Taylor and forward looking rule specifications, following an anticipated 25 basis point shock. In the period the shock is realized it turns positive and equal to 16 and 19 basis points in the two cases. When persistence is high ($\rho = 0.75$), the impact decline is equal to 33 and 40 basis points in the Taylor and forward looking rule models. Moreover, the nominal relative interest rate returns to the stationary steady-state in the quarter the shock is realized and remains constant thereafter.

Finally, when persistence is low ($\rho = 0.5$), the nominal exchange rate appreciates in the initial periods, consistently with equation (5.7) and the nominal UIP condition (5.6), and subsequently converges to its long run equilibrium value. The appreciation in the Taylor and forward looking rule specifications is respectively equal to 63 and 80 basis points in the shock realization period and to 32 and 41 basis points in the equilibrium steady-state. There is overshooting, although the peak response occurs after T quarters in the case of an anticipated shock. When persistence is high ($\rho = 0.75$), the nominal exchange rate appreciates in the initial quarters and reaches the equilibrium value in the shock realization period. The steady-state appreciation is equal to 108 and 146 basis points respectively in the

Taylor and forward looking rule specifications.[14]

6 Statistical properties

In order to provide some further assessment, we calculated volatility statistics for the following variables: the nominal and real relative interest rates $i_{jt} - i_{it}$ and $r_{jt} - r_{it}$, the relative inflation rate $\pi_{jt} - \pi_{it}$ and the real exchange rate q_{ijt}. Tables 1 and 2 report the statistics compiled with the Taylor and forward looking rule models respectively. In each model, we let the monetary policy rule slope parameter α take the values 0.2, 0.5 and 1, the correlation coefficient ρ the values 0.5 and 0.75 and assume the standard deviation of the innovation term is equal to 25 basis points in each economy.[15]

Tables 1 and 2 show that the volatility of the model stationary variables decreases with the monetary policy slope parameter α, in both the low and high persistence specifications ($\rho = 0.5$ and $\rho = 0.75$). This finding illustrates an important property of the leaning against the wind specification for the monetary policy rule. Moreover, in both the Taylor and forward looking rule models, for each value of the slope parameter α the volatility of the real exchange and relative inflation rates is greater, when persistence is high. In addition, for $\rho = 0.75$ the nominal relative interest rate has zero volatility. Finally, for each value of the slope and persistence parameters α and ρ, the volatility statistics in the Taylor rule model are lower than those compiled for the forward looking rule one.

The above described patterns are broadly consistent with the impulse response functions for a contemporaneous monetary policy shock depicted in Figure 1.

[14]While the nominal and real exchange rate dynamics is consistent with the corresponding UIP conditions in the foreign exchange market, the decrease of nominal and real relative interest rates entails an expansionary effect of the contractionary monetary policy in the quarters before the shock realization period. This is a general feature of dynamic stochastic general equilibrium models based on the New Keynesian Phillips curve that has raised serious criticism on their foundations, as summarized for all in Mankiw (2001).

[15]We restricted the computations to the model variables that are stationary in the neighbourhood of the symmetric steady-state. The volatility statistics calculations are described in appendix D.

7 Concluding remarks

We reviewed the implications of a dynamic stochastic open economy model for the study of exchange rate dynamics, drawing on recent advances in international economics. Our choices generalize former approaches along several dimensions.

In the symmetric multi-country economy, each country produces a differentiated product variety, which is traded in domestic and foreign good markets. The consumer intertemporal utility function takes a CES form, with separate inter- and intratemporal elasticities of substitution, and preferences over the differentiated product varieties produced in each country are represented with the SDS aggregator. There is monopolistic competition in each market and nominal rigidities are modelled assuming staggered price setting with dominant currency pricing. In addition, we assume a competitive labour market.

We clarified that our results are robust to more general preference specifications, as multi-stage formulations with an elasticity of substitution between the aggregate consumption of the goods produced in any two economies different from the one between the differentiated varieties produced in each economy, forms for the differentiated product variety aggregator that imply a countercyclical mark-up and to the assumption of monopolistic power and staggered wage setting in labour markets.

Most importantly, the main findings regarding the relation between PPP and UIP conditions in good and foreign exchange markets hold both with complete and incomplete financial markets.

We assume monetary policy follows a strict inflation targeting rule in each economy and present results holding with either a Taylor or a forward looking rule specification. The assumption of strict inflation targeting allows us to reduce the multi-country model to a two equation linear dynamical system, with the real exchange and relative inflation rates between any two economies as endogenous variables. A well-defined calibration of the monetary policy, preference and price-setting parameters implies that the linear rational expectations model fulfils the stability condition, required for the existence of a forward solution. We restrict the analysis to the fundamental solution, which rules out rational speculative bubbles.

