Welfare in Multi-Unit Winner-Pays-Bid Auctions

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Abstract

This paper performs a robust analysis of the multi-unit winner-pays-bid auction welfare (aka, the price of anarchy). We employ a novel smoothed analysis framework, one which analyzes equilibrium bid distributions, to establish lower bounds on the price of anarchy.

The smoothness framework identifies properties of equilibrium bid distributions that drive the convergence of the price of anarchy in a k-unit first-price auction to 1 as k increases. This approach offers additional insights into the structure of equilibrium bid distributions and the mechanisms underlying their convergence. Specifically, we identify a regularity property on equilibrium bid distributions which implies a convergence rate of $1-O(1/k^{0.5-\delta})$ for any constant $\delta \in (0, 0.5)$. We also prove key monotonicity and continuity properties of equilibrium bid distributions — a result that may be of independent interest. We conclude with a conjecture about the worst-case bid distributions of the multi-unit winner-pays-bid auction with a finite number of units.

1 Introduction

Many economic settings involve the sale of multiple identical goods. A prominent example is online marketplaces, where firms sell products from large inventories. A central question in such environments is: what welfare guarantees do commonly used mechanisms provide in large markets? In this paper, we study the efficiency of a winner-pays-bid, highest-bidders-win auction with unit-demand agents, focusing on the asymptotic regime where the number of available units grows large.

A winner-pays-bid, highest-bidders-win auction with unit-demand agents — commonly known as a multi-unit first-price auction — proceeds as follows: each agent submits a bid for a single unit of the good. The available items are allocated to the agents with the highest bids, up to the total supply. Only the winning agents — those who receive an item — pay their respective bids.

In this auction format, agents bid below their true valuations. This strategic behavior can lead to inefficiencies in the allocation of items, as an agent with a higher value may not be allocated an item if they bid less aggressively than a lower-value competitor. A standard objective for quantifying such inefficiency is the *price of anarchy* (see the comprehensive survey by Roughgarden et al., 2017). The price of anarchy is defined as the worst-case ratio of the welfare achieved in equilibrium to the optimal social welfare. In this paper, we study the price of anarchy of the multi-unit first-price auction as the number of units grows asymptotically and agents have independent value distributions.

Solving for equilibrium in multi-unit first-price auctions poses significant analytical challenges. We approach this problem by studying the properties of equilibrium bid distributions. Our main

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contribution is to investigate the price of anarchy question through the lens of the smoothness framework, developed by works such as Roughgarden (2012), Syrgkanis and Tardos (2013), and Hartline et al. (2014). Our framework highlights essential structural properties of equilibrium bid distributions in multi-unit auctions, which, in turn, yield a quantifiable rate at which the price of anarchy converges to 1 as the number of units become large. Our main result states that if a regularity condition on equilibrium bids is satisfied, the price of anarchy converges to 1 at a rate $1 - O(1/k^{0.5-\delta})$ where k is the number of items and δ is any constant in $(0, 0.5)$.

The analysis focuses on characterizing bidder behavior across different bid intervals:

(a) **Extreme bid interval**: We analyze the allocation probabilities for bids that are either very high or very low. Agents have a similar probability of allocation at these extreme bids.

(b) **Moderate bid interval**: We identify a regularity condition on the equilibrium bid distribution within an interval bounded by extreme bids. Intuitively, this condition requires that an interval of bids surrounded by extreme bids has an interval of valuations of proportional length with equilibrium bids in that bid interval.

In addition to these structural properties, we establish that equilibrium bid distributions are continuous and strictly increasing. This result is derived by applying a novel inductive argument on equilibrium bid distributions. It may be of independent interest, as it offers a general characterization of equilibrium behavior in multi-unit first-price auctions.

Swinkels (1999) establishes that the price of anarchy converges to 1 at a rate determined jointly by the number of units and the number of agents, requiring in particular that the number of units k grows at least as fast as $n^{0.5+\epsilon}$ for some constant $\epsilon > 0$, where n denotes the number of agents. In contrast, our analysis dispenses with this dependence on the growth rate of the agent population, requiring only that the number of units becomes asymptotically large.

The smoothness framework we adopt provides further insight into the structure of equilibrium bid distributions and the mechanisms driving their convergence. A key step in this framework is the application of a Chernoff bound, which relies critically on the continuity and monotonicity of equilibrium bid distributions — properties that, as described above, we establish from first principles. This facilitates a convergence that is independent of the number of agents in the auction. [1]

The assumption of independence of agent values is vital for our analysis. If agent values are not independent, the price of anarchy does not improve as the number of units increases, as we show in the result below:

Theorem 1. *In a k-unit winner-pays-bid, highest-bidders-win auction with unit-demand agents whose values can be arbitrarily correlated, the worst-case ratio of expected welfare in Bayes-Nash equilibrium to the expected welfare of an optimal allocation is $1 - \frac{1}{e}$.*

Proof sketch. Consider an auction with two groups of agents:

- Group 1 contains k agents who all deterministically have value 1 and bid 0.
- Group 2 contains $k + 1$ agents with perfectly correlated values drawn from $[0, 1 - 1/e]$, with cumulative distribution function $F(v) = \frac{1}{e(1-v)}$, and who always bid their value.

[1] Swinkels (1999) employ a Chebyshev inequality, and their method does not permit the use of a standard Chernoff bound.

If ties are always broken in favor of Group 1 (which is without loss of generality, see the full proof), this is a valid Bayes-Nash equilibrium that obtains a $1 - \frac{1}{e}$ fraction of the expected optimal welfare. Lastly, an argument by individual and competitive efficiency shows that the above setting is worst-case. □

We provide a complete proof in the Appendix. Theorem 1 extends a known result that applies to the single-unit case ($k = 1$) as shown by Syrgkanis (2014). This extension highlights an important limitation: for the price of anarchy to improve as the number of units increases, some form of independence among agent values is essential. In this paper, we assume agent values are independent and agents bid in Bayes-Nash equilibrium.

"Price of anarchy" guarantees of winner-pays-bids auctions are typically bounded above by $1-\frac{1}{e}$ due to the worst-case *individual efficiency*, which quantifies the loss in welfare due to uncertainty in the competing bid in each bidder's best response problem. The main barrier to showing that the price of anarchy improves in the number of units is showing that the individual efficiency improves in the number of units. The main contribution of this paper gets around this barrier by proving that the competing bid distributions become sharply concentrated around a single point, and quantifies their rate of concentration.

Finally, we turn our attention to characterizing the worst-case value distributions for the price of anarchy in a multi-unit first-price auction with a finite number of items — marking a shift from the asymptotic analysis considered earlier. A recent breakthrough by Jin and Lu (2023) characterizes the worst-case efficiency of the first-price auction in the single-unit setting, showing that its price of anarchy is precisely $1 - 1/e^2$. Building on this line of work, we prove a technical result that we conjecture captures the behavior of a subset of the worst-case bid distributions in the multi-unit setting:

Theorem 10. *Let $X_1, ..., X_n$ be independent non-negative random variables. Then, there exist i.i.d. non-negative random variables $Y_1, ..., Y_n$ where the maximum value in $\{X_1, ..., X_n\}$ has the same distribution as the maximum value in $\{Y_1, ..., Y_n\}$, but where every order statistic of $\{Y_1, ..., Y_n\}$ first-order stochastic dominates the corresponding order statistic of $\{X_1, ..., X_n\}$.*

We conjecture that the worst-case instance involves a (asymptotically infinite) subset of low-value i.i.d. bidders, whose bids have order statistics with a specific closed form. This would line up with a critical feature of the worst-case instance in the single-unit winner-pays-bid auction proved by Jin and Lu (2023). A more detailed discussion of this conjecture is provided later in Section 6.

Related Work

As noted in the introduction, there is a rich body of work on the price of anarchy in auction settings; see Roughgarden et al. (2017) for a comprehensive survey. A central technique in this literature is the smoothness framework, introduced by Roughgarden (2012), Syrgkanis and Tardos (2013), and Roughgarden (2015), which has proven effective for analyzing efficiency guarantees across a range of mechanisms. Hartline et al. (2014) further refine this approach by decomposing smoothness into two key components—individual efficiency and competitive efficiency—which highlight structural properties of auctions that ensure near-optimal outcomes. Our work builds directly on this decomposition, leveraging these properties to analyze the multi-unit winner-pays-bid auction in a large market setting.

A broader generalization of the smoothness framework, developed by Feldman et al. (2016), has been applied to study inefficiencies in large games, particularly in mechanisms like the simultaneous uniform price auction and greedy combinatorial auctions. We view their contributions as complementary to ours, given the shared focus on large-scale market environments.

There is a long line of work in the economics literature considering a large market setting (see Wilson (1977), Rustichini et al. (1994), Swinkels (1999) and Swinkels, 2001). Closest to our work is that of Swinkels (1999) who studies the inefficiency of a multi-unit first-price auction in a large market setting. Our general smoothness framework offers additional insights into equilibrium bid distributions while also highlighting the underlying factors governing the rate of convergence.

The price of anarchy in finite auction settings has also attracted significant attention. Notable works, including Feldman et al. (2013), Bhawalkar and Roughgarden (2011), and Markakis and Telelis (2015), analyze the efficiency of uniform-price and simultaneous item auctions. The single-unit first-price auction, in particular, has been the subject of extensive study (see Kaplan and Zamir, 2012; Hoy et al., 2018). A central challenge across these analyses is the characterization of Bayes-Nash equilibria.

In a recent breakthrough, Jin and Lu (2023) show that the price of anarchy for the single-unit first-price auction is $1 - 1/e^2$. Their analysis characterizes the worst-case equilibrium bid distribution by solving a differential equation. Extending this type of characterization to the multi-unit setting presents substantial analytical challenges. Nevertheless, building on their work, we establish continuity and monotonicity properties of equilibrium bid distributions—crucial elements in the application of our framework. Proving these properties requires novel inductive arguments that, to the best of our knowledge, have not previously been employed to establish similar results in the economics literature.

In orthogonal work on multi-unit mechanisms that are not winner-pays-bid, there is growing evidence that such mechanisms tend to yield better welfare outcomes than their single-unit counterparts. For instance, Yan (2011) shows, via a correlation gap argument, that in posted-price mechanisms with independent values, the price of anarchy converges asymptotically to 1 at a rate of $1 - O(1/\sqrt{k})$. This bound is tight when agent values are identically distributed. Similarly, Chawla et al. (2023) analyzes anonymous posted-price mechanisms and demonstrates that, under independent values, the worst-case distribution of agent demand is Poisson. This insight enables the computation of welfare approximation ratios for any finite number of units. Jiang et al. (2023) establishes that the precise rate of convergence of the welfare approximation ratio is $1-\Theta(\sqrt{\ln(k)/k})$.

2 Model and Preliminaries

Consider an auction $\mathcal{A}$ with n single-dimensional agents, represented by the set $\mathcal{N} = \{1, 2, \ldots, n\}$, each with quasi-linear utility. Each agent i's valuation is drawn independently from a distribution $\mathcal{V}_i$ that satisfy the following assumption:

Assumption 1. *The collection of value distributions satisfy the following properties:*

(a) (Bounded Values) There exists a constant $H < \infty$ such that for any agent i, their value v_i is at most H.

(b) There exists a constant $\kappa > 0$ such that for any k and any agent $i \in \mathcal{N}^{(k)}$, for any $\epsilon > 0$ and any value $v \in [0, H - \epsilon]$, the measure of values

$$Pr\left[V_i \in (v, v + \epsilon)\right] > \kappa\epsilon.$$

Uniform Value Support implies that, asymptotically, there exists a positive mass of values in any interval of any length. A useful consequence of Assumption 1 is that there exists a constant $\zeta > 0$ such that for any k and any agent $i \in \mathcal{N}^{(k)}$, the expected value $\mathbb{E}\left[\mathcal{V}_i\right] \geq \zeta$.

Agents submit a single real-valued bid b_i and the outcome of the auction is determined by an allocation rule $\widetilde{\boldsymbol{x}} : \mathbb{R}^n_{\geq 0} \to \mathcal{X}$ and payment rule $\widetilde{\boldsymbol{p}} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$. The allocation rule $\widetilde{\boldsymbol{x}}(\cdot)$ assigns a feasible allocation of items based on the bid profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, while the payment rule $\widetilde{\boldsymbol{p}}(\cdot)$ specifies agent payments.

An agent's strategy is a mapping from their value to a corresponding bid. Given such a strategy, the distribution over values induces a distribution over bids, determining an interim allocation rule and interim payment rule. Specifically, the interim allocation rule for agent i is defined by the function $x_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which specifies the probability of agent i being allocated the item upon bidding b_i. Similarly, the interim payment rule $p_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ specifies the expected payment associated with a given bid.

The interim utility of agent i given valuation v_i and bid b_i is

$$u_i\left(v_i, b_i\right) = v_i x_i(b_i) - p_i(b_i).$$

We assume agents play according to a Bayes-Nash equilibrium. In equilibrium, the strategy adopted by agent i induces a corresponding bid distribution $\mathcal{B}_i$. The equilibrium utility of agent i with value v_i is $u_i(v_i)$.

Definition 1 (Price of Anarchy)**.** The price of anarchy of auction $\mathcal{A}$ is defined as

$$\mathrm{PoA}(\mathcal{A}) := \min_{\boldsymbol{\mathcal{V}}} \frac{\mathcal{A}(\boldsymbol{\mathcal{V}})}{\mathrm{OPT}(\boldsymbol{\mathcal{V}})}$$

where $\mathcal{A}(\boldsymbol{\mathcal{V}})$ is the welfare of auction $\mathcal{A}$ when agents play a Bayes-Nash equilibrium and $\mathrm{OPT}(\boldsymbol{\mathcal{V}})$ is the expected optimal welfare.

2.1 Individual Efficiency and Competitive Efficiency

In this section we outline preliminary concepts required for analyzing auction efficiency within the smoothed framework as defined in Hartline et al. (2014).

We define the concept of a *critical bid*. Consider a deterministic bid profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. Agent i is allocated an item when her bid exceeds the critical bid, denoted by $\hat{b}_i$, which depends explicitly on the bid profile $\mathbf{b}$. The expected critical bid, with randomization over bid distributions $(\mathcal{B}_i)_{i \in \mathcal{N}}$, is defined as $\widehat{B}_i = \mathbb{E}[\hat{b}_i]$.

An agent's allocation probability and critical bid are closely linked. Specifically, the probability that agent i is allocated an item when submitting bid b_i is $x_i(b_i) = Pr[\hat{b}_i < b_i]$, where the probability is computed over the randomization of equilibrium bids. In scenarios where the equilibrium bid distributions are continuous, the allocation rule directly corresponds to the cumulative distribution function of the critical bid. Thus, the expected critical bid can be explicitly expressed as:

$$\widehat{B}_i = \int_0^\infty \left(1 - x_i(b_i)\right) db_i.$$

We define the individual efficiency of an auction.

Definition 2 (Individual Efficiency)**.** An auction $\mathcal{A}$ is η-individually efficient if for any Bayes-Nash equilibrium, for any agent $i \in \mathcal{N}$ and any value v_i:

$$u_i(v_i) + \widehat{B}_i \geq \eta v_i,$$

where $u_i(v_i)$ is the expected equilibrium utility of agent i when their valuation is v_i and $\widehat{B}_i$ is the expected critical bid.

The individual efficiency of an auction is the largest value of η such that the auction is η-individually efficient.

The definition above highlights two competing forces: either the equilibrium utility of an agent is large *or* the expected critical bid is large indicating strong competition in the auction environment. When competition among agents leads to high seller revenue, the auction allocation recovers almost all of the available welfare. This concept is formalized with the notion of *competitive efficiency.*

Definition 3 (Competitive Efficiency)**.** An auction $\mathcal{A}$ is μ-competitive efficient if for any profile of bids $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and any feasible allocation $\mathbf{y} \in \mathcal{X}$,

$$\mu\text{Revenue}(\mathbf{b}) \geq \sum_{i \in \mathcal{N}} \hat{b}_i y_i,$$

where Revenue($\mathbf{b}$) is the revenue of the auction on bids $\mathbf{b}$ and $\hat{b}_i$ is the critical bid of agent i on bid profile $\mathbf{b}_{-i}$.

