Diversity in Choice as Majorization

Federico Echenique* Teddy Mekonnen† M. Bumin Yenmez‡

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Abstract

We develop a framework of comparative diversity in which one population is more diverse than another if its distribution across identity groups is less concentrated, with concentration formalized by a modification of majorization that accommodates arbitrary distributional objectives. We apply this framework to study how a school should allocate its limited number of seats while balancing priorities and diversity considerations. We provide an axiomatic characterization of choice rules that are optimal: no alternative rule can simultaneously yield a more diverse student body and admit higher-priority students. Our approach has three advantages: majorization delivers a natural notion of diversity; the priority ordering is endogenously recovered rather than exogenously imposed; and compared to reserves and quotas, our choice rules adapt more flexibly to the applicant pool.

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*Department of Economics, UC Berkeley. Contact: fede@econ.berkeley.edu
†Department of Economics, Brown University. Contact: mekonnen@brown.edu
‡Department of Economics, WUSTL and Business School, Durham University (UK). Contact: bumin@wustl.edu.

1. Introduction

Diversity and inclusion are important admissions criteria for many schools. We introduce ideas from the theory of majorization (Hardy et al. (1952); Marshall et al. (2010)), and Schur concavity, to model and measure diversity objectives. In particular, we consider the admissions criteria that emerge from adopting majorization as a criterion for diversity.

Our approach is axiomatic. We propose normative principles, formulated as mathematical axioms, that a school district could decide to either adopt or reject. If the district finds the principles reasonable, then a unique choice procedure is obtained. Our axioms deal with two crucial aspects of school's choices.

The first is a concern for diversity. Many school districts and university admissions offices care deeply about improving the diversity of its student bodies. While such affirmative action policies have been disputed in courts, the current legal framework still provides ways to achieve a school's distributional goals. Recent Supreme Court decisions deal with higher education (*SFFA v. Harvard* and *SFFA v. USC*) and do not necessarily limit the affirmative action policies followed in K-12 education. The 2007 Supreme Court decision (*Parents Involved in Community Schools v. Seattle School District No. 1*) defines a broad framework (through justice Kennedy's opinion) that was used during the Obama administration to guide efforts to pursue diversity goals; see "Guidance on the voluntary use of race to achieve diversity and avoid racial isolation in elementary and secondary schools" by the Department of Education's Civil Rights Division (2011). The guidance was substituted in 2017 with a document from the Department of Education ("Improving Outcomes for All Students: Strategies and Considerations to Increase Student Diversity," 2017) that discusses different policies, and specifically mentions controlled school choice as a viable and legal mechanism to enhance student diversity. In 2023, the American Civil Liberties Union (ACLU) listed twelve policies that can be legally used to address the diversity of student bodies.[1]

Our paper uses majorization to model the comparative diversity of a student body. The connection between majorization and diversity is motivated by a characterization of majorization through so-called Robin Hood transfers. Consider, for example, the distribution of gender or socioeconomic status for a student body. If there are more

[1]See "Moving Beyond the Supreme Court's Affirmative Action Rulings" by S. Hinger on the ACLU's webpage, posted in July 2023.

students of type A than of type B, then a "Robin Hood transfer" decreases the number of type A students and increases the number of type B students *by exactly one* without changing the number of other types or reversing the inequality in favor of type B. Such transfers improve the diversity of the distribution we started from. One can show that if a distribution majorizes another, then the second distribution can be obtained from the first by performing a sequence of Robin Hood transfers and permutations.

Majorization naturally leads to a uniform distribution over types as the ideal distribution. The uniform distribution is restrictive, so we adapt the theory of majorization to account for other ideal distributions. A prominent objective in school choice is to target the population distribution in the relevant geographical community (see Alves and Willie (1987), for example); the population-level racial, or socioeconomic, distribution is usually very different from uniform. Our notion of *r-targeting diversity* encapsulates the resulting flexible distributional objective: it works qualitatively like majorization, but in pursuit of an arbitrary (given) ideal distribution r. The resulting axiom "promotes r-targeting diversity" expresses a preference for diversity as captured by our adapted notion of majorization.

The second aspect is a priority, or merit, ordering over students. Priority refers to a composite score that may reflect, for example, the child's performance on tests, the distance of the child's home from the school, whether the child has an older sibling at the same school, etc. Hence, a desirable property of school choice is that each school should allocate its limited number of seats to children who wish to attend that school and have the highest priorities at the school.

Priority orderings are not assumed exogenously in our result, but rather obtained endogenously from a school's choice procedure. This is important, because it means that our axioms are not parameterized by an arbitrary priority ordering. Instead, the axioms determine a class of choice functions, and each choice function in our class is consistent with some priority ordering over students. The priority ordering can be backed out from the choice rule, in a manner that resembles revealed preference theory. The key axiom "r-targeting Schur revealed preference," imposes the usual acyclicity notion from revealed preference theory but qualified by diversity considerations.

We have three main results: First, we show that an efficiency axiom (nonwastefulness) together with the two axioms we have mentioned (on diversity and priorities) uniquely pin down a choice rule that we term the *r-targeting Schur choice*

rule. Second, we show that the choice rule is optimal in the sense that any choice rule that does not coincide with the r-targeting Schur choice rule must either be wasteful or must admit a student body with less diversity or lower priority. Finally, we show that our r-targeting Schur choice rule coincides with a greedy algorithm that admits students based on priority subject to a diversity constraint.

It is important to note that, while we frame our analysis as a controlled school choice problem, it is not the only salient application of our model and results. Our framework applies to various settings in which diversity considerations play an important role in decision making: when a firm hires employees, when a grant-giving foundation funds proposals, when health providers ration vaccines, or when immigration officials resettle refugees.

1.1. Related Literature

The literature has mostly addressed questions of diversity by taking a mechanism design approach to school choice. This approach was started by Abdulkadiroğlu and Sönmez (2003), who brought the ideas and methods of modern mechanism design to the study of school choice; they also started the formal study of controlled school choice. The literature has since developed around certain models of incorporating diversity considerations into school choice. The focus is, for the most part, on models of school choice via quotas and reserves. See, for example, Abdulkadiroğlu (2005), Kojima (2012), Hafalir et al. (2013), and Ehlers et al. (2014). Within the controlled school choice literature, the paper closest to ours is Echenique and Yenmez (2015), who introduced the axiomatic approach to controlled school choice. They provide characterizations of choice rules based on quotas and reserves using axioms on efficiency and diversity. Relatedly, Imamura (2020) and Abdulkadiroğlu and Grigoryan (2021) also provide axiomatic characterizations of choice rules with reserves and quotas.

We differ from the existing literature in one key aspect. We introduce, and adapt, the theory of majorization as a framework for diversity in school choice. The resulting theory yields a natural diversity criteria. The degree of freedom afforded by the incompleteness of the majorization order allows for priority to play a role whenever majorization does not impose a unique choice decision. Consequently, the r-targeting Schur choice rule that we characterize is more flexible than what is implied by quotas and reserves.

In more recent work, Hafalir et al. (2022) study choice rules based on ordinally concave diversity functions. Ordinal concavity does not represent the majorization order. This is best seen by comparing with Schur concave functions, that are defined by the property of being monotone with respect to majorization. It is easy to find examples that are ordinally but not Schur concave, and vice versa.[2] Whereas Hafalir et al. (2022) focus on choice rules that promote diversity guided by ordinally concave functions, we do not have diversity functions but instead use majorization to provide an incomplete, and rather coarse, diversity ranking of distributions. Therefore, our approach is conceptually different than their approach. However, our Theorem 2 is similar to their result that compares choice rules based on priority. Other priority comparison results have been studied by Sönmez and Yenmez (2022) and Aygün and Turhan (2023). Some of these results, like ours, rely on a matroid structure but the reason for why such a structure arises is very different.

As we noted earlier, our framework is relevant beyond the school choice literature. Thus, the choice rule that we characterize may prove useful in matching markets with regional distributional constraints such as the Japanese residency matching (Kamada and Kojima, 2015), matching public servants with institutions (Thakur, 2018), refugee resettlement (Andersson, 2019; Delacrétaz et al., 2023), and pandemic rationing (Pathak et al., 2022).

Finally, our r-targeting choice rule is equivalent to a greedy algorithm that optimizes priority subject to a majorization constraint. While such an optimization problem may be reminiscent of the recent application of majorization to mechanism and information design, for example, Kleiner et al. (2021), Nikzad (2022), and Arieli et al. (2023), these papers consider a constrained optimization problem where the objective and the constraint are defined over the same domain (in their case, the space of monotone functions). In contrast, our constrained optimization problem features an objective (priority) that is independent of our constraint (diversity). Hence, we see these papers as relying on similar mathematical tools but conceptually distinct.

[2]For example, the *Shannon diversity index* (Shannon, 1948) is Schur concave but not ordinally concave whereas the *saturated diversity index* (Hafalir et al., 2022) is ordinally concave but not Schur concave.

2. Model

2.1. Preliminaries

We use $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$ to denote the set of nonnegative and positive integers, respectively. Similarly, we use $\mathbb{R}_+$ and $\mathbb{R}_{++}$ to denote the set of nonnegative and positive reals. For any $n \in \mathbb{Z}_{++}$ and any vector $x \in \mathbb{R}^n$, let $\langle x \rangle \coloneqq \sum_{i=1}^n x_i$ denote the sum of the coordinates of x, and let $x_{[i]}$ denote the i^{th} largest coordinate of x so that $x_{[1]} \geq \ldots \geq x_{[n]}$. For any $i \in \{1, \ldots, n\}$, let $\chi_i \in \mathbb{Z}_+^n$ denote the unit vector where the i-th coordinate is one and all other coordinates are zero. The $(n-1)$-dimensional simplex in $\mathbb{R}^n$ is written as $\Delta^{n-1} \coloneqq \{x \in \mathbb{R}_+^n : \langle x \rangle = 1\}$.

