A Theory of Payments-Chain Crises*

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Abstract

This paper introduces an endogenous network of payments chains into a business cycle model. Motivated by evidence of linked payments in Costa Rica, we present a model where agents order production in bilateral relations. Some payments are executed immediately. Other payments, chained payments, are delayed until other payments are executed. Because production starts only after orders are paid, chained payments induce production delays. In equilibrium, agents choose the amount of chained payments given interest rates and access to internal funds or credit lines. This choice determines the payments-chain network and aggregate total-factor productivity (TFP). The paper characterizes equilibrium dynamics and their innate inefficiencies. Agents internalize the direct costs of their payment delays, but do not internalize the costs induced onto others. This externality produces novel policy insights and rationalizes permanent reductions in TFP under excessive debt.

Keywords: Payments, Networks, Business Cycles

JEL Classification: E32, E42, G01

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1 Introduction

During financial crises, productive resources often remain idle—not because of technical constraints or lack of demand, but because agents must wait for others to pay, who in turn are waiting on others. A simple example illustrates the idea. Consider a general contractor hired to build a commercial property. The contractor relies on a network of subcontractors—electricians, plumbers, and carpenters—who agree to begin or continue work only after receiving partial payment. The contractor also places orders with material suppliers, who may customize some of the materials and are unwilling to ship them until payment is completed or, at minimum, a proof of funds is provided. However, the contractor is waiting for a payment from the project's developer, who is waiting for a loan installment or the proceeds from the sale of another property. During crises, banks may delay loan payments and property sales may slowdown, disrupting the developer's funding sources, and therefore, the entire chain: the contractor cannot pay the electrician or the supplier; the electrician postpones work; the supplier holds off on shipping; and the project stalls. The slowdown is not due to a lack of labor, materials, or technical capacity, but rather to each party waiting to be paid by someone else. These cascading delays can leave productive resources idle, and the national accounts will record the outcome as a decline in productivity.

Despite how natural such payment delays seem, they are rarely featured in macroeconomic models. Production is typically centralized with spot market exchanges and payment frictions abstracted away. While credit constraints are sometimes modeled, the usual assumption is binary: agents are either constrained or unconstrained. What is missing is a tractable framework where economic activity stalls simply because agents must wait for upstream transactions to clear.

This paper develops a theory of payment-chain disruptions and studies their macroeconomic implications. We show theoretically and empirically that payment timing—not just credit availability—can generate substantial productivity losses during financial crises.

Our empirical contribution uses detailed microdata on all interbank transfers in Costa Rica, along with the universe of transaction-level sales receipts. We document that over half of firms engage mainly in chained payments, and that sectors facing longer delays feature less chained payments. Exploiting a natural experiment where bank accounts were suddenly frozen for some firms, we show that payment disrup-

tions lead to real effects on output, and that these effects depend critically on whether the affected firms participated in payment chains.

Guided by this evidence, we introduce a payment-chain production network that explicitly links transaction timing to production timing. In this framework, some orders are paid upfront (spot), while others are chained—paid only after preceding payments clear. As the fraction of chained orders rises, production delays cascade through longer chains, reducing measured total factor productivity according to a tractable formula linking micro-level payment structure to macro-level outcomes.

We embed this network into a business-cycle model with borrowers and savers. When borrowers face limited short-term credit access, they increase chained orders to maintain consumption, not internalizing how this delays other transactions. The model generates three regions: efficient steady states with minimal delays, temporary payment-chain crises that resolve as debt falls, and permanent crises with debt overhang where deleveraging incentives are insufficient.

The framework yields novel policy insights. A Ramsey planner recognizing payment externalities finds both that savers spend too little on spot orders and borrowers spend too much on chained orders during crises. Regarding fiscal policy, government spending has positive multipliers during payments-chain crises—but only if the government makes spot expenditures. If government spending is chained to tax receipts, it proves detrimental. The mechanism works, not through aggregate demand stimulus, but by speeding up payments, offering a monetarist reinterpretation of fiscal multipliers.

Literature Review. The paper connects with theories that underscore the sharp declines in aggregate TFP during crises. The link between financial crisis and TFP is not at all obvious because financial crises can manifest themselves through distorting labor-market outcomes, not productivity. One branch of the literature explains declines in aggregate TFP through increased misallocation—see Pratap and Urrutia (2012) or Oberfield (2013). However, a common finding is that heterogeneity can only partially explain TFP declines once models are disciplined with data on input use and heterogeneity. Here, the channel is utilization. Other studies also explain declines in TFP as aggravated financial conditions increase the cost of utilizing capital—see Meza and Quintin (2007). The mechanism here is different: financial conditions impact TFP through a slowdown of payments.

Beyond the focus on financial crises, the paper falls at the crossroads of several ar-

eas. Namely, the monetary-payments literature, the economic-networks literature, and the literature on aggregate-demand externalities. The issue of how payment instruments affect production is a classic theme: Lucas and Stokey (1987) analyzes a stochastic cash-in-advance economy; Kiyotaki and Wright (1989) studies trade with indivisible tokens; Lagos and Wright (2005) a model with divisible money and explicit trading arrangements.[1] Recent work focuses on how the distribution, and not the instruments per se, affects production—see Lippi et al. (2015), Rocheteau et al. (2016), and Brunnermeier and Sannikov (2017). Like this literature, the distribution of funding affects allocations, and the main distinction is that we focus on delays in sequential payments.

Sequential payments appear in many other studies. The payments-chain network is inspired by the model in Kiyotaki and Moore (1997a). Kiyotaki and Moore (1997a) focus on finding micro-foundations of production disruptions in a single chain. The contribution here is to present a network of transactions whose structure depends on payment decisions. Furthermore, we embed the payments-chain network into a standard business-cycle model. Other models of sequential payments include Townsend (1980), which studies sequences of payments with spatial separation, Freeman (1996) and Green (1999), which study sequential transactions in overlapping generation environments, La'O (2015), which studies a circular flow of transactions, or Guerrieri and Lorenzoni (2009), which studies sequential transactions in a Lagos and Wright-type environment. Recent work by Hardy et al. (2022) and Bocola (2022) contrasts payments funded externally against trade credit.[2] Relative to these papers, there are two distinctions: here, transactions are formed in a network, and the network is endogenous to expenditure-savings decisions. The importance of this body of theoretical work is substantiated by a body of empirical evidence found in a number of recent papers: Boissay and Gropp (2007), Jacobsen (2015), Barrot (2016), and Costello (2020) among others.

With respect to the economic networks literature, the paper connects with models with endogenous network formation. The contribution relative to that literature is modest, as network formation is not strategic. By contrast, in Oberfield (2018), a network is formed through strategic partnerships. In Kopytov et al. (2022) and Elliott et al. (2022) firms form strategic links, being aware of possible supply-chain breakdowns. Here, the network is randomly formed, but the distribution of chains is en-

[1] See Shi (1997); Lagos et al. (2011); Lagos and Rocheteau (2009); Li et al. (2012); Nosal and Rocheteau (2011); Rocheteau (2011) for many other directions in that area.

[2] See also Biais and Gollier (1997).

dogenous to financial decisions. Like in Elliott et al. (2014), Alvarez and Barlevy (2021) and Taschereau-Dumouchel (2022), there are externalities here too. In those models, externalities occur when individual defaults provoke subsequent defaults. Here, externalities occur through payment delays. Bigio and La'O (2013) considers the propagation of financial shocks that induce misallocation in a production network. Here, there is no misallocation in the production network, but there are production delays.

Finally, the paper connects with models of aggregate demand externalities. An early model of these externalities is Diamond (1982) where, via search, consumption decisions affect output. In most of the literature, demand externalities result from nominal rigidities. There has been a recent interest in coupling nominal rigidities with financial constraints—for example, Eggertsson and Krugman (2012) and Guerrieri and Lorenzoni (2017). Recent papers have further introduced sequential transactions into environments with nominal rigidities—for instance, Woodford (2022) and Guerrieri et al. (2022). In those models, demand externalities occur when agents cut back on any form of expenditure. The nature of demand externalities here is different. In particular, the type of expenditures by the private or public sector matters: spot orders may stimulate output, but chained orders depress it. The demand externalities provoked by the slowdown of payments are part of a classic narrative.

2 Empirical Motivation

This section documents empirical evidence on payment chains, their prevalence, and their effects on firm-level productivity.

Data. We leverage two exceptionally rich data sources. The first is Sinpe—Costa Rica's National Electronic Payments System—which processes and settles every interbank transfer.[3] In 2024 alone, Sinpe recorded over 62 million transfers, capturing, for each transaction, the precise timestamp, unique sender and receiver identifiers, the banks involved, and the exact transferred amount. The coverage spans the universe of interbank activity for 243,234 distinct firms, with an average transaction of USD 4,507.[4] The second dataset derives from Costa Rica's VAT (Value-Added Tax) reporting system,

[3] Sinpe is an acronym for Costa Rica's National Electronic Payments System (*Sistema Nacional de Pagos Electrónicos*), and is operated by the Central Bank of Costa Rica.

[4] For more details related to Costa Rica's firm network, see Méndez and Van Patten (2025); Argente et al. (2025). Alvarez et al. (2023) focuses on a particular branch of Sinpe— Sinpe Móvil—which pertains to peer-to-peer mobile money transfers.

and encompasses the electronic invoices for every formal sale. For each invoice, the registry records the sale price, product code, quantity sold, and flags whether payment was settled on the spot or was deferred. Deferred payments are recorded because, until the seller receives full payment, classifying the transaction as deferred allows her to postpone the VAT remittance on the unpaid portion for up to 90 days. For deferred payments, the system also records the contractual repayment window. This feature provides direct, time-stamped insight into short-term financing arrangements and payment timing at a level of granularity rarely available.

Documented Regularities. We exploit the universe of interbank transfers among firms to test for systematic payment–chain behavior. Absent such chains, there should be no systematic relationship between the payment inflows and outflows of a firm. Thus, to empirically detect chained payments, we first examine if firm j is more likely to issue a payment within 24 hours of receiving one. Specifically, we estimate:

$$sent_{jt} = received_{j,t-1} + \lambda_t + \lambda_{j\times d} + \varepsilon_{jt}, \tag{1}$$

where $sent_{jt} = 1$, if firm j made a payment on business day t, and $received_{j,t-1} = 1$ if firm j received a payment in the preceding 24 hours.[5] To prevent this relationship from being purely mechanical, we include both time fixed effects and firm$\times$day-of-week fixed effects.[6]

Table 1 reports the results. Column (1) shows that, on average, firms are significantly more likely to send a payment within 24 hours of receiving a payment. Reassuringly, payments are only triggered when the amount sent is *less* than the amount received, as shown in Table A.1. Columns (2)–(4) reveal heterogeneity: the payment–chain effect is strongest among small firms (5 workers or less) and attenuated for firms with higher asset-to-sales ratios or greater loan access relative to sales.

Columns (1)–(4) focus on a one-day window to approximate chained-payment responses, as a narrow time frame strengthens the inference of a direct link between inflows and outflows. To trace the temporal profile of this effect, we implement a local

[5] Payments received on weekends or holidays are assigned to the next business day.

[6] Time fixed effects capture calendar-driven patterns—e.g., firms that always pay or receive on month-end or the 15th. Firm × day-of-week fixed effects absorb idiosyncratic weekly rhythms—e.g., some firms systematically pay on Thursdays, others on Fridays—thereby isolating only the abnormal deviations linked to incoming payments. The latter fixed effect would also discipline variation, for instance, from a firm that tends to make payments every day, and instead would force the variation to come from out-of-the-ordinary responses correlated with receiving a payment.

Table 1: Triggered Payments Across Firms

Dependent variable: Probability that firm j makes a payment at time t

	(1)	(2)	(3)	(4)	(5)	(6)
$received_{j,t-1}$	0.030	0.010	0.039	0.038		0.030
	(0.0005)***	(0.0008)***	(0.0008)***	(0.0008)***		(0.0005)***
$Small_j \times received_{j,t-1}$		0.017				
		(0.001)***				
$\frac{Assets_j}{Sales_j} \times received_{j,t-1}$			-0.019			
			(0.001)***			
$\frac{Liabilities_j}{Sales_j} \times received_{j,t-1}$				-0.018		
				(0.001)***		
$received_{j,t-5}$					0.011	0.010
					(0.0004)***	(0.0004)***
Adjusted R^2	0.483	0.446	0.483	0.483	0.483	0.483
Observations	10,001,248	5,290,408	8,399,599	8,399,599	10,001,248	10,001,248
Clusters	222,306	74,566	174,896	174,896	222,306	222,306
$t, j \times$ d FE	Yes	Yes	Yes	Yes	Yes	Yes

Notes: The independent variable in column (1) is an indicator that equals one if firm j received a payment from another firm in the last 24 hours. This variable is interacted with firm j's size, assets-to-sales, and liabilities-to-sales in columns (2), (3), and (4), respectively. Column (5)'s independent variable is an indicator that equals one if firm j received a payment within 5 days of time t. Column (6) includes both indicators described above. Robust standard errors, adjusted for clustering by firm, are in parentheses. We include time and firm×weekday fixed-effects. Data is daily and spans 2024.

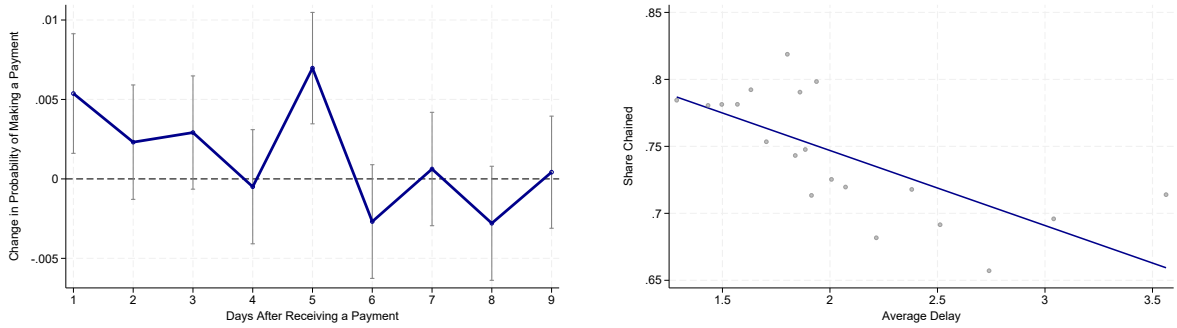
projection (Jordà, 2005; Jordà and Taylor, 2025). Panel (a) of Figure 1 shows that, while payments are triggered within one business day, there is also a second peak around five business days (one week) after receipt of funds. Moreover, coefficients in columns (1), (5), and (6) are remarkably stable, suggesting that payments unfold in two orthogonal waves—an immediate response and a delayed one around business day five.[7]

This two-wave structure reflects both the speed of Costa Rica's modern payment infrastructure and residual frictions. In 2024, when Sinpe processes most transfers electronically, same-day responses are technologically feasible.[8] However, even with a rapid payments network in place, a subset of firms continues to rely on paper checks, which take several business days to clear. This check-clearance lag produces the pronounced second spike at business day five. Consistent with this mechanism, firms exhibiting the five-day payment lag are roughly 60% more likely to pay by check than the

[7] We also examine the intensive margin in Table A.2, which yields results consistent with the extensive-margin analysis: firms are more likely to make payments once they receive transfers.

[8] By contrast, in the 1990s, most payments settled by check, and checks used to require up to 15 days to clear—so that the fastest possible response was about two weeks (Cerdas and Melegatti, 2014).

rest. When we exclude these check-using firms, the day-five peak in Panel (a) of Figure 1 disappears, confirming that check-clearing times underlie the longer delays.



(a) Triggered Payments: Local Projection (b) Chained Payments vs. Average Delays

Figure 1: Dynamics of Chained Payments

Notes: Panel (a) shows a local projection (Jordà, 2005; Jordà and Taylor, 2025), where the dependent and independent variables have 13 lags, as determined by a VAR-based optimal lag selection criteria. Panel (b) shows the average share of firms classified as chained by sector against the average payment delay in the sector. Sectors correspond with ISIC sections. Observations are weighted by sales. Data is monthly and spans 2024.

Placebo Exercises Relations are not mechanical. We conduct placebo tests, where we change the timing so that the independent variable represents payments received h days *after* making a payment, with $h \in \{1, 2, 3\}$. As shown in Table A.3, results are consistently negative or insignificant, speaking against spurious temporal correlations.

Constructing Payments-Chains. Leveraging the comprehensive ACH and wire transfers data, we aim to identify firms where the receipt of a payment triggers the issuance of subsequent payments. Our goal is to classify firms into three distinct groups: (i) those that tend to make chained payments without delay; (ii) those that tend to make chained payments with delay; and (iii) those that tend to make spot payments.

We proceed in two stages. First, to isolate firms chained without delay (type (i)), we compute the monthly frequency with which receiving any payment leads to issuing a payment *within one business day* for each firm. As shown in Panel (a) of Figure A.1, this distribution has large masses of firms concentrated in a value of one (firms with perfect same-day response rates) and in a value of zero (firms with no same-day responses). Given this bimodal distribution, we use k-means clustering to split the sample into two groups: firms type (i) and other firms.

Second, among those firms that were *not* classified as type (i), and informed by the local projection in Panel (a) of Figure 1, we consider the distribution of monthly oc-

currences in which receiving a payment led a firm to make a payment five days later. This distribution, shown in Panel (b) of Figure A.1, shows that relatively few firms are chained with delay and allows us to further split this subsample into two groups: firms concentrated in zero, which correspond with type (iii)—spot, for whom payments sent are uncorrelated with payments received, and a few firms which are concentrated in one and are therefore type (ii)—chained with delay. This two-stage clustering yields a firm-month classification into three types; although firm types are highly persistent, our approach allows for type-switching from month to month.[9] Once firms are classified, we construct payments-chain networks.

Stylized Facts on Payments-Chains Several stylized facts emerge. First, on a given month, 39% of firms tend to make spot payments, while the remaining 61% of firms engage mainly in chained payments. Among the chained firms, 80% make payments without delay and 20% make payments with delay.[10]

In other words, chained firms are prevalent in the economy, a prevalence likely driven by the generally short delays most firms incur. If payment delays raise the cost of participating in a chained network, it is natural to ask whether longer delays deter firms from becoming chained. To explore this, Panel (b) of Figure 1 plots each sector's share of chained firms against its average payment delay. For instance, sectors in which firms tend to rely more on paper checks would have a longer average delay. The figure reveals a strong, negative relationship: sectors with longer delays host proportionally fewer chained participants.[11]

A second observation is that interactions across firm types are common. As shown in panel (a) of Figure 2, 26% of all links in the payments network are "mixed," involving both a spot and a chained firm.[12] Third, if we focus on transactions between same-type firms, network structures differ significantly. On the one hand, Panel (b) of Figure 2 shows the network for transactions between firms classified as chained, blue (green)

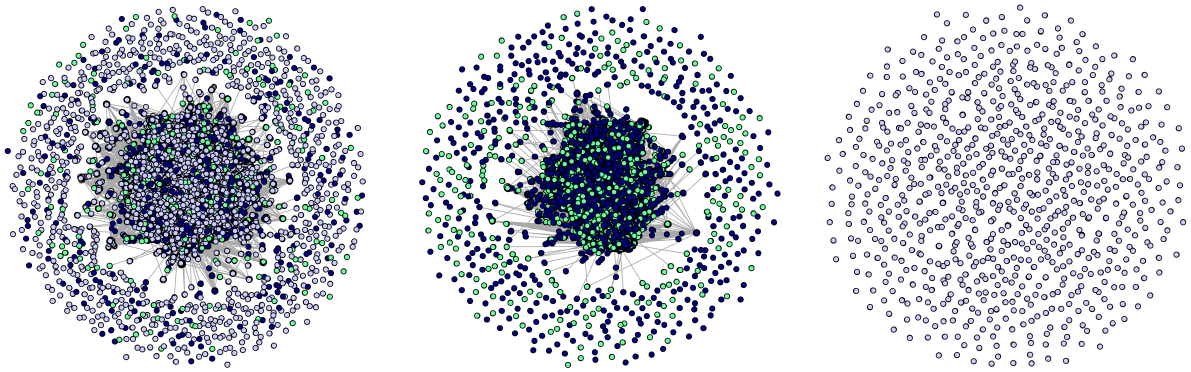
[9] As an alternative, one can estimate each firm's fitted probability of triggered payments from the specification in column (6) of Table 1 and then apply k-means clustering with $k = 3$. The resulting spot vs. chained classification highly correlates with our baseline method, with a correlation of 0.89.

[10] The k-means clustering partitions firms at a fitted probability threshold of 0.42 when considering chained without delay vs. other firms, and of 0.15 when considering chained with delay vs. spot firms.

[11] Panel (b) of Figure 1 categorizes sectors by their one-digit ISIC sections. These broad groupings are chosen deliberately to have an input-output matrix with strongly dominant diagonals.

[12] This representation is based on December 2024. However, these shares are very stable: across all months in 2024, these shares are statistically equal and vary less than one percentage point from month to month, as shown in Table A.4.

dots representing firms chained without (with) delay; about 70% of all links. These links are particularly relevant as consecutive chained links ultimately form the payment chains. The network structure shown is a dense core with isolated nodes. The core cluster represents a critical mass of firms in tight transactional relationships and a circular dependency loop. Firms in this network's core are interdependent, which would relate to high systemic risk: if one core firm delays or defaults, it could ripple through the network. Finally, Panel (c) shows the transactions among firms classified as spot.[13] The network structure is more diffuse and decentralized.



(a) All (b) Chained-to-chained (c) Spot-to-spot

Figure 2: Network Structure by Firm Type

Notes: The figures show the network structure by link type. Panel (a) shows all links between firms. Panel (b) shows links between firms classified as chained. Panel (c) focuses on links between firms classified as spot. Data is collapsed for December 2024 to construct the network representation.

With firm classifications in hand, we construct payment chains as network paths that begin and end at firms making spot payments, with chained firms potentially occupying the intermediate nodes. For each pair of spot firms, we define the chain's length as the number of links (chained firms) separating them.[14] Figure A.2 displays the distribution of chain sizes with an average chain of 5 links.[15]

We further exploit the VAT registry of electronic invoices—linked at the firm level to Sinpe transfers—to observe prices and timing of payment—immediate or deferred,

[13] These circles look less numerous than in Panel (a), as spot firms trade mostly with chained firms.

[14] To avoid double counting of overlapping fragments and to maintain computational feasibility at a monthly frequency, we measure chain length by the shortest path between two spot firms using Dijkstra's algorithm.

[15] This figure corresponds with the distribution for December 2024. Results are nearly identical in other months, exhibiting only minimal month-to-month variation.

and if deferred, for how long—for firms classified as either chained or spot. Because sellers must issue an electronic invoice upon receipt of any partial payment, deferred transactions in the VAT system naturally capture chained-payment behavior. Indeed, firms classified as chained are 26% more likely to engage in deferred payments than spot-payment firms, with an average deferral of 34 days. Leveraging this rare level of detail, we find that, for the same product, invoice prices are 21% higher when payment is deferred, and that this premium is even larger when the purchasing firm is one that regularly makes chained payments. Appendix A.5 presents details of these results.

We also leverage daily data on deferred payments and itemized invoices to study how payment timing affects product delivery. Focusing on retail firms—where the items sold correspond exactly to the items purchased without any in-house transformation —allows us to match each input purchase directly to its downstream sale.[16] For these firms, we isolate new-product transactions with deferred payment and ask: does the retailer start selling the product as soon as the invoice is issued (potentially when receiving a partial payment), or only once the full payment arrives (recall the average deferral is 34 days)? In roughly 70% of cases, sales of the product start only *after* the upstream seller has received payment in full. The timing of such transactions is shown in Figure A.3—an event study centered on the date when *full* payment is due, where the plotted coefficients measure store-level sales (our proxy for delivery).

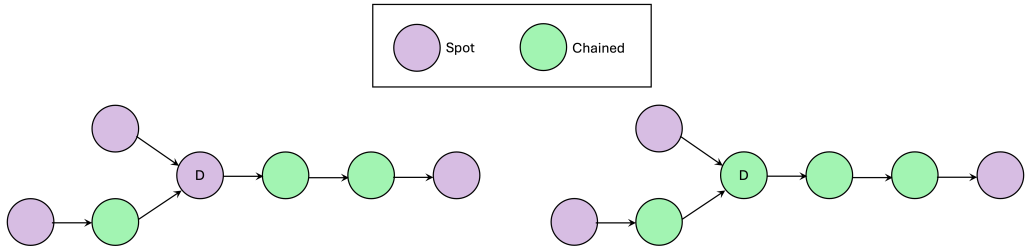
A Payments-Chain Crisis. A payments-chain crisis may arise when shocks to short-term liquidity propagate through chained transactions and impair downstream payments. We exploit a natural experiment to assess the empirical relevance of such crises.

In August 2024, Costa Rica's *Financiera Desyfin*—a financial institution catering mainly to enterprises and business clients—was abruptly and unexpectedly intervened by the financial regulator, who immediately froze all customer accounts due to insolvency concerns. The regulator then declared the institution unviable and placed it into a resolution process under the Organic Law of the Central Bank of Costa Rica, leaving those funds inaccessible for several months.

Using Sinpe data, we can identify the directly shocked firms and trace how disruptions ripple through their surrounding payment networks. To build intuition, we will provide an example. Consider the pre-freeze network in panel (a) of Figure 3, in which arrows represent flows of payments. Firm D is classified as a spot payer, issuing trans-

[16] For example, an automaker buys tires and parts but transforms them into cars, so input and output dates don't align neatly.

fers to chained firms downstream (to its right in the figure) with no systematic link between payments it received and sent. After the Desyfin freeze, however, firm D's outflows started happening only after receiving a payment, so that it "switched" to being classified as chained, as depicted in panel (b).



(a) Chain prior to accounts' freeze (b) Chain after accounts' freeze

Figure 3: Example: Firm D with a Frozen Account "Switches" from Spot to Chained

Notes: The figures show an example of how the accounts' sudden freeze might have impacted a chain in which one, formerly spot, firm "switched" to being classified as chained. Arrows represent payment flows, so circles positioned farther to the right correspond to firms that are further downstream.

With this example in mind, let $Desyfin_i = 1$ if firm i maintained an account at Financiera Desyfin during 2024, and let $Shock_t = 1$ for $t \geq$ August 2024, when the freeze occurred. We also define $Switch2chained_{it} = 1$ if firm i changes from being classified as spot to being classified as chained in month t.[17]

We first explore if firms which had an account at Desyfin become more likely to "switch" to being classified as chained after the freeze. Column (1) of Table 2 reports that this occurs; the probability of switching from spot to chained increased by 0.016 pp—equivalent to a 9% rise relative to the baseline.[18] The estimation has province×industry and time fixed effects, and is therefore not driven by regional or sectoral composition or aggregate shocks that year.

Next, we examine how a "switch" to chained status impacts a directly shocked firm (D, in our example above). Namely, we consider the indicator $Switch2chained_{it}$ as an independent variable and study the behavior of a firm i that had an account at Desyfin *and* switched to being classified as chained. We find that firm i made 18% fewer purchases after the shock, which would align with a hindered ability to make payments, as shown in column (2) of Table 2. The same column shows that the term $Desyfin_i \times$

[17] Recall that the classification of firms by type is month-specific and allows for type-switching across time. While the classification is flexible, firm types tend to be highly persistent.

[18] This result is also reassuring in that it validates our classification measure as capturing meaningful payment behavior.

$Shock_t$ has, on its own, a positive coefficient, consistent with positive pre-shock selection into Desyfin, which catered to corporate clients. In line with fewer purchases, column (3) shows sales fell, and column (4) confirms a fall in value added of about 20%.

We finally ask how chained suppliers fare when selling to a directly shocked firm. Define $Selling2Desyfin_{jt}$ as an indicator equal to one if chained firm j was *selling to* a directly shocked "Desyfin" firm. Panel (B) of Table 2 shows an 18% reduction in purchases after the freeze, along with a fall in sales and value added of a similar magnitude (columns 5-7), *but only when selling to a firm that switched to chained payments and held a Desyfin account.* These results underscore how payment chains can undermine economic resilience. Just as a project developer's lack of funding can trigger cascading delays for the general contractor, subcontractors, and so on, a shock—such as a bank account freeze—that disrupts an initial spot payment can cause delays and hinder production for firms down the payments chain.

This effect on "one-link away" firms is persistent. While none of these firms was directly hit by a shock, Figure 4 shows that the effect remains at least four months after the shock occurred—the figure again focuses on firms selling to a Desyfin firm *which switched* to being classified as chained; i.e., we plot the interaction $Selling2Desyfin_{jt} \times Switch2chained_{it} \times Desyfin_i \times Shock_t$. The effect is also persistent along a chain. As documented in Figure A.6, the effect propagates at least 3 links upstream from a directly hit Desyfin firm that switched to chained.[19]

3 Payments-Chains and Productivity

Motivated by the empirical evidence, this section presents a framework to map a set of income and expenditures in the economy into a payments-chain network. When transaction delays entail an economic loss, the payments-chain network translates into TFP losses. We then embed this module into a dynamic business-cycle model.

Bilateral Relations. Consider an economy where production is organized through bilateral relations where each customer orders a single unit of goods from a supplier owning a single input. As in our introduction, we can think of these inputs as construction materials or services and, more generally, as specially engineered products

[19]The sample size becomes smaller as we increase the number of links of separation, because fewer observations feature chains that long (see Figure A.2). Thus, results beyond 3 links are too noisy to be conclusive.

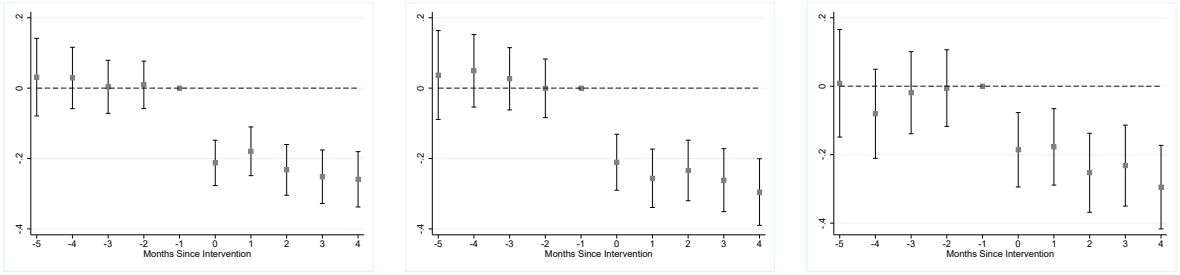
Table 2: A Payments-Chains Crisis: Frozen Bank Accounts

A. Directly Shocked Firm i				
	$Switch2chained_{it}$	ln $Purchases_{it}$	ln $Sales_{it}$	ln $Value\ Added_{it}$
	(1)	(2)	(3)	(4)
$Switch2chained_{it} \times Desyfin_i \times Shock_t$		-0.176	-0.233	-0.197
		(0.064)***	(0.086)***	(0.109)***
$Switch2chained_{it} \times Shock_t$		-0.004	0.019	-0.017
		(0.577)	(0.010)**	(0.012)
$Desyfin_i \times Shock_t$	0.016	0.159	0.195	0.204
	(0.007)***	(0.048)***	(0.037)***	(0.048)***
%Δ w.r.t. mean [mean dep. var]	9% [0.18]			
Observations	1,200,404	1,192,436	939,984	636,485
Province×industry, t FE	Yes	Yes	Yes	Yes
B. Firm j Selling to a Directly Shocked Firm i				
		ln $Purchases_{jt}$	ln $Sales_{jt}$	ln $Value\ Added_{jt}$
		(5)	(6)	(7)
$Selling2Desyfin_{jt} \times Switch2chained_{it}$		-0.176	-0.189	-0.130
$\times Desyfin_i \times Shock_t$		(0.029)***	(0.033)***	(0.038)***
$Switch2chained_{it} \times Shock_t$		0.038	0.017	0.000
		(0.016)**	(0.019)	(0.023)
$Selling2Desyfin_{jt} \times Switch2chained_{it}$		0.651	0.669	0.661
		(0.040)***	(0.044)***	(0.047)***
$Desyfin_i \times Shock_t$		0.035	0.027	0.000
		(0.016)**	(0.019)	(0.023)
Observations		704,087	555,098	382,319
Province×industry and t FE		Yes	Yes	Yes

Notes: The dependent variable in column (1) is an indicator equal to 1 if firm i switched from being classified as a spot firm to a chained firm in period t. Columns (2), (3), and (4) investigate the impact of the freeze on firms who held accounts at Desyfin, before and after the freeze, differentiating between impacts if the firm switched to being classified as chained or not. Columns (5), (6), and (7) instead focus on firms that were selling to a firm directly connected to Desyfin, before and after the shock. Robust standard errors, clustered by firm, are in parentheses. We include province×industry (ISIC-4) and time fixed-effects. Data is monthly and spans 2024.

or a commissioned piece of art. For simplicity, here we assume that all input goods are identical and represent the numeraire. Furthermore, we assume that payments are identical across orders, a condition we explain in the following section.

There are two types of orders: *spot* and *chained* orders. Spot orders are paid immediately. Chained orders are contingent on previous payments; the customer pays for his order contingent on receiving payment from another transaction where he is the supplier. This definition is in line with the one in our empirical exploration in Section 2. There are N^s and N^x spot and chained orders. The sum of orders equals the amount



(a) $\ln Purchases_{jt}$ (b) $\ln Sales_{jt}$ (c) $\ln Value\ Added_{jt}$

Figure 4: Persistent Effect Along the Chain

Notes: The figure shows an event study where time zero corresponds with the month when Desyfin accounts were frozen. The dependent variables in panels (a), (b), and (c) are the (log) of purchases, of sales, and of value added, respectively, of firm j in month t. The plotted coefficients correspond with the triple interaction of $Selling2Desyfin_{jt} \times Switch2chained_{it} \times Desyfin_i \times Shock_t$, and therefore captures the impact of being one link upstream from a Desyfin firm *and* switching to being classified as chained *after* the freeze. Robust standard errors are clustered by firm. We include province×industry (ISIC-4) and time fixed-effects. Data is monthly and spans 2024.

N of input goods, $N = N^s + N^x$, a notion of market-clearing that will emerge in equilibrium in the next section. We also provide an interpretation for spot and chained transactions once we introduce financial decisions in that section.