The competitive efficiency of an auction is the smallest μ such that the auction is μ-competitive efficient.

An auction that exhibits high individual efficiency and low competitive efficiency provides strong welfare guarantees. The following result demonstrates how these two properties jointly bound the auction's competitive ratio.

Theorem 2 (Hartline et al., 2014)**.** *Consider any auction $\mathcal{A}$ with η individual efficiency and μ competitive efficiency. The auction is $\frac{\eta}{\mu}$ efficient, that is, for any value distributions $\boldsymbol{\mathcal{V}} = (\mathcal{V}_i)_{i \in \mathcal{N}}$, the ratio*

$$\frac{\mathcal{A}(\boldsymbol{\mathcal{V}})}{\text{OPT}(\boldsymbol{\mathcal{V}})} \geq \frac{\eta}{\mu},$$

where $\mathcal{A}(\boldsymbol{\mathcal{V}})$ is the welfare of auction $\mathcal{A}$ when agents play a Bayes-Nash equilibrium and OPT($\boldsymbol{\mathcal{V}}$) *is the expected optimal welfare.*

In the remainder of the paper, we analyze the individual and competitive efficiency of the multi-unit first-price auction. We begin by examining its competitive efficiency. Hartline et al. (2014) established that the multi-unit first-price auction has a competitive efficiency of 1. For completeness, we include a proof of the result below.

Lemma 1 (Hartline et al., 2014)**.** *The multi-unit highest-bids-win winner-pays-bid auction has a competitive efficiency of* 1*.*

Proof. Consider a multi-unit first-price auction with k units. To show that the auction has a competitive efficiency of 1, we must show that for every bid profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and any feasible allocation $\boldsymbol{y}$,

$$\text{Revenue}(\mathbf{b}) \geq \sum_{i \in \mathcal{N}} \hat{b}_i y_i.$$

Suppose allocation $\boldsymbol{y}^*$ belongs to the set

$$\boldsymbol{y}^* \in \arg\max_{y \in \mathcal{X}} \sum_{i \in \mathcal{N}} \hat{b}_i y_i.$$

The value of the right hand side for allocation $\boldsymbol{y}^*$ is the sum of the k highest elements in the collection of critical bids $\widehat{\boldsymbol{b}} = \left(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n\right)$.

Consider any agent i and their corresponding critical bid $\hat{b}_i$. This critical bid is at most the k^{th} highest bid in profile $\mathbf{b}$. By definition, the k highest bids in $\mathbf{b}$ are weakly greater than any critical bid. Thus,

$$\begin{aligned}\text{Revenue}(\mathbf{b}) &\geq \sum_{i \in \mathcal{N}} \hat{b}_i y_i^*, \\ &\geq \sum_{i \in \mathcal{N}} \hat{b}_i y_i,\end{aligned}$$

where $\boldsymbol{y}$ is any feasible allocation. The statement of the result follows. □

Next, we examine the individual efficiency of the multi-unit first-price auction. A standard analysis establishes that the auction achieves an individual efficiency of $1 - \frac{1}{e}$. However, a key limitation of that analysis is that it does not assume the agent is best-responding to an allocation rule arising from an equilibrium; instead, the agent may respond to an arbitrary function. In contrast, our analysis relies on the observation that the agent is best-responding to an allocation rule that is part of an equilibrium.

By directly analyzing equilibrium bid distributions, we identify sufficient conditions under which the individual efficiency converges to 1 at a quantifiable rate. These conditions, along with the convergence rate, are formalized in Section 4. A key property we derive is that equilibrium bid distributions induce a strictly increasing and continuous allocation rule — this structural result is presented in Section 5.

Before presenting these results, we first develop a general framework for analyzing the price of anarchy in multi-unit, highest-bids-win, winner-pays-bid auctions. Within this framework, we show that the price of anarchy converges to 1 as the number of items $k \to \infty$.

3 Framework For Efficiency Using A Smoothed Analysis

This section presents a general framework for proving that the price of anarchy in a multi-unit first-price auction converges to 1 as the number of units increase. Additionally, it establishes an explicit rate of convergence characterized as a function of both the number of items and the number of agents. The framework builds upon the concepts of *individual efficiency* and *competitive efficiency*.

We define some key objects that will be used throughout this section. Consider a sequence of winner-pays-bid highest-bids-win auctions $\left(\mathcal{A}^{(k)}\right)_{k=1}^{\infty}$ with $\mathcal{A}^{(k)}$ having k units and $n^{(k)}$ agents. The auction has a collection of agent value distributions $\boldsymbol{\mathcal{V}}^{(k)} = (\mathcal{V}_i)_{i \in n^{(k)}}$.

Let $\mathcal{E}^{(k)}$ be a Bayes-Nash equilibrium. The sequence of auctions and corresponding equilibrium determines a sequence of equilibrium $\left(\mathcal{E}^{(k)}\right)_{k=1}^{\infty}$. For every agent i, the sequence of equilibrium define a sequence of interim allocation rules $\left(x_i^{(k)}(\cdot)\right)_{k=1}^{\infty}$.

The framework aims to identify a pair of bids that provide large individual efficiency guarantees to an agent. To this end, consider a sequence of bid pairs $\left(\mathcal{P}_i^{(k)}\right)_{k=1}^{\infty}$ where each pair is given by $\mathcal{P}_i^{(k)} = \left(\underline{b}_i^{(k)}, \overline{b}_i^{(k)}\right)$ with $\underline{b}_i^{(k)} \leq \overline{b}_i^{(k)}$. We denote the full sequence of bid pairs for agent i by $\boldsymbol{\mathcal{P}}_i$. We now define what it means for such a sequence to be *extremizing*.

Definition 4 (Extremizing Sequence Of Bid Pairs)**.** Consider a sequence of bid pairs $\boldsymbol{\mathcal{P}}_i = \left(\mathcal{P}_i^{(k)}\right)_{k=1}^{\infty}$ where $\mathcal{P}_i^{(k)} = \left(\underline{b}_i^{(k)}, \overline{b}_i^{(k)}\right)$ and the bids $\underline{b}_i^{(k)} \leq \overline{b}_i^{(k)} \leq H$ for agent i. The collection of sequences $\{\boldsymbol{\mathcal{P}}_i\}$ is said to be *extremizing* if the following conditions are satisfied:

(a) The limit of the minimum allocation probability at $\overline{b}_i^{(k)}$ satisfies

$$\lim_{k\to\infty} \min_{i\in\mathcal{N}^{(k)}} x_i^{(k)}\left(\overline{b}_i^{(k)}\right) = 1.$$

(b) The limit of the maximum allocation probability at $\underline{b}_i^{(k)}$ satisfies

$$\lim_{k\to\infty} \max_{i\in\mathcal{N}^{(k)}} x_i^{(k)}\left(\underline{b}_i^{(k)}\right) = 0.$$

We can now state the main theorem of this section, Theorem 3, which provides conditions under which the price of anarchy of the multi-unit winner-pays-bid auction converges to 1 as the number of units becomes asymptotically large.

Theorem 3. *Consider a sequence of winner-pays-bid highest-bids-win auctions $\left(\mathcal{A}^{(k)}\right)_{k=1}^{\infty}$ and a corresponding sequence of Bayes-Nash equilibria $\left(\mathcal{E}^{(k)}\right)_{k=1}^{\infty}$. Suppose the interim allocation rules are continuous and there exists a collection of sequences of bid pairs $\{\boldsymbol{\mathcal{P}}_i\}$ satisfying the following properties:*

(a) The collection $\{\boldsymbol{\mathcal{P}}_i\}$ is extremizing.

(b) The bid pairs in the collection $\{\boldsymbol{\mathcal{P}}_i\}$ satisfy the condition:

$$\lim_{k\to\infty} \max_{i\in\mathcal{N}^{(k)}} \left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) = 0.$$

Then, the price of anarchy of a k-unit highest-bids-win winner-pays-bid auction converges to 1 as $k \to \infty$. Moreover, the rate of convergence to 1 is determined by the rate of convergence of the extremizing bid pairs and the rate of convergence of their allocation probabilities.

Proof. As we described in Section 2.1, our focus is on solving for the individual efficiency of the auction. Our analysis works with a variant of the definition that accounts for an additive error.

Consider any agent i and value v_i. The utility it receives in equilibrium $u_i(v_i)$ is at least

$$u_i(v_i) \geq u_i\left(v_i, \overline{b}_i^{(k)}\right), \tag{1}$$

$$= \left(v_i - \overline{b}_i^{(k)}\right) x_i\left(\overline{b}_i^{(k)}\right), \tag{2}$$

where $u_i\left(v_i, \overline{b}_i^{(k)}\right)$ is the utility the agent receives upon bidding $\overline{b}_i^{(k)}$.

We now solve for the expected critical bid $\widehat{B}_i$. By assumption, the interim allocation rule $x_i(\cdot)$ is continuous, which implies that $x_i(\cdot)$ is the cumulative distribution function of the critical bid for agent i. Therefore, by the tail sum formula for expectations,

$$\widehat{B}_i = \int_0^{\infty} (1 - x_i(b_i))\, db_i.$$

By monotonicity of the interim allocation rule, for any bid $b \leq \underline{b}_i^{(k)}$ the allocation $x_i(b) \leq x_i\left(\underline{b}_i^{(k)}\right)$. It follows that

$$\widehat{B}_i \geq \left(1 - x_i\left(\underline{b}_i^{(k)}\right)\right) \underline{b}_i^{(k)}. \tag{3}$$

Combining (2) and (3), we get

$$u_i(v_i) + \widehat{B}_i \geq v_i x_i\left(\overline{b}_i^{(k)}\right) - \left(\overline{b}_i^{(k)} x_i\left(\overline{b}_i^{(k)}\right) - \left(1 - x_i\left(\underline{b}_i^{(k)}\right)\right)\underline{b}_i^{(k)}\right), \tag{4}$$

$$\geq v_i x_i\left(\overline{b}_i^{(k)}\right) - \left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) - \underline{b}_i^{(k)} x_i\left(\underline{b}_i^{(k)}\right). \tag{5}$$

Consider a valuation profile $\mathbf{v} = (v_1, v_2, \ldots, v_{n^{(k)}})$ and agent i. Suppose $\boldsymbol{y}^*(\mathbf{v})$ is a welfare maximizing allocation. Since $y_i^*(\mathbf{v}) \in [0,1]$, we can multiply (5) by $y_i^*(\mathbf{v})$ to obtain

$$u_i(v_i) + \widehat{B}_i y_i^*(\mathbf{v}) \geq x_i\left(\overline{b}_i^{(k)}\right) v_i y_i^*(\mathbf{v}) - \left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) y_i^*(\mathbf{v}) - \underline{b}_i^{(k)} x_i\left(\underline{b}_i^{(k)}\right) y_i^*(\mathbf{v}).$$

Taking a sum of the above equation over all agents i, we get

$$\sum_{i\in\mathcal{N}^{(k)}} u_i(v_i) + \sum_{i\in\mathcal{N}^{(k)}} \widehat{B}_i y_i^*(\mathbf{v}) \geq \sum_{i\in\mathcal{N}^{(k)}} x_i\left(\overline{b}_i^{(k)}\right) v_i y_i^*(\mathbf{v}) - \sum_{i\in\mathcal{N}^{(k)}} \left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) y_i^*(\mathbf{v}) - \sum_{i\in\mathcal{N}^{(k)}} \underline{b}_i^{(k)} x_i\left(\underline{b}_i^{(k)}\right) y_i^*(\mathbf{v}).$$

The feasibility constraint of the auction implies $\sum_{i\in\mathcal{N}^{(k)}} y_i^*(\mathbf{v}) \leq k$. Also, the bid $\underline{b}_i^{(k)}$ is bounded by H. Substituting this in the equation above gives

$$\sum_{i\in\mathcal{N}^{(k)}} u_i(v_i) + \sum_{i\in\mathcal{N}^{(k)}} \widehat{B}_i y_i^*(\mathbf{v}) \geq \min_{i\in\mathcal{N}^{(k)}} x_i\left(\overline{b}_i^{(k)}\right)\left(\sum_{i\in\mathcal{N}^{(k)}} v_i y_i^*(\mathbf{v})\right) - k \max_{i\in\mathcal{N}^{(k)}}\left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) - Hk \max_{i\in\mathcal{N}^{(k)}} x_i\left(\underline{b}_i^{(k)}\right). \tag{6}$$

It follows from Lemma 1 that $\mathbb{E}\left[\text{Revenue}(\mathbf{b})\right] \geq \mathbb{E}\left[\sum_{i\in\mathcal{N}^{(k)}} \widehat{B}_i y_i^*(\mathbf{v})\right]$. Substituting this in the equation above gives

$$\sum_{i\in\mathcal{N}^{(k)}} u_i(v_i) + \mathbb{E}\left[\text{Revenue}(\mathbf{b})\right] \geq \min_{i\in\mathcal{N}^{(k)}} x_i\left(\overline{b}_i^{(k)}\right)\left(\sum_{i\in\mathcal{N}^{(k)}} v_i y_i^*(\mathbf{v})\right) - k \max_{i\in\mathcal{N}^{(k)}}\left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) - Hk \max_{i\in\mathcal{N}^{(k)}} x_i\left(\underline{b}_i^{(k)}\right). \tag{7}$$

Taking an expectation with respect to the value distributions $\boldsymbol{\mathcal{V}} = (\mathcal{V}_i)_{i\in\mathcal{N}^{(k)}}$ gives the equation

$$\frac{\mathcal{A}(\boldsymbol{\mathcal{V}})}{\text{OPT}(\boldsymbol{\mathcal{V}})} \geq \min_{i\in\mathcal{N}^{(k)}} x_i\left(\overline{b}_i^{(k)}\right) - \frac{k \max_{i\in\mathcal{N}^{(k)}}\left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right)}{\text{OPT}(\boldsymbol{\mathcal{V}})} - \frac{Hk \max_{i\in\mathcal{N}^{(k)}} x_i\left(\underline{b}_i^{(k)}\right)}{\text{OPT}(\boldsymbol{\mathcal{V}})}.$$

By Assumption 1, there exists a constant $\zeta > 0$ such that $\text{OPT}(\boldsymbol{\mathcal{V}}) \geq k\zeta$. It follows that

$$\frac{\mathcal{A}(\boldsymbol{\mathcal{V}})}{\text{OPT}(\boldsymbol{\mathcal{V}})} \geq \min_{i\in\mathcal{N}^{(k)}} x_i\left(\overline{b}_i^{(k)}\right) - \frac{1}{\zeta}\max_{i\in\mathcal{N}^{(k)}}\left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) - \frac{H}{\zeta}\max_{i\in\mathcal{N}^{(k)}} x_i\left(\underline{b}_i^{(k)}\right). \tag{8}$$

The fact that the collection of sequence of bid pairs is extremizing and by assumption

$$\lim_{k\to\infty} \max_{i\in n^{(k)}}\left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right) = 0$$

implies the price of anarchy converges to 1 as $k \to \infty$.

Analyzing (8), the rate of convergence to 1 is governed by the following: the rate of convergence of $\min_{i \in \mathcal{N}^{(k)}} x_i^{(k)} \left(\overline{b}_i^{(k)}\right)$ to 1; the rate of convergence of $\max_{i \in \mathcal{N}^{(k)}} \left(\overline{b}_i^{(k)} - \underline{b}_i^{(k)}\right)$ to 0; the rate of convergence of $\max_{i \in \mathcal{N}^{(k)}} x_i^{(k)} \left(\underline{b}_i^{(k)}\right)$ to 0. □

4 Application of Smoothed Framework

In this section, we apply the framework to demonstrate that the price of anarchy converges to 1 as the number of units $k \to \infty$. Specifically, we establish sufficient conditions on equilibrium bid distributions under which this convergence occurs at an explicit, quantifiable rate.