In the dynamic linear rational expectations model the stochastic properties of

the nominal exchange rate follow from the monetary policy shocks. When monetary policy shocks are transitory, the nominal exchange rate and the relative price level between any two economies are nonstationary $I(1)$ processes, whereas the real exchange and relative inflation rates are weakly stationary, or $I(0)$. Therefore, there exists a cointegrating relation between the nominal exchange rate and the price levels and a purchasing power parity condition holds for any pair of domestic and foreign economies. When monetary policy shocks are permanent, the nominal exchange rate and the relative price level are $I(2)$ processes, while the real exchange and relative inflation rates are $I(1)$. Although there exists a cointegrating relation between the nominal exchange rate and the price levels, the purchasing power parity condition does not hold, because the real exchange rate is a nonstationary process.

In the case of transitory monetary policy shocks, additional results follow from their persistence. When monetary policy shocks have a first order autoregressive structure and persistence is low ($\rho = 0.5$), there is exchange rate overshooting, in both the Taylor and forward looking rule specifications. Conversely, when persistence is high ($\rho = 0.75$), the nominal exchange rate converges to the steady-state equilibrium on impact following a transitory shock.

Further, the volatility statistics for the model stationary variables suggest that the dynamical patterns described by the model simulations hold for different values of the monetary policy slope parameter, in both the Taylor and the forward looking rule specifications.

Finally, the impulse response functions obtained for an anticipated monetary policy shock illustrate the role of expectations. The results confirm prior findings and suggest further research on the microeconomic foundations of nominal rigidities is required, in order to improve our understanding in this field.

Tables and figures

Table 1: Volatility statistics - Taylor rule

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 1$	
	$\rho = 0.5$	$\rho = 0.75$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 0.5$	$\rho = 0.75$
			Variance			
$i_{jt} - i_{it}$	0.0755	0.0000	0.0645	0.0000	0.0509	0.0000
$\pi_{jt} - \pi_{it}$	0.0124	0.1984	0.0106	0.1270	0.0083	0.0714
$r_{jt} - r_{it}$	0.1092	0.1116	0.0933	0.0714	0.0736	0.0402
q_{ijt}	0.4366	1.7857	0.3731	1.1429	0.2943	0.6429
			Std. dev.			
$i_{jt} - i_{it}$	0.2748	0.0000	0.2540	0.0000	0.2256	0.0000
$\pi_{jt} - \pi_{it}$	0.1112	0.4454	0.1028	0.3563	0.0913	0.2673
$r_{jt} - r_{it}$	0.3304	0.3341	0.3054	0.2673	0.2713	0.2004
q_{ijt}	0.6608	1.3363	0.6109	1.0690	0.5425	0.8018

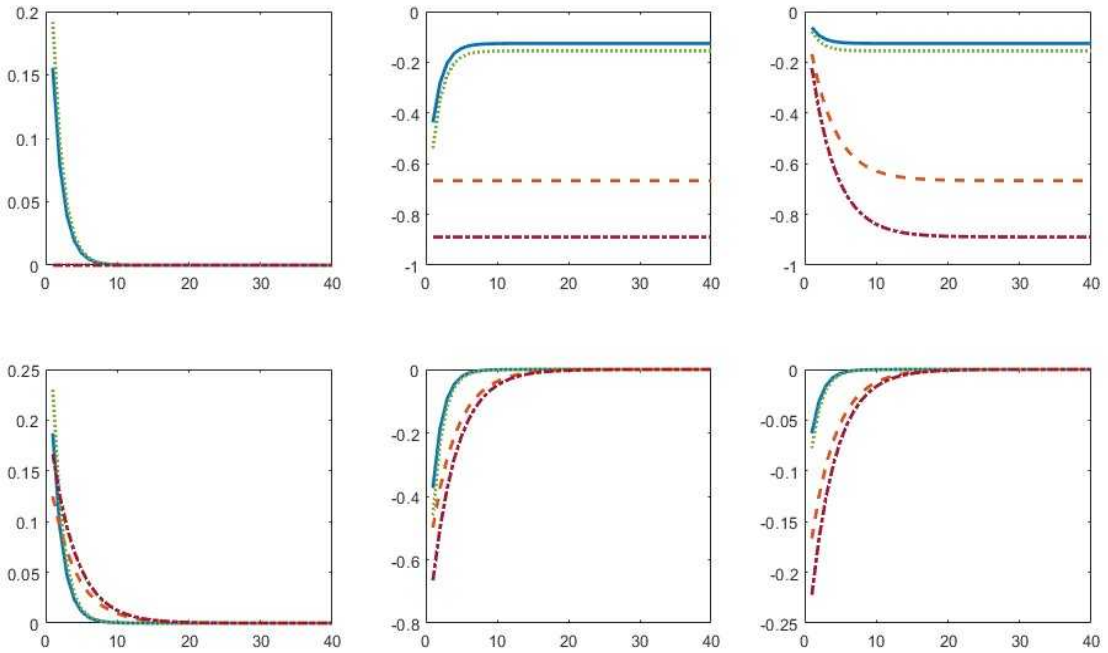
Note: Taylor rule specification: $i_{jt} - i_{it} = (1 + \alpha)(\pi_{jt} - \pi_{it}) + \varepsilon_{jt} - \varepsilon_{it}$. Monetary policy shock distributed as a first order autoregressive process: $\varepsilon_{jt} - \varepsilon_{it} = \rho(\varepsilon_{jt-1} - \varepsilon_{it-1}) + \nu_{jt} - \nu_{it}$. Disturbance term independent and identically distributed with mean zero and variance $2\sigma_\nu^2$: $\nu_{jt} - \nu_{it} \sim i.i.d.\left(0, 2\sigma_\nu^2\right)$ and $\sigma_\nu^2 = 0.0625$.

