The First-Price Principle of Maximizing Economic Objectives*

Byeong-hyeon Jeong† Marek Pycia‡

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Abstract

We show that the first-price auction format—when enriched by restrictions on bid space—is sufficiently flexible to maximize in equilibrium any standard objective, including revenue, welfare, bidder surplus, equality, and their combinations. As this first-price principle is valid with no distributional assumptions needed, it enables us to analyze problems that are beyond the scope of Myersonian mechanism design. The resulting optimal auctions might take surprising forms: not only reserve prices but also bid caps or gaps in bid space might be optimal. We provide tools for the optimal design of these auctions and establish conditions for equilibrium existence and uniqueness.

1 Introduction

First-price auctions have been used to sell objects throughout history. While it is known that the first-price auction format can be used to achieve objectives such as welfare maximization or revenue maximization, the auctioneer may be interested in other types of objectives. For instance, the U.S. Federal Trade Commission often cites the consumer welfare as the objective: "The true measure of our contribution to the economy is our progress in increasing

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†Department of Economics, University of Zurich (email:byeonghyeon.jeong@econ.uzh.ch)

‡Department of Economics, University of Zurich (email:marek.pycia@econ.uzh.ch)

consumer welfare overall. Thus, well-conceived competition and consumer protection policies should take complementary paths to the goal of promoting consumer welfare."[1]

An auctioneer with distributional concerns may want to allocate the objects in a way that minimizes inequality among bidders with different valuations, or that balances inequality measures and revenue maximization. In this paper, we show that a large class of objectives, that includes the above examples, can be maximized using first-price auctions. This insight relies on the objective satisfying three standard conditions: anonymity, convexity of the maximand set, and quasi-convexity.

We study a general class of first-price auctions, in which bidders submit bids from a set of allowed bids, the object is allocated to one of the highest bidders, the winning bidder pays their bid, and the remaining bidders pay nothing. As we focus on designer's objectives that are symmetric across bidders, without loss of generality we can focus on symmetric auctions in which each bidder chooses from the same bid space and has the same chance of winning the object in tie-breaking (see Lemma 2).

Allowing arbitrary restrictions on the set of allowed bids, makes the class of first-price auctions we study surprisingly flexible: our main theorem shows that any objective from the large class we analyze can be achieved through a first-price auction (Theorem 1). A reserve price is an example of bid space restriction that has been known to be sometimes optimal (Myerson, 1981). As we show, however, optimality may sometimes require other restrictions on bid spaces, for instance, a maximum price that can be binding. Such more general restrictions are used in practice. For example, Tsuchiura City, Japan, introduced in 2009 a minimum price rule disqualifying bids below the estimated contract price in their procurement auctions.[2] One of our examples shows that such a cap is optimal if the city's objective is to balance revenue and equality concerns.

We prove this "first-price principle" in a general private values environment in which multiple buyers participate in a mechanism to win an object. Our environment incorporates, as a special case, the standard environment of Myerson (1981) and relaxes two of Myerson's assumptions. We allow type distributions with arbitrary supports, finite or infinite, rather than requiring the support to be an interval; we also allow arbitrary type distributions, not only Lebesgue continuous ones. By showing that restricting attention to first-price auctions is sufficient for optimal design, we make all such design problems tractable.

We complement our main theorem by developing new tools that facilitate the design of first-price auctions so they maximize objectives that take into account revenues, welfare, and

[1] Timothy J. Muris, then Chairman of the Federal Trade Commission in a speech at the Aspen Summit.

[2] A minimum price in procurement auction corresponds, of course, to a maximum price in a regular auction. For more details on Tsuchiura City auctions, see Chassang and Ortner (2019).

inequality. We illustrate the usefulness of these tools and of our main theorem by finding optimal mechanisms for a variety of problems that are beyond the scope of prior work. For instance, we show that when equity considerations are taken into account, the optimal bid space might have not only a reserve price but also an exogenous upper bound on the allowable bids. In particular, the optimal bid space might contain just one bid level: even with a continuum of bidder types a simple mechanism posting merely one price might be optimal. Furthermore, optimality might require the bid space to consist of disjoint intervals of bids.

As, to the best of knowledge, our work is the first to study first price auctions with general bid spaces, we also provide additional results to facilitate the analysis of such auctions.[3] In particular, we prove the existence of monotonic pure-strategy equilibria for first-price auctions with compact bid spaces (Theorem 2). We show that any equilibrium in a first-price auction with any bid space remains an equilibrium if we replace the bid space by its closure (Theorem 3). Thus, from a design perspective, the compactness assumption in our existence results is without loss of generality when the type space is bounded. We also show that many objectives can be optimized using interval bid spaces—with both a reserve and a price cap—provided the bidder value distribution is atomless and convex (Theorem 5).

Applying the design tools we develop, we show how to optimize auctions for a mixture of revenue and equality objectives, and show that bidding caps are then optimal. The above-mentioned practice of setting bidding caps in Japanese municipal auctions might hence be optimal; note that equalizing bidders' chances might have pro-competitive effect on down-stream markets. First price auctions with bid caps were earlier studied by Chassang and Ortner (2019), but they left open the question of why they are used.

As another application of our design tools, we show how to design optimal mechanisms when value distributions have atoms (mass points). We find that gaps in the bid space are an optimal response to such mass points. The mass points arise naturally in auction environments when, responding to regulatory environments that differ for firms of various sizes, firms bunch at or near the threshold sizes. For instance, in France, additional administrative obligations apply once firms exceed 50 employees, creating bunching at the threshold as observed by Garicano, Lelarge, and Van Reenen (2016). Jones and Lee (2018) document that firms in Korea bunch in order to maintain small and medium sized business

[3]Specific restrictions have been studied, primarily the above-mentioned reserve prices (cf. Riley and Samuelson (1981) and Myerson (1981)). Of relevance is also the literature on bidders' budget constraints in first-price auctions (cf. Che and Gale (1998b), Pai and Vohra (2014)); the main differences with this literature are that we allow the designer to optimize the bid space and that we allow a much broader class of bid spaces. Beyond first-price auctions, there is also work on bid caps in all pay auctions, cf. Che and Gale (1998a), Szech (2015), Olszewski and Siegel (2019).

status and Harju, Matikka, and Rauhanen (2016) find analogous bunching behavior among Finnish firms. Other size thresholds include the regulation in the US Affordable Care Act that mandates employee health insurance for firms with more than 50 employees.

Such mass points break the smoothness assumptions underlying the traditional Myersonian approach, and our focus on the first-price principle is at the source of our ability to relax these smoothness assumptions. Unlike Myerson-inspired mechanism design analysis, we do not use incentive compatibility to pin the slope of participants' mappings from types to payoffs. It is this slope determination that restricts Myersonian analysis to Lebesgue continuous distributions with interval supports; a restriction our approach avoids. Instead, we derive our first-price principle in the following two steps. In the first step we restrict attention to finite type spaces, and show that any symmetric mechanism is a lottery over first-price auctions. This auxiliary result can be restated as showing that the set of extreme points of symmetric mechanisms consists of first-price auctions.[4] Methodologically, our analysis of discrete type spaces blends in a new way Border's (1991) analysis of interim allocations and Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi's (2013) approach to Bayesian-Dominant-strategy allocation equivalence. In the second step, we treat the general type space as a limit case of discrete type spaces and show that the validity of the first-price principle for discrete type spaces is preserved in the limit.