Assumption 2 (Asymptotic Proportional Extremal Bidding)**.** *There exists a positive constant $\beta > 0$ satisfying the following property: Consider any sequence of bid intervals $\left(\left(\underline{b}^{(k)}, \overline{b}^{(k)}\right)\right)_{k=1}^{\infty}$. There exists a $k^* > 0$ such that for any sequence of agents $\left(i^{(k)}\right)_{k=1}^{\infty}$ satisfying the condition*

$$\lim_{k \to \infty} x_{i^{(k)}} \left(\underline{b}^{(k)}\right) = 0 \text{ and } \lim_{k \to \infty} x_{i^{(k)}} \left(\overline{b}^{(k)}\right) = 1,$$

for every $k > k^*$ there is an interval of valuations $V'_{i^{(k)}}$ such that every value in the interval has an equilibrium bid in $\left(\underline{b}^{(k)}, \overline{b}^{(k)}\right)$. Moreover,

$$\textit{length}\left(V'_{i^{(k)}}\right) \geq \beta \left(\overline{b}^{(k)} - \underline{b}^{(k)}\right).$$

In other words, for a given sequence of bid intervals, for large k, agents with a large difference in allocation probabilities between the end points of this interval have a proportional (in terms of interval length) set of values with equilibrium bids in this interval. A consequence of this assumption is that a constant interval length of values have equilibrium bids in an interval of constant length where the allocation for a bid at the lower end of the interval converges to 0 while at the upper end of the interval converges to 1. Our main result is a rate of convergence on the price of anarchy, subject to Assumption 2.

We construct a collection of bid pair sequences that satisfy the conditions of Theorem 3. We define some objects that will be used throughout the rest of the paper.

Consider auction $\mathcal{A}^{(k)}$ along with a corresponding equilibrium $\mathcal{E}^{(k)}$. Let $\mathcal{B}_i$ denote the equilibrium bid distribution of agent i, and let $\mathcal{B}^{(k)}$ denote the distribution of the k^{th} highest bid. As we establish in Section 5, it is without loss of generality to assume that the following condition holds:

$$\inf(\text{supp}(\mathcal{B}^{(k)})) = 0.$$

Let $\overline{\lambda}^{(k)}$ denote the supremum of the support of this order statistic, that is,

$$\overline{\lambda}^{(k)} := \sup(\text{supp}(\mathcal{B}^{(k)})).$$

We now construct a sequence of bid pairs. Let $\mathcal{T}_i(b)$ denote the Bernoulli random variable which takes value 1 when agent i has an equilibrium bid strictly above b. The total equilibrium bids above b is

$$\mathcal{T}(b) = \sum_{i \in \mathcal{N}^{(k)}} \mathcal{T}_i(b).$$

The total number of equilibrium bids strictly above a threshold b for agents other than i is

$$\mathcal{T}_{-i}(b) = \sum_{l \in \mathcal{N}^{(k)} \setminus \{i\}} \mathcal{T}_l(b).$$

Fix any constant δ in the interval $(0, 0.5)$. We define the bid $\overline{\psi}_i^{(k)}$ as follows

$$\overline{\psi}_i^{(k)} := \min \left\{ b \geq 0 \middle| \, \mathbb{E}[\mathcal{T}_{-i}(b)] \leq k - k^{0.5+\delta} - 1 \right\}.$$

The existence of $\overline{\psi}_i^{(k)}$ is guaranteed by the right-continuity of the function $\mathbb{E}[\mathcal{T}_{-i}(b)]$. Similarly, we define the bid $\underline{\psi}_i^{(k)}$ as

$$\underline{\psi}_i^{(k)} := \min \left\{ b \geq 0 \middle| \, \mathbb{E}[\mathcal{T}_{-i}(b)] \leq k + k^{0.5+\delta} - 1 \right\}.$$

We define the sequence of bid pair collections $\boldsymbol{\mathcal{P}}^{\psi} = \{\boldsymbol{\mathcal{P}}_k^{\psi}\}_{k=1}^{\infty}$ as follows:

$$\boldsymbol{\mathcal{P}}_k^{\psi} = \left\{ \left(\underline{\psi}_i^{(k)}, \overline{\psi}_i^{(k)} \right) \right\}_{i \in \mathcal{N}^{(k)}}.$$

The following result establishes that the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi}$ is extremizing. The proof leverages a law of large numbers argument to show that the allocation probability converges to 0 when agent i bids $\underline{\psi}_i^{(k)}$, and to 1 when bidding $\overline{\psi}_i^{(k)}$.

Lemma 2. *Suppose $\underline{\psi}_i^{(k)} > 0$ and $\overline{\psi}_i^{(k)} < \overline{\lambda}^{(k)}$ for every i and k. Then, the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi}$ is extremizing.*

Proof. Consider any k and any agent $i \in \mathcal{N}^{(k)}$. First, we analyze the probability of allocation at $\overline{\psi}_i^{(k)}$. This involves solving for the probability of the event at most $k-1$ equilibrium bids are strictly above $\overline{\psi}_i^{(k)}$. Consider the following sequence of inequalities:

$$\begin{aligned}
\mathbb{P}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right) \leq k-1\right] &= \mathbb{P}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right) - \left(k - k^{0.5+\delta} - 1\right) \leq k^{0.5+\delta}\right], \\
&\geq \mathbb{P}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right) - \mathbb{E}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right)\right] \leq k^{0.5+\delta}\right], \\
&\geq \mathbb{P}\left[\left|\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right) - \mathbb{E}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right)\right]\right| \leq k^{0.5+\delta}\right], \\
&\geq 1 - 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}.
\end{aligned}$$

The first inequality follows from the definition of $\overline{\psi}_i^{(k)}$ for which

$$\mathbb{E}\left[\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right)\right] = k - k^{0.5+\delta} - 1. \tag{9}$$

The equality in (9) follows from Theorem 8 in Section 5, which states no equilibrium bid distribution can have a point mass in the interval $\left(0, \overline{\lambda}^{(k)}\right)$. The final inequality follows from applying the Chernoff bound to $\mathcal{T}_{-i}\left(\overline{\psi}_i^{(k)}\right)$ which is a sum of independent random variables.

We will now analyze the probability of allocation at $\overline{\psi}_i^{(k)}$. Consider an $\epsilon > 0$ with a corresponding bid $b_i = \overline{\psi}_i^{(k)} + \epsilon$. We have shown that with high probability at most $k-1$ bids are strictly above $\overline{\psi}_i^{(k)}$. Thus, the probability of allocation on bidding b_i is at least

$$x_i(b_i) \geq 1 - 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}.$$

By the continuity of an agents allocation rule (see Theorem 9 in Section 5), the probability of allocation at $\overline{\psi}_i^{(k)}$ is at least

$$x_i\left(\overline{\psi}_i^{(k)}\right) \geq 1 - 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}.$$

Since the above inequality is satisfied for every agent i, it follows that

$$\lim_{k \to \infty} \min_{i \in \mathcal{N}^{(k)}} x_i\left(\overline{\psi}_i^{(k)}\right) = 1.$$

Now we analyze the probability of allocation at $\underline{\psi}_i^{(k)}$. Observe that if agent i was allocated an item for bidding $\underline{\psi}_i^{(k)}$, then it must have been the case that at most $k-1$ bids are strictly greater than $\underline{\psi}_i^{(k)}$. This implies the inequality

$$x_i\left(\underline{\psi}_i^{(k)}\right) \leq \mathbb{P}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right) \leq k-1\right]. \tag{10}$$

The rest of this proof analyzes the event at most $k-1$ bids are strictly greater than $\underline{\psi}_i^{(k)}$. Consider the following sequence of inequalities:

$$\begin{aligned}
\mathbb{P}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right) \leq k-1\right] &= \mathbb{P}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right) - \left(k + k^{0.5+\delta} - 1\right) \leq -k^{0.5+\delta}\right], \\
&= \mathbb{P}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right) - \mathbb{E}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right)\right] \leq -k^{0.5+\delta}\right], \\
&\leq \mathbb{P}\left[\left|\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right) - \mathbb{E}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right)\right]\right| \geq k^{0.5+\delta}\right], \\
&\leq 2e^{-\frac{1}{3} \cdot \frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}.
\end{aligned}$$

The second equality is true because $\underline{\psi}_i^{(k)} > 0$, where Theorem 8 implies

$$\mathbb{E}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}\right)\right] = k + k^{0.5+\delta} - 1.$$

The final inequality follows from applying the Chernoff bound to a sum of independent random variables. Since the above inequality is satisfied for every agent i, it follows that

$$\lim_{k \to \infty} \max_{i \in \mathcal{N}^{(k)}} x_i\left(\underline{\psi}_i^{(k)}\right) = 0.$$

The statement of the result follows. □

In order to apply Theorem 3 to $\boldsymbol{\mathcal{P}}^{\psi}$, it remains to show

$$\lim_{k\to\infty} \max_{i\in\mathcal{N}^{(k)}} \left(\overline{\psi}_i^{(k)} - \underline{\psi}_i^{(k)}\right) = 0.$$

To facilitate this analysis, we introduce an alternative sequence of extremizing bids. Define

$$\underline{\psi}_*^{(k)} := \max_{i\in\mathcal{N}^{(k)}} \underline{\psi}_i^{(k)} \quad \text{and} \quad \overline{\psi}_*^{(k)} := \min_{i\in\mathcal{N}^{(k)}} \overline{\psi}_i^{(k)}.$$

Using these bids, we define the sequence of bid pair collections $\boldsymbol{\mathcal{P}}^{\psi^*} = \{\boldsymbol{\mathcal{P}}_k^{\psi^*}\}_{k=1}^{\infty}$ as follows:

$$\boldsymbol{\mathcal{P}}_k^{\psi^*} := \left\{\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right)\right\}_{i\in\mathcal{N}^{(k)}}.$$

This sequence is analytically convenient because, for any given auction involving k units, the bids are identical across agents.

Lemma 3. *Suppose $\underline{\psi}_i^{(k)} > 0$ and $\overline{\psi}_i^{(k)} < \overline{\lambda}^{(k)}$ for every i and k. Then, the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi^*}$ is extremizing.*

Proof. We begin by analyzing the allocation under the bid $\overline{\psi}_*^{(k)}$ in an auction with k units. For notational simplicity, we suppress the dependence on k throughout the proof. Consider an agent

$$i^* \in \arg\min_{i\in\mathcal{N}} \overline{\psi}_i.$$

We now show that the allocation probability at $\overline{\psi}_*^{(k)}$ is similarly large for all other agents. Consider any $i \in \mathcal{N}$ and the difference in allocation probability

$$\left|x_{i^*}\left(\overline{\psi}_*\right) - x_i\left(\overline{\psi}_*\right)\right|.$$

Define

$$\mathcal{T}_{-ii^*}(b) := \sum_{j\in\mathcal{N}\setminus\{i,i^*\}} \mathcal{T}_j(b).$$

By Lemma 3.2 in Swinkels (2001), the difference in probability can be bounded as

$$\left|x_{i^*}\left(\overline{\psi}_*\right) - x_i\left(\overline{\psi}_*\right)\right| \leq \Pr\left[k-2 \leq \mathcal{T}_{-ii^*}(\overline{\psi}_*) \leq k\right]. \tag{11}$$

By the definition of $\overline{\psi}_*$, the assumption that $\overline{\psi}_* \in \left(0, \overline{\lambda}^{(k)}\right)$, and the continuity of agents' allocation rules (see Theorem 9 in Section 5), it follows that

$$\mathbb{E}\left[\mathcal{T}_{-i^*}\left(\overline{\psi}_*\right)\right] = k - k^{0.5+\delta} - 1.$$

This expression can be decomposed as

$$\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] + \Pr\left[\mathcal{B}_i > \overline{\psi}_*\right] = k - k^{0.5+\delta} - 1,$$

which implies the bound $\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \in \left[k - k^{0.5+\delta} - 2, k - k^{0.5+\delta} - 1\right]$. We will analyze the right-hand side of (11). Consider the following chain of inequalities:

$$\begin{aligned}
\Pr\left[k-2 \leq \mathcal{T}_{-ii^*}(\overline{\psi}_*) \leq k\right] &= \Pr\left[k-2-\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \leq \mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right) - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \leq k - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right]\right], \\
&\leq \Pr\left[k^{0.5+\delta} - 1 \leq \mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right) - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \leq k^{0.5+\delta} + 2\right], \\
&\leq \Pr\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right) - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \geq k^{0.5+\delta} - 1\right].
\end{aligned}$$

Applying a Chernoff bound to $\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)$ while using the lower bound

$$\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \geq k - k^{0.5+\delta} - 2,$$

we obtain the following tail inequality

$$\Pr\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right) - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\overline{\psi}_*\right)\right] \geq k^{0.5+\delta} - 1\right] \leq e^{-\frac{k^{2\delta} - \frac{2}{k^{0.5-\delta}} + \frac{1}{k}}{2 - \frac{1}{k^{0.5-\delta}} - \frac{3}{k}}}.$$

Thus,

$$\left|x_{i^*}\left(\overline{\psi}_*\right) - x_i\left(\overline{\psi}_*\right)\right)\right| \leq e^{-\frac{k^{2\delta} - \frac{2}{k^{0.5-\delta}} + \frac{1}{k}}{2 - \frac{1}{k^{0.5-\delta}} - \frac{3}{k}}}. \tag{12}$$

It is important to note that the inequality above if for any agent $i \in \mathcal{N}$. Recall, from the proof of Lemma 2, that the allocation to agent i^* under $\overline{\psi}_*$ satisfies

$$x_{i^*}(\overline{\psi}_*) \geq 1 - 2e^{-\frac{1}{3}\cdot\frac{(k)^{2\delta}}{1 - \frac{1}{k^{0.5-\delta}} - \frac{1}{k}}}. \tag{13}$$

Combining equations (12) and (13), we conclude that

$$\lim_{k\to\infty} \min_{i\in\mathcal{N}^{(k)}} x_i^{(k)}\left(\overline{\psi}_*^{(k)}\right) = 1.$$

That is, as the number of units grows, every agent receives an allocation arbitrarily close to one under the bid $\overline{\psi}_*^{(k)}$.

A similar analysis establishes the difference in allocation at $\underline{\psi}_*$:

$$\left|x_{i_*}\left(\underline{\psi}_*\right) - x_i\left(\underline{\psi}_*\right)\right| \leq e^{-\frac{k^{2\delta} - \frac{4}{k^{0.5-\delta}} + \frac{4}{k}}{2 + \frac{2}{k^{0.5-\delta}} - \frac{1}{k}}},$$

where $i_* \in \arg\max_{i\in\mathcal{N}} \underline{\psi}_i$. We provide a proof for completeness. By the definition of $\underline{\psi}_*$, the assumption that $\underline{\psi}_i \in (0, \overline{\lambda}^{(k)})$, and the continuity of agents' allocation rules (see Theorem 9 in Section 5), it follows that

$$\mathbb{E}\left[\mathcal{T}_{-i^*}\left(\underline{\psi}_*\right)\right] = k + k^{0.5+\delta} - 1.$$

This expression can be decomposed as

$$\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\underline{\psi}_*\right)\right] + \Pr\left[\mathcal{B}_i > \underline{\psi}_*\right] = k - k^{0.5+\delta} - 1,$$

which implies the bound $\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\underline{\psi}_*\right)\right] \in \left[k + k^{0.5+\delta} - 2, k + k^{0.5+\delta} - 1\right]$. An analysis identical to that for $\overline{\psi}_*$ establishes that

$$\begin{aligned}\left|x_{i_*}\left(\underline{\psi}_*\right) - x_i\left(\underline{\psi}_*\right)\right| &\leq \Pr\left[\mathcal{T}_{-ii^*}\left(\underline{\psi}_*\right) - \mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\underline{\psi}_*\right)\right] \leq -\left(k^{0.5+\delta} - 2\right)\right], \\ &\leq e^{-\frac{k^{2\delta} - \frac{4}{k^{0.5-\delta}} + \frac{4}{k}}{2 + \frac{2}{k^{0.5-\delta}} - \frac{1}{k}}},\end{aligned}$$

where we apply a Chernoff bound while using the lower bound on $\mathbb{E}\left[\mathcal{T}_{-ii^*}\left(\underline{\psi}_*\right)\right]$. It is important to note that the inequality above if for any agent $i \in \mathcal{N}$.