Table 2: Volatility statistics - Forward looking rule

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 1$	
	$\rho = 0.5$	$\rho = 0.75$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 0.5$	$\rho = 0.75$
			Variance			
$i_{jt} - i_{it}$	0.1079	0.0000	0.0981	0.0000	0.0845	0.0000
$\pi_{jt} - \pi_{it}$	0.0177	0.3527	0.0161	0.2257	0.0138	0.1270
$r_{jt} - r_{it}$	0.1560	0.1984	0.1418	0.1270	0.1221	0.0714
q_{ijt}	0.6239	3.1746	0.5672	2.0317	0.4884	1.1429
			Std. dev.			
$i_{jt} - i_{it}$	0.3285	0.0000	0.3132	0.0000	0.2906	0.0000
$\pi_{jt} - \pi_{it}$	0.1330	0.5939	0.1268	0.4751	0.1176	0.3563
$r_{jt} - r_{it}$	0.3950	0.4454	0.3766	0.3563	0.3494	0.2673
q_{ijt}	0.7899	1.7817	0.7531	1.4254	0.6989	1.0690

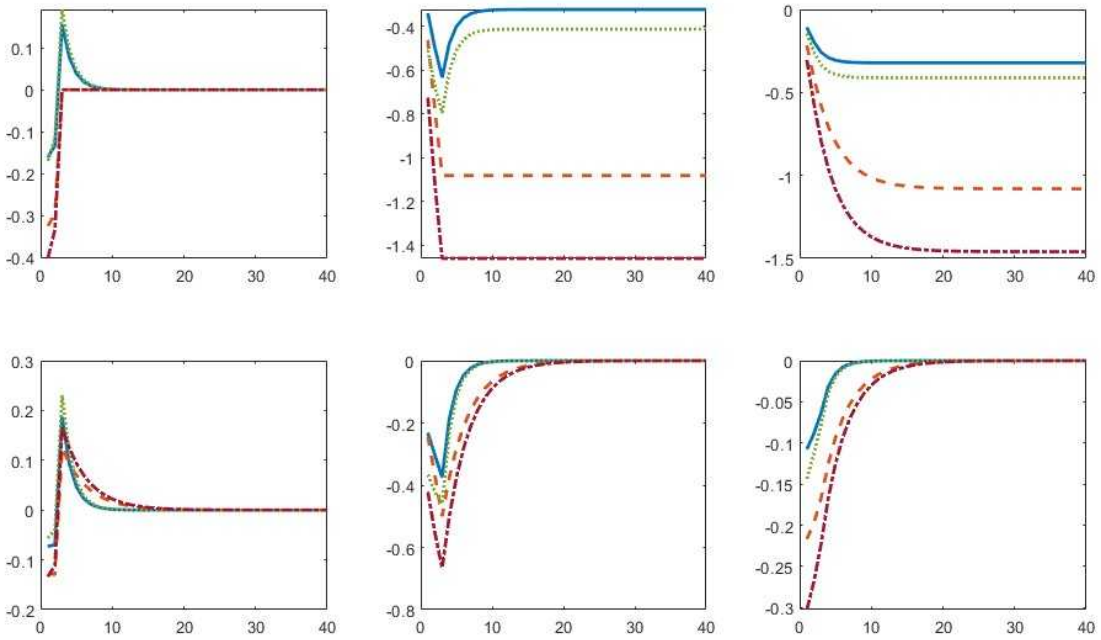
Note: Forward looking rule specification: $i_{jt} - i_{it} = (1+\alpha)\mathbb{E}_t\left(\pi_{jt+1} - \pi_{it+1}\right) + \varepsilon_{jt} - \varepsilon_{it}$. Monetary policy shock distributed as a first order autoregressive process: $\varepsilon_{jt} - \varepsilon_{it} = \rho\left(\varepsilon_{jt-1} - \varepsilon_{it-1}\right) + \nu_{jt} - \nu_{it}$. Disturbance term independent and identically distributed with mean zero and variance $2\sigma_\nu^2$: $\nu_{jt} - \nu_{it} \sim i.i.d.\left(0, 2\sigma_\nu^2\right)$ and $\sigma_\nu^2 = 0.0625$.

