Learning and Trading with Heterogeneous Misspecified Models

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Abstract

We study learning and market outcomes in a dynamic setting, where market participants lack complete knowledge of the market structure and have heterogeneous, misspecified models about the market. Market participants include high-frequency algorithmic traders who use Linear Factor Models (LFM) to predict future prices, as well as long-run investors who maintain constant beliefs about future prices. Algorithmic traders are Bayesian, and update their beliefs in the LFM coefficients upon observing new prices. We prove that market dynamics converge to a Berk-Nash equilibrium. We use this framework to provide novel insights and explanations of a number of market phenomena, including bubble formation, market fragility and patterns of different market returns for algorithmic vs. long-run investors. Our results highlight the role of algorithmic trading in various market inefficiencies.

1 Introduction

In financial markets, real-time prices are determined by the collective decisions of all market participants, who base their investment choices on expected future returns. The literature has extensively studied how financial markets aggregate the private information of agents, and how agents update their beliefs about future returns based on equilibrium prices. In the classic Rational Expectations Equilibrium (REE) framework, agents are assumed to have common knowledge of each other's information structures and to rationally infer others' private information from market prices. In practice, however, it is challenging to identify who the market participants are, let alone consistently extract and interpret their information from prices. This has motivated extensive research on deviations from REE in which agents have *incomplete knowledge* of others' beliefs and actions (Hirshleifer, 2015; Hong and Stein, 2007). The challenge in this literature is to find models that retain a sufficient degree of rationality to tether the market to fundamentals, while maintaining tractability to ensure that market dynamics and outcomes can be analyzed. In particular, a major challenge is a lack of tractable, dynamic market models that capture the behavior of algorithmic traders,

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who, rather than inferring others' strategies, rely on statistical models trained on historical data to form expectations about future returns, determine their long or short positions based on these expectations, and update their models as new stock price data becomes available (Balasubramanian and Yang, 2019).

We build a dynamic model that incorporates these common algorithmic trading practices and allows us to study the traders' belief formation and updating process, as well as market outcomes. We focus on a relatively short time span where market outcomes are decided by market participants' subjective beliefs rather than fundamentals of assets. In our model, traders' beliefs about stock prices are based on Linear Factor Models (LFM), in which asset prices are assumed to be well represented by a linear combination of a small number of price vectors called *features*. Traders represent market prices by the feature coefficients, and learn by updating these coefficients. We assume that each trader incorrectly believes that their features span the space of prices, whereas in reality these features capture only a partial picture, and so traders are misspecified. This misspecification persists in the learning process, driving abnormalities in market prices; nevertheless, we show that traders are sufficiently rational for this process to converge to an equilibrium. The analysis of these equilibria yields novel insight into various market inefficiencies, and into how these outcomes depend on the models used by the traders.

In our model, time is infinite and discrete. There are N many assets in the market and market participants trade all assets simultaneously at each time period. There are two types of market participants - algorithmic traders (traders, henceforth) and long-run investors (investors, henceforth). Each trader i adopts $F_i < N$ many features to forecast asset prices for all assets, using a Linear Factor Model (LFM). They update the coefficients of the LFM once asset prices are newly realized. Features adopted by these traders are potentially different, making their models fundamentally different. As opposed to the traders, each investor holds a belief about future asset prices that does not change with time; this assumption reflects our focus on short time spans in which fundamentals do not change significantly.

Given their beliefs about future prices, myopic market participants with mean-variance preferences place contingent orders to maximize their expected utility payoff in the next period, specifying how much they will hold for each asset given any hypothetical current price. The realized price observed by traders is the market clearing price of these collective contingent orders plus an independent noisy demand by exogenous factors. Subsequently, traders update the coefficients in their linear models using Bayes rule, adjusting for the difference between realized prices and their prior forecasts. This completes one cycle of trading and belief updates.

The discrepancy between traders' subjective LFM and the actual market environment leads to misspecified learning in our model. Each trader's belief about market prices is constrained by their parametric space, which is determined by the features they are using, while the actual market prices exist in a higher dimensional space that cannot be fully expanded by their parametric model. Consequently, the classic results from the REE framework do not apply to our setting. We therefore introduce an equilibrium concept tailored to our setting

and show that it is equivalent to the Berk-Nash equilibrium[1] of Esponda and Pouzo (2016a).

In Section 3, we establish both the existence and uniqueness of an equilibrium. In equilibrium, a trader's market forecast may mismatch the actual market prices; her market forecast is the closest point in her parametric space to the actual market prices. We provide a characterization of the equilibrium, which depends on the features used by traders and the beliefs of the long-run investors. Next, in Section 4, we prove the convergence of the market dynamics to this unique equilibrium. In our market dynamics, the investors provide an anchoring effect, which pulls the traders' beliefs towards the investors' beliefs, to the extent that the parametric spaces of traders permit, resulting in traders' beliefs converging to the same equilibrium, regardless of their starting point.

Section 5 discusses the empirical implications of our model. In Section 5.1, we provide a mechanism for bubble formation. Suppose that a long-run investor irrationally believes in a sunspot feature that is independent of the fundamentals of assets. We find that even when this deviation from rationality is very small, a bubble can form, because the deviations from fundamentals are amplified by the traders' use of LFM. In particular, the better that traders' features capture the sunspot feature, the larger the bubble. In Section 5.2, we formulate the *feature-weight downward spiral channel* for stock market crashes: a crash is triggered by the entry of a new trader who barely perceives a feature that is commonly recognized by existing traders in the market. As a result, this feature receives less weight in the new market price, prompting the incumbent traders to revise downward their beliefs about its importance—further reducing its weight in market prices. The downward spiral can be especially severe when the share of long-run investors is small, as their anchoring effect becomes negligible. Lastly, in Section 5.3, we find that traders incur losses that are gains to investors in the learning process, as traders adjust their beliefs that come closer to those of investors. Yet learning is fruitful for traders because in equilibrium, traders who make more accurate market forecasts achieve higher Sharpe ratios.

Finally, Section 7 concludes the paper and outlines possible future directions of research. All mathematical proofs are provided in Appendix B.

1.1 Literature Review

A collection of models deviates from the Rational Expectations Equilibrium (REE) by dropping the assumption of common knowledge in the market. Instead, agents are allowed to have different priors about market uncertainties or to hold different subjective models of the information structure of other participants; this leads to persistent disagreement among agents. Earlier work has already identified significant consequences of such disagreement. Under a short-sale constraint, Miller (1977) finds overvaluation of stocks in a static model, and Harrison and Kreps (1978) show the emergence of speculative bubbles in a dynamic setting. Harris and Harris and Raviv (1993) find that agents' differing interpretations of public information can explain various patterns in trading volume. More recent literature tends

[1]Berk-Nash is a general game-theoretical equilibrium concept, which allows for disparities between players' subjective perceptions of the game environment and the objective reality.

to explore behavioral foundations for agents holding subjective models that deviate from objective distributions. Scheinkman and Xiong (2003) propose overconfidence as a source of heterogeneous beliefs and revisit the mechanism of speculative bubbles. Collin-Dufresne et al. (2017) find that incorporating generational learning bias can generate high risk premia. Our paper contributes to this literature on heterogeneous agents with persistent disagreement by proposing new channels of misperception. There is also a substantial body of work that builds behavioral models with representative agents; we refer readers to Hirshleifer (2015) and Barberis (2018).

Our model builds on the literature on misspecified learning, which originates from the seminal work of Berk (1966). More recently, Esponda and Pouzo (2016b) formalize Berk-Nash equilibrium as a solution concept in a static game where (potentially heterogeneous) players have a constrained set of subjective distributions over payoff-relevant consequences as a function of actions and information. In the same paper, they provide a learning foundation for the Berk-Nash equilibrium, with myopic players repeatedly playing a slightly perturbed version of the game. They show that if the strategy profile converges, the limit must be a Berk-Nash equilibrium. However, whether convergence occurs from an arbitrary prior remains an open question in general.

Subsequent work on misspecified learning shows convergence of learning dynamics in various single-agent settings (Frick et al. (2022), Fudenberg et al. (2017a, 2021), Heidhues et al. (2018), and Molavi (2023)), with the limiting behavior often characterized by the Berk-Nash equilibrium. In multi-agent settings, however, the outcome in each period depends on the actions of all agents, making it difficult to track the repeated interactions of misspecified learners. Recently, Murooka and Yamamoto (2023, 2025) study a two-agent model in which each agent has not only biased beliefs about the physical environment but also biased beliefs about the other agent's bias. They provide conditions for convergence and non-convergence and show that even small biases can significantly affect the learning process. Our model contributes to this literature by analyzing the convergence of misspecified learning dynamics in settings with an arbitrary number of heterogeneous agents.

In the finance literature, there are also a handful of studies on learning with misspecified models (Adam et al. (2016), Marcet and Nicolini (2003), and Marcet and Sargent (1989a, 1989b)). More recently, Molavi (2023) proposes a general framework that extends Berk-Nash equilibrium to recursive general-equilibrium macroeconomic settings and explores convergence to equilibrium. However, our model of heterogeneous agents does not fit within the framework of Molavi (2023), as their economy features only a representative agent. Our model most closely resembles that of Molavi et al. (2024), in which agents use low-dimensional statistical models to approximate a high-dimensional data-generating process. We study the learning of parameters in a linear factor model (LFM) with endogenous features. Our focus on how these features affect asset prices and generate market anomalies distinguishes our work from the existing literature.

Unlike most of the literature, which develops models for a single stock, we consider a market-wide stock selection scenario in which investors forecast returns for all stocks in the

market and implement a long-short strategy based on the "alpha" they generate. Market-wide selection strategies are widely adopted in the finance industry, as they help mitigate the risks associated with individual assets. Moreover, factor models—the workhorse for implementing such strategies—have been extensively developed in two main directions: the hand-crafting of economically interpretable factors (e.g., Fama and French (1992), Fama and French (2015)) and the application of machine learning algorithms to automatically generate predictive factors (e.g., Kelly et al. (2019), Gu et al. (2021)). Our work builds on Balasubramanian and Yang (2019), which studies market outcomes arising from traders' use of linear factor models (LFM) in a static setting; our model provides a dynamic extension of theirs.

Lastly, since the Bayesian learning process under a linear factor model can be interpreted as ridge regression (as we demonstrate in Section 2), our work is also connected to the recent literature on machine learning and asset pricing, which models algorithmic trading (e.g., Molavi et al. (2024), Mondria et al. (2022), Dugast and Foucault (2023)). Unlike these studies, which often incorporate computational costs and non-Bayesian learning paradigms, our model assumes that traders continue to engage in Bayesian learning. Moreover, our primary focus is on characterizing the learning dynamics when the stochastic process agents attempt to learn is itself endogenously determined.

2 Setting

2.1 Belief & Belief Updating

There are N assets in the market. The market participants consist of **high-frequency traders** (she) (abbreviated as traders), **long-run investors** (he) (abbreviated as investors), and **noise traders**. We label the $K \geq 1$ many traders by $\{1, \ldots, K\}$ and $M \geq 0$ many investors by $\{K+1, \ldots, K+M\}$. A **feature** of assets is a N-dimensional vector, and each trader i observes F_i many features of assets. The **feature matrix** X_i of trader i is the $N \times F_i$ dimension matrix that collects all the features of assets trader i observes. The linear span of the features trader i observe, denoted by $\text{Span}(X_i)$, will be called the **dataset** of trader i. We also introduce the notation $\text{Proj}_{X_i}(\cdot)$ as the projection to the space $\text{Span}(X_i)$.

Time is infinite and the market opens at $t = 1, 2, 3, \ldots$ and so on. We use P_t to denote the vector of asset prices at time t and use $P_{t,j}$ to denote the price of asset j in the same period. Importantly, we assume that each feature matrix X_i is exogenous and remains constant over time. Connecting the model to the real world, a single period ranges from minutes to hours, while a long time span, extending from 0 to a large time T in the model, should be interpreted as spanning from days to a month.[2]

We make the following assumptions about market participants' learning and trading process. Traders and investors both have subjective belief about future asset prices. The salient difference between the traders and investors is their belief formation process.

Assumption 1 (**Investors' Beliefs**). *Each long-run investor k believes that asset prices in the next time period follow the normal distribution $P_{t+1} \sim N(q_k, \Sigma_e)$, where q_k does not change over time. The term Σ_e represents their perceived market noise, and is shared for all investors (and traders as well)* [3].

Long-run investors in our model represent the value investors in the market who make their investment decisions based on the fundamentals of assets. In a short time span, fundamentals of assets are unlikely to change significantly, and therefore, their beliefs about future asset prices remain relatively stable. In particular, they do not react to instantaneous stock prices as quick as high frequency traders do, which we now discuss.

Traders, however, constantly update their beliefs upon the realization of market prices. They use the features about assets X_i to forecast asset prices, but they are unsure about the "weights" of these features, and these are what traders constantly update.

[2] In high-frequency or mid-frequency setting, the change in asset features are modest. If there are changes in assets' feature, they can be thought as shuffling the rows of feature matrix: if we use $T_t(\cdot)$ to represent a shuffling operator and $X_{i,t} = T_t(X_i)$, then all the results in the paper holds and mathematically it equivalents to X_i is constant overtime. The selection of features is not our focus either as we are mainly concerned with the learning of coefficients of these features.

[3] The assumption does not require that investors/traders perceive noise correctly, but according to assumptions made later, their perceived market noise equals actual market noise in the long run. This simplification aids modeling and is not overly restrictive, as price volatility is generally more predictable.

Assumption 2 (**Linear Factor Model**). *Each trader i uses a Linear Factor Model (LFM) to forecast asset prices at the next period: she holds a subjective belief over the coefficients in the linear factor model g_i at time t and forecasts the price at the next period as*

$$P = X_i g_i + e, \quad \textit{with} \quad e \sim N(0, \Sigma_e)$$

Given the coefficients g_i, the investor's belief about the price follows $P|g_i \sim N(X_i g_i, \Sigma_e)$. Henceforth, we call g_i the ***LFM coefficients****.*

Assumption 3 (**Initial Beliefs of Traders**). *For each trader i at time 0, she holds a prior belief about the LFM coefficient g_i that is normally distributed:*

$$g_i \sim N(\mu_i, \Sigma_i)$$

Assumption 4 (**Belief Updating by Traders**). *Whenever a new market price P is newly realized, a trader i updates her belief about g_i using Bayesian inference.*

With these assumptions, traders' Bayesian learning process under LFM can be characterized by the following:

Lemma 1. *After the realization of the price in the first period P_1, the posterior belief of trader i about the LFM coefficients g_i is normally distributed with mean*

$$\mu_i^1 := (X'\Sigma_e^{-1}X + \Sigma_i^{-1})^{-1}(X'\Sigma_e^{-1}P_1 + \Sigma_i^{-1}\mu_i) \tag{2.1}$$

and covariance matrix

$$\Sigma_i^1 := (X'\Sigma_e^{-1}X + \Sigma_i^{-1})^{-1} \tag{2.2}$$

After observing the prices $P_1, \ldots, P_T$, trader i updates her posterior belief about LFM coefficients to a normal distribution with mean μ_i^T and covariance matrix Σ_i^T satisfying

$$\mu_i^T = (X'\Sigma_e^{-1}X + \frac{1}{T}\Sigma_i^{-1})^{-1}(X'\Sigma_e^{-1}\bar{P}_T + \frac{1}{T}\Sigma_i^{-1}\mu_i) \tag{2.3}$$

$$\Sigma_i^T = (TX'\Sigma_e^{-1}X + \Sigma_i^{-1})^{-1}, \tag{2.4}$$

where $\bar{P}_T := \frac{1}{T}(P_1 + \ldots + P_T)$.

We emphasize that deviating from the classical asset pricing literature, we assume that the assets do not pay dividends. This assumption is reasonable in our setting, as we consider a relatively short time window where high-frequency traders primarily engage in arbitrage rather than trading for dividends. The return from holding one unit of asset j from time t to $t+1$ is simply given by $P_{t+1,j} - P_{t,j}$.

Assumption 5 (**Utility**). *In each unit time interval, both traders and investors have mean-variance preferences for each asset. More precisely, in making the investment decision at time t, if a trader or an investor (subjectively) believes that holding a certain amount of*

an asset has mean return r and variance σ^2, their demand for that asset[4] *is determined by maximizing the utility function*

$$U_i(r, \sigma^2) = r - \frac{1}{2} A_i \sigma^2, \tag{2.5}$$

where A_i is the risk preference parameter of trader or investor i.

Here we take a parsimonious approach and assume the simplest utility functional form to make the model tractable. Firstly, we recall that market participants in our model either base their price beliefs on LFM or hold constant beliefs, acting as price takers rather than anticipating others' actions and trying to manipulate market prices. When the dependence of market prices across periods and their trading actions is ignored, maximizing long-run payoffs is equivalent to maximizing payoffs period-by-period, so here we just assume that market participants are all myopic.