Each input unit is assigned an identifier, $i \in \mathcal{N} = \{1, 2, \ldots, N\}$. Likewise, each order is assigned a unique identifier, $i \in \mathcal{N}$. We partition $\mathcal{N}$ into two sets, $\mathcal{N}^s$ and $\mathcal{N}^x$, denoting the set of identifiers of spot and chained orders, respectively.[20]

We define two relations that together determine a payments-chain network. First, $\mathcal{P} : \mathcal{N} \to \mathcal{N}$ is a one-to-one assignment from an order to an input unit, where by assumption $i \neq \mathcal{P}(i)$. $\mathcal{P}(i)$ indicates the input unit that fulfills order i. Second, a chained income-expenditure relation associates a chained order with an input unit whose sale is the source of funds to pay for the order. This relation is the identity function defined on $\mathcal{N}^x$. That is, order i uses the payments to input unit i if $i \in \mathcal{N}^x$, as a source of funds to execute order i. The idea is that although order i is not externally funded, the customer who places order i owns input unit i. Hence, order i can be funded after order $j = \mathcal{P}^{-1}(i)$ executes a payment. The assumption that the income-expenditure relation is the identity is without loss of generality.[21]

We do not initially impose restrictions on the assignment $\mathcal{P}$. Different specifications

[20] $\mathcal{N}^s$ and $\mathcal{N}^x$ are a partition of $\mathcal{N}$: $\mathcal{N}^s \cap \mathcal{N}^x = \emptyset$ and $\mathcal{N}^s \cup \mathcal{N}^x = \mathcal{N}$ with $N^s = |\mathcal{N}^s|$, $N^x = |\mathcal{N}^x|$.

[21] Formally, we can define the chained income-expenditure relation $\mathcal{X}$ as the identity function on $\mathcal{N}^x$, that is $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}^x$ such that $\mathcal{X}(i) = i$. The idea is that $i \in \mathcal{N}^x$ obtains funds from input unit $\mathcal{X}(i) = i$. Indeed, the identity function $\mathcal{X}(i)$ can be replaced by any injective function $\mathcal{X} : \mathcal{N}^x \to \mathcal{N}$ so that input units and associated chained orders do not have the same identifiers. Changing identifiers does not affect the results.

of $\mathcal{P}$ lead to different network structures, which we explore as special cases below. $\mathcal{P}$ can result from a random assignment of orders and inputs or from economic considerations that we explore below.

To understand how these relations induce a payments-chain network, consider input unit j assigned to order i, where $\mathcal{P}(i) = j$. The client placing order i must pay for input j. This creates a payment link from i to j. In turn, if order j is spot, $j \in \mathcal{N}^s$, the funds used to pay for order i are not used in further payments. However, if order j is chained, $j \in \mathcal{N}^x$, the funds are used again to pay for input $k = \mathcal{P}(j)$. In other words, when $j \in \mathcal{N}^x$, there is a flow of payments from i to j, and from j to k. If order k is also chained, the same funds are used to pay input $\mathcal{P}(k)$, and so on. The payments chain continues until a final order in a chain is placed on some input unit i not associated with a chained order. Since every order is paired with an input unit, the economy features an entire network of transactions forming a collection of payment chains. The network captures the economy-wide transactions described in the empirical analysis, albeit simplifying things to make all transactions for the same price and quantities.

As long as there are no shocks, the payments network is inconsequential if there are no delays in transferring funds. Indeed, if every payment was executed immediately, the transfer of funds in the entire network would be instantaneous. In the extreme, a single dollar could finance an entire economy's transactions if passed quickly enough—clearly absurd. Yet, even if transfers are not immediate, the network would be inconsequential if no resources were lost with payment delays. As documented in our empirical analysis, we observe a prevalence of transaction delays, dependence on income flows to execute expenditures, and non-trivial economic losses—the last firm in the average chain was found to suffer a delay of between 5-25 business days a year, equivalent to a delay and productivity loss of between 2% and 10% per year, with longer chains being associated with longer delays. Therefore, when there are delays and economic consequences, an agent's position in a payment chain—–how many transactions separate them from the original source of funds—becomes crucial for determining outcomes. Moreover, given liquidity shocks, we documented sizable economic consequences for those firms that rely on contingent payments. Overall, the payments-chain network determines who and by how much bears those losses.

To formalize the notion of delays and losses, we define an order's **position** n in a payment chain as the number of transactions that must take place before that transaction is executed. Naturally, spot orders occupy position $n = 0$, chained orders re-

ceiving payment from spot orders occupy position $n = 1$, chained orders receiving payment from orders in position $n = 1$ occupy position $n = 2$, and so on. To capture the delays, let $\tau(n)$ denote the time at which payment is executed for position n, with $\tau(0) = 0 < \tau(1) < \tau(2) < \cdots < 1$. Transaction delays can arise due to limited technology to process transactions, like the empirically documented delays due to the reliance on paper checks, or as a protocol to withhold transactions until product inspection or delivery. To capture the economic losses, let y_n denote the amount of final goods received by an order at position n, where $y_0 = 1 > y_1 > y_2 > \cdots > 0$. This degradation can arise from depreciation or reduced production times. We provide examples below.

In what follows, we first analyze the general framework with arbitrary y_n, deriving the distribution of chain lengths and aggregate implications. We then consider two special cases that provide economic microfoundations for specific functional forms of y_n: a random network where frictions generate exponential decay, and an efficient network that minimizes aggregate losses subject to constraints.

Payments-Chain Network. Given the assignment $\mathcal{P}$ and the partition of orders into spot and chained, we can construct a network of payment chains as follows: each payment chain starts from a spot order, followed by a sequence of chained orders in which those initial funds are transferred from one transaction to another, ending with an order for a unit that is not the source of funds to any order. The number of chained orders defines the chain length—the length can be zero if the initial spot order is followed by a spot order. A payments-chain network is the collection of all payment chains, encompassing the universe of transactions. Formally:

Definition 1. *A **payments-chain network** $\mathcal{K}$ is an acyclical directed network with degree 1 with nodes $\mathcal{N} = \{1, 2, ..., N\}$ and links $\mathcal{V} = \{(i, j)|\mathcal{P}(i) = j, j \in \mathcal{N}^x\}$, $\mathcal{K} = (\mathcal{N}, \mathcal{V})$. A **payments chain** of length n is a finite sequence of nodes $\{i_k\}_{k=0}^{n}$ such that the sequence starts with some $i_0 \in \mathcal{N}^s$ and $\forall k \in \{1, \ldots, n\}$, $\mathcal{P}(i_{k-1}) = i_k \in \mathcal{N}^x$. By convention, if $i \in \mathcal{N}^s$ and $j = \mathcal{P}(i) \notin \mathcal{N}^x$, i and j define a chain of length zero.*

Notice that in the definition, nodes in the payments-chain network represent both orders and input units. The directed links represent the direction of the flows of funds. As transactions are bilateral, at most one link stems from each node—the degree distribution is one. A link from i to j indicates that i orders from input unit j and that j is a chained order, $\mathcal{P}(i) = j \in \mathcal{N}^x$. The source of funds for order j is the payment from order i. In turn, if a node does not receive a link, it represents a spot order. Furthermore,

if no link is directed toward order $j = \mathcal{P}(i)$, then order j is also spot—in which case i and j form a zero-length chain. In this construction, each order is guaranteed a source of funds, either internally or externally. Any maximal path of links defines a payment chain. The collection of payment chains is the payments-chain network.

The assumption that the network is acyclical rules out cycles where chained orders form closed loops without any spot order. In settings where $\mathcal{P}$ follows from random assignment, this may be an issue for the finite N case, but the assumption is not restrictive for the case where $N \to \infty$, the focus of our paper.[22]

Examples. Let us provide an example. Set $\mathcal{N} = \{1, 2, \ldots, 8\}$ and let the subset of spot orders be $\mathcal{N}^s = \{1, 3, 7\}$. Also, define the assignment relation as follows: let $\{i_n\}_{n \in \mathcal{N}} = \{1, 5, 7, 4, 6, 2, 3, 8\}$ such that $i_{n+1} = \mathcal{P}(i_n)$ and $i_1 = \mathcal{P}(i_N)$. Thus, the links in this payments-chain network are $\mathcal{V} = \{(1, 5), (7, 4), (4, 6), (6, 2), (3, 8)\}$.

Several graphs are associated with this example. The left panel of Figure 5 depicts the chained income-expenditure relation. In that panel, we split each node into counterparts: the input units $\{u_n\}$ and orders $\{o_n\}$ for nodes $n \in \{1, 2, ..., 8\}$. The links represent the flow of funds from input units to their corresponding chained orders, defined by the chained income-expenditure relation.

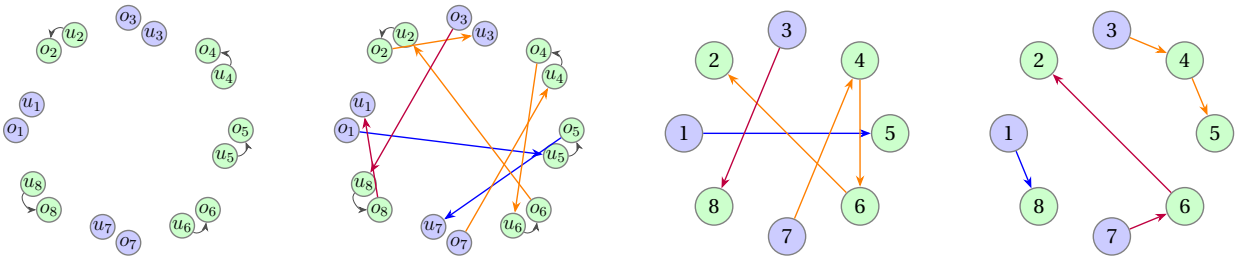
The middle panel adds the links to the flow of payments for inputs, corresponding to $\mathcal{P}$. That is, the links in that panel add the payments to input units. Adding the links of the chained income-expenditure and assignment relations allows us to trace funds. Notice that the links from orders to input units with the same color share a common source of funds.

The right panel depicts the payments-chain network. In this example, there are three chains with lengths $(1, 3, 1)$: the first chain from 1 to 5 (length 1), the second chain from 7 through 4, 6, to 2 (length 3), and the third chain from 3 to 8 (length 1).

Note that this network generates unequal losses: order 2 at position $n = 3$ receives only y_3, while orders at positions $n = 1$ receive $y_1 > y_3$. The network could be reorganized to reduce aggregate losses. For instance, reassigning the links to create chains of lengths $(1, 2, 2)$ would improve output for order 2 (from y_3 to y_2) while only slightly reducing output for one other order. This observation motivates our analysis of efficient

[22] In the random assignment with $N \to \infty$, such cycles would theoretically emerge but with vanishing probability. The shortest cycle length grows at rate $\log(N)$ (Erdős and Rényi, 1959, 1960; Bollobás, 2001), meaning cycles would contain infinitely many orders in the limit. Moreover, the probability that any given node belongs to a finite cycle vanishes as $N \to \infty$ (Janson, Łuczak, and Ruciński, 2000), meaning that the set of orders in those cycles is of measure zero.

networks below, where we characterize assignments $\mathcal{P}$ that minimize aggregate losses.



(a) Income relation (b) Network in detail (c) Random Network (d) Efficient Network

Figure 5: Components of the Payments-Chain Network

Chain-Length Distributions and Aggregate Output. We now derive key moments of the payment-chain network as the number of transactions increases, $N \to \infty$, while keeping the ratio $\mu \equiv N^x/N$ constant. Aggregate output per order depends on the distribution of positions that orders occupy in payment chains. This position distribution is induced by the distribution of chain lengths, which in turn is determined by the assignment $\mathcal{P}$. Let $g(n;\mu,\mathcal{P})$ denote the probability mass function of chain length under assignment $\mathcal{P}$, and let $G(n;\mu,\mathcal{P}) = \sum_{m=0}^{n} g(m;\mu,\mathcal{P})$ denote its cumulative distribution function.

Since position n is found only in chains of length n or greater, the fraction of orders at position n is proportional to $1 - G(n-1;\mu,\mathcal{P})$ leading to:

$$H(n;\mu,\mathcal{P}) = \frac{1 - G(n-1;\mu,\mathcal{P})}{\sum_{k=0}^{\infty}[1 - G(k-1;\mu,\mathcal{P})]} \tag{2}$$

where we define $G(-1) = 0$. Hence, aggregate output per order, given the output function y_n, is given by:

$$\mathcal{Y}(\mu,\mathcal{P}) = \sum_{n=0}^{\infty} H(n;\mu,\mathcal{P}) \cdot y_n = (1-\mu) + \mu \cdot \mathcal{A}(\mu,\mathcal{P})$$

where $\mathcal{A}(\mu,\mathcal{P}) \equiv \mathbb{E}[y_n|\text{chained order}]$ is the expected output among chained orders. The key insight is that different assignments $\mathcal{P}$ generate different distributions $g(n;\mu,\mathcal{P})$ and thus different aggregate outcomes, even holding μ fixed.

For now, examine three types of assignments that span a spectrum of possible network structures, leaving other variations to the extensions below. On one end, we have a **random assignment** where any input unit can be matched to any order with equal

probability—serving as a benchmark representing decentralized markets without coordination. On the other end, the **efficient assignment** that maximizes $\mathcal{A}(\mu, \mathcal{P})$ given μ—this represents either the social planner's solution when the fraction of chained orders is taken as given, or a market outcome where orders can be reallocated to minimize position in chains.

For the output function, we consider the exponential form $y_n = \delta^n$ where $\delta \in (0,1)$. In Appendix [XXX], we provide several possible mechanisms stemming from the following ideas. First, it can represent time decay: if payments take a unit of time to clear, then $\tau(n) = n$ and δ^n can capture time discounting or the value of final goods net of depreciation. Second, the function can emerge when intermediate products require time to be transformed into final goods, and transfers happen after an inspection that takes a fraction $(1-\delta)$ of the period. In that case, $\tau(n) = 1-\delta^n$, and production time is $1-\tau(n) = \delta^n$, so again $y_n = \delta^n$. This latter interpretation is convenient as it ensures all transactions are completed within a unit time interval.

For the assignments and output functions under consideration, we obtain:

Proposition 1 (Random versus Efficient Networks)**.** *Consider an economy with a fraction μ of chained orders and output function $y_n = \delta^n$. As $N \to \infty$, we characterize two benchmark assignments:*

Assignment	**Auxiliary Objects**	**Distribution** $g(n;\mu,\mathcal{P})$	**Chained Output** $\mathcal{A}(\mu;\delta,\mathcal{P})$
Random	–	$(1-\mu)\mu^n$	$\delta\frac{1-\mu}{1-\delta\mu}$
Efficient	$\underline{n} = \lfloor \mu/(1-\mu) \rfloor$ $\overline{n} = \lceil \mu/(1-\mu) \rceil$ $p = \overline{n} - \mu/(1-\mu)$	$\begin{cases} p \text{ if } n = \underline{n} \\ 1-p \text{ if } n = \overline{n} \\ 0 \text{ otherwise} \end{cases}$	$\delta\frac{1-[p\delta^{\underline{n}}+(1-p)\delta^{\overline{n}}]}{1-\delta}\frac{1-\mu}{\mu}$

Furthermore, under both assignments, $\mathcal{A}$ is concave and decreasing in μ with limits:

$$\lim_{\mu\to 0}\mathcal{A}(\mu;\delta) = \delta, \quad \lim_{\mu\to 1}\mathcal{A}(\mu;\delta) = 0, \quad \lim_{\delta\to 0}\mathcal{A}(\mu;\delta) = 0, \quad \lim_{\delta\to 1}\mathcal{A}(\mu;\delta) = 1$$

For the random assignment, these properties are strict, but weak in the efficient case.

The geometric distribution emerges naturally in random networks from the matching process. Consider building a payment chain starting from a spot order. This order is matched to an input unit. The chain continues if and only if this unit corresponds to

a chained order, which occurs with probability μ. If it is chained, it links to another unit that is also chained with probability μ, and so on. The chain terminates when reaching a spot order, occurring with probability $(1-\mu)$. Thus, the probability of a chain having exactly length n is $g(n;\mu) = (1-\mu)\mu^n$—the probability of n consecutive chained orders followed by one spot order. For the geometric chain length distribution, the distribution of positions is also geometric, $H(n;\mu) = (1-\mu)\mu^n$. For this assignment, the formula for output follows immediately from $\mathcal{Y} = \sum_{n=0}^{\infty} H(n;\mu)\cdot y_n = \sum_{n=0}^{\infty}(1-\mu)\mu^n \cdot \delta^n = \frac{1-\mu}{1-\mu\delta}$.

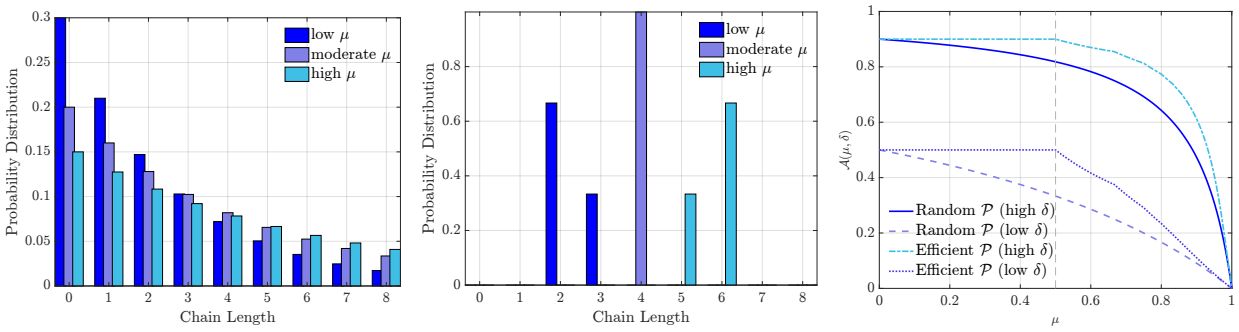
The efficient assignment minimizes output losses by equalizing chain lengths as much as the integer nature of the chains permits. To see why this is optimal, consider any assignment with chains of varying lengths. If we have chains of lengths n_1 and n_2 such that $n_1 + 1 < n_2$, we could move an order from position n_2 in the longer chain to extend the shorter chain to length n_1+1. This reallocation strictly increases total output since the relocated order gains $y_{n_1+1} - y_{n_2} = \delta^{n_1+1} - \delta^{n_2} > 0$. Such improvements are possible whenever chains differ by more than one in length. Therefore, the efficient assignment must concentrate all chains on at most two consecutive integers: chains of length $\underline{n}$ and $\overline{n} = \underline{n} + 1$. The weight p on chains of length $\underline{n}$ ensures that the average ratio of spot to chained orders across chains yields the ratio across all orders $\mu/(1-\mu)$.

The average output per chained order, $\mathcal{A}$, plays a critical role once we consider problems where orders follow from agent decisions. For both assignments, $\mathcal{A}$ is concave and decreasing in μ with well-behaved limits. As $\mu \to 1$, chains become arbitrarily long and the average final goods received per chained order tends to zero. Chain lengths become increasingly long as $\mu \to 0$, so chained orders receive zero units of final goods. Similarly, as $\delta \to 0$, chained orders receive nothing, whereas any losses vanish as $\delta \to 1$.

Figure 6 illustrates the key differences between random and efficient assignments. Panel (a) shows the geometric distribution of chain lengths under random assignment for three values of μ. The increase in μ shifts probability mass toward longer chains. Panel (b) shows the efficient assignment for the same parameters. In that case, chains are concentrated on only two adjacent lengths, eliminating both very short and very long chains. Panel (c) compares $\mathcal{A}(\mu)$ across both assignments for $\delta \in \{0.5, 0.9\}$. Output per chained order decreases in μ for both assignments, but the efficient assignment substantially outperforms the random assignment, with the gap widening as μ increases. Notice that for $\mu \leq 1/2$, $A(\mu) = \delta$ under the efficient assignment because it is possible to assign each chained order to a spot order.

The main insight from this section is that economic losses can result from payment

delays and the possible disorder of payments. It is worth distinguishing the sources of these losses from other environments. Unlike search models, there are no congestion externalities—the production assignment is one-to-one. Unlike sticky-price models, the fact that payments are the same across orders does not have allocative implications. Instead, economic losses arise exclusively from payment delays, which is clear because the economy is at full capacity when all transactions are spot or $\delta \to 1$. This is true under both assignments of payments, but importantly, losses are magnified where payments are disorganized by the random assignment of orders. The efficient assignment minimizes these losses by concentrating all chains at similar lengths.



(a) g(n) (random assignment) (b) g(n) (efficient assignment) (c) Output $\mathcal{A}(\mu)$

Figure 6: Chain-Length Distributions and Average Output per Chained Order

Notes: Panels (a) and (b) show distributions $g(n;\mu)$ for $\mu \in \{0.7, 0.8, 0.85\}$ under random and efficient assignments. Panel (c) shows $\mathcal{A}(\mu)$ for both assignments with $\delta \in \{0.5, 0.9\}$.

3.1 Interpretations and Extensions

Economic Mechanisms for Network Formation. The random and efficient assignments analyzed above can emerge from different market structures and environments. Random assignments arise naturally in centralized markets where a Walrasian mechanism, as we show below, clears the market for inputs—equating supply and demand for inputs at given prices—but cannot coordinate the specific assignment of orders to inputs to speed up payment flows. In the market-clearing process, agents must anticipate the random arrival of payments, knowing they will be supplied the good, while foreseeing that the arrival time will depend on the distribution of payment chains.

In contrast, an efficient assignment can emerge from sequential market organization. If orders are placed sequentially as existing chains are observed, agents placing chained orders will choose the shortest available chain. This sequential allocation

mechanism naturally generates chains of nearly equal length, minimizing aggregate losses. This resembles queuing systems where arrivals join the shortest queue.

Intermediate Assignments and Empirical Patterns. The evidence from Section 2 suggests that real payment networks lie between these polar cases. The observed distribution is concentrated around chains of length 4-5, with a bell-shaped curve, which is inconsistent with the geometric distribution's monotonic decline. Moreover, a mass of transactions are spot-to-spot (i.e., chains of length zero), which cannot arise in the efficient assignment where all spot orders are utilized to minimize chain lengths. These patterns motivate us to study other assignments in Appendix C.

A first extension in Appendix C.1 explores **correlated assignments**, where chained orders have a higher probability of being linked to other chained orders than in the i.i.d. case; the probability is $\mu + \rho$. We construct payment-chains using Markov chains and show that the chain length distribution, for chains of length above 1, is geometric with parameter $\mu + \rho$, and $\mathcal{A}(\mu; \delta, \rho) = \delta \frac{1-(\mu+\rho)}{1-\delta(\mu+\rho)}$. Thus, when chained orders are less likely to connect with chained orders, $\rho < 0$, the economy achieves greater efficiency than under baseline i.i.d. random assignment, though falling short of full efficiency.

We further consider a **pooled assignment**—Appendix C.2—a hybrid between random and efficient allocations. Under this assignment, the share of spot-to-spot links remains random, preserving the baseline inefficiency caused by spot-to-spot links. However, chained orders are efficiently assigned among themselves, excluding the spot-to-spot links. The formula for $\mathcal{A}$ matches that of the efficient assignment, but replaces the ratio of chained to spot orders $\mu/(1-\mu)$ with the ratio of chained to available spot orders $1/(1-\mu)$. This pooled assignment captures income pooling in scenarios where agents manage multiple orders: whenever a chained order receives payment, those funds are immediately redeployed to finance other unfunded chained orders, creating chains of approximately equal length. The relevance of this assignment becomes clear when we introduce agent decisions and trace their income flows from different orders.

Another possibility is costly **reassignment**—Appendix C.3——a hybrid between random and efficient chains. This assignment arises when orders are initially assigned randomly, but chain orders can be reallocated at a cost Ξ. We solve the case where reassigned orders sequentially fill the shortest available slots in existing chains. The cost Ξ can be thought of as negotiation costs, and a threshold η emerges endogenously from the trade-off between reassignment costs and reduced payment delays. The resulting

distribution exhibits efficient clustering at short lengths followed by the natural geometric distribution up to η. This assignment showcases an intermediate case, while also highlighting the information requirements necessary to increase efficiency and illustrates why random chains remain a natural benchmark.

Variations to the environment. We consider two extensions that do not modify the assignment of orders, but the production within the chains. In Appendix C.4, we consider **default and cascading**, where orders at position $n > \eta$ default, keeping their funds rather than executing payments. This captures situations where the losses from delays are very large. Importantly, once a transaction fails, subsequent transactions also fail due to a lack of funds. The chain-length distribution remains geometric, but production is truncated: chains longer than η only generate output up to position η. The resulting TFP is the baseline TFP scaled by $(1-\delta^{\eta}p^{\eta})$, where the scaling factor captures the direct loss from truncated positions and the probability mass of affected chains. Unlike the reassignment extension, which requires information on the network's transactions, defaulting is a unilateral decision that only requires knowing the passage of time.

Motivated by the empirical evidence on different delays, a final extension—Appendix C.5— introduces **heterogeneity in delays**, where in a chain of length n, the first k transactions occur immediately—with probability q of experiencing a delay. This captures payment environments where instant payments coexist with traditional settlement systems, such as electronic payments alongside checks. The resulting TFP formula elegantly decomposes into two components:

$$\mathcal{A}_q = (1-q) + \delta \frac{q(1-\mu)}{1-\mu(1-q+\delta q)}. \tag{3}$$

The first term $(1-q)$ is the contribution from instant payments, while the second term captures the reduced productivity from delayed transactions. When $q = 0$ (all instant), $\mathcal{A}_q = 1$; when $q = 1$ (all delayed), we recover the baseline formula. The extension reveals how expanding instant payment infrastructure (reducing q) can substantially improve aggregate productivity, without changing the underlying chain structure.

3.2 From Payments-Chain Networks to Portable Budget Constraints

The previous section maps from the fraction of chained orders μ to the average value of final goods $\mathcal{A}$, obtained from different assignment rules. We now demonstrate how these payment-chain dynamics manifest into reduced-form budget constraints that

can be embedded into standard economic models. The key insight –is that payment delays create an effective price wedge $q(\mu) > 1$ for goods obtained through chained orders, where this price depends on the economy-wide fraction μ, potentially creating pecuniary externalities. This allows us to study how individual payment decisions affect aggregate outcomes without explicitly modeling the underlying network. The price wedge also aligns with our empirical finding that, even for the same good, unit prices for chained orders are higher than for spot orders.

Timing. Consider a dynamic model with discrete time periods $t \in \{0, 1, 2, \ldots\}$. Expenditure and savings decisions are made at the start of each period. All transactions occur within each unit time interval, following the payment-chain network above, in the times $\tau(n) = (1-\delta)\delta^n \in [0, 1]$ corresponding to position $n \in \{0, 1, 2, \ldots\}$ in a payment chain. The numéraire is in goods.

We focus on a single agent, which could be a household or a firm, that has y goods available for sale in the network at the start of each period t. This endowment is composed of N units of goods, each of size $1/N$, sold through the network. The agent also begins the period with financial wealth W and must decide how to allocate expenditures between spot orders (requiring immediate payment) and chained orders (contingent on receiving income). Let E^s and E^x denote the agent's expenditures on spot and chained orders corresponding to N^s spot orders and N^x chained orders, respectively.

Since inputs are fixed in the time period, $\mathcal{Y}$ is both output and TFP. However, as efficiency losses are driven by utilization among chained orders, we refer to $\mathcal{A}$ as TFP.

Income Flows and Transaction Timing. Recall that each of the N goods the agent sells is assigned a position within a payment chain. Index each good by $j \in \{1, 2, \ldots, N\}$ and let $u_j \in \{0, 1, 2, \ldots\}$ denote the random position at which good j is sold: position 0 corresponds to the good being sold in a spot sale. Since all goods are sold at one unit of numéraire, the price is $1/N$. Hence, total income equals the endowment:

$$y = \frac{1}{N} \sum_{n=0}^{\infty} \sum_{j=1}^{N} \mathbb{I}[u_j = n]. \tag{4}$$

Similarly, each chained order is associated with a payment time. Enumerate each chained order by $i \in \{1, 2, \ldots, N^x\}$ and let $o_i \in \{1, 2, 3, \ldots\}$ denote the position at which chained expenditure i is executed. The final goods obtained from a chained order i is

$(1-\delta)\delta^{o_i}$ per unit, due to the delay, so the total goods obtained from chained orders is:

$$X = \frac{1}{N}\sum_{i=1}^{N^x}(1-\delta)\delta^{o_i}, \tag{5}$$

where X is random given the randomness in o_i. Summing across positions, total chained expenditures equal:

$$E^x = \frac{1}{N}\sum_{n=1}^{\infty}\sum_{i=1}^{N^x}\mathbb{I}[o_i = n]. \tag{6}$$

From Payments-Chain Network Budgets to Period Budgets. Wealth evolves within each period as transactions are executed at different moments $\tau(n) \in [0,1]$ within period t. Let $W_{\tau(n)}$ denote wealth immediately after all transactions at position n have been completed. Spot expenditures and spot sales occur at $\tau(0)$, so that

$$W_{\tau(0)} = W + \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}[u_j = 0] - E^s. \tag{7}$$

This budget adds the income from goods sold in spot transactions and subtracts spot expenditures from the initial wealth W, yielding $W_{\tau(0)}$. At each later position n, the agent receives income from goods sold and pays for executed chained orders:

$$W_{\tau(n)} = W_{\tau(n-1)} + \frac{1}{N}\sum_{j=1}^{N}\mathbb{I}[u_j = n] - \frac{1}{N}\sum_{i=1}^{N^x}\mathbb{I}[o_i = n], \quad \forall n \geq 1. \tag{8}$$

The first term represents carried-over wealth, the second term is income from sales at position n, and the third term is payments for chained orders executed at position n.

We sum across all sequential budgets to obtain a period budget constraint:

$$W_{\tau(\infty)} = W + \underbrace{\frac{1}{N}\sum_{n=0}^{\infty}\sum_{j=1}^{N}\mathbb{I}[u_j = n]}_{=y} - E^s - \underbrace{\frac{1}{N}\sum_{n=1}^{\infty}\sum_{i=1}^{N^x}\mathbb{I}[o_i = n]}_{=E^x}. \tag{9}$$

Thus, we obtain an intertemporal budget constraint:

$$\frac{W'}{R} + E^s + E^x = y + W, \tag{10}$$

where $W'/R = W_{\tau(\infty)}$ represents the wealth carried over to the next period at gross in-

terest rate R. Budget constraint (10) encodes the evolution of transactions within the period. Importantly, an agent placing chained orders would face uncertainty about the final goods X obtained from these orders. However, as the number of orders becomes large ($N \to \infty$), positions become predictable. Taking the limit as $N \to \infty$, while holding total expenditures fixed, we obtain:

$$\lim_{N\to\infty} X = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N^x} (1-\delta)\delta^{o_i} = E^x \sum_{n=1}^{\infty} (1-\delta)\delta^n G(n) = \mathcal{A}(\mu)E^x, \tag{11}$$

where $\mathcal{A}(\mu)$ is the aggregate productivity factor and μ is the economy-wide fraction of chained orders. The budget constraint (10) becomes:

$$\frac{W'}{R} + S + q(\mu)X = y + W, \tag{12}$$

where $q(\mu) \equiv \mathcal{A}(\mu)^{-1}$ is the average price of goods obtained through chained orders relative to spot orders. To derive this condition we used the fact that inputs are the numeraire and spot orders obtain the full value, $S = E^s$.

The large transactions limit $N \to \infty$ makes the problem tractable, as budget constraints become deterministic. The cost of placing chained orders, reflected in reduced final goods received, is captured by the average price $q(\mu) \equiv \mathcal{A}(\mu)^{-1} > 1$. Crucially, the specific assignment rule determines how agents interpret this price and budget:

When network positions are unknown, agents act as if they expect to receive the average value $\mathcal{A}(\mu)E^x$ from chained orders. This formulation implicitly relies on a "big family" assumption in the tradition of Lucas and Stokey (1987), where individual transaction risk is diversified away but income from spot transactions is not pooled to finance pending chained orders. Note that under random assignment, cash inflows from spot sales remain with the agent rather than being redeployed to fund chained orders—explaining why the pooled assignment can improve efficiency.

Within the "big family" assumption, we may conceive that $\mathcal{A}(\mu)$ corresponds to the pooled. When agents can pool income across multiple orders, spot income is immediately redeployed to finance chained orders. The effective price becomes $q^{pool}(\mu) = \mathcal{A}^{pool}(\mu)^{-1}$, reflecting the efficiency gains from internal coordination.

If intermediaries centralize all orders, they can induce an efficient assignment that optimizes the entire network. However, in that case, there should be a market for spot and chained orders. These prices re-emerge when we study an efficient allocation.

Pecuniary Externality. Regardless of the assignment rule, depending on whether individual agents take $q(\mu)$ as given or prices are internalized, their collective payment choices determine μ and thus $q(\mu)$. This opens the door to a pecuniary externality, creating a divergence between private and social incentives: agents do not internalize how their payment choices affect network efficiency. The magnitude of this externality depends on the assignment mechanism—it is largest under random assignment and smallest under efficient assignment, where the intermediary partially internalizes network effects.

Discussion: Reduction of Economic Complexity. We deliberately abstract from more realistic features to isolate the core mechanism. Real economies exhibit greater complexity: transactions vary in size (100 vs 200), production networks involve multilateral relationships, and payment timing is more nuanced. When transactions differ in size, an agent receiving \$100 but owing \$200 faces complex cash flow problems that require sophisticated optimization—challenges that even large firms struggle with, analogous to NP-hard problems in supply chain logistics. The random assignment captures this inherent complexity, suggesting that statistical approaches may be more realistic than assuming perfect optimization. The pooled and efficient assignments represent increasing levels of coordination capability, from simple income pooling to full network optimization. This hierarchy of assignments—random, pooled, efficient—provides a spectrum for analyzing how payment infrastructure and coordination technology affect economic outcomes.

Regardless of the assignment, the framework provides an explicit mapping from the financial decision variable μ to measurable efficiency losses under various institutional arrangements. The next section leverages this property to analyze agents' endogenous decisions between placing spot and chained orders.

4 Endogenous Expenditure Decisions

The previous section shows how, given an amount of chained expenditures, E^x, an agent obtains final goods from chained transactions that depend on the network structure and the ratio of economy-wide chained transactions encoded in $q(\mu)$. So far, we set aside the decision of placing chained orders, but now turn to behavior. We develop two dynamic decision problems in which agents decide on savings and the number of orders of each size to place. A common theme emerges: when short-term means

of payment (e.g., credit lines) are scarce, agents optimally resort to chained orders despite their higher cost, e.g., $q > 1$. Moreover, unlike standard financial frictions models, agents do not always have incentives to delever to escape their constraints.