Recall, from the proof of Lemma 2, that the allocation to agent i_* under $\underline{\psi}_*$ satisfies

$$x_{i_*}(\underline{\psi}_*) \leq 2e^{-\frac{1}{3}\cdot\frac{(k)^{2\delta}}{1-\frac{1}{k^{0.5-\delta}}-\frac{1}{k}}}. \tag{14}$$

Combining (14) and the bound on $\left|x_{i_*}\left(\underline{\psi}_*\right) - x_i\left(\underline{\psi}_*\right)\right|$, we conclude that

$$\lim_{k\to\infty}\max_{i\in\mathcal{N}^{(k)}} x_i^{(k)}\left(\underline{\psi}_*^{(k)}\right) = 0.$$

That is, as the number of units grows, every agent receives an allocation arbitrarily close to zero under the bid $\underline{\psi}_*^{(k)}$. The statement of the result follows. □

We now verify that condition (b) of Theorem 3 holds. In particular, under Assumption 2 we establish that

$$\lim_{k\to\infty}\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) = 0.$$

The proof proceeds by contradiction. Suppose the quantity $\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right)$ does not converge to 0. Then equilibrium bids satisfying asymptotic proportional extremal bidding will have in expectation a linear number of equilibrium bids in the interval $\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right)$ for a large enough k and some agent i. This contradicts the observation that $O(k^{0.5+\delta})$ equilibrium bids are expected to lie in this region.

Lemma 4. *Consider the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi}$. Suppose $\underline{\psi}_i^{(k)} > 0$ and $\overline{\psi}_i^{(k)} < \overline{\lambda}^{(k)}$ for every i and k. The following condition is satisfied:*

$$\lim_{k\to\infty}\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) = 0.$$

Moreover, the rate at which it converges to 0 *is*

$$\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) \leq O\left(\frac{1}{k^{0.5-\delta}}\right).$$

Proof. Suppose $\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right)$ does not converge to 0 as $k \to \infty$. That is to say, there exists an $\epsilon > 0$ and a $K > 0$ such that $\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) > \epsilon$ for all $k > K$. For a large enough $k > K$, consider an agent $i_*^{(k)} \in \arg\max_{i\in\mathcal{N}^{(k)}}\left(\overline{\psi}_i^{(k)} - \underline{\psi}_i^{(k)}\right)$.

Recall, by definition

$$\overline{\psi}_*^{(k)} = \min_{i\in\mathcal{N}^{(k)}} \overline{\psi}_i^{(k)} \quad \text{and} \quad \underline{\psi}_*^{(k)} = \max_{i\in\mathcal{N}^{(k)}} \underline{\psi}_i^{(k)}.$$

By the continuity of equilibrium bid distributions and definition of $\overline{\psi}_i^{(k)}$ and $\underline{\psi}_i^{(k)}$, the expected number of competing equilibrium bids strictly above $\underline{\psi}_*^{(k)}$ but at most $\overline{\psi}_*^{(k)}$ is at most

$$\mathbb{E}\left[\mathcal{T}_{-i}\left(\underline{\psi}_*^{(k)}\right)\right] - \mathbb{E}\left[\mathcal{T}_{-i}\left(\overline{\psi}_*^{(k)}\right)\right] \leq 2\,(k)^{0.5+\delta}. \tag{15}$$

As we have established in Lemma 3, the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi^*}$ is extremizing. By Assumption 2, for a k large enough, every agent $i \in \mathcal{N}^{(k)}$ will have a valuation interval of length at least $\beta\epsilon$ such that every valuation in that interval has an equilibrium bid in $\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right)$. We will denote the number of competing equilibrium bids in the interval $\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right)$ by $\mathcal{T}_{-i}\left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right)$. By Uniform Value Support, the expected value of this random variable is at least

$$\begin{aligned}\mathbb{E}\left[\mathcal{T}_{-i}\left(\underline{\psi}_i^{(k)}, \overline{\psi}_i^{(k)}\right)\right] &\geq \sum_{j \in \mathcal{N}^{(k)} \setminus \{i\}} \epsilon\beta\kappa, \\ &\geq \left(n^{(k)} - 1\right)\epsilon\beta\kappa.\end{aligned}$$

But this is a contradiction to (15) because $n^{(k)} \geq k$. Thus, it must be that

$$\lim_{k\to\infty}\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) = 0.$$

Furthermore, the rate of convergence is $\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) < O(\frac{1}{k^{0.5-\delta}})$. To see this, by Assumption 2, Uniform Value Support and (15), it must be that

$$\left(n^{(k)} - 1\right)\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right)\beta\kappa \leq 2k^{0.5+\delta},$$

and consequently

$$\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right) = O\left(\frac{k^{0.5+\delta}}{n^{(k)}}\right).$$

Substituting $n^{(k)} \geq k$ establishes the desired rate of convergence. □

Finally, we prove the main result of this section.

Theorem 4. *Consider a sequence of winner-pays-bid highest-bids-win auctions $\left(\mathcal{A}^{(k)}\right)_{k=1}^{\infty}$ and a corresponding sequence of Bayes-Nash equilibrium $\left(\mathcal{E}^{(k)}\right)_{k=1}^{\infty}$. Consider any $\delta \in (0, 0.5)$. Suppose the equilibria satisfy Assumption 2 and the bid pairs $\left(\underline{\psi}_i^{(k)}, \overline{\psi}_i^{(k)}\right)$ are such that $\underline{\psi}_i^{(k)} > 0$ and $\overline{\psi}_i^{(k)} < \overline{\lambda}^{(k)}$ for every i and k. The price of anarchy is at least $1 - O(1/k^{0.5-\delta})$, that is, for any values distributions $\boldsymbol{\mathcal{V}}$ satisfying Assumption 1, the ratio*

$$\frac{\mathcal{A}^{(k)}(\boldsymbol{\mathcal{V}})}{\mathrm{OPT}(\boldsymbol{\mathcal{V}})} \geq 1 - O\left(\frac{1}{k^{0.5-\delta}}\right).$$

Proof. Consider the collection of sequence of bid pairs $\boldsymbol{\mathcal{P}}^{\psi^*} = \{\boldsymbol{\mathcal{P}}_i^{\psi^*}\}$ where

$$\boldsymbol{\mathcal{P}}_i^{\psi^*} = \left(\mathcal{P}_i^{(k)}\right)_{k=1}^{\infty} \text{ with } \mathcal{P}_i^{(k)} := \left(\underline{\psi}_*^{(k)}, \overline{\psi}_*^{(k)}\right).$$

By Lemma 3, this collection is extremizing. By Lemma 4, condition (b) of Theorem 3 is satisfied. And finally, we show continuity and strict monotonicity of the competing bid distributions in Section 5. An application of Theorem 3 implies the price of anarchy converges to 1 as $k \to \infty$.

Moreover, the rate of convergence to 1 is governed by the rate of convergence of the quantity $\left(\overline{\psi}_*^{(k)} - \underline{\psi}_*^{(k)}\right)$ to 0. This is because the rate of convergence of allocation probabilities in the extremizable sequence is exponential in k (see the proof of Lemma 3). Thus the rate of convergence is $O(\frac{1}{k^{0.5-\delta}})$. □

5 Continuity and Monotonicity of Equilibrium Bid Distributions

In the previous section, our smoothness proofs relied on the assumption that equilibrium bids are continuous and strictly increasing. We now establish that this assumption holds. Section 5.1 derives some standard properties of equilibrium bid distributions in the multi-unit setting. Section 5.2 applies an extension of a neighborhood deviation argument to show this bid distribution is continuous and strictly increasing.

5.1 Properties of Equilibrium Bid Distributions

Before proving our main result, we introduce key definitions used throughout this section. Let $\mathcal{B}_i$ denote the equilibrium bid distribution of agent i. We define γ_i as the infimum of the support of agent i's equilibrium bid distribution:

$$\gamma_i := \inf(\text{supp}(\mathcal{B}_i)) \text{ and } \lambda_i := \sup(\text{supp}(\mathcal{B}_i)).$$

Additionally, we define $\mathbf{\Gamma}$ as the collection of γ_i values, sorted in descending order:

$$\mathbf{\Gamma} := \{\gamma_i\}_{i\in\mathcal{N}}.$$

We denote the i^{th} element in this ordered set by $\gamma_{(i)}$.

Finally, let $\mathcal{B}^{(k)}$ represent the distribution of the k^{th} highest equilibrium bid. We define its infimum as:

$$\gamma^{(k)} := \inf(\text{supp}(\mathcal{B}^{(k)})).$$

Lemma 5. *The infimum of the support of the bid distribution of the k^{th}-highest agent is the k^{th} element in the order $\mathbf{\Gamma}$, that is,*

$$\gamma^{(k)} = \gamma_{(k)}.$$

Proof. Consider any agent i such that γ_i is in the top k elements of the order $\mathbf{\Gamma}$. By definition of γ_i, this agent almost surely has a bid $b \geq \gamma_i$. Since this holds for any agent corresponding to a top k element in the order $\mathbf{\Gamma}$, the bid of the k^{th} highest agent is almost surely at least $\gamma_{(k)}$. Thus it must be that

$$\gamma^{(k)} \geq \gamma_{(k)}.$$

We will now show that in fact $\gamma^{(k)} = \gamma_{(k)}$. Let us denote the agent with the k^{th} highest element in $\mathbf{\Gamma}$ by i^*, that is, $\gamma_{i^*} = \gamma_{(k)}$. By the definition of γ_{i^*}, agent i^* bids in the interval $\left[\gamma_{(k)}, \gamma^{(k)}\right)$ with strictly positive probability. Suppose $\gamma^{(k)} > \gamma_{(k)}$. This implies that the bid of the k^{th} highest agent lies in the interval $\left[\gamma_{(k)}, \gamma^{(k)}\right)$ with probability 0. This is a contradiction to agent i^* having their γ_{i^*} in the top k elements of $\mathbf{\Gamma}$. □

The result above demonstrated an equivalence of the infimum of equilibrium bid distributions and the distribution of the k^{th} highest bid. In addition to this, we prove a crucial result which shows that the $k+1$ highest suprema of equilibrium bid distributions must all be the same.

Lemma 6 (Support upper bound equivalence)**.** *Suppose $\lambda_{(i)}$ is the i^{th} highest value in the set $\{\lambda_j\}_{j\in\mathcal{N}}$. Then,*

$$\lambda_{(1)} = ... = \lambda_{(k+1)}.$$

Proof. Suppose for a contradiction it is not true that $\lambda_{(1)} = ... = \lambda_{(k+1)}$. Then, since $\lambda_{(1)} \geq ... \geq \lambda_{(k+1)}$, we must have

$$\lambda_{(k+1)} < \lambda_{(1)}.$$

This implies the following: $\lambda_{(k+1)}$ is finite and the region $(\lambda_{(k+1)}, \lambda_{(1)}] \subseteq \mathbb{R}$ is non-empty. In addition to this, there must exist an agent i with $\lambda_i = \lambda_{(1)}$ who bids in $(\lambda_{(k+1)}, \lambda_{(1)}]$ with positive probability, and for all $z \in (\lambda_{(k+1)}, \lambda_{(1)}]$, this agent's interim allocation is $x_i(z) = 1$.

This permits agent i to adjust their bidding strategy such that, whenever they would have bid $b_i \in (\lambda_{(k+1)}, \lambda_{(1)}]$, they instead bid

$$b_i^* := \lambda_{(k+1)} + (b_i - \lambda_{(k+1)})/2.$$

This keeps their interim allocation allocation $x_i(b_i^*) = x_i(b_i) = 1$ constant, but strictly decreases their bid, since $b_i^* < b_i$. Thus, this new strategy strictly increases agent i's ex-ante utility, and so their original bidding strategy must not have been optimal, which is a contradiction. □

We now present results that extend the equilibrium properties established by Jin and Lu (2023) from the single-unit setting to the multi-unit setting.

The first result concerns an agent's interim allocation rule and shows several standard properties. Specifically, an agent's interim allocation weakly increases in their bid. Additionally, in equilibrium, a bid above $\gamma^{(k)}$ secures a unit with strictly positive probability, while a bid below $\gamma^{(k)}$ has a zero probability in receiving a unit.

Lemma 7. *The interim allocation rule $x_i(b)$ for any agent i satisfies the following properties:*

(a) $x_i(b)$ is weakly increasing in b.

(b) $x_i(b)$ is bounded as

$$\sup\{\mathcal{B}_{-i}^{(k)}(t) : t < b\} \leq x_i(b) \leq \mathcal{B}_{-i}^{(k)}(b) \quad \text{for all } b.$$

(c) Consider any bid b in the support of the bid distribution $\mathcal{B}_i$. It has interim allocation

(i) $x_i(b) = 0$ when $b < \gamma^{(k)}$.
(ii) $x_i(b) > 0$ when $b > \gamma^{(k)}$.

Next, we examine the equilibrium strategy of an agent. The agent obtains positive utility if and only their value is strictly greater than the infimum $\gamma^{(k)}$. Furthermore, they submit a bid at or above this threshold if their value exceeds it strictly.

Lemma 8. *Consider an agent i and value $v \in \text{supp}(V_i)$. The equilibrium strategy $s_i(\cdot)$ for any agent i satisfies the following properties:*

(a) If $v \leq \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) = 0$ and the equilibrium bid $s_i(v) \leq \gamma^{(k)}$ almost surely over the random strategy $s_i(\cdot)$.

(b) If $v > \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) > 0$ and the equilibrium bid $\gamma^{(k)} \leq s_i(v) < v$ almost surely over the random strategy $s_i(\cdot)$.

Furthermore, we reference a result from Chawla and Hartline (2013), which establishes that an agent obtains strictly higher utility with a larger value, and, almost surely, submits a higher bid. For completeness, we include a proof of this result in the appendix.

Lemma 9 (Chawla and Hartline, 2013)**.** *Consider values v, w in the support of value distribution V_i such that $v > w > \gamma^{(k)}$. The following properties hold:*

(a) The optimal utility at these values is ordered as

$$\mathcal{U}_i(v) > \mathcal{U}_i(w) > 0.$$

(b) The equilibrium bids are ordered as $s_i(v) \geq s_i(w)$ almost surely over the random strategy $s_i(\cdot)$.

5.2 An Application of Neighborhood Deviation Arguments

This section establishes our main continuity and monotonicity result. We utilize a standard neighborhood deviation argument (see Jin and Lu, 2023) as a basis for the rest of our proof. They establish the continuity and monotonicity of equilibrium bid distributions in the single-unit setting. A key advantage of that setting is it only requires analyzing the highest equilibrium bid, which corresponds to the first-order statistic.

In the multi-unit setting however, a fundamental challenge arises: changes in an agent's equilibrium bid distribution need not necesssarily impact the competing bid distribution of another agent. This is because an agent's allocation is determined solely by the k^{th}-highest bid among the remaining agents. Addressing this issue requires a different set of arguments. Before we prove the main result of this section, we provide the neighborhood deviation argument for completeness.