Secondly, we assume mean-variance utility function in market participants' investment decisions in each period. Theoretically, in making a *one-time investment decision*, the mean-variance utility function is strictly justified when returns are normally distributed and decision makers have CARA utility function, or their preferences are truly quadratic (Feldstein, 1969; Merton, 1969; Tobin, 1958); though both are hard to be justified, mean-variance utility is still believed to be a good approximation for expected utility maximization (Levy and Markowitz, 1979).

To justify the use of mean-variance utility function in the multiple periods setting, we first recall the approximation in the one-period setting. Suppose that a market participant i has an increasing, concave, and sufficiently smooth utility function U_i over his or her wealth, and at the be beginning of some time T, he or she has wealth W_T. If holding some amount of the risky asset has a random payoff $X \sim N(r, \sigma^2)$, then we can approximate the expected utility payoff by the second-order Taylor expansion,

$$\mathbb{E}U_i(W_T + X) \approx U_i(W_T) + U_i'(W_T)r + \frac{1}{2} U_i''(W_T)\sigma^2.$$

Maximizing the second-order approximation is equivalent to maximizing,

$$r - \frac{U_i''(W_T)}{2U_i'(W_T)}\sigma^2,$$

and setting the risk preference parameter

$$A_i := \frac{U_i''(W_T)}{2U_i'(W_T)}$$

[4]Moreover, in this paper, while market participants perform optimization, they do not account for the correlation of prediction errors across different assets, as we assume that random shocks to each asset are independent. Consequently, when traders update their beliefs over a long time horizon, the prediction errors across assets exhibit no correlation. However, in the short run, due to correlated estimation errors in their pricing coefficients, some correlation in prediction errors may arise. For simplicity, we disregard this minor effect in our modeling and assume that agents optimize their utility for each asset independently.

gives us back to Equation 2.5. This analysis is flawed in the multiple periods utility maximization problem because the wealth of the market participant i may change over time yet we take A_i to be a constant. However, we are considering a relative short time span (hours to days in total) over which the wealth of market participants change very little, so that the approximation

$$A_i \approx \frac{U_i''(W_T)}{2U_i'(W_T)}$$

remains accurate in all periods given that U_i is smooth enough.

Empirically, literature has verified that the second-order approximation is almost perfect for CARA utility and most CRRA utility functions (when the CRRA coefficient is less than 3), in portfolio selections Hlawitschka, 1994; Levy and Markowitz, 1979 and hedge funds practices Fung and Hsieh, 1999.[5]

2.2 Linear Ridge Regression

Though Bayesian updating in the linear factor model already seems normative appealing, we provide another micro foundation of Bayesian learning in the linear factor model from the perspective of machine learning algorithm and demonstrate their equivalence. This connection is also illustrated in Hoerl and Kennard (1970), Balasubramanian and Yang (2019), and Martin and Nagel (2022).

Assume that a trader has a prior "single-point" best guess μ_i over the linear model coefficients. Upon observing the stock prices in the first T periods $P_1, \ldots, P_T$, her best guess would be updated from μ_i to some $g = g^*$ that minimizes the loss function

$$\mathcal{L}(g) = \sum_{t=1}^{T} (P_t - Xg)'(P_t - Xg) + \lambda ||g - \mu_i||_2^2.$$

By the first order condition, the optimal g^* is:

$$g^* = (TX'X + \lambda I)^{-1}(\lambda \mu_i + \sum_{t=1}^{T} X'P_t) \tag{2.6}$$

Finding the optimal g^* to minimize the loss function is known as **the Linear Ridge Regression**. To see this corresponds to Bayesian updating, assume that Σ_e and Σ_i in the Assumption 2 and Assumption 3 has diagonal form, with $\Sigma_e = \sigma_e^2 I$ and $\Sigma_i = \sigma_g^2 I$. Then (2.3) becomes:

$$\mu_i^T = (X'X + \frac{1}{T}\frac{\sigma_e^2}{\sigma_g^2} I)^{-1}(X'\bar{P}_T + \frac{\sigma_e^2}{\sigma_g^2}\mu_i)$$

with $\frac{\sigma_e^2}{\sigma_g^2}$ as the noise to signal ratio. If we set $\frac{\sigma_e^2}{\sigma_g^2} = \lambda$, the posterior mean is exactly the minimizer of the loss function $\mathcal{L}$. To summarize, an investor who uses Linear Ridge Regression to

[5]Their method is to take many mutual funds, portfolios consisting of randomly selected large stocks, or hedge funds and calculated the correlation between the average of the utility payoffs under U across some time span and the second-order approximation of the utility payoffs.

generate the coefficient given historical data can be seen as Bayesian Learner with diagonal covariance matrices for coefficients and the error term with λ equal to the signal to noise ratio.

We therefore assume that both the noise term Σ_e and traders' belief covariance matrix Σ_i are multiples of the identity matrix. Moreover, we normalize the noise term to be the identity matrix. In addition, we assume that the features that traders observe are orthonormal, i.e. $X_i'X_i = I$; this is without loss of generality as we can apply eigenvalue decomposition or Cholesky decomposition on X_i to make the column space orthonormal, without changing the datasets Span(X_i) of traders, or traders' beliefs over market prices.

Assumption 6 (**Simplification**). *We assume that the perceived market noise is normally distributed with the covariance matrix $\Sigma_e = I$. For each trader i, we assume that her dataset satisfies $X_i'X_i = I$ and the covariance matrix of her initial belief about LFM coefficients is $\Sigma_i = \frac{1}{\lambda_i} I$ for some $\lambda_i > 0$.*

2.3 Trading and Market Prices

So far, we have seen how traders update their beliefs given realized market prices. We now discuss how market participants trade and what determine the market prices. We let $\psi_{i,j}^T(\cdot)$ represent the **contingent plan** that a market participant i makes regarding their position in stock j at time T; that is, for any hypothetical price of asset j at time T, the market participant shall decide on their holding of the asset. We use $\psi_i^T(\cdot) : R^N \to R^N$ to denote the vector of contingent plans across all assets:

$$\psi_i^T(\cdot) = (\psi_{i,1}^T(\cdot), \ldots, \psi_{i,j}^T(\cdot), \ldots, \psi_{i,N}^T(\cdot))'.$$

Since there is no dividend in the market, the payoff that agent i receives from holding position j is given by:

$$\psi_{i,j}^T(P_{T,j})[P_{T+1,j} - P_{T,j}] \tag{2.7}$$

where we recall that $P_{T,j}$ is the price of asset j at time T and $P_{T+1,j}$ is its price at time $T+1$.

Trader i's subjective belief over the price P_{T+1} in the next period is derived from the the linear model $P = X_i g + e$, where she believes that g is distributed as $N(\mu_i^T, \Sigma_i^T)$. Therefore, her subjective belief over the return from purchasing one unit of asset j at price P_j is thus normally distributed with:

$$N\left((X_i\mu_i^T)_j - P_j, (X_i\Sigma_i^T X_i' + \Sigma_e)_j\right).$$

From her mean-variance preference, trader i's demand for asset j is

$$\psi_{i,j}^T(P_j) = \frac{(X_i\mu_i^T)_j - P_j}{A_i(X_i\Sigma_i^T X_i' + \Sigma_e)_j}.^6 \tag{2.8}$$

[6]When investors engage in hypothetical thinking about possible prices, they do not treat these hypothetical prices as real inputs for Bayesian updating. This is because inputs are not actual observations, and computation cost of belief updating for each hypothetical price is high.

Long-run investor k, on the other hand, always believes that P_{T+1} is distributed as $N(q_k, \Sigma_e)$. By the same reasoning, the contingent plan submitted by investor k is

$$\psi_{k,j}^T(P_j) = \frac{q_{k,j} - P_j}{A_k} \tag{2.9}$$

where $q_k \equiv (q_{k,1}, \ldots, q_{k,j}, \ldots, q_{k,N})'$.

Noise traders at time T have a random demand of $\xi_T = (\xi_{T,1}, \ldots, \xi_{T,N})$. We assume that there is zero net supply of each asset in the market, and market participants can hold any amount of asset, so given the demand of the asset j, the **market price** $P_{T,j}$ of asset j is determined by the **market clearing condition**:

$$\xi_{T,j} + \sum_{i=1}^{M+K} \psi_{i,j}^T(P_{T,j}) = 0.$$

Note that for any $\xi_{T,j}$, there exists a unique $P_{T,j}$ satisfying the market clearing condition, as each $\psi_{i,j}^T$ is a monotonic decreasing linear function.

Let P_T denote the vector of asset prices $P_T = (P_{T,1}, \ldots, P_{T,N})$. Finally, market participants trade according to P_T: trader or investor i's holding of asset j is $\psi_{i,j}^T(P_{T,j})$. We allow unrestricted holding and shorting of assets in the market, under the assumption that active participants collectively hold a net zero position in stocks after aggregation. This assumption is satisfied if, for each asset, there is a broker with substantial financial resources (e.g., being capable of holding and managing a significant inventory of the asset). Active traders and investors, who are initially assumed to hold no asset, submit their contingent plans to the broker. The broker then determines the market-clearing price based on these plans. [7] The next assumption about noise traders guarantee that investors/traders correctly perceive the market noise in the long run.

Assumption 7 (**Market Noise**). *The random demands of noisy traders $\{\xi_T\}_T$ are i.i.d. according to $N(0, \bar{\lambda}^2 \cdot I_N)$, where*

$$\bar{\lambda} := \sum_{i=1}^{K+M} \frac{1}{A_i}$$

We shall interpret $\bar{\lambda}$ as the total weighted number of investors and traders. By setting the random demand to have standard deviation exactly to the weighted number, we shall

[7] From a mathematical perspective, when there are long-run investors in the market, having nonzero aggregate supply of assets does not affect our main findings. Let s be the total supply of assets, and then the market clearing price satisfies

$$\sum_{i=1}^{M+K} \psi_{i,j}^T(P_{T,j}^*) = s.$$

If there is at least one long-run investor in the market, picking investor i for example, then the market clearing price would be the same as the price in the market when the total supply of assets is zero and investor i's belief is $q_i + B_i s$ instead. Hence, our main results including Proposition 1, Theorem 1, and analysis in Section 5 remain true.

show that investors and traders are asymptotically correct about the magnitude of the market noise. More rigorously, we let $P^*_{T,j}$ be uniquely given by

$$\sum_{i=1}^{M+K} \psi^T_{i,j}(P^*_{T,j}) = 0.$$

and call $P^*_{T,j}$ the **efficient price** of asset j at time T; this is the hypothetical market price if there were no noise traders in the market. Let P^*_T be the vector collecting all the efficient prices. The noisy part of the market price is then $e_T := P_T - P^*_T$, which we refer to as the **market noise**.

Lemma 2. *The market noise e_T is normally distributed according to $N(0, f_T \cdot I_N)$ for some deterministic constant number f_T, and is independent across time. Moreover, the constants satisfy $f_T = 1 + O(\frac{1}{T})$ as $T \to \infty$.*

Recall from Assumption 1, 2 and 6 that investors and traders subjective believe the market noise is distributed as $N(0, I_N)$. We remark that investors and traders asymptotically have the correct belief about the magnitude of the market noise; we will show in the next section that P^*_T converges a.s., so that the empirical distribution of $P_T - P^*_T$ over a long period is exactly $N(0, I_N)$, which is consistent with investors' and traders' subjective believes over the market noise. However, in a finite time, investors and traders have misspecifications about the market noise, under which traders update their beliefs at the end of every period. We argue that this bounded rationality is because traders are not aware of others' belief distributions, and thus cannot compute the magnitude of the market noise in every period explicitly. Instead, they use the long-run market noise as a proxy for short-run market noise.

The timeline for the learning and trading cycle is summarized in the following table.



- End of time $T-1$: P_{T-1} is realized
- Begin of time T: Each trader update belief about their coefficients g in LFM given P_{T-1} and form a new belief over P_{T+1}.
- Mid of time T: Given the updated belief about P_{T+1}, market participant i proposes the contingent plan $\psi^T_{i,j}(\cdot) : \mathbb{R} \to \mathbb{R}$ for any asset j. The random demand $\xi_{T,j}$ is realized.
- End of time T: The market price P_T is determined by the market clearing condition and is observed by all the market participants. Market participant i's holding from time T to $T+1$ is $\psi^T_{i,j}(P_T)$.

2.4 Market forecast

Trader i's subjective belief over market prices consist of the deterministic part $X_i g$ plus the noisy part $e \sim N(0, \Sigma_e)$. We have seen from Lemma 2 that traders are asymptotically correct about the distribution of the market noise $P_T - P_T^*$. Hence, we interpret $X_i g$ as trader i's belief over the efficient price P_T^*. *To avoid confusions, from now on we will exclusively use "traders' beliefs" for their subjective beliefs over LFM coefficients g, and use "market forecast" for their subjective beliefs over the efficient price.* Arithmetically, there is a one-to-one correspondence between trader i's belief g over LFM coefficients and her market forecast $X_i g$; given a market forecast P, we can restore her belief as $X_i' P$.

One key feature of our model is that trader i's market forecast is supported on her dataset $\text{Span}(X_i)$, which is often a proper subset of $\mathbb{R}^N$. However, since the real efficient price is determined by the aggregation of all participants' beliefs, the real market-clearing price vector will likely lie outside $\text{Span}(X_i)$ for each i. This features exists partly due to the multiple assets market participants trade at the same time, and represents a fundamental deviation from the previous literature about learning in financial markets. Our market participants not only have "misspecified" subjective model of the market (in opposition to for example Townsend (1978)), but also intrinsically have difficulty in understanding market prices (in opposition to Bray (1982)).

How do traders perceive a realized market price that does not lie in $\text{Span}(X_i)$? We have assumed that traders use Bayesian rule to update their beliefs upon realization of a new market price P_T, and an explicit formula is given by Lemma 1. Recall that $\text{Proj}_{X_i}(P) = X'XP$ is the projection operator onto the space X_i. The following is a useful and intuitive fact.

Lemma 3. *For any normally distributed prior belief of traders, observing a market price P_T induces the same posterior belief as observing a market price $Proj_{X_i}(P_T)$.*

Proof. By Lemma 1, the posterior belief is also normally distributed with the variance does not depend on the realized market price. The posterior mean, as given by (2.6), would remain the same if we replace P_T by $\text{Proj}_{X_i}(P_T)$. □

The intuition is that as traders do not believe the market price would be outside $\text{Span}(X_i)$, they find the closet point in their dataset, $\text{Proj}_{X_i}(P_T)$, as the reference point for updating their belief. The remaining part $P_T - \text{Proj}_{X_i}(P_T)$ is attributed to the market noise.

3 Equilibrium concepts

3.1 Stationary equilibrium

So far we have defined the dynamics of learning and trading. In this section, we move on to characterize the long-run dynamics of the market. Because the random demands ξ_T are

independent across time, the market prices will not converge. However, it is possible that the efficient price P_T^*, only depending on the beliefs of market participants, might stabilize over time if traders' beliefs stabilize. We leave the question whether prices and beliefs stabilize to the next section, and explore the limiting behavior assuming limits exist.

Suppose that the efficient price eventually converges to some price vector P_e, then the market prices observed by traders will eventually be centered around P_e. By Lemma 3, traders will perceive these prices as around $\text{Proj}_{X_i}(P_e)$. Over time, she will be very confident in her forecast of future prices as the posterior variance Σ_i^T is close 0 by (2.4), and her market forecast will converge to the delta mass at $\text{Proj}_{X_i}(P_e)$; equivalently her belief about LFM coefficients will converge to $X_i' P_e$. On the other hand, the beliefs of traders will also pin down the efficient price by their contingent plan. Motivated by these discussions, we introduce the notion of the stationary equilibrium. We use the notation δ_x as the probability measure that attaches x with probability 1.

Definition 1. A **stationary equilibrium** consists of the contingent plans $\{\psi_i\}_i$ of all the market participants, a price vector P_e and subjective beliefs about the LFM coefficients $\{\nu^i \in \Delta(\mathbb{R}^{F_i})\}_i$ of traders in the market such that

(1) The price P_e satisfies the market clearing condition without noise traders:

$$\sum_i \psi_i(P_e) = 0$$

(2) Trader i's subjective belief is $\nu^i = \delta_{X_i' P_e}$.

(3) For a market participant i, the contingent plan $\psi_i(P)$ maximizes his or her utility payoff given market participants' beliefs μ_i (for traders) or q_i (for investors) and a hypothetical price P at the current period.

We call P_e a **stationary equilibrium price**.