4.1 Consumption-Saving Decisions

We first study the problem of entrepreneurs who buy and sell goods in the payments-chain network. Time is discrete time, $t = 0, 1, 2, \ldots$. At the start of each period, entrepreneurs receive a unit endowment of non-storable goods that are available for sale and purchase goods produced by others for consumption. Entrepreneurs have log-utility over final consumption goods. At the start of each period, entrepreneurs place and receive purchase orders, as above. There is perfect foresight. The network's effects are summarized by an exogenous sequence $\{q_t\}_{t\geq 0}$ and they taking between period interest rates, $\{R_{t+1}\}_{t\geq 0}$ as given.

At the start of period t, the entrepreneur chooses *spot* purchases $S_t \geq 0$ (requiring immediate funds) and *chained* goods $X_t \geq 0$ (purchased through chained orders). Thus, the goods entering their period utility, are $S_t + X_t$ obtained from making total expenditures $E_t = S_t + q_t X_t$.

The entrepreneur enters period t with debt B_t (corresponding to negative wealth in the previous section) and carries over to next-period debt $B_{t+1} \leq \bar{B} \equiv 1/(1-\beta)$. Importantly, spot purchases must be financed with up-front funds. A *spot borrowing line* (SBL) $\tilde{B}_t$ limits intraperiod liquidity:

$$S_t \leq \bar{S}_t \equiv \max\{\tilde{B}_t - B_t,\, 0\}.$$

The entrepreneur chooses its total expenditures and its split, between spot and chained expenditures, to maximize its utility over consumption streams $\sum_t \beta^t \log(S_t + X_t)$. Given total expenditures E_t, since $q_t \geq 1$, by cost minimization, the entrepreneur makes as many spot expenditures as the SBL permits. Thus, given E_t, spot and chained expenditures are respectively:

$$S_t = \min\left\{\bar{S}_t, E_t\right\}, \qquad X_t = \left(E_t - \min\left\{\bar{S}_t, E_t\right\}\right)/q_t. \tag{13}$$

Given this optimal expenditure split, by the principle of optimality, the entrepreneur's intertemporal utility maximization can be cast into a dynamic programming problem:

Problem 1. (Entrepreneur's Recursive Problem): *Given* B_0 *and* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$*, the entrepreneur chooses debt holdings,* $\{B_{t+1}\}_{t\geq 0}$*, that solve:*

$$V_t(B) = \max_{B'\leq \bar{B}} \log(S+X) + \beta V_{t+1}(B')$$

where S *and* X *are given by* (13) *and total expenditures by* $E = B'R_{t+1}^{-1} + 1 - B$*.*

The time index in the value function reflects the evolution of $\{R_{t+1}, \tilde{B}_t, q_t\}$. The Bellman equation expresses the entrepreneur's problem as a choice over intertemporal debt. The choice of carried-over debt, B', determines the period's expenditures, through (10), and the optimal split (13) yields S and X given the level of expenditures.

Interpretation of Financial Constraints. The spot borrowing line (SBL), $\bar{S}_t$, represents short-term working capital. The sources can be credit lines, external supply-chain finance facilities, or even trade credit.[23]

Notice that $\bar{S}_t$ represents within-period credit, which carries no interest, and differs from the carried-over debt B_t, which does. The SBL constraint, $\tilde{B}_t$, differs fundamentally from the hard debt limit, $\bar{B}$, that governs intertemporal borrowing B_{t+1}. While $\bar{B}$ represents the maximum sustainable debt level, $\tilde{B}_t$ applies only to within-period short-term funding. Critically, agents can roll over SBL usage into future debt B_{t+1} without immediate principal repayment, akin to having a limit on credit-card borrowings that accrue interest only upon carrying over an end-of-period balance. A motivation for these differences is that financial intermediaries often set different limits as a way to tighten short-term credit lines without forcing costly defaults by suddenly and unfeasibly demanding principal repayment.

Characterization. We identify two debt thresholds that determine whether entrepreneurs can purchase all of their consumption through spot orders or, conversely, whether they can only purchase goods via chained orders, given the lack of short-term funds.

Lemma 1 (Expenditure Threshold Points)**.** $S_{t+1} = 0$ *if and only if* $B_{t+1} > \tilde{B}_{t+1}$*. Define the efficiency threshold,* $B^{\star}_{t+1} \equiv R_{t+1}(\tilde{B}_t - 1)$*. Then:* $X_t > 0$ *if and only if* $B_{t+1} > B^{\star}_{t+1}$*.*

[23] Practical examples include Business Credit Lines, Standby Letters of Credit, and Supplier Finance Programs. Under a Supplier Finance Program, buyers defer payment while suppliers receive immediate cash from banks against receivables. Recall that chained orders are contracts on goods that have not been delivered; they are delivered only upon payment.

The following period's SBL $\tilde{B}_{t+1}$ determines whether entrepreneurs lose access to spot purchases in the following period. In turn, the efficiency threshold $B^{\star}_{t+1}$ marks a threshold for carried-over debt above which entrepreneurs begin using chained purchases in the current period.

These thresholds are critical for the characterization because debt levels affect the availability of spot expenditures and, thus, the price at which goods can be bought at the margin. Thus, the threshold debt levels determine *marginal prices*, the relevant prices for marginal decisions:

Definition 2 (Marginal and Average Prices)**.** *The marginal expenditure price is* $\tilde{q}^{E}_{t}(B') \equiv 1 + (q_t - 1)\mathbb{I}_{[B' \geq B^{\star}_{t+1}]}$. *The marginal borrowing price is* $\tilde{q}^{B}_{t+1}(B') \equiv 1 + (q_{t+1} - 1)\mathbb{I}_{[B' > \tilde{B}_{t+1}]}$. *Average prices are* $Q_t \equiv E_t / (S_t + X_t)$.

Marginal prices capture the marginal cost of financing expenditures. If current debt forces chained purchases, reducing debt requires sacrificing $1/\tilde{q}^{E}_{t}$ units of consumption. If future debt exceeds the spot borrowing line, savings translate into chained rather than spot purchases, yielding only $1/\tilde{q}^{B}_{t+1}$ additional goods per unit of debt reduction. These marginal prices capture the consumption-saving tradeoffs that determine the entrepreneur's debt trajectory via the following generalized Euler equation, which adjusts the usual intertemporal condition with discontinuous marginal price shifts:

Proposition 2 (Entrepreneur's First-Order Condition)**.** *Any solution* $\{B_{t+1}\}_{t \geq 0}$ *to the entrepreneur's problem satisfies:*

$$\frac{E_{t+1}}{\beta E_t}\frac{Q_t}{Q_{t+1}} = \frac{R_{t+1}}{\tilde{q}^{B}_{t+1}(B_{t+1})/\tilde{q}^{E}_{t}(B_{t+1})} \text{ if } B_{t+1} \neq B^{\star}_{t+1}, \quad \beta q_t R_{t+1} \geq \frac{E_{t+1}}{E_t} \geq \beta R_{t+1} \text{ if } B_{t+1} = B^{\star}_{t+1}.$$

Despite its unconventional representation, this Euler equation retains a familiar interpretation: at interior solutions ($B_{t+1} \neq B^{\star}_{t+1}$), marginal rates of substitution across periods equal the return on savings corrected for price differences between spot and chained consumption. The left side represents the ratio of marginal rates of substitution (expressed as the ratio of expenditures over average prices yielding the ratio of consumption consistent with log utility), while the right side captures returns adjusted by "inflation" in terms of marginal prices. In turn, the inequalities follow when future debt is the efficient debt level, a corner solution.

This Euler condition is necessary but not sufficient, as the value function is not concave in B'. Hence, multiple sequences of future debt may satisfy the condition starting

from an initial state. This non-convexity opens the door to debt-overhang leading to hysteresis in general equilibrium.

Stationary Dynamics. We now characterize the stationary dynamics of the entrepreneur's problem, i.e., when the variables he takes as given are constant, $\tilde{B}_t = \tilde{B}_{ss}$, $q_t = q_{ss}$, and $R_t = R_{ss} = 1/\beta$. We focus on the case $\tilde{B}_{ss} > 1$ and define $B^\star_{ss} \equiv B^\star(1/\beta, \tilde{B}_{ss})$. We obtain the following:

Proposition 3 (Stationary Entrepreneur Problem)**.** *In a stationary solution: If $B_0 \leq B^\star_{ss}$, then $B_t = B_0$ for all t. Moreover, there exists $B^h > \tilde{B}_{ss}$ such that: If $B_0 < B^h$, then $B_t \to B^\star_{ss}$ in finite time. If $B_0 > B^h$, then $B_t = B_0$ for all t.*

Figure 7 brings together the theoretical insights of the proposition. To illustrate the forces at play, we overlay the entrepreneur's value function with two benchmark cases: $\bar{V}$, the value function when the spot borrowing limit is unbounded, ($\tilde{B}_t = \infty$) and the entrepreneur can finance any desired of spot expenditures, and, $\underline{V}$, where no liquidity is available ($\tilde{B}_t = 0$) and the entrepreneur can consume only via chained expenditures.

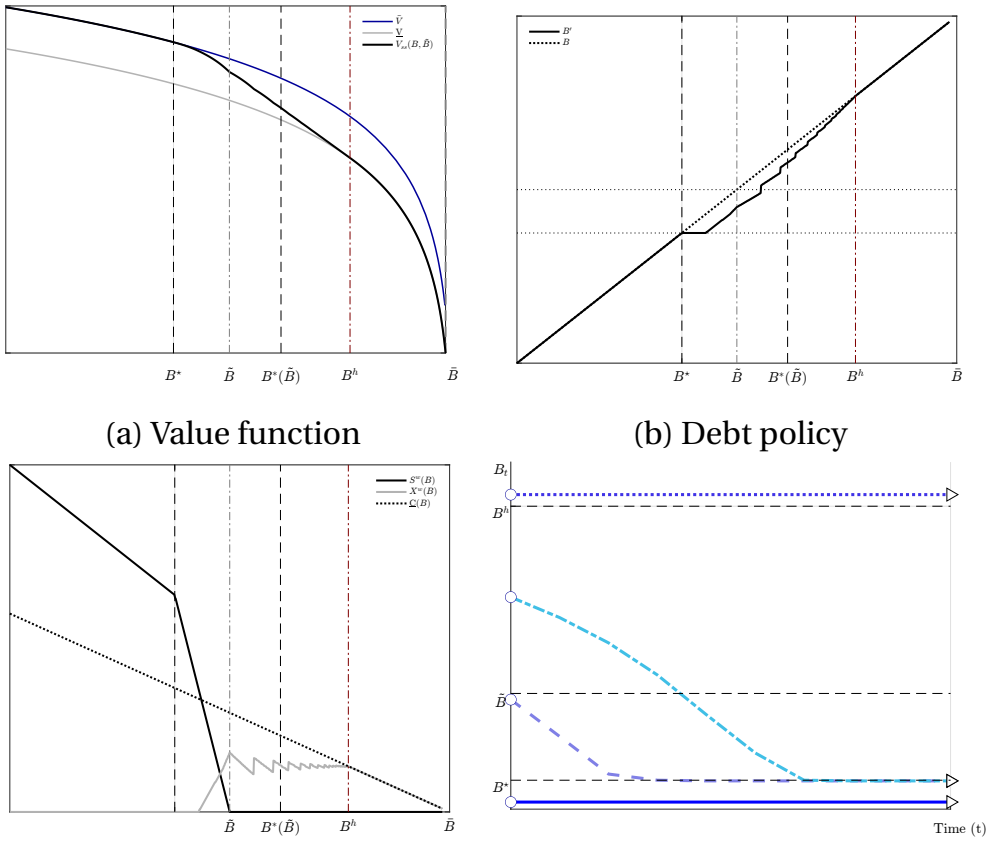
Panel (a) demonstrates how the actual value function deviates from both perfect liquidity ($\bar{V}$) and zero liquidity ($\underline{V}$) benchmarks for intermediate debt levels. For debt levels below $B^\star$, the value function V coincides with $\bar{V}$: entrepreneurs are unconstrained and consume the annuity value of their endowment net of interest payments entirely via spot purchases, showcasing that for sufficiently low debt levels, the SBL is irrelevant. When debt exceeds the high threshold B^h, the value function lies on $\underline{V}$, indicating that entrepreneurs consume exclusively via chained purchases forever. Crucially, even when highly indebted, the entrepreneur could potentially save to access better prices in the future. However, he chooses not to because the benefits of deleveraging come too far in the future, requiring an excessive short-run sacrifice.

Consistent with Panel (a), Panel (b) shows non-monotonicity in the optimal debt policy function. Entrepreneurs with debt below $B^\star_{ss}$ have constant debt levels. Likewise, entrepreneurs with debt above B^h remain permanently at their initial debt levels. Only those with moderate debt in $(B^\star_{ss}, B^h]$ gradually deleverage toward the efficient threshold $B^\star$ by choosing future debt (solid line) below current debt (dashed line).

Panel (c) illustrates how consumption policies differ across debt regions. When debt is below (above) $B^\star_{ss}$ (B^h), the entrepreneur consumes the annuity of his total wealth annuity spot (chained) consumption. For debt levels slightly below B^h, we observe a reduction in overall consumption, which is how entrepreneurs escape the expensive

chained consumption of those high debt levels. The chain-saw pattern of the optimal policy is a feature: it reflects a trade-off between smoothing consumption and shortening the number of periods to arrive at the efficient debt level.

Panel (d) traces the debt dynamics that emerge from the non-monotonic policy functions. Starting from different initial debt levels, entrepreneurs either start and remain in debt levels that allow consuming the period income in spot consumption, remain permanently consuming the period income in expensive chained goods, or temporarily delever to unconstrained debt levels. Appendix XXX provides a complete characterization of B^h and the periods needed to arrive at $B^\star_{ss}$, including a continuous-time solution for which the saw-like pattern disappears. The properties have important general equilibrium consequences.



(a) Value function

(b) Debt policy

(c) Consumption policy

(d) Transitions

Figure 7: Bellman Equation and Policy Functions: Entrepreneur's Problem

Note: Figures computed using value function iteration with $\beta = 0.xxx$, $q = xxx$, and $\tilde{B} = xxx \cdot \bar{B}$.

4.2 Investment Decisions

We can also study how payment chains can affect investment and production by incorporating these features into the entrepreneur's problem. Here, we study a problem where both consumption and investment must be purchased in the payment network. The problem is as before, except that yesterday's capital purchases determine today's goods available for sale:

Problem 2. (Entrepreneur's Problem with Production): *Given* B_0 *and* $\left\{\tilde{B}_t, R_{t+1}, q_t\right\}_{t\geq 0}$, *the entrepreneur chooses debt-holdings and investment* $\{B_{t+1}, K_{t+1}\}_{t\geq 0}$, *the solution to:*

$$V_t(B,K) = \max_{B'\leq \bar{B}, K'\geq 0} \log(S + X - K') + \beta V_{t+1}(B', K')$$

where S *and* X *are given by* (13) *and total expenditures by* $E = B'R_{t+1}^{-1} + f(K) - B$.

The problem is as above, but instead of an endowment, the entrepreneur has $f(K)$ goods produced with input (e.g., capital or inventory) K acquired in the previous period. In turn, currently purchased goods, $S+X$, are distributed between consumption, C, and inputs acquired for next-period production, K', assuming full depreciation to keep the analysis as simple as possible.

Recall that in the endowment problem above, the entrepreneur's stationary problem reaches two types of steady-state debt levels: either all-spot-expenditures or all-chained-expenditures steady states. The same two types of steady states are reached in the problem with investment. Furthermore, away from steady state, the same incentive to reduce debt and confront better prices for moderate debt levels is also present in this problem. The payment-chain frictions create two domains of attraction, and the inclusion of investment decisions does not alter this fundamental property.

The key insight of the version with production is that the higher cost of chained goods also distorts the optimal investment scale. In particular, if the entrepreneur makes only spot expenditures, the return on investment equals the rate of interest: $f'(K_{ss}) = R = \frac{1}{\beta}$. However, if all expenditures are chained, the first-order condition for inputs becomes: $f'(K_{ss}) = qR = \frac{q}{\beta}$. Hence, the marginal product of capital exceeds the rate of return on saving and investment, as in other distorted economies.

Figure 9 illustrates the domains of attraction toward the two classes of steady states and the rich dynamics that emerge in a setting with production. Panel (a) shows the

phase portrait[24] with arrows indicating the direction of change for both state variables (B, K) of the model's solution together with some trajectories. The shaded region represents the domain of attraction toward the disrupted steady state with only chained expenditures and inefficient investment. Four representative paths demonstrate the range of possible dynamics. Path A begins with low debt and excess capital: convergence to the efficient steady state quickly as excess income is used to reduce debt, reaching equilibrium with undistorted investment while maintaining the pattern of only spot expenditures. Paths B and C start with high initial debt but different capital levels—both converging to the steady state with low investment and only chained expenditures. The most interesting case, Path D, starts with intermediate debt and low capital. Initially, the entrepreneur accumulates both debt and capital, but once debt falls sufficiently to enable some spot purchases, investment is reduced to accelerate deleveraging—consistent with the incentive to escape the high marginal prices of chained transactions.

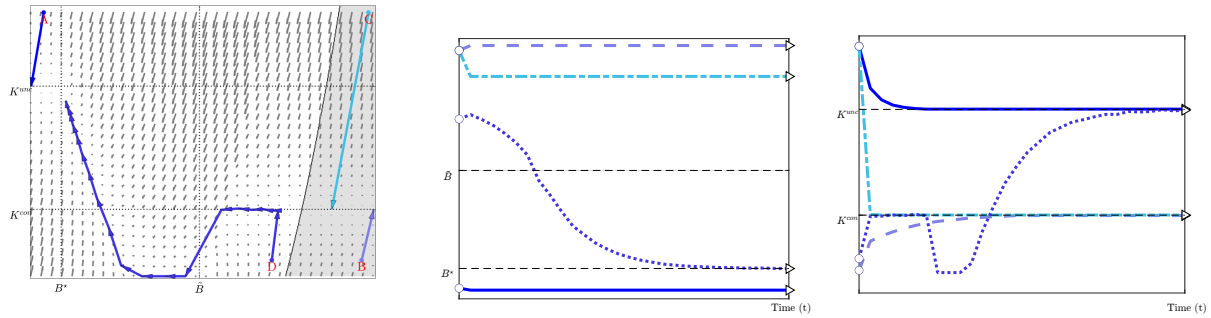
Panels (b) and (c) trace these debt and capital paths over time, confirming that the same two-domain pattern holds with production: entrepreneurs either converge quickly to efficiency or remain trapped in a state of persistent underinvestment. The same pattern of debt overhang, when debt starts high but deleveraging when debt is low, is observed in panel (b). Panel (c) shows how capital evolves depending on the marginal prices.

This section showed that entrepreneurs' incentives to remain at high debt levels also arise in problems with investment. Having shown this robust pattern, we now move to general equilibrium implications in the simplest setting, abstracting from investment.

5 General Equilibrium

The previous section analyzes how entrepreneurs choose between spot and chained purchases given prices and interest rates. We now embed these decisions into a general equilibrium where prices and interest rates are endogenous. This transition reveals how individual payment choices create aggregate externalities: when one agent uses chained purchases, they contribute to network congestion that affects the effective prices faced by others. These externalities, absent from partial equilibrium analy-

[24] A phase portrait draws the discrete vector field—where arrows indicate the direction and magnitude of change from each point—one (B, K) to (B', K').



(a) Phase Portrait (b) Path of Debt (c) Path of Capital

Figure 8: Investment and Leverage Decisions

Note: Figures are calculated using value function iteration with $\beta = xxx$, $\delta = xxx$, and $\tilde{B} = xxx \cdot \bar{B}$. Panels (a) and (b) xxx. The shaded area in panel (c) corresponds to the hysteresis region.

sis, drive the model's key insights about payment-chain crises and hysteresis.

Environment. As in Kiyotaki and Moore (1997b), the economy consists of two groups of agents who interact through both goods and financial markets. *Entrepreneurs* are the same agents studied in the previous section, specializing their endowment to $y = 1$. The new agents are *creditors* who hold positive financial wealth but receive no endowment. They have the same preference as entrepreneurs. Creditors always have positive wealth, D_t, earning the gross return R_{t+1} between periods. Hence, creditors serve as the economy's savers, providing the funds that entrepreneurs borrow. Creditors play a crucial role beyond simply closing asset markets: Their access to spot purchases provides a "liquidity buffer" that prevents the entire economy from collapsing when entrepreneurs can only place chained transactions.

Timing remains identical to the partial equilibrium problems above. Creditors solve:

Problem 3 (Creditor's (Recursive) Problem)**.** *Given D_0 and $\{R_{t+1}\}_{t\geq 0}$, the creditor chooses deposit holdings $\{D_{t+1}\}_{t\geq 0}$ to solve:*

$$V_t(D) = \max_{D' \geq 0} \log(C) + \beta V_{t+1}(D')$$

subject to the intertemporal budget constraint: $C + R_{t+1}^{-1} D' = D$.

The solution is standard for logarithmic preferences: $C_t = (1-\beta)D_t$ and $D_{t+1} = \beta R_{t+1} D_t$ for all $t \geq 0$. Their total expenditure each period is $C_t = (1-\beta)D_t$. Implicitly, we used the result that since creditors always have positive wealth, they exclusively make spot purchases, so we abstract away from their expenditure decisions—their budget

constraint also takes the form of (12).

Equilibrium. Clearing in the asset market requires the creditors' deposits exactly equal entrepreneurs' debt: $D_t = B_t$. Spot expenditures come from both creditors (who only make spot purchases) and entrepreneurs' spot purchases: $E_t^s = C_t + S_t$. Chained expenditures come only from entrepreneurs: $E_t^x = q_t X_t$. Total expenditures must equal the economy's endowment, since this is the numeraire. Since entrepreneurs receive $y = 1$ and creditors receive nothing: $E_t^s + E_t^x = 1$.

From the payments-chain technology (Theorem **??**), chained expenditures are $\mu_t = q_t X_t$, which determines the effective price through $q_t = \mathcal{A}^{-1}(\mu_t)$.

Definition 3 (Symmetric Competitive Equilibrium)**.** *Given a sequence of borrowing limits $\{\tilde{B}_t\}_{t\geq 0}$, sequence $\{B_t, C_t, S_t, X_t, R_t, q_t\}_{t\geq 0}$ is a **symmetric competitive equilibrium** it satisfies: **(I. Optimization)** Given $\{R_t, q_t\}_{t\geq 0}$, the sequences $\{B_t, S_t, X_t\}_{t\geq 0}$ solve the entrepreneur's problem and $\{C_t\}_{t\geq 0}$ solves the creditor's problem, **(II. Asset and Goods market clearing)** $D_t = B_t$, $C_t + S_t + X_t = \mathcal{Y}(q_t X_t)$ for all t, **(III. Price consistency)** $q_t = \mathcal{A}^{-1}(q_t X_t)$ for all t.*

The equilibrium definition is standard, except that q_t and $\mathcal{Y}_t$ depend on the amount of chained expenditures. Yet, the equilibrium definition conceals the transactions and delays that occur in the payments-chain network. Despite the presence of network features, the economy operates exactly as an economy with consumption externalities on $\mathcal{Y}$ and q aggravated when entrepreneurs rationally choose to make chained purchases.

Equilibrium Dynamics: Recursive Representation. A virtue of this setting is that it renders a tractable way to characterize the equilibrium recursively. Given that $D_t = B_t$, B_t is the single endogenous aggregate state. The state variable is the vector $\{B, \tilde{B}, \tilde{B}'\}$. A recursive equilibrium is a mapping from the state to the vector of prices R_{t+1}, q_t, and allocations, $\{C_t, S_t, X_t\}$, and future endogenous state B_t. Since only B is endogenous, we seek a mapping $\mathcal{B}$ that determines tomorrow's debt as a function of today's state: $B' = \mathcal{B}(B, \tilde{B}, \tilde{B}')$.

We begin by establishing that the time t allocation is a only function of B and $\tilde{B}$:

Proposition 4 (Equilibrium Allocation given $\{B, \tilde{B}\}$)**.** *Given debt B and spot borrowing limit $\tilde{B}$, the equilibrium allocation is characterized by:*

1. ***Creditor consumption**: $C(B) = (1-\beta)B$*

2. ***Entrepreneur expenditures:*** $E(B) = 1 - (1-\beta)B$

3. ***Spot purchases:*** $S(B, \tilde{B}) = \min\{\max\{0, \tilde{B} - B\}, E(B)\}$

4. ***Chained expenditures:*** $\mu(B, \tilde{B}) = E(B) - S(B, \tilde{B})$

5. ***Network price:*** $q(B, \tilde{B}) = \mathcal{A}^{-1}(\mu(B, \tilde{B}))$

6. ***Chained goods:*** $X(B, \tilde{B}) = \mu(B, \tilde{B})/q(B, \tilde{B})$

7. ***Average price:*** $Q(B, \tilde{B}) = E(B)/(S(B, \tilde{B}) + X(B, \tilde{B}))$

The debt mapping $\mathcal{B}$ emerges from combining the entrepreneur's and creditor's Euler equations with the market-clearing conditions. From the creditor's Euler equation, the gross real interest rate can be expressed as a function of B and B'; namely, $R(B, B') = B'/(\beta B)$. Thus, we can replace this gross rate in the entrepreneur's interior Euler equation, Proposition 2, to obtain

$$\underbrace{\frac{B}{E(B)} \cdot Q(B, \tilde{B})}_{\equiv \mathcal{E}(B;\tilde{B})} = \underbrace{\frac{B'}{E(B')} \cdot Q(B', \tilde{B}') \cdot \frac{q^E(B'; B, \tilde{B}, \tilde{B}')}{q^B(B'; \tilde{B}')}}_{\equiv \mathcal{E}'(B';B,\tilde{B},\tilde{B}')} \tag{14}$$

where $q^E(B'; B, \tilde{B}, \tilde{B}') = 1 + (q(B, \tilde{B}) - 1)\mathbb{I}_{[B' \geq B^\star(\tilde{B}')]}$ is the marginal expenditure price and $q^B(B'; \tilde{B}') = 1 + (q(B', \tilde{B}') - 1)\mathbb{I}_{[B' > \tilde{B}']}$ is the marginal borrowing price.

The debt mapping is determined by solving $\mathcal{E}(B; \tilde{B}) = \mathcal{E}'(B'; B, \tilde{B}, \tilde{B}')$ for B' when interior solutions are valid. Due to the discontinuous nature of payment choices, this equation may have multiple solutions. However, the partial equilibrium problem studied above tells us how to select the appropriate solution:

$$\mathcal{B}(B, \tilde{B}, \tilde{B}') = \max\left\{B^\star(\tilde{B}), \arg\min_{B'}\{\mathcal{E}(B; \tilde{B}) = \mathcal{E}'(B'; B, \tilde{B}, \tilde{B}')\}\right\}.$$

This mapping ensures that entrepreneurs never choose debt levels below the threshold $B^\star(\tilde{B})$ where they would be unconstrained, and among multiple solutions, they select the one that maintains continuity in their value function.

This recursive representation provides a recipe for solving the model: first, we solve for the equilibrium allocation and price functions as a function of $\{B, \tilde{B}\}$. Then, we solve for $\mathcal{B}$. We exploit this feature to obtain several equilibrium implications.

5.1 Implication I: Transitional Dynamics and Hysteresis

The aggregate Euler condition allows us to characterize the equilibrium dynamics, the possibility of hysteresis, and analyze payments-chain crises in response to shocks.

Steady States. In a steady state with constant exogenous credit limit $\tilde{B}_{ss}$, the endogenous debt level must also be constant: $B_{ss} = B_t$ for all t. From the creditor's solution, we know that a necessary condition for any steady state is that the real interest rate remains constant at $R_{ss} = \beta^{-1}$. A specific debt level constitutes a steady state if, given this interest rate and the resulting equilibrium price q, entrepreneurs optimally choose to maintain that same debt level. Since entrepreneurs face a corner solution at the steady-state rate $R_{ss} = 1/\beta$—either making all purchases through spot markets or all through chained markets—exactly two types of steady states can emerge.

In an *undisrupted steady state*, debt remains sufficiently low that entrepreneurs exclusively make spot purchases. This yields $q = 1$, immediate trade, and the economy operates at full potential with $\mathcal{Y}_{ss} = 1$. In a *disrupted steady state*, entrepreneurs are forced to rely entirely on chained purchases, resulting in $q > 1$, delayed trade, and output below potential: $\mathcal{Y}_{ss} < 1$.

The inefficiency in the disrupted steady state arises from a debt-overhang equilibrium: high debt levels force entrepreneurs into chained purchases, which raises the effective price q, making the consumption sacrifice required for deleveraging even more costly. This creates a self-reinforcing trap where entrepreneurs find it optimal to remain constrained rather than undertake the costly deleveraging process. Formally, these are given by the following regions:

Corollary 1 (Steady-State Characterization)**.** *Fix $\tilde{B}_t = \tilde{B}_{ss} > 0$. Then, for all t:*

i The economy is in an undisrupted steady state when $B_t \leq B^\star(\beta^{-1}, \tilde{B}_{ss})$.

ii The economy is in a disrupted steady state when $B_t \geq B^h(\beta^{-1}, \tilde{B}_{ss})$.

The disrupted steady state represents a hysteresis region where excessive debt permanently impairs the economy's payment efficiency. This mechanism provides a theoretical foundation for understanding persistent periods of low productivity observed during extended economic stagnation, such as Japan's lost decades or the prolonged aftermath of the European sovereign debt crisis.

Domain of Attraction toward Efficient Steady States. For debt levels between the two steady-state regions, the economy may experience transitional dynamics toward the undisrupted steady-state region. The domain of attraction toward undisrupted steady states is characterized by the condition that the aggregate Euler equation $\mathcal{E}(B;\tilde{B}) = \mathcal{E}'(B';B,\tilde{B},\tilde{B}')$ has a solution where $B' < B$—that is, entrepreneurs choose to deleverage over time. This domain has an upper bound solving:

$$B^*(\tilde{B}) \equiv \tilde{B}/(S(\tilde{B},\tilde{B}) + X(\tilde{B},\tilde{B}) + C(\tilde{B})) \geq \tilde{B},$$

where the inequality follows from total consumption being less than total expenditures when payments are delayed.

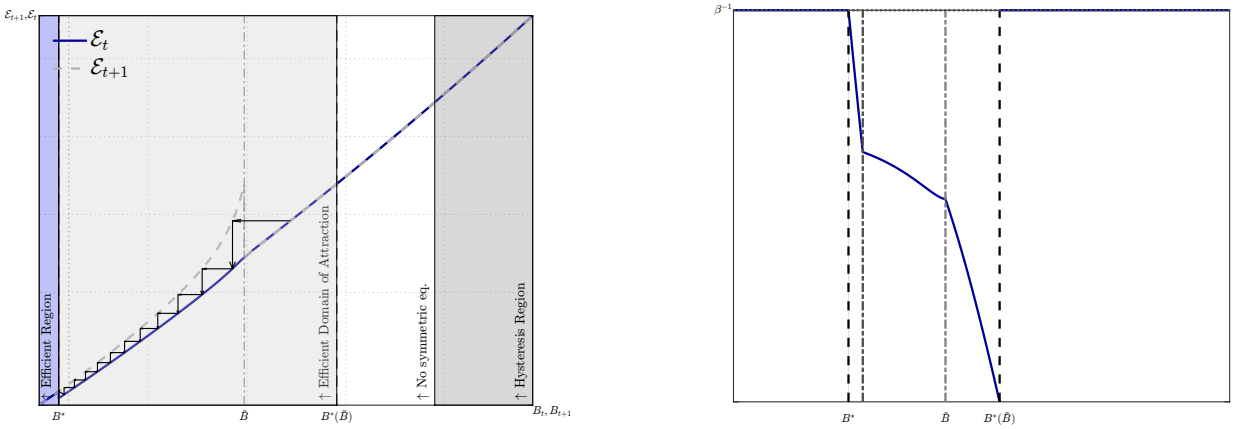
Corollary 2 (Convergence to Efficiency)**.** *Let $\tilde{B}_t = \tilde{B}_{ss}$. For any $B_0 < B^*(\tilde{B}_{ss})$:*

1. *If $B_t \in (B^\star(\beta^{-1},\tilde{B}_{ss}), B^*(\tilde{B}_{ss}))$, then $B_{t+1} < B_t$ (deleveraging occurs)*
2. *If $B_t \leq B^\star(\beta^{-1},\tilde{B}_{ss})$, then $B_{t+1} = B_t$ (steady state reached)*
3. *If $\tilde{B}_{ss} > 1$, the economy converges to an undisrupted steady state in finite time*

Thus, a payments-chain crisis is only *temporary* when debt starts in the domain of attraction toward efficient steady states. In this region, entrepreneurs find it optimal to gradually reduce their debt burden despite the short-term consumption sacrifice, because the future benefits of accessing spot markets justify the deleveraging costs.

Figure 9 illustrates these dynamics. Panel (a) plots the functions $\mathcal{E}$ (solid blue) and $\mathcal{E}'$ (dashed gray) against debt levels, holding $\tilde{B}$ fixed. The arrows trace the equilibrium path of debt: when the functions intersect below the 45-degree line, entrepreneurs deleverage ($B' < B$). The convergence region extends from $B^\star$ to B^*, while the gray shaded area represents the hysteresis region where no deleveraging occurs. Panel (b) shows how equilibrium interest rates vary across debt levels, remaining at β^{-1} in both steady-state regions but falling below this threshold during transitions.

Finally, we note that the domains of attraction toward efficient steady states and the hysteresis region are disconnected. When debt falls in the intermediate range $\{B^*, B^h\}$, symmetric competitive equilibria do not exist. In this middle region, entrepreneurs individually would want to deleverage, but in general equilibrium, this would require interest rates below β^{-1}, for which no solutions exist to the equilibrium condition. We leave the analysis of asymmetric equilibria for future work.