Figure 1: Impulse response functions - Contemporaneous monetary policy shock



Note: Monetary policy shock distributed as a first order autoregressive process: $\varepsilon_{jt} - \varepsilon_{it} = \rho\left(\varepsilon_{jt-1} - \varepsilon_{it-1}\right) + \nu_{jt} - \nu_{it}$. Disturbance term independent and identically distributed with mean zero and variance $2\sigma_\nu^2$: $\nu_{jt} - \nu_{it} \sim i.i.d.\left(0, 2\sigma_\nu^2\right)$. The bold and dotted lines respectively report the impulse responses to a 25 basis point shock for the Taylor and forward looking rule specifications when $\rho = 0.5$, the dashed and dashed-dotted lines the corresponding ones when $\rho = 0.75$. Responses of the nominal relative interest rate $i_{jt} - i_{it}$, the nominal exchange rate s_{ijt} and the relative price level $p_{jt} - p_{it}$ in the first row, from left to right, and of the real relative interest rate $r_{jt} - r_{it}$, the real exchange rate q_{ijt} and the relative inflation rate $\pi_{jt} - \pi_{it}$ in the second one.

Figure 2: Impulse response functions - Anticipated monetary policy shock



Note: Monetary policy shock distributed as a first order autoregressive process: $\varepsilon_{jt} - \varepsilon_{it} = \rho\left(\varepsilon_{jt-1} - \varepsilon_{it-1}\right) + \nu_{jt} - \nu_{it}$. Disturbance term independent and identically distributed with mean zero and variance $2\sigma_\nu^2$: $\nu_{jt} - \nu_{it} \sim i.i.d.\left(0, 2\sigma_\nu^2\right)$. The bold and dotted lines respectively report the impulse responses to a 25 basis point shock for the Taylor and forward looking rule specifications when $\rho = 0.5$, the dashed and dashed-dotted lines the corresponding ones when $\rho = 0.75$. Responses of the nominal relative interest rate $i_{jt} - i_{it}$, the nominal exchange rate s_{ijt} and the relative price level $p_{jt} - p_{it}$ in the first row, from left to right, and of the real relative interest rate $r_{jt} - r_{it}$, the real exchange rate q_{ijt} and the relative inflation rate $\pi_{jt} - \pi_{it}$ in the second one. Monetary policy shock anticipated two quarters.

A Symmetric steady-state equilibrium

In order to obtain the symmetric steady-state aggregate consumption and labour supply, substitute the equilibrium wage rate and aggregate price level in the marginal condition (2.9) and rearrange terms to yield:

$$\overline{C} = \left(\frac{\kappa\phi}{\phi - 1}\right)^{-\sigma} \overline{L}^{-\frac{\sigma}{\eta}} \tag{A.1}$$

Substituting (A.1) in the consumer budget constraint (2.5) and applying the definition of the real gross profit function (2.10) we obtain the following:

$$\frac{\phi}{\phi - 1}\left(\frac{\kappa\phi}{\phi - 1}\right)^{-\sigma} \overline{L}^{-\frac{\sigma}{\eta}} = \overline{L} + \left(\frac{\phi}{\phi - 1} - 1\right)\overline{L} \tag{A.2}$$

The required results follow from equations (A.1) and (A.2).