By allowing a wide range of objective functions, our results are related to Condorelli (2013)—who shows that maximizing a linear combination of assignment values leads to assigning objects to agents with the highest willingness to pay—and Akbarpour, Dworczak, and Kominers (2020)—who shows that mechanisms maximizing a linear combination of revenue and welfare combine assortative (or market) allocation and uniform randomization. The objectives they study are standard in our sense and hence our results are applicable to their environment. Ours and their work rely on different methodologies: we establish our results for finite type spaces (in which revenue equivalence might fail) and extend them to more general type spaces, while the above three papers study continuum type spaces (in which revenue equivalence obtains).

Combining our results with those of Akbarpour and Li (2020) study of credibility, we show any standard objective that is a function of revenue and social welfare, and that is increasing in each argument, can be credibly achieved because it can be achieved through a first-price auction.

Our focus on the first-price auction is not merely driven by the canonical nature of this mechanism class: we show that an analogue of our main result fails if we replace first-

[4]In finite type spaces, we additionally show that lotteries over first-price auctions can achieve any anonymous objective whose maximand set is convex.

price auctions with second-price auctions (Section 7). Indeed, we show that there are value distributions in which even such standard objectives as revenue maximization cannot be achieved by second-price auctions.

Our results contribute to the literature by creating new tools for mechanism design. Most related is Kleiner, Moldovanu, and Strack (2021). They focus on characterizing optimal allocations while we characterize the space of optimal mechanisms. While they maintain the standard Myersonian continuity and support assumptions, our approach allows us to relax them. In an earlier work, Carbajal and Ely (2013) maintain the continuity assumptions but relax the indivisibility assumption. Nikzad (2022) maintains the continuity assumptions but allows additional constraints on the design problem.

Earlier works on mechanism design with discrete type spaces focused on extending the slope determination behind revenue equivalence, and explored its limits, e.g. Lovejoy (2006). Beyond discrete spaces, the limits of the Myersonian slope approach were explored in Che and Gale (2006), who showed that first-price auctions yield higher expected revenue when risk-averse bidders are financially constrained, and in Chung and Olszewski (2007) and Heydenreich et al. (2009), who characterized the restrictions on the primitives that lead to the revenue equivalence result. Our work takes a different approach by allowing more restrictions on the bid space and shows that the first-price format dominates the second-price auction format in terms of the expected revenue.

Our existence result complements other papers establishing existence of equilibria in first-price auctions with unconstrained (or constrained only by the reserve price) bid spaces, e.g. Riley and Samuelson (1981). Reny (2011) proved existence of monotonic pure-strategy equilibria in finite Bayesian games. His result implies the existence of equilibria in our auctions when the bid space is finite and we build on his existence result to prove existence for more general bid spaces.[5]

2 An Example

Myerson (1981) famously showed that in regular environments he studied first-price auctions with reserve prices maximize revenue as well as efficiency (when the reserve price is non-

[5]Our extension builds on the ideas from Reny and Zamir (2004), who show the existence of monotone pure-strategy equilibria in first-price auctions for standard bid space with bidder-specific reserve prices. They first show the existence of an equilibrium when the bid space is finite and there is no tie-breaking which can be ensured by not allowing common bids for different bidders, and show that the limit of the equilibria via approximation is still an equilibrium. We also employ a limit argument made more complex because we need to take care of exogenous tie-breaking. For other related limit arguments, see, e.g., Woodward (2016). Olszewski, Reny, and Siegel (2023) analyze the existence of equilibria in first-price auctions with asymmetric type distributions.

binding). His analysis hinged on the value distribution being Lebesgue absolutely continuous distribution with interval support; his analysis also hinged on the specific objectives he studied. For other objectives or for general value distributions, first price auctions with an optimal reserve price may fail to maximize revenue, and neither are they sufficient to achieve many other objectives. The following simple example illustrates this.

Example 1. Consider two bidders whose types are drawn independently from the identical distribution π on two-type support $\Theta = \{\theta_L, \theta_H\}$ where $\theta_H > \theta_L \geq 0$. Let $p_L = \pi(\theta_L)$ and $p_H = 1 - p_L = \pi(\theta_H)$ and suppose that p_L is sufficiently close to 1.[6]

Can an individually rational first price auction with a reserve price maximize revenue? It is enough to consider two potential reserves: θ_H and θ_L. The former reserve gives us expected revenue $(1 - p_L^2)\theta_H$. With the latter reserve, in any equilibrium low type bidders randomize between bidding θ_L and staying out, while high type bidders mix on $[\theta_L, p_H\theta_H + p_L\theta_L]$ with a cumulative distribution function $F_H(b) = \frac{p_L(b-\theta_L)}{p_H(\theta_H - b)}$. In effect, in any equilibrium, the expected payoff of low-type bidders is 0 while the expected payoff of high-type bidders is $p_L(\theta_H - \theta_L)$. In the revenue-maximizing equilibrium within this auction, low-type bidders always bid θ_L with probability 1 and the allocation is efficient: the item is always allocated, and whenever a high type bidder is present, a high type bidder wins with probability 1. The seller's revenue is $p_L^2\theta_L + (1 - p_L^2)\theta_H - 2p_L(\theta_H - \theta_L)$. For sufficiently high p_L, the expected revenue generated by reserve θ_L is higher than the expected revenue generated by reserve θ_H.

This expected revenue can be increased by restricting the bid space to $\mathcal{B} = \{\theta_L, b_H\}$, where $b_H = \frac{\theta_H + p_L\theta_L}{1+p_L}$. There is an equilibrium in which low-type bidders bid θ_L and high-type bidders bid b_H. Indeed, low-type bidders are indifferent between staying out and bidding θ_L (both yield 0 payoff). High-type bidders are indifferent between bidding θ_L and bidding b_H. The resulting allocation is again efficient, while bidders' expected payoffs are lower than in the reserve-only auction and strictly lower for high types. Hence expected revenue is higher than with interval bid space and equals

$$p_L^2\theta_L + \left(1 - p_L^2\right) b_H = p_L\theta_L + p_H\theta_H.$$

This is the maximum revenue among mechanisms that serve both types, because the individual rationality of high-type bidders and the incentive compatibility of high-type bidders are binding, minimizing the expected payoffs.

First price auctions with reserve prices are also not sufficient for many other designer's objectives. For instance, if the designer's goal is to equalize the interim probability of winning

[6] E.g., we can assume that $p_L\theta_L \geq 2(\theta_H - \theta_L)$.

of bidders, then a simple mechanism in which each bidder wins with probability $\frac{1}{2}$ is optimal and strictly dominates any first price auction with reserve. At the same time, this simple mechanism can be implemented as a first price auction with just one allowed bid level θ_H.