(a) Transition plot

(b) Equilibrium rates

Figure 9: Equilibrium Phase Diagram and Hysteresis Region

Note: Figures are calculated using value function iteration with $\beta = 0.95$, $\delta = 0.9$, and $\tilde{B} = 0.2 \cdot \bar{B}$. Panel (a) shows the phase diagram of B_t constructed using $\mathcal{E}(B'; B)$ and $\mathcal{E}'(B'; B, \tilde{B}, \tilde{B}')$. Panel (b) plots $R(B, \tilde{B}, \tilde{B}')$ over $[B^\star - 1,\ B^\star + 2]$. The shaded area corresponds to the hysteresis region.

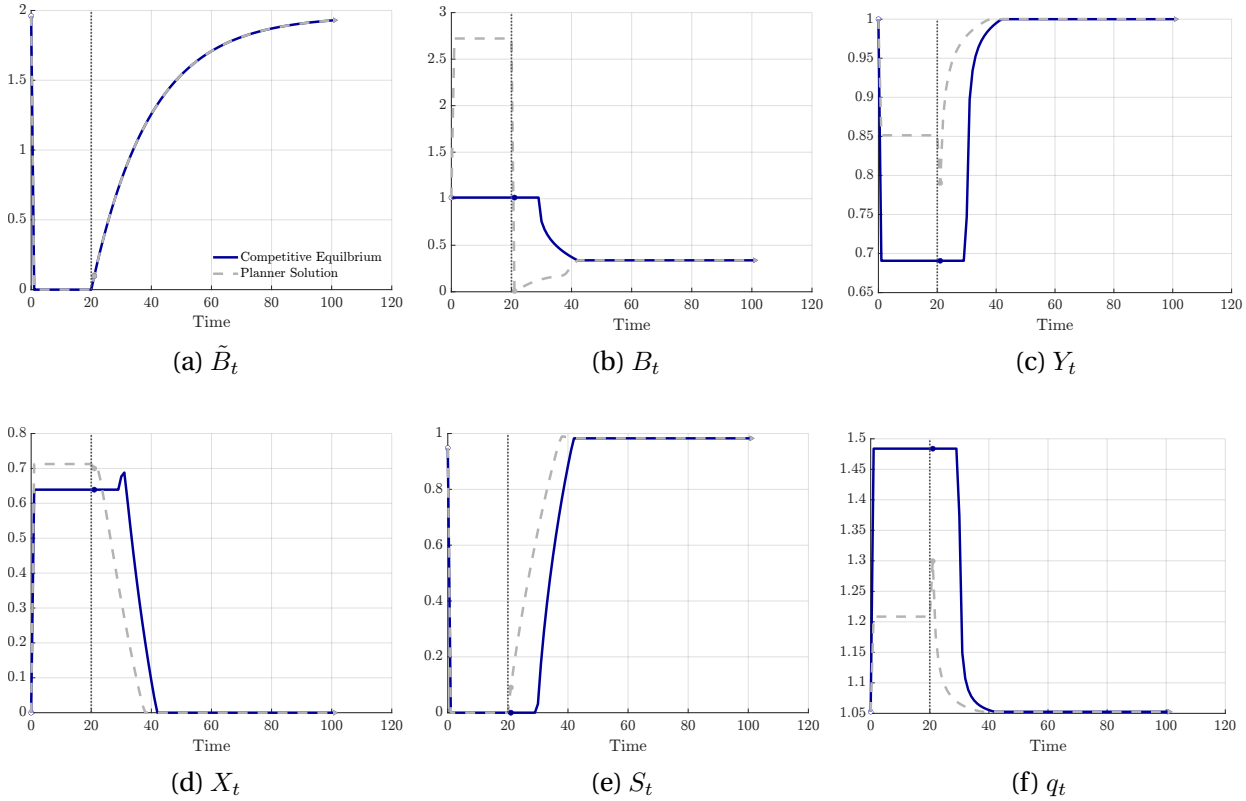
5.2 Implication II: Credit Crunches and Resilience Paradox

Credit Crunches. We now analyze the economy's response to temporary disruptions in credit availability. Consider a credit crunch modeled as a temporary tightening of borrowing limits $\{\tilde{B}_t\}$ that mean-reverts to its steady-state level. The debt mapping $\mathcal{B}$ from the aggregate Euler equation allows us to characterize the full transition path. As long as initial debt satisfies $B_0 \leq B^*(\tilde{B}_{ss})$ and $\tilde{B}_{ss} > 1$, the economy eventually reverts to an undisrupted steady state, but the adjustment process exhibits distinct phases.

Figure 10 illustrates a typical transition following a credit crunch. The solid line shows the competitive equilibrium response, which exhibits four distinct phases: The *extreme phase* occurs when $\tilde{B}_t$ falls to zero, eliminating access to short-term credit. Observed TFP remains depressed at its lowest value as entrepreneurs rely entirely on chained orders. Debt levels remain roughly constant during this phase because entrepreneurs focus on maintaining consumption rather than deleveraging, as the temporary nature of the shock reduces incentives for costly debt reduction. The *smoothing phase* begins once $\tilde{B}_t$ starts recovering toward its steady-state level. Entrepreneurs, anticipating future access to credit that will enable spot expenditures, increase current chained expenditures to smooth consumption across the transition. This forward-looking behavior temporarily raises debt levels despite the long-term goal of deleveraging, reflecting the tension between consumption smoothing and debt management. In the *recovery phase*, the borrowing limit has recovered sufficiently that entrepreneurs

begin trading off consumption smoothing against debt repayment. They gradually deleverage to expand their future credit capacity and regain access to spot markets. This phase sees the most active debt reduction as the benefits of lower debt become tangible. The *convergence phase* sees the economy reach the efficient steady state with permanently lower debt and fully restored TFP. All transactions occur through spot markets, and the final debt level is lower than the pre-shock level, reflecting the entrepreneurs' learned caution about over-leveraging.

The dashed line represents the constrained planner's solution, discussed in the next section, which provides a benchmark for evaluating the efficiency of the market response. This transition pattern holds generically for any credit shock that leaves the economy within the convergence region.



(a) $\tilde{B}_t$ (b) B_t (c) Y_t

(d) X_t (e) S_t (f) q_t

Figure 10: Efficient and Competitive Transition after a Credit Crunch

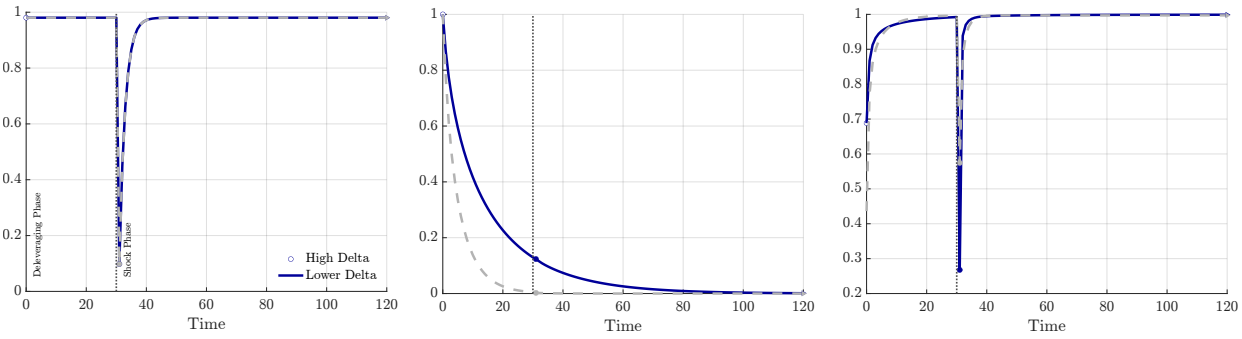
Note: This figure reports a numerical example of a credit-crunch episode. Figures are calculated using value function iteration with $\beta = 0.95$, $\delta = 0.9$, and $\tilde{B}_{ss} = 0.1 \cdot B^{\star}$. The crunch happens at period $t = 10$. For $t \in \{11, \ldots, 29\}$, $\tilde{B}_t = 0$. Then $\tilde{B}_t$ returns to steady state according to an AR(1) process with coefficient 0.9. The transition ends when $\tilde{B}_t = \tilde{B}_{ss}$.

The Resilience Paradox. A natural question arising from the credit crunch analysis is: what determines an economy's resilience to such shocks? We investigate how the

degree of payment delays—captured by parameter δ—affects economic resilience to credit line reductions. This analysis reveals a striking paradox: economies with lower δ (greater payment delays) may exhibit superior resilience to credit shocks, despite experiencing larger losses for any given level of chained purchases.

The mechanism underlying this resilience paradox operates through differential deleveraging incentives. Lower δ means higher payment delays in the payments-chain network, making chained purchases more costly relative to spot purchases. This creates stronger incentives for entrepreneurs to reduce debt levels during normal times to avoid being forced into expensive chained purchases during future shocks. Thus, lower δ induces a fundamental trade-off: greater payment inefficiency in the short run, but reduced vulnerability to credit disruptions through precautionary deleveraging.

Figure 11 demonstrates this paradox through a calibrated comparison. We examine two economies that begin with identical debt levels but differ in their payment efficiency parameter δ. After allowing sufficient time for each economy to reach its preferred debt level, we subject both to an identical credit shock—the reduction in $\tilde{B}_t$ shown in Panel (a).



(a) $\tilde{B}_t$ (b) B_t (c) Y_t

Figure 11: Comparison Resilience: xxx.

Note: This figure reports a numerical example of a credit crunch episode. Figures are calculated using value function iteration: $\beta = xxx$, $\delta = xxx$, and $\tilde{B}_{ss} = xxx \cdot B^{\star}$. The crunch happens at period t=xxx. For $t \in \{xxx\}$,$\tilde{B}_t = xxx$. Then $\tilde{B}_t$ returns to steady state according to an AR(1) process with coefficient xxx. The transition ends at t=xxx when $\tilde{B}_t = \tilde{B}_{ss}$.

Panel (b) reveals the key mechanism: when the shock hits, the low-δ economy (solid line) has achieved significantly lower debt levels than the high-δ economy (dashed line) due to stronger deleveraging incentives from costlier chained purchases. Panel (c) shows the paradoxical outcome: despite being less efficient at processing payments, the low-δ economy experiences a smaller output decline during the credit crunch because its lower debt levels reduce reliance on chained purchases when credit tightens.

This resilience advantage comes with important caveats. The paradox only manifests when the low-δ economy has had sufficient time to deleverage following the initial adjustment. For extremely severe shocks that eliminate credit lines entirely ($\tilde{B}_t \to 0$), the low-δ economy will necessarily perform worse due to its inherent payment inefficiencies. Additionally, both economies must begin in the deleveraging region; if initial debt levels place both in either the hysteresis region (too high) or efficient steady state (too low), no differential deleveraging occurs.

The resilience paradox captures a deeper economic principle: payment system inefficiencies, while costly during normal times, can induce precautionary behavior that builds shock resilience. Economies with more efficient payment systems may become overly comfortable with higher debt levels, leaving them vulnerable when credit conditions deteriorate. The latter resembles the Costa Rica case, in which the relatively short delays faced by many participants led to a prevalence of chained transactions, which was less pronounced in segments of the economy that featured longer delays. Moreover, in the Desyfin experiment, the abundance of chained participants paired with the liquidity freeze exacerbated the economic losses given the shock. Thus, both the theoretical and empirical evidence suggest that some degree of "beneficial inefficiency" in payment systems may serve as a natural stabilizer against financial crises.

5.3 Implication III: Constrained Inefficiency

In Section 3, we demonstrated that the organization of payments leads to productive inefficiencies—a planner would reorganize payments so that each chain is of equal length if it could, a Pareto improving re-organization. In practice, governments may not be able to directly reorganize payments. However, they can influence aggregate expenditures via policy interventions.[25] This section studies a planner problem that respects the transactions technology and constraints of entrepreneurs, but can directly choose the path of debt. This analysis allows us to outline constrained inefficiencies stemming from aggregate expenditure decisions, treating the payments-chain network as given. The spirit of our exercise is to clarify the pecuniary externalities which differ from the organizational inefficiencies of the payments technology.

[25] Traffic regulation offers an analogy: governments cannot assign drivers to a lane, but they can tax vehicle purchases to influence traffic patterns.

The Primal Planner Problem. To study constrained inefficiency, we consider a social planner with Pareto weights θ on entrepreneurs and $(1-\theta)$ on creditors. We endow the planner with weights that guarantee the same allocation of some undisrupted steady-state competitive equilibrium. Since the economy is efficient in an undisrupted steady state, this allows us to compare inefficiencies along the transition where payment-chains are relevant. This requires setting θ such that:

$$\frac{S_{ss}}{C_{ss}} = \frac{Y - (1-\beta)\, B_{ss}}{(1-\beta)\, B_{ss}} = \frac{\theta}{1-\theta}, \tag{15}$$

for some undisrupted steady-state level of debt $B_s s$.

We analyze a primal planner who directly chooses debt sequences $\{B_t\}_{t\geq 0}$ to maximize welfare, respecting the transactions technology and payment-chain frictions. The planner's problem is:

Problem 4. (Primal Planner Problem): *Taking* $\left\{\tilde{B}_t\right\}$ *as given:*

$$\max_{\{B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \mathcal{P}\left(B_t, \tilde{B}_t\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) \equiv (1-\theta)\log\left((1-\beta)B\right) + \theta \log\left(S\left(B, \tilde{B}\right) + X\left(B, \tilde{B}\right)\right), \tag{16}$$

and taking as given $S\left(B, \tilde{B}\right)$ *and* $X\left(B, \tilde{B}\right)$ *as defined by Proposition 4.*

The planner takes the payment-chain structure as given through $S(B, \tilde{B})$ and $X(B, \tilde{B})$. This formulation reveals the pecuniary externalities distinct from the organizational inefficiencies of the payments technology.

The objective is time-separable because current debt determines creditor expenditures through log utility, with entrepreneur expenditures following residually. As the objective is time-separable, the optimal debt at t depends only on the current SBL $\tilde{B}_t$:

$$B_t = B^p(\tilde{B}_t) = \arg\max_{B\in[0,\bar{B}]} \mathcal{P}\left(B, \tilde{B}_t\right).$$

The allocation can be implemented through a sequence of debt and income taxes—see Appendix F.2.

The Planner Solution. The planner's solution reveals that during credit crunches, debt may be either inefficiently high or low. The planner has two ways to increase productive efficiency: redistribute wealth toward entrepreneurs (reducing debt to free credit lines) or toward creditors (increasing debt to stimulate spot expenditures). This ambivalent nature leads to four qualitatively different regimes depending on the tightness of credit constraints.

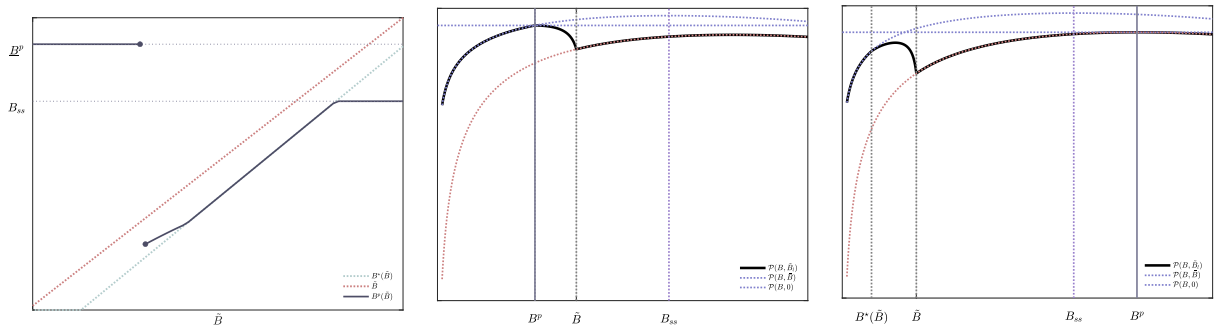
When $\tilde{B}$ is sufficiently high, the economy operates efficiently with $\mathcal{A} = 1$ and optimal social insurance. As $\tilde{B}$ falls below the efficiency threshold, the planner faces a trade-off between productive efficiency and social insurance. In moderate constraint regions, the planner sacrifices insurance but maintains efficiency. In severe constraint regions, the planner accepts some productive inefficiency, balancing efficiency gains against insurance costs. Most strikingly, when constraints become extreme, the planner reverses strategy entirely: rather than helping the constrained entrepreneur, the planner redistributes wealth toward creditors, *the unconstrained agent*. This seemingly paradoxical result occurs because marginal increases in creditor wealth generate more spot expenditures than the efficiency losses from constraining entrepreneurs.

This ambivalent nature distinguishes payment-chain crises from typical models with pecuniary externalities, where social insurance and efficiency typically reinforce each other. Here, the planner may switch between complementary and conflicting policies as constraint severity varies.

Figure 10 illustrates how the planner's solution and the competitive equilibrium differ during a payment-chain crisis. Initially, the planner implements regressive policies—by increasing debt which constraints further the entrepreneur—because increased creditor wealth generates spot expenditures that raise TFP. Despite higher dent, entrepreneurs actually increase their consumption through lower chained-good prices. As recovery progresses, the planner reverses course, reducing debt relative to the competitive equilibrium.

5.4 Implication IV: Fiscal Policy (Bocola Effect)

Government expenditures have strikingly different effects depending on their payment timing. When governments pay for expenditures upfront (spot), they inject immediate liquidity into payment chains. When governments chain expenditures to future tax receipts, they extend payment chains and worsen delays. We call this the Bocola effect.



(a) Planner debt choice (b) $\mathcal{P}\left(\cdot, \tilde{B}\right)$ (moderate $\tilde{B}$) (c) $\mathcal{P}\left(\cdot, \tilde{B}\right)$ (tight $\tilde{B}$)

Figure 12: Primal Planner Solution

Note: This figure reports a numerical example of the planner's solution. Figures are calculated using value function iteration: $\beta = 0.95$, $\delta = 0.9$, and $\theta = 0.75$. Panel (a) plots B^p as a function of values of $\tilde{B}$ in the range $[0.1, \theta + 0.1] * B^{star}$. Panels (b) and (c) plot the objective of the planner for different values of B in the range $[0, 1.8]$. Panel (b) fixes $\tilde{B}$ at 0.15 and Panel (c) at 0.08.

To formalize this distinction, consider spot government expenditures G^s that require immediate payment, analogous to household spot purchases. The government can also make chained expenditures G^x contingent on receiving tax revenues, treating tax receipts as income flows similar to labor income. For simplicity, assume the government raises labor taxes to finance expenditures under a balanced budget constraint, and that all government spending represents resource waste. The key insight emerges from how these expenditures affect the aggregate fraction of chained transactions:

$$\mu_t = q_t X_t^w + G_t^x \quad \text{and} \quad q_t = \mathcal{A}(\mu_t)^{-1}.$$

Spot government expenditures reduce the chained expenditure ratio by substituting for private chained purchases, while chained government expenditures increase μ_t.

Proposition 5. (Government Spending Multipliers): *Consider infinitesimal government expenditures starting from* $(G^x, G^s) = (0, 0)$. *Consumption and output multipliers are:*

Efficient Region *($B \leq B^*(\tilde{B})$): Both expenditure types reduce welfare:*

$$\frac{\partial C^w}{\partial G^x} = \frac{\partial C^w}{\partial G^s} = -1, \quad \frac{\partial \mathcal{Y}}{\partial G^s} = 0, \quad \frac{\partial \mathcal{Y}}{\partial G^x} = \mathcal{A}(\mu)(1 + \epsilon_\mu^{\mathcal{A}}) - 1 < 0.$$

Crisis Region *($B > B^*(\tilde{B})$): Spot and chained expenditures have opposite effects:*

$$\frac{\partial C^w}{\partial G^x} = -\mathcal{A}(\mu), \quad \frac{\partial C^w}{\partial G^s} = -\mathcal{A}(\mu)(1 + \epsilon_\mu^{\mathcal{A}}), \quad \frac{\partial \mathcal{Y}}{\partial G^x} = 0, \quad \frac{\partial \mathcal{Y}}{\partial G^s} = 1 - \mathcal{A}(\mu)(1 + \epsilon_\mu^{\mathcal{A}}) > 0.$$

The economic mechanism is clear. In efficient regions, any government spending wastes resources without improving payment chain efficiency. Spot spending merely transfers efficient expenditures from private to public use, yielding zero output effects. Chained spending worsens efficiency by lengthening payment chains.

During payment-chain crises, the effects reverse dramatically. Chained government expenditures remain harmful, adding to payment delays without productivity benefits. However, spot government expenditures create positive externalities by reducing average chain lengths. The elasticity $\epsilon_{\mu}^{\mathcal{A}}$ captures this effect: when sufficiently negative (which occurs in deep crises), spot spending can actually increase entrepreneur consumption despite higher taxes. The mechanism works through reduced prices of chained goods, which more than offset the tax burden.

The output multiplier for spot expenditures is always positive during crises because the government substitutes inefficient private chained expenditures for efficient public spending that shortens payment chains. Welfare increases when this multiplier exceeds unity, which occurs precisely when payment delays are most severe.

This analysis yields a novel perspective on fiscal stimulus. The conventional view that governments can stimulate aggregate demand simply by spending misses a crucial element: the timing of payments matters as much as the amount spent. A government that finances expenditures through future tax receipts—the typical approach—may inadvertently reduce welfare and have no output effects. Effective stimulus requires paying upfront to break payment chains rather than extend them. This dimension differs fundamentally from standard aggregate demand arguments. Here, fiscal policy works through a transactions channel: it stimulates output only when it speeds transactions.

6 Conclusion

Financial crises have traditionally been understood through the lens of coordination failures and demand externalities, yet what makes these episodes uniquely susceptible to such problems remains unclear. The contribution of this paper is to highlight payment delays as a central mechanism. We present empirical evidence showing that many firms depend on upstream payments to execute their own transactions, and demonstrate through a natural experiment how payment disruptions propagate through these chains during financial stress.

Building on this evidence, we develop a theory where the core economic problem is

that credit constraints lead to an inefficient distribution of funds, causing payment delays that disrupt production. Crises in this framework feature demand externalities and coordination failures that arise endogenously from the payment structure. Although our policy recommendations resemble those in environments with price rigidity, we underscore the critical importance of accelerating transactions through appropriate fund transfers.

For tractability, we make several key simplifications. We assume all transactions are bilateral and of identical size, and that delays occur within periods rather than across periods. We also abstract from risk considerations. Developing extensions with richer transaction sequences and heterogeneous delays will be important for quantitative applications. Nonetheless, we believe the core lessons about payment-chain disruptions are robust to such generalizations.

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Appendix (nNot intended for publication)

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A Appendix to the Empirical Analysis - Section 2

A.1 Additional Results

Table A.1: Heterogeneous Effect by Size of Amounts Sent/Received

Dependent variable: Prob. firm j makes a payment at time t

	(1)
$received_{j,t-1} \times smaller\ sent_{jt}$	0.526
	(0.002)***
$received_{j,t-1}$	-0.109
	(0.001)***
Adjusted R^2	0.535
Observations	10,001,248
clusters	222,306
$t, j \times$ d FE	Yes

Notes: The independent variable $received_{j,t}$ is an indicator equal to one if firm j received a transfer from another firm within the last 24 hours. This variable is interacted with $smaller\ sent_{jt}$, which equals one if the amount received by firm j at time $t-1$ is less than the amount sent by firm j at time t, and zero otherwise. Robust standard errors, clustered by firm, are in parentheses. We include time and firm×weekday fixed-effects. Data is daily and spans 2024.

A.2 Intensive Margin Regressions

We also study the intensive margin relationship between payments made and received. We follow equation (1), but now considering the (inverse hyperbolic sine of the) amounts sent and received by firm j. Results are presented in Table A.2 consistent with those of the extensive margins; firms become more likely to make payments once they receive money transfers. The elasticity across all firms in column (1) implies that a 1% increase in the amounts received by a firm on a given day increases the amounts sent the next day by approximately 0.04%.

Table A.2: Triggered Payments Across Firms—Intensive Margin

Dependent variable: Payments made by firm j at time t (asinh)

	(1)	(2)	(3)	(4)	(5)	(6)
$\ln amount\ received_{j,t-1}$	0.036	0.017	0.045	0.044		0.036
(*asinh*)	(0.0005)***	(0.001)***	(0.001)***	(0.001)***		(0.0005)***
$Small_j \times \ln amount\ received_{j,t-1}$		0.016				
		(0.001)***				
$\frac{Assets_j}{Sales_j} \times \ln amount\ received_{j,t-1}$			-0.019			
			(0.001)***			
$\frac{Liabilities_j}{Sales_j} \times \ln amount\ received_{j,t-1}$				-0.018		
				(0.001)***		
$\ln amount\ received_{j,t-5}$					0.012	0.012
(*asinh*)					(0.0004)***	(0.0004)***
Adjusted R^2	0.496	0.464	0.497	0.497	0.495	0.496
Observations	10,001,248	5,290,408	8,399,599	8,399,599	10,001,248	10,001,248
Clusters	222,306	74,566	174,896	174,896	222,306	222,306
$t, j \times$ d FE	Yes	Yes	Yes	Yes	Yes	Yes

Notes: The independent variable in column (1) is the total amount (in colones) received by firm j from another firm within the last 24 hours, transformed using the inverse hyperbolic sine function. This variable is interacted with firm j's size, assets-to-sales and liabilities-to-sales in columns (2), (3), and (4) respectively. Columns (5) and (6) add an independent variable equal to the total amount (in colones) received by firm j from another firm 5 days before time t, transformed using the inverse hyperbolic sine function. Robust standard errors, clustered by firm, are in parentheses. We include time and firm×weekday fixed-effects. Data is daily and spans 2024.

A.3 Placebo Exercises

Table A.3: Placebos—Receiving Payments Right After Making a Transfer

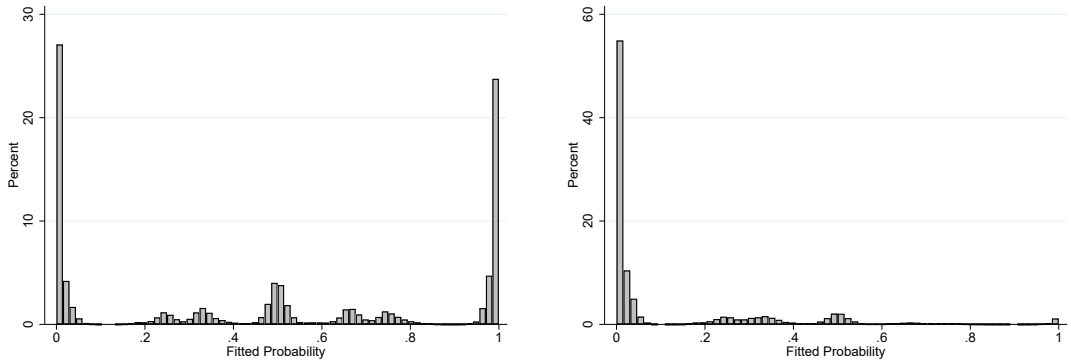
Dependent variable: Payment made by firm j at time t

	A. Extensive margin				B. Intensive margin (*asinh*)		
	$h=1$	$h=2$	$h=3$		$h=1$	$h=2$	$h=3$
	(1)	(2)	(3)		(4)	(5)	(6)
$received_{j,t+h}$	-0.001	-0.003	-0.001	$\ln amount$	-0.001	-0.002	0.000
	(0.0004)***	(0.0004)***	(0.0004)	$received_{j,t+h}$	(0.0004)	(0.0004)***	(0.0004)
Adjusted R^2	0.483	0.483	0.483		0.495	0.495	0.495
Observations	10M	10M	10M		10M	10M	10M
Clusters	222,306	222,306	222,306		222,306	222,306	222,306
$t, j \times$ d FE	Yes	Yes	Yes		Yes	Yes	Yes

Notes: In panel A, the independent variables are indicators equal to one if firm j received a payment in $t+h$. In panel B, the independent variables are the total amounts (in colones) received by firm j from another firm in $t+h$, transformed using the inverse hyperbolic sine function. Robust standard errors, clustered by firm, are in parentheses. We include time and firm×weekday fixed-effects. Data is daily and spans 2024.

A.4 Results on Firm Classification and Chains

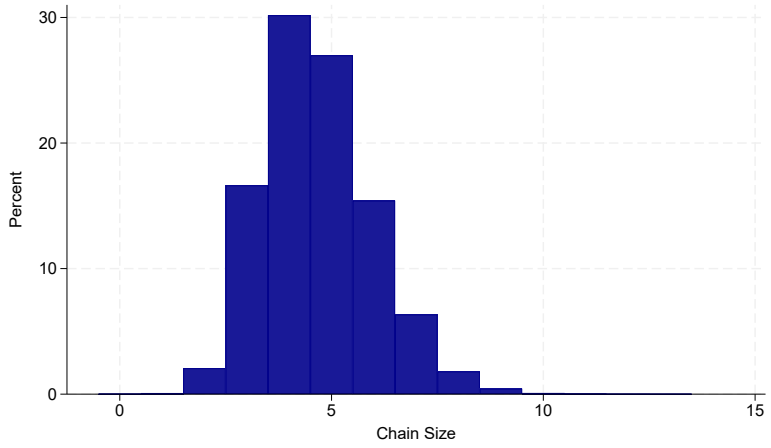
Figure A.1: Histogram of Fitted Probabilities of Triggered Payments by Firm



(a) Payments within 24 hours (b) Payments five days after

Notes: The figures show the distribution of monthly average triggered payments by firm based on an extensive margin estimation. Firms with a value of zero have a payment behavior that is orthogonal to receiving money transfers, while firms with a value of one have a perfect correlation between receiving a transfer and making a payment either the day after (panel a) or five days later (panel b). Data is monthly and spans 2024.

Figure A.2: Distribution of Chain Sizes



Notes: The figure shows the distribution of chain sizes based on a 5% random sample of spot firms in the economy used as initial nods. Data is collapsed for December 2024 to construct the chains.

Table A.4: Classification of Transactions by Month

Month	S→S	X→X	Mixed	Total	Month	S→S	X→X	Mixed	Total
1	1%	72%	28%	277,758	7	1%	72%	28%	291,866
2	1%	71%	28%	278,278	8	1%	72%	27%	276,308
3	1%	72%	28%	262,887	9	1%	72%	28%	280,320
4	1%	72%	28%	291,672	10	1%	71%	28%	290,711
5	1%	72%	28%	293,231	11	1%	71%	28%	282,262
6	1%	72%	28%	267,484	12	1%	72%	27%	287,585

Notes: The table shows the stability of the classification of transactions across months, depending on the buyer's and seller's type. Link types are divided into spot-to-spot (S→S), chained-to-chained (X→X), or mixed (S→X or X→S).

A.5 Chained Payments and Prices

We consider the following regression to explore the role of deferred payments on prices:

$$\ln Price_{vpjt} = \delta + \gamma\, Deferred_{vpjt} + \lambda_p + \lambda_j + \lambda_t + \varepsilon_{vpjt}$$

for transaction v, where $Deferred_{vpjt} = 1$ if sale v has a deferred payment and zero otherwise, and we include product, firm, and time fixed effects. We can also interact $Deferred_{vpjt}$ with an indicator, $Chained_{kt} = 1$, which depends on whether firm k—which made payment v to firm j—was classified as chained.

Column (1) of Table A.5 shows that prices for a given product are 21% higher in transactions with a deferred payment. Column (2) shows this effect is larger for firms classified as chained. Column (3) further breaks down transactions depending on the type of both the receiver (j) and sender (k) of the payment. Reassuringly, a higher price is charged only if the payment is made by a chained firm, regardless of the receiver's type.

Table A.5: Chained Payments and Prices

Dependent variable: Price of transaction v for good p sold by firm j to firm k at time t (log)

	(1)	(2)	(3)
$Deferred_{vpjt}$	0.189	0.111	0.181
	(0.006)***	(0.006)***	(0.006)***
$Chained_{kt} \times Deferred_{vpjt}$		0.091	
		(0.006)***	
$Chained_{jt}$&$Chained_{kt} \times Deferred_{vpjt}$			0.065
			(0.002)***
$Spot_{jt}$&$Chained_{kt} \times Deferred_{vpjt}$			0.058
			(0.003)***
$Spot_{jt}$&$Spot_{kt} \times Deferredt_{vpjt}$			0.006
			(0.004)
Adjusted R^2	0.682	0.682	0.682
Observations	45,801,998	45,801,998	45,801,998
Clusters	175,829	175,829	175,829
p, j, t FE	Yes	Yes	Yes

Notes: The independent variable in column (1) is an indicator equal to 1 if transaction v;s payment was deferred, as opposed to having an immediate payment. This variable is interacted with a dummy that depends on firm k's classification as chained or spot in column (2). Column (3) further decomposes transactions depending on whether firm j and k were chained or spot at t; the omitted category has firm j as chained and firm k as spot. Robust standard errors, clustered by firm, are in parentheses. We include product, firm, and time fixed-effects. Data is by sale and spans 2024.

A.6 Payments and Timing of Delivery

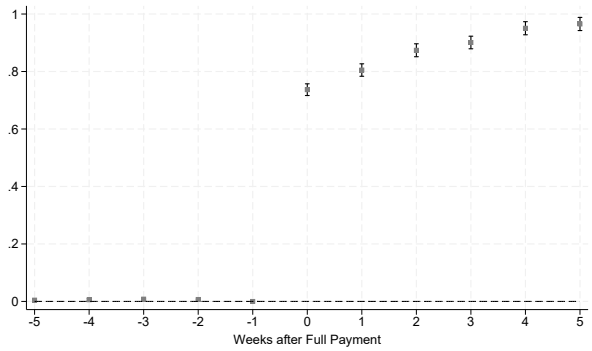
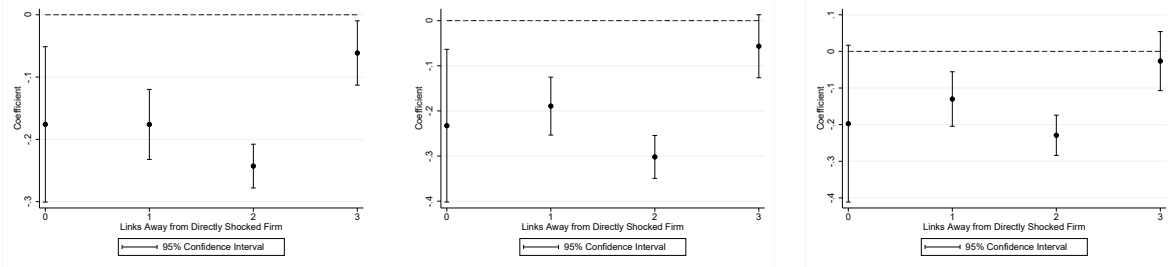


Figure A.3: Timing of Product Delivery

Notes: The figure shows an event study for orders with deferred payment in which product sales become positive after the *full* payment is due. Time zero corresponds with the date in which the full payment is due and the coefficient represents the sales of the product at the store.