From the definition, if P_e is a stationary equilibrium price, then trader i's market forecast is $\text{Proj}_{X_i}(P_e)$ and by (2.8), trader i's contingent plan $\psi_i(P_e)$ is $\frac{\text{Proj}_{X_i}(P_e) - P_e}{A_i}$. As for long-run investor k, his contingent plan is always $\psi_k(P_e) = \frac{q_k - P_e}{A_k}$. We plug these into the market clearing condition to get

$$\sum_{i=1}^{K} \frac{\text{Proj}_i(P_e) - P_e}{A_i} + \sum_{k=K+1}^{M+K} \frac{q_k - P_e}{A_k} = 0 \tag{3.1}$$

In fact, a price P satisfying (3.1) uniquely defines a stationary equilibrium, as the stationary equilibrium price determines traders' beliefs, which then pin down the contingent plans. When there are long-run investors in the market, we are able to prove the uniqueness of the P satisfying (3.1) so we get:

Proposition 1. *When there are long-run investors in the market, i.e. $M \geq 1$, then there exists a unique stationary equilibrium.*

In contrast, when there are no long-run investors in the market, (3.1) is simplified to

$$\sum_{i=1}^{K} \frac{\text{Proj}_i(P_e) - P_e}{A_i} = 0, \tag{3.2}$$

and any $P_e \in \text{Span}(X_i)$ is a solution.

Proposition 2. *When there is no long-run investor in the market, i.e. $M = 0$, the set of all possible stationary equilibrium prices is $\cap Span(X_i)$. In particular, if $\cap Span(X_i) \neq \{0\}$, there are infinitely many stationary equilibria.*

The deciding factor for the different conclusion in the two lemmas above is the *anchoring effect* of the investors. With the constant beliefs of investors in the market serving as the reference point, the stationary equilibrium price cannot be too far away from the average belief of investors. In contrast, in the absence of investors in the market, stationary equilibrium prices are self-fulfilling, meaning that if all the traders forecast the efficient prices to be P_e, then indeed P_e will be realized as the efficient prices.

3.2 Connections to Berk-Nash Equilibrium

The notion of the stationary equilibrium differs from the standard Nash equilibrium in that, some traders use a *misspecified* subjective model. A useful framework to study misspecified models is the **Berk-Nash equilibrium**. We give a brief introduction to the Berk-Nash equilibrium in Appendix A and refer readers to Esponda and Pouzo (2016a) and Fudenberg et al. (2017b) for details. In this subsection, we prove that the stationary equilibria are equivalent to Berk-Nash equilibria in a properly defined game, which we now introduce.

Back to our learning and trading processes we developed in Section 2, though the process has infinite periods, since market participants are myopic, we can think of them as playing a two-period game, where their payoff depends on their asset positions in the first period and the asset prices in both periods. Since we are interested in the long-run dynamics of the game, the equilibrium we are interested in is in a steady state. We translate the setup into the language of mis-specified models.

The game $\mathcal{G}$ is composed of an objective game and a subjective model. The **objective game** is as the following.

- Players are the $K + M$ market participants.
- The payoff-relevant states are the noisy traders' demand in the first and second period ξ_1 and ξ_2 respectively.
- There are no signals in the market.
- ξ_1 and ξ_2 follow assumption 7.
- The action of market participant i is his or her contingent plan ψ_i.

- The observable consequence is the market price P_1 in the first period and the market price P_2 in the second period.
- The realized market price for each single stock is given by

$$\xi_{t,j} + \sum_{i=1}^{M+K} \psi_{i,j}(P_{t,j}) = 0$$

for $t \in \{1, 2\}$. [8]

- Market participant i's monetary payoff is $(P_2 - P_1) \cdot \psi_i(P_1)$ which is the dot product of price changes and their position holdings given contingent plan. i's utility payoff is given by $\pi^i : \sum_{n=1}^{N} [1 - e^{-A_i(P_2^n - P_1^n)\psi_{i,n}(P_1^n)}]$ [9]

For simplicity we restrict to deterministic actions of market participants, so that a strategy of a market participant i is a contingent plan ψ_i. A strategy profile $\sigma = (\psi_i)_i$ consists of the contingent plans of all the market participants. We denote the objective distribution of Player i's observable consequences P_1 and P_2 by Q_σ^i. The next paragraph introduces the **subjective environment**: the game that players think they are playing.

Trader i's parameter space $\Theta^i = \mathbb{R}^{F_i}$, where we recall that F_i is the number of features that trader i uses. If her belief about the LFM coefficients has point mass on $\theta^i \in \Theta^i$, then her belief about the efficient price P_e has a point mass at $X^i\theta^i$. Given $\theta^i \in \Theta^i$ and a contingent plan ψ_i, we assume her subjective belief $Q_{\theta^i}^i(\cdot|\psi_i)$ over P_1 and P_2 are independently distributed as $N(X_i\theta^i, I_N)$. The degeneracy of $Q_{\theta_i}^i$ in ψ_i means market participants do not realize their actions will affect the market prices. Investor k has a singleton parameter space $\Theta_k := \{q_k\}$, and her belief about P_1 and P_2 are independently distributed as $N(q_k, I_N)$.

Additionally, given that traders and investors have point mass belief, by Lemma 2, the objective distribution of observable consequence Q_σ^i follows $N(P_e, I_N)$, where P_e solves

$$\sum_{i=1}^{M} \frac{X_i\theta^i - P_e}{A_i} + \sum_{k=M+1}^{K} \frac{q_k - P}{A_k} = 0$$

In the equation, $\frac{X_i\theta^i - P}{A_i}$ and $\frac{q_k - P}{A_k}$ represent the optimal contingent plans $\psi_i(P)$ and $\psi_k(P)$ for traders and investors separately given their point mass beliefs.

[8] In our original model, the market clearing price P_T^* is decided by the contingent plans submitted in the Tth period. Since we model a steady state here, we assume that market participants submit the same contingent plans to both periods.

[9] This formulation indicates that each agent exhibits constant absolute risk aversion (CARA) utility for each stock, with total utility represented as the sum of individual stock utilities. This representation aligns with the setting in previous sections, as CARA preferences under normally distributed returns are equivalent to mean-variance preferences. Moreover, it is consistent with our earlier assumption that participants evaluate stocks independently.

Given a strategy profile $\sigma = (\psi_i)_i$ that gives the efficient price P_e, for a trader i with a point mass belief on $\theta^i \in \Theta^i$, the **Kullback-Leibler divergence** between $Q^i_{\theta_i}$ and Q^i_σ is: [10]

$$\begin{aligned} & K^i(\sigma, \theta^i) \quad with \quad Q^i_\sigma \sim N(P_e, I),\ Q^i_{\theta^i} \sim N(X_i\theta^i, I) \\ &= \frac{1}{2} E_{Q^i_\sigma}[2P_e'X - 2(X_i\theta^i)X] - \frac{1}{2}P_e'P_e + \frac{1}{2}(X_i\theta^i)'X_i\theta^i \\ &= \frac{1}{2}P_e'P_e - (X_i\theta^i)'P_e + \frac{1}{2}(X_i\theta^i)'(X_i\theta^i) \\ &= \frac{1}{2}(P_e - X_i\theta^i)'(P_e - X_i\theta^i). \end{aligned}$$

As a result, $\arg\min_{\theta^i \in \Theta^i} K(\sigma, \theta^i) = \{X_i'P_e\}$.

Berk-Nash Equilibrium

Having set up the game, we can study its Berk-Nash equilibria. A Berk-Nash equilibrium consists of a strategy profile $\sigma = (\psi_i)_i$ and $\mu^i \in \Delta(\Theta^i)$ such that

(1) ψ_i is optimal given μ^i, and

(2) If θ is in the support of μ^i, then $\theta \in \arg\min_{\theta^i \in \Theta^i} K(\sigma, \theta^i)$.

Given $\arg\min_{\theta^i \in \Theta^i} K(\sigma, \theta^i) = \{X_i'P_e\}$, where P_e is the efficient price satisfying $\sum_{i=1}^{K+M} \psi_i(P_e) = 0$, the second condition immediately gives $\mu_i = \delta_{X_i'P_e}$ for trader i. We immediately have its following connection to the stationary equilibrium.

Proposition 3. *There is a one-to-one correspondence between stationary equilibria and Berk-Nash equilibria of the game $\mathcal{G}$: a strategy profile $\{\psi_i\}_i$, the efficient price P_e and the traders' parameters $\{\mu^i\}$ in a Berk-Nash equilibrium define a stationary equilibrium. Conversely, contingent plans $\{\psi_i\}_i$ of all the market participants, traders' beliefs about the LFM coefficients $\{\mu^i\}_i$ and inventors' constant beliefs define a Berk-Nash equilibrium.*

4 Convergence to Equilibrium

In the last section, we have proven the existence of a stationary equilibrium; in other words, there are steady states in the markets when the beliefs of the traders and the efficient prices no longer evolve. We move on to ask: after repeated interactions of the market participants, does the dynamics of the market eventually converge to a stationary equilibrium? We give positive answers both in the case of the absence and in the case of the presence of investors.

Suppose that there are long-run investors in the market. As we have seen in Proposition 1, there is a unique stationary equilibrium. We prove the convergence of the market dynamics to the equilibrium in the following sense.

[10] In this setup, both the realized price distribution and player's anticipation of realized price in the first period and the second period are the same, as a result, we only need to calculate the KL divergence of a single period.

Theorem 1. *Suppose that there are long-run investors in the market, and let P_e be the unique stationary equilibrium price given by Equation 3.1. Then*

- *The efficient price P_T^* converges to P_e a.s and in L_2. Furthermore, if we define*

$$\alpha := \frac{\sum_{i=K+1}^{K+M} \frac{1}{A_i}}{\sum_{i=1}^{K+M} \frac{1}{A_i}}$$

  to be the share of investors in the market, then

$$|||P_T^* - P_e|||_{L^2} := O(\frac{\log T}{T^{\min\{1/2,\alpha\}}}),$$

- *Trader i's belief distribution about the LFM coefficient g converges to $\delta_{X'P_e}$ a.s., and her market forecast $X_i g$ converges to $\delta_{Proj_{X_i}(P_e)}$ a.s.*

To understand why the limits do not depend on the initial beliefs of traders, imagine for a second that there is no market noise, and the market price is simply the average of traders' and investors' belief. While traders update their beliefs towards the market price, investors hold their beliefs unchanged. A recurrent theme in the learning literature is that those who are reluctant to update their beliefs will have their beliefs prevalent among the social group. Therefore, the long-run investors consistently pull the market forecast of the traders towards the belief of themselves until the traders forget their initial beliefs in the limit. The market noise can affect traders' beliefs in the short run, yet by some variant of law of large numbers, since the market noise is independent across periods, the aggregate effect of the noise is close enough to zero and thus is negligible compared to the effect of the investors.

Without the anchoring effect of the long-run investors in the market, Proposition 1 shows that there are infinitely many stationary equilibria (unless $\text{Span}(X_i) = \emptyset$). We again prove the convergence of the dynamics of the market, yet the limit is not deterministic.

Theorem 2. *When there is no long-term investor, then there exists a random variable P_e supported on $\in \cap Span(X_i)$ such that P_T^* converges to P_e a.s. Moreover, the center of trader i's market forecast $X_i\mu_i^T$ satisfies $\|X_i\mu_i^T - P_T^*\|_{l^2} \to 0$ a.s.*

To see why eventually the efficient price lies in the intersection of traders' datasets X_i, if all people except for one, say trader j, observe a feature that is orthogonal to all other features, then trader j will only attribute the price discrepancy driven by other traders' belief about this feature to noise; this drags down the market belief about this feature. Thus, other traders will recalibrate their beliefs about this feature downward, and this makes this feature even less important in predicting market prices. The downward learning and trading cycle keeps going until everyone deems this feature to be irrelevant to the market. Again, this finding is consistent with the conventional wisdom that agents who are more reluctant to learn from others will have more influence on others' opinions.

In the absence of long-run investors, the market noise has a long-lasting effect on the market, as all the traders include the market noise in their belief updating process and the

beliefs of traders are self-fulfilled. Furthermore, the initial beliefs of traders do affect the distribution of the stationary equilibrium price P_e. The following is a special case where we explicitly calculate the distribution of equilibrium prices in each time.

Example 1. *Suppose that there are K high-frequency traders and no long-run investors in the market. The traders share the same feature matrix X and trader i has a prior belief $N(\mu_i, \frac{1}{\lambda}I)$, where λ is common for all traders. Then the asset prices at each time t is consistent with the linear factor model with:*

$$P_t = X\beta_t + e_t$$

where the dynamic of β_t follows

$$\beta_{t+1} = \beta_t + \frac{1}{t+\lambda}X'e_t \text{ with } \beta_1 = \frac{1}{\sum \frac{1}{A_i}} \sum_{i=1}^{K} \frac{\mu_i}{A_i}.$$

Furthermore, the difference between traders' pricing kernel μ_i^t and the actual market pricing kernel β_{t+1} shrinks at a fixed rate:

$$\mu_i^t - \beta_{t+1} = \frac{\lambda}{t+\lambda}(\mu_i - \beta_1)$$

As we can see, the market pricing kernel β_1 in the first period aggregates the initial beliefs of traders, and then it evolves recursively according to $\beta_{t+1} = \beta_t + \frac{1}{t+\lambda}X'e_t$, following a random walk with a shrinking step size. The initial noise terms are given greater weight because, when traders are less confident in their beliefs, they tend to update those beliefs more. The limit of the efficient price can take any possibly value in the dataset of traders.

5 Empirical Implication

In this section, we utilize our framework to provide a theoretical explanation of bubble formation as well as patterns in traders' returns complementing with simulation study using synthetic data. Our model is able to provide new perspectives on these subjects largely due to its statistical learning feature.

5.1 Bubble Formation

In our model, market participants do not have rational expectations and they have misspecified models about markets. The market prices entirely depend on market participants' beliefs about future market prices, and we do not yet make any assumption on the relations between their beliefs and fundamental values of assets. Therefore, when market participants' hold "irrational" beliefs about market prices, we expect stationary equilibrium prices not to align with fundamental values of assets. Our focus is then not the emergence of bubbles, but rather, how large the bubble is. The main finding is that, traders' statistical learning behavior provides an amplification channel that exacerbates abnormal pricing in the market: when traders' datasets match the irrational beliefs of investors better, the stationary equilibrium price will deviate more from the fundamental values.

Suppose that F features about assets capture all the fundamental aspects of assets, e.g. their expected cash flow and risk premium. Let X denote the $N \times F$ grand feature matrix where N stands for all the assets in the market. There are three groups of market participants—rational investors, irrational investors, and traders—all of whom share the same risk aversion parameters. Each individual within a given group has access to the same dataset, and investors within each group hold identical beliefs about market prices. We assume that investors' belief about future prices are also derived from the LFM model, and they put the same weights on the fundamental features. We let $q \in \text{Span}(X)$ denote the constant belief of the rational investor. The irrational investor, however, believes that a sunspot feature x_{sun} that is completely irrelevant to the fundamentals is also important to pricing in the long run. We assume that x_{sun} is orthogonal to X and the irrational investor's belief about future prices is $q + cx_{sun}$ for some constant c. The common q both investors share is interpreted as the fundamental value of assets. The trader does not know the sunspot feature the irrational investor uses, but she observes another feature x_{trader} that she believes to be potentially useful. We assume that $x_{trader} := \rho \cdot x_{sun} + \sqrt{1-\rho^2} \cdot x_{alter}$, where x_{alter} is orthogonal to both X and x_{sun} and $\rho \in [0, 1]$. Here ρ represents how well traders can perceive the irrational investors sunspot feature.

We let α, β, γ be the ratios of traders, rational investors, and irrational investors in the total population and we normalize $\alpha + \beta + \gamma = 1$. We recall that the stationary equilibrium price does not depend on the initial beliefs of traders so what we have assumed is sufficient to derive the stationary equilibrium price using (3.1).