For any two vectors $x, y \in \mathbb{R}^n$, we write $x \geq y$ if $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$, and we write $x > y$ if $x \geq y$ and $x_i > y_i$ for some $i \in \{1, \ldots, n\}$. Additionally, we say that x *majorizes* y, denoted as $x \succ y$, if

1. $\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}$ for every $k \in \{1, \ldots, n-1\}$, and
2. $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$.

We say that x *strictly majorizes* y, denoted as $x \succ_s y$, if $x \succ y$ but $y \not\succ x$. Finally, for any finite set X, let 2^X denote the power set, and $|X|$ the cardinality, of X.

2.2. Setup

We consider an environment with one school and a finite, nonempty set of students $\mathcal{S}$. The school has a capacity $q \in \mathbb{Z}_{++}$ and a priority ranking P over $\mathcal{S}$. The ranking P is a complete, transitive, and antisymmetric binary relation over $\mathcal{S}$. For example, the priority ranking could be obtained from a composite score for each student that reflects the child's performance on standardized tests, the distance of the child's home from the school, whether the child has an older sibling at the same school, etc. In our main result, the priority ranking is not imposed exogenously; instead, the result characterizes the choice rules consistent with some priority ranking.

The school cares about both the priority ranking and the diversity of its student body. A central element of our model is the classification of students into different types, which can capture attributes such as disability status, ethnicity, gender, race, socioeconomic status, caste, or veteran status. We focus on mutually exclusive types with each student belonging to exactly one of $n \in \mathbb{Z}_{++}$ possible types. Although

the assumption of mutually exclusive types is restrictive, it captures two important settings. First, a school may focus narrowly on diversity along a single attribute, such as socioeconomic status. In this case, each student can be uniquely assigned to one of, for example, three types: low, middle, or high income. Second, a school may adopt a broader approach that considers the intersection of multiple attributes. For instance, if a school values both socioeconomic status and gender, each student can be uniquely assigned to one of six intersectional types: low-income male, middle-income male, high-income male, low-income female, middle-income female, or high-income female.

Given the n possible types, we index each type by $i \in \mathcal{N} \coloneqq \{1, \ldots, n\}$. We can then map each nonempty set of students $S \subseteq \mathcal{S}$ into a vector $\xi(S) \in \mathbb{Z}_+^n$, where $\xi_i(S)$ denotes the number of type-i students in set S. We refer to $\xi(S)$ as the *type distribution* of students in S. Throughout the paper, we identify the space $\mathbb{Z}_+^n$ of nonnegative, integral, n-dimensional vectors with the set of all *feasible* type distributions that can arise in a school's choice problem.

3. Diversity as Majorization

We develop a notion of comparative diversity by evaluating any two student bodies according to their respective type distributions. Formally, we begin by defining a binary relation over the space of all type distributions $\mathbb{R}_+^n$, and subsequently restrict this relation to the subset of feasible type distributions $\mathbb{Z}_+^n$.

As a benchmark, imagine that a maximally diverse student body has an equal fraction of each type of students. Formally, let $u \in \Delta^{n-1}$ denote the uniform measure, with $u_i = 1/n$ for all $i \in \mathcal{N}$. Given $\lambda > 0$ students, the maximally diverse distribution is then given by $\lambda u \in \mathbb{R}_+^n$. Of course, if $\lambda u \notin \mathbb{Z}_+^n$, such a type distribution is infeasible in our school choice setting. Nevertheless, we may treat it as the target distribution that a school with λ students aspires to approximate.

Now consider any type distribution $x \in \mathbb{R}_+^n$ such that $x_i - \langle x \rangle u_i > x_j - \langle x \rangle u_j$ for some types $i, j \in \mathcal{N}$. In words, type i is overrepresented relative to type j with respect to the uniform measure. Let us construct a new distribution $y \in \mathbb{R}_+^n$ by transferring a mass $\delta \geq 0$ from type i to type j while leaving all other types intact, i.e., $y = x + \delta\chi_j - \delta\chi_i$. Since $\langle x \rangle = \langle y \rangle$, the relevant target distribution for both x and y is given by $\langle x \rangle u$. If $\delta \in (0, x_i - x_j)$, the constructed y is closer than x to the target

distribution,[3] which suggests that y is more diverse than x under the uniform-measure benchmark. By contrast, if $\delta = x_i - x_j$, then y is merely a permutation of x in which the mass of types i and j are swapped, suggesting that x and y are equally diverse.

This notion of improving a distribution towards a uniform ideal by transferring mass from a relatively overrepresented type to an underrepresented one—referred to as a Robin Hood transfer by Arnold (2012)—was first suggested by Pigou (1912) and formalized by Dalton (1920) in the context of income inequality. Moreover, Hardy et al. (1952) show that for any two distributions $x, y \in \mathbb{R}^n_+$, it is possible to obtain y from x through a sequence of Robin Hood transfers if and only if x majorizes y. Thus, under a uniform-measure benchmark, majorization provides a natural binary relation for comparing diversity.

For our purposes, however, a notion of diversity based on the uniform measure is too restrictive. For example, a common consideration for a school is to match the type distribution of its student body with the type distribution of a broad population (Alves and Willie, 1987), and the latter is unlikely to have an equal fraction of each type. We, therefore, introduce a generalized notion of diversity that allows for an arbitrary benchmark measure $r \in \Delta^{n-1}$, where r_i may represent, for example, the fraction of type-i individuals in the general population residing within the school's geographical location. Given $\lambda > 0$ students, the maximally diverse distribution is now given by $\lambda r \in \mathbb{R}^n_+$. Once again, such a type distribution is infeasible if $\lambda r \notin \mathbb{Z}^n_+$, but we may treat it as a target distribution for a school with λ students.

Definition 1. Given a measure $r \in \Delta^{n-1}$ and two type distributions $x, y \in \mathbb{R}^n_+$, we say that y is obtained from x via an *r-targeting Robin Hood transfer* if there exist types $i, j \in \mathcal{N}$ with $x_i - \langle x \rangle r_i \geq x_j - \langle x \rangle r_j$, and a constant $0 \leq \delta \leq x_i - x_j - \langle x \rangle (r_i - r_j)$ such that $y = x + \delta\chi_j - \delta\chi_i$.

Definition 1 is a straightforward extension of the Robin Hood transfer from the uniform measure to an arbitrary measure r: the distribution y is constructed by moving mass from some relatively overrepresented type i to an underrepresented type j so that y is (weakly) closer than x to the target distribution $\langle x \rangle r$. It is clear to see that we recover the standard definition of a Robin Hood transfer in the special case when $r = u$.

[3] Formally, $||y - \langle y \rangle u||_2 < ||x - \langle x \rangle u||_2$. Moreover, y is (weakly) closer than x to the target distribution in the ℓ^1-norm, which in our setting is equivalent to the total-variation norm.

Definition 2. Given a measure $r \in \Delta^{n-1}$ and two type distributions $x, y \in \mathbb{R}^n_+$, we say that y is *more r-diverse* than x, denoted by $y \trianglerighteq_r x$, if there exists a finite sequence of distributions $\{z_k\}_{k=0}^K$ with $z_0 = x$, $z_K = y$, and for each $k = \{1, \dots, K\}$, z_k is obtained from z_{k-1} via an r-targeting Robin Hood transfer. Furthermore, we say that y is *strictly more r-diverse* than x, denoted by $y \triangleright_r x$, if $y \trianglerighteq_r x$ and $x \not\trianglerighteq_r y$.

We refer to this notion of comparative diversity as an *r-targeting diversity*. Intuitively, $y \trianglerighteq_r x$ if we can reach y from x through a series of bilateral transfers across types that also moves x closer to the target distribution $\langle x \rangle r$. Notice that our notion of comparative diversity applies only if x and y are equal-sized, i.e., $\langle x \rangle = \langle y \rangle$.

Proposition 1. *Fix a measure $r \in \Delta^{n-1}$, and let $T^r : \mathbb{R}^n \to \mathbb{R}^n$ be given by*

$$T^r(x) \coloneqq x + \langle x \rangle (u - r).$$

For any two type distributions $x, y \in \mathbb{R}^n_+$, y is more r-diverse than x if and only if $T^r(x)$ majorizes $T^r(y)$.

Proposition 1 formalizes the equivalence between r-targeting diversity and generalized majorization. The equivalence also implies that the binary relation $\trianglerighteq_r$ is reflexive and transitive, but it is neither antisymmetric nor complete. Intuitively, the transformation T^r linearly translates distributions so that, given $\lambda > 0$ students, the target distribution under measure r is mapped to the target distribution under the uniform measure, i.e., $T^r(\lambda r) = \lambda u$. Thus, for any distribution $x \in \mathbb{R}^n_+$ with $\langle x \rangle = \lambda$, we have $T^r(x) \succ T^r(\lambda r)$, confirming that λr serves as the maximally diverse distribution for λ students. Moreover, T^r preserves student-body size: for any distribution x, we have $\langle x \rangle = \langle T^r(x) \rangle$. Finally, when $r = u$, T^r reduces to the identity function and we recover the standard equivalence between majorization and Robin Hood transfers with respect to the uniform measure.

The following example illustrates that the choice of r can alter the comparative diversity ordering: distributions that are equivalent or incomparable under u may become strictly ranked once evaluated relative to r.