In this two-type example, it is straightforward to see that the first-price auction with properly pre-selected bid levels can maximize revenue or equality. One might think that this optimality of the first-price auctions is driven by specifics of the objectives or by there being two types.[7] In the sequel we show that this optimality obtains for any distribution of types and for a large class of standard objectives including revenue maximization and equality. We furthermore leverage this first-price principle to offer solutions to mechanism design problems beyond the reach of the Myersonian approach (or other approaches we discuss in Introduction).

3 Model: Environment, Auctions, and Objectives

We study an environment with n bidders indexed by $i \in N = \{1, \ldots, n\}$. Each bidder i has a utility type $\theta_i \in \Theta$, where Θ is a compact subset of $\mathbb{R}$. The discrete case $\Theta = \{\theta^1, \ldots, \theta^T\}$ will play a special role in our proofs. We assume that each bidder's types are distributed according to distribution π and that the types are distributed i.i.d. across bidders. For brevity we also use π to refer to the joint distribution, and—in the discrete case—we write $\pi(\theta_1, ..., \theta_n) = \pi(\theta_1) ... \pi(\theta_n)$. In the discrete case, we index the types so that θ^t is strictly increasing in t.[8]

We study the allocation of a single indivisible object. A pure outcome is a vector $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, where $\mathbf{a} = (a_1, ..., a_n) \in \{0,1\}^n$ such that $\sum_{i \in N} a_i = 1$ is the vector of allocations and $\mathbf{p} = (p_1, ..., p_n) \in \left[\underline{p}, \infty\right)^n$ is the vector of payments for some $\underline{p} \leq 0$.[9] When $a_i = 1$ and $p_i = p$ we say that agent i receives the object and pays p. The space of pure outcomes is $\mathbf{X}$; an individual outcome (a_i, p_i) of each bidder i belongs to $X = \{0,1\} \times \mathbb{R}$. An **outcome** is a lottery over pure outcomes. Given a pure outcome $\mathbf{x} = (\mathbf{a}, \mathbf{p})$, the utility of each bidder

[7] For instance, as we show in Appendix A, in this example we could also maximize revenue using a second-price auction with two bid levels, but this optimality of the second-price auction does not extend beyond two types as we show in an example we discuss in Section 7.

[8] The strict monotonicity is without loss because we can represent the case in which two types are equal, $\theta^t = \theta^{t+1}$, by using only one of these types and assigning it probability $\pi(\theta^t) + \pi\left(\theta^{t+1}\right)$ for each profile of other bidders' types. In order to simplify exposition, when using superscripts (and subscripts) as above, we suppose that $\pi\left(\theta^{T+1}\right)$ and similar undefined expressions take value 0. In the main text, we study symmetric environments and anonymous objectives; for an extension to asymmetric environments and objectives, see Appendix D.

[9] Bounding the possible payments from below means that we bound the potential subsidies the seller might offer to the bidders. We may interpret this bound as reflecting the seller participation or budget constraint. Imposing this is not needed when we study objectives such as revenue-maximization but it simplifies our analysis of more general objectives by ensuring the space of relevant mechanisms is compact.

i is given by $a_i\theta_i - p_i$; an agent utility from an outcome is the resulting expected utility. A **social rule** is a mapping from the space of type profiles Θ^N to the space of outcomes $\triangle(\mathbf{X})$. Given a social rule $\mathbf{y}$ and a profile of types θ, we denote by $\mathbf{q}(\theta) = (q_i(\theta))_{i\in N}$ the vector of resulting **expected allocations** and by $\mathbf{t}(\theta) = (t_i(\theta))$ the vector of resulting **expected payments**, where each $q_i(\theta)$ is the expected value of a_i and $t_i(\theta)$ is the expected value of p_i calculated using the lottery $\mathbf{y}(\theta)$.

Bidders participate in a mechanism that allocates a single indivisible object and collects payments from the bidders. A mechanism is given by the space of actions and a choice function. The space of actions is a set $\mathcal{B}$; because auctions play a focal role in our analysis, we refer to actions $b \in \mathcal{B}$ as **bids** and to $\mathcal{B}$ as the **bid space**. For simplicity of exposition, we assume that $\mathcal{B}$ is the same for all bidders; an assumption that is without loss of generality in the study of Bayesian Nash equilibria.[10] A **outcome function** $\mathbf{f}$ maps any profile of bids $\mathbf{b} = (b_i)_{i\in N}$ into a lottery $\mathbf{f}(\mathbf{b})$ over the space of pure outcomes $\mathbf{X}$; using standard notation, we denote the space of the resulting lotteries by $\triangle(\mathbf{X})$. We say that a mechanism **generates** a social rule if there exists an equilibrium $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ such that $\mathbf{f}(\boldsymbol{\sigma}(\theta))$ is the same as the social rule applied to type profile θ for every profile in Θ^n. A social rule is **implementable** if there exists a mechanism that generates it and is **individually rational** if the expected utility of each bidder is weakly greater than 0 at any outcome in the range of the social rule.

A canonical example of a mechanism is a **direct mechanism**, in which $\mathcal{B} = \Theta$. We are also interested in auctions. An **auction** is a mechanism such that $\mathcal{B} \subseteq [0, \infty) \cup \{\varnothing, r^-\}$ and $\varnothing \in \mathcal{B}$. Submitting a bid $\varnothing$ means that the bidder has opted out of the auction; by assuming that $\varnothing \in \mathcal{B}$, we guarantee that bidders' participation constraints are satisfied. We interpret bid r^- as strictly lower than $\inf(\mathcal{B}\cap\mathbb{R})$.[11] The outcome function $\mathbf{f}$ has the following property: if at least one bidder submits a bid in $\mathbb{R} \cup \{r^-\}$, then exactly one bidder i, whose bid is weakly higher than all other submitted real bids, receives allocation $a_i = 1$, and all other bidders $j \neq i$ receive allocation $a_j = 0$. Otherwise, if all bidders have opted out, then $a_i = 0$ for all bidders. A **first-price auction** is an auction such that $p_i = b_i$ if $b_i \in \mathbb{R}$ and $a_i = 1$; $p_i = \inf(\mathcal{B} \cap \mathbb{R})$ if $b_i = \varnothing^+$ and $a_i = 1$; and $p_i = 0$ otherwise, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and $\mathbf{b} \in \mathcal{B}^n$. A **second-price auction** is an auction such that, for any $\mathbf{x} \in \text{supp}(\mathbf{f}(\mathbf{b}))$ and

[10] Indeed, for any equilibrium in any mechanism in which each agent i has a space $\mathcal{B}_i$, we can find an outcome function and strategies in a mechanism in which each agent has the space $\mathcal{B} = \cup_{i\in N}\mathcal{B}_i$ such that the strategies are in equilibrium and the mechanism with action spaces all being $\mathcal{B}$ implements the same mapping from types to outcomes as the original mechanism with action spaces $\mathcal{B}_i$.