Lemma 10 (Neighborhood Deviation Argument, Jin and Lu, 2023)**.** *Consider any agent i with interim allocation rule $x_i(\cdot)$. The following conditions are satisfied for any bid b:*

(a) Suppose there is a sequence of bids $(b_m)_{m=1}^{\infty}$ such that $\lim_{m\to\infty} b_m = b$. Moreover, there is a $\delta > 0$ for which $x_i(b_m) \geq x_i(b) + \delta$ for any m. Then b is not an equilibrium bid.

(b) Suppose there is a sequence of bids $(b_m)_{m=1}^{\infty}$ and a $\delta > 0$ such that $b - b_m \geq \delta$ for every m. Moreover, the allocation probability $\lim_{m\to\infty} x_i(b_m) = x_i(b)$. Then b is not an equilibrium bid.

Proof. We first prove part (a). Take a bid from the sequence $(b_m)_{m=1}^{\infty}$ that is arbitrarily close to b. This would give an agent strictly better utility than bidding b because $x_i(b_m) \geq x_i(b) + \delta$ for $\delta > 0$.

We next prove part (b). Take a bid from the sequence $(b_m)_{m=1}^{\infty}$ such that its allocation probability is arbitrarily close to $x_i(b)$. This would give an agent strictly better utility than bidding b because $b - b_m \geq \delta$ but their allocation probabilities are arbitrarily close. □

We establish a key result below demonstrating that at most one agent can have a probability mass at any bid within the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. The argument used to prove this result will serve as a foundation for several proofs later on.

Lemma 11. *Consider any bid b in $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. The bid distribution of at most 1 agent places strictly positive probability mass at b.*

Proof. Suppose there are 2 agents, i and j, such that their bid distributions $\mathcal{B}_i$ and $\mathcal{B}_j$ have probability mass at b. Our goal is to show that the interim allocation for one of these agents has a discontinuity at b. A sufficient condition for this is if the following properties hold:

(i) With strictly positive probability, b is the bid of the k^{th} highest agent.

(ii) With strictly positive probability, exactly $k-1$ agents other than i and j bid strictly above b.

The rest of the proof will focus on showing the properties above. Define L as

$$L := \{l : \mathcal{B}_l(b) = 0\}.$$

The agents in L bid above b almost surely. It must be the case that $|L| \leq k-1$ otherwise we would have $\gamma^{(k)} \geq b$ which is not possible. Next, define T as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of Lemma 6 tells us that $|T| \geq k+1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i, j\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k+1$ and $|L| \leq k-1$. We now consider the set $\mathcal{S} = L \cup T'$. We will show that agents in $\mathcal{S}$ bid above b with strictly positive probability. By construction, the size of $\mathcal{S}$ is $k-1$. The agents in L bid above b almost surely. Any agent $l \in T'$ has $\lambda_l > b$ and thus bids above b with strictly positive probability. It follows that with strictly positive probability, agents in $\mathcal{S}$ bid above b with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{B}_l(b) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid at most b with strictly positive probability. Since agents i and j have probability mass at b, it is the bid of the k^{th} highest agent with strictly positive probability. Moreover, exactly $k-1$ agents bid strictly above b with positive probability. The statement follows. □

We establish one of the main results of this section. Recall, we denote by $\mathcal{B}^{(k)}$ the distribution of the k^{th} highest equilibrium bid. The distribution $\mathcal{B}^{(k)}$ has a strictly increasing CDF in the closed interval $\left[\gamma^{(k)}, \lambda^{(k)}\right]$. The proof applies an inductive argument on the agents to show that the distribution of the k^{th} highest bid cannot be flat in any open interval of $\left[\gamma^{(k)}, \lambda^{(k)}\right]$.

Theorem 5. *The distribution of the k^{th} highest bid $\mathcal{B}^{(k)}$ has a strictly increasing CDF on the interval $\left[\gamma^{(k)}, \lambda^{(k)}\right]$.*

Proof. We will first show that the bid distribution of the k^{th} highest agent $\mathcal{B}^{(k)}$ is strictly increasing on the interval $[\gamma^{(k)}, \lambda^{(k)}]$. We will state an important lemma that will be used in the proof below.

Lemma 12. *Suppose the bid distribution of the k^{th} highest agent $\mathcal{B}^{(k)}$ is flat in (α, β). Then, no agent has a point mass in their bid distribution at any bid in (α, β).*

Proof. Suppose agent i has a point mass at some bid b in the interval (α, β). We will follow the approach in the proof of Lemma 11. Our goal is to show that with strictly positive probability there are exactly $k-1$ agents who bid strictly above b and the rest bid strictly below b. This will imply that the bid of the k^{th} highest agent is b with strictly positive probability which is a contradiction to the assumption that $\mathcal{B}^{(k)}$ is flat in (α, β).

Define L as

$$L := \{l : \mathcal{B}_l(b) = 0\}.$$

The agents in L bid above b almost surely. It must be the case that $|L| \leq k-1$ otherwise we would have $\gamma^{(k)} \geq b$ which is a contradiction to $\gamma^{(k)} < \alpha$. Next, define T as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of Lemma 6 tells us that $|T| \geq k+1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k+1$ and $|L| \leq k-1$. We now consider the set $\mathcal{S} = L \cup T'$. We will show that agents in $\mathcal{S}$ bid above b with strictly positive probability. By construction, the size of $\mathcal{S}$ is $k-1$. The agents in L bid above b almost surely. Consider any agent l in T'. This agent would bid strictly above b with positive probability because $\lambda_l > b$. It follows that agents in $\mathcal{S}$ bid above b with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{B}_l(b) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid at most b with strictly positive probability. Since agent i has a probability mass at b, it is the bid of the k^{th} highest agent with strictly positive probability.

This is a contradiction to the assumption that $\mathcal{B}^{(k)}$ is flat in (α, β). The statement of the result follows. □

Suppose $\mathcal{B}^{(k)}$ has no probability density or mass on an open interval (α, β). It is without loss for the interval to be maximal in the sense that for any $\delta > 0$, $\mathcal{B}^{(k)}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. Thus, we have that $\mathcal{B}^{(k)}(\beta) \geq \mathcal{B}^{(k)}(\alpha) > 0$. Consider the following cases:

(i) *Case* 1: Suppose $\mathcal{B}^{(k)}(\beta) > \mathcal{B}^{(k)}(\alpha) > 0$. It must be the case that $\mathcal{B}^{(k)}$ has a point mass at β. And thus, there is some agent i_0 that has probability mass at β in their equilibrium bid distribution. By Lemma 11, no other agent has a point mass at β in their bid distribution. The competing bid distribution $\mathcal{B}^{(k)}_{-i_0}$ is thus continuous at β.

If it is the case that agent i_0's competing bid distribution $\mathcal{B}^{(k)}_{-i_0}$ has no probability density or probability mass in the open interval (α, β), then agent i_0's interim allocation is constant in $(\alpha, \beta]$. By Lemma 10, it is suboptimal for i_0 to have probability mass at β which is a contradiction to $\beta \in \text{supp}(\mathcal{B}_{i_0})$.

Thus, it must be that $\mathcal{B}^{(k)}_{-i_0}$ has probability density or probability mass at some point in (α, β). But, because the distribution of the k^{th} highest bid was flat in (α, β) it must be the case that

$$\gamma_{i_0} \geq \sup\left\{(\alpha, \beta) \cap \text{supp}\left(\mathcal{B}^{(k)}_{-i_0}\right)\right\}. \tag{16}$$

This is because, if it were the case that $\gamma_{i_0} < \sup\left\{(\alpha, \beta) \cap \text{supp}\left(\mathcal{B}^{(k)}_{-i_0}\right)\right\}$, then the bid of the k^{th} highest agent apart from i_0 becomes the bid of the k^{th} highest agent overall with strictly positive probability. Moreover, such bids would lie in (α, β) which is a contradiction to the assumption that the distribution of the k^{th} highest bid overall is flat in (α, β).

Recall, we are analyzing a scenario in which $\mathcal{B}^{(k)}_{-i_0}$ has probability density or probability mass at some point in (α, β). There must be an agent i_1 (clearly not the same as i_0 because we are analyzing the competing bid distribution) and a corresponding bid $\beta_1 \in (\alpha, \beta)$ such that i_1 has probability density on β_1 and $\gamma_{i_0} \geq \beta_1$ (an application of Lemma 12 tells us that this agent will not have a probability mass at β_1). Moreover, since no agent has a point mass at β_1, as shown in Lemma 12, agent i_1's competing bid distribution is continuous at β_1.

We will inductively apply the above argument to show there are k agents whose $\gamma_i > \alpha$. This is a contradiction to $(\alpha, \beta) \subset [\gamma^{(k)}, \lambda^{(k)}]$.

Inductive hypotheses: Suppose we have a set of agents $\mathcal{I}_l = \{i_0, \ldots, i_{l-1}\}$, where $|\mathcal{I}_l| < k$. In addition, suppose this set satisfies the conditions stated below:

(a) There is a corresponding set of bids $\mathbb{B}_l := \{\beta_j\}_{j=0}^{l-1}$ where $\beta_0 = \beta$ and $\beta_j \in (\alpha, \beta)$ for any $j \in [l-1]$. Moreover, the bids are monotonically decreasing, that is,

$$\beta_j > \beta_{j+1} \quad \text{for all } j \in \{0, \ldots, l-1\}.$$

(b) For every agent in $\mathcal{I}_l$, the competing bid distribution $\mathcal{B}_{-i_j}^{(k)}$ has probability density at some bid in (α, β_j).

(c) For every agent i_j in $\mathcal{I}_l$, the infimum of the support of the equilibrium bid distribution satisfies

$$\gamma_{i_j} \geq \sup\left\{(\alpha, \beta_j) \cap \operatorname{supp}\left(\mathcal{B}_{-i_j}^{(k)}\right)\right\}.$$

(d) For any agent i_j, where $j \in \{0, \ldots, l-2\}$,

$$\gamma_{i_j} \geq \beta_{j+1}.$$

(e) There is an agent $i_l \notin \mathcal{I}_l$ and a bid β_l in its support such that

$$\alpha < \beta_l < \beta_{l-1} \text{ and } \gamma_{i_{l-1}} \geq \beta_l.$$

Base case: Recall, we have shown that the conditions above are satisfied for $l = 1$. The set in consideration is $\mathcal{I}_l = \{i_0\}$. Part (a) is satisfied vacuously with the set of bids $\{\beta\}$. Part (b) is satisfied because we are analyzing the scenario where $\mathcal{B}_{-i_0}^{(k)}$ has probability density or probability mass in (α, β). Part (c) is satisfied by (16). Part (d) holds vacuously. Part (e) is satisfied by agent i_1: there is a bid $\beta_1 \in \operatorname{supp}(\mathcal{B}_{i_1})$ such that $\beta_1 \in (\alpha, \beta)$ and $\gamma_{i_0} \geq \beta_1$.

Inductive argument: Suppose the conditions hold for some l where $1 < l < k$. We will analyze the competing bid distribution of agent i_l. Consider the case $\mathcal{B}_{-i_l}^{(k)}$ is flat in (α, β_l). By Lemma 12, no agent has a point mass at β_l and thus $\mathcal{B}_{-i_l}^{(k)}$ is continuous at β_l. An immediate consequence of this is i_l has a constant allocation rule in the interval $(\alpha, \beta_l]$. But, by Lemma 10, this is a contradiction to the fact that $\beta_l \in \operatorname{supp}(\mathcal{B}_{i_l})$.

Thus, it must be the case that $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l) (at the expense of being repetitive, by Lemma 12, $\mathcal{B}_{-i_l}^{(k)}$ cannot have a point mass in this interval). Let us define the set

$$\mathcal{I}_{l+1} := \mathcal{I}_l \uplus \{i_l\}.$$

By Part (e), $i_l \notin \mathcal{I}_l$, and thus you are adding a unique element to $\mathcal{I}_l$. Let us also consider the corresponding set of bids

$$\mathbb{B}_{l+1} := \{\beta_j\}_{j=0}^{l-1} \cup \{\beta_l\}.$$

By Part (e), $\beta_l < \beta_{l-1}$ and thus, applying Part (a),

$$\beta_j > \beta_{j+1} \text{ for all } j \in \{0, \ldots, l\}.$$

Thus $\mathcal{I}_{l+1}$ and $\mathbb{B}_{l+1}$ satisfies Part (a) for $l+1$ agents. Part (b) is satisfied for $(l+1)$ agents as well because we are analyzing the case $\mathcal{B}_{-i_l}^{(k)}$ has density at some point in (α, β_l).

We will now show that Part (c), (d) and (e) are also satisfied for $(l+1)$ agents. The competing bid distribution $\mathcal{B}^{(k)}_{-i_l}$ has density at some point in (α, β_l). However, the distribution of the bid of the k^{th} highest agent is flat in (α, β). Thus, it must be the case that

$$\gamma_{i_l} \geq \sup\left\{(\alpha, \beta_l) \cap \operatorname{supp}\left(\mathcal{B}^{(k)}_{-i_l}\right)\right\}. \tag{17}$$

This is because, if it were the case that $\gamma_{i_l} < \sup\left\{(\alpha, \beta_l) \cap \operatorname{supp}\left(\mathcal{B}^{(k)}_{-i_l}\right)\right\}$, then the bid of the k^{th} highest agent apart from i_l becomes the bid of the k^{th} highest agent overall with strictly positive probability. Moreover, such bids would lie in (α, β) which is a contradiction to the assumption that the distribution of the k^{th} highest bid overall is flat in (α, β). Since, by the inductive hypotheses, this condition also holds for agents $j \in \{0, \ldots, l-1\}$, Part (c) is satisfied for $(l+1)$ agents. Next, Part (d) is satisfied for $(l+1)$ agents because in Part (e) of the induction hypotheses, we had that $\gamma_{i_{l-1}} \geq \beta_l$.

Now, $\mathcal{B}^{(k)}_{-i_l}$ has density at some point in (α, β_l). Thus, there must exist an agent i_{l+1} and a corresponding bid $\beta_{l+1} \in (\alpha, \beta_l)$ such that $\beta_{l+1} \in \operatorname{supp}(\mathcal{B}_{i_{l+1}})$. In addition to this, an implication of (17) is that $\gamma_{i_l} \geq \beta_{l+1}$. Finally, the induction hypotheses tells us that for any agent in $\mathcal{I}_l$, the infimum of the support of their bid distribution

$$\gamma_{i_j} \geq \beta_{j+1} \geq \beta_l.$$

Thus it cannot be that agent i_{l+1}, who has probability density in (α, β_l), lies in the set $\mathcal{I}_l$. This satisfies Part (e).

To summarize the conclusions, at each index $l \in [k-1]$, either we have a contradiction in the case $\mathcal{B}^{(k)}_{-i_l}$ is flat in (α, β_l) or the conditions of the inductive hypotheses are satisfied for index $(l+1)$. If the latter is satisfied at each index, we have a set of k agents $\mathcal{I}_k$ such that

$$\gamma_{i_j} \geq \beta_{j+1} > \alpha \quad \text{for all } j \in \{0, \ldots, k-1\}.$$

But, this is a contradiction to the infimum of bid distribution of the k^{th} highest agent $\gamma^{(k)}$ being strictly smaller than α.

(ii) *Case* 2: Suppose $\mathcal{B}^{(k)}(\beta) = \mathcal{B}^{(k)}(\alpha) > 0$. First, we will show that no agent can have a point mass at β. To see this, consider the argument in the proof of Lemma 12. Suppose there exists an agent, labeled i, who has a point mass at β. Define L as

$$L := \{l : \mathcal{B}_l(\beta) = 0\}.$$

The agents in L bid above β almost surely. It must be the case that $|L| \leq k-1$ otherwise we would have $\gamma^{(k)} \geq \beta$ which is a contradiction to $\gamma^{(k)} < \alpha$. Next, define T as

$$T := \{l : \lambda_l = \lambda^{(k)}\}.$$

An application of Lemma 6 tells us that $|T| \geq k+1$. Consider a set T' such that

$$T' \subseteq T \setminus (\{i\} \cup L) \quad \text{and} \quad |T'| = k - 1 - |L|.$$

Such a set exists because $|T| \geq k+1$ and $|L| \leq k-1$. We now consider the set $\mathcal{S} = L \uplus T'$. We will show that agents in $\mathcal{S}$ bid above β with strictly positive probability. By construction,

the size of $\mathcal{S}$ is $k-1$. The agents in L bid above β almost surely. Any agent $l \in T'$ has $\lambda_l > \beta$. Thus, he bids strictly above β with positive probability. It follows that agents in $\mathcal{S}$ bid above β with strictly positive probability.