Example 1. Let $n = 3$. Consider the following three type distributions: $x = (1, 4, 1)$, $y = (4, 1, 1)$, and $z = (3, 0, 3)$. Notice that $x \succ y$ and $y \succ x$ whereas $x \not\succ z$ and $z \not\succ x$. Thus, under the uniform measure, $x \trianglerighteq_u y$ and $y \trianglerighteq_u x$, while x and z are incomparable

to each other in terms of u-targeting diversity. If the target measure is instead $r = (1/6, 1/2, 1/3)$, the transformed distributions are $T^r(x) = (2, 3, 1)$, $T^r(y) = (5, 0, 1)$, and $T^r(z) = (4, -1, 3)$. Notice now that $T^r(z) \succ_s T^r(x)$ and $T^r(y) \succ_s T^r(x)$, whereas $T^r(z) \not\succ T^r(y)$ and $T^r(y) \not\succ T^r(z)$. Hence, under measure r, $x \rhd_r y$ and $x \rhd_r z$, but y and z are now incomparable to each other in terms of r-targeting diversity.

For the remainder of the paper, we fix a target measure $r \in \Delta^{n-1}$, and consider the restriction of the associated r-targeting diversity preorder to the subset of all feasible distributions $\mathbb{Z}^n_+$.

4. School Choice Rules and Diversity

A school can only admit students from among its applicants, so the type distributions it can attain for its admitted student body is constrained by the type distribution of its applicants. If the applicant type distribution is given by $x \in \mathbb{Z}^n_+$, then the set of attainable student-body distributions is given by the *budget set*:

$$B(x) \coloneqq \big\{y \in \mathbb{Z}^n_+ : y \leq x \text{ and } \langle y \rangle \leq q\big\}.$$

Within this budget set, the subset of distributions that are maximal with respect to both the size and the diversity of the student body is given by the *r-targeting Schur frontier*:

$$F_r(x) \coloneqq \big\{y \in B(x) : \nexists z \in B(x) \text{ s.t. } z > y \text{ or } z \rhd_r y\big\}.$$

In other words, the r-targeting Schur frontier is the subset of feasible distributions beyond which the school can neither admit strictly more students nor choose a strictly more r-diverse student body.[4]

We now turn to our main question: **How should a school allocate its limited number of seats in a way that balances diversity and priority considerations?** We address this question by following an axiomatic approach.

A *choice rule* is a function $C : 2^{\mathcal{S}} \to 2^{\mathcal{S}}$ that maps each nonempty set of applicants $S \subseteq \mathcal{S}$ into a set of admitted students $C(S)$. We impose the following natural restriction on the choice rule: the school can only admit students up to its capacity

[4]In Appendix A.1, we provide several properties of the Schur frontier that we use to prove our main results.

from among its applicants. Formally, for each $S \subseteq \mathcal{S}$, the choice rule must satisfy $C(S) \subseteq S$ and $|C(S)| \leq q$.

Our first axiom says that the choice rule should admit as many students as possible, until the school's capacity constraint binds. In other words, no school seat shall go to waste.

Axiom 1. A choice rule C is **nonwasteful** if, for every $S \subseteq \mathcal{S}$, $|C(S)| = \min\{q, |S|\}$.

Our second axiom says that, given the choice rule, the diversity of the admitted student body cannot be "locally" improved upon by swapping the admission of one student for another.

Axiom 2. A choice rule C **promotes r-targeting diversity** if for every $S \subseteq \mathcal{S}$, $s \in C(S)$ and $s' \in S \setminus C(S)$ imply $\xi\Big(\big(C(S) \setminus \{s\}\big) \cup \{s'\}\Big) \not\trianglerighteq_r \xi(C(S))$.

We use the term "locally" because Axiom 2 does not restrict the possibility that diversity can be improved by swapping the admission decisions of multiple students. Nevertheless, the following lemma shows that our two axioms together imply that the diversity of the admitted student body cannot be "globally" improved.

Lemma 1. *A choice rule C is nonwasteful and promotes r-targeting diversity if and only if $\xi(C(S)) \in F_r(\xi(S))$ for all $S \in \mathcal{S}$.*

To understand our third axiom, consider an applicant set $S \subseteq \mathcal{S}$ and two students: an admitted student $s \in C(S)$ and a rejected student $s' \in S \setminus C(S)$ such that

$$\xi(C(S)) \not\trianglerighteq_r \xi\left(\big(C(S) \setminus \{s\}\big) \cup \{s'\}\right).$$

In words, the school admits s over s' even though doing so does not improve the diversity of the student body. That is, if we counterfactually swapped the admission of s and s' while holding all else fixed, the resulting diversity would be either the same or incomparable. In such cases, the school's decision to admit s rather than s' should be understood as evidence that s has higher priority than s'. Our third axiom states that the priority ranking uncovered by such a revealed preference exercise should not lead to a cycle.

Axiom 3. A choice rule C satisfies the **r-targeting Schur-revealed preference axiom** if there are no sequences of students $\{s_k\}_{k=1}^K$ and sets $\{S_k\}_{k=1}^K$ such that, for every $k \in \{1, \ldots, K\}$,

$$s_k \in C(S_k),\ s_{k+1} \in S_k \setminus C(S_k), \text{ and } \xi(C(S_k)) \not\trianglerighteq_r \xi\Big(\big(C(S_k) \setminus \{s_k\}\big) \cup \{s_{k+1}\}\Big).$$

where $s_{K+1} = s_1$.

We believe that our three axioms capture normative principles that are particularly well-suited for school choice. The first axiom reflects an efficiency concern: scarce educational resources should be fully utilized, and no seat should remain empty when there are willing applicants. In practice, a school may also face legal or institutional constraints that require it to fill all available seats. The second and third axioms together articulate one possible way to reconcile two competing objectives of school admissions. The second axiom reflects the institution's diversity objective: conditional on filling its seats, the admitted class should be as r-diverse as possible, at least in the sense that no simple substitution of one student for another could yield a strictly more diverse student body. While it may be infeasible for administrators to evaluate how diverse the student body would have been under entirely different admissions, our axiom provides a tractable and transparent yardstick for evaluating diversity. Finally, the third axiom reflects the idea that, in the absence of diversity considerations, admissions decisions should maximize priorities in a consistent manner. Taken together, the axioms can be interpreted as selecting an outcome on the diversity–priority Pareto frontier.

Having specified our axioms, we next characterize the class of choice rules that satisfy them. To this end, we define the *r-targeting Schur choice rule* for a given priority ranking P over $\mathcal{S}$, which we denote by C_r^P.[5]

> **r-targeting Schur Choice Rule**
> **Input:** A set of students $S \subseteq \mathcal{S}$. Let $k \coloneqq |S|$. Label students in S such that $s_1\ P \ldots P\ s_k$.
> **Initialization:** Set $S_0 \coloneqq \emptyset$.
> **Steps** $i \in \{1, \ldots, k\}$**:** If $\xi\big(S_{i-1} \cup \{s_i\}\big) \leq x$ for some $x \in F_r\big(\xi(S)\big)$, then set $S_i \coloneqq S_{i-1} \cup \{s_i\}$. Otherwise, set $S_i \coloneqq S_{i-1}$.
> **Output:** $S_k = C_r^P(S)$.

In other words, an r-targeting Schur choice rule admits the students with the highest priority as long as doing so is consistent with some distribution that is maximally

[5]The reference to Schur in the name of the choice rule alludes to "Schur concavity," a property of functions that are monotone with respect to the majorization order (Marshall et al., 2010).

r-diverse among the feasible distributions in the budget set. An r-targeting Schur rule depends on two parameters, a target r and a priority ranking P. Our first main result shows that a choice rule that satisfies our three axioms must be an r-targeting Schur choice rule for some priority ranking.

Theorem 1. *A choice rule C is nonwasteful, promotes r-targeting diversity, and satisfies r-targeting Schur-revealed preference if and only if $C = C_r^P$ for some priority ranking P over $\mathcal{S}$.*

Theorem 1 characterizes the choice rules for which there exists some priority ranking that makes the rule equal to an r-targeting Schur rule. It is a revealed preference exercise, in the sense that the priority ranking is inferred from the choice rule. Once a priority ranking of students is fixed, either exogenously or endogenously via Theorem 1, we can use it to compare the admitted student body under different choice rules. The comparison notion is analogous to first-order stochastic dominance.

Definition 3. Given a priority ranking P and two distinct sets of students $S, S' \subseteq \mathcal{S}$, S *P-priority dominates* S', if $|S| \geq |S'|$ and, for every $k \leq |S'|$, if s_k is the student with the k-th highest priority ranking in S and s'_k is the student with the k-th highest priority ranking in S', then $s_k \; P \; s'_k$ or $s_k = s'_k$.

In words, S P-priority dominates S' if S has at least as many students as S', and when the two sets are compared student-by-student in order of priority, the student in S always has weakly higher priority than the corresponding student in S'.

Our second main result, Theorem 2, compares the outcomes of the r-targeting Schur choice rule with those of any alternative choice rule. Whenever the r-targeting Schur rule admits a different set of students than the alternative choice rule, then either it admits more students, or a more r-diverse set of students, or a priority-dominating set of students.

Theorem 2. *Fix a priority ranking P over $\mathcal{S}$. Let C be any choice rule such that $C(S) \neq C_r^P(S)$ for some $S \subseteq \mathcal{S}$. Then either*

(i) $|C_r^P(S)| > |C(S)|$, *or*

(ii) $\xi(C_r^P(S)) \rhd_r \xi(C(S))$, *or*

(iii) $C_r^P(S)$ *P-priority dominates* $C(S)$.