[11] In the first-price and second-price auctions described next, bidding r^- allows the bidder to win only if no bidders submit bids from $\mathbb{R}$; if the bidder wins, the winner pays $\inf(\mathcal{B}\cap\mathbb{R})$. We introduce this low bid to allow equilibria in which, by bidding r^-, some bidders communicate their indifference between winning and losing at price $\inf(\mathcal{B}\cap\mathbb{R})$.

for any $\mathbf{b} \in \mathcal{B}^n$, if $a_i = 1$ then

$$p_i = \begin{cases} \inf(\mathcal{B} \cap \mathbb{R}) & \text{if no bidder } j \neq i \text{ submitted a bid in } \mathbb{R} \\ \max\{b_j | j \in N - \{i\} \text{ and } b_j \in \mathbb{R}\} & \text{if some bidder } j \neq i \text{ submitted a bid in } \mathbb{R} \end{cases}$$

and if $a_i = 0$ then $p_i = 0$. Note that $\inf(\mathcal{B} \cap \mathbb{R})$ is usually called the reserve price. Note also that in both first-price and second-price auction, we allow the design of the bid space that goes beyond imposing a reserve price; in examples below, we show that this additional degree of design freedom is necessary to achieve standard objectives. We also study the broader classes of auctions, the **first-price auction with subsidy** and the **second-price auction with subsidy** in which there is a fixed $s \geq 0$ such that all prices p_i are decreased by s. Thus s is a subsidy granted to all bidders, regardless of who wins and the bid profile.

Fix a social rule $\mathbf{y} = (y_1, \ldots, y_n)$. Each bidder i with a type θ_i can compute the **interim expected outcome** (Q_i, T_i) by $\sum_{\theta_{-i} \in \Theta_{-i}} y_i(\theta_i, \theta_{-i})\pi(\theta_i, \theta_{-i})$ and we say that Q_i is the **interim expected allocation** and T_i is the **interim expected payment.** We call $\mathbf{Q} = (Q_i)_{i \in N}$ the **interim allocation rule** and $\mathbf{T} = (T_i)_{i \in N}$ the **interim payment rule**. We call $(\mathbf{Q}, \mathbf{T})$ a **reduced-form** of $\mathbf{y}$ and we also say that a mechanism $(\mathcal{B}, \mathbf{f})$ **implements** a reduced-form mechanism $(\mathbf{Q}, \mathbf{T})$ if it generates a social rule $\mathbf{y}$ such that $(\mathbf{Q}, \mathbf{T})$ is a reduced form of $\mathbf{y}$. Whenever we refer to a reduced-form mechanism, we always assume that there exists a mechanism $(\mathcal{B}, \mathbf{f})$ that implements it. We then say that an interim allocation rule $\mathbf{Q}$ is **feasible** if there exists a reduced form mechanism $(\mathbf{Q}, \mathbf{T})$ and, a priori, we allow interim allocation rules that are not necessarily feasible. A reduced form mechanism $(\mathbf{Q}, \mathbf{T})$ is **incentive compatible** if $Q_i(\theta)\theta - T_i(\theta) \geq Q_i(\hat{\theta})\theta - T_i(\hat{\theta})$ for all $i \in N$ and $\theta, \hat{\theta} \in \Theta$; notice that the incentive compatibility requires that agents are willing to report their true types to the reduced form mechanism; in this sense the reduced from mechanism is a direct reduced form mechanism. A reduced form mechanism $(\mathbf{Q}, \mathbf{T})$ is **individually rational** if $Q_i(\theta)\theta - T_i(\theta) \geq 0$ for all $i \in N$ and $\theta, x \in \Theta$. An interim allocation rule $\mathbf{Q}$ is **symmetric** if $Q_i(\theta) = Q_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A reduced form mechanism $(\mathbf{Q}, \mathbf{T})$ is **symmetric** if $\mathbf{Q}$ is symmetric and $T_i(\theta) = T_j(\theta)$ for all $i, j \in N$ and $\theta \in \Theta$. A social rule $\mathbf{y}$ is called **symmetric** if $y_i(\boldsymbol{\theta}) = y_1(\tau^i(\boldsymbol{\theta}))$ for all i and $\boldsymbol{\theta}$, where τ^i is the transposition that interchanges the first and i-th coordinates of $\boldsymbol{\theta}$.

3.1 Objectives

The goal of the allocation is to maximize an objective such as welfare maximization, revenue maximization or inequality minimization. Given the distribution of types π, an **objective** U_π is a mapping from the set of feasible, incentive compatible, and individually rational

mechanisms $(\mathcal{Q}, \mathcal{T})$ to $\mathbb{R}$. We say that a mechanism **maximizes** the objective if it is feasible, individually rational, incentive compatible, and there is no other mechanism in $(\mathcal{Q}, \mathcal{T})$ that maps to a higher value. We say that an objective is **standard** if, for any feasible, incentive compatible, and individually rational $(\mathbf{Q}, \mathbf{T})$, $(\mathbf{Q}_1, \mathbf{T}_1)$, and $(\mathbf{Q}_2, \mathbf{T}_2)$, the following properties are satisfied:[12]

(i) Anonymity: If $(\mathbf{Q}, \mathbf{T})$ maximizes the objective then so does $\tau^{-1} \circ (\mathbf{Q}, \mathbf{T}) \circ \tau$ for any bijection $\tau : N \to N$.

(ii) Convexity of the maximand set: If $(\mathbf{Q}_1, \mathbf{T}_1)$ and $(\mathbf{Q}_2, \mathbf{T}_2)$ maximize the objective then so does $\frac{1}{2}(\mathbf{Q}_1, \mathbf{T}_1) + \frac{1}{2}(\mathbf{Q}_2, \mathbf{T}_2)$.

(iii) Quasi-convexity: For any symmetric $(\mathbf{Q}_1, \mathbf{T}_1)$ and $(\mathbf{Q}_2, \mathbf{T}_2)$, if $\lambda(\mathbf{Q}_1, \mathbf{T}_1)+(1-\lambda)(\mathbf{Q}_2, \mathbf{T}_2)$ maximizes the objective for some $\lambda \in (0,1)$, then either $(\mathbf{Q}_1, \mathbf{T}_1)$ or $(\mathbf{Q}_2, \mathbf{T}_2)$ (or both) are feasible, incentive-compatible, and maximize the objective.

(iv) Continuity: the mapping from $(\pi, \mathbf{Q}, \mathbf{T})$ to $U_\pi(\mathbf{Q}, \mathbf{T})$ is continuous in topology of pointwise convergence on $(\mathbf{Q}, \mathbf{T})$ and Prokhorov topology on π.[13]

The expected welfare and revenue maximization are standard in the above sense. The class of standard objectives also includes the seller's expected profit-maximization for any cost function of the seller, as well as linear combinations of expected welfare, revenue, and profits.

Downstream consumer welfare is also standard provided the mapping from auction outcomes to the induced downstream consumer welfare is anonymous and satisfies mild regularity assumptions. Other examples of standard objectives include maximizing or minimizing the sum of payoffs of all bidders or of bidders of a specific type or the probability of that bidders with intermediate values win. Maximizing the weighted sum of payments, where weights depend on the bidders' types, is also a standard objective. In contrast, maximizing the variance of payments is an example of an objective that, in general, is not standard in the above sense because it violates our condition (ii).

Many objectives minimizing inequality are also standard. Among them is the mini-

[12]See Bergson (1938) and Samuelson (1956) for the convexity assumptions; see May (1952), Sen (1970), and Pycia (2019) for anonymity. Because of upper semicontinuity, property (ii) implies that if $\mathbf{y}_1$ and $\mathbf{y}_2$ maximize the objective then so does $\lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2$ for any $\lambda \in [0,1]$. In the main text, we study symmetric environments and anonymous objectives; for an extension to asymmetric environments and objectives, see Appendix D.