Proposition 4. *Suppose that there are a trader, a rational investor, and an irrational in-*

vestor whose beliefs are given as above. Then the stationary equilibrium price is

$$P_e = q + a_1 x_{sun} + a_2 x_{alter},$$

where

$$a_1 = \left[1 - \alpha \frac{\rho}{\rho + \sqrt{1-\rho^2}} \left(1 + \frac{\alpha\sqrt{1-\rho^2}}{\rho + (1-\alpha)\sqrt{1-\rho^2}}\right)\right]^{-1} \gamma c. \tag{5.1}$$

$$a_2 = \frac{\alpha\sqrt{1-\rho^2}}{\rho + (1-\alpha)\sqrt{1-\rho^2}} a_1.$$

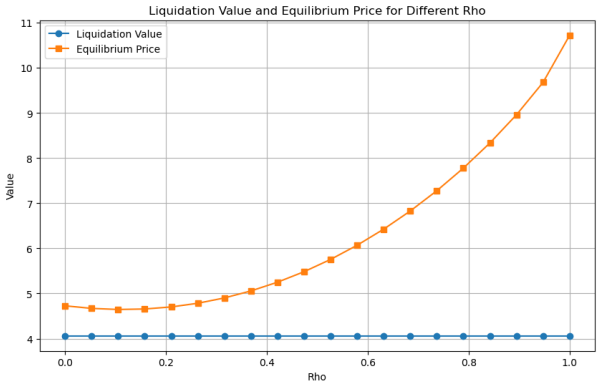
As we can see, the stationary equilibrium price P_e deviates from the fundamental value q. We interpret γc in (5.1) as the contribution to this deviation from irrational investors, and the coefficient in the front as the amplifying effect from the trader. The coefficient is strictly increasing in α and ρ, meaning that if there are more traders in the market, or if traders' dataset can better capture the sunspot feature, then the sunspot feature is weighted more in the equilibrium price. Intuitively, if ρ is bigger, i.e. when the feature x_{trader} is more aligned with the sunspot feature, x_{trader} can better explain the market prices and the traders will put more weight on x_{trader}, thus pushing the stationary equilibrium price to deviate more from the fundamental value.

We cross check our result in a simulation study. We randomly generate a 1000×15 matrix with orthogonal columns to serve as the feature matrix X, representing 1000 assets each having 15 features. Additionally, we randomly generate x_{sun} and x_{alter}, which are mutually orthonormal and orthonormal to X. The financial market simulation involves 900 traders (i.e. $\alpha = 0.9$), 90 rational investors (i.e. $\beta = 0.09$) and 10 rational investors (i.e. $\gamma = 0.01$). Rational investors share a common belief $q \in \text{Span}(X)$, while irrational investors share a belief $q + cx_{sun} \in \text{Span}([X, x_{sun}])$ for some constant c. Each trader, in contrast, has a distinct randomly generated initial belief but relies on the same feature matrix $[X, x_{alter}]$ to forecast asset prices.

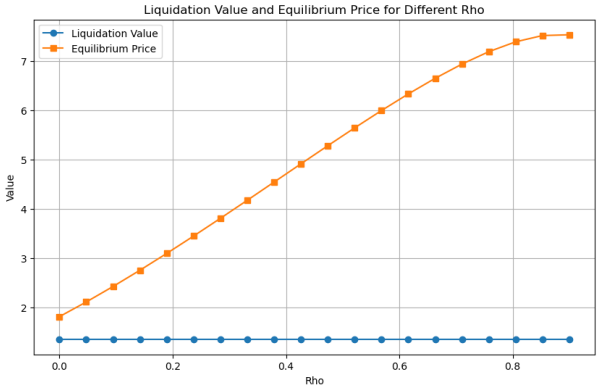
Figure 5.1 plots the fundamental values (blue curve) and the stationary equilibrium prices (orange curve) of four assets as ρ varies between 0 and 1. Starting from $\rho = 0$, when traders cannot comprehend the sunspot feature and the stationary equilibrium price only slightly deviates from the fundamental value, a clear increasing trend in this deviation can be observed as ρ increases to 1. The orange curves representing the stationary equilibrium prices exhibit different shapes because the stationary equilibrium prices also depend on the confounding feature x_{alter}.

5.2 Market Fragility

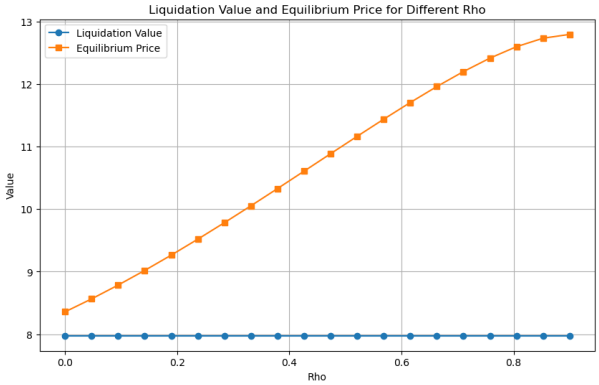
Algorithmic traders have become dominant participants in modern financial markets, executing a substantial share of equity and futures transactions. While proponents argue that algorithms enhance liquidity and price efficiency under normal conditions, critics contend that their behavior can make markets more fragile and prone to instability. A central concern is that many algorithmic strategies do not rely on information about fundamentals such



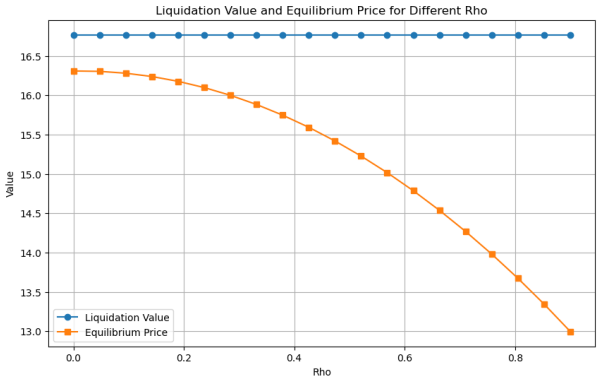
(a) Stock 1: Liquidation Value and Equilibrium price for different ρ



(b) Stock 2: Liquidation Value and Equilibrium price for different ρ



(c) Stock 3: Liquidation Value and Equilibrium price for different ρ



(d) Stock 4: Liquidation Value and Equilibrium price for different ρ

Figure 5.1: Liquidation Value VS Stationary Equilibrium price for different ρ

as cash flows or economic conditions; instead, they exploit short-term price patterns. Empirically, Kirilenko et al., 2017 examine the Flash Crash in 2010 and find that the withdrawal of high-frequency traders from the buy side led to the crash of the market. Pyrgiotakisa et al., 2024 find that globally the entry of high-frequency traders significantly increases stock price crash risk.

Theoretical studies on market crashes caused by algorithmic trading often attribute them to the correlated responses of algorithmic traders to shocks and to feedback loops that amplify price movements through the progressive withdrawal of liquidity. We propose a channel that is based on the learning of LFM parameters in our model and the misperception of the market pries.

In our model, in the absence of long-run investors, the equilibrium market price is entirely driven by traders' beliefs and market noise. Recall from Theorem 2 that the stationary equilibrium price lies in the intersection of all traders' datasets $\cap_i \text{Span}(X_i)$.

Suppose, hypothetically, that the market is in a stationary equilibrium in the sense that traders are nearly certain about their beliefs, and these beliefs match the efficient market price. Consider a scenario in which there is a feature F that all traders share and to which they assign positive weight in their LFM. Now suppose a new trader enters the market whose features are all orthogonal to F. To illustrate the **feature–weight downward spiral channel**, imagine for a moment that there is no market noise. The new market price, which aggregates the beliefs of all traders including the newcomer, will reflect the importance of feature F slightly less than before. As a result, the incumbent traders will slightly reduce their belief in F. Due to this drift in beliefs, the market price in the next period will depend even less on F, prompting another downward revision in beliefs. When market noise is present, the price fluctuates around the efficient price; nonetheless, traders can still effectively infer the efficient price by averaging market prices over time. Thus, the downward spiral of beliefs in F is accompanied by local fluctuations.

The learning process ends when all traders assign zero weight to feature F. Notably, regardless of how strong the initial beliefs of the incumbent traders in F are—or how small the scale of the new trader—the belief in F always converges to zero. In the most extreme case, where the new trader's features are orthogonal to $\cap_i \text{Span}(X_i)$, the efficient market price collapses to zero. To summarize, in the absence of long-run investors, the arrival of any new trader can trigger a complete market crash.

We formulate the general result as a corollary of Theorem 2.

Corollary 2.1. *Suppose that there are K traders and no investors in the market and that the market is in a hypothetical stationary equilibrium with the equilibrium price P_e. Suppose a new trader, labeled $K+1$, enters the market. Then the new stationary price will be*

$$Proj_{\cap_{i=1}^{K+1} Span(X_i)}(P_e),$$

the projection of the old equilibrium price P_e onto the common datasets of all traders.

The anchoring effect of long-run investors contributes to market stability, in contrast to markets without them. However, when the investor's share is small, the market price remains vulnerable to small perturbations from traders. We illustrate this in the following extreme example.

Example 2. *Suppose that there are K traders and one investor with a constant belief q and a risk aversion parameter A in the market. Suppose further that for each trader i's dataset X_i, we have $q \in Span(X_i)$. Then the stationary equilibrium price is*

$$P_e = q.$$

Now assume a new trader, labeled $K+1$, enters the market with a risk aversion parameter of $\frac{A}{\lambda}$ (i.e., her scale is λ times that of the investor). In addition, assume that her dataset X_{K+1} is orthogonal to q. Then by Proposition 2, the new equilibrium price is the solution to

$$(I - \frac{1}{\sum_{i=1}^{K} \frac{1}{A_i} + \frac{1}{A} + \frac{\lambda}{A}} \sum_{i=1}^{K} \frac{1}{A_i} Proj_{X_i} - \frac{\frac{\lambda}{A}}{\sum_{i=1}^{K} \frac{1}{A_i} + \frac{1}{A} + \frac{\lambda}{A}} Proj_{X_{K+1}})P = \frac{\frac{1}{A}}{\sum_{i=1}^{K} \frac{1}{A_i} + \frac{1}{A} + \frac{\lambda}{A}} q.$$

Hence, the new stationary equilibrium price is

$$P_e' = \frac{1}{\lambda + 1} q.$$

That is, the larger the scale of the new trader, the more the equilibrium price shrinks.

We expect the impact of a new trader to the equilibrium price to be generic; the effect should be significant when the share of the investor is small and the incumbent traders can nearly infer the investor's belief, while the new trader cannot.

5.3 Traders' returns

5.3.1 Returns in a stationary equilibrium

Consider a market with investors so that there is a unique stationary equilibrium characterizing the long-run dynamics of the market, and we let P_e denote the stationary equilibrium price. We study a hypothetical market at time ∞, where traders' market forecasts are exactly the same as their forecasts in the stationary equilibrium and other assumptions in the model remain unchanged; this is a proxy for studying the market after a long time. By Definition 1, trader i with dataset X_i has belief $\delta_{\mathrm{Proj}_{X_i}(P_e)}$ over the efficient price.

While traders stop updating their beliefs and the efficient prices are static at time ∞, the realized market prices keep fluctuating because of the market noise. Upon the realization of the market price $P_e + e$, trader i's holding of asset j is

$$\frac{1}{A_i} \left(\mathrm{Proj}_{X_i}(P_e) - (P_e + e) \right)_j .$$

From this, we note that if asset j's price is lower than usual, i.e. e_j is negative, the trader holds more of asset j; since the market noise is independent across periods, the asset price will be likely to increase from this period to the next one. In short, the trader correctly has a higher position when the asset price is more likely to increase so she makes profits from the market fluctuation. Indeed, the expected (period by period) return of trader i is

$$\mathbb{E} \sum_j \frac{1}{A_i} \left(\mathrm{Proj}_{X_i}(P_e) - (P_e + e) \right)_j (e' - e)_j = \frac{1}{A_i} \mathbb{E} \left\| e \right\|_{l^2}^2 > 0,$$

where e' is the market noise in the next period.

The expected return is independent of traders' dataset or their subjective beliefs about prices. However, if her subjective belief $\mathrm{Proj}_{X_i}(P_e)$ about the efficient price is far away from P_e, then the trader will either hold lots of assets or in a deep short position. Either way, the trader is exposed to a high risk in the market. This intuition is validated in the following proposition.

Proposition 5. *In a stationary equilibrium, the sharpe ratio of trader i is*

$$\frac{\mathbb{E} \left\| e \right\|_{l^2}^2}{\sqrt{2 \left\| Proj_{X_i}(P_e) - P_e \right\|_{l^2}^2 + 3}},$$

which is strictly decreasing in $\left\|Proj_{X_i}(P_e) - P_e\right\|_{l^2}$. *In particular, if trader* j*'s dataset is a superset of trader* i*'s, then the sharpe ratio of trader* j *is higher than that of trader* i.

5.3.2 Returns in a learning process

When traders' market forecasts are deviating from the market price, they will update their market forecasts towards the market prices and thus moving the market equilibrium prices. We find that in this learning process, traders are likely to lose money to investors. We start with a simple model to illustrate the idea.

Example 3. *Suppose that there are one investor and one trader with the same risk preferences in the market trading one asset. The investor's belief over assets prices in the next period is distributed as* $N(q, 1)$. *The trader observes one feature (which we normalize to* 1 *about the asset so that her belief about the coefficient of the feature is the same as the market forecast about the asset.*

At some time T*, let the trader's market forecast be distributed as* $N(\mu, \frac{1}{\lambda})$. *The trader's return between time* T *and* $T+1$ *is then*

$$R_{T,T+1} := \frac{\lambda}{\lambda+1}(\mu - P_T)(P_{T+1} - P_T),$$

where $\frac{\lambda}{\lambda+1}(\mu' - P_T)$ *is her holding at time* T *and* $P_{T+1} - P_T$ *is the return per share. We decompose* $R_{T,T+1}$ *into two parts:*

$$R_{T,T+1} = \underbrace{\left[\frac{\lambda}{\lambda+1}(\mu - P_T)(P^*_{T+1} - P^*_T)\right]}_{R^{investors}_{T,T+1}} + \underbrace{\left[\frac{\lambda}{\lambda+1}(\mu - P_T)(e_{T+1} - e_T)\right]}_{R^{noise\ traders}_{T,T+1}}.$$

Because the efficient price is the hypothetical market price without noise traders, we interpret the first term as the trader's net profits from trading with the investor, and the second term as the net profits from trading with the noise traders.

As in the case of a stationary equilibrium, the trader has positive expected returns from trading with noise traders:

$$\begin{aligned} \mathbb{E}R^{noise\ traders}_{T,T+1} &= \mathbb{E}\frac{\lambda}{\lambda+1}(\mu - P^*_T - e_T)(e_{T+1} - e_T) \\ &= \frac{\lambda}{\lambda+1}\mathbb{E}e^2_T. \end{aligned}$$

The returns from trading with the investor $R^{investors}_{T,T+1}$ *is more complicated. Assume that* $\mu > q$. *Then the efficient price is*

$$P^*_T = \frac{\frac{\lambda}{\lambda+1}\mu + q}{\frac{\lambda}{\lambda+1} + 1} < \mu.$$

Let μ' *be the posterior mean so that the trader's market forecast is distributed as* $N(\mu', \frac{1}{1+\lambda})$. *If* $P_T = P^*_T + e_T < \mu$, *(which is more likely when the prior mean* μ *deviates more from* q*),*

then the trader will downwardly update their market forecast: $\mu' < \mu$. It is true, under a week condition, that the more pessimistic market forecast of the trader will lead to a decreasing efficient prices so that the trader with positive holding will have returns $R^{investors}_{T,T+1} < 0$. We formalize this idea in the following proposition.

Proposition 6. *Under the same setting as in the previous example, we have*

$$R^{investors}_{T,T+1} = (1+\lambda)(\mu-\mu')\left[\frac{\lambda}{2\lambda+1}(\mu'-\mu) + \frac{1}{(3+2\lambda)(2\lambda+1)}(\mu'-P^*)\right].$$

Suppose in addition that $\mu > \mu' > P^*$, then we have $R^{investors}_{T,T+1} < 0$. Moreover, for fixed μ, λ, $R^{investors}_{T,T+1}$ is strictly increasing in μ'.

A similar proposition is true if $q < \mu$, in which case the trader loses money to the investor when upwardly updating her belief. In both cases, the more the trader's belief drifts towards that of the investor's, the more the trader will lose.

In the presence of multiple traders in the market trading multiple assets, the analysis of the returns of traders becomes much more complicated; the dynamics of equilibrium prices depends on how **each** investor updates her belief and thus becomes very untractable. Nevertheless, we give a lower bound on investors' returns from the changes in market prices.

Proposition 7. *Under the general setting as in Section 2, consider an investor k with the risk preference parameter A_k and the belief distribution $N(q_k, \Sigma_e)$ of future market prices. Let P_1 and P_T be the market prices in the 1st and Tth period separately. Then the total profits of investor k between period 1 and T is at least*

$$\frac{1}{2A_k}\left[\|q_k - P_1\|^2_{l_2} - \|q_k - P_T\|^2_{l_2}\right].$$

The total profits of all investors between period 1 and n is at least

$$\frac{1}{2A}\left[\|q - P_1\|^2_{l_2} - \|q - P_T\|^2_{l_2}\right],$$

where

$$A = (\sum_k \frac{1}{A_k})^{-1}$$

is the aggregated risk aversion parameter and

$$q = A\sum_k \frac{1}{A_k} q_k$$

is the weighted average of investors' belief.