In other words, once we adopt our notion of comparative diversity, Theorem 2 establishes that the r-targeting Schur choice rule always selects an outcome on the diversity–priority Pareto frontier. That is, any attempt to improve upon the outcome in terms of diversity or priority necessarily requires either worsening along the other dimension or else making an inefficient allocation in which some seats remain unfilled.

The proof of Theorem 2 builds on a structural connection to greedy algorithms in matroid theory. Specifically, we show that the maximally r-diverse student bodies that can be admitted from a given applicant pool form the bases of a matroid. This structure guarantees that greedy methods yield optimal admission policies in the sense of Theorem 2. To make this connection concrete, we introduce an alternative greedy algorithm and show that it coincides with the r-targeting Schur choice rule.

To this end, we first introduce some useful notation and terminology. A *matroid* is a pair $(\mathcal{S}, \mathcal{B})$ where $\mathcal{B}$, known as the *set of bases*, is a collection of subsets of $\mathcal{S}$ satisfying the following *matroid base axioms*:

(B1) $\mathcal{B}$ is nonempty and

(B2) for every $S \in \mathcal{B}$, $S' \in \mathcal{B}$, and $s \in S \setminus S'$, there exists $s' \in S' \setminus S$ such that $(S \setminus \{s\}) \cup \{s'\} \in \mathcal{B}$ and $(S' \setminus \{s'\}) \cup \{s\} \in \mathcal{B}$.

Lemma 2. *Let $S \subseteq \mathcal{S}$. Then $\mathcal{B}(S) \coloneqq \{S' \subseteq S : \xi(S') \in F_r(\xi(S))\}$ satisfies the matroid base axioms.*

We now define the r-targeting *greedy choice rule* for a given priority ranking P over $\mathcal{S}$, which we denote $G_r^P : 2^{\mathcal{S}} \to 2^{\mathcal{S}}$.

> **r-targeting Greedy Choice Rule**
> **Input:** A set of students $S \subseteq \mathcal{S}$. Let $k \coloneqq |S|$. Label students in S such that $s_1 \; P \; \ldots \; P \; s_k$.
> **Initialization:** Set $S_0 \coloneqq \emptyset$.
> **Steps** $i \in \{1, \ldots, k\}$**:** If $S_{i-1} \cup \{s_i\} \subseteq S'$ for some $S' \in \mathcal{B}(S)$, then set $S_i \coloneqq S_{i-1} \cup \{s_i\}$. Otherwise, set $S_i \coloneqq S_{i-1}$.
> **Output:** $S_k = G_r^P(S)$.

The next result says that the two choice rules that we have introduced are one and the same.

Proposition 2. *Fix a priority ranking P over $\mathcal{S}$. The r-targeting greedy choice rule is the same as the r-targeting Schur choice rule, i.e., $G_r^P = C_r^P$.*

We conclude this section with a simple illustration of the r-targeting Schur choice rule for a fixed priority ranking.

Example 2. Suppose there are $n = 2$ types: blue ($i = 1$) and red ($i = 2$). Let the target be uniform $r = u$. The school has capacity $q = 3$, and five students apply: $S = \{s_1, s_2, s_3, s_4, s_5\}$, where s_1, s_2, and s_3 are blue; s_4 and s_5 are red. Thus $\xi(S) = (3, 2)$ and

$$B(\xi(S)) = \{y \in \mathbb{Z}_+^2 : y_1 \leq 3,\ y_2 \leq 2,\ \text{ and }\ y_1 + y_2 \leq 3\}.,$$

Suppose the priority ranking is: $s_1\ P\ s_2 \ldots P\ s_5$. A choice rule that admits students based only on the priority would yield $\{s_1, s_2, s_3\}$, which consists of only blue-type students.

On the other hand, the u-targeting Schur frontier is given by

$$F_u(\xi(S)) = \{(2, 1), (1, 2)\},$$

which approximates an equal representation of red and blue students given the budget set. Thus, $C_u^P(S) = G_u^P(S) = \{s_1, s_2, s_4\}$. In other words, once s_1 and s_2 are admitted, the algorithm skips over s_3 in favor of s_4 to achieve greater diversity, even though s_3 has strictly higher priority than s_4.

5. Comparison to Reserves and Quotas

Much of the existing literature and many real-world policies have relied on reserves or quotas to reconcile schools' dual objectives. Such mechanisms impose universal constraints on the choice rule, regardless of the applicant pool, which limits flexibility and makes it difficult to adapt admissions outcomes to the specific composition of applicants.

Given our notion of comparative diversity, Theorem 2 already establishes that no other choice rule, including reserves or quotas, can outperform our r-targeting Schur choice rule along both diversity and priority dimensions. Two natural questions arise:

is the r-targeting Schur rule equivalent to a choice rule with quotas? Is it equivalent to a choice rule with reserves?

The answer to the first question is immediate. Given any quota, one can construct an applicant pool in which the quota binds and produces unfilled seats. Since our rule is nonwasteful, it cannot be equivalent to a choice rule with quotas.

Our choice rule is also not equivalent to any choice rule with reserves. To see this, suppose there are $n = 3$ types and consider the uniform target $r = u$. Let priority P order all type-1 students above all type-2 students, and all type-2 students above all type-3 students. The school has capacity $q = 5$.

A reserve policy specifies a vector of type-specific reserved seats $R \in \mathbb{Z}^3_+$ with $\langle R \rangle \leq q = 5$. Following Echenique and Yenmez (2015), the rule first admits $\min\{x_i, R_i\}$ of each type i (based on P), then fills the remaining

$$5 - \sum_{i \in \mathcal{N}} \min\{x_i, R_i\}$$

open seats purely by priority.

We consider three different sets of applicants: S_1 with $\xi(S_1) = (5, 5, 0)$, S_2 with $\xi(S_2) = (5, 0, 5)$, and S_3 with $\xi(S_3) = (5, 3, 2)$. Suppose, for the sake of a contradiction, that the reserve policy coincides with the u-targeting Schur rule in all three cases.

1. For S_1, the budget set is

$$B(\xi(S_1)) = \{y \in \mathbb{Z}^3_+ : y_1 \leq 5,\ y_2 \leq 5,\ y_3 = 0 \text{ and } \ y_1 + y_2 \leq 5\},$$

   and the u-targeting Schur frontier is $F_u(\xi(S_1)) = \{(3, 2, 0), (2, 3, 0)\}$, which approximates an equal representation of types 1 and 2 given the budget set. Since type-1 students all have a higher priority than type-2 students, $\xi(C(S_1)) = (3, 2, 0)$.

   On the other hand, the reserve policy would allocate all open seats to type-1 students and this leads to a distribution $(5 - R_2, R_2, 0)$. Thus, the reserve policy coincides with the Schur choice rule if and only if $R_2 = 2$.

2. For S_2, the budget set is

$$B(\xi(S_2)) = \{y \in \mathbb{Z}^3_+ : y_1 \leq 5,\ y_2 = 0,\ y_3 \leq 5 \text{ and } \ y_1 + y_3 \leq 5\},$$

and the u-targeting Schur frontier is $F_u(\xi(S_2)) = \{(3,0,2),(2,0,3)\}$, which approximates an equal representation of types 1 and 3 given the budget set. Since type-1 students all have a higher priority than type-3 students, $\xi(C(S_2)) = (3,0,2)$.

On the other hand, the reserve policy would allocate all open seats to type-1 students and this leads to a distribution $(5-R_3, 0, R_3)$. Thus, the reserve policy coincides with the Schur choice rule if and only if $R_3 = 2$.

3. For S_3, the budget set is

$$B(\xi(S_3)) = \{y \in \mathbb{Z}^3_+ : y_1 \leq 5,\ y_2 \leq 3,\ y_3 \leq 2 \text{ and } y_1 + y_2 + y_3 \leq 5\},$$

and the u-targeting Schur frontier is $F(\xi(S_3)) = \{(2,2,1),(1,2,2),(2,1,2)\}$. Since type-1 students all have a higher priority than type-2 students, and since type-2 students all have a higher priority than type-3 students, our choice rule yields $\xi(C(S_3)) = (2,2,1)$.

On the other hand, the reserve policy with $R_2 = R_3 = 2$ would yield a distribution $(1,2,2)$, leading to a contradiction!

Hence, in general, no fixed reserve vector R can replicate the outcome of the u-targeting Schur rule across all applicant sets. The key limitation is that reserves must be specified ex-ante, before observing the type distribution of applicants. While unused reserves can revert to open seats, this mechanism lacks the adaptability of the Schur rule, which tailors the allocation to the realized applicant pool.

A. Appendix

A.1. On the r-targeting Schur frontier

We provide several results that characterize and detail the properties of the r-targeting Schur frontier. These results are important in the proofs of our main results, and may be of independent interest because the frontier captures the maximally r-diverse distributions that a school is able to achieve.

Lemma 3. *Let $x \in \mathbb{Z}_+^n$. For all $y \in F_r(x)$, $\langle y \rangle = \min\{q, \langle x \rangle\}$.*

Proof of Lemma 3. Consider any $y \in F_r(x)$ and, for the sake of a contradiction, suppose there exists $z \in B(x)$ such that $\langle z \rangle > \langle y \rangle$. Equivalently, $\langle z \rangle \geq \langle y \rangle + 1$. Then, there exists an $i \in \mathcal{N}$ such that $z_i \geq y_i + 1$. Define a new distribution $\tilde{y} = y + \chi_i$. By construction, $\tilde{y}_i \leq z_i \leq x_i$ and $\tilde{y}_j = y_j \leq x_j$ for all $j \neq i$, so $\tilde{y} \leq x$. Additionally, $\langle \tilde{y} \rangle \leq \langle z \rangle \leq q$. Hence, $\tilde{y} \in B(x)$ and $\tilde{y} > y$, which would contradict the fact that $y \in F_r(x)$. Consequently, $\langle y \rangle = \max_{z \in B(x)} \langle z \rangle = \min\{q, \langle x \rangle\}$. □

Lemma 4. *Let $x \in \mathbb{Z}_+^n$ and suppose $|F_r(x)| > 1$. Then there exists a constant $a \in \mathbb{R}$ such that for any $y \in \mathbb{Z}_+^n$, the following are equivalent:*

(i) $y \in F_r(x)$.