[13]The continuity assumption plays a role in our analysis of general type spaces; in the special case of finite Θ, this assumption however can be completely relaxed. A sequence of functions $(f_k)_{k\in\mathbb{N}}$ converges to f in topology of pointwise convergence if and only if $(f_k)_{k\in\mathbb{N}}$ converges to f pointwise; a special case of product topology. Prokhorov topology is given by the Prokhorov metric, which measures distances between Borel probability measures on a metric space: Prokhorove metric between two probability measure μ and ν on a metric space (M,d) is $\inf\{\varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all measurable } A \subseteq M\}$, where $A^\varepsilon = \{x \in M \mid \exists y \in A, d(x,y) < \varepsilon\}$. For more on pointwise convergence, product topology, and Prokhorov metric see, e.g., Folland (1984) and Billingsley (1999).

mization of the Gini coefficient with respect to any strictly positive measure of individual outcomes, x_i:

$$\frac{\sum_{i,j}\int_\Theta\int_\Theta |x_i(\theta)-x_j(\theta')|\,d\pi(\theta)\,d\pi(\theta')}{\sum_i\int_\Theta x_i(\theta)d\pi(\theta)};$$

for instance, x_i can be equal to allocation Q_i, or transfer T_i, or the resulting utility $U_i(\theta) = \theta Q_i(\theta) - T_i(\theta)$ of bidder i, or any convex combination of these variables of interest. This follows from Lemma 1—that we discuss below—because incentive compatibility ensures that all these parameters (and their convex combinations) are monotonic in type θ. Similarly, the minimization of Gini mean difference

$$\sum_{i,j}\int_\Theta\int_\Theta |x_i(\theta)-x_j(\theta')|\,d\pi(\theta)\,d\pi(\theta')$$

is a standard objective, as is the minimization of the Palma ratio

$$\frac{\sum_i\int_{\{\theta\in\Theta|F(\theta)\geq 1-\gamma\}} x_i(\theta)\,d\pi(\theta)}{\sum_i\int_{\{\theta\in\Theta|F(\theta)\leq\beta\}} x_i(\theta)\,d\pi(\theta)},$$

where $0<\beta<1-\gamma<1$ and F is the cumulative distribution function of π.

The above claims build on the fact that linear combinations of standard and convex objectives are standard, as are are ratios of such objectives when the denominator is affine and bounded away from zero. We define these concepts as follows: An objective U_π is convex if

$$U_\pi(\lambda(Q^1,T^1)+(1-\lambda)(Q^1,T^1)) \leq \lambda U_\pi(Q^1,T^1)+(1-\lambda)U_\pi(Q^1,T^1)$$

for all $\lambda\in[0,1]$. An objective is affine in symmetric (Q,T) if $U_\pi(aQ+b,aT+b) = aU_\pi(Q,T)+b$ for all $(a,b)\in\mathbb{R}^2$. An objective is strictly positive if it always takes strictly positive values.

The following then obtains:

Lemma 1. *If U_π^1 and U_π^2 are standard and convex then so is any linear combination $\lambda_1 U_\pi^1 + \lambda_2 U_\pi^2$ with nonnegative coefficients $\lambda_1,\lambda_2\geq 0$ is standard and convex.*

Let T be a compact set endowed with probability measure λ. If U_π^τ are standard and convex for all $\tau\in T$ then $\int_T U_\pi^\tau d\lambda(\tau)$ is also standard and convex.

If U_π is standard and convex while V_π is standard, strictly positive, and affine in symmetric (Q,T), then $\frac{U_\pi}{V_\pi}$ is standard.

This lemma is of independent interest as policy makers are often interested in multiple objectives. The proof of this lemma and the proofs of all other results not proven directly in the main text are in Appendix B.

4 The First-Price Principle

Our main result shows that first-price auctions and identical subsidies for all bidders are sufficient to implement all standard objectives. Furthermore, the subsidies are not needed if the objective U_π is weakly increasing in revenue in the following sense: for every $\varepsilon > 0$ we have $U_\pi(\mathbf{Q}, (T_1 + \varepsilon, \dots, T_n + \varepsilon)) \geq U_\pi(\mathbf{Q}, (T_1, \dots, T_n))$.

Theorem 1. ***[First-Price Principle]*** *For any standard objective, there exists a bid space $\mathcal{B}$ and subsidy $s \geq 0$ such that the objective is maximized in a pure-strategy equilibrium in the first-price auction with bid space $\mathcal{B}$ and subsidy s. Furthermore, if the objective is weakly increasing in revenue, then $s = 0$ is an optimal subsidy, and hence the objective is maximized in a pure-strategy equilibrium of the first-price auction.*

An analogous argument shows that if the designer is constraint to non-wasteful allocations—that is allocations satisfying $\sum_{i \in N} \mathbb{E}_\pi [Q_i(\theta)] = 1$—then an analogue of Theorem 1 holds: there exists a first-price auction with subsidy that achieves the objective; furthermore, in this case the minimum bid allowed is weakly below the minimum of support of bidders' values.[14]

This result implies that, for standard objectives, the allocation is weakly assortative and it is uniformly random on intervals of types that submit the same bid. Let $(I, \prec)$ be any linearly ordered index set (it may be finite, countable, or uncountable). A family $\{\Theta_i\}_{i \in I}$ is a **monotone partition** of Θ if:

1. (Partition) They are pairwise disjoint and $\bigcup_{i \in I} \Theta_i = \Theta$.
2. (Monotonicity) For all $i \prec j$ and all $x \in \Theta_i,\ y \in \Theta_j$ we have $x < y$.

Corollary 1. *For all standard objectives, there is a monotone partition of the type space such that the object is assigned to a bidder with type in the highest element of the partition and all such bidders have the same probability of obtaining the object.*

In particular, this corollary contains the main insight of Akbarpour, Dworczak, and Kominers (2020).[15] In addition, it implies similar insights for other standard objectives, e.g.

[14] While the theorem is formulated as partial implementation theorem (that is it only asserts that there exists an equilibrium that maximizes the objective), Theorem 4 below shows an essential uniqueness of equilibria in first-price auctions with interval bid spaces and Theorem 5, shows that in many natural problems we cam maximize the objective using an interval bid space. Taken together, Theorems 1, 4, and 5 give us an essential full implementation.

[15] Our Theorem 1 is independent of their work and the proof approaches in ours and their work are completely different.

for the maximization of any linear combination of revenue, welfare, and the ex-ante Gini mean difference.