A.7 Persistence of Desyfin Effect Along the Chain

Table A.6: Desyfin Effect Along the Chain: Firms Upstream From Directly Hit Firm



(a) $\ln Purchases_{jt}$ (b) $\ln Sales_{jt}$ (c) $\ln Value\ Added_{jt}$

Notes: The figure shows how the impact of being upstream from a Desyfin firm evolves as increase the number of links of separation along the chain. The dependent variables in panels (a), (b), and (c) are the (log) of purchases, sales, and value added, respectively, of firm j in month t. The plotted coefficients correspond with the triple interaction of $Links(n)_{jt} \times Switch2chained_{it} \times Desyfin_i \times Shock_t$, and captures the impact of being n links away from a Desyfin firm i ($Links(n)_{jt}$) *and* firm i switching to being classified as chained *after* the freeze. Robust standard errors are clustered by firm. We include province×industry (ISIC-4) and time fixed-effects. Data is monthly and spans 2024.

B Appendix to Payments-Chain Model - Section 3

B.1 Microfoundations for formula for y_n

We introduce two financial frictions that delay payments and, consequently, production. The first friction is that production can only start once the buyer shows proof of funds. If an order is spot, by definition, proof of funds is immediate since the order is funded externally. By contrast, chained orders can provide evidence of funds only after the buyer receives a payment from another transaction and the money enters their account. Production requires proof of funds to deter fraudulent behavior; otherwise, buyers could promise payments they know will never occur. Since production is customized, this would lead to a disadvantageous ex-post renegotiation.

We need a reason to delay transactions: here, we invoke limited commitment. The idea is that after the customer shows proof of funds, the funds are only released after the fraction $1-\delta$ of the output is inspected. Without inspection, The producer could produce a good for himself, taking away the funds. Since there is no way to verify a customization, inspection is necessary. The delay in transactions provokes production losses. Namely, the payments-chain network determines production: production occurs within a unit time interval. Each order starts at some endogenous time $\tau \in [0,1]$ when the payment is executed. Once production starts, it cannot stop. Production is linear in time: if production starts at τ, production is $1-\tau$. The delay in the time to execute payments then translates into idle resources which map into TFP losses.

Placing these frictions together, proof of funds and limited commitment, we obtain a production structure where the greater the number of chained orders, the more payment delays and the longer the production delays. To produce predictions about aggregate TFP, we first formalize the definition of the payments chain network and derive statistical properties that help us map it into production losses.

The length distribution is convenient for obtaining a measure of economic losses. For that, we turn to our specific micro-foundation.

We use this distribution to obtain aggregate production: we first derive the production in a chain of arbitrary length. In a chain of length zero, there is always a spot order whose production begins immediately. In a chain of length one, the production of the first spot order begins immediately, but there is a delay in the second order. For the second order, the funds are received after $1-\delta$ of the production of the first order is

finished. This happens at time $\tau = 1 - \delta$. Hence, this leaves only δ time to produce the second order in the chain. For the second order, production is δ of which the $1 - \delta$ fraction must be inspected. If there is a third order in the chain, the transfer of funds occurs $(1 - \delta)\,\delta$ time after the first transfer at $1-\delta$. Adding these consecutive delays, the production of the third order can only start by $(1 - \delta) + \delta\,(1 - \delta) = 1 - \delta^2$. This leaves δ^2 time to produce that third order.

We can deduce a pattern by forward induction.[26] In a chain of length n, the production vector of $n + 1$ consecutive orders in a chain is $\{1, \delta, \delta^2, \ldots, \delta^n\}$. Using the distribution of chain lengths to compute expectations, we obtain aggregate production.

B.2 Random and Efficient Assignment Outcomes (Proof of Proposition 1)

Proposition 1 presents several results for random and efficient assignments. We start with the formulas for random assignments

B.2.1 Random Assignments.

Chain-Length Distribution. We first show that the distribution of payment-chain lengths is geometric.

Lemma 2. *For the random assignment, the chain-length distribution is geometric.*

Proof. Recall that the order assignment is random and nodes are either spot orders ($\mathcal{N}^s$) or chained ($\mathcal{N}^x$). As $N \to \infty$, each node has an outgoing link with probability μ (the order buys from a product committed to another chained order) or no incoming link with probability $1 - \mu$, the link is with a spot order.

By construction, every chained order must be funded through exactly one production link, creating a directed path that ultimately terminates at a spot order. This structure ensures that every node belongs to exactly one payment chain, and each chain originates from a unique spot order. Therefore, we have exactly N^s payment chains, which we index by $i \in \mathcal{N}^s$.

[26] If the k-th node initiates production at time $1 - \delta^{k-1}$, the delay from its inspection is $(1 - \delta)\,\delta^{k-1}$, which added to previous delays leads to a transfer of funds only by time $1 - \delta^{k-1} + (1 - \delta)\,\delta^{k-1} =: 1 - \delta^k$. This, leaves δ^k time for production to the subsequent unit. Since we computed the delay for $k = 1, 2, \ldots$. the productions in a chain of length n are $\{1, \delta, \delta^2, \ldots, \delta^n\}$.

For chain i originating at spot order i, we define its length as the number of chained orders it contains (equivalently, the number of edges in the directed path). We now derive the distribution of chain lengths.

Consider chain i: It is of **length 0** if spot order i orders from a node associated with a spot order, which occurs with probability $1-\mu$. It is of **length 1** the first order in the chain orders from a chained order, which happens with probability μ, whereas that order is not linked, which occurs with probability $1-\mu$. Thus, the overall probability is $\mu(1-\mu)$. Proceeding by induction, for a chain of **length** n, extending this reasoning, there must be n consecutive chained orders, followed by a termination (no further links). By independence, this occurs with probability $\mu^n(1-\mu)$. Therefore, if L_i denotes the length of chain i, we have:

$$\mathbb{P}(L_i = n) = \mu^n(1-\mu) \quad \text{for } n \in \{0, 1, 2, \ldots\}$$

, i.e., the geometric distribution with parameter $1-\mu$ (using the convention where support begins at 0). □

Output and TFP. Next, we derive the distribution of positions for the geometric distribution.

Lemma 3. *When the chain length distribution is geometric, the distribution of positions is also geometric.*

Proof. For the geometric distribution:

$$(n-1) = (1-\mu)\sum_{m=0}^{n-1}\mu^n = (1-\mu^n), \quad n \geq 1$$

. Therefore, we have that, $1 - G(n-1) = \mu^n$, and, thus,

$$\sum_{n}^{\infty}(1 - G(n-1)) = \sum_{n=0}^{\infty}\mu^n = \frac{1}{1-\mu}.$$

Thus, we have that:

$$h(n) = \frac{\mu^n}{1/(1-\mu)} = (1-\mu)\mu^n.$$

Thus, the distribution of positions follows a geometric distribution with the same parameter $1-\mu$, completing the proof. □

Next, we derive the formula for TFP, $\mathcal{A}$, and output $\mathcal{Y}$ using two approaches. We first do so by employing the distribution of positions.

Lemma 4. *For the random assignment we have that* $\mathcal{Y} = \frac{1-\mu}{1-\delta\mu}$ *and* $\mathcal{A} = \delta\frac{1-\mu}{1-\delta\mu}$.

Calculations using the Position Distribution h.

Proof. Given that the distribution of positions $h(n) = (1-\mu)\mu^n$ the average quantity of final goods,

$$\mathcal{Y} \equiv \mathbb{E}[y_n] = \mathbb{E}[\delta^n] = \sum_{n=0}^{\infty}(1-\mu)\mu^n\delta^n = \frac{1-\mu}{1-\delta\mu}.$$

Then, it follows that $\mathcal{Y} = (1-\mu) + \mu\mathcal{A} \to \mathcal{A} = \delta\frac{1-\mu}{1-\delta\mu}$. □

Calculation using the Length Distribution.

Alternative Proof via Weighted Averaging. We demonstrate a general approach for deriving position distributions from chain length distributions, applicable even when direct computation is complex. We illustrate with the geometric case where $\mathbb{P}(L = n) = (1-\mu)\mu^n$.

Step 1: Expected production per chain. For a chain of length $n \geq 1$, the cumulative production discount is:

$$E[y_n|\text{length} = n] = \sum_{i=1}^{n}\delta^i = \frac{\delta(1-\delta^n)}{1-\delta}$$

where $\delta \in (0,1)$ represents the production efficiency at each stage.

Step 2: Average chain length. The expected number of chained orders per chain (excluding spot orders) is:

$$\bar{n}_x = \sum_{n=1}^{\infty} n \cdot \mathbb{P}(L = n) = \sum_{n=1}^{\infty} n(1-\mu)\mu^n \tag{17}$$

$$= (1-\mu)\mu\sum_{n=1}^{\infty} n\mu^{n-1} \tag{18}$$

$$= (1-\mu)\mu \cdot \frac{d}{d\mu}\left(\frac{1}{1-\mu}\right) \tag{19}$$

$$= (1-\mu)\mu \cdot \frac{1}{(1-\mu)^2} = \frac{\mu}{1-\mu} \tag{20}$$

Step 3: Position distribution via weighted averaging. To find the distribution of positions, we weight each chain length by its contribution to the total mass of chained orders:

$$\text{Weight for length } n = \frac{n \cdot \mathbb{P}(L = n)}{\bar{n}_x}$$

The probability that a randomly selected chained order is at position k equals:

$$\mathbb{P}(X = k | X > 0) = \sum_{n=k}^{\infty} \frac{(1-\mu)\mu^n \cdot 1}{\mu/(1-\mu)} = (1-\mu)^2 \sum_{n=k}^{\infty} \mu^{n-1} = (1-\mu)\mu^{k-1}$$

Accounting for spot orders (position 0), we have:

$$\mathbb{P}(X = k) = \begin{cases} 1-\mu & \text{if } k = 0 \\ \mu \cdot (1-\mu)\mu^{k-1} = (1-\mu)\mu^k & \text{if } k \geq 1 \end{cases}$$

This confirms that positions follow a geometric distribution with parameter $1-\mu$.

□

Part 2. Limits. We first consider the limit as $\mu \to 0$:

$$\lim_{\mu \to 0} \mathcal{A}(\mu; \delta) = \delta \lim_{\mu \to 0} \left(\frac{1-\mu}{1-\delta\mu} \right) = \delta.$$

For output:

$$\lim_{\mu \to 0} \mathcal{Y}(\mu; \delta) = \lim_{\mu \to 0} (1-\mu) + \mu \mathcal{A}(\mu; \delta) = 1.$$

Next, we consider the limit as $\mu \to 1$:

$$\lim_{\mu \to 1} \mathcal{A}(\mu; \delta) = \delta \lim_{\mu \to 1} \left(\frac{1-\mu}{1-\delta\mu} \right) = 0.$$

For output:

$$\lim_{\mu \to 1} \mathcal{Y}(\mu) = \lim_{\mu \to 1} (1-\mu) + \lim_{\mu \to 1} \mu \lim_{\mu \to 1} \mathcal{A}(\mu) = 0.$$

Next, we consider the limit as $\delta \to 0$:

$$\lim_{\delta \to 0} \mathcal{A}(\mu; \delta) = \lim_{\delta \to 0} \frac{\delta - \delta\mu}{1-\mu\delta} = 0.$$

For output,

$$\lim_{\delta\to 0}\mathcal{Y}(\mu)=(1-\mu)+\mu\lim_{\delta\to 0}\mathcal{A}(\mu;\delta)=(1-\mu).$$

Finally, we consider the limit as $\delta\to 1$:

$$\lim_{\delta\to 1}\mathcal{A}(\mu;\delta)=\lim_{\delta\to 1}\delta\cdot\lim_{\delta\to 1}\frac{1-\mu}{1-\delta\mu}=1.$$

For output,

$$\lim_{\delta\to 1}\mathcal{Y}(\mu)=(1-\mu)+\mu\lim_{\delta\to 1}\mathcal{A}(\mu;\delta)=1.$$

Part 3. Monotonicity and concavity. We establish the following result:

Lemma 5. *$\mathcal{Y}(\mu),\mathcal{A}(\mu)$ are strictly decreasing and concave in μ.*

Proof. Next, we investigate the monotonicity of $\mathcal{A}$ and $\mathcal{Y}$. We do so by signing the derivatives:

$$\begin{aligned}\mathcal{Y}_\mu &= \mathcal{Y}(\mu)\left(\frac{\delta}{1-\delta\mu}-\frac{1}{1-\mu}\right)\\ &= -\frac{1-\delta}{(1-\delta\mu)^2}<0.\end{aligned}$$

From here, we also have that, $\mathcal{A}_\mu=\delta\mathcal{Y}_\mu<0$.

Next, we perform the convexity analysis by computing second derivatives. Using (21) we have that:

$$\mathcal{Y}_{\mu\mu} = -2\delta\frac{1-\delta}{(1-\delta\mu)^3}<0.$$

Thus, $\mathcal{Y}_\mu$ is strictly concave. From here, we also have that, $\mathcal{A}_{\mu\mu}=-2\delta^2\frac{1-\delta}{(1-\delta\mu)^3}<0$.

Thus, we establish that both functions are strictly monotonically decreasing and concave in μ.

□

We also have monotonicity in δ, but a different convexity property.

Lemma 6. *$\mathcal{Y}(\mu),\mathcal{A}(\mu)$ are strictly increasing in δ. In turn, $\mathcal{Y}(\mu)$ is strictly concave in δ whereas $\mathcal{Y}(\mu)$ is strictly convex in δ.*

Proof. Note that:

$$\mathcal{Y}_\delta \;=\; \mathcal{Y}(\mu)\left(\frac{\mu}{1-\delta\mu}\right) > 0.$$

Thus, $\mathcal{A}_\delta = \mathcal{Y}(\mu) + \delta\mathcal{Y}_\delta = \mathcal{Y}(\mu)\left(\frac{1}{1-\delta\mu}\right) > 0$.

As for the convexity analysis:

$$\mathcal{Y}_{\delta\delta} \;=\; -2\mathcal{Y}(\mu)\frac{\mu^2}{(1-\delta\mu)^2} < 0.$$

From here, we have that, $\mathcal{A}_{\delta\delta} = \mathcal{Y}_\delta\left(\frac{1}{1-\delta\mu}\right) + \mathcal{Y}(\mu)\left(\frac{\mu}{1-\delta\mu}\right) > 0$. This establishes the strict monotonicity and convexity properties. □

B.2.2 Efficient Assignment

We now prove the formulas for the chain-size distribution for the efficient assignment.

Lemma 7. *Let,* $n = \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$ *and* $p = n + 1 - \frac{\mu}{1-\mu}$. *Then, under the efficient assignment* $g(n) = p, \quad g(n+1) = 1-p$.

Proof. Consider the problem of allocating $N^x = \mu N$ chained orders across $N^s = (1-\mu)N$ chains to maximize total output.

Step 1: Marginal productivity and reallocation.

Consider any assignment with two chains of lengths n and n' where $n' > n+1$. We can strictly increase output by moving the order from the final position in the chain of length n' to adding a link to the chain of length n. The final goods obtain by that order is:

$$\delta^{n+1} - \delta^{n'} = \delta^{n+1}(1 - \delta^{n'-n-1}) > 0.$$

The final goods of no other order is not affected.

Step 2: Characterization of efficient allocation. By Step 1, in any efficient allocation, chain lengths can differ by at most 1. Therefore, the distribution must be:

$$g(k) = \begin{cases} p & \text{if } k = n \\ 1-p & \text{if } k = n+1 \\ 0 & \text{otherwise} \end{cases}$$

for some integer $n \geq 0$ and $p \in [0,1]$. Otherwise, it's possible to move a mass of orders and increase the final output.

Step 3: Determining n and p. The total number of chained orders must equal N^x:

$$N^s[pn + (1-p)(n+1)] = N^x.$$

Dividing both sides by N^s:

$$pn + (1-p)(n+1) = \frac{N^x}{N^s} = \frac{\mu N}{(1-\mu)N} = \frac{\mu}{1-\mu}.$$

Simplifying:

$$n + (1-p) = \frac{\mu}{1-\mu}.$$

Therefore:

$$p = n + 1 - \frac{\mu}{1-\mu}.$$

Step 4: Integer constraints. For $p \in [0,1]$, we need:

$$0 \leq n + 1 - \frac{\mu}{1-\mu} \leq 1. \tag{21}$$

This implies:

$$n \leq \frac{\mu}{1-\mu} < n + 1$$

.

Hence, $n = \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$, $n+1 = \left\lceil \frac{\mu}{1-\mu} \right\rceil$, and $p = n + 1 - \frac{\mu}{1-\mu}$.

Step 5: Final distribution. The efficient allocation has:

- A fraction $p = \left\lceil \frac{\mu}{1-\mu} \right\rceil - \frac{\mu}{1-\mu}$ of chains with length $n = \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$
- A fraction $1 - p = \frac{\mu}{1-\mu} - \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$ of chains with length $n+1$

Note that for $\mu < \frac{1}{2}$, the mass of chains of length 1 is $1 - p = \frac{\mu}{1-\mu} = \frac{N^x}{N^s}$, precisely the ratio of chained orders to spot orders.

This completes the characterization of the efficient chain-length distribution. □

The distribution of positions $h(n)$ is thus given by:

$$h(k) = \begin{cases} \frac{1}{n+(1-p)} & \text{if } k = n \\ \frac{1-p}{n+(1-p)} & \text{if } k = n+1 \\ 0 & \text{otherwise} \end{cases}$$

for $n = \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$.

Next, we derive the formula for $\mathcal{A}$ under the efficient assignment.

Lemma 8. *Under the efficient assignment:*

$$\mathcal{A}(\mu;\delta) = \delta \left(p\frac{1-\delta^n}{1-\delta} + (1-p)\frac{1-\delta^{n+1}}{1-\delta} \right) \frac{1-\mu}{\mu}.$$

Furthermore:

$$\mathcal{Y}(\mu;\delta) = (1-\mu)\left(1 + \delta\left(p\frac{1-\delta^n}{1-\delta} + (1-p)\frac{1-\delta^{n+1}}{1-\delta} \right)\right).$$

Note that in a chain of length n, the average output among chained orders: $\frac{1}{n}\sum_{i=1}^{n}\delta^i = \frac{\delta}{n}\frac{1-\delta^n}{1-\delta}$.

The mass of chained orders is: $pn + (1-p)n = \frac{\mu}{1-\mu}$. From here, we compute output per chained order:

$$\mathcal{A}(\mu) = \delta \left(p\frac{1-\delta^n}{1-\delta} + (1-p)\frac{1-\delta^{n+1}}{1-\delta} \right) / \left(\frac{\mu}{1-\mu} \right), \tag{22}$$

yielding the formula.

Monotonicity. Next we prove the established monotonicity and continuity.

Lemma 9. *$\mathcal{A}(\mu)$ and $\mathcal{Y}(\mu)$ are continuous and monotone decreasing under the efficient allocation.*

Proof. Recall that under efficient allocation, average output per chained order is:

$$\mathcal{A}(\mu) = \frac{\delta}{1-\delta}\left(p(1-\delta^n) + (1-p)(1-\delta^{n+1})\right)\frac{1-\mu}{\mu}$$

where $n = \left\lfloor \frac{\mu}{1-\mu} \right\rfloor$ and $p = n + 1 - \frac{\mu}{1-\mu}$.

Step 1: Continuity at integer values. When $\frac{\mu}{1-\mu} = k$ for integer k, all chains have length k, giving:

$$\mathcal{A}(\mu) = \frac{\delta(1-\delta^k)}{1-\delta} \cdot \frac{1-\mu}{\mu}$$

As μ approaches this integer point from below, $n = k-1$ and $p \to 0$, yielding the same limit. Thus $\mathcal{A}(\mu)$ is continuous everywhere.

Step 2: Derivative between integer points. For $\frac{\mu}{1-\mu} \notin \mathbb{Z}$, the floor function is locally constant, so:

$$\frac{\partial \mathcal{A}}{\partial \mu} = \frac{\delta}{1-\delta} \cdot \frac{\partial}{\partial \mu}\left(\frac{1-\mu}{\mu}\right) \cdot [(1-\delta^n) - n(1-\delta)\delta^n]$$

Since $\frac{\partial}{\partial \mu}\left(\frac{1-\mu}{\mu}\right) = -\frac{1}{\mu^2} < 0$, we need to show:

$$(1-\delta^n) - n(1-\delta)\delta^n > 0$$

Step 3: Verifying the inequality. Rearranging:

$$(1-\delta^n) - n(1-\delta)\delta^n = 1 - (1+n)\delta^n + n\delta^{n+1}$$

We need to prove: $1 + n\delta^{n+1} > (1+n)\delta^n$, or equivalently:

$$\frac{1}{\delta^n} + n\delta > 1 + n$$

Since $\delta < 1$ and $n \geq 1$: - The term $\frac{1}{\delta^n} > 1$ - The inequality $n\delta < n$ is compensated by the excess $\frac{1}{\delta^n} - 1 > 0$

More formally, define $f(\delta) = 1 + n\delta^{n+1} - (1+n)\delta^n$. We have: - $f(0) = 1 > 0$ - $f(1) = 0$ - $f'(\delta) = n(n+1)\delta^n - n(1+n)\delta^{n-1} = n(1+n)\delta^{n-1}(\delta - 1) < 0$

Since f is strictly decreasing from positive to zero on $[0,1]$, we have $f(\delta) > 0$ for all $\delta \in (0,1)$.

Step 4: Global monotonicity. Although the derivative has discontinuities at integer values of $\frac{\mu}{1-\mu}$, the function is continuous and decreasing between these points. At integer points, the left and right limits coincide, ensuring global monotonicity.

Therefore, $\mathcal{A}(\mu)$ is strictly decreasing in μ for all $\mu \in (0,1)$. □

B.3 Related Results Used Elsewhere

In this section, we derived properties that are used later in the text.

Part 5. Inverse productivity. Now, we study the inverse of productivity. Let

$$q(\mu;\delta) = \mathcal{A}^{-1}(\mu;\delta).$$

The function has limits:

$$\lim_{\mu\to 0} q(\mu;\delta) = \delta^{-1}, \qquad \lim_{\mu\to 1} q(\mu;\delta) = \infty, \qquad \lim_{\delta\to 0} q(\mu;\delta) = \infty, \qquad \lim_{\delta\to 1} q(\mu;\delta) = 1.$$

We also have that:

$$q_\mu = -\frac{\mathcal{A}_\mu}{\mathcal{A}^2} = \delta\frac{1-\delta}{(1-\mu)^2} > 0.$$

The convexity is guranateed by using:

$$q_{\mu\mu} = -\frac{\mathcal{A}_{\mu\mu}}{\mathcal{A}^2} + \frac{\mathcal{A}_\mu}{\mathcal{A}^3} > 0.$$

Hence, q is convex in μ.

Part 6. Elasticity of $\mathcal{A}$. A useful object in later derivations is the elasticity of $\mathcal{A}$ with respect to μ:

$$\epsilon_\mu^{\mathcal{A}} \equiv \frac{\mathcal{A}_\mu(\mu)\,\mu}{\mathcal{A}(\mu)}.$$

Using the derivations above:

$$\frac{\mathcal{A}_\mu(\mu)\,\mu}{\mathcal{A}(\mu)} = -\frac{(1-\delta)\mu}{(1-\delta\mu)(1-\mu)} < 0.$$

Note that:

$$\frac{\partial}{\partial\mu}[\mathcal{A}(\mu)\,\mu] = \mathcal{A}'(\mu)\,\mu + \mathcal{A}(\mu) = \mathcal{A}(\mu)(1+\epsilon_\mu^{\mathcal{A}})$$

We have that:

$$1+\epsilon_\mu^{\mathcal{A}} = \frac{(1-\mu)(1-\delta\mu)-(1-\delta)\mu}{(1-\delta\mu)(1-\mu)} = \frac{1-\mu+\delta\mu^2-\mu}{(1-\delta\mu)(1-\mu)} = \frac{1}{1-\delta\mu} - \frac{\mu}{1-\mu}.$$

In later proofs, we require obtaining the sign of $1+\epsilon_\mu^{\mathcal{A}}$ and its limits. We know $\epsilon_\mu^{\mathcal{A}} < 0$. Thus, the sign of $1+\epsilon_\mu^{\mathcal{A}}$ is the sign of:

$$\frac{1}{1-\delta\mu} - \frac{\mu}{1-\mu}.$$

The limits of the function that governs the sign are:

$$\lim_{\mu \to 0} \left[\frac{1}{1-\delta\mu} - \frac{\mu}{1-\mu} \right] = 1 > 0,$$

and

$$\lim_{\mu \to 1} \left[\frac{1}{1-\delta\mu} - \frac{\mu}{1-\mu} \right] = -\infty.$$

Since the function is continuous in μ, the sign is ambiguous.

Next, to establish monotonicity, notice that

$$\frac{\partial}{\partial \mu} \left[\frac{1}{1-\delta\mu} - \frac{\mu}{1-\mu} \right] = \left[\frac{\delta}{(1-\delta\mu)^2} - \frac{\mu}{1-\mu} \left(\frac{1}{1-\mu} - \frac{1}{\mu} \right) \right] = \frac{\delta}{(1-\delta\mu)^2} - \frac{1}{(1-\mu)^2} < 0.$$

The numerator of the first term is lower than that of the second, whereas the opposite is true about the denominators. Hence, the function is monotone decreasing. Thus, there's a unique crossing point where the function $1 + \epsilon_{\mu}^{\mathcal{A}}$ becomes negative.

C Extensions to Other Assignments

C.1 Correlated Assignments

In the main text, we considered payment chains where spot and chained orders have equal probabilities of being linked to orders of the same type. However, our empirical analysis in Table XXX reveals that payment types exhibit significant correlation: chained orders predominantly connect to other chained orders, while spot orders tend to cluster together. This correlation in order types affects the distribution of chain lengths and ultimately productivity. To examine this, we allow the probability that chained orders link to other chained orders to differ from the probability that spot orders connect to chained orders. We formalize this using a Markov chain representation.

Markov Chain Representation. Consider a payment-chain network where order types (spot or chained) exhibit correlation along production chains. To ease the notation, let $\mu_x = \mu$ denote the fraction of chained orders and $\mu_s = 1 - \mu_x$ the fraction of spot orders. Starting from a spot order, the entire payment network is characterized by a sequence $S_t = \{s, x, s, x, s, \ldots\}$ where $\{s, x\}$ in position i represents whether the i-th order is spot or chained, respectively. If links across chained and spot orders only depend on the current order type, the sequence can be generated by a Markov chain:

$$Q = \begin{bmatrix} p_{ss} & 1 - p_{xx} \\ 1 - p_{ss} & p_{xx} \end{bmatrix} \tag{23}$$

where p_{ss} is the probability that a spot order is linked to a spot, and p_{xx} the corresponding probability that a chained order is linked to a chained order.

For a large N, the stationary distribution must satisfy:

$$\boldsymbol{\mu} = Q\boldsymbol{\mu}, \quad \text{where } \boldsymbol{\mu} = \begin{bmatrix} \mu_s \\ \mu_x \end{bmatrix}. \tag{24}$$

That is, the transition matrix must guarantee that population shares aggregate.

To see how the construction works, we verify the formulas for the i.i.d. case in which

transition probabilities equal unconditional probabilities:

$$Q^{\text{iid}} = \begin{bmatrix} \mu_s & \mu_s \\ \mu_x & \mu_x \end{bmatrix}. \tag{25}$$

This trivially satisfies $\boldsymbol{\mu} = Q^{\text{iid}}\boldsymbol{\mu}$ since each column sums to one.

To introduce correlation, we parameterize deviations from independence as:

$$p_{ss} = \mu_s\kappa_s, \quad p_{xx} = \mu_x\kappa_x \tag{26}$$

where $\kappa_s, \kappa_x > 0$ are correlation factors. When the corresponding κ is positive, orders have a greater chance of matching with orders of the same type than under an i.i.d. assignment; when κ is negative, the random assignment overrepresents mixed links.

For positive values of κ, the transition matrix becomes:

$$Q = Q^{\text{iid}} + \tilde{Q} \tag{27}$$

where the deviation matrix is:

$$\tilde{Q} = \begin{bmatrix} \mu_s(\kappa_s - 1) & -\mu_x(\kappa_x - 1) \\ -\mu_s(\kappa_s - 1) & \mu_x(\kappa_x - 1) \end{bmatrix}. \tag{28}$$

There are some parameter constraints: We require a stationarity condition on $\tilde{Q}\boldsymbol{\mu} = \mathbf{0}$ that guarantees the preservation of population shares:

$$\mu_s^2(\kappa_s - 1) = \mu_x^2(\kappa_x - 1). \tag{29}$$

Defining $q \equiv \mu_s(\kappa_s - 1)$ and using the constraint, we obtain:

$$\mu_x(\kappa_x - 1) = \frac{\mu_s}{\mu_x}q. \tag{30}$$

Setting $\rho \equiv \frac{\mu_s}{\mu_x} q$, the transition matrix simplifies to:

$$Q(\rho) = \begin{bmatrix} \mu_s + q & 1 - (\mu_x + \rho) \\ 1 - (\mu_s + q) & \mu_x + \rho \end{bmatrix}. \tag{31}$$

To ensure valid probabilities ($0 \leq p_{ij} \leq 1$), we require: if $\mu_s > 1/2$, then $-\mu_s \leq q \leq 1-\mu_s$; if $\mu_x > 1/2$, then $-\mu_x \leq \frac{\mu_s}{\mu_x} q \leq 1 - \mu_x$. Combining these constraints:

$$-\min\{\mu_x, \mu_s\} \leq \rho \leq \min\{\mu_x, 1 - \mu_x\}. \tag{32}$$

Chain Length Distribution. Conditional on featuring some chained orders, the length of a chain (consecutive chained orders) follows a geometric distribution. A chain of length n occurs with probability:

$$\Pr(n|n \geq 1) = P(\text{chain ends}) \cdot P(\text{continue chain})^{n-1}. \tag{33}$$

Given the parameterization:

$$P(\text{continue chain}) = p_{xx} = \mu_x + \rho \tag{34}$$

$$P(\text{chain ends}) = 1 - p_{xx} = 1 - \mu_x - \rho \tag{35}$$

Therefore:

$$\Pr(n|n \geq 1) = (1 - \mu_x - \rho)(\mu_x + \rho)^{n-1}. \tag{36}$$

This is a geometric distribution with success probability $p = \mu_x + \rho$.

The expected chain length conditional on some chained orders is:

$$\mathbb{E}[n|n \geq 1] = \frac{1}{1-p} = \frac{1}{1 - \mu_x - \rho}. \tag{37}$$

Thus, the unconditional distribution is:

$$\mathbb{E}[n] = \mathbb{E}[n|n \geq 1] \Pr[n \geq 1] = \frac{\mu_x - q}{1 - \mu_x - \rho} = \frac{\mu_x - \frac{\mu_x}{\mu_s}\rho}{1 - \mu_x - \rho},$$

where we used that $q \equiv \frac{\mu_x}{\mu_s}\rho$. Notice that when $\rho > 0$, we have positive correlation—chained orders cluster together, increasing average chain length; when $\rho < 0$, the correlation

decreases average chain length. When $\rho = 0$ (independence), we recover $\mathbb{E}[L] = \frac{\mu}{1-\mu}$.

Output and TFP. Using the same steps as we do for the i.i.d. case, using the chain-length distribution method, we obtain:

$$\mathcal{A}(\mu; \delta, \rho) = \delta \frac{1 - (\mu_x + \rho)}{1 - \delta(\mu_x + \rho)}. \tag{38}$$

Output is computed by weighting this distribution:

$$\mathcal{Y}(\mu; \delta, \rho) = (1 - \mu) + \mu\mathcal{A} = 1 - \mu \frac{1 - \delta}{1 - \delta(\mu + \rho)}. \tag{39}$$

Again, this verifies the consistency of the formulas when $\rho = 0$.

C.2 Pooled Chained Orders

Next, we consider assignments where chained orders can be efficiently reassigned to chains of approximately equal length, while the mass of spot-to-spot orders remains unchanged. Under this mechanism, the income from every chained order is immediately used to finance another chained order as long as unfunded chained orders exist. This captures scenarios where agents placing multiple chained orders pool their funds, ensuring no cash remains idle while chained orders await funding.

Setup with Correlation. As an the case with correlated assignments, the probability that a spot order is linked to a chained order is $(1 - \mu_s - q)$, where q is the correlation parameter from our analysis above. While spot-to-spot matches follow a random process, any spot order that connects to a chained order enters a pool where chains are optimally formed to maximize total production. Thus, the mass of spot orders that can potentially match with chained orders is constant:

$$\bar{N}_s = (1 - \mu)(1 - \mu_s - q)N = (1 - \mu)(\mu - q)N. \tag{40}$$

The ratio of chained orders to available spot orders is:

$$\xi(q) = \frac{N_x}{\bar{N}_s} = \frac{\mu N}{(1 - \mu)(\mu - q)N} = \frac{\mu}{(1 - \mu)(\mu - q)} \tag{41}$$

When $q > 0$ (positive correlation): Fewer spot orders connect to chained orders, increasing ξ; when $q < 0$ (negative correlation): More spot orders connect to chained orders, decreasing ξ toward the efficient ratio. When $q = 0$ (independence), we obtain $\xi(0) = \frac{1}{1-\mu}$, thus the ratio of chained to spot orders exceeds $\frac{\mu}{1-\mu}$. This reveals that the assignment is less efficient than the fully efficient case, since some spot orders are inefficiently matched with other spot orders. When $q = -(1-\mu)$, we have $\xi(-(1-\mu)) = \frac{\mu}{1-\mu}$, recovering the fully efficient assignment case.