(ii) *For each distribution $z \in F_r(x)$ with $z \neq y$, there exists an involutory permutation $\pi : \mathcal{N} \to \mathcal{N}$ such that for all $i \in \mathcal{N}$:* [6]

  (a) $T_i^r(y) = T_{\pi(i)}^r(z)$.

  (b) $T_i^r(y) > T_i^r(z)$ *implies* $T_i^r(y) = T_i^r(z) + 1 = a$.

  (c) $T_i^r(y) < T_i^r(z)$ *implies* $T_i^r(y) + 1 = T_i^r(z) = a$.

Proof of Lemma 4. Fix any distribution $x \in \mathbb{Z}_+^n$ such that $|F_r(x)| > 1$, and let $y \in F_r(x)$. Pick any distribution $z \in F_r(x)$ with $y \neq z$. Since $\langle y \rangle = \langle z \rangle$ by Lemma 3, there exist types $i, j \in \mathcal{N}$ with $i \neq j$ such that $y_i > z_i$ and $y_j < z_j$. Define the nonempty and disjoint sets $\mathbf{A} \coloneqq \{i \in \mathcal{N} : y_i > z_i\}$ and $\mathbf{B} \coloneqq \{i \in \mathcal{N} : y_i < z_i\}$.

Claim 1. *For all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $y_i = y_j + 1 + \lfloor \langle y \rangle (r_i - r_j) \rfloor$ and $z_j = z_i + 1 + \lfloor \langle y \rangle (r_j - r_i) \rfloor$.*

[6] A permutation $\pi : \mathcal{N} \to \mathcal{N}$ is an *involution* if $\pi(\pi(i)) = i$ for all $i \in \mathcal{N}$.

Proof of Claim 1. We first show that for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $T_i^r(y) - T_j^r(y) \leq 1$. To see why, suppose $T_{i^*}^r(y) - T_{j^*}^r(y) > 1$ for some $i^* \in \mathbf{A}$ and $j^* \in \mathbf{B}$. Notice that the distribution $y + \chi_{j^*} - \chi_{i^*} \in B(x)$ and $T^r(y + \chi_{j^*} - \chi_{i^*}) = T^r(y) + \chi_{j^*} - \chi_{i^*}$. Thus, $T^r(y) \succ_s T^r(y + \chi_{j^*} - \chi_{i^*})$, or equivalently, $y + \chi_{j^*} - \chi_{i^*} \rhd_r y$. We would then conclude that $y \notin F_r(x)$, yielding a contradiction. Hence, for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $T_i^r(y) - T_j^r(y) \leq 1$, or equivalently, $y_i - y_j \leq 1 + \langle y \rangle (r_i - r_j)$. Since $y_i - y_j \in \mathbb{Z}^n$, we can conclude that $y_i - y_j \leq 1 + \lfloor \langle y \rangle (r_i - r_j) \rfloor$. By a symmetric argument, we have $T_j^r(z) - T_i^r(z) \leq 1$, or equivalently, $z_j - z_i \leq 1 + \lfloor \langle y \rangle (r_j - r_i) \rfloor$.

We next show that for any $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $T_i^r(y) - T_j^r(y) > 0$. Again, suppose there exists $i^* \in \mathbf{A}$ and $j^* \in \mathbf{B}$ such that $T_{i^*}^r(y) - T_{j^*}^r(y) \leq 0$. By definition of the sets $\mathbf{A}$ and $\mathbf{B}$, we would then have

$$T_{j^*}^r(z) > T_{j^*}^r(y) \geq T_{i^*}^r(y) > T_{i^*}^r(z),$$

which would imply $T_{j^*}^r(z) - T_{i^*}^r(z) \geq 2$, leading to a contradiction of the already proven claim that $T_{j^*}^r(z) - T_{i^*}^r(z) \leq 1$. Thus, for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $T_i^r(y) - T_j^r(y) > 0$, or equivalently, $y_i - y_j > \langle y \rangle (r_i - r_j)$. Since $y_i - y_j \in \mathbb{Z}^n$, we can conclude that $y_i - y_j \geq 1 + \lfloor \langle y \rangle (r_i - r_j) \rfloor$. By a symmetric argument, we have $T_j^r(z) - T_i^r(z) > 0$, or equivalently, $z_j - z_i \geq 1 + \lfloor \langle y \rangle (r_j - r_i) \rfloor$. This concludes the proof to Claim 1. □

Claim 2. *For all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $\langle y \rangle (r_i - r_j) \in \mathbb{Z}$.*

Proof of Claim 2. From Claim 1, for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$,

$$\begin{aligned} 0 &< y_i - z_i \\ &= y_j - z_j + 2 + \lfloor \langle y \rangle (r_i - r_j) \rfloor + \lfloor \langle y \rangle (r_j - r_i) \rfloor \\ &= \begin{cases} y_j - z_j + 2 & \text{if} \quad \langle y \rangle (r_i - r_j) \in \mathbb{Z} \\ y_j - z_j + 1 & \text{if} \quad \langle y \rangle (r_i - r_j) \notin \mathbb{Z} \end{cases}. \end{aligned}$$

Notice that if $\langle y \rangle (r_i - r_j) \notin \mathbb{Z}$ for some $i \in \mathbf{A}$ and $j \in \mathbf{B}$, then $y_j - z_j < 0$ and $y_j - z_j + 1 > 0$, which is not feasible as $y_j - z_j \in \mathbb{Z}$. Hence, for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $\langle y \rangle (r_i - r_j) \in \mathbb{Z}$. □

Claim 3. *For all $i \in \mathbf{A}$, $T_i^r(y) = T_i^r(z)+1$. Similarly, for all $j \in \mathbf{B}$, $T_j^r(z) = T_j^r(y)+1$. Consequently, $|\mathbf{A}| = |\mathbf{B}|$.*

Proof of Claim 3. From Claim 2, we can conclude that for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$,

$$0 < y_i - z_i = y_j - z_j + 2.$$

Therefore, we have $0 < z_j - y_j < 2$ for all $j \in \mathbf{B}$, which implies that $z_j - y_j = 1$ and $y_i - z_i = 1$. We can then conclude that $T_i^r(y) = T_i^r(z) + 1$ for all $i \in \mathbf{A}$ and $T_j^r(z) = T_j^r(y) + 1$ for all $j \in \mathbf{B}$.

Additionally, since $\langle y \rangle = \langle z \rangle$, we have

$$\begin{aligned} 0 &= \langle y \rangle - \langle z \rangle \\ &= \sum_{i \notin \mathbf{A} \cup \mathbf{B}} \underbrace{(y_i - z_i)}_{=0} + \sum_{i \in \mathbf{A}} \underbrace{(y_i - z_i)}_{=1} + \sum_{j \in \mathbf{B}} \underbrace{(y_j - z_j)}_{=-1} \\ &= |A| - |B|, \end{aligned}$$

giving us the desired conclusion. □

Claim 4. *There exists a constant $a \in \mathbb{R}$ such that $T_i^r(y) = T_j^r(z) = a$ for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$.*

Proof of Claim 4. We first show that for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$, $T_i^r(y) = T_j^r(z)$. To see why, note that for all $i \in \mathbf{A}$ and $j \in \mathbf{B}$,

$$\begin{aligned} T_i^r(y) &= T_j^r(y) + 1 + \lfloor \langle y \rangle (r_i - r_j) \rfloor - \langle y \rangle (r_i - r_j) \\ &= T_j^r(z) + \lfloor \langle y \rangle (r_i - r_j) \rfloor - \langle y \rangle (r_i - r_j) \\ &= T_j^r(z), \end{aligned}$$

where the first equality comes from Claim 1, the second equality from Claim 3, and the last equality from Claim 2.

We then show that for all $i \in \mathbf{A}$, $T(y)_i = a$ for some $a \in \mathbb{R}$. Such a statement is vacuously true if $|A| = 1$. Let $|A| > 1$. Suppose there exist $i, k \in \mathbf{A}$ with $i \neq k$ such

that $T_i^r(y) > T_k^r(y)$, which implies that for any $j \in \mathbf{B}$,

$$\begin{aligned} y_i &\geq y_k + 1 + \lfloor \langle y \rangle (r_i - r_k) \rfloor \\ &= y_j + 2 + \lfloor \langle y \rangle (r_k - r_j) \rfloor + \lfloor \langle y \rangle (r_i - r_k) \rfloor \\ &> y_j + 1 + \lfloor \langle y \rangle (r_i - r_j) \rfloor, \end{aligned}$$

where the equality follows from applying Claim 1 to k and j, and the last inequality follows from Claim 2 and the fact that for any $a \in \mathbb{R}$, $1 + \lfloor a \rfloor > a \geq \lfloor a \rfloor$. The last inequality however contradicts Claim 1 applied to i and j. Hence, $T_i^r(y) = T_k^r(y)$ for all $i, k \in \mathbf{A}$, and a symmetric argument establishes that $T_j^r(z) = T_k^r(z)$ for all $j, k \in \mathbf{B}$. □

We have thus far shown that there exist a constant $a \in \mathbb{R}$ and a bijective mapping $\hat{\pi} : \mathbf{A} \to \mathbf{B}$ such that

(a) For all $i \notin \mathbf{A} \cup \mathbf{B}$, $T_i^r(y) = T_i^r(z)$, and

(b) For all $i \in \mathbf{A}$, $T_i^r(y) = T_i^r(z) + 1 = T_{\hat{\pi}(i)}^r(z) = T_{\hat{\pi}(i)}^r(y) + 1 = a$.