A key step in our analysis shows that any symmetric reduced-form mechanism that is incentive compatible and individually rational is equivalent to a lottery over first-price auctions with subsidy, and any such lottery is a symmetric mechanism (we exchangeably use the terms a lottery over mechanisms and a convex combination of the mechanisms). Establishing the decomposition insight for symmetric mechanism is sufficient to prove our opening claim because of the following:

Lemma 2. *Any standard objective is maximized by a symmetric mechanism.*

Proof. Given a standard objective, suppose that a social rule $\mathbf{y}$ maximizes the objective and let $\mathcal{T}$ be the space of all permutations over the set of agents, N. Then the social rule $\frac{1}{|N|!}\sum_{\tau\in\mathcal{T}}\tau^{-1}\circ(\mathbf{Q},\mathbf{T})\circ\tau$ is symmetric, individually rational, and implementable. Furthermore, the properties (i) and (ii) of standard objectives ensure that this symmetric social rule maximizes the objective. □

This lemma extends the straightforward observation that efficient allocations are symmetric and the classical insight that the allocations of revenue-maximizing mechanisms are symmetric as well. The latter insight goes back to Myerson (1981).[16]

We prove Theorem 1 by first establishing it for discrete type spaces and then showing that the result remains true for general type space by approximating them as a limit of finite type spaces.[17]

For finite type spaces, we show

Lemma 3. *For finite type spaces and any anonymous objective with a convex maximand set, there is a lottery over first-price auctions with subsidy such that this objective is maximized in a pure strategy equilibrium of the resulting mechanism.*

Note that the designer maximizing a standard objective can without loss restrict the bid space to bids chosen on equilibrium path. Furthermore, when a pure strategy equilibrium is played by bidders with finite type space, then a finite subset of bid space is used on equilibrium path. Hence, if we restrict attention to pure strategy equilibria then considering finite bid spaces is sufficient.[18]

[16]Cf. also Lovejoy (2006) and Bergemann and Pesendorfer (2007) who proved this symmetry insight in the context of discrete type spaces.

[17]This approach to mechanism design is unusual but has some forerunners; see Reny (1999), Reny (2011), and Woodward (2016) for the analysis of discontinuous games.

[18]A similar claim for mixed strategies relies on a more subtle argument and we omit it because we are not relying on it.

We prove Lemma 3 for finite type spaces, building on Lemma 4 below. Lemma 3 and Lemma 2 imply Theorem 1 for finite type spaces together with quasi-convexity. We then extend Theorem 1 from finite to any general type spaces by approximating them as a limit of finite type spaces. We use Helly's selection theorem to select a convergent sequence of first-price auctions with subsidy that are optimal a sequence of finite type spaces and we show that the limit of this sequence is a first-price auction with subsidy that maximizes the objective.

Lemma 4. *If the type space is finite, then every symmetric, incentive compatible, and individually rational mechanism is a lottery over equilibria of first-price auctions with subsidy.*

In the theorem we could replace a lottery over equilibria with an equilibrium of the mechanism in which bidders submit bids conditional on the first-price auction drawn by the lottery.[19] We prove Lemma 4 in Section A.1 by analyzing the extreme points in the space of all mechanisms (see Lemma 7); in particular we show there that first-price auctions are such extreme points (Lemma 6).

The need to allow the special bid r^- is illustrated by the following:

Example 2. (Necessity of allowing the special bid r^-). Suppose there are two bidders with types drawn independently from $\Theta = \{1, 2\}$ with the probability $\pi(1) = \pi(2) = 0.5$. The objective is to maximize

$$U_\pi(\mathbf{Q}, \mathbf{T}) = \sum_{i=1}^{2} \sum_{\theta \in \Theta} \theta Q_i(\theta) \pi_i(\theta) - \alpha \sum_{i=1}^{2} |T_i(2) - T_i(1)|,$$

where $\alpha > 0$ measures how much the designer cares about minimizing the expected payment gap between two different types of bidders. In this example, we need the special bid level r^- if we care sufficiently about surplus ($\alpha < .25$) as then having bidder of type 2 bid 1 and bidder of type 1 bid r^- maximizes surplus while minimizing the differences of expected payments of the two types.[20]

[19] Such conditioning is natural in our analysis because we allow different bid spaces in different first-price auctions.

[20] Indeed, by 1, it is without loss to restrict attention to first-price auctions (with possible subsidies). Moreover, any universal subsidy adds the same amount to $T_i(2)$ and $T_i(1)$, so the gap $|T_i(2) - T_i(1)|$ is unchanged; hence we may set subsidies to zero without loss. Let b_H and b_L to denote the equilibrium bid in a first-price auction for $\theta = 2$ and $\theta = 1$, respectively. There are four possible cases: (i) $b_H > b_L > \varnothing$, (ii) $b_H = b_L > \varnothing$, (iii) $b_H > b_L = \varnothing$, and (iv) $b_H = b_L = \varnothing$. In cases (i) and (iii), type-2 bidders win the auction against type-1 bidders and win half the time against type-2. Hence, the interim allocation for type-2 bidders is $Q(2) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$. In case (i), type-1 bidders win the auction only against type-1 bidders via tie-breaking, hence $Q(1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. In case (iii), type-1 bidders never win, hence $Q(1) = 0$. In case (ii), both types only win via tie-breaking, hence $Q(2) = Q(1) = \frac{1}{2}$. The interim transfers are bids

5 Tools to Design and Analyze First Price Auctions

The existence of equilibria in First Price Auctions with standard bid space (which allow all bids above some reserve price) have been proven by Riley and Samuelson (1981). Does their existence remain valid for other bid spaces? For atomless type spaces we show that equilibria exist provided bid spaces are compact; as the spaces of relevant bids is always bounded, compactness can be weakened to the bid space being topologically closed. Furthermore, it is easy to construct examples in which the pure-strategy equilibrium fails to exist because the bid space is not topologically closed.

Theorem 2. ***[Existence]*** *If the type distribution is atomless and the bid space is compact, then there exists a monotone pure-strategy symmetric equilibrium in the first-price auction with subsidy.*

If the bid space is finite, the existence of a pure-strategy symmetric equilibrium is implied by Reny (2011). Appendix B provides the proof for the general case in which we use the equilibria in the finite bid spaces to construct an equilibrium for the general case.[21]

The theorem restricts attention to compact bid spaces and the following complementary lemma shows that this restriction is without loss of generality in our analysis of First Price Auctions.[22]

Theorem 3. ***[Sufficiency of Closed Bid Spaces]*** *If a symmetric profile of bidding strategies $b(\cdot)$ is an equilibrium in the First Price Auction with bid space $\mathcal{B}$, then the same profile of strategies is in equilibrium in the First Price Auction with bid space $cl(\mathcal{B})$, where $cl(\mathcal{B})$ is the closure of $\mathcal{B}$.*

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multiplied by interim allocations. Therefore, we have (i) $U_\pi(b_H, b_L) = \left(\frac{3}{4} \times 2 + \frac{1}{4} \times 1\right) - 2\alpha\left(\frac{3}{4}b_H - \frac{1}{4}b_L\right)$, (ii) $U_\pi(b_H, b_L) = \left(\frac{1}{2} \times 2 + \frac{1}{2} \times 1\right)$, and (iii) $U_\pi(b_H, b_L) = \left(\frac{3}{4} \times 2\right) - 2\alpha\left(\frac{3}{4}b_H\right)$. In particular, properly designed case (i) dominates case (iii); furthermore properly designed case (ii) dominates case (iv). Thus, choosing bids of type (iii) or (iv) cannot be optimal. Within (i), $U_\pi(b_H, b_L) = \left(\frac{3}{4} \times 2 + \frac{1}{4} \times 1\right) - 2\alpha\left(\frac{3}{4}b_H - \frac{1}{4}b_L\right)$ is maximized when $b_H = 1$ and $b_L = r^-$ with $\mathcal{B} = \{\varnothing, r^-, 1\}$, where b_L is interpreted as 1 when b_L is the highest bid with the maximized value $U_\pi = 1.75 - \alpha$. Furthermore, $b_H = 1$ and $b_L = r^-$ constitutes an equilibrium because type-1 bidders are indifferent between two bid levels as their net payoffs are 0 in either case and type-2 bidders strictly prefer bidding b_H to win more often. Importantly, this outcome cannot be replicated without the special bid r^-. Within case (ii), it is optimal to have both bidder types bid 1, which yields $U_\pi = 1.5$. In effect, the optimal bid space is $\mathcal{B} = \{\varnothing, r^-, 1\}$ for $\alpha < 0.25$ and $\mathcal{B} = \{\varnothing, 1\}$ for $\alpha > 0.25$.