The proposition says that investors make profits from price changes towards their beliefs. The price drift from P_1 to P_T consists of two parts: the contribution from the noise traders e_1 and e_T, and the contribution from traders' belief updating. While the noises e_1 and e_T are entirely random, the traders' market forecasts are likely to drift towards investors' belief.

6 Conclusion

The focus of our model is on hedge fund professionals who use asset features to predict prices and trade multiple assets simultaneously. In some markets, they contribute to a significant portion of the trading volume, yet their trading strategies and the resulting market outcomes remain under-explored. We model them as mis-specified high-frequency traders who make market forecasts using Linear Factor Models and update their model coefficients via Bayesian rules upon observing new market prices. We demonstrate that Bayesian learning in our setting is equivalent to algorithmic trading, where traders perform linear ridge regression to forecast market prices. Our framework allows traders to observe different features, reflecting real-world practices in the financial sector, where practitioners increasingly rely on statistical models that incorporate unconventional data.

We define stationary equilibrium in our model and prove its existence. Additionally, we relate our problem to the literature on mis-specification, showing that the stationary equilibrium in our infinite setting corresponds to a Berk-Nash equilibrium in a comparable static setting. We characterize the long-run market dynamics both in the presence and absence of long-run investors and obtain convergence results. Departing from the Rational Expectations Equilibrium literature, our work contributes to the understanding of learning with mis-specified models in financial markets. Based on our model, we offer a theoretical framework to analyze traders' returns and provide explanations for bubble formation and market fragility.

We highlight two directions for future research. First, while we focus on a short time span and assume the features are static, in practice, features can evolve, for instance, due to the release of financial statements. Investigating how hedge funds learn in a dynamic setting and respond to changing features, as well as the resulting market outcomes, remains an open question. Second, our model assumes that all market participants are myopic, but an alternative framework could explore scenarios where some traders hold higher-order beliefs. Whether these traders can exploit others is a potentially interesting avenue for further study.

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A Berk-Nash Framework

In the Mis-specified Model, agents are not required to have correctly specified view of the Environment. Each player is characterized by a subjective model, which describes the set of beliefs over payoff-relevant consequences as a function of action.
Game $\mathcal{G} =< \mathcal{O}, \mathcal{Q} >$ is composed of objective game $\mathcal{O}$ and a subjective model $\mathcal{Q}$.

A.1 Objective Model

A.1.1 Notation

$\mathcal{O} =< I, \Omega, S, P, X, Y, f, \pi >$.

- I represents the set of players.
- Ω is the set of payoff relevant states.
- $S = \times_{i \in I} S^i$ is the set of profiles of private signal. S^i is the set of signal of player i.
- P is the probability distribution over $\Omega \times S$. $P_{\Omega|s^i}(\cdot|s^i)$ is the conditional distribution of player i given observing s^i.
- $X = \times_{i \in I} X^i$ is the set of profile of action. X^i is the action space of player i.
- $Y = \times_{i \in I} Y^i$ is the set of profile of observable consequence. Y^i is the consequence space for player i.
- $f = \{f^i\}_{i \in I}$ where $f^i : X \times \Omega \to Y^i$ is the consequence function for player i.
- $\pi = \{\pi^i\}_{i \in I}$ where $\pi^i : X^i \times Y^i \to R$ is the payoff function for player i.

A.1.2 Strategy

A strategy of player i is a mapping $\sigma^i : S_i \to \Delta(X^i)$. $\sigma^i(x_i|s^i)$ representing the probability of choosing x_i after observing signal s^i. $\sigma = \{\sigma^i\}_{i \in I}$ represents a strategy profile, Σ denote the space of all strategy profile. For each strategy profile σ, there is an objective distribution over player i's consequence.

$$Q^i_\sigma : S^i \times X^i \to \Delta(Y^i)$$

$$Q^i_\sigma(y^i|s^i, x^i) = \sum_{\{(\omega, x_{-i}) : f_i(x^i, x_{-i}, \omega) = y^i\}} \sum_{s_{-i}} \prod_{j \neq i} \sigma^j(x^j|s^j) \times P_{\Omega \times S^{-i}}(\omega, s^{-i}|s^i)$$

$Q^i_\sigma(y^i|s^i, x^i)$ represent the true distribution over consequence given strategy profile is σ and agent i implement action x^i after observe s^i.

A.2 Subjective Model

$\mathcal{Q} =< \Theta, \{Q_\theta\}_{\theta \in \Theta} >$.

A.2.1 Notation

- $\Theta = \times_{i\in I}\Theta^i$, Θ^i represents player i's parameter space. For each $\theta^i \in \Theta$, there is a conditional distribution over player i's consequence parameterized by θ^i, $Q^i_{\theta^i}(\cdot|s^i, x^i) : S^i \times X^i \to \Delta(Y^i)$. $Q^i_{\theta^i}(\cdot|\cdot,\cdot)$ represents player's perception of their environment. And Θ^i represents all the perception of environment player i can think of.

A.2.2 Kullback-Leibler divergence

Given a strategy profile σ, we need a distance notion to represent how far a perceived environment is away from a true environment. The Kullback-Leibler divergence of σ and θ^i is defined in following way:

$$K^i(\sigma,\theta^i) = \sum_{(s^i,x^i)\in S^i\times X^i} E_{Q^i_\sigma(\cdot|s^i,x^i)}\left[\ln\frac{Q^i_\sigma(Y^i|s^i,x^i)}{Q^i_{\theta^i}(Y^i|s^i,x^i)}\right]\sigma^i(x^i|s^i)p_{S^i}(s^i).$$

We use $\Theta^i(\sigma) = \underset{\theta^i\in\Theta^i}{argmin} K^i(\sigma,\theta^i)$, to represent the perception that is closest to the real environment, given strategy profile is σ.

A.2.3 Equilibrium Concept

For player i, A strategy σ^i is optimal given $\mu^i \in \Delta(\Theta^i)$ if $\sigma^i(x^i|s^i) > 0$ iff

$$x^i \in \arg\max_{x^i\in X^i} E_{\tilde{Q}^i_{\mu^i}(\cdot|s^i,x^i)}\left[\pi^i(x^i,y^i)\right],$$

with

$$\tilde{Q}_{\mu^i}(\cdot|s^i,x^i) = \int_{\Theta^i} Q^i_{\theta^i}(\cdot|s^i,x^i)\mu^i(d\theta^i)$$

Definition 2. A strategy profile σ is a *Berk–Nash equilibrium* of game $\mathcal{G}$ if, for all players $i \in I$, there exists $\mu^i \in \Delta(\Theta^i)$ such that

(i) σ^i is optimal given μ^i, and

(ii) $\mu^i \in \Delta(\phi^i(\sigma))$, that is, if $\hat{\theta}^i$ is in the support of μ^i, then

$$\hat{\theta}^i \in \underset{\theta^i\in\Theta^i}{\operatorname{argmin}} K^i(\sigma,\theta^i).$$

B Proofs

In the following, if $v \in \mathbb{R}^N$ is a vector, then $\|v\|$ denotes its l^2-norm. If Φ is a linear operator, $\|\Phi\|$ denotes its operator norm with respect to l^2:

$$\|\Phi\| := \sup_{\|v\|=1} \|\Phi v\|.$$

In case P is a projection operator, then $\|P\| = 1$. Some useful facts are

$$\|\Phi+\Psi\| \le \|\Phi\| + \|\Psi\| \text{ and } \|\Phi\circ\Psi\| \le \|\Phi\|\,\|\Psi\|$$

for any two linear operators.

B.1 Proof of Lemma 1

The Bayesian updating with normally distributed prior is a well-known fact. We give the proof here for completion.

Given that the prior of g is distributed $N \sim (\mu_i, \Sigma_i)$, we can rewrite g as $g = \mu_i + e_i$ with $e_g \sim N(0, \Sigma_i)$. Trader i believes that the market price P_1 is distributed as $P_1 = Xg + e$ with $e \sim N(0, \Sigma_e)$. By Bayes's rule, after observing P_1, her posterior belief over g is

$$\begin{aligned}
&Pr(g|P_1) \propto Pr(P_1|g)Pr(g) \\
=&\exp\left[-\frac{1}{2}((P_1 - Xg)^T\Sigma_e^{-1}(P_1 - Xg) + (g-\mu_i)^T\Sigma_i^{-1}(g-\mu_i))\right] \\
=&\exp\left[-\frac{1}{2}(P_1^T\Sigma_e^{-1}P_1 - P_1^T\Sigma_e^{-1}Xg - g^TX'\Sigma_e^{-1}y_1 + g^TX'\Sigma_e^{-1}Xg)\right] \\
&\times\exp\left[-\frac{1}{2}(g^T\Sigma_i^{-1}g - g^T\Sigma_i^{-1}\mu_i - \mu_i^T\Sigma_i^{-1}g + \mu_i^T\Sigma_i^{-1}\mu_i)\right] \\
=&\exp\left[-\frac{1}{2}(g-Xb)^TB^{-1}(g-Xb) + (P_1^T\Sigma_e^{-1}P_1 + \mu_i^T\Sigma_i^{-1}\mu_i)\right] \\
\propto&\exp\left[-\frac{1}{2}(g-Xb)^TB^{-1}(g-Xb)\right]
\end{aligned}$$

with

$$\begin{aligned}
B^{-1} &= X'\Sigma_e^{-1}X + \Sigma_i^{-1} \\
b &= X'\Sigma_e^{-1}P_1 + \Sigma_i^{-1}\mu_i.
\end{aligned}$$

Therefore, the posterior belief is also normally distributed with mean b and variance B. By iterations of the one-period updating rule we just proved, we easily obtain (2.3) to (2.4).

B.2 Proof of Lemma 2

Recall the market price of asset j at time T is given by

$$\sum_{i=1}^{M}\frac{(X_i\mu_i^T)_j - P_{T,j}}{A_i(1+(X_i\Sigma_i^TX_i')_j)} + \sum_{k=K+1}^{K+M}\frac{q_{k,j} - P_{T,j}}{A_k} + \xi_j^T = 0. \tag{B.1}$$

and the efficient price $P_{T,j}^*$ is the solution of

$$\sum_{i=1}^{M}\frac{(X_i\mu_i^T)_j - P_{T,j}^*}{A_i(1+(X_i\Sigma_i^TX_i')_j)} + \sum_{k=K+1}^{K+M}\frac{q_{k,j} - P_{T,j}^*}{A_k} = 0. \tag{B.2}$$

By subtracting (B.1) from (B.2), we have

$$\sum_{i=1}^{M}\frac{P_{T,j} - P_{T,j}^*}{A_i(1+(X_i\Sigma_i^TX_i')_j)} + \sum_{k=K+1}^{K+M}\frac{P_{T,j} - P_{T,j}^*}{A_k} = \xi_j^T.$$

We then simplify this to

$$e_{T,j} = P_{T,j} - P_{T,j}^* = \xi_j^T\frac{1}{\sum_{i=1}^{M}\frac{1}{A_i(1+(X_i\Sigma_i^TX_i')_j)} + \sum_{k=K+1}^{K+M}\frac{1}{A_k}}.$$

The first observation is that $e_{T,j}$s are independent across both T and j as ξ_j^Ts are independent. Furthermore, from (1) and Assumption 6, we note that

$$\left\|\Sigma_i^T\right\| = \left\|\left(T+\lambda_i\right)^{-1} I\right\|^{-1} = O(\frac{1}{T})$$

so that

$$e_{T,j} = (1+O(\frac{1}{T}))\xi_j^T \frac{1}{\sum_{i=1}^{M}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}.$$

It then follows from the distribution of ξ_j^T that $e_{T,j}$ is distributed as $N(0, 1+O(\frac{1}{T}))$.

B.3 Proof of Proposition 1

As a stationary equilibrium price uniquely pins down a stationary equilibrium, it suffices to show there is a unique solution to (3.1). We rewrite (3.1) as

$$\left(\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i}\text{Proj}_i(\cdot) - I\right)P = -\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{k=K+1}^{K+M}\frac{1}{A_k}q_k \tag{B.3}$$

We claim that

$$Q := \left(\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i}\text{Proj}_i(\cdot) - I\right),$$

is an invertible matrix, then there is a unique solution to $(B.3)$. Assume that x is such that $Qx = 0$. Then

$$\begin{aligned}
\|x\| &= \left\|\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i}\text{Proj}_i(x)\right\| \\
&\leq \frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i}\|\text{Proj}_i(x)\| \\
&\leq \frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i}\|x\|
\end{aligned}$$

Since there is at least one long-run investor, the coefficient $\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{1}{A_i} < 1$. For the inequality to hold, we must have $x = 0$ so that Q is invertible.

B.4 Proof of Proposition 2

Recall from (3.2) that a price P is a stationary equilibrium price if and only if

$$\frac{1}{\sum_i \frac{1}{A_i}}\sum_{i=1}^{K}\frac{1}{A_i}\text{Proj}_i(P) = P \tag{B.4}$$

For such a P satisfying (B.4), we have

$$\begin{aligned}\sum_i^K \frac{1}{A_i}\|P\| &= \left\|\sum_{i=1}^K \frac{1}{A_i}\mathrm{Proj}_i(P)\right\| \\ &\le \sum_{i=1}^K \frac{1}{A_i}\|\mathrm{Proj}_i(P)\| \\ &\le \sum_{i=1}^K \frac{1}{A_i}\|P\|,\end{aligned}$$

where the two inequalities are equalities iff $\mathrm{Proj}_i(P) = P$. In other words, $P \in \cap_i \mathrm{Span}(X_i)$.

On the other hand, if $P \in \cap_i \mathrm{Span}(X_i)$, then it is straightforward to check (B.4) is true, the so set of equilibrium prices is exactly $\cap_i \mathrm{Span}(X_i)$.

C Proof of convergence theorems

C.1 Recursive formula for the price

We first prove the following central limit theorem for the market noise $\{e_T\}$, which is asymptotically i.i.d.

Lemma 4. *We have*

$$\frac{1}{T}(e_1 + \cdots + e_T) \to 0 \text{ a.s.}$$

Furthermore, $\left\|\left\|\frac{1}{T}(e_1 + \cdots + e_T)\right\|\right\|_{L^2} = O(\frac{1}{\sqrt{T}})$.

Proof. On each entry j, note that

$$\sum_{k=1}^T \frac{Var(e_{k,j})}{k^2} = \sum_{k=1}^T \frac{(1+O(\frac{1}{k}))^2}{k^2} = O\left(\sum_{k=1}^T \frac{1}{k^2}\right) < \infty.$$

Therefore, by Kolmogorov's SLLN, we have $\frac{1}{T}(e_{1,j} + \cdots + e_{T,j}) \to 0$ a.s.

For the second part, again entrywise,

$$Var\left(\frac{1}{T}(e_{1,j} + \cdots + e_{T,j})\right) = \sum_{k=1}^T Var\left(\frac{1}{T}e_{k,j}\right) = \sum_{k=1}^T \frac{1}{T^2}(1 + O(\frac{1}{T})) = O(\frac{1}{T}).$$

Since l_2-norm and l_1-norm are equivalent, we have

$$\begin{aligned}
&\left\|\left\|\frac{1}{T}(e_{1,j}+\cdots+e_{T,j})\right\|\right\|_{L^2} \\
=\;& O\left(\left\|\sum_{j=1}^{N}\left|\frac{1}{T}(e_{1,j}+\cdots+e_{T,j}\right|\right\|_{L^2}\right) \\
=\;& O\left(N\cdot\left\|\frac{1}{T}(e_{1,j}+\cdots+e_{T,j})\right\|_{L^2}\right) \\
=\;& O\left(\sqrt{Var\left(\frac{1}{T}(e_{1,j}+\cdots+e_{T,j})\right)}\right) \\
=\;& O(\frac{1}{\sqrt{T}}).
\end{aligned}$$

□

To proceed with deriving the recursive formula for the price, we rewrite (B.2) into the following:

$$P^*_{T+1,j}=\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i(1+(X_i\Sigma_i^TX_i')_j}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\left[\sum_{i=1}^{K}\frac{(X_i\mu_i^T)_j}{A_i(1+(X_i\Sigma_i^TX_i')_j)}+\sum_{k=K+1}^{K+M}\frac{q_{k,j}}{A_k}\right].$$

from (1) and Assumption 6, we know

$$\mu_i^T=(1+\frac{\lambda_i}{T})^{-1}(X_i'\bar{P}_T+\frac{\lambda_i}{T}\mu_i)$$

and

$$\Sigma_i^T=(T+\lambda_i)^{-1}I.$$

We thus write the evolution of price in a recursive manner:

$$P^*_{T+1}=L_{T+1}(\bar{P}_T)+r_T, \tag{C.1}$$

where L_{T+1} is the linear operator with each entry given by

$$(L_{T+1}(P))_j:=\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i(1+(X_i\Sigma_i^TX_i')_j)}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\sum_{i=1}^{K}\frac{(X_i(1+\frac{\lambda_i}{T})^{-1}X_i'P)_j}{A_i(1+(X_i\Sigma_g^TX_i')_j)},$$

and r_T is the constant vector

$$(r_T)_j:=\frac{1}{\sum_{i=1}^{K}\frac{1}{A_i(1+(X_i\Sigma_i^TX_i')_j)}+\sum_{k=K+1}^{K+M}\frac{1}{A_k}}\left[\sum_{i=1}^{K}\frac{(X_i(1+\frac{\lambda_i}{T})^{-1}\frac{\lambda_i}{T}\mu_i)}{A_i(1+(X_i\Sigma_i^TX_i')_j)}+\sum_{k=K+1}^{K+M}\frac{q_{k,j}}{A_k}\right].$$

As $T \to \infty$, we have $\Sigma_i^T = (T + \lambda_i)^{-1} I \to 0$ so that

$$\begin{aligned} L_T &\to L_\infty := \frac{1}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{A_k}} \sum_{i=1}^K \frac{X_i X_i'}{A_i} \\ &= \frac{1}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{A_k}} \sum_{i=1}^K \frac{1}{A_i} \mathrm{Proj}_{X_i} \end{aligned}$$

in the operator norm and

$$r_T \to r := \frac{1}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{B_i}} \sum_{k=K+1}^{K+M} \frac{q_{k,j}}{A_k}.$$

We need finer estimates on how quick the convergence of L_T and r_T takes place. The following results can be easily derived using the asymptotic bound $\left\|\Sigma_i^T\right\| = O(\frac{1}{T})$ and $\left\|\Sigma_i^T - \Sigma_i^{T+1}\right\| = O(\frac{1}{T^2})$.