B Sectoral New Keynesian Phillips curve

The derivation of the sectoral New Keynesian Phillips curve is relatively standard. Let $I^{\perp} \subset [0,1]$ and $I^{\top} = [0,1] \setminus I^{\perp} \subset [0,1]$ respectively represent the subsets of firms that reset and do not reset price in country i and period t and note that $I^{\perp} \cap I^{\top} = \varnothing$ and $I^{\perp} \cup I^{\top} = [0,1]$. The price index for the differentiated product varieties of each country i may be represented as $P_{ijt} = \left[\int_{I^{\top}} P_{ijt-1}(\omega)^{1-\phi}\, d\omega + \int_{I^{\perp}} P_{ijt}(\omega)^{1-\phi}\, d\omega\right]^{1/(1-\phi)}$, where we separate between the firms that reset or do not reset price in period t. For firms that do not reset at time t the price is equal to the period $t-1$ price and, because in the Calvo (1983) staggered price setting equilibrium the probability of price setting is independent across firms and equal to $1-\delta$ in each time period, the price distribution for the fraction δ of firms that do not reset price does not change between periods $t-1$ and t. Moreover, we denote with P^{*}_{ijt} the optimal reset price for each firm that resets price in period t, converted in the local currency j using the law of one price. Therefore, the differentiated product varieties price index P_{ijt} takes the form:

$$P_{ijt} = \left[\delta P_{ijt-1}^{1-\phi} + (1-\delta)\, P_{ijt}^{*1-\phi}\right]^{\frac{1}{1-\phi}} \tag{B.1}$$

for each $i, j = 1, \ldots, N$.

A log-linear Taylor expansion of equation (B.1) around the stationary steady-state yields:

$$p_{ijt} = \delta p_{ijt-1} + (1 - \delta)\, p^*_{ijt} \tag{B.2}$$

where $p^*_{ijt} = \log\left(P^*_{ijt}\right)$.

For country i firms, either producing for the domestic market or exporting differentiated product varieties to country j, as stated in the main text the law of one price implies the following recursive equation holds as a log-linear Taylor expansion:

$$p^*_{ijt} = (1 - \delta\beta)\left(\mu + w_{it} - a_{it} - s_{jit}\right) + \delta\beta\mathbb{E}_t\left(p^*_{ijt+1}\right) \tag{B.3}$$

In equation (B.3) we account for the circumstance that the optimal reset price is constant across country i firms $\omega \in I^{\perp}$, for all $i, j = 1, \ldots, N$.

Next, from equation (B.2) we may easily get:

$$\delta\pi_{ijt} = (1 - \delta)\left(p^*_{ijt} - p_{ijt}\right) \tag{B.4}$$

In turn, subtracting the log price index p_{ijt} from both sides of (B.3) we have:

$$p^*_{ijt} - p_{ijt} = (1 - \delta\beta)\left(\mu + w_{it} - a_{it} - s_{jit} - p_{ijt}\right) + \delta\beta\mathbb{E}_t\left(p^*_{ijt+1} - p_{ijt}\right) \tag{B.5}$$

Substituting equation (B.5) in (B.4):

$$\begin{aligned}\delta\pi_{ijt} = (1 - \delta)(1 - \delta\beta)\left(\mu + w_{it} - a_{it} - s_{jit} - p_{ijt}\right) + \\ (1 - \delta)\delta\beta\mathbb{E}_t\left(p^*_{ijt+1} - p_{ijt}\right)\end{aligned} \tag{B.6}$$

Finally, from equation (B.2) it also follows that:

$$\pi_{ijt+1} = (1 - \delta)\left(p^*_{ijt+1} - p_{ijt}\right) \tag{B.7}$$

Substituting (B.7) in (B.6) and rearranging terms yields the Phillips curve equation (3.10).

C Characteristic polynomials and forward solutions

In order to analyse the solutions to the dynamic rational expectations model, for both the Taylor and forward looking rule specifications it is necessary to compute the eigenvalues of the endogenous variable matrix A.