Optimal Chain Length Distribution. Following the arguments of the efficient case, the efficient assignment creates chains of two adjacent lengths for chains of length greater than or equal to one:

$$n(q) = \lfloor \xi(q) \rfloor \tag{42}$$

$$n(q) + 1 = \lceil \xi(q) \rceil \tag{43}$$

The fraction of chains with the shorter length is:

$$p(q) = \lceil \xi(q) \rceil - \xi(q) \tag{44}$$

The resulting distribution is:

$$g(0) = 1 - (\mu + q) \quad \text{(spot-only transactions)} \tag{45}$$

$$g(n(q)) = p(q)(\mu + q) \quad \text{(shorter chains)} \tag{46}$$

$$g(n(q) + 1) = (1 - p(q))(\mu + q) \quad \text{(longer chains)} \tag{47}$$

Clearly, the pooling mechanism partially corrects the inefficiency from random matching, but maintains the inefficiencies provoked by the assignment of spot-to-spot links.

Production and TFP. TFP is in this case given by:

$$\mathcal{A}(\mu, q) = \frac{p(q) \cdot Y(n(q)) + (1 - p(q)) \cdot Y(n(q) + 1)}{p(q) \cdot n(q) + (1 - p(q)) \cdot (n(q) + 1)} \tag{48}$$

where $Y(n) = \delta \frac{1-\delta^n}{1-\delta}$.

Expected production per order follows the baseline formula, $\mathcal{Y}(\mu, q) = (1 - \mu) + \mu\mathcal{A}(\mu, q)$.

C.3 Endogenous Reallocation

The pooled assignment above assumes costless reallocation of funds across chained orders. However, it is natural to expect that reallocation only occurs for orders in positions exceeding a certain threshold. We now examine a model where chains beyond a threshold length η must reallocate their excess orders to shorter chains, while chains below this threshold operate normally.

Setup. Consider a payment network where chains longer than the threshold η must reallocate their excess orders. We define the **orders needing reallocation** (N^r) as the total number of chained orders in positions beyond η:

$$N^r \equiv \sum_{n=\eta+1}^{\infty} (n-\eta) \cdot g(n) = \frac{\mu^\eta}{1-\mu} \tag{49}$$

This counts all orders in chains of length $n > \eta$ that exceed the threshold.

The **available slots** (N^a) in other chains is the total unused capacity in chains of length $n \leq \eta$:

$$N^a \equiv \sum_{n=0}^{\eta} (\eta - n) \cdot g(n) = \eta - \left(\frac{\eta\mu^{\eta+1}}{1-\mu}\right)(2-\mu) - \frac{1}{1-\mu} + \frac{(1+\eta)\mu^\eta}{1-\mu} \tag{50}$$

Each chain of length $n \leq \eta$ can potentially accept $(\eta - n)$ additional orders before reaching the threshold.

The ratio of reallocating orders to available slots determines the efficient assignment structure:

$$\xi(\eta) = \frac{N^r}{N^a} = \frac{\mu^\eta}{(\eta-1)(1-\mu) + \mu^\eta[1+\eta(1-\mu)^2]} \tag{51}$$

This ratio determines the optimal chain lengths for efficient assignment:

$$n(\eta) = \lfloor \xi(\eta) \rfloor \tag{52}$$

$$n(\eta) + 1 = \lceil \xi(\eta) \rceil \tag{53}$$

with fraction $p(\eta) = \lceil \xi(\eta) \rceil - \xi(\eta)$ assigned to the shorter length.

In this setting, orders are reallocated sequentially by approaching the customers and suppliers in the last position of an existing chain. They offer to buy from the sup-

plier (corresponding to a spot order) and sell to the last customer order (corresponding to a chain order). When abandoning the chain, the final customer buys from the preceding customer. The reallocation has a cost of Ξ.

Chain-Length Distribution. Because the assignment is sequential, reassigned orders will be placed at the shortest available slots, including those from spot orders. As a result, the distribution will be efficient among the reassigned orders.

Let $g(k) = (1-\mu)\mu^{k-1}$ for $k \geq 1$ denote the baseline geometric distribution for chain lengths, and $G(k) = \sum_{j=1}^{k} g(j) = 1 - \mu^k$ its cumulative distribution function. Note that in the baseline model, spot orders have mass $(1-\mu)$.

After the re-assignment, the resulting hybrid distribution for chain lengths is:

$$\hat{g}(k;\eta,\mu) = \begin{cases} \alpha \cdot p(\eta) & k = n(\eta) \\ \alpha \cdot (1 - p(\eta)) & k = n(\eta)+1 \\ g(k) & n(\eta)+2 \leq k \leq \eta - 1 \\ \mu^{\eta} & k = \eta \text{ (truncated chains)} \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

where $\alpha = G(n(\eta)+1) = 1 - \mu^{n(\eta)+1}$ represents all chains that would have been length $\leq n(\eta)+1$ in the baseline. This distribution has three key features: (i) all spot orders have been converted into chains through the efficient assignment, (ii) the original distribution between lengths $n(\eta)+1$ and $\eta - 1$, is preserved reflecting that orders in those positions are not re-assigned, (iii) the mass at η equals μ^{η}, capturing that all chains with length above η become chains of length η.

Endogenous Choice. Given the cost Ξ and the sequential nature of the re-assignment, the value of η can be found by solving for the value for which:

$$\Xi < \delta^{n(\eta)+1} - \delta^{\eta+1}, \qquad \Xi \geq \delta^{n(\eta)+1} - \delta^{\eta}.$$

The left inequality ensures reallocation from position $\eta+1$ to the last position with positive mass $n(\eta)+1$ is profitable, while the right inequality ensures reallocation from position η is not.

C.4 Default and Cascading

The reassignment model above assumes agents can identify and relocate to shorter chains. However, this requires substantial information about the network. We now consider a simpler response: when chains exceed a threshold η, orders beyond this point are withdrawn rather than reassigned. This captures situations where extended payment chains become unsustainable. Orders in positions $n > \eta$ simply default, keeping their funds. When they do so, all of the subsequent transactions experience a cascading sequence of breakdowns.

Chain Length Distribution. As before, chain length follows a geometric distribution:

$$g(n) = (1-p)p^{n-1} \tag{55}$$

where p generalizes to include correlation effects: $p = \mu + \rho$ in the correlated case.

We capture default by modifying the output function:

$$y_x(n) = \begin{cases} \delta \frac{1-\delta^n}{1-\delta} & \text{if } n \leq \eta \\ \delta \left(\frac{1-\delta^\eta}{1-\delta} \right) & \text{if } n > \eta. \end{cases} \tag{56}$$

The second that transactions above position η get truncated.

Output and TFP. Average production per link in a chain of length n is:

$$\bar{y}_x(n) = \frac{y_x(n)}{n} = \frac{\delta}{n} \frac{1-\delta^n}{1-\delta} - \bar{\gamma}(n) \tag{57}$$

where the default adjustment is:

$$\bar{\gamma}(n) = \begin{cases} 0 & \text{if } n \leq \eta \\ \frac{1}{n} \frac{\delta^\eta}{1-\delta} (1-\delta^{n-\eta}) & \text{if } n > \eta. \end{cases} \tag{58}$$

Using the methods presented for the baseline case, the formula for TFP becomes:

$$\mathcal{A}_\eta = \frac{\sum_{n=1}^{\infty} g(n) \bar{y}_x(n)}{\sum_{n=1}^{\infty} g(n) n}. \tag{59}$$

Combining all terms we obtain that:

$$\mathcal{A}_\eta = \frac{\delta(1-p)}{1-p\delta} - \frac{\delta(1-p)}{1-p\delta}\delta^\eta p^\eta = \mathcal{A}(1-\delta^\eta p^\eta) \tag{60}$$

where $\mathcal{A} = \frac{\delta(1-p)}{1-p\delta}$ is the TFP without default. The truncation factor $(1-\delta^\eta p^\eta)$ captures the productivity loss from default. This loss increases the lower threshold η (more chains are truncated). When $\eta \to \infty$, we recover $\mathcal{A}_\eta \to \mathcal{A}$, the case without default. Output follows the formula for the correlated case, replacing $\mathcal{A}_\eta$ with $\mathcal{A}$.

C.5 Partial Delays

We now consider a hybrid payment system in which each transaction in a chain has a probability q of occurring with a delay. This captures heterogeneity in payment processing—some transactions use sophisticated instant payment systems while others require traditional settlement with delays.

Modified Production Structure. For a chain of length n where k transactions happen, production depends critically on the *location* of delays. We assume delays occur sequentially from the beginning of the chain (positions 1 through k), yielding:

$$y(k,n) = (n-k) + \delta\frac{1-\delta^k}{1-\delta} \tag{61}$$

where $(n-k)$ represents immediate payments and the second term captures delayed payments.

Distribution of Delays and TFP. The number of delayed payments in a chain of length n follows a binomial distribution:

$$M(k,n) = \binom{n}{k} q^k (1-q)^{n-k}. \tag{62}$$

The expected production for a chain of length n is:

$$\mathbb{E}[y(k,n)|n] = \sum_{k=0}^{n} y(k,n) \times M(k,n). \tag{63}$$

To evaluate this expectation, we use three key properties of the binomial distribution:

$$\sum_{k=0}^{n} M(k,n) = 1 \tag{64}$$

$$\sum_{k=0}^{n} k \cdot M(k,n) = qn \tag{65}$$

$$\sum_{k=0}^{n} \delta^k M(k,n) = (1-q+\delta q)^n. \tag{66}$$

The third property follows from the binomial theorem.[27] Substituting these properties:

$$\mathbb{E}[y(k,n)|n] = \sum_{k=0}^{n}(n-k)M(k,n) + \frac{\delta}{1-\delta}\left[1 - \sum_{k=0}^{n}\delta^k M(k,n)\right] \tag{68}$$

$$= (1-q)n + \frac{\delta}{1-\delta}\left[1-(1-q+\delta q)^n\right]. \tag{69}$$

Average production across all chain lengths, weighted by the geometric distribution $g(n) = (1-p)p^{n-1}$ is given by:

$$\mathcal{A}_q = \frac{\sum_{n=1}^{\infty} g(n) \cdot \mathbb{E}[y(k,n)|n]}{\sum_{n=1}^{\infty} g(n) \cdot n} \tag{70}$$

The denominator equals $\frac{1}{1-p}$. For the numerator:

$$(1-p)\sum_{n=1}^{\infty} p^{n-1}\mathbb{E}[y(k,n)|n] \tag{71}$$

$$= \frac{1-q}{1-p} + \frac{\delta}{1-\delta}\left[1-(1-p)\frac{1-q+\delta q}{1-p(1-q+\delta q)}\right] \tag{72}$$

Simplifying yields:

$$\mathcal{A}_q = (1-q) + \delta\frac{q(1-p)}{1-p(1-q+\delta q)} \tag{73}$$

[27] The assumption that delays occur sequentially from the beginning of the chain is crucial. If delays were randomly distributed throughout the chain, the production function would be more complex, as non-delayed segments would be interrupted by delayed ones. The sequential structure ensures that all immediate payments (positions $k+1$ through n) can be realized without interruption. For the third property, recall that

$$\sum_{k=0}^{n}\binom{n}{k}(\delta q)^k(1-q)^{n-k} = (1-q+\delta q)^n. \tag{67}$$

This generalizes our baseline formula: When $q = 1$ (full delays): $\mathcal{A}_q = \frac{\delta(1-p)}{1-p\delta}$. When $q = 0$ (no delays): $\mathcal{A}_q = 1$ When $p = \mu$ and $q = 1$, it recovers the baseline uncorrelated case.

D Proofs Regarding the Partial Equilibrium Analysis - Section 4

D.1 Entrepreneur's Problem: Details

This appendix provides a detailed characterization of the entrepreneur's problem, including formal statements and proofs of the key results presented in the main text.

Thresholds. The following lemma reproduces Lemma 1 in the main text, which identifies two debt threshold points that determine whether an entrepreneur's expenditure is a mix between spot and chained purchases or whether expenditures are concentrated in one type of order.

Lemma 10. (Expenditure Threshold Points): *Define the* ***efficiency threshold****,* $B^{\star}_{t+1} \equiv R_{t+1}\left(\tilde{B}_t - 1\right)$*. Then:*

(i) $S_{t+1} = 0$ *if and only if* $B_{t+1} > \tilde{B}_{t+1}$.

(ii) $X_t > 0$ *if and only if* $B_{t+1} > B^{\star}_{t+1}$.

Proof. Part (i): If $B_{t+1} > \tilde{B}_{t+1}$, then $\bar{S}_{t+1} = \max\{\tilde{B}_{t+1} - B_{t+1}, 0\} = 0$. By conditions (13), $S_{t+1} = \min\{\bar{S}_{t+1}, E_{t+1}\} = 0$. Conversely, if $S_{t+1} = 0$, then either $\bar{S}_{t+1} = 0$ or $E_{t+1} = 0$. Since entrepreneurs must have positive endowments at all t consume at all periods, $E_{t+1} > 0$, implying $\bar{S}_{t+1} = 0$, which occurs if and only if $B_{t+1} > \tilde{B}_{t+1}$.

Part (ii): From equation (13), chained expenditures are positive, $X_t > 0$, if and only if expenditures exceed the limit on spot expenditures, $E_t > \bar{S}_t = \tilde{B}_t - B_t$. Using the budget constraint $E_t = B_{t+1}R^{-1}_{t+1} + 1 - B_t$, this condition becomes:

$$B_{t+1}R^{-1}_{t+1} + 1 - B_t > \tilde{B}_t - B_t$$

Simplifying yields $B_{t+1} > R_{t+1}(\tilde{B}_t - 1) = B^{\star}_{t+1}$. □

As stated in the text, the efficiency threshold $B^{\star}_{t+1}$ marks the debt level above which entrepreneurs must resort to chained purchases in the current period due to insufficient spot borrowing capacity. The threshold $\tilde{B}_{t+1}$ determines whether entrepreneurs lose access to spot purchases entirely in the following period.

Marginal Prices and Optimality Conditions. The presence of two types of purchases—spot and chained—with different prices creates discontinuities in the marginal cost of buying goods. We formalize this through the concept of marginal prices, reproducing the definition in the main text.

Definition 4. (Average and Marginal Prices):

I. *The **average price** per unit of consumption is* $Q_t \equiv E_t/(S_t + X_t)$.

II. *The **marginal expenditure price** at* t *given debt choice* B' *is:*

$$\tilde{q}_t^E(B') \equiv 1 + (q_t - 1)\,\mathbb{I}_{\left[B' \geq B_{t+1}^\star\right]}$$

III. *The **marginal borrowing price** at* $t+1$ *given debt* B' *is:*

$$\tilde{q}_{t+1}^B(B') \equiv 1 + (q_{t+1} - 1)\,\mathbb{I}_{\left[B' > \tilde{B}_{t+1}\right]}$$

The marginal expenditure price $\tilde{q}_t^E(B')$ represents the price at which the entrepreneur purchases goods at the margin in period t. If the choice of future debt exceeds the efficiency threshold, $B' \geq B_{t+1}^\star$, the marginal price of goods equals q_t, the value of chained goods; otherwise, it equals 1 (the spot price). Similarly, $\tilde{q}_{t+1}^B(B')$ captures the price at which saved resources translate into consumption in period $t+1$. If future debt exceeds the SBL ($B' > \tilde{B}_{t+1}$), any savings yield chained purchases at price q_{t+1} because no spot goods can be purchased; otherwise, they yield spot purchases at price 1 because any additional unit of savings allows an additional unit of spot consumption on the margin.

D.2 Proof of Proposition 2 (Entrepreneur's Euler Equation)

We begin by formulating the entrepreneur's problem in sequential form:

Problem 5. *Given* B_0 *and* $\left\{R_{t+1}, \tilde{B}_t\right\}_{t\geq 0}$,

$$\max_{\{S_t, X_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C_t\right),$$

subject to the budget constraint, $B_t + E_t = R_{t+1}^{-1} B_{t+1} + 1, \forall t \geq 0$, *conditions* (13) *and* $C_t = S_t + X_t$, *and to the intra- and inter-period constraints,* $S_t \leq \bar{S}_t$ *and* $B_t \leq \bar{B}$.

We derive the entrepreneur's Euler equation as a necessary condition for optimality. The entrepreneur's total expenditures as a function of B_{t+1} are:

$$E_t = E(B_t, B_{t+1}, R_{t+1}) \equiv 1 - B_t + \frac{B_{t+1}}{R_{t+1}}.$$

Given total expenditures, spot expenditures are:

$$S\left(B_t, \tilde{B}_t, B_{t+1}\right) = \min\left\{\max\left\{\tilde{B}_t - B_t, 0\right\}, 1 - B_t + \frac{B_{t+1}}{R_{t+1}}\right\},$$

and chained expenditures are:

$$q_t X_t = E(B_t, B_{t+1}, R_{t+1}) - S\left(B_t, \tilde{B}_t, B_{t+1}\right).$$

The entrepreneur's consumption (total goods obtained) is:

$$\begin{aligned} C_t &= S_t + X_t \\ &= \frac{E(B_t, B_{t+1}, R_{t+1}) - S\left(B_t, \tilde{B}_t, B_{t+1}\right)}{q_t} + S\left(B_t, \tilde{B}_t, B_{t+1}\right). \end{aligned}$$

Thus, we can write current consumption as:

$$C_t = C(B_t, B_{t+1}, \tilde{B}_t, q_t, R_{t+1}) \equiv \frac{E(B_t, B_{t+1}, R_{t+1})}{q_t} + \left(1 - \frac{1}{q_t}\right) S\left(B_t, \tilde{B}_t, B_{t+1}\right).$$

The entrepreneur's intertemporal optimization problem can be written entirely in

terms of a sequence of debt levels:

$$\max_{\{B_{t+1}\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \log\left(C(B_t, B_{t+1}, \tilde{B}_t, q_t, R_{t+1})\right)$$

with B_0 given.

Consider the choice of B_{t+1}. Define:

$$\begin{aligned}\Upsilon_t\left(B_{t+1}; B_t, B_{t+2}, \tilde{B}_t, \tilde{B}_{t+1}, R_{t+1}, R_{t+2}, q_t, q_{t+1}\right) \equiv\ & \log\left(C(B_t, B_{t+1}, \tilde{B}_t, q_t, R_{t+1})\right) \\ & +\beta \log\left(C(B_{t+1}, B_{t+2}, \tilde{B}_{t+1}, q_{t+1}, R_{t+2})\right).\end{aligned}$$

The function Υ_t is the piece that contains the influence of B_{t+1} in the objective. Then, we have that Υ_t features kinks at the threshold points $\left\{B^\star_{t+1}, \tilde{B}_{t+1}\right\}$. These kinks arise from the kinks in the functions E and S–reflecting forced switches between spot and chained purchases.

We analyze the optimality conditions by studying the effect of a marginal change in B_{t+1}. Recall the relation between $\tilde{B}_t$ and $B^\star_{t+1}$:

$$B^\star_{t+1} = R_{t+1}\left(\tilde{B}_t - 1\right).$$

We do so splitting the marginal conditions depending on whether some spot expenditures are available at t:

Case I: $B_t \geq \tilde{B}_t$ (No spot purchases at t). We distinguish subcases depending on whether debt levels permit spot expenditures tomorrow:

I.a. If $B_{t+1} < \tilde{B}_{t+1}$ (spot purchases available at $t+1$):

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t}\frac{1}{q_t R_{t+1}} - \beta\frac{1}{C_{t+1}}.$$

I.b. If $B_{t+1} > \tilde{B}_{t+1}$ (no spot purchases at $t+1$):

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t}\frac{1}{q_t R_{t+1}} - \beta\frac{1}{C_{t+1}}\frac{1}{q_{t+1}}$$

At $B_{t+1} = \tilde{B}_{t+1}$, the derivative features a discontinuity:

$$\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon_t'(B') < \lim_{B' \downarrow \tilde{B}_{t+1}} \Upsilon_t'(B'),$$

where the inequality follows from q_{t+1}. The condition implies that the marginal increase in utility is greater to the right $\tilde{B}_{t+1}$. Since a solution at the kink would require:

$$\lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon_t'(B') > 0 > \lim_{B' \downarrow \tilde{B}_{t+1}} \Upsilon_t'(B'),$$

no solution will feature $B_{t+1} = \tilde{B}_{t+1}$.

Case II: $B_t < \tilde{B}_t$ (Possible spot purchases at t). We distinguish subcases based on the debt level relative to the thresholds:

II.a. If $B_{t+1} < B^\star_{t+1}$ (only spot at t, some spot at $t+1$):

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}$$

II.b. If $B_{t+1} \in \left(B^\star_{t+1}, \tilde{B}_{t+1}\right)$ (mixed at t, spot at $t+1$):

$$\Upsilon_t'(B_{t+1}) = \frac{1}{q_t}\frac{1}{C_t} \cdot \frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}$$

II.c. If $B_{t+1} > \tilde{B}_{t+1}$ (mixed at t, only chained at $t+1$):

$$\Upsilon_t'(B_{t+1}) = \frac{1}{C_t} \cdot \frac{1}{q_t}\frac{1}{R_{t+1}} - \beta \frac{1}{C_{t+1}}\frac{1}{q_{t+1}}$$

At the threshold points we have the following discontinuities:

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon_t'(B') > \lim_{B' \downarrow B^\star_{t+1}} \Upsilon_t'(B'), \quad \lim_{B' \uparrow \tilde{B}_{t+1}} \Upsilon_t'(B') < \lim_{B' \downarrow \tilde{B}_{t+1}} \Upsilon_t'(B') \tag{74}$$

Using the definition of average price $Q_t = E_t/C_t$ and marginal prices, we express Υ_t' in a compact form:

$$\Upsilon_t'(B_{t+1}) = \frac{Q_t}{E_t} \cdot \frac{1}{R_{t+1}\tilde{q}_t^E(B_{t+1})} - \beta \frac{Q_{t+1}}{E_{t+1}} \cdot \frac{1}{\tilde{q}_{t+1}^B(B_{t+1})}$$

Setting $\Upsilon_t'(B_{t+1}) = 0$ at interior solutions yields:

$$\frac{E_{t+1}/E_t}{\beta} \cdot \frac{Q_t}{Q_{t+1}} = \frac{R_{t+1}}{\tilde{q}^B_{t+1}(B_{t+1})/\tilde{q}^E_t(B_{t+1})}. \tag{75}$$

We already established that $B_{t+1} = \tilde{B}_{t+1}$ is not a possible outcome. Hence, the only relevant corner solution is $B_{t+1} = B^\star_{t+1}$. Optimality at that corner solution is given by:

$$\lim_{B' \uparrow B^\star_{t+1}} \Upsilon_t'(B') \geq 0 \geq \lim_{B' \downarrow B^\star_{t+1}} \Upsilon_t'(B')$$

Opening up the condition translates to:

$$\beta q_t R_{t+1} \geq \frac{E_{t+1}}{E_t} \geq \beta R_{t+1}. \tag{76}$$

Arriving at equations (76) and (75) complete the proof of Proposition 2. □

The Euler equation captures how the entrepreneur's consumption-saving decision depends on both current and future marginal prices, which vary discontinuously with debt levels due to the spot borrowing line constraint.

D.3 Proof of Proposition 3 (Entrepereneur's Stationary Solution)

In the stationary version of the entrepreneur's problem, $R_t = \beta^{-1}$, the SBL is constant, $\tilde{B}_t = \tilde{B}_{ss}$, and the chained-goods price is constant, $q_t = q_{ss}$. Throughout this proof, we assume $\tilde{B}_{ss} > 1$. For convenience, we suppress the arguments of $B^\star(1/\beta, \tilde{B}_{ss})$ and denote its steady-state value simply as $B^\star_{ss}$.

We establish the following results:

I. If $B_0 \in [0, B^\star_{ss}]$, then $B_t = B_0$ for all t.

II. There exists a threshold $B^h > \tilde{B}_{ss}$ such that:

II.a If $B_0 < B^h$, then $B_t \to B^\star_{ss}$ in finite time.

II.b If $B_0 > B^h$, then $B_t = B_0$ for all t.

The proof strategy involves: First, establishing two benchmark value functions, $\bar{V}(B)$ and $\underline{V}(B)$, that bound the entrepreneur's value function. Second, showing that for $B_0 \leq B^\star_{ss}$, a constant debt policy is optimal and achieves the upper bound $\bar{V}(B)$. Third,

proving that for $B_0 \in (B^\star_{ss}, \tilde{B}_{ss}]$, optimal deleveraging occurs in finite time with debt converging to $B^\star_{ss}$. Finally, establishing the existence and uniqueness of $B^h > \tilde{B}_{ss}$ such that entrepreneurs with debt above B^h maintain constant debt and $V(B) = \underline{V}(B)$.

Value Function Bounds. Consider two auxiliary problems that provide bounds for the entrepreneur's value function. First, the unconstrained entrepreneur problem:

Problem 6 (Upper Bound)**.**

$$\bar{V}(B) = \max_{B' \leq \bar{B}} \ln(E) + \beta \bar{V}(B')$$

subject to: $B + E = 1 + \frac{B'}{R}$.

Second, an entrepreneur problem where only chained goods are consumed:

Problem 7 (Lower Bound)**.**

$$\underline{V}(B) = \max_{B' \leq \bar{B}} \ln(E/q_{ss}) + \beta \underline{V}(B')$$

subject to: $B + E = 1 + \frac{B'}{R}$.

Lemma 11. *The solutions to the benchmark problems are:*

$$\bar{V}(B) = \frac{\ln(1 - (1-\beta)B)}{1-\beta}$$

and

$$\underline{V}(B) = \frac{\ln(1 - (1-\beta)B) - \ln(q_{ss})}{1-\beta}.$$

In both cases, optimal expenditures are $E = 1 - (1-\beta)B$.

Proof. For $\underline{V}$, we guess and verify $\underline{V}(B) = \frac{\ln(1-(1-\beta)B)-\ln(q_{ss})}{1-\beta}$. Substituting into the Bellman equation and taking first-order conditions with respect to B' yields:

$$\frac{1}{1 - B + B'/R} \frac{1}{R} = \frac{\beta}{1 - (1-\beta)B'}.$$

Using $\beta = R^{-1}$, we obtain $B = B'$, confirming constant debt. Expenditures are $E = 1 - (1-\beta)B$, verifying the conjecture. Setting $q_{ss} = 1$ yields $\bar{V}(B)$. □

Note that $V(B) \in [\underline{V}(B), \bar{V}(B)]$ since these correspond to more constrained and relaxed versions of the original problem.

Proof of Part I.

Lemma 12. *For $B \leq B^{\star}_{ss}$, $V(B) = \bar{V}(B)$ and $E = 1-(1-\beta)B$.*

Proof. We verify that setting $E = 1-(1-\beta)B$ in the original problem yields the upper bound value. Note that:

$$\begin{aligned} E &\leq 1-(1-\beta)B^{\star}_{ss} \\ &= 1+\beta B^{\star}_{ss} - B^{\star}_{ss} \\ &= 1+\beta\beta^{-1}(\tilde{B}_{ss}-1) - B^{\star}_{ss} \\ &= \tilde{B}_{ss} - B^{\star}_{ss} \\ &\leq \tilde{B}_{ss} - B. \end{aligned}$$

Thus, consuming only through spot purchases is feasible. With $B' = \beta^{-1}(B+E-1) = B$ and consumption $C = S_t + X_t = 1-(1-\beta)B$ constant across periods, $V(B)$ attains the upper bound. □

Proof of Part II - Preliminary Results. Define the set of debt levels from which setting $B' = B^{\star}_{ss}$ is optimal:

$$\mathcal{B}_0 \equiv \left[B^{\star}_{ss}, B^{\star}_{ss} + (1-\frac{1}{q_{ss}})(\tilde{B}_{ss} - B^{\star}_{ss}) \right]. \tag{77}$$

Lemma 13. *If $B' = B^{\star}_{ss}$, then $C = \tilde{B}_{ss} - B$.*

Proof. When $B' = B^{\star}_{ss}$, by definition $Q = 1$. Thus:

$$C = 1 + \frac{B^{\star}_{ss}}{R} - B = 1 + \frac{(\tilde{B}_{ss}-1)}{\beta R} - B = \tilde{B}_{ss} - B,$$

using $B^{\star}_{ss} = 1/\beta(\tilde{B}_{ss}-1)$. □

Proposition 6. *$B' = B^{\star}_{ss}$ if and only if $B \in \mathcal{B}_0$.*

Proof. We find values of B such that the sub-differential condition holds:

$$\frac{C(B^{\star}_{ss})}{C(B)} \in [1, q_{ss}],$$

where marginal prices are both 1 since $B' \leq B^\star_{ss} < \tilde{B}_{ss}$ and $\beta R = 1$. This yields:

$$\frac{\tilde{B}_{ss} - B^\star_{ss}}{\tilde{B}_{ss} - B} \geq 1 \iff B \geq B^\star_{ss},$$

and

$$\frac{\tilde{B}_{ss} - B^\star_{ss}}{\tilde{B}_{ss} - B} \leq q_{ss} \iff B \leq \tilde{B}_{ss} - \frac{1}{q_{ss}}(\tilde{B}_{ss} - B^\star_{ss}) < \tilde{B}_{ss}.$$

□

Proposition 7. *Let $\gamma \equiv \beta/q_{ss}$ and define:*

$$\Gamma(t) \equiv \frac{1-\gamma^t}{1-\gamma} - q_{ss}^{-t}\frac{1-\beta^t}{1-\beta}, \quad t \geq 0.$$

Let T be the unique positive integer satisfying $\Gamma(T) < 1 < \Gamma(T+1)$.

Any decreasing sequence $\{B_{-t}\}$ for $t \in \{0, T\}$ satisfying the entrepreneur's Euler equation with terminal condition $B_0 \in \mathcal{B}_0$ satisfies $B_{-t} \in \mathcal{B}_{-t}$ where

$$\mathcal{B}_{-t} \equiv (B^\ell_{-t}, B^\ell_{-t-1}] \cap [0, \tilde{B}_{ss}],$$

with

$$B^\ell_{-t} \equiv B^\star_{ss} + (\tilde{B}_{ss} - B^\star_{ss})\Gamma(t).$$

Moreover, $\cup_{t=0}^{T} \mathcal{B}_{-t} = (B^\star_{ss}, \tilde{B}_{ss}]$.

Proof. For any $B \in (B^\star_{ss}, \tilde{B}_{ss}]$, the entrepreneur makes both spot and chained purchases, so the ratio of marginal prices is q_{ss}. Consider a decreasing sequence of debt levels converging to some $B_0 \in \mathcal{B}_0$. For any $B_{-t} \in (B^\star_{ss}, \tilde{B}_{ss}]$, the entrepreneur's Euler equation yields:

$$\frac{C_{-t+1}}{C_{-t}} \cdot \frac{Q_{-t}}{Q_{-t+1}} = \beta R \cdot \frac{\tilde{q}^E_{-t}(B_{-t+1})}{\tilde{q}^B_{-t+1}(B_{-t+1})} = 1/q_{ss},$$

where we used that $\beta R = 1$ in steady state and the ratio of marginal prices equals q_{ss} for debt in this range. This gives us the difference equation:

$$C_{-t} = \frac{1}{q_{ss}} C_{-t+1} \text{ for } B_{-t+1} \in (B^\star_{ss}, \tilde{B}_{ss}],$$

with terminal condition $C_0 = \tilde{B}_{ss} - B_0$ (from the previous lemma).

Shifting forward the equation, we obtain:

$$C_0/C_{-t} = q_{ss}^t.$$

Using the budget constraint, consumption at $-t$ is:

$$C_{-t} = \underbrace{\tilde{B}_{ss} - B_{-t}}_{S_{-t}} + \underbrace{\frac{1}{q_{ss}}(1 + \beta B_{-t+1} - \tilde{B}_{ss})}_{X_{-t}}.$$

Combining these expressions:

$$\tilde{B}_{ss} - B_{-t} + q_{ss}^{-1}(1 + \beta B_{-t+1} - \tilde{B}_{ss}) = q_{ss}^{-t}(\tilde{B}_{ss} - B_0).$$

Rearranging:

$$B_{-t} = \tilde{B}_{ss} - q_{ss}^{-t}(\tilde{B}_{ss} - B_0) + q_{ss}^{-1}(1 + \beta B_{-t+1} - \tilde{B}_{ss}).$$

Since $1 + \beta B_{-t+1} - \tilde{B}_{ss} = \beta(B_{-t+1} - B_{ss}^\star)$ (using the definition of $B_{ss}^\star$), we get:

$$B_{-t} = \tilde{B}_{ss} - q_{ss}^{-t}(\tilde{B}_{ss} - B_0) + \frac{\beta}{q_{ss}}(B_{-t+1} - B_{ss}^\star).$$

For the boundary sequence with $B_0 = B_0^\ell = B_{ss}^\star$, solving this difference equation yields:

$$B_{-t}^\ell = B_{ss}^\star + (\tilde{B}_{ss} - B_{ss}^\star)\sum_{\tau=0}^{t-1}\gamma^\tau(1 - q_{ss}^{-t+\tau}),$$

where $\gamma = \beta/q_{ss}$. This sum simplifies to $\Gamma(t)$ as defined.

Since $B_{-1}^\ell = B_{ss}^\star + (1 - 1/q_{ss})(\tilde{B}_{ss} - B_{ss}^\star) = B_0^u$ (the upper bound of $\mathcal{B}_0$), the intervals $\mathcal{B}_{-t}$ form a partition of $(B_{ss}^\star, \tilde{B}_{ss}]$.

The condition $\Gamma(T) < 1 < \Gamma(T+1)$ ensures that T is the last period for which $B_{-T}^\ell \leq \tilde{B}_{ss}$. Since $\Gamma(1) = 1 - q_{ss}^{-1} < 1$ and $\lim_{t\to\infty}\Gamma(t) = 1/(1-\gamma) > 1$, with Γ increasing in t, there exists a unique such T. □

Existence and Uniqueness of B^h. Following the analysis for sequences above $\tilde{B}_{ss}$, we establish:

Corollary 3. *There exists a maximal debt level*

$$\underline{B} = \frac{1 - q_{ss}^{-T}(1 - (1-\beta)B_{ss}^{\star})}{1-\beta},$$

such that deleveraging toward $B_{ss}^{\star}$ following the Euler equation is infeasible.

For values above $\underline{B}$, the policy $B' = B$ is the unique solution consistent with the Euler equation. Since $V(\tilde{B}_{ss}) > \underline{V}(\tilde{B}_{ss})$ while $V(\underline{B}) = \underline{V}(\underline{B})$, by continuity there exists $B^h \in [\tilde{B}_{ss}, \underline{B}]$ such that $V(B) = \underline{V}(B)$ for all $B > B^h$.