Claim 1. *We have*

$$\begin{aligned} \|L_\infty - L_T\| &= O(\frac{1}{T}) \\ \|L_{T+1} - L_T\| &= O(\frac{1}{T^2}) \\ \|r_T\| &= O(\frac{1}{T}) \\ \left(r_T - \frac{T-1}{T} r_{T-1}\right) &= O(\frac{1}{T^2}) \end{aligned}$$

We also need a lemma that gives a sufficient condition for a sequence to vanish in the limit.

Lemma 5. *Suppose $\{a_i\}$ is a sequence of non-negative numbers satisfying*

$$a_{t+1} \le (c + o(1)) \left(\frac{1}{t} \sum_{i=1}^t a_i\right) + o(1), \tag{C.2}$$

for some $c < 1$ and any t, then $a_i \to 0$.

Proof. We first show that the sequence is uniformly bounded. We let D be large enough to satisfy

$$D \ge (c + \frac{1}{D})D + \frac{1}{D}.$$

There exists some T such that for all $t > T$, it holds that

$$a_{t+1} \le (c + \frac{1}{D}) \frac{1}{t} \sum_{i=1}^t a_i + \frac{1}{D}.$$

Then use induction on t starting from T, we get

$$a_t \leq \sup_{1\leq i\leq T} a_i + D$$

for any t.

We then let $S := \limsup_t a_t < \infty$. Taking limsup on both sides of (C.2) yields

$$S \leq cS.$$

Therefore, $S = 0$ and equivalently, $a_i \to 0$. □

C.2 Proof of Theorem 1

In the presense of long-run investors, we note that

$$\begin{aligned}
\|L_\infty\| &\leq \frac{1}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{A_k}} \sum_{i=1}^K \left\| \frac{X_i X_i'}{A_i} \right\| \\
&= \frac{1}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{A_k}} \sum_{i=1}^K \left\| \frac{\text{Proj}_{X_i}}{A_i} \right\| \\
&= \frac{\sum_{i=1}^K \frac{1}{A_i}}{\sum_{i=1}^K \frac{1}{A_i} + \sum_{k=K+1}^{K+M} \frac{1}{A_k}} < 1.
\end{aligned}$$

We let P_e denote the unique stationary equilibrium price as given by Proposition 1. Then P_e satisfies

$$P_e = L_\infty(P_e) + r.$$

We then write

$$\begin{aligned}
P^*_{T+1} - P_e &= L_{T+1}(\bar{P}_T) - L_\infty(P_e) + r_T - r \\
&= \left[L_{T+1}(\bar{P}_T) - L_{T+1}(P_e) - L_{T+1}(\frac{1}{T}(e_1 + \cdots + e_T)) \right] \\
&\quad + (L_{T+1}(P_e) - L_\infty(P_e)) + L_{T+1}(\frac{1}{T}(e_1 + \cdots + e_T)) + (r_T - r) \\
&= L_{T+1}\left(\frac{1}{T} \sum_{i=1}^T (P^*_i - P_e) \right) \\
&\quad + (L_{T+1} - L_\infty)(P_e) + L_{T+1}(\frac{1}{T}(e_1 + \cdots + e_T)) + (r_T - r) \qquad \text{(C.3)}
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\| P^*_{T+1} - P_e \right\| &\leq \|L_{T+1}\| \left(\frac{1}{T} \sum_{i=1}^T \|P^*_i - P_e\| \right) \\
&\quad + \|L_{T+1} - L_\infty\| \|P_e\| + \|L_{T+1}\| \left\| \frac{1}{T}(e_1 + ... + e_T) \right\| + \|r_T - r\|.
\end{aligned}$$

By the law of large numbers, $\left\|\frac{1}{T}(e_1 + ... + e_T)\right\| \to 0$ a.s. On the event that $\left\|\frac{1}{T}(e_1 + ... + e_T)\right\| \to 0$, which happens with probability 1 by Lemma 4, we have

$$\begin{aligned}\left\|P^*_{T+1} - P_e\right\| &\leq (\|L_\infty\| + o(1))\left(\frac{1}{T}\sum_{i=1}^{T}\|P^*_i - P_e\|\right) \\ &\quad + o(1) + o(1) + o(1).\end{aligned}$$

It follows immediately from Lemma 5 that $\|P^*_T - P_e\| \to 0$.

The average of realized market prices

$$\bar{P}_T = \frac{1}{T}\sum_{i=1}^{T} P^*_T + \frac{1}{T}\sum_{i=1}^{T} e_i$$

then converges to P_e on the event that $\left\|\frac{1}{T}(e_1 + ... + e_T)\right\| \to 0$. Thus, trader i's posterior mean over LFM coefficients is

$$\mu_i^T = (1 + \frac{\lambda_i}{T})^{-1}(X_i'\bar{P}_T + \frac{\lambda_i}{T}\mu_i) \to X_i'P_e.$$

Recall that the variance of her subjective belief over LFM coefficients is $\Sigma_i^T = (T + \lambda_i)^{-1} I$, which converges to 0 as $T \to \infty$. Thus, trader i's belief distribution of LFM coefficients converges to $\delta_{X_i'P_e}$, and her market forecast converges to $\delta_{\mathrm{Proj}_{X_i}(P_e)}$ a.s.

To prove the convergence of P^*_T in L^2, we let $a_T = \|P^*_T - P_e\|_{l^2}$. Then (C.3) gives

$$a_T \leq \|L_{T+1}\|\left(\frac{1}{T}\sum_{i=1}^{T} a_i\right) + \|L_{T+1} - L_\infty\|\,\|P_e\| + \|L_{T+1}\|\left\|\frac{1}{T}(e_1 + \cdots + e_T)\right\| + \|r_T - r\|. \tag{C.4}$$

Taking the L^2-norm of (C.4) and evoking Lemma 4, we have

$$\begin{aligned}\|a_T\|_{L^2} &\leq (\alpha + O(\frac{1}{T}))\left(\frac{1}{T}\sum_{i=1}^{T}\|a_i\|_{L^2}\right) + O(\frac{1}{T})\|P_e\| + (\|L_\infty\| + O(\frac{1}{T}))\sqrt{T} + O(\frac{1}{T}) \\ &= (\alpha + O(\frac{1}{T}))\left(\frac{1}{T}\sum_{i=1}^{T}\|a_i\|_{L^2}\right) + O(\frac{1}{\sqrt{T}})\end{aligned}$$

We apply Lemma 5 to the sequence $\{\|a_T\|_{L^2}\}$ to get $\|a_T\|_{L^2} \to 0$. Therefore, we can further simplify the last inequality to

$$\|a_T\|_{L^2} \leq \alpha\frac{1}{T}\sum_{i=1}^{T}\|a_i\|_{L^2} + O(\frac{1}{\sqrt{T}}).$$

We let $\beta = \min\{\frac{1}{2}, \alpha\}$. Then the following is also true:

$$\|a_T\|_{L^2} \leq \beta\frac{1}{T}\sum_{i=1}^{T}\|a_i\|_{L^2} + O(\frac{1}{\sqrt{T}}).$$

Let C be large enough such that

$$\|a_T\|_{L^2} \le \beta \frac{1}{T} \sum_{i=1}^{T} \|a_i\|_{L^2} + C\frac{1}{\sqrt{T}}$$

for any T. Let N be such that $\frac{\ln t}{t^\beta}$ is decreasing for $t \ge N$, and let D be such that

$$D \ge \sum_{i=1}^{N} \|a_i\|_{L^2}$$

and for any $T \ge N$,

$$D\frac{\ln T}{T^\beta} + (1-\beta)\frac{1}{T}D - \frac{DT^{-\beta}}{1-\beta} + C\frac{1}{\sqrt{T}} \le D\frac{\ln(T+1)}{(T+1)^\beta}.$$

Such a D exists because asymptotically

$$\left(\frac{\ln T}{T^\beta} - \frac{\ln(T+1)}{(T+1)^\beta}\right) = o(T^{-\beta}), C\frac{1}{\sqrt{T}} = O(T^{-\beta}).$$

We claim that for $T \ge N$, we have $a_T \le D\frac{\ln T}{T^\beta}$. Suppose that this has been proven up to some T', then

$$\begin{aligned}
\|a_{T'+1}\|_{L^2} &\le (1-\beta)\frac{1}{T'}\sum_{i=1}^{T'} \|a_i\|_{L^2} + C\frac{1}{\sqrt{T'}} \\
&\le (1-\beta)\frac{1}{T'}D + (1-\beta)D\frac{1}{T'}\sum_{t=N}^{T'} \frac{\ln t}{t^\beta} + C\frac{1}{\sqrt{T'}} \\
&= (1-\beta)\frac{1}{T'}D + (1-\beta)D\frac{1}{T'}\int_{t=N}^{T'} \frac{\ln t}{t^\beta} dt + C\frac{1}{\sqrt{T'}} \\
&\le (1-\beta)\frac{1}{T'}D + (1-\beta)D\frac{1}{T'}\left(\frac{1}{1-\beta}T'^{1-\beta}\ln T' - \frac{T'^{1-\beta}}{(1-\beta)^2}\right) + C\frac{1}{\sqrt{T'}} \\
&\le D\frac{\ln T'}{T'^\beta} + (1-\beta)\frac{1}{T'}D - \frac{DT'^{-\beta}}{1-\beta} + C\frac{1}{\sqrt{T'}} \\
&\le D\frac{\ln(T'+1)}{(T'+1)^\beta},
\end{aligned}$$

completing the induction step.

C.3 Proof of Theorem 2

Since the proof is involved, we make an outline of the proof below.

Step 1. We study a sequence of vectors $\{a_t\}$ satisfying some random recursive relations that are slightly simpler than the dynamics of $\{P_T^*\}$. Under some regularity conditions, we prove that $\|a_t\| = O(\ln t)$. Under stronger conditions, we prove that $\{a_t\}$ converges a.s.

Step 2. We prove that L_∞ is an identity operator on $\cap\text{Span}(X_i)$ and is a strict contraction on $(\cap\text{Span}(X_i))^\perp$.

Step 3. We decompose each P_T^* into $A_T^* + B_T^*$, where each $A_T^* \in \cap\text{Span}(X_i)$ and $B_T^* \in (\cap\text{Span}(X_i))^\perp$. We apply the first part of the lemma in Step 1 to $\{P_T^*\}$ to show that $\|A_T^*\| = O(\ln t)$. Then from the recursive relation of $\{P_T^*\}$ we obtain a recursive relation of $\{B_T^*\}$ and prove that $B_T^* \to 0$ a.s.

Step 4. We write down the dynamics of A_T^* recursively and apply the second part of the lemma in Step 2 to show that A_T^* converges a.s.

The main text of the proof starts from here.

Step 1:

Lemma 6. *Let $\{\Gamma_t\}_{t=1}^\infty, \{\Phi_t\}_{t=1}^\infty, \{\zeta_t\}_{t=1}^\infty$ be deterministic linear operators and $\{c_t\}_{t=1}^\infty$ be deterministic vectors such that $\|\Gamma_t\| = 1 + O(\frac{1}{t^2})$, $\|\Phi_t\| = O(\frac{1}{t})$, $\|\zeta_t\| = O(\frac{1}{t^2})$ and $c_t = O(\frac{1}{t^2})$. Let $\{e_t\}_{t=1}^\infty$ be a sequence of independent random variables with e_t distributed as $N(0, 1 + O(\frac{1}{t}))$. If a_1 is a constant vector and $\{a_t\}$ is a sequence of vector-valued random variables satisfying the recursive formula*

$$a_{t+1} = \Gamma_t a_t + \Phi_t e_t + \zeta_t \left[\frac{1}{t}(a_1 + \cdots + a_{t-1}) + \frac{1}{t}(e_1 + \cdots + e_{t-1})\right] + c_t.$$

Then for all almost every realization of $\{e_i\}$, we have $\|a_t\| = O(\log t)$. If in addition that $\|\Gamma_t - Id\| = O(\frac{1}{t^2})$, then a_t converges to a finite vector-valued random variable a.s.

Proof. Let $C > 1$ be such that those coefficients satisfy $\|\Gamma_t\| - 1, \|\zeta_t\|, \|c_t\| \le \frac{C}{t^2}, \|\Phi_t\| \le \frac{1}{t}$ and $\|a_1\| \le C$. Let

$$C' := C\prod_{k=1}^{\infty}(1 + \frac{3C}{k^2}) < \infty$$

We first prove that there exists a deterministic constant C', linear operators $(x_i^j)_{0<i<j}$ and vectors $\left\{x_0^j\right\}_j$ such that

$$a_t = x_0^t + x_1^t e_1 + \cdots + x_{t-1}^t e_{t-1},$$

where for $i \ge 1$,

$$\left\|x_i^t\right\| \le \frac{C}{i}\prod_{k=1}^{t-1}(1 + \frac{3C}{k^2}) \le \frac{1}{i}C'$$

and

$$\left\|x_0^t\right\| \le C\prod_{k=1}^{t-1}(1 + \frac{3C}{k^2}) \le C'.$$

Basis: $t = 1$. $x_0^1 = \|a_1\| \le C$. Suppose that the bounds on coefficients hold for $a_1, \ldots, a_t$. We now bound the coefficients in front of a_{t+1}. Immediately from the recursive relation we

have

$$\begin{aligned}
\left\|x_0^{t+1}\right\| &\leq \left\|(1+\Gamma_t)x_0^t\right\| + \sum_{j=1}^{t-1}\left\|\zeta_t\frac{1}{t}x_0^j\right\| + \|c_t\| \\
&\leq (1+\frac{C}{t^2})C\prod_{k=1}^{t-1}(1+\frac{3C}{k^2}) + \frac{C}{t^2}\frac{1}{t}\sum_{j=1}^{t-1}C\prod_{k=1}^{j-1}(1+\frac{3C}{k^2}) + \frac{C}{t^2} \\
&\leq (1+\frac{C}{t^2})C\prod_{k=1}^{t-1}(1+\frac{3C}{k^2}) + \frac{C}{t^2}C\prod_{j=1}^{t-1}(1+\frac{3C}{j^2}) + \frac{C}{t^2} \\
&\leq C\prod_{k=1}^{t}(1+\frac{3C}{k^2}).
\end{aligned}$$

Similarly, for $i = 1, \ldots, t-1$, we have

$$\begin{aligned}
\left\|x_i^{t+1}\right\| &\leq \left\|(1+\Gamma_t)x_0^t\right\| + \sum_{j=i+1}^{t-1}\frac{1}{t}\left\|\zeta_t x_i^j\right\| + \frac{1}{t}\left\|\zeta_t\right\| \\
&\leq (1+\frac{C}{t^2})\frac{C}{i}\prod_{k=1}^{t-1}(1+\frac{3C}{k^2}) + \sum_{j=i+1}^{t-1}\frac{1}{t}\frac{C}{t^2}\frac{C}{i}\prod_{k=1}^{j-1}(1+\frac{3C}{k^2}) + \frac{C}{i}\frac{C}{t^2} \\
&\leq (1+\frac{C}{t^2})\frac{C}{i}\prod_{k=1}^{t-1}(1+\frac{3C}{k^2}) + \frac{C}{t^2}\frac{C}{i}\prod_{k=1}^{t-1}(1+\frac{3C}{k^2}) + \frac{C}{i}\frac{C}{t^2} \\
&\leq \frac{C}{i}\prod_{k=1}^{t}(1+\frac{3C}{k^2})
\end{aligned}$$

And lastly,

$$\left\|x_t^{t+1}\right\| = \|\Phi_t\| \leq \frac{C}{t} \leq \frac{C}{t}\prod_{k=1}^{t}(1+\frac{3C}{k^2}).$$

This completes the induction step.