Simple algebra shows that in the Taylor rule specification the characteristic polynomial is equal to $p(\lambda) = \lambda^2 - (1+\beta+\theta)\lambda/\beta + [1+\theta(1+\alpha)]/\beta$, with discriminant $\Delta = \left\{(1+\beta+\theta)^2 - 4\beta[1+\theta(1+\alpha)]\right\}/\beta^2$. We prove that for $\beta \to 1$ and a well-defined specification of the monetary policy rule and probability of price setting parameters α and δ the discriminant is negative. Notice, that $\beta \to 1$ implies that the discriminant $\Delta \to \theta(\theta - 4\alpha)$ and hence we require $\theta < 4\alpha$. In addition, it is easy to show that $\beta \to 1$ implies $\theta \to (1-\delta)^2/\delta$. Thus, suppose $\delta \geq 1/2$, which implies that for each firm the average length of a price spell is greater than or equal to two quarters, then for any value of $\alpha > 1/8$ the discriminant is negative. Next, we show that the stability condition is fulfilled. Actually, with $\Delta < 0$ the characteristic polynomial has two complex conjugate roots, with modulus equal to the square root of $[1+\theta(1+\alpha)]/\beta > 1$ for $\beta < 1$.

In the forward looking rule specification the characteristic polynomial is $p(\lambda) = \lambda^2 - (1+\beta-\alpha\theta)\lambda/\beta + 1/\beta$, with discriminant $\Delta = \left[(1+\beta-\alpha\theta)^2 - 4\beta\right]/\beta^2$. For $\beta \to 1$ the discriminant $\Delta \to \alpha\theta(\alpha\theta - 4)$ and hence we require $\theta < 4/\alpha$. For $\delta \geq 1/2$ and $\alpha < 8$ the latter condition is always satisfied. Moreover, as with $\Delta < 0$ the complex conjugate roots have modulus equal to the squared root of $1/\beta > 1$ for $\beta < 1$, the stability condition is fulfilled.

The forward solution to the dynamical system may be derived with the method of undetermined coefficients as follows. In the case of a contemporaneous transitory shock suppose the solution takes the form:

$$z_t = \sum_{k=0}^{+\infty} \Lambda_k \nu_{t-k} \tag{C.1}$$

Substituting (C.1) and the monetary policy shock term in the stochastic difference equation (4.4) and taking the conditional expectation on the left hand side we obtain the dynamical system:

$$\sum_{k=0}^{+\infty} \Lambda_{k+1} \nu_{t-k} = A \sum_{k=0}^{+\infty} \Lambda_k \nu_{t-k} + \sum_{k=0}^{+\infty} \Psi_k \nu_{t-k} \tag{C.2}$$

Equating the coefficient matrices that multiply each innovation vector ν_{t-k} on the left and right hand sides of (C.2) yields the first order difference equation:

$$\Lambda_{k+1} = A\Lambda_k + \Psi_k \tag{C.3}$$

in the unknown matrix series Λ_k for $k = 0, \ldots, +\infty$.

Multiplying both sides by the inverse matrix A^{-1}, rearranging terms and solving the resulting equation with forward recursive substitution we obtain the required result:

$$\Lambda_k = -\sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau} \tag{C.4}$$

which holds for all $k = 0, \ldots, +\infty$.

In (C.4) we assume the transversality condition $\lim_{\tau \to +\infty} A^{-(\tau+1)} \Lambda_{k+\tau+1} = 0$ is satisfied.

In order to prove that the multivariate stochastic process described by equations (C.1) and (C.4) is well-defined and weakly stationary, notice that the stability condition on the dynamical system (4.4) implies $\sum_{\tau=0}^{+\infty} \left| A^{-(\tau+1)} \right| < +\infty$ and from the absolute summability of the moving average coefficients in the monetary policy shock process (3.2) it follows that $\sum_{\tau=0}^{+\infty} |\Psi_{k+\tau}| \leq \sum_{\tau=0}^{+\infty} |\Psi_\tau| < +\infty$.

The moving average coefficient matrices in equation (C.1) are finite and absolutely summable, since the latter inequalities imply $|\Lambda_k| = \left| \sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau} \right| \leq \sum_{\tau=0}^{+\infty} \left| A^{-(\tau+1)} \right| |\Psi_{k+\tau}| \leq \left(\sum_{\tau=0}^{+\infty} \left| A^{-(\tau+1)} \right| \right) \left(\sum_{\tau=0}^{+\infty} |\Psi_{k+\tau}| \right) < +\infty$ for all k and moreover $\sum_{k=0}^{+\infty} |\Lambda_k| = \sum_{k=0}^{+\infty} \left| \sum_{\tau=0}^{+\infty} A^{-(\tau+1)} \Psi_{k+\tau} \right| \leq \sum_{\tau=0}^{+\infty} \sum_{k=0}^{+\infty} \left| A^{-(\tau+1)} \right| |\Psi_{k+\tau}| \leq$

$\left(\sum_{\tau=0}^{+\infty}\left|A^{-(\tau+1)}\right|\right)\left(\sum_{k=0}^{+\infty}|\Psi_k|\right)<+\infty$.