Now, any agent l that does not belong to $\mathcal{S}$ has $\mathcal{B}_l(\beta) > 0$. Thus, agents that do not belong to the set $\mathcal{S}$ bid strictly smaller than β with positive probability. It follows that β is the bid of the k^{th} highest agent with positive probability because agent i bids β with positive probability. This is a contradiction to the assumption that $\mathcal{B}^{(k)}(\alpha) = \mathcal{B}^{(k)}(\beta)$. Thus, no agent can have a point mass at β.

We will show that the induction argument presented in *Case 1* extends to this case as well.

Recall, the interval (α, β) is maximal in the following sense: for any $\delta > 0$, $\mathcal{B}^{(k)}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. Thus, there must exist an agent i_0 who has β in the support of his equilibrium bid distribution $\mathcal{B}_{i_0}$.

If it is the case that agent i_0's competing bid distribution $\mathcal{B}^{(k)}_{-i_0}$ has no probability density or probability mass in the open interval (α, β), then agent i_0's interim allocation is constant in $(\alpha, \beta]$. This is because no agent has a point mass at β and thus $\mathcal{B}^{(k)}_{-i_0}$ is continuous at that point. By Lemma 10, it is suboptimal for i_0 to have probability mass at β which is a contradiction to $\beta \in \text{supp}(\mathcal{B}_{i_0})$.

Thus, it must be that $\mathcal{B}^{(k)}_{-i_0}$ has probability density or probability mass at some point in (α, β). However, because the distribution of the k^{th} highest bid was flat in (α, β) it must be the case that

$$\gamma_{i_0} \geq \sup\left\{(\alpha, \beta) \cap \text{supp}\left(\mathcal{B}^{(k)}_{-i_0}\right)\right\},$$

i.e. the inequality in 16, holds.

Moreover, there must be an agent i_1 (clearly not the same as i_0 because we are analyzing the competing bid distribution) and a corresponding bid $\beta_1 \in (\alpha, \beta)$ such that i_1 has probability density on β_1 and $\gamma_{i_0} \geq \beta_1$ (an application of Lemma 12 tells us that this agent will not have a probability mass at β_1). Moreover, since no agent has a point mass at β_1, as shown in Lemma 12, agent i_1's competing bid distribution is continuous at β_1.

The arguments above show that the base case for the induction argument is satisfied. The conclusion of the argument is the same and we thus have a contradiction.

Both of these cases imply that there does not exist an open interval in $[\gamma^{(k)}, \lambda^{(k)}]$ such that $\mathcal{B}^{(k)}$ has no probability density or mass in it. Thus, the bid distribution $\mathcal{B}^{(k)}$ is strictly increasing on $[\gamma^{(k)}, \lambda^{(k)}]$. □

Theorem 6. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}^{(k)}_{-i}$ has a strictly increasing CDF on the interval $[\gamma^{(k)}, \lambda^{(k)}]$.*

Proof. Consider any agent $i \in \mathcal{N}$. Suppose the competing bid distribution $\mathcal{B}^{(k)}_{-i}$ is flat in an open interval $(\alpha, \beta) \subset [\gamma^{(k)}, \lambda^{(k)}]$. It is without loss for this interval to be maximal: for any $\delta > 0$, distribution $\mathcal{B}^{(k)}_{-i}$ has probability density or mass in $(\alpha - \delta, \alpha]$ and $[\beta, \beta + \delta)$. By Theorem 5, however, the bid of the k^{th} highest agent is strictly increasing in (α, β). This implies there is a bid b in the interval (α, β) such that b is in the support of agent i's equilibrium bid distribution.

Since $b \in (\alpha, \beta)$, the competing bid distribution is flat in $(\alpha, b]$. This means that agent i has a constant interim allocation $x_i(\cdot)$ in $(\alpha, b]$. By Lemma 10, this is a contradiction to $b \in \text{supp}(\mathcal{B}_i)$. Thus, the competing bid distribution $\mathcal{B}^{(k)}_{-i}$ is strictly increasing in $[\gamma^{(k)}, \lambda^{(k)}]$. □

The previous results established that the distribution of the k^{th} highest bid $\mathcal{B}^{(k)}$ and the competing bid distribution $\mathcal{B}_{-i}^{(k)}$ are strictly increasing in the interval $[\gamma^{(k)}, \lambda^{(k)}]$. In addition to this, we determine these bid distribution are continuous on $\left(\gamma^{(k)}, \lambda^{(k)}\right]$.

Theorem 7. *The bid distribution $\mathcal{B}^{(k)}$ of the k^{th} highest agent is continuous on the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$.*

Proof. Suppose $\mathcal{B}^{(k)}$ is discontinuous at some bid b^* in $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. There must be some agent i who has a point mass at b^* in the support of his equilibrium bid distribution. By Lemma 11, only agent i has a point mass at this bid. But since, by Theorem 6, his competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is strictly increasing at b^*, there is a different agent i' such that b^* is in the support of the equilibrium bid distribution $\mathcal{B}_{i'}$. An argument similar to the one used in the proof of Lemma 11 will imply that the interim allocation $x_{i'}(\cdot)$ is discontinuous at b^*. This means there exists a $\delta' > 0$ such that

$$x_{i'}(b') \geq x_{i'}(b^*) + \delta' \quad \text{for all } b' > b^*. \tag{18}$$

We will show that this is a contradiction to the following statement: there are bids smaller than b^* and arbitrarily close to it that lie in the support of the equilibrium distribution of some agent.

(i) Suppose for any $\delta > 0$, the bid distribution $\mathcal{B}_{i'}$ has positive probability in $(b^* - \delta, b^*]$. Consider a sequence of bids $(b_l)_{l=1}^{\infty}$ such that each element b_l lies in the interval $\left(b^* - \frac{1}{l}, b^*\right)$ and $b_l \in \text{supp}(\mathcal{B}_{i'})$. The existence of such a bid b_l, for every $l \in \mathbb{N}$, is guaranteed by the assumption that the distribution has positive probability in the interval $(b^* - \delta, b^*]$ for every $\delta > 0$. The sequence (b_l) converges to b^*. If we consider the utility of i' for each bid, it will become arbitrarily close to

$$(v - b^*) \lim_{l \to \infty} x_{i'}(b_l)$$

where v is his value for which b^* is in the support of $\mathcal{B}_{i'}$. The limit is upper bounded by $(v - b^*)x_{i'}(b^*)$. Because of the discontinuity in the allocation rule at $b*$, it follows by Lemma 10 that there exists a bid b' strictly greater than b^* which gives the agent a strictly better utility. A contradiction to bids arbitrarily close to b^* belonging to the support of $\mathcal{B}_{i'}$.

(ii) Suppose there exists a $\delta > 0$ such that $\mathcal{B}_{i'}$ has 0 probability in $(b^* - \delta, b^*]$. But because $b^* \in \text{supp}(\mathcal{B}_{i'})$ it must be that for any $\delta > 0$, bid distribution $\mathcal{B}_{i'}$ has positive probability in $[b^*, b^* + \delta)$. We will show there exists an agent $i^\dagger$, not the same as i and i', such that for all $\delta_{i^\dagger} > 0$, he has positive probability in $(b^* - \delta_{i^\dagger}, b^*]$. Suppose that apart from i and i', no agent satisfies the above condition. By assumption, there exists a $\delta > 0$ such that $\mathcal{B}_{i'}$ has 0 probability in $(b^* - \delta, b^*]$. Thus, the competing bid distribution of agent i $\mathcal{B}_{-i}^{(k)}$ is flat in $(b^* - \delta_i, b^*]$ for some $\delta_i > 0$. This is a contradiction to the result that $\mathcal{B}_{-i}^{(k)}$ is strictly increasing in the interval $[\gamma^{(k)}, \lambda^{(k)}]$ (which was stated as Theorem 5).

Thus, there exists an agent $i^\dagger$, not the same as i and i', such that for all $\delta_{i^\dagger} > 0$, he has positive probability in $(b^* - \delta_{i^\dagger}, b^*]$. A consequence of this is that $b^* \in \text{supp}(\mathcal{B}_{i^\dagger})$ Moreover, this agent will have a discontinuous interim allocation rule at b^*. Applying the argument in Part (i), we get a contradiction to bids arbitrarily close to b^* belonging to the support of $\mathcal{B}_{i^\dagger}$.

Since we get a contradiction in both cases, it cannot be the case that $\mathcal{B}^{(k)}$ is discontinuous in $(\gamma^{(k)}, \lambda^{(k)})$. □

Theorem 8. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}_{-i}^{(k)}$ is continuous on the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. Moreover, the agents bid distribution $\mathcal{B}_i$ is also continuous in the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$.*

Proof. Suppose $\mathcal{B}_{-i}^{(k)}$ is discontinuous at some bid b^* in the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. There must be some agent j (not the same as i) who has a point mass at b^* in the support of his equilibrium bid distribution. The statement then follows from the argument given in the proof of Theorem 7.

We will now analyze the bid distribution $\mathcal{B}_i$. Suppose the bid distribution is discontinuous at some bid b^* in the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. Thus, agent i has a point mass at b^*. The continuity of $\mathcal{B}_i$ follows from the argument given in the proof of Theorem 7. □

The previous results established continuity of the bid distributions in the open interval $\left(\gamma^{(k)}, \lambda^{(k)}\right)$. We extend this result to show that the distribution of the k^{th} highest bid and critical bid distributions $\mathcal{B}_{-i}^{(k)}$ for every agent i is continuous in the interval $\left(\gamma^{(k)}, \lambda^{(k)}\right]$.

Lemma 13. *Consider any profile of equilibrium bid distributions $\{\mathcal{B}_i\}_{i\in\mathcal{N}}$. At most $k-1$ agents have a point mass at $\lambda^{(k)}$ in their equilibrium bid distribution.*

Proof. We will first show at most k agents can have a point mass at $\lambda^{(1)}$ in their equilibrium bid distributions. Suppose at least $(k+1)$ agents have a point mass at $\lambda^{(1)}$. We will show there must be an agent who has a discontinuity in their allocation at $\lambda^{(1)}$. Formally, there exists an agent i and a corresponding $\delta_i > 0$ satisfying

$$x_i(b) \geq x_i(\lambda^{(1)}) + \delta_i \quad \text{for all } b > \lambda^{(1)}.$$

The allocation probability of agent i for bid b is

$$x_i(b) = \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b]\,\mathbb{P}[\text{agent } i \text{ wins in a tie-break} \mid \mathcal{B}_{-i}^{(k)} = b].$$

Because at least $(k+1)$ agents have a point mass at $\lambda^{(1)}$, and there are only k items available, there must exist an agent i for which

$$\mathbb{P}[\text{agent } i \text{ wins in a tie-break} \mid \mathcal{B}_{-i}^{(k)} = \lambda^{(1)}] < 1.$$

The allocation rule is discontinuous at $\lambda^{(1)}$ for this agent. A neighborhood deviation argument (as stated in Lemma 10) implies that $\lambda^{(1)}$ cannot be a best response bid of agent i. This is a contradiction to $\lambda^{(1)}$ being in the support of their equilibrium bid distribution. Thus, at most k agents can have a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution.

Suppose exactly k agents have a point mass at $\lambda^{(1)}$. By Lemma 6, there exists an agent $i^\dagger$ such that the equilibrium bid distribution $\mathcal{B}_{i^\dagger}$ does not have a point mass at $\lambda^{(1)}$ but where $\lambda^{(1)}$ belongs to its support. It must be that

$$\mathbb{P}[\text{agent } i^\dagger \text{ wins in a tie-break} \mid \mathcal{B}_{-i^\dagger}^{(k)} = \lambda^{(1)}] < 1. \tag{19}$$

If this is not the case, then there must exist an agent i^* who has a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution and

$$\mathbb{P}[\text{agent } i^* \text{ wins in a tie-break} \mid \mathcal{B}_{-i^*}^{(k)} = \lambda^{(1)}] < 1.$$

However, this implies i^* has a discontinuity in their allocation rule at $\lambda^{(1)}$. Applying the neighborhood deviation argument, as was applied in the paragraph above, would lead to a contradiction. (19) implies the allocation rule of $i^\dagger$ is discontinuous at $\lambda^{(1)}$. This is a contradiction to the fact that $\lambda^{(1)}$ is in the support of $\mathcal{B}_{i^\dagger}$. Thus, at most $k-1$ agents have a point mass at $\lambda^{(1)}$ in their equilibrium bid distribution. □

A direct implication of the preceding result is the continuity of competing bid distributions on the interval $\left[\gamma^{(k)}, \lambda^{(k)}\right]$.

Theorem 9. *Consider any agent $i \in \mathcal{N}$. The competing bid distribution $\mathcal{B}_{-i}^{(k)}$ has a continuous cumulative distribution function on the interval $\left[\gamma^{(k)}, \lambda^{(k)}\right]$.*

And finally, by Uniform Value Support, it must be that $\gamma^{(k)} = 0$.

6 Towards a Worst-Case Instance

Thus far, our analysis has focused on the asymptotic behavior of multi-unit winner-pays-bid auctions. We now shift attention to the finite-unit setting. Identifying the exact worst-case bid distributions for any number of units k is considerably more complex than analyzing their asymptotic behavior. In this section, we establish characterizations of the order statistics of random variables that we conjecture form a subset of the worst-case bids in the multi-unit winner-pays-bid auction.

In the single-unit setting, Jin and Lu (2023) exactly characterize the worst-case bid distributions. Roughly speaking, the instance behaves as follows:

- There is a single high-value bidder whose value is deterministic, but who risks not winning the item by bidding randomly in a region below their value.
- There is a large (asymptotically infinite) horde of low-value bidders whose values are random, and whose bids are a non-random function of their values.

The optimal allocation always provides the item to the high-value bidder. But, a winner-pays-bid auction sometimes allocates the item to one of the many low-value bidders. A particular bid distribution for each of these two groups produces the tight $1 - \frac{1}{e^2}$ approximation.

Extending this analysis to the multi-unit setting requires a careful characterization of the low-value bidders. In the single-unit case, since only one low-value bidder can potentially win the item, it is sufficient to consider the distribution of the *maximum* bid among these bidders. This distribution determines the probability that a low-value agent outbids the high-value agent and secures the item.

The multi-unit setting necessitates a characterization not only of the *maximum* of these low-value bids, but of their *order statistics*. This is because in a k-unit auction, the joint distribution of the top k bids among the low-value agents is required to determine the probability that a given number of units are allocated to this group rather than to higher-value bidders.

Welfare is reduced when the multi-unit winner-pays-bid auction allocates more units to low-value bidders; however, the bid distribution of the low-value bidders also has a secondary effect, in that the bid-to-value mappings of other agents also depends on the bid distribution of the low-value bidders. Therefore, increasing the low-value bidders bid CDFs in the first-order stochastic dominance (FOSD) order has the potential to both reduce expected welfare by allocating more frequently to low-value bidders, and sometimes increase expected welfare via changes in the profile of bid-to-value mappings. In the single-unit setting, the first effect dominates, and the worst-case instance involves low-value bid distributions as high in the FOSD order as possible, subject to a fixed distribution of the maximum low-value bidders' bid. This involves making the low-value bids identically distributed.