From the bounds of the coefficients, we notice for any realization of $\{e_i\}$ that

$$\|a_t\| \leq \left\|x_0^t\right\| + \sum_{i=1}^{t-1}\left\|x_i^t\right\|\|e_i\| \leq C'\left(1+\sum_{i=1}^{t-1}\frac{1}{i}\|e_i\|\right).$$

We let $\mu_i := \mathbb{E}\|e_i\|$ and define

$$b_t := \sum_{i=1}^{t-1}\frac{1}{i}\left(\|e_i\| - \mu_i\right).$$

Then every b_t is a sum of independent terms $\frac{1}{i}\left(\|e_i\| - \mu_i\right)$ with

$$\sum_{i=1}^{\infty}\mathbb{E}\left(\frac{1}{i}\left(\|e_i\| - \mu_i\right)\right) = 0$$

and

$$\begin{aligned}\sum_{i=1}^{\infty} Var\left(\frac{1}{i}\left(\|e_i\| - \mu_i\right)\right) &= \sum_{i=1}^{\infty} \frac{1}{i^2} Var(\|e_i\|) \\ &= \sum_{i=1}^{\infty} \frac{1}{i^2} \mathbb{E}\left(\sum_{j=1}^{N} e_{i,j}^2\right) \\ &= N \sum_{i=1}^{\infty} \frac{1}{i^2} O(1 + \frac{1}{i}) \\ &< \infty.\end{aligned}$$

By the Kolmogorov's two-series theorem, b_t converges a.s. On the event that b_t converges,

$$O(1) = b_t = \sum_{i=1}^{t-1} \frac{1}{i}\left(\|e_i\| - \mu\right) = \sum_{i=1}^{t-1} \frac{1}{i}\|e_i\| - O(\ln t)$$

so that $\sum_{i=1}^{t-1} \frac{1}{i}\|e_i\| = O(\ln t)$ as desired.

Next we prove that $t \mapsto x_i^t$ converges when $\|\Gamma_t\| = 1 + O(\frac{1}{t^2})$. In fact, for $t > i > 0$, we have

$$\begin{aligned}\left\|x_i^{t+1} - x_i^t\right\| &\leq \left\|(\Gamma_t - Id)x_i^t\right\| + \sum_{j=1}^{t-1} \frac{1}{t}\left\|\zeta_t x_i^j\right\| + \frac{1}{t}\|\zeta_t\| \\ &\leq \frac{C}{t^2}\frac{C'}{i} + \frac{C}{t^2}\frac{C'}{i} + \frac{1}{t}\frac{C}{t^2} \\ &= O(\frac{1}{it^2}).\end{aligned} \tag{C.5}$$

Similarly, the constant terms satisfy

$$\begin{aligned}\left\|x_0^{t+1} - x_0^t\right\| &\leq \left\|(\Gamma_t - Id)x_0^t\right\| + \sum_{j=1}^{t-1} \frac{1}{t}\left\|\zeta_t x_0^j\right\| + \|c_t\| \\ &\leq \frac{C}{t^2}C' + \frac{C}{t^2}C' + \frac{C}{t^2} \\ &= O(\frac{1}{t^2}).\end{aligned}$$

We let $x_i^\infty := \lim_t x_i^t$. As for $i \geq 1$, we have the bound $\|x_i^t\| \leq \frac{1}{i}C'$, the same bound holds in the limit:

$$\|x_i^\infty\| \leq \frac{1}{i}C'.$$

Similarly,

$$\|x_0^\infty\| \leq C'.$$

Define

$$a_\infty := x_0^\infty + \sum_{i=1}^{\infty} x_i^\infty e_i,$$

then this is a.s. well-defined and finite because $\sum_{i=1}^{\infty} x_i^{\infty} e_i$ is a sum of independent normally distributed random variables satisfying

$$\sum_{i=1}^{\infty} Var(x_i^{\infty} e_i) \leq \sum_{i=1}^{\infty} \frac{1}{i^2} C'^2 (1 + \frac{1}{i}) < \infty$$

We claim that $a_k \to a_\infty$ a.s. We write

$$a_\infty - a_k = \left(x_0^{\infty} - x_0^k\right) + \left(\sum_{i=1}^{k-1} (x_i^{\infty} - x_i^k) e_i\right) + \left(\sum_{i=k}^{\infty} x_i^{\infty} e_i\right).$$

As the infinite sum $\sum_{i=1}^{\infty} x_i^{\infty} e_i$ converges a.s, we have the partial sum $\sum_{i=k}^{\infty} x_i^{\infty} e_i$ converges to 0 a.s. It remains to show $\sum_{i=1}^{k-1} (x_i^{\infty} - x_i^k) e_i$ converges a.s. From (C.5), we have

$$\left\| x_i^{\infty} - x_i^k \right\| = \sum_{t=k}^{\infty} O(\frac{1}{t^2}) \frac{1}{i} = O(\frac{1}{ki}).$$

Then for any $l \in \mathbb{Z}^+$ and $k > l$, we have

$$\left\| \sum_{i=1}^{k-1} (x_i^{\infty} - x_i^k) e_i \right\| \leq \sum_{i=1}^{k-1} O(\frac{1}{ki}) \left\| e_i \right\| \leq \sum_{i=1}^{l} O(\frac{1}{ki}) \left\| e_i \right\| + \sum_{i=l+1}^{k-1} O(\frac{1}{kl}) \left\| e_i \right\|$$

Sending $k \to \infty$, we have

$$\limsup_k \left\| \sum_{i=1}^{k-1} (x_i^{\infty} - x_i^k) e_i \right\| \leq O(\frac{1}{l}) \limsup_k \frac{1}{k} \sum_{i=l+1}^{k} \left\| e_i \right\|.$$

We let $\{f_i\}$ be a sequence of i.i.d. $N(0,1)$ random variables with such that $\|e_i\| \leq 2 \|f_i\|$ for all i large enough; such a sequence exists because $\{e_i\}$ is a sequence of independent random variables with $e_i \sim N(0, 1 + O(\frac{1}{i}))$. Then

$$O(\frac{1}{l}) \limsup_k \frac{1}{k} \sum_{i=l+1}^{k} \left\| e_i \right\| \leq O(\frac{1}{l}) \limsup_k \frac{1}{k} 2 \sum_{i=l+1}^{k} \left\| f_i \right\| = O(\frac{1}{l}) 2\mathbb{E} \left\| f_i \right\| \text{ a.s.}$$

Since l is arbitrary, this implies $\sum_{i=0}^{k-1} (x_i^{\infty} - x_i^k) e_i \to 0$ a.s., and thus $a_k \to a_\infty$ a.s. □

Step 2:

On this step, we prove the following lemma about L_∞.

Lemma 7. *On $\cap Span(X_i)$, $L_\infty = Id$. Furthermore, there exists some constant $c < 1$ such that for any $x \in (\cap Span(X_i))^{\perp}$, we have*

$$\left\| L_\infty x \right\| \leq c \left\| x \right\|$$

and

$$L_\infty x \in (\cap Span(X_i))^{\perp}.$$

Proof. It is straightforward to check $L_\infty = \mathrm{Id}$ on $\cap \mathrm{Span}(X_i)$ and we focus on the rest part of the lemma. We construct an orthonormal basis for $\mathbb{R}^N$ as follows. Let B_0 be an orthonormal basis for $\cap \mathrm{Span}(X_i)$ and let B_1 be an orthonormal basis for $\mathrm{Span}(X_1)^\perp$. For $2 \le n \le K$, let B_n be an orthonormal basis of $\left(\cap_{i=1}^{n-1} \mathrm{Span}(X_i)\right) \cap \mathrm{Span}(X_n)^\perp$. Then B_i's are disjoint from each other and $\cup B_i$ is an orthonormal basis for $\mathbb{R}^N$.

Let $x \in (\cap \mathrm{Span}(X_i))^\perp$ be a vector of unit length. We write $x = \sum_{v \in \cup_{i=1}^K B_i} c_v v$, where $c_v \in \mathbb{R}$. Since

$$\left|\cup_{i=1}^K B_i\right| \le \left|\cup_{i=0}^K B_i\right| = N,$$

and $\sum c_v^2 = 1$, there exists some coefficient c_u with $c_u^2 \ge \frac{1}{N}$. For this $u \in \cup_{i=1}^K B_i$, there exists some j such that $u \in \mathrm{Span}(X_j)^\perp$. Then

$$\left\|X_j X_j' x\right\| \le \left\| X_j X_j' \left(\sum_{v \in \cup_{i=1}^K B_i} c_v v \right) \right\| = \left\| X_j X_j' \left(\sum_{v \in \cup_{i=1}^K B_i, v \ne u} c_v v \right) \right\| \le \sqrt{\sum_{v \in \cup_{i=1}^K B_i, v \ne u} c_v^2} \le \sqrt{\frac{N-1}{N}}.$$

Therefore,

$$\begin{aligned}
\|L_\infty x\| &\le \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K \left\| \frac{X_i X_i'}{A_i} x \right\| \\
&\le \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K \frac{1}{A_i} - \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \frac{1}{A_j} \left(1 - \sqrt{\frac{N-1}{N}}\right) \\
&= 1 - \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \frac{1}{A_j} \left(1 - \sqrt{\frac{N-1}{N}}\right)
\end{aligned}$$

We let

$$c := \max_j \left[1 - \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \frac{1}{A_j} \left(1 - \sqrt{\frac{N-1}{N}}\right) \right] < 1.$$

It then follows that

$$\|L_\infty x\| \le c \|x\|$$

for $x \in (\cap \mathrm{Span}(X_i))^\perp$.

For any $x \in (\cap \mathrm{Span}(X_i))^\perp$ and $y \in \cap \mathrm{Span}(X_i)$, we have

$$\begin{aligned}
\langle L_\infty x, y \rangle &= \left\langle \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K \frac{1}{A_i} \mathrm{Proj}_{X_i}(x), y \right\rangle \\
&= \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \left[\sum_{i=1}^K \frac{1}{A_i} \left\langle \mathrm{Proj}_{X_i}(x), y \right\rangle \right] \\
&= \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \left[\sum_{i=1}^K \frac{1}{A_i} \left\langle x, \mathrm{Proj}_{X_i}(y) \right\rangle \right] \\
&= \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \left[\sum_{i=1}^K \frac{1}{A_i} \langle x, y \rangle \right] \\
&= \langle x, y \rangle \\
&= 0,
\end{aligned}$$

where we use Proj_{X_i} as a Hermitian operator in the third equality and $y \in \cap\text{Span}(X_i)$ in the fourth equality. Hence, $L_\infty x \in (\cap\text{Span}(X_i))^\perp$. □

Step 3:

We are ready to analyze the dynamics of $\{P_T^*\}$. Since there are no long-run investors, the recursive relation of prices (C.1) is simplified to

$$P_{T+1}^* = L_{T+1}(\bar{P}_T) + r_T,$$

where $r_T \in \mathbb{R}^N$ is such that

$$(r_T)_j = \frac{1}{\sum_{i=1}^K \frac{1}{A_i(1+X_i\Sigma_i^T X_i')_j}} \left[\sum_{i=1}^K \frac{1}{A_i(1+X_i\Sigma_i^T X_i')_j} X_i(1+\frac{\lambda_i}{T})^{-1}\frac{\lambda_i}{T}\mu_i\right]$$

and L_{T+1} be the linear operator on $\mathbb{R}^N$ defined by

$$[L_{T+1}(P)]_j := \frac{1}{\sum_{i=1}^K \frac{1}{A_i(1+X_i\Sigma_i^T X_i')_j}} \sum_{i=1}^K \frac{1}{A_i(1+X_i\Sigma_i^T X_i')_j} X_i(1+\frac{\lambda_i}{T})^{-1}(X_i'P)_j.$$

We further write P_{T+1}^* into

$$\begin{aligned}
P_{T+1}^* &= L_{T+1}(\bar{P}_T) + r_T. \\
&= L_{T+1}\left(\frac{1}{T}(P_1 + \cdots + P_T)\right) + r_T \\
&= L_{T+1}\left(\frac{1}{T}(P_T^* + e_T)\right) + L_{T+1}\left(\frac{1}{T}(P_1 + \cdots + P_{T-1})\right) + r_T \\
&= L_{T+1}(\frac{1}{T}P_T^*) + L_{T+1}(\frac{1}{T}e_T) + L_T\left(\frac{1}{T}(P_1 + \cdots + P_{T-1})\right) + \frac{T-1}{T}r_{T-1} \\
&\quad +(L_{T+1} - L_T)\left(\frac{1}{T}(P_1 + \cdots + P_{T-1})\right) + (r_T - \frac{T-1}{T}r_{T-1}). \\
&= (\frac{1}{T}L_{T+1} + \frac{T-1}{T})P_T^* + L_{T+1}(\frac{1}{T}e_T) \\
&\quad +(L_{T+1} - L_T)\left[\frac{1}{T}(P_1^* + \cdots + P_{T-1}^*) + \frac{1}{T}(e_1 + \cdots + e_{T-1})\right] + (r_T - \frac{T-1}{T}r_{T-1}) \qquad \text{(C.16)}
\end{aligned}$$

We recall from Claim 1 that $\|L_{T+1} - L_T\| = O(\frac{1}{T^2})$ and $r_T - \frac{T-1}{T}r_{T-1} = O(\frac{1}{T^2})$. Furthermore, $\|L_\infty\| = 1$ so that $\left\|\frac{1}{T}L_{T+1} + \frac{T-1}{T}Id\right\| = 1 + O(\frac{1}{T^2})$. Therefore, we are able to apply the first part of Lemma 6 on the sequence $\{P_t^*\}$ to get $\|P_t^*\| = O(\log t)$ for almost every realization of $\{e_i\}$.

We apply the orthogonal decomposition to write P_T^* into $P_T^* = A_T^* + B_T^*$, where $A_T^* \in$

$\cap\text{Span}(X_i)$ and $B_T^* \in (\cap\text{Span}(X_i))^\perp$. Then

$$\begin{aligned} P_{T+1}^* &= L_{T+1}\left(\frac{1}{T}(P_1+\cdots+P_T)\right)+r_T \\ &= L_\infty(\frac{1}{T}(P_1+\cdots+P_T))+(L_{T+1}-L_\infty)(\frac{1}{T}(P_1+\cdots+P_T))+O(\frac{1}{T}) \\ &= L_\infty(\frac{1}{T}\sum_{i=1}^T A_i^*+\frac{1}{T}\sum_{i=1}^T B_i^*+\frac{1}{T}\sum_{i=1}^T e_i)+(L_{T+1}-L_\infty)(\frac{1}{T}(P_1+\cdots+P_T))+O(\frac{1}{T}). \end{aligned}$$

Recall from Lemma 7 that L_∞ is the identity map on $\cap\text{Span}(X_i)$ and $(\cap\text{Span}(X_i))^\perp$ is invariant under L_∞. It follows that

$$B_{T+1}^* = L_\infty(\frac{1}{T}\sum_{i=1}^T B_i^*+\frac{1}{T}\text{Proj}_{\cap\text{Span}(X_i)}\sum_{i=1}^T e_i)+\text{Proj}_{\cap\text{Span}(X_i)}(L_{T+1}-L_\infty)(\frac{1}{T}(P_1+\cdots+P_T))+O(\frac{1}{T}).$$

On the event that $\|P_t^*\| = O(\log t)$ and $\frac{1}{T}\sum_{i=1}^T e_i \to 0$, which happens with probability 1, we have

$$\begin{aligned} \left\|\frac{1}{T}(P_1+\cdots+P_T)\right\| &= \left\|\frac{1}{T}(P_1^*+\cdots+P_T^*)+\frac{1}{T}\sum_{i=1}^T e_i\right\| \\ &\le \frac{1}{T}\sum_{i=1}^T\|P_i^*\|+o(1) \\ &= O(\log T)+o(1). \end{aligned}$$

Therefore by taking l^2-norm in the recursive formula of B_T^* and using Lemma 7 as well as estimates in Claim 1, we have

$$\begin{aligned} \left\|B_{T+1}^*\right\| &\le (c+O(\frac{1}{T}))\left(\frac{1}{T}\sum_{i=1}^T\|B_i^*\|+o(1)\right)+O(\frac{1}{T})[O(\log T)+o(1)]+O(\frac{1}{T}) \\ &\le (c+O(\frac{1}{T}))\frac{1}{T}\sum_{i=1}^T\|B_i^*\|+o(1). \end{aligned} \tag{C.7}$$

By applying Lemma 5 to the sequence $\{\|B_i^*\|\}$, we have $\|B_i^*\| \to 0$ a.s., and equivalently, $B_i \to 0$ a.s.