In addition, for a contemporaneous permanent shock $\Psi_k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for all $k = 0, \ldots, +\infty$. The method of undetermined coefficients yields the solution (C.1) with $\Lambda_k = -A^{-1}\left(I - A^{-1}\right)^{-1}$ for each k.

In the case of a transitory shock anticipated T periods in advance suppose:

$$z_t = \sum_{k=0}^{+\infty} \Lambda_k \nu_{t+T-k} \tag{C.5}$$

Substituting (C.5) and the monetary policy shock process in the stochastic difference equation (4.4) and taking the conditional expectation on the left hand side we obtain the dynamical system:

$$\sum_{k=0}^{+\infty} \Lambda_{k+1} \nu_{t+T-k} = A \sum_{k=0}^{+\infty} \Lambda_k \nu_{t+T-k} + \sum_{k=0}^{+\infty} \Psi_k \nu_{t-k} \tag{C.6}$$

Equating the coefficient matrices that multiply each innovation vector on the left and right hand sides of equation (C.6) we obtain a first order difference equation system, defined by two sets of equalities. The first equality:

$$\Lambda_{k+1} = A\Lambda_k \tag{C.7}$$

must be fulfilled for $k = 0, \ldots, T-1$.

The second one:

$$\Lambda_{k+1} = A\Lambda_k + \Psi_{k-T} \tag{C.8}$$

has to be verified for $k = T, \ldots, +\infty$.

In order to solve the dynamical system, note that it is relatively straightforward to apply an inductive argument and show that equation (C.7) implies:

$$\Lambda_k = A^{-(T-k)}\Lambda_T \tag{C.9}$$

for $k = 0, \ldots, T-1$.

Next, multiplying both sides of equation (C.8) by A^{-1}, rearranging terms and

solving with forward recursive substitutions we obtain the following partial result:

$$\Lambda_k = -\sum_{\tau=0}^{+\infty} A^{-(\tau+1)}\Psi_{k+\tau-T} \tag{C.10}$$

which holds for $k = T, \ldots, +\infty$.

In (C.10) we assume the transversality condition $\lim_{\tau\to+\infty} A^{-(\tau+1)}\Lambda_{k+\tau+1} = 0$ is satisfied.

Finally, substituting equation (C.10) evalutated at lag $k = T$ in (C.9) yields:

$$\Lambda_k = -\sum_{\tau=T-k}^{+\infty} A^{-(\tau+1)}\Psi_{k+\tau-T} \tag{C.11}$$

for $k = 0, \ldots, T-1$.

Notice that $|\Lambda_k| = \left|\sum_{\tau=T-k}^{+\infty} A^{-(\tau+1)}\Psi_{k+\tau-T}\right| \leq \sum_{\tau=T-k}^{+\infty} \left|A^{-(\tau+1)}\right| |\Psi_{k+\tau-T}| \leq \left(\sum_{\tau=T-k}^{+\infty} \left|A^{-(\tau+1)}\right|\right)\left(\sum_{\tau=T-k}^{+\infty} |\Psi_{k+\tau-T}|\right) < +\infty$ for $k = 0, \ldots, T-1$ and the results derived for equations (C.1) and (C.4) imply $|\Lambda_k| < +\infty$ for $k = T, T+1 \ldots, +\infty$ and $\sum_{k=T}^{+\infty} |\Lambda_k| < +\infty$. Therefore, the moving average matrices in (C.5) are finite for all $k = 0, \ldots, +\infty$ and absolutely summable. The multivariate stochastic process defined in equations (C.5), (C.10) and (C.11) is well-defined and weakly stationary.

D Volatility statistics

In order to compile the volatility statistics we need some preliminary definitions. We let $\Sigma = \mathbb{E}\left(\nu_t \nu_t'\right) = \begin{bmatrix} 2\sigma_\nu^2 & 0 \\ 0 & 0 \end{bmatrix}$ denote the variance of the monetary policy innovation vector ν_t, where $\mathbb{E}$ represents the expectation operator. Notice that Σ is a singular matrix, a feature that has to be taken into account in dealing with the endogenous variable and monetary policy shock stochastic processes z_t and ε_t. In addition, we use $z_{t+1}^t = \mathbb{E}_t\left(z_{t+1}\right)$ as a shorthand notation for the conditional expectation of the one period ahead endogenous variable vector.