[21] Allowing for risk aversion and interdependent values, Reny and Zamir (2004) established the existence of a monotone pure-strategy equilibrium for finite bid spaces, and by taking the limit, for standard bid space with bidder-specific reserve prices. Our proof also starts with finite bid spaces. By taking the limit we then extend the existence to all compact bid spaces; in the process we need to handle the additional subtleties arising from the nonconvexity of the bid space. Cf. also the classic analysis of Riley and Samuelson (1981) who characterized the pure-strategy symmetric equilibrium under standard bid space with a reserve price, as well as Woodward's (2016) limit analysis of finite type spaces in multi-unit auctions.

[22] The converse does not necessarily hold true. One can construct a type space and bid space $\mathcal{B}$ for which there is a symmetric pure-strategy equilibrium in the First Price Auction with bid space $cl(\mathcal{B})$ and in this equilibrium some types submit bids from $cl(\mathcal{B}) - \mathcal{B}$.

Our next tool-result characterizes equilibria of first-price auctions with interval bid spaces and establishes their essential uniqueness.

Theorem 4. ***[Essential Uniqueness]*** *Suppose $\underline{\theta}, \bar{\theta}, \underline{b}, \bar{b} \in \mathbb{R}_+$, $\underline{\theta} < \bar{\theta}$, $\underline{b} \leq \bar{b}$, and bidders' types are drawn from an atomless distribution with interval support $[\underline{\theta},\bar{\theta}]$ and the bid space is $\{\varnothing\} \cup \left[\underline{b}, \bar{b}\right]$. Then, all bidder types are indifferent between all equilibria.*

Furthermore, there are three cases:

** If $\bar{b} \leq \underline{\theta}$, then all bidders bidding $\bar{b}$ is an equilibrium.*

** If $\underline{\theta} < \bar{b} < \bar{\theta} - \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1}\, dx$, then the equation:*

$$\left[\hat{\theta} - \bar{b}\right] \int_{\hat{\theta}}^{\bar{\theta}} F(x)^{n-1} dF(x) = \left[1 - F(\hat{\theta})\right] \int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx$$

has a unique solution in $[\bar{b}, \bar{\theta})$, and the symmetric profile of bidding strategies $(b_i)_{i \in N}$ given by

$$b_i(\theta) = \begin{cases} \varnothing & \text{if } \theta \leq \underline{b} \\ \theta - \frac{\int_{\underline{b}}^{\theta} F(x)^{n-1} dx}{F(\theta)^{n-1}} & \text{if } \underline{b} < \theta < \hat{\theta}, \\ \bar{b} & \text{if } \theta \geq \hat{\theta}, \end{cases}$$

forms an equilibrium.

** If $\bar{b} \geq \bar{\theta} - \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1}\, dx$, then the symmetric profile of bidding strategies $(b_i)_{i \in N}$ given by*

$$b_i(\theta) = \begin{cases} \varnothing & \text{if } \theta \leq \underline{b}, \\ \theta - \frac{\int_{\underline{b}}^{\theta} F(x)^{n-1} dx}{F(\theta)^{n-1}} & \text{if } \theta > \underline{b}, \end{cases}$$

forms an equilibrium.

A weaker version of this result was established by Chassang and Ortner (2019), who restricted attention to symmetric equilibria in pure strategies; in contrast, our approach does not rely on any symmetry assumptions and allows for mixed strategies. In particular, we prove that substantially mixed or asymmetric equilibria cannot exist; only mixing and asymmetries that do not break the essential uniqueness established in the theorem are possible. These mentioned asymmetries are rather limited. In the first case of the theorem, asymmetries can only arise if $\bar{b} = \underline{\theta}$; then type $\underline{\theta}$ is indifferent between bidding $\varnothing$ and $\bar{b}$ and can adapt any mixed strategy with support $\left\{\varnothing, \bar{b}\right\}$. In the second case, bidders with type $\underline{b}$ are indifferent between bidding $\underline{b}$ and $\varnothing$ (staying out) and bidders with type $\hat{\theta}$ are indifferent between bidding $\bar{b}$ and bidding $\hat{\theta} - \frac{\int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx}{F^{n-1}(\hat{\theta})}$. As the type distribution is atomless, there

exist asymmetric equilibria where bidders at the cutoff type behave differently and these cutoff types can mix over the relevant two bid levels. In all these equilibria, all bidder types have the same expected utility (and the expected revenue and welfare are also the same). Furthermore, in any pure strategy equilibrium, the bids of types $\theta \neq \underline{b}, \hat{\theta}$ are given by $b_i(\theta)$ above while $b_i(\underline{b}) \in \{\varnothing, \underline{b}\}$ and $b_i(\hat{\theta}) \in \{\hat{\theta} - \frac{\int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx}{F^{n-1}(\hat{\theta})}, \bar{b}\}$. An analogous situation arises in the third case of the theorem, in which only bids of bidders of type $\underline{b}$ are not fully determined and can take either of the two values in $\{\varnothing, \underline{b}\}$ or any lottery over them.

Our final tool-result shows that interval bid spaces are sufficient for many distributions and objectives:

Theorem 5. ***[Sufficient Condition for Interval Bid Spaces]*** *Suppose bidders type are drawn from a convex and atomless c.d.f. with interval support $[\underline{\theta}, \bar{\theta}]$. Let $\mathcal{B}$ be a compact bid space and $b(\cdot)$ be a symmetric equilibrium bid strategy for bid space $\mathcal{B}$. Let $\hat{\mathcal{B}} = \{\varnothing\} \cup \left[r, \bar{b}\right]$, where $r = \inf\{\theta \mid b(\theta) > \varnothing\}$ and $\bar{b} = b(\bar{\theta})$, and $\hat{b}(\cdot)$ be the symmetric equilibrium bid strategy for bid space $\hat{\mathcal{B}}$. Then, the revenue in equilibrium $\hat{b}$ is weakly higher than in equilibrium b. Furthermore, for any bidder type θ, the expected utility in equilibrium $\hat{b}(\cdot)$ is weakly lower than the expected utility in equilibrium $b(\cdot)$.*

6 Applications

6.1 Balancing Revenue and Equality

Minimum bid price systems, in which the winning bid is selected among the lowest bids above a certain threshold, are a common feature of public works procurement in Japan and are widely implemented in competitive auctions (Ministry of Land, Infrastructure, Transport and Tourism, 2005). Such rules have also been studied by Chassang and Ortner (2019). Our Theorem 1 illustrates that such a restriction is optimal if the policy objective is to balance the equality and the revenue.[23] While the Japanese case arises in a procurement setting, the following example is cast in the standard auction environment—where a seller offers an item for sale—because our model is formulated in the language of regular auctions rather than procurement auctions.