Step 4:

Lastly we prove the convergence of A_T^*. Continuing from (C.6), we write

$$\begin{aligned} P_{T+1}^* &= (\frac{1}{T}L_{T+1}+\frac{T-1}{T})P_T^*+L_{T+1}(\frac{1}{T}e_T) \\ &\quad +(L_{T+1}-L_T)\left[\frac{1}{T}(P_1^*+\cdots+P_{T-1}^*)+\frac{1}{T}(e_1+\cdots+e_{T-1})\right]+(r_T-\frac{T-1}{T}r_{T-1}). \\ &= (\frac{1}{T}L_\infty+\frac{T-1}{T})P_T^*+\frac{1}{T}(L_{T+1}-L_\infty)P_T^*+L_{T+1}(\frac{1}{T}e_T) \\ &\quad +(L_{T+1}-L_T)\left[\frac{1}{T}(P_1^*+\cdots+P_{T-1}^*)+\frac{1}{T}(e_1+\cdots+e_{T-1})\right]+(r_T-\frac{T-1}{T}r_{T-1}). \end{aligned}$$

We let Q denote the projection operator onto $\cap\text{Span}(X_i)$. By Lemma 7, we can write the dynamics of A_T^* recursively as

$$\begin{aligned} A_{T+1}^* &= A_T^* + \frac{1}{T}Q(L_{T+1} - L_\infty)P_T^* + QL_{T+1}(\frac{1}{T}e_T) \\ &\quad + Q(L_{T+1} - L_T)\left[\frac{1}{T}(P_1^* + \cdots + P_{T-1}^*) + \frac{1}{T}(e_1 + \cdots + e_{T-1})\right] + Q(r_T - \frac{T-1}{T}r_{T-1}). \end{aligned}$$

Recall that we know $B_T^* \to 0$ a.s., then on this event, we have

$$\begin{aligned} A_{T+1}^* &= A_T^* + \frac{1}{T}Q(L_{T+1} - L_\infty)A_T^* + QL_{T+1}(\frac{1}{T}e_T) \\ &\quad + Q(L_{T+1} - L_T)\left[\frac{1}{T}(A_1^* + \cdots + A_{T-1}^*) + \frac{1}{T}(e_1 + \cdots + e_{T-1})\right] + Q(r_T - \frac{T-1}{T}r_{T-1}) \\ &\quad + o\left(\left\|\frac{1}{T}Q(L_{T+1} - L_\infty)\right\|\right) + o\left(\|Q(L_{T+1} - L_T)\|\right). \end{aligned}$$

Applying the bounds on the operator norms from Claim 1, we simplify A_{T+1}^* to

$$\begin{aligned} A_{T+1}^* &= A_T^* + \frac{1}{T}Q(L_{T+1} - L_\infty)A_T^* + QL_{T+1}(\frac{1}{T}e_T) \\ &\quad + Q(L_{T+1} - L_T)\left[\frac{1}{T}(A_1^* + \cdots + A_{T-1}^*) + \frac{1}{T}(e_1 + \cdots + e_{T-1})\right] + O(\frac{1}{T^2}) \\ &\quad + o(\frac{1}{T^2}) + o(\frac{1}{T^2}) \end{aligned}$$

Applying Lemma 6 to $\{A_T^*\}$ proves the almost surely pointwise convergence of the sequence. Since $P_T^* = A_T^* + B_T^*$ and we have proven the convergence of both parts, we conclude that P_T^* converges a.s.

C.4 Proof of Example 1

We first claim that for $t \geq 0$,

$$\beta_{t+1} = \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K \frac{\mu_i^t}{A_i}, \tag{C.8}$$

where we define $\mu_i^0 := \mu_i$. Recall from Lemma 1 for trader i that $\Sigma_i^t = (tX'X + \lambda I)^{-1} = (t+\lambda)^{-1}$ and her contingent plan at time $t+1$ is

$$\psi_i^{t+1}(P) = \frac{X\mu_i^t - P}{A_i(1 + \frac{1}{t+\lambda})}.$$

Thus, from the market clearing condition,

$$\sum_{i=1}^K \psi_i^{t+1}(P_{t+1}^*) = \sum_{i=1}^K \frac{X\mu_i^t - P_{t+1}^*}{A_i(1 + \frac{1}{t+\lambda})} = 0$$

so that

$$P^*_{t+1} = \frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K X\mu_i^t = X\left(\frac{1}{\sum_{i=1}^K \frac{1}{A_i}} \sum_{i=1}^K \frac{\mu_i^t}{A_i}\right),$$

thus verifying (C.8).

Given the belief at time $t-1$ and the realized market price P_t, from Lemma 1, the posterior mean is

$$\mu_i^t = (tX'X + \lambda I)^{-1}(X'P_t + (\lambda I + (t-1)X'X)\mu_i^{t-1}).$$

We then write β_{t+1} as:

$$\begin{aligned}
\beta_{t+1} &= \frac{1}{\sum_{i=1}^K} \sum_{i=1}^K \frac{\mu_i^t}{A_i} \\
&= \frac{1}{\sum_{i=1}^K} \sum_{i=1}^K \frac{1}{A_i}(tX'X + \lambda I)^{-1}(X'P_t + (\lambda I + (t-1)X'X)\mu_i^{t-1}) \\
&= (tX'X + \lambda I)^{-1}(X'X\beta_t + X'e_t) + (tX'X + \lambda I)^{-1}(\lambda I + (t-1)X'X\beta_t) \\
&= (tX'X + \lambda I)^{-1}(tX'X\beta_t + \lambda I)\beta_t + (tX'X + \lambda I)^{-1}X'e_t \\
&= \beta_t + (tX'X + \lambda I)^{-1}X'e_t \\
&= \beta_t + \frac{1}{t+\lambda}X'e_t
\end{aligned}$$

verifying the recursive formula of β_t.

Lastly, we compute

$$\begin{aligned}
& \mu_i^t - \beta_{t+1} \\
= \; & \frac{1}{t+\lambda}(X'P_t + (\lambda + (t-1))\mu_i^{t-1}) - (\beta_t + \frac{1}{t+\lambda}X'e_t) \\
= \; & \frac{1}{t+\lambda}(X'P_t^*) + \frac{\lambda + t - 1}{t+\lambda}\mu_i^{t-1} - \beta_t \\
= \; & \frac{1}{t+\lambda}\beta_t + (1 - \frac{1}{t+\lambda})\mu_i^{t-1} - \beta_t \\
= \; & (1 - \frac{1}{t+\lambda})(\mu_i^{t-1} - \beta_t).
\end{aligned}$$

C.5 Proof of Proposition 4

We aim to find a stationary equilibrium price of the form $P = q + a_1 x_{sun} + a_2 x_{alter}$. We write the condition (3.1) into

$$P = \alpha \text{Proj}_{[X, trader]}(P) + \beta q + \gamma(q + c x_{sun}). \tag{C.9}$$

Given the equilibrium price $P = q + a_1 x_{sun} + a_2 x_{alter}$, the trader would find a coefficient e in front of x_{trader} to minimize the distance $\|e x_{trader} - a_1 x_{sun} - a_2 x_{alter}\|$. That is, e minimizes

$$(e\rho - a_1)^2 + (e\sqrt{1-\rho^2} - a_2)^2$$

Taking derivative in e, we have

$$2(e\rho - a_1) + 2(e\sqrt{1-\rho^2} - a_2) = 0.$$

Then

$$e = \frac{a_1 + a_2}{\rho + \sqrt{1-\rho^2}}.$$

The trader's market forecast in an equilibrium is thus

$$q + \frac{a_1 + a_2}{\rho + \sqrt{1-\rho^2}}(\rho x_{sun} + \sqrt{1-\rho^2} x_{alter}).$$

We plug this back to (C.9), we have

$$\begin{aligned} a_1 &= \gamma c + \alpha \frac{a_1 + a_2}{\rho + \sqrt{1-\rho^2}} \rho \\ a_2 &= \alpha \frac{a_1 + a_2}{\rho + \sqrt{1-\rho^2}} \sqrt{1-\rho^2}. \end{aligned}$$

This is a linear system of equations of two variables, and the solution is

$$a_1 = \left[1 - \alpha \frac{\rho}{\rho + \sqrt{1-\rho^2}} \left(1 + \frac{\alpha\sqrt{1-\rho^2}}{\rho + (1-\alpha)\sqrt{1-\rho^2}} \right) \right]^{-1} \gamma c.$$

and

$$a_2 = \frac{\alpha\sqrt{1-\rho^2}}{\rho + (1-\alpha)\sqrt{1-\rho^2}} a_1.$$

C.6 **Proof of Proposition 5**

Since the market noise is independent across assets, the variance of the total return is the sum of variance of individual returns:

$$\begin{aligned} & Var\left(\sum_j \frac{1}{A_i} \left(\text{Proj}_{X_i}(P_e) - (P_e + e) \right)_j (e' - e)_j \right) \\ = & \sum_j Var\left(\frac{1}{A_i} \left(\text{Proj}_{X_i}(P_e) - (P_e + e) \right)_j (e' - e)_j \right) \\ = & \frac{1}{A_i^2} \sum_j \left[\mathbb{E}(\text{Proj}_{X_i}(P_e) - P_e - e)_j^2 (e' - e)_j^2 - (\mathbb{E}e_j^2)^2 \right] \\ = & \frac{1}{A_i^2} \left[2\mathbb{E}e_j^2 \sum_j (\text{Proj}_{X_i}(P_e) - P_e)_j^2 + \mathbb{E}e_j^4. \right] \\ = & \frac{1}{A_i^2} \left(2\mathbb{E}e_j^2 \left\| \text{Proj}_{X_i}(P_e) - P_e \right\|^2 + \mathbb{E}e_j^4 \right) \end{aligned}$$

Hence, the Sharpe ratio is

$$\frac{\frac{1}{A_i}\mathbb{E}\left\|e\right\|^2}{\sqrt{\frac{1}{A_i^2}\left(2\mathbb{E}e_j^2 \left\| \text{Proj}_{X_i}(P_e) - P_e \right\|^2 + \mathbb{E}e_j^4 \right)}} = \frac{\mathbb{E}\left\|e\right\|^2}{\sqrt{2\left\| \text{Proj}_{X_i}(P_e) - P_e \right\|^2 + 3}}.$$

C.7 Proof of Proposition 6 and 7

Proof of Proposition 6. After the realization of $P_T = P_T^* + e_T$, the trader's posterior belief will be updated to

$$N(\frac{P_T + \lambda\mu}{1+\lambda}, \frac{1}{1+\lambda}).$$

so that $\mu' = \frac{P_T+\lambda\mu}{1+\lambda}$. We rewrite this into

$$\frac{1}{1+\lambda}(\mu - P_T) = \mu - \mu'.$$

Given μ', the efficient prize in the next period is

$$P_{T+1}^* = \frac{\frac{1+\lambda}{2+\lambda}\mu' + q}{\frac{1+\lambda}{2+\lambda} + 1}.$$

We can therefore write

$$\begin{aligned}
R_{T,T+1}^{\text{investors}} &= \frac{1}{1+\lambda}(\mu - P_T)(P_{T+1}^* - P_T^*) \\
&= (\mu - \mu')\left(\frac{\frac{1+\lambda}{2+\lambda}\mu' + q}{\frac{1+\lambda}{2+\lambda} + 1} - \frac{\frac{\lambda}{\lambda+1}\mu + q}{\frac{\lambda}{\lambda+1} + 1}\right) \\
&= (\mu - \mu')\left[\frac{(1+\lambda)\mu'}{3+2\lambda} - \frac{\lambda\mu}{2\lambda+1} + \frac{(2+\lambda)q}{3+2\lambda} - \frac{(\lambda+1)q}{2\lambda+1}\right] \\
&= (\mu - \mu')\left[\frac{\lambda}{2\lambda+1}(\mu' - \mu) + \frac{1}{(3+2\lambda)(2\lambda+1)}(\mu' - q)\right].
\end{aligned}$$

Given that $\mu > \mu'$, we have $R_{T,T+1}^{\text{investors}} < 0$ iff

$$\frac{\lambda}{2\lambda+1}(\mu' - \mu) + \frac{1}{(3+2\lambda)(2\lambda+1)}(\mu' - q) < 0.$$

We can equivalently write this into

$$\begin{aligned}
& \lambda(\mu - \mu') + \frac{\mu' - q}{3+2\lambda} < 0 \\
\Leftrightarrow\ & \lambda(\mu - \mu')(3+2\lambda) > \mu' - q \\
\Leftrightarrow\ & \lambda(3+2\lambda)\mu + q > \mu'(1 + \lambda(3+2\lambda)) \\
\Leftrightarrow\ & \lambda(3+2\lambda)\mu + q > \frac{P_T + \lambda\mu}{1+\lambda}(1+\lambda)(1+2\lambda) \\
\Leftrightarrow\ & \lambda(3+2\lambda)\mu + q > (P_T + \lambda\mu)(1+2\lambda) \\
\Leftrightarrow\ & 2\lambda\mu + q > P_T(1+2\lambda) \\
\Leftrightarrow\ & 2\lambda\mu + q > \lambda\mu + (\lambda+1)q + e_T \\
\Leftrightarrow\ & \lambda(\mu - q) > e_T
\end{aligned} \tag{C.10}$$

When $\mu > \mu'$, we have $\mu > P_T$ so that

$$\mu > \frac{\lambda\mu + (\lambda+1)q}{1+2\lambda} + e_T.$$

The above inequality can be simplified to

$$\mu - q > \frac{2\lambda+1}{\lambda+1} e_T,$$

which implies (C.10). Thus, we obtain that $R^{\text{investors}}_{T,T+1} < 0$.

Finally, we notice that $(\mu - \mu')$ is positive and strictly decreasing in μ' and

$$\frac{\lambda}{2\lambda+1}(\mu' - \mu) + \frac{1}{(3+2\lambda)(2\lambda+1)}(\mu' - P^*)$$

is negative and strictly increasing in μ'. Therefore, their product $R^{\text{investors}}_{T,T+1}$ is strictly increasing in μ'. □

Proof of Proposition 7. For investor k, the holding of asset j at time t is

$$\frac{1}{A_k}(q_{k,j} - P_{t,j}).$$

The profits of investor k from trading asset j between period t and $t+1$ is then

$$\begin{aligned} & \frac{1}{A_k}(q_{k,j} - P_{t,j})(P_{t+1,j} - P_{t,j}) \\ = \; & \frac{1}{A_k}(q_{k,j} - \frac{1}{2}P_{t,j} - \frac{1}{2}P_{t+1,j})(P_{t+1,j} - P_{t,j}) + \frac{1}{2A_k}(P_{t+1,j} - P_{t,j})^2 \end{aligned}$$

so that we can lower bound the profits by

$$\frac{1}{A_k}(q_{k,j} - \frac{1}{2}P_{t,j} - \frac{1}{2}P_{t+1,j})(P_{t+1,j} - P_{t,j}) = \frac{1}{2A_k}\left[(q_{k,j} - P_{t,j})^2 - (q_{k,j} - P_{t+1,j})^2\right].$$

The sum of profits of investor k from trading asset j across all the periods is then lower bounded

$$\sum_{t=1}^{T-1} \frac{1}{2A_k}\left[(q_{k,j} - P_{t,j})^2 - (q_{k,j} - P_{t+1,j})^2\right] = \frac{1}{2A_k}\left[(q_{k,j} - P_{1,j})^2 - (q_{k,j} - P_{1,j})^2\right].$$

as desired.

The total profits of investor k from all the assets is then

$$\sum_{k=1}^{N} \frac{1}{2A_k}\left[(q_{k,j} - P_{1,j})^2 - (q_{k,j} - P_{1,j})^2\right] = \frac{1}{2A_k}\left[\|q_k - P_1\|_{l_2}^2 - \|q_k - P_T\|_{l_2}^2\right].$$

To find the total profits of all the investors, we define an aggregated investor to be the investor with the constant belief

$$q = B\sum_k \frac{1}{A_k} q_k$$

and the risk preference parameter

$$B = (\sum_k \frac{1}{A_k})^{-1}.$$

Then for any hypothetical market price P, the aggregated investor's position is

$$\frac{1}{B}(q - P) = \sum_k \frac{1}{A_k} q_k,$$

which is exactly the total holdings of investors in the original model. Therefore, the total profits of all the investors is the same as that in the market where there is an aggregated investor instead.

□