Consequently, $T^r(y)$ and $T^r(z)$ are permutations. Furthermore, since the two properties listed above must hold for any two distributions in $F_r(x)$, the constant a is independent of y and z. Finally, we define the mapping $\pi : \mathcal{N} \to \mathcal{N}$ given by

$$\pi(i) = \begin{cases} i & \text{if} \quad i \notin \mathbf{A} \cup \mathbf{B} \\ \hat{\pi}(i) & \text{if} \quad i \in \mathbf{A} \\ \hat{\pi}^{-1}(i) & \text{if} \quad i \in \mathbf{B} \end{cases},$$

which gives us the statement of Lemma 4. It only remains to show that π is an involution. By construction, for $i \notin \mathbf{A} \cup \mathbf{B}$, $\pi(\pi(i)) = \pi(i) = i$. For $i \in \mathbf{A}$, $\pi(i) = \hat{\pi}(i) \in \mathbf{B}$ and thus $\pi(\pi(i)) = \hat{\pi}^{-1}(\hat{\pi}(i)) = i$. Similarly, for $i \in \mathbf{B}$, $\pi(i) = \hat{\pi}^{-1}(i) \in \mathbf{A}$ and thus $\pi(\pi(i)) = \hat{\pi}(\hat{\pi}^{-1}(i)) = i$. This concludes the proof. □

By Lemma 3, any type distribution in the (r-targeting) Schur frontier is achieved by admitting as many students as the capacity permits. Furthermore, by Lemma 4, any two distributions in a nonsingleton Schur frontier must be "permutations" of each other, and if the two distributions differ in their i^{th} coordinate, then they must do so by only one.

Lemmas 3 and 4 provide substantial structure on the Schur frontier, and allow us to provide a simple characterization that relates majorization to the ℓ^2-norm. This is striking because majorization—and hence, r-targeting diversity—is not a complete binary relation, whereas comparisons of feasible distributions based on their proximity in the Euclidean distance to a target distribution yields a total order over $\mathbb{Z}_+^n$.

To that end, for any $x \in \mathbb{Z}_+^n$, define

$$M_r(x) \coloneqq \arg\min_{y \in B(x)} \sum_{i=1}^{n} \left(y_i - \langle y \rangle\, r_i\right)^2 \quad \text{s.t.} \quad \langle y \rangle = \min\{q, \langle x \rangle\}. \tag{1}$$

Since the constraint set in (1) is nonempty and finite, $M_r(x)$ is nonempty for any $x \in \mathbb{Z}_+^n$. From the following proposition, we can then also establish that $F_r(x)$ is nonempty for any type distribution $x \in \mathbb{Z}_+^n$.

Proposition 3. *For any $x \in \mathbb{Z}_+^n$, $F_r(x) = M_r(x)$.*

Proof of Proposition 3. Fix a feasible distribution $x \in \mathbb{Z}_+^n$, and notice that we can equivalently define $M_r(x)$ as

$$M_r(x) \coloneqq \arg\min_{y \in B(x)} ||T^r(y)||_2 \quad \text{s.t.} \quad \langle y \rangle = \min\{q, \langle x \rangle\}.$$

Let us first show that $M_r(x) \subseteq F_r(x)$. Fix some $y \in M_r(x)$. Take any $z \in B(x)$, which by definition of the budget set, satisfies $\langle z \rangle \leq \min\{q, \langle x \rangle\} = \langle y \rangle$. Thus, $z \not\succ y$.

Suppose $z \rhd_r y$. By Proposition 1, we have $T^r(y) \succ_s T^r(z)$, which further implies that $\langle z \rangle = \langle y \rangle$. Thus, for any continuous and strictly convex function $g : \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{i=1}^{n} g(T_i^r(y)) > \sum_{i=1}^{n} g(T_i^r(z)).$$

However, taking $g(X) = X^2$, the above inequality implies $||T^r(z)||_2 < ||T^r(y)||_2$, which contradicts the fact that $y \in M_r(x)$. To avoid such a contradiction, we must have that $z \not\rhd_r y$. In other words, for each $y \in M_r(x)$, there exists no distribution $z \in B(x)$ with either $z > y$ or $z \rhd_r y$. Therefore, $M_r(x) \subseteq F_r(x)$.

Let us now show the converse: $F_r(x) \subseteq M_r(x)$. We have already established that $M_r(x)$ is nonempty and that $M_r(x) \subseteq F_r(x)$. Pick any $z \in M_r(x)$ and any $y \in F_r(x)$. If $y \neq z$, then from Lemma 4, $T^r(y)$ is a permutation of $T^r(z)$, and hence $y \in M_r(x)$, which establishes $M_r(x) \supseteq F_r(x)$. □

Lemma 5. *Let $x \in \mathbb{Z}_+^n$, and let $y \in B(x)$ such that $\langle y \rangle = \min\{q, \langle x \rangle\}$. If $y \notin F_r(x)$, then*

(i) There exist $i, j \in \mathcal{N}$ such that $y + \chi_i - \chi_j \in B(x)$ and $y + \chi_i - \chi_j \rhd_r y$, and

(ii) For all $z \in F_r(x)$, $z \rhd_r y$.

Proof of Lemma 5. Fix a distribution $x \in \mathbb{Z}_+^n$ such that $F_r(x) \neq B(x)$. Pick any $y \in B(x) \backslash F_r(x)$ such that $\langle y \rangle = \min\{q, \langle x \rangle\}$.

(Proof of point (i)): Consider any $y \in B(x)$ such that $\langle y \rangle = \min\{q, \langle x \rangle\}$ and $y \notin F_r(x)$. Thus, y satisfies the constraints of the minimization problem (1) but, by Proposition 3, $y \notin M_r(x)$. Therefore, there exists a distribution $z \in M_r(x) \subseteq B(x)$ with $||T^r(z)||_2 < ||T^r(y)||_2$.

Let us define the sets $\mathbf{A} \coloneqq \{i : y_i > z_i\}$ and $\mathbf{B} \coloneqq \{i : y_i < z_i\}$. Since $\langle z \rangle = \langle y \rangle$ and $z \neq y$, we know $\mathbf{A}$ and $\mathbf{B}$ are nonempty sets. Let $y_{i^*} \coloneqq \max\{y_i : i \in \mathbf{A}\}$ and $y_{i_*} \coloneqq \min\{y_i : i \in \mathbf{B}\}$. Notice that for all $i \in \mathbf{A}$, $T_i^r(z) + 1 \leq T_i^r(y) \leq T_{i^*}^r(y)$, and hence,

$$T_i^r(y) + T_i^r(z) \leq 2T_{i^*}^r(y) - 1. \tag{3}$$

Similarly, for $i \in \mathbf{B}$, $T_i^r(z) \geq T_i^r(y)_i + 1 \geq T_{i_*}^r(y) + 1$, and hence,

$$T_i^r(y) + T_i^r(z) \geq 2T_{i_*}^r(y) + 1. \tag{4}$$

As a result, we can now express $||T^r(y)||_2 - ||T^r(z)||_2 > 0$ as

$$\begin{aligned}
0 <& \sum_{i=1}^{n} (T_i^r(y))^2 - (T_i^r(z))^2 \\
=& \sum_{i \in \mathbf{A} \cup \mathbf{B}} (T_i^r(y))^2 - (T_i^r(z))^2 \\
=& \sum_{i \in \mathbf{A}} \underbrace{\Big(T_i^r(y) - T_i^r(z)\Big)}_{>0} \Big(T_i^r(y) + T_i^r(z)\Big) + \sum_{i \in \mathbf{B}} \underbrace{\Big(T_i^r(y) - T_i^r(z)\Big)}_{<0} \Big(T_i^r(y) + T_i^r(z)\Big) \\
\leq& \Big(2T_{i^*}^r(y) - 1\Big) \sum_{i \in \mathbf{A}} \Big(T_i^r(y) - T_i^r(z)\Big) + \Big(2T_{i_*}^r(y) + 1\Big) \sum_{i \in \mathbf{B}} \Big(T_i^r(y) - T_i^r(z)\Big) \\
=& 2\Big(T_{i^*}^r(y) - T_{i_*}^r(y) - 1\Big) \sum_{i \in \mathbf{A}} \Big(T_i^r(y) - T_i^r(z)\Big),
\end{aligned}$$

where the first equality follows from the fact that $T_i^r(y) = T_i^r(z)$ for all $i \notin \mathbf{A} \cup \mathbf{B}$, the second inequality follows from (3) and (4), and the last equality follows from the fact that $\sum_{i \in \mathbf{A}} T_i^r(y) - T_i^r(z) = \sum_{i \in \mathbf{B}} T_i^r(z) - T_i^r(y)$ because $\langle T^r(z) \rangle = \langle T^r(y) \rangle$.

The above display equation yields $T_{i^*}^r(y) - T_{i_*}^r(y) > 1$, which implies that $T^r(y) \succ_s T^r(y) + \chi_{i_*} - \chi_{i^*}$, or equivalently, $y + \chi_{i_*} - \chi_{i^*} \rhd_r y$. Furthermore, $y + \chi_{i_*} - \chi_{i^*} \in B(x)$ because we are taking a seat away from a student of type i^* (which is possible because $y_{i^*} > z_{i^*} \geq 0$ as $i^* \in \mathbf{A}$) and giving a seat to a student of type i_* (which is possible because $y_{i_*} < z_{i_*} \leq x_{i_*}$ as $i_* \in \mathbf{B}$). This concludes the proof of Lemma 5-(i).