Theorem 10 replicates and extends this idea to the multi-unit setting:

Theorem 10. *Let $X_1, ..., X_n$ be independent random variables with cumulative distribution functions $\mathcal{B}_1, ..., \mathcal{B}_n$, and let $X^{(\ell)}$ denote the ℓ^{th} highest value of the set $\{X_1, ..., X_n\}$, with cumulative distribution function $\mathcal{B}^{(\ell)}$. Then, there exist i.i.d. random variables $Y_1, ..., Y_n$ such that $\max_{i\in[n]} Y_i \sim \mathcal{B}^{(1)}$ and where for all $\ell \in [n]$, the order statistic $Y^{(\ell)}$ first-order stochastic dominates $X^{(\ell)}$.*

Theorem 10 states that under a fixed distribution of the maximum, the highest possible order statistics occur when all random variables involved are identically distributed. A natural corollary of this fact is that more i.i.d. random variables produces higher order statistics as well; this is illustrated in Corollary 10.1:

Corollary 10.1. *Let $X_1, ..., X_n$ be independent non-negative random variables whose maximum $\max_{i\in[n]} X_i$ has cumulative distribution function $\mathcal{B}^{(1)}$. Then, the cumulative distribution function $\mathcal{B}^{(j)}$ of the j^{th}-highest element of $\{X_1, ..., X_n\}$ is bounded below as follows:*

$$\mathcal{B}^{(j)}(z) \geq \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z) \frac{\ln(1/\mathcal{B}^{(1)}(z))^\ell}{\ell!}$$

Furthermore, this bound is tight.

Proof sketch. This follows directly from Theorem 10; consider that the order statistics of any n non-negative random variables are identical to the order statistics of those n, alongside any additional random variables with a degenerate distribution at 0. Thus, applying Theorem 10 to the set of random variables with these additional degenerate random variables creates an i.i.d. collection of random variables with weakly higher order statistics in the FOSD order, compared to only applying Theorem 10 to the original random variables.

The expression obtained above for the worst-case order statistic CDF is what directly follows from taking the limit of the number of random variables $n \to \infty$ when they are assumed to be identically distributed. □

We conjecture that the worst-case instance of the multi-unit winner-pays-bid auction has a collection of infinite low-value bidders whose order statistics for their bids follows the closed form expressed in Corollary 10.1, for some distribution of the maximum $\mathcal{B}^{(1)}$. This would replicate a key qualitative feature of the single-unit case.

Solving for the exact bid distributions of the worst-case instance is beyond the scope of this paper. Theorem 10 and Corollary 10.1 are intended to be useful tools for future research that aims to solve for the exact worst-case instance.

7 Conclusions and Future Work

In this work, we analyze the price of anarchy for a multi-unit winner-pays-bid mechanism with independently distributed valuations, using the individual and competitive efficiency framework of Hartline et al. (2014). We develop a general approach to show that the price of anarchy converges to 1 as the number of units becomes asymptotically large. Our framework isolates key structural properties of equilibrium bid distributions that drive a convergence rate of $1 - O\left(\frac{1}{k^{0.5-\delta}}\right)$ for any constant $\delta \in (0, 0.5)$.

To support this result, we analyze equilibrium behavior in extreme bidding regions and identify a regularity condition on moderate bids sufficient to ensure the desired convergence rate. Additionally,

we establish crucial continuity and monotonicity properties of equilibrium bid distributions, which play a central role in our analysis.

An important direction for future research is to characterize the value distributions for which the corresponding equilibrium bid distributions satisfy the regularity condition identified in our analysis. This remains a challenging problem, as solving for equilibria in first-price auctions typically involves analyzing solutions to differential equations, which become increasingly complex in the multi-unit setting.

Another valuable extension involves exploring alternative equilibrium concepts. While Jin and Lu (2023) analyze the single-unit case under Bayes-Nash equilibrium, neither the single-unit nor the multi-unit winner-pays-bid auctions have been thoroughly studied under the assumption of independent valuations when bidders play according to correlated or coarse correlated equilibria. We conjecture that in the single-unit case, the price of anarchy under these broader equilibrium notions lies between the known bounds of $1 - \frac{1}{e}$ and $1 - \frac{1}{e^2}$, both of which were derived under the assumption of Bayes-Nash equilibrium. Establishing this conjecture, as well as extending the analysis to the multi-unit setting, represents a promising direction for future work.

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A Proofs for Section 1 (Introduction)

Theorem 1. *In a k-unit winner-pays-bid, highest-bidders-win auction with unit-demand agents whose values can be arbitrarily correlated, the worst-case ratio of expected welfare in Bayes-Nash equilibrium to the expected welfare of an optimal allocation is $1 - \frac{1}{e}$.*

Proof. Our proof will proceed in two steps. First, we will construct a set of agents with specific value distributions, and a Bayes-Nash equilibrium for their bidding strategies, such that the winner-pays-bid, highest-bids-win auction achieves an expected welfare that is a $1 - \frac{1}{e}$ fraction of the expected optimal welfare. Next, we will show by way of an argument via individual and competitive efficiency that any such auction must achieve at least a $1 - \frac{1}{e}$ fraction of the expected optimal welfare, establishing that the $1 - \frac{1}{e}$ bound is tight.

Consider an auction with two groups of agents:

- Group 1 contains k agents who all deterministically have value 1 and bid 0.
- Group 2 contains $k + 1$ agents with perfectly correlated values drawn from $[0, 1 - 1/e]$, with cumulative distribution function $F(v) = \frac{1}{e(1-v)}$, and who always bid their value.

Ties are always broken in favor of Group 1, and ties are broken among members of Group 2 arbitrarily.

To see that this is a Bayes-Nash equilibrium, consider that for any of the agents in Group 1, they always obtain an expected interim utility of $(1 - b)F(b) = \frac{1}{e}$ if they bid anywhere in the interval $[0, 1 - 1/e]$, and strictly less interim utility for bidding above that interval, and so choosing to bid 0 is an optimal bid.

Then, consider that for the agents in Group 2, they always obtain 0 utility for choosing to bid exactly their value, but would (a) obtain negative utility for choosing to bid strictly above their value, and (b) still obtain 0 utility if they chose to bid less than their value, since the other k members of Group 2 would still bid their value and thus take all k of the good, guaranteeing that a member of Group 2 that unilaterally chose to reduce their bid will not be allocated an item.

At this Bayes-Nash equilibrium, the expected welfare of the auction is $k(1 - \frac{1}{e})$. To see this, note that there are two possibilities:

1. All agents in Group 2 bid 0, causing the k agents in Group 1 to receive all k units of the good.
2. All agents in Group 2 bid above 0, meaning all units of the good are solely allocated to members of Group 2.

The welfare of Case 1 times its probability is $k \cdot F(0) = \frac{k}{e}$. The welfare of Case 2 integrated over the probability of all possible non-zero bids of Group 2 is

$$\int_0^{1-\frac{1}{e}} kbF'(b)db = \int_0^{1-\frac{1}{e}} kb\frac{1}{(1-b)^2}db = k\left(1 - \frac{2}{e}\right).$$

Adding the terms of both cases together, and we obtain an expected welfare of $k\left(1 - \frac{1}{e}\right)$. Meanwhile, the optimal allocation is to hand all k to the members of Group 1, producing an expected welfare of $k \cdot 1$. Thus, the ratio of these two expected welfares is $1 - \frac{1}{e}$.

An arbitrarily close ratio to $1 - \frac{1}{e}$ of equilibrium to optimal expected welfare can be obtained without the assumption that ties are always broken in favor of Group 1, by altering the setting so that the point mass in Group 2's bid distribution at 0 is replaced by a continuous distribution on

$[0, \varepsilon]$, while the agents in Group 1 bid between exactly at ε. Since the distribution of Group 2's bids are now continuous, no tie-breaking is relevant.

To show that this bound is tight, we follow the same analysis as Hartline et al. (2014) to claim that a winner-pays-bid auction has a $1 - \frac{1}{e}$ individual efficiency, and then note that the k-unit winner-pays-bid, highest-bids-win auction has a competitive efficiency of 1. This is because for any profile of bids $\vec{b} = (b_1, ..., b_n)$, the critical bid $\hat{b}_i$ of agent i is either equal to the k^{th}-highest bid overall if agent i's bid is not one of the top k bids, or equal to the $(k+1)^{\text{th}}$-highest bid overall if agent i's bid is one of the top k bids. Thus, since Revenue$(\vec{b})$ is the sum of the top k bids, and this is an upper bound on

$$k \cdot \text{ [the } k^{\text{th}}\text{-highest bid overall]}$$

which we have established is at most the sum of the top k critical bids. Therefore, the auction is has a competitive efficiency of 1. Combining both the competitive and individual efficiency, and we have an approximation ratio of $1 - \frac{1}{e}$. Thus, the instance described at the start of the proof was worst-case, and $1 - \frac{1}{e}$ is a tight bound. □

B Proofs for Section 5 (Continuity and Monotonicity of Equilibrium Bid Distributions)

Lemma 7. *The interim allocation rule $x_i(b)$ for any agent i satisfies the following properties:*

(a) $x_i(b)$ is weakly increasing in b.

(b) $x_i(b)$ is bounded as

$$\sup\{\mathcal{B}_{-i}^{(k)}(t) : t < b\} \leq x_i(b) \leq \mathcal{B}_{-i}^{(k)}(b) \quad \text{for all } b.$$

(c) Consider any bid b in the support of the bid distribution $\mathcal{B}_i$. It has interim allocation

(i) $x_i(b) = 0$ when $b < \gamma^{(k)}$.
(ii) $x_i(b) > 0$ when $b > \gamma^{(k)}$.

Proof. The allocation rule of agent i for a bid b is

$$x_i(b) = \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b]\,\mathbb{P}[i \text{ wins in a tie-break at bid } b]. \tag{20}$$

We will use this expression to prove the stated properties.

(a) *Proof of Part (a)*: It is clear from (20) that the interim allocation is bounded as

$$x_i(b) \leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} \leq b].$$

Consider an arbitrary $\epsilon > 0$ and a bid $b' = b + \epsilon$. The interim allocation for bid b' satisfies

$$\begin{aligned} x_i(b') &= \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b'] + \mathbb{P}[\mathcal{B}_{-i}^{(k)} = b']\,\mathbb{P}[i \text{ wins in a tie-break at bid } b'], \\ &\geq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b + \epsilon], \\ &\geq \mathbb{P}[\mathcal{B}_{-i}^{(k)} \leq b], \\ &\geq x_i(b). \end{aligned}$$

The equality follows from (20) and the first inequality follows from the fact that a probability is always non-negative. Since, we show that for any $\epsilon > 0$, the interim allocation $x_i(b+\epsilon) \geq x_i(b)$, the statement in Part (a) follows.

(b) *Proof of Part (b)*: As was shown in the proof of Part (a), an upper bound on the interim allocation is

$$x_i(b) \leq \mathcal{B}_{-i}^{(k)}(b).$$

To show the lower bound, it holds that

$$\mathcal{B}_{-i}^{(k)}(t) \leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b] \quad \text{for all } t < b.$$

It is straightforward to see that

$$\begin{aligned} \sup\{\mathcal{B}_{-i}^{(k)}(t) : t < b\} &\leq \mathbb{P}[\mathcal{B}_{-i}^{(k)} < b], \\ &\leq x_i(b). \end{aligned}$$

The statement of Part (b) follows.

(c) *Proof of Part (c)*: Consider an agent i. We will now analyze its competing bid distribution $\mathcal{B}_{-i}^{(k)}$. Let us denote the infimum of its support by $\gamma_{-i}^{(k)}$. Moreover, let us define the collection

$$\mathbf{\Gamma}_{-i} := \{\gamma_j\}_{j \neq i}$$

An application of Lemma 5 tells us that

$$\gamma_{-i}^{(k)} = \Gamma_{-i}^{(k)}. \tag{21}$$

Suppose agent i is such that $\gamma_i < \gamma^{(k)}$. The infimum of the support of the competing bid distribution satisfies

$$\begin{aligned} \gamma_{-i}^{(k)} &= \Gamma_{-i}^{(k)}, \\ &= \gamma_{(k)}, \\ &= \gamma^{(k)}. \end{aligned}$$

The first equality follows from (21), the second follows from the assumption $\gamma_i < \gamma^{(k)}$ and the third follows from Lemma 5. Consider any bid b in the support of agent i's bid distribution. In the case $b < \gamma^{(k)}$, the interim allocation $x_i(b) = 0$ because the bid lies strictly below the infimum of the support of the competing bid distribution. In the case $b > \gamma^{(k)}$, he is allocated with strictly positive probability because the bid lies strictly above the support of the competing bid distribution.

Suppose agent i is such that $\gamma_i \geq \gamma^{(k)}$. By definition of γ_i, the agent almost surely has a bid $b \geq \gamma_i$. Thus, Part (i) vacuously holds true. The infimum of the support of the competing bid distribution

$$\begin{aligned} \gamma_{-i}^{(k)} &= \Gamma_{-i}^{(k)}, \\ &\leq \gamma_{(k)}, \\ &= \gamma^{(k)}. \end{aligned}$$

The second equality follows from observing that removing an element from the order $\mathbf{\Gamma}$ only decreases the value of entry k. It follows that any bid $b > \gamma^{(k)}$ is allocated with strictly positive probability because the infimum of the competing bid distribution is strictly below the bid.

The 3 parts of the result statement are proved. □

Lemma 8. *Consider an agent i and value $v \in \text{supp}(V_i)$. The equilibrium strategy $s_i(\cdot)$ for any agent i satisfies the following properties:*

(a) If $v \leq \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) = 0$ and the equilibrium bid $s_i(v) \leq \gamma^{(k)}$ almost surely over the random strategy $s_i(\cdot)$.

(b) If $v > \gamma^{(k)}$, the optimal utility $\mathcal{U}_i(v) > 0$ and the equilibrium bid $\gamma^{(k)} \leq s_i(v) < v$ almost surely over the random strategy $s_i(\cdot)$.

Proof. We will consider two cases: one where $\gamma_i < \gamma^{(k)}$ and the other where $\gamma_i \geq \gamma^{(k)}$.

Suppose $\gamma_i < \gamma^{(k)}$. As we have shown in the proof of Lemma 7, the infimum of the support of the competing bid distribution $\gamma_{-i}^{(k)}$ is equal to $\gamma^{(k)}$.

(i) Proof of Part (a): Consider a value $v \leq \gamma^{(k)}$. Let it be the case that $v \leq \gamma^{(k)}$. If the agent were to bid $b > \gamma^{(k)}$, this would give them utility $\mathcal{U}_i(v, b) < 0$ because he is allocated with strictly positive probability. If the agent bid $b < \gamma^{(k)}$, the agent is allocated with probability 0 because the infimum of the support of the competing bid distribution is $\gamma^{(k)}$. It follows that they receive $\mathcal{U}_i(v, b) = 0$ for such a bid. If the agent bid $b = \gamma^{(k)}$, the utility he gets for such a bid is at most 0 (the utility would be 0 in the case he is not allocated an item). Thus, the optimal utility for such a value is $\mathcal{U}_i(v) = 0$. Moreover, the bids which guarantee this utility must be at most $\gamma^{(k)}$. Thus, the equilibrium bid $s_i(v) \leq \gamma^{(k)}$ almost surely.

(ii) Proof of Part (b): Consider a value $v > \gamma^{(k)}$. We will show that the bid $b = \frac{v+\gamma^{(k)}}{2}$ gives the agent a strictly positive utility. Observe that the bid $b > \gamma^{(k)}$ and has allocation $x_i(b) > 0$ because the bid is strictly greater than the infimum of the support of the competing bid distribution $\gamma^{(k)}$. Since the bid $b < v$, the utility for this bid $\mathcal{U}_i(v, b) > 0$. The optimal bid can only do better. Thus, the optimal utility $\mathcal{U}_i(v) > 0$. We will now show that $\gamma^{(k)} \leq s_i(v) < v$ almost surely. If the agent bids $b < \gamma^{(k)}$, his allocation is $x_i(b) = 0$ (the bid is strictly smaller than the infimum of the support of the competing bid distribution). This would give the agent a utility of 0 and thus cannot be optimal. It follows that $\gamma^{(k)} \leq s_i(v) < v$ almost surely (the bound $s_i(v) < v$ follows from the observation that a bid $b \geq v$ will be allocated with strictly positive probability but given a utility $\mathcal{U}_i(v, b) \leq 0$, thus not being optimal).