To establish uniqueness, we show that if $V(B^d) > \underline{V}(B^d)$ for some $B^d > \tilde{B}_{ss}$, then $V(B) > \underline{V}(B)$ for all $B < B^d$. This follows from the decreasing absolute risk aversion property of log utility: starting from any $B < B^d$, the entrepreneur can consume the annuity of the difference $\Delta = B^d - B$ in chained goods and follow the same deleveraging path, yielding strictly higher value than $\underline{V}(B)$.

This completes the proof of Proposition 3. □

E Proofs Regarding the General Equilibrium Analysis - Section 5

E.1 Proof of Proposition 4

Proof. The allocation follows from individual optimization and market clearing:

Step 1: From the creditor's log utility and budget constraint $B_{t+1}/R_{t+1} + C_t = B_t$, optimal consumption is:

$$C(B) = (1-\beta)B$$

This is the annuity value of debt holdings.

Step 2: Market clearing for goods requires total expenditures equal total production (normalized to 1):

$$C(B) + E(B) = 1$$

Hence, entrepreneur expenditures are:

$$E(B) = 1 - C(B) = 1 - (1-\beta)B$$

Step 3: Given total expenditures $E(B)$ and the constraint that $q \geq 1$ (with equality only when $\mu = 0$), the entrepreneur maximizes goods obtained by maximizing spot purchases subject to the SBL:

$$S(B, \tilde{B}) = \min\{\bar{S}, E(B)\} = \min\{\max\{0, \tilde{B} - B\}, 1 - (1-\beta)B\}$$

Step 4: The value of chained expenditures follows as the residual:

$$\mu(B, \tilde{B}) = E(B) - S(B, \tilde{B})$$

Step 5: The equilibrium price of chained goods is determined by the network technology:

$$q(B, \tilde{B}) = \mathcal{A}^{-1}(\mu(B, \tilde{B}))$$

where $\mathcal{A}(\cdot)$ is the network efficiency function.

Step 6: The quantity of chained goods obtained:

$$X(B,\tilde{B}) = \frac{\mu(B,\tilde{B})}{q(B,\tilde{B})} = \mathcal{A}(\mu(B,\tilde{B}))$$

Step 7: The average price per unit of consumption:

$$Q(B,\tilde{B}) = \frac{E(B)}{S(B,\tilde{B}) + X(B,\tilde{B})} = \frac{S(B,\tilde{B}) + q(B,\tilde{B})X(B,\tilde{B})}{S(B,\tilde{B}) + X(B,\tilde{B})}$$

□

E.2 Analysis: Preliminary Observations

Analysis of the Marginal Expenditure Price. The marginal expenditure price depends on both debt and the SBL: $\tilde{q}^E_t = \tilde{q}^E(B_t, \tilde{B}_t)$ where:

$$\tilde{q}^E(B,\tilde{B}) \equiv q(\mu(B,\tilde{B}))\mathbb{I}_{[B\geq B^\star(\tilde{B})]} + (1 - \mathbb{I}_{[B\geq B^\star(\tilde{B})]}).$$

The fraction of chained transactions in equilibrium is:

$$\mu(B,\tilde{B}) = 1 - (1-\beta)B - \min\{\max\{0, \tilde{B} - B\}, 1 - (1-\beta)B\}.$$

This yields:

$$\mu(B,\tilde{B}) = \begin{cases} 0 & B < B^\star(\tilde{B}), \\ 1 + \beta B - \tilde{B} & B \in [B^\star(\tilde{B}), \tilde{B}], \\ 1 - (1-\beta)B & B > \tilde{B}. \end{cases}$$

The function $\mu(B,\tilde{B})$ is continuous, starts at zero, increases up to $B = \tilde{B}$, then decreases. Since q_t is monotone in μ_t, the marginal expenditure price follows the same pattern.

At the efficiency threshold:

$$\lim_{B\downarrow B^\star(\tilde{B})} \tilde{q}^E(B,\tilde{B}) = \lim_{\mu\downarrow 0} q(\mu) = \delta^{-1},$$

where δ is the network efficiency parameter. Thus $\tilde{q}^E$ has a discontinuity at $B^\star$ since $\lim_{B\uparrow B^\star(\tilde{B})} \tilde{q}^E = 1 \neq \delta^{-1}$.

Behavior of Average-to-Marginal Price Ratios. The ratio $Q/\tilde{q}^E$ is crucial for equilibrium dynamics. For $B < B^\star(\tilde{B})$, since $q = 1$, we have $Q/\tilde{q}^E = 1$. For $B \geq \tilde{B}$, also $\tilde{q}^E = Q = q$, so $Q/\tilde{q}^E = 1$.

In the intermediate range $B \in [B^\star(\tilde{B}), \tilde{B}]$:

$$\frac{Q}{\tilde{q}^E} = \frac{1}{q} \cdot \frac{q}{(1 - S/E) + q(S/E)} = \frac{1}{(1 - S/E) + q(\mu(B, \tilde{B}))(S/E)}.$$

At $B = \tilde{B}$, we have $S/E = 0$, so the function is continuous there. However, as $B \downarrow B^\star(\tilde{B})$:

$$\frac{S}{E} \downarrow 1 \text{ and } q \downarrow \delta^{-1} > 1,$$

yielding $Q/\tilde{q}^E = \delta$ at the right limit of $B^\star$ and $Q/\tilde{q}^E = 1$ at the left limit—a discontinuity.

Analysis of the Marginal Borrowing Price. The marginal borrowing price $\tilde{q}^B_{t+1} = \tilde{q}^B(B_{t+1}, \tilde{B}_{t+1})$ where:

$$\tilde{q}^B(B, \tilde{B}) \equiv q(\mu(B, \tilde{B}))\mathbb{I}_{[B \geq \tilde{B}]} + (1 - \mathbb{I}_{[B \geq \tilde{B}]}).$$

This yields:

$$\tilde{q}^B(B, \tilde{B}) = \begin{cases} 1 & B < B^\star(\tilde{B}), \\ 1 & B \in [B^\star(\tilde{B}), \tilde{B}], \\ q & B > \tilde{B}. \end{cases}$$

At $B = \tilde{B}$, the function has a discontinuity:

$$\lim_{B \downarrow \tilde{B}} \tilde{q}^B(B, \tilde{B}) = q(\mu(\tilde{B}, \tilde{B})) = q(1 - (1 - \beta)\tilde{B}) > 1.$$

For the ratio $Q/\tilde{q}^B$:

$$\frac{Q}{\tilde{q}^B} = \begin{cases} 1 & B < B^\star(\tilde{B}), \\ Q & B \in [B^\star(\tilde{B}), \tilde{B}], \\ 1 & B > \tilde{B}, \end{cases}$$

where $\lim_{B \downarrow B^\star(\tilde{B})} Q = 1$.

Average Price Elasticity. The elasticity of average price with respect to debt in general equilibrium (after imposing $E = 1 - (1 - \beta)B$) is critical for policy analysis.

Starting from:

$$Q = \frac{1}{\frac{1}{q}(1 - S/E) + S/E},$$

we obtain:

$$\frac{\partial Q}{\partial B} = -Q \cdot \frac{(1 - S/E)\frac{\partial}{\partial B}[1/q] - (1/q - 1)\frac{\partial}{\partial B}[S/E]}{(1/q(1 - S/E) + S/E)}.$$

For the derivative of the expenditure share:

$$\frac{\partial}{\partial B}\left[\frac{S}{E}\right] = \begin{cases} 0 & B < B^\star(\tilde{B}) \\ \left[\frac{\tilde{B}-B}{1-(1-\beta)B}\right]\left(\frac{(1-\beta)}{1-(1-\beta)B} - \frac{1}{\tilde{B}-B}\right) & B \in [B^\star(\tilde{B}), \tilde{B}) \\ 0 & B \geq \tilde{B} \end{cases}$$

This derivative is negative in the intermediate region and has discontinuities at the boundaries:

$$\frac{\partial}{\partial B}\left[\frac{S}{E}\right]\Big|_{B\uparrow\tilde{B}} = -\frac{1}{1 - (1-\beta)\tilde{B}} < -1,$$

and

$$\frac{\partial}{\partial B}\left[\frac{S}{E}\right]\Big|_{B\downarrow B^\star} = -1.$$

For the price derivative:

$$\frac{\partial}{\partial B}[q] = \frac{\partial q}{\partial \mu} \cdot \frac{\partial \mu}{\partial B} = q_\mu \mu \frac{\frac{\partial E}{\partial B} - \frac{\partial S}{\partial B}}{E},$$

where ϵ_μ^q denotes the elasticity of q with respect to μ. These elasticities determine the general equilibrium feedback effects between debt accumulation and payment chain formation.

E.3 Derivation of the map $\mathcal{B}$

For convenience, we reproduce the aggregate Euler equation (14) here:

$$\underbrace{\frac{B}{1 - (1-\beta)B} \cdot Q\left(B, \tilde{B}\right)}_{\equiv \mathcal{E}\left(B;\tilde{B}\right)} = \underbrace{\frac{B'}{1 - (1-\beta)B'} \frac{Q\left(B', \tilde{B}'\right)}{\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right)}}_{\equiv \mathcal{E}'\left(B';B,\tilde{B},\tilde{B}'\right)}.$$

First, we show that for a subset of $B > \tilde{B}'$ there are two roots B' that solve $\mathcal{E} = \mathcal{E}'$. The roots satisfy $B'_1 < \tilde{B}' < B'_2$ and $B'_2 = B$ thus $R = \beta^{-1}$ and $q = q'$. Then, the proof finishes by show that do to the hysteresis result of stationary problems, for each $B \in \left(\tilde{B}', B^h\right)$ the larger root B'_2 cannot be an individual optimum during a transition.

The first result is summarized by the following Lemma.

Lemma 14. *There exists a threshold B^* such that for any $B \in [\tilde{B}', B^*]$, the equation $\mathcal{E} = \mathcal{E}'$ has two roots B': one root is $B = B'$ and the other satisfies $B' \leq \tilde{B}'$. For any $B > B^*$, only $B = B'$ is a solution. B^* solves:*

$$\mathcal{E}'\left(B^*; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

The interpretation of B^* is obtained from the left panel of Figure 9. B^* is the highest value for which the Euler equation has a solution $B' \in (\tilde{B}', B^*)$.

Proof. Since $\tilde{B}' > \tilde{B}$, if $B' = B > \tilde{B}'$, the worker only spends chained expenditures. Thus,

$$\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right) = 1 \text{ and } Q\left(B, \tilde{B}\right) = Q\left(B', \tilde{B}'\right) = q\left(1 - (1-\beta)B\right).$$

Hence, $B' = B$ is always a solution to the Euler equation when $B' = B > \tilde{B}'$.

Let $B > B^*$. The function $\mathcal{E}(B)$ is monotone increasing. In turn, the function $\mathcal{E}'(B')$ is monotone increasing to the left of $\tilde{B}'$. Hence, for $B > B^*$ a solution must lie at $B' > \tilde{B}'$. However, for any $B' > \tilde{B}'$,

$$\mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \frac{B'}{1-(1-\beta)B'} q\left(1 - (1-\beta)B'\right)$$

which follows from the fact that $B' > \tilde{B}' > B^\star\left(1/\beta, \tilde{B}'\right)$ in which case $\Pi\left(B'; B, \tilde{B}, \tilde{B}'\right) = 1$. This implies that for any $B > B^*$ we have that:

$$\mathcal{E}\left(B; \tilde{B}\right) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right),$$

which implies,

$$\frac{B}{1-(1-\beta)B} \cdot q\left(1 - (1-\beta)B\right) = \frac{B'}{1-(1-\beta)B'} \cdot q\left(1 - (1-\beta)B'\right).$$

By monotonicity, we immediately conclude that $B = B'$ and that is the only solution.

Now consider $B \in \left(\tilde{B}', B^*\right)$. We now look for a solution $B' \in \left(B^\star(\tilde{B}'), \tilde{B}'\right)$. If $B < B^*$, again by monotonicity:

$$\mathcal{E}'\left(B; \tilde{B}, \tilde{B}, \tilde{B}'\right) < \lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

$$\mathcal{E}\left(B; \tilde{B}\right) = \mathcal{E}'\left(B'; B, \tilde{B}, \tilde{B}'\right)$$

$$\mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \downarrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)$$

$$\frac{B'}{1-(1-\beta)B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{B'}{1-(1-\beta)B'} \cdot \frac{q\left(1-(1-\beta)\tilde{B}'\right)}{\lim_{B' \uparrow \tilde{B}'} \Pi\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)}$$

$$\frac{B'}{1-(1-\beta)B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\tilde{B}'} \cdot \frac{q\left(1-(1-\beta)\tilde{B}'\right)}{\frac{q\left(1-(1-\beta)\tilde{B}'\right)}{q\left(1-(1-\beta)\tilde{B}\right)}}$$

$$\frac{B'}{1-(1-\beta)B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\tilde{B}'} \cdot q\left(1-(1-\beta)\tilde{B}\right) > 0$$

Fix $\tilde{B}, \tilde{B}'$. Define $B^* > \tilde{B}$ to be the value that solves:

$$\mathcal{E}'\left(B^*; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right).$$

By definition, if indeed $B^* > \tilde{B}$, then

$$\frac{B^*}{1-(1-\beta)B^*} \cdot q\left(1-(1-\beta)B^*\right) = \lim_{B' \uparrow \tilde{B}'} \frac{B'}{1-(1-\beta)B'} \cdot \frac{Q\left(B', \tilde{B}'\right)}{\Pi\left(B'; B^*, \tilde{B}, \tilde{B}'\right)}$$

which implies

$$B^* = \frac{\tilde{B}'}{\frac{1-(1-\beta)\tilde{B}'}{q\left(1-(1-\beta)\tilde{B}'\right)} + (1-\beta)\tilde{B}'} = \frac{\tilde{B}'}{C^w\left(\tilde{B}', \tilde{B}'\right) + C^s\left(\tilde{B}'\right)} > \tilde{B}'.$$

The last inequality verifies that $B^* > \tilde{B}'$. This result follows from

$$\lim_{B' \uparrow \tilde{B}'} \Pi\left(B'; B^*, \tilde{B}, \tilde{B}'\right) = \left(q\left(1-(1-\beta)\,B^*\right)\right)^{-1}$$

and

$$\lim_{B' \uparrow \tilde{B}'} Q\left(B', \tilde{B}'\right) = q\left(1-(1-\beta)\,\tilde{B}'\right).$$

The inequality holds since $q > 1$. Furthermore, this satisfies $B^* > \tilde{B}$ as required since the SBL sequence is weakly increasing.[28]

Next, we show that for any $B \in \left(\tilde{B}', B^*\right)$, the equation $\mathcal{E} = \mathcal{E}'$ has two roots; one above $\tilde{B}'$ and one below. The solution above $\tilde{B}'$ is trivial: $B = B' > \tilde{B}' \geq \tilde{B}$ satisfies the equation. For the solution below, we use continuity and monotonicity to show that there is a unique $B' < \tilde{B}'$ such that

$$\mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right) = \lim_{B' \downarrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)$$

$$\frac{B'}{1-(1-\beta)\,B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{B'}{1-(1-\beta)\,B'} \cdot \frac{q\left(1-(1-\beta)\,\tilde{B}'\right)}{\lim_{B' \uparrow \tilde{B}'} \Pi\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)}$$

$$\frac{B'}{1-(1-\beta)\,B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\,\tilde{B}'} \cdot \frac{q\left(1-(1-\beta)\,\tilde{B}'\right)}{\frac{q\left(1-(1-\beta)\tilde{B}'\right)}{q\left(1-(1-\beta)\tilde{B}\right)}}$$

$$\frac{B'}{1-(1-\beta)\,B'} \cdot Q\left(B', \tilde{B}'\right) = \frac{\tilde{B}'}{1-(1-\beta)\,\tilde{B}'} \cdot q\left(1-(1-\beta)\,\tilde{B}\right) > 0$$

Lets call this B', $\underline{B}$ and think of the interval $\left(\tilde{B}', B^*\right)$. The interpretation of $\underline{B}$ is that it is the small root for the debt level "$\tilde{B}' + \varepsilon$". Meaning, if $\tilde{B}'$ is the small root for B^*(the end of the interval) then $\underline{B}$ is the small root for the start of the interval. Now, notice that the RHS is a constant with respect to $\tilde{B}'$. To establish existence of $\underline{B}$ further notice that the LHS tends to $\lim_{B' \uparrow \tilde{B}'} \mathcal{E}'\left(B'; B^*, \tilde{B}, \tilde{B}'\right)$ as $B' \uparrow \tilde{B}'$ (this was shown in the previous step) and this is larger than the RHS just by comparing magnitudes ($\tilde{B}' \geq \tilde{B}$ and $q\left(1-(1-\beta)\,B^*\right) > 1$). Then notice that as $B' \downarrow 0$ the LHS goes to zero which is lower than the RHS. By continuity of the LHS we can apply the intermediate value theorem for existence. Uniqueness is granted since the LHS is increasing in B'. The fraction is

[28] This is important because the relevant points to study are those for which the all chained equilibrium is possible. These are the cases where $B^* > \tilde{B}' \geq \tilde{B}$.

clearly increasing in B', the average price too because as $B' \uparrow \tilde{B}'$ the share of chained expenditure increases and also its price does so. This statement also uses the fact that Q is the weighted harmonic mean of prices with expenditure weights. Since the LHS is continuous and increasing it maps the interval $\left(\underline{B}, \tilde{B}'\right)$ onto

$$\left(\lim_{B' \downarrow \tilde{B}'} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right), \lim_{B' \uparrow B^*} \mathcal{E}'\left(B'; \tilde{B}, \tilde{B}, \tilde{B}'\right)\right).$$

So (since $\mathcal{E}'$ is increasing and continuous) it covers all the image of $\left(\tilde{B}', B^*\right)$. This proves that for $B \in \left(\tilde{B}', B^*\right)$ there exist two roots that solve $\mathcal{E} = \mathcal{E}'$. One is $B'_2 = B > \tilde{B}'$ and the other is a $B'_1 \in \left(\underline{B}, \tilde{B}'\right)$.

To make affairs clearer, suppose that $B^* < B^h$ then for all $B \in \left(\tilde{B}', B^*\right)$ the larger root $B' = B$ is not an equilibrium and there is a region $\left(B^*, B^h\right)$ that does not have a symmetric competitive equilibrium because the only root that solves equation (**??**) provides allocations and prices that are not an equilibrium. Now suppose that $B^* > B^h$, then for $B \in \left(\tilde{B}', B^h\right)$ the larger root $B' = B$ is not an equilibrium and for $B \in \left(B^h, \overline{B}\right)$. So summarizing both cases. For $B_0 < B^h\left(\tilde{B}_0\right)$, if a (symmetric competitive) equilibrium exists then it is given by the smaller root of equation **??**.

□

E.4 Proof of Corollary 1

At $B_t \leq B^\star\left(\beta^{-1}, \tilde{B}_{ss}\right)$ if $R = \beta^{-1}$ then marginal and average prices are equal to 1. Then it is evident that $B_{t+1} = B_t$ solves equation **??** and the expenditure is $1 - (1-\beta) B_t \leq \tilde{B}_{ss} - B_t$ by assumption. As a consequence, the steady state is non-disrupted.

E.5 Proof of Corollary 2

That $B_{t+1} < B_t$ if $B_t \in \left(\tilde{B}, B^*\left(\tilde{B}\right)\right)$ is immediate from the result proved in Proposition **??** since we are choosing the smaller root and the larger root is $B_{t+1} = B_t$. For $B_t > B^\star$ it is enough to show that $\beta R_{t+1} < 1$. This was done in step 3 of the proof of Proposition **??**.

E.6 Proof of Corollary ??

The proof follows immediately from Proposition **??**. If $B_0 > B^h$, there exists an equilibrium with $R_t = 1/\beta$ and $B_t = B_0$. Recall that if $R_t = 1/\beta$, and $B_t = B_{t+1}$ is a solution to $\mathcal{E}_t = \mathcal{E}_{t+1}$. Since for $R_t = 1/\beta$, $B_0 = B_t$ is a solution to the worker's problem, both Euler equations are satisfied and the asset clearing condition holds, sequences with $R_t = 1/\beta$ and $B_t = B_0$ are consistent with the equilibrium definition. Since at $B_t > B^h > B^\star$, $q_t > 1$. Hence, the economy remains permanently disrupted.

F Constrained Efficiency Analysis - Section 5

F.1 Characterization of the Primal Problem

This appendix provides the complete characterization of the Primal Problem solution from Section 5.3. We first present the problem in its general form, then derive the solution across different regimes of the spot borrowing limit.

Since the maximization is static (time-separable), we can solve the problem period by period. The Primal Planner's problem at each date is:

Problem 8. (Static Primal Problem):

$$\mathcal{P}^{\theta}\left(\tilde{B}\right) = \max_{B \in \left[0, \bar{B}\right]} \mathcal{P}\left(B, \tilde{B}\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) = (1-\theta)\log\left((1-\beta)B\right) + \theta \log\left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\mu\left(B, \tilde{B}\right) + S\left(B, \tilde{B}\right)\right)$$

subject to:

$$\mu\left(B, \tilde{B}\right) \equiv 1 - (1-\beta)B - \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta)B\right\}$$

$$S\left(B, \tilde{B}\right) = \min\left\{\max\left\{\tilde{B} - B, 0\right\}, 1 - (1-\beta)B\right\}.$$

This is a problem where the planner can choose debt at will, but respects the constraints. Next, we proceed to solve this problem. Naturally, welfare depends on whether and how the planner may want to distort TFP to provide insurance. There are multiple policy regimes that depend on the SBL, $\tilde{B}$. We make use of two auxiliary problems, the best and worst value problems:

$$\underline{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}(0) \text{ and } \bar{\mathcal{P}}^{\theta} \equiv \mathcal{P}^{\theta}\left(\bar{B}\right).$$

Proposition 8. (Solution of the Primal Problem):

Let the solution to this problem be $\mathcal{B}^{\theta}$. Then, the solution to the primal planner's problem is $B_t = \mathcal{B}^{\theta}\left(\tilde{B}_t\right)$. The function $\mathcal{B}^{\theta}$ satisfies:

I. Efficiency Threshold. *For $\tilde{B} \geq \frac{1-\theta\beta}{1-\beta}$, $\mathcal{B}^{\theta} = B_{ss}$. Moreover, for this debt level $X = 0$.*

II. Inefficiency Threshold. *For $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$ the planner's solution induces TFP losses. The solution to the Primal Planner's problem in this region depends on the threshold SBL, $\tilde{B}^i$.*

II.a Social Insurance and Productive Efficiency. *For $\tilde{B} \in \left[\tilde{B}^i, \frac{1-\theta\beta}{1-\beta}\right]$, we have that $\mathcal{B}^\theta = B^\star\left(\tilde{B}\right)$:*

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)} \leq \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}$$

II.b Social Insurance Complements Productive Efficiency. *If $\tilde{B}^s < \tilde{B}^i$, for $\tilde{B} \in \left[\tilde{B}^s, \tilde{B}^i\right]$, we have that $\mathcal{B}^\theta$ solves:*

$$\frac{1-(1-\beta)\,B^\theta}{(1-\beta)\,B^\theta} = \frac{\theta}{1-\theta}\frac{Q\left(B^\theta,\tilde{B}\right)}{q\left(B^\theta,\tilde{B}\right)}\left(\frac{q\left(B^\theta,\tilde{B}\right)-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B^\theta,\tilde{B}\right)\right)\right)}{1-\beta}\right).$$

Moreover, for this debt level $X, S > 0$.

II.c Social Insurance Conflicts Productive Efficiency. *For $\tilde{B} \in \left[0, \tilde{B}^s\right]$, we have that $\mathcal{B}^\theta$ is the unique constant solution $B^\theta > B_{ss}$ to the equation:*

$$\frac{1-(1-\beta)\,B^\theta}{(1-\beta)\,B^\theta} = \frac{\theta}{1-\theta}\left(1+\epsilon_\mu^{\mathcal{A}}\left(1-(1-\beta)\,B^\theta\right)\right).$$

Moreover, for this debt level $S = 0$.

Threshold value $\tilde{B}^i$. *$\tilde{B}^i$ is the unique solution to:*

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}^i\right)}{(1-\beta)\,B^\star\left(\tilde{B}^i\right)} = \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}$$

Threshold value $\tilde{B}^s$. *Let $\underline{\mathcal{P}}^\theta \equiv \mathcal{P}^\theta(0)$ and $\bar{\mathcal{P}}^\theta \equiv \mathcal{P}^\theta\left(\bar{B}\right)$. The threshold $\tilde{B}^s$ solves:*

$$\bar{\mathcal{P}}^\theta = \underline{\mathcal{P}}^\theta + \int_{\tilde{B}^s}^{\frac{1-\theta\beta}{1-\beta}}\left(\mathcal{P}_{\tilde{B}} + \mathcal{P}_B|_{B=B^\star(\tilde{B})}\cdot B^\star_{\tilde{B}}\left(\tilde{B}\right)\right)d\tilde{B}.$$

where for $\tilde{B} \in \left[\tilde{B}^i, \frac{1-\theta\beta}{1-\beta}\right]$ *we have that* $\mathcal{P}^{\theta}_{\tilde{B}}\left(\tilde{B}\right)$ *equals:*

$$\frac{\theta}{1-(1-\beta)\,B^p(\tilde{B})}\frac{Q\left(B^p(\tilde{B}),\tilde{B}\right)}{q\left(B^p(\tilde{B}),\tilde{B}\right)}\left(\mathcal{A}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)-\left(1+\epsilon^{\mathcal{A}}_{\mu}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)\right)\right)$$

and

$$\mathcal{P}_B|_{B=B^\star(\tilde{B})} = \frac{1-\theta}{(1-\beta)\,B^\star\left(\tilde{B}\right)} - \frac{\theta\,(1-\beta\delta)}{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}$$

Case 1. Values of $\tilde{B}$ such that all consumption is spot. Ideally, the planner wants to maximize spot consumption and set $\mu = 0$. The unconstrained solution to the Primal Planner's problem is given by the ratio of Pareto weights:

$$\frac{1-(1-\beta)\,B^o}{(1-\beta)\,B^o} = \frac{\theta}{1-\theta} \to B^o = \frac{1-\theta}{1-\beta} = B_{ss}.$$

This yields the same value as $\bar{\mathcal{P}}^{\theta}$. This level of debt must satisfy the condition that all spot consumption must be feasible:

$$\max\left\{\tilde{B}-B^o,0\right\} \geq 1-(1-\beta)\,B^o > 0.$$

Thus, we need

$$\tilde{B} \geq B^o$$

and that:

$$\tilde{B} \geq 1+\beta B^o.$$

Combining both constraints we have:

$$\tilde{B} \geq B^o + \max\left\{1-(1-\beta)\,B^o, 0\right\}.$$

We know that the optimal debt B^o must be less than the natural borrowing limit. Hence, the inequality is just:

$$\tilde{B} \geq 1+\beta B^o = 1+\frac{\beta}{1-\beta}(1-\theta) = \frac{1-\theta\beta}{1-\beta}. \tag{78}$$

Thus, for these levels of the SBL, the planner can achieve the unconstrained solution. This corresponds to the debt in the efficient steady state level of the competitive equilibrium that produces the planner's Pareto weights. Now consider the case where the constraint binds, $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$. In this case, the planner cannot achieve the unconstrained solution. The amount of chained expenditures are therefore positive:

$$\mu\left(B,\tilde{B}\right) = 1-(1-\beta)B - \min\left\{\max\left\{\tilde{B}-B,0\right\},1-(1-\beta)B\right\} > 0.$$

We have critical values.

Proof.

- If the planner chooses $B^p \geq B_h\left(\tilde{B}\right) = \tilde{B}$, there is no spot consumption.
- If the planner chooses $B^p < B_l\left(\tilde{B}\right) < \tilde{B}$, there is no chained consumption. This threshold $B_l\left(\tilde{B}\right)$ solves,

$$\max\left\{\tilde{B}-B_l,0\right\} = 1-(1-\beta)B_l$$

when since $B_l < \tilde{B}$, we obtain:

$$\tilde{B} = 1+\beta B_l \rightarrow B_l\left(\tilde{B}\right) = \max\left\{0,\beta^{-1}\left(\tilde{B}-1\right)\right\} = B^{\star}\left(\tilde{B}\right).$$

Obviously, $B^{\star}\left(\tilde{B}\right) < B^o$ since we are in the constrained region.

Case 2. Values of $\tilde{B}$ such that some consumption is chained. Consider now the planner problem that restricts choices to at least some of both goods are consumed by the entrepreneur:

Problem 9. *The Primal Planner's problem restricted to both types of consumption is:*

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) = \max_{B}\left\{(1-\theta)\log\left((1-\beta)B\right)+\theta\log\left(\mathcal{A}\left(\mu\left(B,\tilde{B}\right)\right)\cdot\mu\left(B,\tilde{B}\right)+S\left(B,\tilde{B}\right)\right)\right\}$$

subject to:

$$B \in \left[B_l\left(\tilde{B}\right),B_h\left(\tilde{B}\right)\right]$$

$$\mu\left(B,\tilde{B}\right) \equiv 1+\beta B-\tilde{B}$$

and

$$S\left(B,\tilde{B}\right) = \tilde{B}-B.$$

We have the following Lemma.

Lemma 15. *For any $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$, the original planner problem satisfies:*

$$\mathcal{P}^{\theta}\left(\tilde{B}\right) = \max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right), \underline{\mathcal{P}}^{\theta}\right\}.$$

Proof. Indeed, in the region $B^p \in \left[0, B_l\left(\tilde{B}\right)\right]$ the objective of the planner is equivalent to the objective when the SBL is most relaxed, $\mathcal{P}\left(B, \bar{B}\right)$. Thus, since $\tilde{B} < \frac{1-\theta\beta}{1-\beta}$, the planner's objective is increasing in the region $\left[0, B_l\left(\tilde{B}\right)\right]$. Thus, the planner's solution must fall in between $B^p \in \left[B_l\left(\tilde{B}\right), \bar{B}\right]$. For any $B^p \geq B_h\left(\tilde{B}\right) = \tilde{B}$, the objective function in $\mathcal{P}\left(B, \tilde{B}\right)$ is independent of $\tilde{B}$ and hence, must coincide with the value of $\underline{\mathcal{P}}^{\theta}$. Hence, we can partition $\mathcal{P}^{\theta}\left(\tilde{B}\right)$ according to the Lemma. □

To prove the main result, we solve the problems $\underline{\mathcal{P}}^{\theta}$ and $\tilde{\mathcal{P}}\left(\tilde{B}\right)$.

Auxiliary Problem $\underline{\mathcal{P}}^{\theta}$: no spot consumption. The planner's problem with the tightest SBL $\underline{\mathcal{P}}^{\theta} = \mathcal{P}^{\theta}(0)$ is given by:

$$\underline{\mathcal{P}}^{\theta} = \max_{B \in \left[0, \bar{B}\right]} \underline{\mathcal{P}}(B, 0)$$

where

$$\underline{\mathcal{P}}^{\theta} = \max_{B \in \left[0, \bar{B}\right]} \left\{(1-\theta)\log\left((1-\beta)B\right) + \theta\log\left(\mathcal{A}\left(\mu\left(B,0\right)\right)\mu\left(B,0\right)\right)\right\}$$

subject to:

$$\mu(B,0) \equiv 1 - (1-\beta)B.$$

To solve this problem, we perform some calculations. First, note that:

$$\frac{\partial\left[\mathcal{A}(\mu)\mu\right]}{\partial\mu} = \mathcal{A}(\mu)\left(1 + \epsilon^{\mathcal{A}}\right)$$

where,

$$\epsilon_{\mu}^{\mathcal{A}} \equiv \frac{\partial\mathcal{A}(\mu)}{\partial\mu}\frac{\mu}{\mathcal{A}(\mu)}.$$

The derivative $\underline{\mathcal{P}}_B$ is therefore given by:

$$\underline{\mathcal{P}}_B = (1-\theta)\frac{(1-\beta)}{(1-\beta)B} + \theta\frac{\mathcal{A}(\mu)\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mu_B(B,0)}{\mathcal{A}(\mu)\mu} = \frac{(1-\theta)}{B} - \theta\frac{\left(1+\epsilon_\mu^{\mathcal{A}}\right)(1-\beta)}{1-(1-\beta)B}.$$

The second equation uses that $\mu_B(B,0) \equiv -(1-\beta)$.

The first term in $\underline{\mathcal{P}}_B$, $(1-\theta)/B$, is decreasing in B. The second term,

$$\theta\frac{\left(1+\epsilon_\mu^{\mathcal{A}}\right)(1-\beta)}{1-(1-\beta)B}, \tag{79}$$

is increasing. We know this because the denominator is decreasing in $B \in \left[0, \bar{B}\right]$ and the elasticity of TFP is itself decreasing in μ,

$$\epsilon_{\mu\mu}^{\mathcal{A}} = \frac{\partial}{\partial\mu}\left[\frac{\mathcal{A}'(\mu)\mu}{\mathcal{A}(\mu)}\right] = \frac{\mathcal{A}''(\mu)\mu}{\mathcal{A}(\mu)} + \frac{\mathcal{A}'(\mu)}{\mathcal{A}(\mu)} - \frac{\left[\mathcal{A}'(\mu)\right]^2}{\left[\mathcal{A}(\mu)\right]^2}\mu < 0.$$

Hence, $\epsilon_{\mu\mu}^{\mathcal{A}}\mu_B(B,0) > 0$, since the product of two negative numbers is positive, thus the numerator of the second term (79) is increasing in B. Thus, $\underline{\mathcal{P}}$ is concave and therefore $\underline{\mathcal{P}}^\theta$ has a unique solution:

$$\frac{1-(1-\beta)B}{(1-\beta)B} = \frac{\theta}{1-\theta}\left(1+\epsilon_\mu^{\mathcal{A}}(\mu(B,0))\right)$$

and recall that $\mu(B,0) = 1-(1-\beta)B$. We call this solution $\underline{B}^p$: the planner debt level under the most tight SBL. We have the following Lemma:

Lemma 16. *The solution $\underline{B}^p > B^o$.*

Proof. The proof is immediate from $1+\epsilon_\mu^{\mathcal{A}} < 1$ and the fact that

$$\frac{1-(1-\beta)B^o}{(1-\beta)B^o} = \frac{\theta}{(1-\theta)}.$$

□

Next, we solve $\tilde{\mathcal{P}}\left(\tilde{B}\right)$.