With the above definitions, equation (4.6) and the assumption that the monetary policy shock term follows a first order autoregressive process in (5.1) imply that the endogenous variable vector variance is equal to:

$$\begin{aligned}\mathbb{E}\left(z_t z_t'\right) &= \sum_{k=0}^{+\infty} \rho^{2k} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma \left(I - \rho A^{-1}\right)^{-1\prime} A^{-1\prime} \\ &= \frac{1}{1-\rho^2} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma \left(I - \rho A^{-1}\right)^{-1\prime} A^{-1\prime}\end{aligned} \tag{D.1}$$

and the autocovariance function is:

$$\begin{aligned}\mathbb{E}\left(z_t z_{t+j}'\right) &= \sum_{k=0}^{+\infty} \rho^{2k+j} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma \left(I - \rho A^{-1}\right)^{-1\prime} A^{-1\prime} \\ &= \frac{\rho^j}{1-\rho^2} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma \left(I - \rho A^{-1}\right)^{-1\prime} A^{-1\prime}\end{aligned} \tag{D.2}$$

for all $j = 0, \ldots, +\infty$.

We should note in passing that equations (D.1) and (D.2) imply $\mathbb{E}\left(z_t z_{t+j}'\right) = \rho^j \mathbb{E}\left(z_t z_t'\right)$. This result follows from the autoregressive structure of the endogenous variable vector process, which is described in the main text by equation (5.2).

Moreover, the covariance between the endogenous variable and monetary policy shock vectors z_t and ε_t is equal to:

$$\begin{aligned}\mathbb{E}\left(z_t \varepsilon_t'\right) &= -\sum_{k=0}^{+\infty} \rho^{2k} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma \\ &= -\frac{1}{1-\rho^2} A^{-1} \left(I - \rho A^{-1}\right)^{-1} \Sigma\end{aligned} \tag{D.3}$$

and the monetary policy shock vector variance is:

$$\mathbb{E}\left(\varepsilon_t \varepsilon_t'\right) = \sum_{k=0}^{+\infty} \rho^{2k} \Sigma = \frac{1}{1-\rho^2} \Sigma \tag{D.4}$$

Further, the variance of the one period ahead endogenous variable vector conditional expectation z_{t+1}^t is:

$$\mathbb{E}\left(z_{t+1}^{t}z_{t+1}^{t\prime}\right)=\sum_{k=1}^{+\infty}\rho^{2k}A^{-1}\left(I-\rho A^{-1}\right)^{-1}\Sigma\left(I-\rho A^{-1}\right)^{-1\prime}A^{-1\prime} \tag{D.5}$$
$$=\frac{\rho^{2}}{1-\rho^{2}}A^{-1}\left(I-\rho A^{-1}\right)^{-1}\Sigma\left(I-\rho A^{-1}\right)^{-1\prime}A^{-1\prime}$$

and its covariance with the monetary policy shock vector is equal to:

$$\mathbb{E}\left(z_{t+1}^{t}\varepsilon_{t}^{\prime}\right)=-\sum_{k=0}^{+\infty}\rho^{2k+1}A^{-1}\left(I-\rho A^{-1}\right)^{-1}\Sigma \tag{D.6}$$
$$=-\frac{\rho}{1-\rho^{2}}A^{-1}\left(I-\rho A^{-1}\right)^{-1}\Sigma$$

For completeness, notice that the autoregressive form of the endogenous variable vector process implies $\mathbb{E}\left(z_{t+1}^{t}z_{t+1}^{t\prime}\right)=\rho^{2}\mathbb{E}\left(z_{t}z_{t}^{\prime}\right)$ and $\mathbb{E}\left(z_{t+1}^{t}\varepsilon_{t}^{\prime}\right)=\rho\mathbb{E}\left(z_{t}\varepsilon_{t}^{\prime}\right)$. In addition, an application of the law of iterated expectations shows that the covariance between the endogenous variable vector and its one period ahead conditional expectation is: $\mathbb{E}\left(z_{t}z_{t+1}^{t\prime}\right)=\mathbb{E}\left[\mathbb{E}_{t}\left(z_{t}z_{t+1}^{\prime}\right)\right]=\mathbb{E}\left(z_{t}z_{t+1}^{\prime}\right)$. The latter quantity may therefore be calculated from equation (D.2).

Finally, the real exchange and relative inflation rate variances are the terms in the main diagonal of matrix (D.1). The nominal and real relative interest rate variances may be compiled from (D.1)-(D.6), applying standard univariate variance and covariance formulas.

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