(Proof of point (ii)): Consider any $z \in F_r(x)$, which implies $\langle z \rangle = \min\{q, \langle x \rangle\} = \langle y \rangle$. Since $z \in F_r(x)$, we have that $y \not\rhd_r z$. Additionally, since $y \notin F_r(x)$, we have that y and z are not equally r-diverse, i.e., it is not the case that $y \trianglerighteq_r z \trianglerighteq_r y$. Thus, there are only two mutually exclusive possibilities remaining: Either y and z are not comparable in terms of r-targeting diversity, or $z \rhd_r y$. We show that the first case is not possible, and hence, the only possibility is that $z \rhd_r y$.

For the sake of a contradiction, suppose y and z are not comparable in terms of r-targeting diversity, i.e., $T^r(y)$ and $T^r(z)$ cannot be ordered by majorization. Let $y_0 \equiv y$. Since $y_0 \notin F_r(x)$, we know from Lemma 5-(i) that there exists $i_i, j_1 \in \mathcal{N}$ such that $y_1 \coloneqq y_0 + \chi_{i_1} - \chi_{j_1} \in B(x)$ and $y_1 \rhd_r y_0$. Notice that $z \not\trianglerighteq_r y_1$; Otherwise, $z \rhd_r y_0$ by transitivity, which goes against the initial assumption that y_0 and z are incomparable. Furthermore, $y_1 \not\trianglelefteq_r z$; Otherwise, since z cannot be strictly less r-diverse, we would have that $T^r(y_1)$ and $T^r(z)$ are permutations and thus, $z \rhd_r y_0$ by transitivity. However, this once again would go against the initial assumption that y_0 and z are incomparable. Finally, $y_1 \notin F_r(x)$; Otherwise, $T^r(y_1)$ and $T^r(z)$ would be permutations by Lemma 4, which once again would go against the initial assumption that y_0 and z are incomparable., Therefore, we conclude that there exists a distribution $y_1 \in B(x)$ such that $y_1 \rhd_r y_0$, y_1 and z are incomparable, and $y_1 \notin F_r(x)$.

Now suppose we have established that for some $k > 1$ that there exists a distribution $y_k \in B(x)$ such that $y_k \rhd_r y_{k-1}$, y_k and z are incomparable, and $y_k \notin F_r(x)$. The above argument establishes that there exists a distribution $y_{k+1} \in B(x)$ such that $y_{k+1} \rhd_r y_k$, y_{k+1} and z are incomparable, and $y_{k+1} \notin F_r(x)$. Hence, there would exist an infinite sequence of distinct distributions $\{y_k\}_{k=0}^{\infty}$ with each $y_k \in B(x)$ and $y_{k+1} \rhd_r y_k$. However, it is not possible to have such an infinite sequence of distinct distributions that are ranked by r-targeting diversity when there are only a finite number of feasible distributions in $B(x)$. Thus, to avoid this contradiction, we must

have $z \rhd_r y$. $\square$

Lemma 6. *Let $R, R' \subseteq S \subseteq \mathcal{S}$ with $R \neq R'$. Suppose $\xi(R)$ and $\xi(R')$ are in $F_r(\xi(S))$. Then for each student $s \in R \setminus R'$, there exists a student $s' \in R' \setminus R$ such that $\xi((R \setminus \{s\}) \cup \{s'\}) \in F_r(\xi(S))$ and $\xi((R' \setminus \{s'\}) \cup \{s\}) \in F_r(\xi(S))$.*

Proof of Lemma 6. Since both $\xi(R)$ and $\xi(R')$ are in $F_r(\xi(S))$, we have $|R| = |R'|$. Additionally, since $R \neq R'$, the sets $R \setminus R'$ and $R' \setminus R$ are both nonempty. Pick any student $s \in R \setminus R'$ and let $i^* \in \mathcal{N}$ be the student's type.

We first consider the case in which $\xi_{i^*}(R) = \xi_{i^*}(R')$. In other words, R contains the same *number* of type-i^* students but not the same type-i^* students as R'. In this case, there exists a student $s' \in R' \setminus R$ with type $i_{s'}$ such that $i_{s'} = i^*$. The statement of the lemma then follows trivially.

Next consider the case in which $\xi_{i^*}(R) \neq \xi_{i^*}(R')$. Without loss of generality, suppose $\xi_{i^*}(R) > \xi_{i^*}(R')$. Since $|R| = |R'|$, there exists a type $i' \in \mathcal{N}$ with $i' \neq i^*$ such that $\xi_{i'}(R) < \xi_{i'}(R')$. Thus, there exists a student $s' \in R' \setminus R$ with type $i_{s'}$ such that $i_{s'} = i'$. Since both $\xi(R) \in F_r(\xi(S))$ and $\xi(R') \in F_r(\xi(S))$, by Lemma 4, $T^r_{i^*}(\xi(R)) = T^r_{i^*}(\xi(R')) + 1 = T^r_{i'}(\xi(R')) = T^r_{i'}(\xi(R)) + 1$.

Construct a new distribution $\xi((R \setminus \{s\}) \cup \{s'\}) = \xi(R) + \chi_{i'} - \chi_{i^*}$. Then notice that, by construction,

$$T^r_k\big(\xi((R \setminus \{s\}) \cup \{s'\})\big) = \begin{cases} T^r_{i'}(\xi(R)) & \text{if} \quad k = i^* \\ T^r_{i^*}(\xi(R)) & \text{if} \quad k = i' \\ T^r_k(\xi(R)) & \text{if} \quad k \notin \{i^*, i'\} \end{cases}.$$

In other words, $T^r\big(\xi((R \setminus \{s\}) \cup \{s'\})\big)$ is a permutation of $T^r(\xi(R))$, and by Proposition 1, $\xi(R) \trianglerighteq_r \xi((R \setminus \{s\}) \cup \{s'\}) \trianglerighteq_r \xi(R)$. Moreover, $\xi((R \setminus \{s\}) \cup \{s'\}) \in B(x)$ because $\langle \xi((R \setminus \{s\}) \cup \{s'\}) \rangle = \langle \xi(R) \rangle \leq \min\{q, \langle x \rangle\}$, and because

$$0 \leq \xi_{i^*}(R') = \xi_{i^*}((R \setminus \{s\}) \cup \{s'\}) < \xi_{i^*}(R) \leq x_{i^*}$$

and

$$0 \leq \xi_{i'}(R) < \xi_{i'}((R \setminus \{s\}) \cup \{s'\}) = \xi_{i'}(R') \leq x_{i'}.$$

We can therefore conclude that $\xi((R \setminus \{s\}) \cup \{s'\}) \in F_r(\xi(S))$. A symmetric argument shows that $\xi((R' \setminus \{s'\}) \cup \{s\}) \in F_r(\xi(S))$. $\square$

A.2. Proofs of Section 3

Proof of Proposition 1. Consider two distributions $z, z' \in \mathbb{R}^n_+$ such that z' is obtained from z via a Robin Hood transfer with respect to r. Then there exist types $i, j \in \mathcal{N}$ with $z_i - \langle z \rangle r_i \geq z_j - \langle z \rangle r_j$, and a constant $0 \leq \delta \leq z_i - z_j - \langle z \rangle (r_i - r_j)$ such that

$$z' = z + \delta\chi_j - \delta\chi_i.$$

By definition of T^r, we have:

(a) $\langle z \rangle = \langle T^r(z) \rangle$,

(b) $z_i - \langle z \rangle r_i \geq z_j - \langle z \rangle r_j \iff T^r_i(z) - \langle T^r(z) \rangle u_i \geq T^r_j(x) - \langle T^r(z) \rangle u_j$,

(c) $\delta \leq z_i - z_j - \langle z \rangle (r_i - r_j) \iff \delta \leq T^r_i(z) - T^r_j(z)$, and

(d) $z' = z + \delta\chi_j - \delta\chi_i \iff T^r(z') = T^r(z) + \delta\chi_j - \delta\chi_i$.

Thus, z' is obtained from z via a Robin Hood transfer with respect to r if and only if $T^r(z')$ is obtained from $T^r(z)$ via a Robin Hood transfer with respect to u.

Consequently, for any two distributions $x, y \in \mathbb{R}^n_+$, y is more r-diverse than x if and only $T^r(y)$ is more u-diverse than $T^r(x)$. Moreover, from Hardy et al. (1952) (see for example, Theorem B.2 of Marshall et al. (2010)), $T^r(y)$ is more u-diverse than $T^r(x)$ if and only if $T^r(x)$ majorizes $T^r(y)$, which yields the desired conclusion. □

A.3. Proofs of Section 4

Proof of Lemma 1.
(*If* direction:) Let $\xi(C(S)) \in F_r(\xi(S))$ for all $S \in \mathcal{S}$. From Lemma 3, we have $\langle \xi(C(S)) \rangle = \min\{q, \langle \xi(S) \rangle\}$. By noting that $\langle \xi(C(S)) \rangle = |C(S)|$ and $\langle \xi(S) \rangle = |S|$, we can conclude that C is a nonwasteful choice rule. Furthermore, by the definition of the r-targeting Schur frontier, any $z \in B(\xi(S))$ satisfies $z \not\succcurlyeq_r \xi(C(S))$. Hence, C promotes r-targeting diversity.