Suppose $\gamma_i \geq \gamma^{(k)}$. We will state two observations which will be helpful in proving both parts of the statement. First, the infimum of the support of $\mathcal{B}_i$ is at least $\gamma^{(k)}$. Thus, the equilibrium bid of this agent is at least $\gamma^{(k)}$ almost surely. Second, as we have shown in the proof of Lemma 7, the infimum of the support of the competing bid distribution $\gamma_{-i}^{(k)}$ is at most $\gamma^{(k)}$.

(i) Proof of Part (a): Consider a value $v \leq \gamma^{(k)}$. We will show that it must be the case equilibrium bids $s_i(v)$ give the agent 0 utility. Since the equilibrium strategy is such that $s_i(v) \geq \gamma^{(k)}$ almost surely, the agent must be getting at most 0 utility. It cannot be the case that an agent gets strictly negative utility at equilibrium. Thus, the optimal utility is 0. We will now analyze the equilibrium bid strategy $s_i(v)$. For this value v, it cannot be the case that a bid b in the support of $s_i(v)$ lies strictly above $\gamma^{(k)}$. The agents utility conditional on such a bid is strictly negative. The agent can strictly increase utility by moving probability measure around b to a bid that gives him non-negative utility (such a bid must exist because the optimal utility is 0). Thus, the only possibility at equilibrium is $s_i(v) \leq \gamma^{(k)}$ almost surely.

(ii) Proof of Part (b): Consider a value $v > \gamma^{(k)}$. Consider a value $v > \gamma^{(k)}$. We will show that the bid $b = \frac{v+\gamma^{(k)}}{2}$ gives the agent a strictly positive utility. Observe that the bid $b > \gamma^{(k)}$ and has allocation $x_i(b) > 0$ because the bid is strictly greater than $\gamma_{-i}^{(k)}$. To be precise, $\gamma_{-i}^{(k)} \leq \gamma^{(k)}$ for agent i. Since the bid $b < v$, the utility for this bid $\mathcal{U}_i(v, b) > 0$. The optimal bid can only do better. Thus, the optimal utility $\mathcal{U}_i(v) > 0$. Moreover, by definition of γ_i, the equilibrium bid satisfies $\gamma^{(k)} \leq s_i(v) < v$ almost surely.

The statement of the result holds. □

Lemma 9 (Chawla and Hartline, 2013)**.** *Consider values v, w in the support of value distribution V_i such that $v > w > \gamma^{(k)}$. The following properties hold:*

(a) The optimal utility at these values is ordered as

$$\mathcal{U}_i(v) > \mathcal{U}_i(w) > 0.$$

(b) The equilibrium bids are ordered as $s_i(v) \geq s_i(w)$ almost surely over the random strategy $s_i(\cdot)$.

Proof. (i) Proof of Part (a): At value v, agent i could bid according to random strategy $s_i(w)$. Because the allocation is exactly what the agent would get if the value were w, such a strategy would give a strictly higher utility than $\mathcal{U}_i(w)$. The optimal strategy for value v can only do weakly better and thus $\mathcal{U}_i(v) > \mathcal{U}_i(w)$. The fact that both of these are strictly greater than 0 follows from Lemma 8.

(ii) Proof of Part (b): The proof of this is exactly as that in the 1-unit setting.
The statement of the result holds. □

C Proofs for Section 6 (Towards a Worst-Case Instance)

Theorem 10. *Let $X_1, ..., X_n$ be independent random variables with cumulative distribution functions $\mathcal{B}_1, ..., \mathcal{B}_n$, and let $X^{(\ell)}$ denote the ℓ^{th} highest value of the set $\{X_1, ..., X_n\}$, with cumulative distribution function $\mathcal{B}^{(\ell)}$. Then, there exist i.i.d. random variables $Y_1, ..., Y_n$ such that $\max_{i\in[n]} Y_i \sim \mathcal{B}^{(1)}$ and where for all $\ell \in [n]$, the order statistic $Y^{(\ell)}$ first-order stochastic dominates $X^{(\ell)}$.*

Proof. To demonstrate this, we establish a "path" through the space of CDFs that starts at $(\mathcal{B}_1, ..., \mathcal{B}_n)$, and ends at the identical CDFs $(\mathcal{B}^{(1)}(\cdot)^{1/n}, ..., \mathcal{B}^{(1)}(\cdot)^{1/n})$. These identical CDFs correspond to the distributions of $Y_1, ..., Y_n$, and possess the property that their maximum

$$\max_{i\in[n]} Y_i \sim \prod_{i=1}^{n} \mathcal{B}^{(1)}(\cdot)^{1/n} = \mathcal{B}^{(1)}.$$

We will demonstrate that this path produces order statistics that are increasing in the FOSD order.

To begin, let us define the functions

$$Q_i(t, z) := (\mathcal{B}^{(1)}(z)^{1/n})^t \mathcal{B}_i(z)^{1-t}$$

for each $i \in \mathcal{N}$, and note that for any fixed $t \in [0, 1]$, the mapping $Q_i(t, \cdot)$ is a CDF (i.e. is weakly increasing, non-negative, bounded above by 1, and right-continuous). Furthermore the product of

these CDFs is

$$\begin{aligned}\prod_{i=1}^{n} Q_i(t,z) &= \prod_{i=1}^{n} (\mathcal{B}^{(1)}(z)^{1/n})^t \mathcal{B}_i(z)^{1-t} \\ &= \left[\prod_{i=1}^{n} \mathcal{B}^{(1)}(z)^{1/n}\right]^t \left[\prod_{i=1}^{n} \mathcal{B}_i(z)\right]^{1-t} \\ &= \left[\mathcal{B}^{(1)}(z)\right]^t \left[\mathcal{B}^{(1)}(z)\right]^{1-t} \\ &= \mathcal{B}^{(1)}(z).\end{aligned}$$

Thus, if each $Q_i(t,\cdot)$ were the CDF of an independent random variable, the maximum of those random variables would have CDF $\mathcal{B}^{(1)}(z)$, and this holds for all $t \in [0,1]$.

Note also that $Q_i(0,\cdot) = \mathcal{B}_i(\cdot)$ and $Q_i(1,\cdot) = \mathcal{B}^{(1)}(\cdot)^{1/n}$. Thus, as t goes from 0 to 1, the collection of CDFs $(Q_1(t,z),...,Q_n(t,z))$ smoothly interpolates from the initial CDFs $(\mathcal{B}_1,...,\mathcal{B}_n)$ and ends at the identical CDFs $(\mathcal{B}^{(1)}(\cdot)^{1/n},...,\mathcal{B}^{(1)}(\cdot)^{1/n})$ as desired.

Fix any $z \in \mathbb{R}$. We will show that all CDFs of all order statistics associated with $(Q_1(t,z),...,Q_n(t,z))$ *decrease* as t goes from 0 to 1.

We consider two cases: either $\mathcal{B}^{(1)}(z) = 0$ or $\mathcal{B}^{(1)}(z) > 0$. The first case is trivial; if $\mathcal{B}^{(1)}(z) = 0$, then there must exist at least one $i \in \mathcal{N}$ such that $\mathcal{B}_i(z) = 0$, in which case moving from $t = 0$ to $t = 1$ makes $Q_i(1,z) = 0$ for all $i \in \mathcal{N}$, rather than a subset of $\mathcal{N}$. Thus, all CDFs $(Q_1(t,z),...,Q_n(t,z))$ are decreasing in t, so the CDFs of all order statistics must also be decreasing in t.

The second case is harder to analyze. We begin by defining the CDF of the ℓ^{th} order statistic at t as

$$Q^{(\ell)}(t,z) := \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq\ell-1}} \left[\prod_{i\in S}(1-Q_i(t,z))\right]\left[\prod_{i\in\mathcal{N}\setminus S} Q_i(t,z)\right]$$

and write as shorthand

$$P_S(t,z) := \left[\prod_{i\in S}(1-Q_i(t,z))\right]\left[\prod_{i\in\mathcal{N}\setminus S} Q_i(t,z)\right]$$

so that we can write the CDF of the ℓ^{th} order statistic as

$$Q^{(\ell)}(t,z) = \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq\ell-1}} P_S(t,z).$$

We then take the derivative of both sides of this equation with respect to t:

$$\begin{aligned}\frac{dQ^{(\ell)}(t,z)}{dt} &= \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq\ell-1}} \frac{dP_S(t,z)}{dt} \\ &= \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq\ell-1}} \left[\left(\sum_{i\in\mathcal{N}\setminus S} \frac{d\ln(Q_i(t,z))}{dt}\right) + \left(\sum_{i\in S} \frac{d\ln(1-Q_i(t,z))}{dt}\right)\right] P_S(t,z)\end{aligned}$$

We note that the derivative of $Q_i(t,z) = (\mathcal{B}^{(1)}(z)^{1/n})^t \mathcal{B}_i(z)^{1-t}$ with respect to t is $\ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) Q_i(t,z)$. Substituting this into our expression for $\frac{dQ^{(\ell)}(t,z)}{dt}$, we have

$$\begin{aligned}
\frac{dQ^{(\ell)}(t,z)}{dt} &= \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \left[\left(\sum_{i\in\mathcal{N}\setminus S} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)\right) - \left(\sum_{i\in S} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)\frac{Q_i(t,z)}{1-Q_i(t,z)}\right)\right] P_S(t,z) \\
&= \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \left[\sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)\left(\mathbb{1}(i\notin S) - \mathbb{1}(i\in S)\frac{Q_i(t,z)}{1-Q_i(t,z)}\right)\right] P_S(t,z) \\
&= \sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) \sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \left[\mathbb{1}(i\notin S) - \mathbb{1}(i\in S)\frac{Q_i(t,z)}{1-Q_i(t,z)}\right] P_S(t,z) \\
&= \sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) \left[\left[\sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \mathbb{1}(i\notin S) P_S(t,z)\right] - \left[\sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \mathbb{1}(i\in S)\frac{Q_i(t,z)}{1-Q_i(t,z)} P_S(t,z)\right]\right] \\
&= \sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) \left[\left[\sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-1}} \mathbb{1}(i\notin S) P_S(t,z)\right] - \left[\sum_{\substack{S\subseteq\mathcal{N}:\\ |S|\leq \ell-2}} \mathbb{1}(i\notin S) P_S(t,z)\right]\right] \\
&= \sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) \left[\sum_{\substack{S\subseteq\mathcal{N}:\\ |S|= \ell-1}} \mathbb{1}(i\notin S) P_S(t,z)\right]
\end{aligned}$$

Observe that each term $\sum_{S\subseteq\mathcal{N}:|S|=\ell-1} \mathbb{1}(i\notin S)P_S(t,z)$ is non-negative, and thus the above derivation of $\frac{dQ^{(\ell)}(t,z)}{dt}$ can be viewed as a weighted sum of $\ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)$ terms, which may themselves be either negative or non-negative. Observe also that $\ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)$ and $\sum_{S\subseteq\mathcal{N}:|S|=\ell-1} \mathbb{1}(i\notin S)P_S(t,z)$ are anti-correlated in the choice of the agent i; thus, $\frac{dQ^{(\ell)}(t,z)}{dt}$ would be larger if it were an unweighted average of $\ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right)$ terms. Finally, note that

$$\sum_{i\in\mathcal{N}} \ln\left(\frac{\mathcal{B}^{(1)}(z)^{1/n}}{\mathcal{B}_i(z)}\right) = 0$$

which implies $\frac{dQ^{(\ell)}(t,z)}{dt} \leq 0$, and so $Q^{(\ell)}(t,z)$ is decreasing in t. Therefore, since this holds for the CDFs of all order statistics, all order statistics are increasing in the FOSD order as t increases. □

Corollary 10.1. *Let $X_1, ..., X_n$ be independent non-negative random variables whose maximum $\max_{i\in[n]} X_i$ has cumulative distribution function $\mathcal{B}^{(1)}$. Then, the cumulative distribution function $\mathcal{B}^{(j)}$ of the j^{th}-highest element of $\{X_1, ..., X_n\}$ is bounded below as follows:*

$$\mathcal{B}^{(j)}(z) \geq \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z)\frac{\ln(1/\mathcal{B}^{(1)}(z))^\ell}{\ell!}$$

Furthermore, this bound is tight.

Proof. We know immediately from Theorem 10 that the CDF of any order statistic is lowest when all distributions of $X_1, ..., X_n$ are identical. Suppose we were to instead consider a set of random variables

$$X_1, ..., X_n, X_{n+1}, ..., X_{n'}$$

for some n' much greater than n, such that for all $i \in \{n, n+1, ..., n'\}$, the random variable X_i has a degenerate distribution with all its probability mass at 0. These additional random variables $X_{n+1}, ..., X_{n'}$ are not distributed identically to the random variables $X_1, ..., X_n$, yet the united collection $X_1, ..., X_{n'}$ produces identical order statistics to the original collection $X_1, ..., X_n$. Therefore, Theorem 10 implies that a *larger collection* of i.i.d. random variables will have order statistics which first-order stochastically dominate a smaller collection, subject to the constraint that the maximum of both collections has the same distribution.

To prove our claim that

$$\mathcal{B}^{(j)}(z) \geq \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z) \frac{\ln(1/\mathcal{B}^{(1)}(z))^\ell}{\ell!},$$

and that this bound is tight, we will show that the right-hand side is exactly the order statistic CDF which emerges when we make the number of random variables n asymptotically large and all random variables are identically distributed.

For any finite n and i.i.d. random variables $X_1, ..., X_n$, we have

$$\begin{aligned}
\mathcal{B}^{(j)}(z) &= \sum_{\substack{S \subseteq \mathcal{N}: \\ |S| \leq j-1}} \left[\prod_{i \in S} (1 - \mathcal{B}_i(z))\right] \left[\prod_{i \in \mathcal{N} \setminus S} \mathcal{B}_i(z)\right] \\
&= \sum_{\substack{S \subseteq \mathcal{N}: \\ |S| \leq j-1}} \left[\prod_{i \in S} (1 - \mathcal{B}^{(1)}(z)^{1/n})\right] \left[\prod_{i \in \mathcal{N} \setminus S} \mathcal{B}^{(1)}(z)^{1/n}\right] \\
&= \sum_{\substack{S \subseteq \mathcal{N}: \\ |S| \leq j-1}} \left[1 - \mathcal{B}^{(1)}(z)^{1/n}\right]^{|S|} \left[\mathcal{B}^{(1)}(z)^{1/n}\right]^{|\mathcal{N} \setminus S|} \\
&= \sum_{\ell=0}^{j-1} \frac{n!}{\ell!(n-\ell)!} \left[1 - \mathcal{B}^{(1)}(z)^{1/n}\right]^{\ell} \left[\mathcal{B}^{(1)}(z)^{1/n}\right]^{n-\ell} \\
&= \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z) \frac{n!}{\ell!(n-\ell)!} \left[\mathcal{B}^{(1)}(z)^{-1/n} - 1\right]^{\ell}
\end{aligned}$$

then taking the limit as $n \to \infty$ produces

$$\mathcal{B}^{(j)}(z) = \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z) \frac{\ln(1/\mathcal{B}^{(1)}(z))^\ell}{\ell!}.$$

Therefore, the inequality of

$$\mathcal{B}^{(j)}(z) \geq \sum_{\ell=0}^{j-1} \mathcal{B}^{(1)}(z) \frac{\ln(1/\mathcal{B}^{(1)}(z))^\ell}{\ell!}$$

not only holds for any collection of non-negative random variables, but is tight as the number of random variables becomes asymptotically large. □