Auxiliary Problem $\tilde{\mathcal{P}}\left(\tilde{B}\right)$: spot and chained consumption. Consider now the planner problem where at least some of both goods are consumed by the entrepreneur:

$$\tilde{\mathcal{P}}\left(\tilde{B}\right) = \max_{B \in \left[B^{\star}\left(\tilde{B}\right), \tilde{B}\right]} \tilde{\mathcal{P}}\left(B, \tilde{B}\right)$$

where

$$\tilde{\mathcal{P}}\left(B, \tilde{B}\right) \equiv (1-\theta)\log\left((1-\beta)B\right) + \theta \log\left(\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right) \cdot \mu\left(B, \tilde{B}\right) + S\left(B, \tilde{B}\right)\right),$$

subject to:

$$\mu\left(B, \tilde{B}\right) \equiv 1 + \beta B - \tilde{B}$$

and

$$S\left(B, \tilde{B}\right) = \tilde{B} - B.$$

The derivative of the objective in $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is:

$$\tilde{\mathcal{P}}_B\left(B, \tilde{B}\right) = (1-\theta)\frac{1}{B} + \theta \frac{\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\left(1 + \epsilon_{\mu}^{\mathcal{A}}\left(\mu\left(B, \tilde{B}\right)\right)\right)\beta - 1}{\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\mu\left(B, \tilde{B}\right) + \tilde{B} - B}.$$

Recall that,

$$C\left(B, \tilde{B}\right) = \mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)\mu\left(B, \tilde{B}\right) + \tilde{B} - B.$$

and

$$Q \cdot C\left(B, \tilde{B}\right) = E\left(B, \tilde{B}\right) = 1 - (1-\beta)B$$

Hence, using the definition of Q and q we rewrite:

$$\tilde{\mathcal{P}}_B\left(B, \tilde{B}\right) = (1-\theta)\frac{1}{B} - \theta Q \frac{1 - \beta\left(1 + \epsilon_{\mu}^{\mathcal{A}}\left(\mu\left(B, \tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)}{1 - (1-\beta)B}$$

We can multiply both sides by the ratio of $1-(1-\beta)B$ and divide by $(1-\beta)(1-\theta)$ and obtain:

$$\tilde{\mathcal{P}}_B\left(B, \tilde{B}\right)\frac{1-(1-\beta)B}{(1-\theta)(1-\beta)B} = \frac{1-(1-\beta)B}{(1-\beta)B} - \frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right).$$

This function must have the same sign as $\tilde{\mathcal{P}}_B\left(B,\tilde{B}\right)$, since it was obtained by multiplication of positive numbers. The first term is decreasing in B. In turn, $Q_\mu \mu_B$ is increasing in B. Hence, as long as

$$\mathcal{A}(\mu)\left(1+\epsilon_\mu^{\mathcal{A}}\right)=\mathcal{A}(\mu)+\mathcal{A}'(\mu)\mu$$

is decreasing in B, the second term is increasing. The second term is indeed decreasing in μ since its derivative is:

$$2\mathcal{A}'(\mu)+\mathcal{A}''(\mu)\mu<0,$$

where the sign follows immediately from the concavity and monotone decreasing properties of $\mathcal{A}$. Hence, the objective function $\tilde{\mathcal{P}}\left(B,\tilde{B}\right)$ is concave in B. Furthermore, since

$$Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right)>1,$$

and we have an interior maximum in the region $B\in\left[B^\star\left(\tilde{B}\right),\tilde{B}\right]$, B must solve:

$$\frac{1-(1-\beta)B}{(1-\beta)B}=\frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right),$$

and must be such that $B<B^o$.

Next, we establish properties regarding the limits of this function at $B^\star\left(\tilde{B}\right)$ and $\tilde{B}$. We start with $B^\star\left(\tilde{B}\right)$:

$$\lim_{B\downarrow B^\star\left(\tilde{B}\right)}\frac{1-(1-\beta)B}{(1-\beta)B}-\frac{\theta}{1-\theta}Q\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\right)\mathcal{A}(\mu)}{1-\beta}\right)=$$

$$\frac{1-(1-\beta)B^\star\left(\tilde{B}\right)}{(1-\beta)B^\star\left(\tilde{B}\right)}-\frac{\theta}{1-\theta}\left(\frac{1}{1-\beta}-\frac{\beta\left(1+\lim_{\mu\downarrow 0}\epsilon_\mu^{\mathcal{A}}\right)\lim_{\mu\downarrow 0}\mathcal{A}(\mu)}{1-\beta}\right),$$

where we used $\lim_{\mu\downarrow 0}Q=1$. Appendix B, shows that $\lim_{\mu\downarrow 0}1+\epsilon_\mu^{\mathcal{A}}=0$ and that $\lim_{\mu\downarrow 0}\mathcal{A}(\mu)=$

δ. Thus, the limit of the objective function at the left boundary has the sign of:

$$\frac{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)}-\frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}=\frac{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)}-\frac{1-(1-\beta)\,B^{o}}{(1-\beta)\,B^{o}}\frac{1-\beta\delta}{1-\beta}.$$

The function $B^{\star}\left(\tilde{B}\right)<B^{o}$ but $\frac{1-\beta\delta}{1-\beta}>1$, hence the sign is ambiguous. The solution is at this corner if the function is negative.

Next, we consider the limit of the derivative of the objective at $\tilde{B}$. Evaluating the limits is immediate. Hence, if

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}}\geq\frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu\right)\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)\Bigg|_{\mu=\left(1-(1-\beta)\tilde{B}\right)}$$

the solution is $B=\tilde{B}$. Moreover, we know that since

$$\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)>1,$$

the corner solution $\tilde{B}$ is chosen only if

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}}>\frac{1-(1-\beta)\,B^{o}}{(1-\beta)\,B^{o}}.$$

Collecting the results up to this point, we have the following Lemma.

Lemma 17. *The solution B_{ℓ}^{p} to $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is as follows.*

I. $B_{\ell}^{p}=\tilde{B}$ *if*

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}}\geq\frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu\right)\right)\mathcal{A}\left(\mu\right)}{1-\beta}\right)\Bigg|_{\mu=\left(1-(1-\beta)\tilde{B}\right)},$$

II. $B_{\ell}^{p}=B^{\star}\left(\tilde{B}\right)$ *if*

$$\frac{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)}\leq\frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}.$$

III. *Otherwise, B_ℓ^p solves:*

$$\frac{1-(1-\beta)\,B_\ell^p}{(1-\beta)\,B_\ell^p} = \frac{\theta}{1-\theta} Q\left(B_\ell^p, \tilde{B}\right) \left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B_\ell^p, \tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B_\ell^p, \tilde{B}\right)\right)}{1-\beta} \right)\Bigg|_{\mu=\left(1+\beta B_\ell^p - \tilde{B}\right)}.$$

The following Lemma characterizes threshold values for $\tilde{B}$ corresponding to the Lemma above:

Lemma 18. *The condition*

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)} \leq \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}$$

holds for all $\tilde{B} \geq \tilde{B}^i$ such that:

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}^i\right)}{(1-\beta)\,B^\star\left(\tilde{B}^i\right)} = \frac{\theta}{1-\theta}\frac{1-\beta\delta}{1-\beta}.$$

The reverse inequality holds for $\tilde{B} \leq \tilde{B}^a$ such that:

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}^a\right)}{(1-\beta)\,B^\star\left(\tilde{B}^a\right)} = \frac{\theta}{1-\theta}\left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}(\mu)\right)\mathcal{A}(\mu)}{1-\beta} \right)\Bigg|_{\mu=\left(1-(1-\beta)\tilde{B}^a\right)}.$$

Proof. The proof follows immediately from the fact that:

$$\frac{1-(1-\beta)\,B^\star\left(\tilde{B}\right)}{(1-\beta)\,B^\star\left(\tilde{B}\right)},$$

is decreasing in $\tilde{B}$ and the function:

$$\frac{\theta}{1-\theta} Q\left(B, \tilde{B}\right) \left(\frac{1-\beta\left(1+\epsilon_\mu^{\mathcal{A}}\left(\mu\left(B, \tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B, \tilde{B}\right)\right)}{1-\beta} \right)\Bigg|_{\mu=(1-(1-\beta)B)}$$

increasing in B. □

Overall Solution. We showed above that:

$$\mathcal{P}^{\theta}\left(\tilde{B}\right)=\max\left\{\tilde{\mathcal{P}}\left(\tilde{B}\right),\underline{\mathcal{P}}^{\theta}\right\}.$$

The following Lemma shows that the solution B^p to the Planner's problem is never at $B^p=\tilde{B}$.

Lemma 19. *The planner never chooses* $B^p=\tilde{B}$.

Proof. To prove this Lemma observe that the left limit as $B\uparrow\tilde{B}$ satisfies

$$\frac{q-1-\epsilon^{\mathcal{A}}}{1-\beta}+1+\epsilon^{\mathcal{A}}>1+\epsilon^{\mathcal{A}},$$

where the inequality follows from $q>1$ and $\epsilon^{\mathcal{A}}<0$. As a consequence,

$$\frac{1-(1-\beta)\,\tilde{B}}{(1-\beta)\,\tilde{B}}\geq\frac{\theta}{1-\theta}\lim_{B\uparrow\tilde{B}}\Lambda\left(B,\tilde{B}\right)$$

implies

$$\frac{1-(1-\beta)\,\tilde{B}_t}{(1-\beta)\,\tilde{B}_t}>\frac{\theta}{1-\theta}\lim_{B\downarrow\tilde{B}_t}\Lambda\left(B,\tilde{B}\right).$$

Hence, although the derivative of the objective is discontinuous at $\tilde{B}$, we know that if the derivative is weakly increasing from the left and increasing from the right. This implies that $B=\tilde{B}$ is never an optimal choice. □

Next, observe $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ has a compact-valued and continuous constraint correspondence with a continuous objective. Hence, it satisfies the conditions for the Theorem of the Maximum. In addition it is immediate to verify that:

$$\lim_{\tilde{B}\uparrow\frac{1-\theta\beta}{1-\beta}}\tilde{\mathcal{P}}\left(\tilde{B}\right)=\bar{\mathcal{P}}^{\theta}\text{ and }\lim_{\tilde{B}\uparrow\frac{1-\theta\beta}{1-\beta}}B^p(\tilde{B})=B^o.$$

We employ the Envelope Theorem on $\tilde{\mathcal{P}}\left(\tilde{B}\right)$. In the region where the solution to $\tilde{\mathcal{P}}\left(\tilde{B}\right)$ is not at a corner solution, the Envelope Theorem yields:

$$\tilde{\mathcal{P}}_{\tilde{B}}=\frac{\theta}{1-(1-\beta)\,B^p(\tilde{B})}Q\left(B^p(\tilde{B}),\tilde{B}\right)\left(1-\left(1+\epsilon_{\mu}^{\mathcal{A}}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)\right)\mathcal{A}\left(\mu\left(B^p(\tilde{B}),\tilde{B}\right)\right)\right)>0,$$

since $\tilde{B}$ appears directly through μ and S in the objective.

The function is strictly increasing in $\tilde{B}$ since $\epsilon_{\mu}^{\mathcal{A}} < 0$ and $\mathcal{A} < 1$. In the region where the function is at the lower corner of the constraint:

$$B = B^{\star}\left(\tilde{B}\right),$$

the value of this term is:

$$\tilde{\mathcal{P}}_{\tilde{B}} = \frac{\theta}{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}\,(1-\delta) > 0$$

and

$$\tilde{\mathcal{P}}_{B}|_{B=B^{\star}(\tilde{B})} = \frac{1-\theta}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)} - \frac{\theta\,(1-\beta\delta)}{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)}.$$

Thus, we have that the marginal objective value is:

$$\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_{B}\left(B^{p}(\tilde{B})\right)B^{\star}_{\tilde{B}}\left(\tilde{B}\right) = \frac{1-\theta}{(1-\beta)\,B^{\star}\left(\tilde{B}\right)}\beta^{-1} + \frac{\theta\delta}{1-(1-\beta)\,B^{\star}\left(\tilde{B}\right)} > 0.$$

Hence, the envelope condition guarantees that $\tilde{\mathcal{P}}$ is increasing in $\tilde{B}$.

There exists a threshold $\tilde{B}^{s}$ such that:

$$\tilde{\mathcal{P}}\left(\tilde{B}^{s}\right) = \underline{\mathcal{P}}^{\theta}.$$

In particular, it solves:

$$\bar{\mathcal{P}}^{\theta} = \underline{\mathcal{P}}^{\theta} + \int_{\tilde{B}^{s}}^{\frac{1-\theta\beta}{1-\beta}} \left(\tilde{\mathcal{P}}_{\tilde{B}} + \tilde{\mathcal{P}}_{B}\left(B^{p}(\tilde{B})\right)B^{\star}_{\tilde{B}}\left(\tilde{B}\right)\right)d\tilde{B}.$$

Notice that $\underline{\mathcal{P}}^{\theta}$ is finite and lower than the value of the problem without constraints. $\tilde{\mathcal{P}}\,(0)$ tends to $-\infty$ and $\tilde{\mathcal{P}}\left(\overline{B}\right)$ tends to the value of problem without constraints. This implies that $\tilde{\mathcal{P}}\,(0) < \underline{\mathcal{P}}^{\theta} < \tilde{\mathcal{P}}\left(\overline{B}\right)$ and from continuity of the value function (Theorem of the Maximum) and the Intermediate Value Theorem, the existence of $\tilde{B}^{s}$ is guaranteed. The solution to $\tilde{B}^{s}$ follows from the fundamental theorem of calculus.

Recall that in the previous Lemma we had showed that $\tilde{B}$ is never a solution to $\tilde{\mathcal{P}}$ and the monotonicity implies the existence of the threshold $\tilde{B}^{i}$. These thresholds segment the intervals as given by the proposition. This concludes the proof of Proposition 8.

□

F.2 Descentralization with Taxes

To characterize the constrained efficient allocation, we demonstrate that Ramsey problem is equivalent to a problem where a planner chooses a path of debt respecting the payments technology and the desired expenditure rules of both agents.

For ease of exposition we reproduce both problems here. The Ramsey problem is:

Problem 10. (Ramsey Problem): *Given* B_0 *and* $\left\{\tilde{B}_t\right\}$*:*

$$\mathcal{V}^{Ramsey} = \max_{\left\{\tau_{t+1}^k, \tau_t^c\right\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \left[(1-\theta)\log\left(C_t\right) + \theta \log\left(S_t + X_t\right)\right],$$

subject to the creditor budget constraint and optimality:

$$\left(1+\tau_{t+1}^k\right) R_{t+1}^{-1} B_{t+1} + \left(1+\tau_t^c\right) C_t = B_t,\ \forall t \geq 0,$$

the entrepreneur budget constraint:

$$B_t + \left(1+\tau_t^c\right) E_t = R_{t+1}^{-1} B_{t+1} + Y,\ \forall t \geq 0,$$

the government's budget constraint:

$$\tau_{t+1}^k R_{t+1}^{-1} B_{t+1} + \tau_t^c \left(C_t + E_t\right) = 0,\ \forall t \geq 0,$$

and the three conditions of a competitive equilibrium: (i) Optimization, (ii) Asset and Goods market clearing, and (iii) the price consistency.

In turn, the Primal Problem is:

Problem 11. (Primal Planner Problem): *Taking* $\left\{\tilde{B}_t\right\}$ *as given:*

$$\mathcal{V}^{Primal} = \max_{\{B_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t \mathcal{P}\left(B_t, \tilde{B}_t\right)$$

where

$$\mathcal{P}\left(B, \tilde{B}\right) \equiv (1-\theta)\log\left((1-\beta)B\right) + \theta \log\left(S\left(B, \tilde{B}\right) + X\left(B, \tilde{B}\right)\right), \tag{80}$$

and taking as given $S\left(B,\tilde{B}\right)$ *and* $X\left(B,\tilde{B}\right)$ *as defined by Proposition 4.*

The Primal Problem differs from the competitive equilibrium in that the planner directly chooses the sequence of debt. The allocation and final goods is the same as the competitive equilibrium.

We will prove the following Proposition:

Proposition 9. (Primal and Ramsey Equivalence): *Let* $\left\{B^p_{t+1}\right\}_{t\geq 0}$ *be a solution to the Primal Planner Problem. The solution to the Ramsey problem achieves the same value as the Primal Planner Problem,* $\mathcal{V}^{Ramsey} = \mathcal{V}^{Primal}$*. The solution to the Primal Planner Problem can be implemented setting* $(1+\tau^c_0) = B_0/B^p_0$ *and a sequence of debt taxes given in the proof below.*

The proposition asserts that the solution to the Ramsey Problem can be obtained from the solution of a Primal Problem where the planner directly chooses the sequence of debt.[29]

If a solution to the primal can be implemented with taxes, it solves the Ramsey Problem. The proposition shows that this is the case. The implementation of the Primal Problem is possible because the Ramsey planner can set capital taxes to induce a desired path of debt $\{B_t\}$. Controlling B_t is key to the implementation.[30]

Before we proceed to the proof, we establish the following useful intermediate result:

Lemma 20 (Creditor's Solution with Taxes)**.** *Let* $\{\tau^k_{t+1},\tau^c_t,\tau^\ell_{t+1}\}_{t\geq 0}$ *be a sequence of taxes in the Ramsey Problem. The solution to the creditor's problem is:*

$$C_t = (1-\beta)B^o_t, \quad (1+\tau^c_t)C_t = (1-\beta)B^o_t$$

where $B^o_t = B_t/(1+\tau^c_t)$ *and* $B^o_{t+1} = \hat{R}_{t+1}\beta B^o_t$ *with:*

$$\hat{R}_{t+1} \equiv \frac{R_{t+1}}{(1+\tau^k_{t+1})}\cdot\frac{1}{(1+\tau^c_{t+1})/(1+\tau^c_t)}$$

[29] This relation follows because the constraint set in the Primal Problem includes the constraints of the Ramsey Problem. This is immediate since market clearing in the asset market and the budget balance, implies, by Walras's law, that the resource constraint holds. In the implementation, capital taxes and a period-zero expenditure tax are required by the Ramsey planner to distort the evolution of debt and replicate the solution to the Primal Problem.

[30] Time separability also implies that expenditure taxes are redundant after $t=0$. All that the Ramsey planner needs is a sequence of capital taxes to control the path of debt and the labor tax to redistribute the capital tax receipts toward the worker. The planner only needs expenditure taxes at $t=0$ because debt is predetermined at $t=0$.

Proof. The creditor's problem with taxes is:

$$V_t = \sum_{s=0}^{\infty} \beta^s \log(C_{t+s})$$

subject to $(1+\tau^k_{t+1})R^{-1}_{t+1}B_{t+1} + (1+\tau^c_t)C_t = B_t$. Dividing both sides by $(1+\tau^c_t)$ and defining $B^o_t \equiv B_t/(1+\tau^c_t)$, we can re-write the budget constraint as:

$$\hat{R}^{-1}_{t+1}B^o_{t+1} + C_t = B^o_t \tag{81}$$

where $\hat{R}_{t+1} \equiv \frac{R_{t+1}}{(1+\tau^k_{t+1})} \cdot \frac{1}{(1+\tau^c_{t+1})/(1+\tau^c_t)}$.

With log utility and a budget constraint like (81), the creditor's optimal expenditure rule is:

$$C_t = (1-\beta)B^o_t$$

which yields $B^o_{t+1} = \hat{R}_{t+1}\beta B^o_t$. Since $B_t = (1+\tau^c_t)B^o_t$, we have $C_t = (1+\tau^c_t)(1-\beta)B^o_t$ as claimed. □

F.3 Proof Ramsey-Primal Equivalence - Proposition 9

Proof. To characterize the constrained efficient allocation, the strategy is to show that any solution to the Ramsey Problem satisfies the constraints of the Primal Problem (Step 1). Then, we show that any solution to the Primal Problem can be induced by a proper tax sequence $\{\tau^k_{t+1}, \tau^c_t, \tau^\ell_{t+1}\}_{t\geq 0}$ (Step 2).

Step 1: From Ramsey to Primal - Deriving the Resource Constraint. Adding the two agents' budget constraints we arrive at:

$$(1+\tau^c_t)(S_t + q_t X_t + C_t) + \tau^k_{t+1}\frac{B_{t+1}}{R_{t+1}} = 1 - \tau^\ell_{t+1}.$$

Subtracting the government budget constraint we arrive at:

$$\begin{aligned}&(1+\tau^c_t)(S_t + q_t X_t + C_t) + \tau^k_{t+1}\frac{B_{t+1}}{R_{t+1}} \\ &\quad - \left(\tau^k_{t+1}\frac{B_{t+1}}{R_{t+1}} + \tau^c_t(C_t + S_t + q_t X_t) + \tau^\ell_{t+1}\right) = 1 - \tau^\ell_{t+1}\end{aligned}$$

simplifying to:

$$S_t + q_t X_t + C_t = 1.$$

From Lemma 20, $C_t = (1-\beta)B_t^o$, given $B_t^o = B_t/(1+\tau_t^c)$. Therefore:

$$S_t + q_t X_t = 1 - (1-\beta)B_t^o. \tag{82}$$

Thus, setting $E_t = S_t + q_t X_t$ we have that the entrepreneurs expenditures are:

$$E_t = 1 - (1-\beta)B_t^o. \tag{83}$$

The optimality of spot and chained expenditures is still given by the conditions in Proposition 4.

Using $q_t = \mathcal{A}(\mu_t)^{-1}$ and $\mu_t = q_t X_t = \mathcal{A}(\mu_t)^{-1} X_t$. These are exactly the constraints of the Primal Problem with B_t^o as the state variable instead of B_t^p.

Step 2: Tax Implementation of the Ramsey problem. Let $\{B_t^p\}_{t\geq 0}$ solve the Primal Problem. We now show how to construct taxes that implement this as a Ramsey equilibrium. Set the initial consumption tax:

$$(1+\tau_0^c) = B_0/B_0^p$$

This ensures $B_0^o = B_0/(1+\tau_0^c) = B_0^p$.

For the Primal solution to be implementable, we need both agents' Euler equations to hold. The creditor's Euler equation:

$$\frac{(1-\beta)B_{t+1}^o/(1-\beta)B_t^o}{\beta} \cdot \frac{(1+\tau_{t+1}^c)}{(1+\tau_t^c)} \cdot (1+\tau_{t+1}^k) = R_{t+1} \tag{84}$$

The entrepreneur's Euler equation (using the notation from Proposition 2):

$$\frac{(S_{t+1}+X_{t+1})/(S_t+X_t)}{\beta} \cdot \frac{(1+\tau_{t+1}^c)}{(1+\tau_t^c)} \cdot \frac{\tilde{q}_{t+1}^B(B_{t+1}^o)}{\tilde{q}_t^E(B_{t+1}^o)} = R_{t+1} \tag{85}$$

where the marginal price ratio captures the payment chain friction.

Eliminating $[(1+\tau_{t+1}^c)/(1+\tau_t^c)]\beta R_{t+1}$ from both equations:

$$(1+\tau_{t+1}^k) = \frac{B_t^o}{B_{t+1}^o} \cdot \frac{S_{t+1}+X_{t+1}}{S_t+X_t} \cdot \frac{\tilde{q}_{t+1}^B(B_{t+1}^o)}{\tilde{q}_t^E(B_{t+1}^o)}$$

Since $S_t + X_t = 1 - (1-\beta)B_t^o$ and using average prices $Q_t = (S_t + q_t X_t)/(S_t + X_t)$:

$$\tau^k(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}) = \frac{B_t^o}{B_{t+1}^o} \cdot \frac{1-(1-\beta)B_{t+1}^o}{1-(1-\beta)B_t^o} \cdot \frac{Q(B_t^o, \tilde{B}_t)}{Q(B_{t+1}^o, \tilde{B}_{t+1})} \cdot \frac{\tilde{q}_{t+1}^B(B_{t+1}^o)}{\tilde{q}_t^E(B_{t+1}^o)} - 1 \tag{86}$$

The equilibrium interest rate follows from (84):

$$R_{t+1} = \frac{(1+\tau_{t+1}^c)}{(1+\tau_t^c)} \cdot \frac{B_{t+1}^o}{\beta B_t^o} \cdot (1 + \tau^k(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1}))$$

Step 3: Indeterminacy of Consumption Taxes Note that consumption taxes $\{\tau_t^c\}_{t\geq 1}$ are indeterminate. To see this, substitute the creditor's Euler equation into their budget constraint:

$$\frac{(1-\beta)B_t^o}{(1-\beta)B_{t+1}^o}\beta\left[\frac{(1+\tau_t^c)}{(1+\tau_{t+1}^c)}\right]B_{t+1} + (1+\tau_t^c)(1-\beta)B_t^o = B_t$$

Since $B_t = (1+\tau_t^c)B_t^o$, this is satisfied for any sequence of consumption taxes. The government budget constraint then determines:

$$\tau_{t+1}^\ell = -\tau_{t+1}^k \frac{B_{t+1}}{R_{t+1}} - \tau_t^c \left((1-\beta)B_t^o + \frac{1-(1-\beta)B_t^o}{Q(B_t^o, \tilde{B}_t)}\right)$$

Setting $\tau_t^c = 0$ for all $t \geq 1$ yields the simple implementation:

$$\tau_{t+1}^\ell = -\frac{\tau^k(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1})}{1+\tau^k(B_t^o, B_{t+1}^o, \tilde{B}_t, \tilde{B}_{t+1})} \cdot \beta B_t^o$$

This completes the proof that $\mathcal{V}^{Ramsey} = \mathcal{V}^{Primal}$. □

G Derivation of the Bocola Effect - Proof of Proposition 5

Let's define the indirect social utility function of the Planner Problem with expenditures $\mathcal{P}^g$:

$$\mathcal{P}^g\left(B,\tilde{B},G^s,G^x\right)=(1-\theta)\log\left(C^s\left(B\right)\right)+\theta\log\left(S^w\left(B,\tilde{B},G^s,G^x\right)+X^w\left(B,\tilde{B},G^s,G^x\right)\right).$$

The following functions determine the allocation:

$$\begin{aligned}
C^s\left(B\right) &\equiv (1-\beta)B\\
E^w\left(B,G^s,G^x\right) &\equiv 1-E^s\left(B\right)-G^x-G^s\\
S^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\}\\
X^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \frac{E^w-\min\left\{\max\left\{\tilde{B}-B,0\right\},E^w\right\}}{q}\\
q\left(B,\tilde{B},G^s,G^x\right) &\equiv \mathcal{A}^{-1}\left(\mu\right)\\
\mu\left(B,\tilde{B},G^s,G^x\right) &\equiv G^x+X^wq.
\end{aligned}$$

The worker's total consumption is:

$$C^w=S^w\left(B,\tilde{B},G^s,G^x\right)+X^w\left(B,\tilde{B},G^s,G^x\right).$$

Because

$$\mathcal{Y}=C^w+C^S+G^s+\frac{G^x}{q\left(\mu\right)},$$

but $C^s\left(B\right)$ is independent of the government's expenditure, we obtain: $d\mathcal{Y}=dC^w+dG^s+d\left(G^x/q\left(\mu\right)\right)$.

Thus, we have that the government's expenditure multiplier for expenditure of type $i=x,s$ relates to the worker's consumption as follow:

$$\mathcal{M}^s\left(B,\tilde{B}\right)\equiv\frac{d\mathcal{Y}}{dG^s}=\frac{dC^w}{dG^s}+1$$

and

$$\begin{aligned}\mathcal{M}^x\left(B,\tilde{B}\right) &\equiv \frac{d\mathcal{Y}}{dG^x}=\frac{dC^w}{dG^x}+\frac{1}{q}-\frac{G^x}{q^2}\frac{dq}{d\mu}\frac{d\mu}{dG^x}\\ &= \frac{dC^w}{dG^x}+\mathcal{A}\left(\mu\right).\end{aligned}$$

where the last equation is because we are interested in obtaining the government's infinitesimal multiplier, evaluated at $G^x = G^s = 0$. We have defined these multipliers as:

$$\mathcal{M}^i\left(B,\tilde{B}\right)\equiv \left.\frac{d\mathcal{Y}}{dG^i}\right|_{G^x=G^s=0},$$

and relating to the indirect social utility function, the change in the objective is:

$$\frac{\theta}{C^w}\frac{dC^w}{dG^i},$$

for expenditure i.

Case 1. All spot consumption $B < B^\star$. If there is only spot consumption:

$$\begin{aligned}C^s\left(B\right)&\equiv\left(1-\beta\right)B\\ C^w\left(B,\tilde{B},G^s,G^x\right)&\equiv 1-\left(1-\beta\right)B-G^x-G^s\\ X^w\left(B,\tilde{B},G^s,G^x\right)&\equiv 0\\ q\left(B,\tilde{B},G^s,G^x\right)&\equiv\mathcal{A}^{-1}\left(\mu\right)\\ \mu\left(B,\tilde{B},G^s,G^x\right)&\equiv G^x.\end{aligned}$$

Since worker consumption is independent of q, we have that:

$$\frac{dC^w}{dG^i}=-1$$

for both $i\in\{x,s\}$. In turn, we have that

$$\mathcal{M}^s\left(B,\tilde{B}\right)=\frac{dC^w}{dG^s}+1=0.$$

Likewise, for chained expenditures we have:

$$\mathcal{M}^x\left(B,\tilde{B}\right)=\frac{dC^w}{dG^x}+\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right)=-\left(1-\mathcal{A}\left(\mu\right)\right)+\epsilon_\mu^{\mathcal{A}}<0.$$

Case 2. Some chained consumption. If there are some chained expenditures:

$$\begin{aligned}
C^s\left(B\right) &\equiv \left(1-\beta\right)B\\
E^w\left(B,G^s,G^x\right) &\equiv 1-\left(1-\beta\right)B-G^x-G^s\\
C^w\left(B,\tilde{B},G^s,G^x\right) &\equiv S^w+X^w\\
S^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \max\left\{\tilde{B}-B,0\right\}\\
X^w\left(B,\tilde{B},G^s,G^x\right) &\equiv \frac{E^w-\max\left\{\tilde{B}-B,0\right\}}{q}\\
q\left(B,\tilde{B},G^s,G^x\right) &\equiv \mathcal{A}^{-1}\left(\mu\right)\\
\mu\left(B,\tilde{B},G^s,G^x\right) &\equiv G^x+X^w q.
\end{aligned}$$

Rewriting the last three identities using $\mathcal{A}\left(\mu\right)$ instead of q we have

$$\begin{aligned}
X^w &\equiv \mathcal{A}\left(\mu\right)\left(E^w-\max\left\{\tilde{B}-B,0\right\}\right)\\
\mu &\equiv G^x+E^w-\max\left\{\tilde{B}-B,0\right\}.
\end{aligned}$$

Substituting E^w we have, naturally,

$$\mu=1-\underbrace{\left(\left(1-\beta\right)B+G^s+\max\left\{\tilde{B}-B,0\right\}\right)}_{\text{spot exp.}}. \tag{87}$$

From here we obtain that:

$$\frac{dX^w}{dG^x}=-\mathcal{A}\left(\mu\right)+\mathcal{A}'\left(\mu\right)\frac{d\mu}{dG^x}\left(E^w-\max\left\{\tilde{B}-B,0\right\}\right).$$

Since $\frac{d\mu}{dG^x}=\frac{dS^w}{dG^x}=0$, we have that:

$$\frac{dC^w}{dG^x}=-\mathcal{A}\left(\mu\right).$$

Hence, the government multiplier for chained expenditures is:

$$\mathcal{M}^x\left(B,\tilde{B}\right)=\epsilon_\mu^{\mathcal{A}}<1.$$

Next, observe that:

$$X^w\equiv\ \mathcal{A}\left(\mu\right)\left(1-\left(1-\beta\right)B-G^x-G^s-\max\left\{\tilde{B}-B,0\right\}\right).$$

Hence,

$$\begin{aligned}dX^w &= -\mathcal{A}\left(\mu\right)dG^s+\mathcal{A}'\left(\mu\right)\left(\frac{X^w}{\mathcal{A}\left(\mu\right)}\right)d\mu\\ &= -\mathcal{A}\left(\mu\right)dG^s+\mathcal{A}\left(\mu\right)\epsilon_\mu^{\mathcal{A}}\left(\frac{X^w}{\mathcal{A}\left(\mu\right)}/\mu\right)d\mu.\end{aligned}$$

The second line follows from:

$$\mathcal{A}\left(\mu\right)\mu\equiv\ G^x+X^w.$$

Also following this condition, we have that:

$$\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right)d\mu\equiv dX^w.$$

Combining the differentials evaluated at $G^x=0$, we obtain:

$$d\mu=-dG^s.$$

Hence,

$$\frac{dC^w}{dG^s}=\frac{dX^w}{dG^s}=-\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right).$$

Following the relationship with the fiscal multiplier, we obtain:

$$\mathcal{M}^s\left(B,\tilde{B}\right)=1-\mathcal{A}\left(\mu\right)-\mathcal{A}\left(\mu\right)\epsilon_\mu^{\mathcal{A}}.$$

Summary. We summarize the results:

$$\frac{dC^w}{dG^x} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}\left(\mu\right) & B > B^\star\left(\tilde{B}\right) \end{cases}$$

and

$$\frac{dC^w}{dG^s} = \begin{cases} -1 & B < B^\star\left(\tilde{B}\right) \\ \\ -\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right) & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Finally, recall that the government multipliers relate to the change in consumption as follows:

$$\mathcal{M}^s\left(B,\tilde{B}\right) \equiv \frac{d\mathcal{Y}}{dG^s} = \frac{dC^w}{dG^s} + 1$$

and

$$\mathcal{M}^s\left(B,\tilde{B}\right) \equiv \frac{dC^w}{dG^x} + \mathcal{A}\left(\mu\right)$$

with:

$$\frac{d\mu}{dG^x} = \begin{cases} 1 & B < B^\star\left(\tilde{B}\right) \\ \\ 0 & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Therefore, adding terms:

$$\mathcal{M}^x\left(B,\tilde{B}\right) = \begin{cases} -\left(1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right)\right) & B < B^\star\left(\tilde{B}\right) \\ \\ 0 & B > B^\star\left(\tilde{B}\right). \end{cases}$$

Hence,

$$\mathcal{M}^s\left(B,\tilde{B}\right)=\begin{cases}0 & B<B^\star\left(\tilde{B}\right)\\ \\ 1-\mathcal{A}\left(\mu\right)\left(1+\epsilon_\mu^{\mathcal{A}}\right) & B>B^\star\left(\tilde{B}\right).\end{cases}$$