(*Only if* direction:) Let C be a nonwasteful choice rule that promotes r-targeting diversity. Since $C(S) \subseteq S$ and $|C(S)| \leq q$, we have $\xi(C(S)) \in B(\xi(S))$. Furthermore, because C is nonwasteful, $\langle \xi(C(S)) \rangle = \min\{q, \langle \xi(S \rangle\}$. It remains to show that $\xi(C(S)) \in F_r(\xi(S))$. For the sake of contradiction, suppose not! Then, by

Lemma 5-(i), there would exist $i, j \in \mathcal{N}$ such that $\xi(C(S)) + \chi_i - \chi_j \in B(\xi(S))$ and $\xi(C(S)) + \chi_i - \chi_j \rhd_r \xi(C(S))$. However, this would contradict the fact that C promotes r-targeting diversity. Thus, to avoid the contradiction, we must have $\xi(C(S)) \in F_r(\xi(S))$ as desired. □

Proof of Theorem 1.

(If direction): Fix some priority ranking P over $\mathcal{S}$. We show that C_r^P satisfies the three axioms. First, by construction, $\xi(C_r^P(S)) \in F_r(\xi(S))$ for all $S \subseteq \mathcal{S}$, which by Lemma 1, implies that C_r^P is nonwasteful and promotes r-targeting diversity.

To show that C_r^P also satisfies r-targeting Schur-revealed preference, let $S \subseteq \mathcal{S}$, $s \in C_r^P(S)$, $s' \in S \setminus C_r^P(S)$, and

$$\xi((C_r^P(S)) \not\rhd_r \xi\Big(\big(C_r^P(S) \setminus \{s\}\big) \cup \{s'\}\Big).$$

From Lemma 5-(ii), we must have $\xi\Big(\big(C_r^P(S) \setminus \{s\}\big) \cup \{s'\}\Big) \in F_r(\xi(S))$. We want to show that in this case, $s \; P \; s'$.

For the sake of contradiction, suppose $s' \; P \; s$. Then s' is considered before s by C_r^P. Let $\overline{S} \subseteq C_r^P(S)$ be the subset of admitted students who are ranked higher than s' and let $\underline{S} \subseteq C_r^P(S)$ be the subset of admitted students who are ranked lower than s'. Thus, $\big(C_r^P(S) \setminus \{s\}\big) \cup \{s'\} = \overline{S} \cup (\underline{S} \setminus \{s\}) \cup \{s'\}$. However, notice that

$$\xi(\overline{S} \cup \{s'\}) \leq \xi(\overline{S} \cup (\underline{S} \setminus \{s\}) \cup \{s'\}) \in F_r(\xi(S)),$$

which implies that when C_r^P considers s', there is a distribution in the Schur frontier that justifies admitting s'. This of course contradicts the initial assumption that $s' \in S \setminus C_r^P(S)$. To avoid this contradiction, it must be that $s \; P \; s'$.

Consequently, there cannot exist sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$ with $s_{K+1} = s_1$ and, for every $k \in \{1, \ldots, K\}$,

$$s_k \in C_r^P(S_k),\; s_{k+1} \in S_k \setminus C_r^P(S_k), \text{ and } \xi(C_r^P(S)) \not\rhd_r \xi\Big(\big(C_r^P(S_k) \setminus \{s_k\}\big) \cup \{s_{k+1}\}\Big)$$

because this would generate a cycle

$$s_1 \; P \; s_2 \; P \; \ldots \; P \; s_K \; P \; s_1$$

contradicting the fact that P is a preference relation.

(Only-if direction): Suppose that C is a choice rule that is nonwasteful, promotes r-targeting diversity, and satisfies r-targeting Schur-revealed preference. From Lemma 1, we know that $\xi(C(S)) \in F_r(\xi(S))$ for all $S \subseteq \mathcal{S}$.

Define the following binary relation P: for any $s, s' \in \mathcal{S}$, say $s \; P \; s'$ whenever there exists $S \subseteq \mathcal{S}$ such that $s \in C(S)$, $s' \in S \setminus C(S)$, and $\xi(C(S)) \not\trianglerighteq_r \xi\big((C(S) \setminus \{s\}) \cup \{s'\}\big)$. The r-targeting Schur-revealed preference implies that P does not have any cycles. Therefore, by Szpilrajn's lemma (see the version in Chambers and Echenique (2016)), it can be completed to a preference relation over $\mathcal{S}$. Let P^* be such a completion, and let $C_r^{P^*}$ be the r-targeting Schur choice rule for priority ranking P^*. By construction, $\xi(C_r^{P^*}(S)) \in F_r(\xi(S))$ for all $S \subseteq \mathcal{S}$.

To prove the *Only-if* direction, we show that $C(S) = C_r^{P^*}(S)$ for all $S \subseteq \mathcal{S}$. Suppose, to the contrary, that there exists a set $S \subseteq \mathcal{S}$ such that $C(S) \neq C_r^{P^*}(S)$. By Lemma 6, there exist students $s \in C(S) \setminus C_r^{P^*}(S)$ and $s' \in C_r^{P^*}(S) \setminus C(S)$ such that $\xi((C(S) \setminus \{s\}) \cup \{s'\}) \in F_r(\xi(S)$. Consequently, $\xi(C(S)) \not\trianglerighteq_r \xi\big((C(S) \setminus \{s\}) \cup \{s'\}\big)$, implying that $s \; P \; s'$ by the definition of the priority ranking P.

Since P^* is a completion, we must also have $s \; P^* \; s'$, which implies that s is considered before s' by the choice rule $C_r^{P^*}$. Furthermore, again by Lemma 6, we have that $\xi((C_r^{P^*}(S) \setminus \{s'\}) \cup \{s\}) \in F_r(\xi(S))$ and, therefore, s should be chosen by $C_r^{p^*}$ when considered, i.e., $s \in C_r^{P^*}(S)$. However, this contradicts the fact that $s \in C(S) \setminus C_r^{P^*}(S)$. To avoid such a contradiction, we must have that $C(S) = C_r^{P^*}(S)$ for all $S \subseteq \mathcal{S}$, which concludes the proof. □

Proof of Theorem 2. Fix a priority ranking P over $\mathcal{S}$. Let C be a choice rule such that for some $S \subseteq \mathcal{S}$, $C(S) \neq C_r^P(S)$. Suppose $|C(S)| \geq \left|C_r^P(S)\right|$ and $\xi(C_r^P(S)) \not\trianglerighteq_r \xi(C(S))$. To prove the theorem, it suffices to show that in this case, $C_r^P(S)$ P-priority dominates $C(S)$.

Consider $\mathcal{B}(S) = \{S' \subseteq S : \xi(S') \in F_r(\xi(S))\}$. By construction, $\xi(C_r^P(S)) \in F_r(\xi(S))$. Additionally, $\xi(C(S)) \in F_r(\xi(S))$ by Lemma 5-(ii). Therefore, $C_r^P(S)$ and $C(S)$ are in $\mathcal{B}(S)$. By Lemma 2, $\mathcal{B}(S)$ is a set of bases of a matroid. Furthermore, by Proposition 2, $C_r^P = G_r^P$, where G_r^P is the greedy choice rule defined on the matroid with the set of bases $\mathcal{B}(S)$. We then get the desired result from the following:

Lemma 7 (Gale (1968))**.** *The greedy choice rule $G_r^P(S)$ P-priority dominates any independent subset of S.*

Thus, by Lemma 7, $C_r^P(S)$ P-priority dominates any independent subset of S. Since $C(S)$ is independent (because $C(S) \in \mathcal{B}(S)$), we get that $C_r^P(S)$ P-priority dominates $C(S)$. □

Proof of Lemma 2. For $S \subseteq \mathcal{S}$, let $\mathcal{B}(S) \coloneqq \{S' \subseteq S : \xi(S') \in F_r(\xi(S))\}$. By Proposition 3, $F_r(\xi(S))$ is nonempty. As a result, $\mathcal{B}(S)$ is nonempty, which means that $\mathcal{B}(S)$ satisfies matroid base axiom B1.

Let $S' \in \mathcal{B}(S)$, $S'' \in \mathcal{B}(S)$, and $s' \in S' \setminus S''$. By definition of $\mathcal{B}(S)$, $\xi(S'), \xi(S'') \in F_r(\xi(S))$. By Lemma 6, there exists $s'' \in S'' \setminus S'$ such that $\xi((S' \setminus \{s'\}) \cup \{s''\}) \in F_r(\xi(S))$ and $\xi((S'' \setminus \{s''\}) \cup \{s'\}) \in F_r(\xi(S))$. These imply that $(S' \setminus \{s'\}) \cup \{s''\} \in \mathcal{B}(S)$ and $(S'' \setminus \{s''\}) \cup \{s'\} \in \mathcal{B}(S)$. Therefore, $\mathcal{B}(S)$ satisfies matroid base axiom B2. □

Proof of Proposition 2. Let $S \subseteq \mathcal{S}$ be the input set. We show that G_r^P and C_r^P coincide at each step i of the respective algorithms by showing that either the same student s_i is added to the set of chosen students or s_i is skipped. Since both algorithms start with the empty set, we get the result that the two choice rules produce the same outcome for input S.

At step i of the r-targeting greedy choice rule, s_i is added to S_{i-1} if and only if $S_{i-1} \cup \{s_i\} \subseteq S'$ for some $S' \in \mathcal{B}(S)$. By definition, $S' \in \mathcal{B}(S)$ if and only if $\xi(S') \in F_r(\xi(S))$. In other words, s_i is added to S_{i-1} if and only if $\xi(S_{i-1} \cup \{s_i\}) \leq x$ for some $x \in F_r(\xi(S))$, which is step i of the r-targeting Schur choice rule. □

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