Suppose there are two bidders and each bidder has a type independently drawn from a uniform distribution on $[0, 1]$. A designer balances maximizing expected revenue and minimizing the disparities in the winning probabilities among different types: the objective U is a mix of the expected revenue and the total variation in the winning probability among

[23] Our analysis of optimality of capped bid spaces has no counterpart in Chassang and Ortner (2019).

different types, $U_\pi(\mathbf{Q},\mathbf{T}) = \mathbb{E}_\pi\left[T_1(\theta_1)+T_2(\theta_2)\right] - \alpha TV\left(Q_1(\theta_1)+Q_2(\theta_2)\right)$. The parameter $\alpha > 0$ indicates the level of concern about the differences in winning probabilities; the higher α is the more designer is focused on minimizing these disparities and less on revenue. Because the total variation of a monotonic function is simply the difference between its values at the start and end points, this objective is standard and even linear on the set of incentive-compatible mechanisms. Thus, the objective can be expressed as $U_\pi(Q,T) = 2\mathbb{E}_\pi\left[T\left(\theta\right)\right] - 2\alpha\left[Q(1)-Q(0)\right]$.

Theorem 1 and Theorem 3 enables us to find an optimal mechanism by looking only at first-price auctions with a compact bid space and without subsidy. Furthermore, the bid r^- is not necessary because the type distribution is atomless; including r^- cannot increase revenue but only may reduce the interim allocation for type 0. Consider an arbitrary compact bid space $\hat{\mathcal{B}}$ with a bidding strategy $\hat{b}\left(\theta\right)$ that forms an equilibrium. Let $\hat{u}\left(\theta\right)$ and $\hat{Q}\left(\theta\right)$ denote the payoff and the interim allocation for type θ in the equilibrium formed by $\hat{b}\left(\theta\right)$. For expositional simplicity, we assume that $\hat{b}\left(\cdot\right)$ is right-continuous. This is without loss because the value distribution is atomless. Let $r = \min\left\{\hat{b}\left(\theta\right)|\hat{b}\left(\theta\right) > \varnothing\right\}$ and $\bar{b} = \hat{b}\left(1\right)$. The minimum exists because $\hat{b}$ is right-continuous. In finding the optimal bid space, we analyze the improvement from $\hat{\mathcal{B}}$ and $\hat{b}\left(\theta\right)$ by considering three cases: $\hat{Q}\left(0\right) = 0$, $\hat{Q}\left(0\right) \in (0, \frac{1}{n})$, and $\hat{Q}\left(0\right) = \frac{1}{\mathrm{n}}$.

Case 1. Suppose that $\hat{Q}\left(0\right) = 0$. We show that $\mathcal{B} = \{\varnothing\} \cup \left[r, \bar{b}\right]$ with the equilibrium $b(\cdot)$ characterized by Theorem 4 achieves a weakly higher objective value. Let $u\left(\theta\right)$ and $Q(\theta)$ denote the payoff and the interim allocation for type θ in the equilibrium formed by $b(\theta)$. Because $\hat{b}\left(r\right) = r$ and $b\left(r\right) = r$, we have $\hat{u}(r) = u\left(r\right) = 0$. Because a uniform distribution is both convex and concave, by Lemma 18 (provided in Appendix B), we must have $u(1) = \hat{u}(1)$. Because $u(1) = Q(1)(1-\bar{b})$ and $\hat{u}(1) = \hat{Q}(1)(1-\bar{b})$, we have $Q(1) = \hat{Q}(1)$. Because $Q(1) - Q(0) = \hat{Q}(1) - \hat{Q}(0)$, the only possible difference between equilibria $b\left(\cdot\right)$ and $\hat{b}\left(\cdot\right)$ lies in the expected revenue. By Theorem 5, the bid space $\mathcal{B}$ with the equilibrium in $b(\cdot)$ revenue dominates the bid space $\hat{\mathcal{B}}$ with the equilibrium in $\hat{b}\left(\cdot\right)$.

Case 2. Suppose that $0 < \hat{Q}\left(0\right) < \frac{1}{n}$. That is, there is a positive mass of bidder types that bid 0 and there is a positive mass of bidders that bid higher than 0. Let $r^+ = \min\left\{b\left(\theta\right)|\hat{b}\left(\theta\right) > 0\right\}$. The minimum exists because $\hat{b}$ is right-continuous and it is bounded away from 0: $\hat{b}$ must jump from 0 to a strictly positive bid due to the tie-breaking at the 0

bid. Then, $\mathcal{B} = \{\varnothing, 0\} \cup \left[\frac{r^+}{2}, \bar{b}\right]$ with the equilibrium

$$b(\theta) = \begin{cases} 0 & \text{if } \theta < r^+ \\ \frac{1}{2}\theta & \text{if } r^+ \leq \theta < \frac{\bar{b}}{1-\bar{b}} \\ \bar{b} & \text{if } \theta \geq \frac{\bar{b}}{1-\bar{b}} \end{cases}$$

dominates $\hat{\mathcal{B}}$ and $\hat{b}$ for the following reasons; because the cutoff type bidders must be indifferent between two bids level, and $\hat{Q}(0) = Q(0)$, we have $\hat{u}(r^+) = u(r^+)$. Hence, by Lemma 18, we have again $\hat{Q}(1) = Q(1)$. Furthermore, by considering an auction with a uniform type distribution on $[r^+, 1]$, we conclude that the expected revenue is weakly higher in $b(\cdot)$ than in $\hat{b}(\cdot)$.

Case 3. Suppose that $\hat{Q}(0) = \frac{1}{n}$. In this case, the item is allocated at random regardless of the type and the value of the objective is 0. This can be achieved by setting $\mathcal{B} = \{\varnothing, 0\}$ with the equilibrium $b(\theta) = 0$ for all θ.

Combining the three cases, it is sufficient to consider the following three bid spaces;

– $\mathcal{B} = \{\varnothing\} \cup \left[r, \bar{b}\right]$,

– $\mathcal{B} = \{\varnothing, 0\} \cup \left[r, \bar{b}\right]$,

– $\mathcal{B} = \{\varnothing, 0\}$.

Theorem 4 then implies the following: given the bid space $\mathcal{B} = \{\varnothing\} \cup \left[r, \bar{b}\right]$, the bidding strategy in the essentially unique equilibrium take the form

$$b(\theta) = \begin{cases} \varnothing & \text{if } \theta < r, \\ \frac{1}{2}\theta + \frac{1}{2\theta}r^2 & \text{if } r \leq \theta < \frac{\bar{b}-r^2}{1-\bar{b}}, \\ \bar{b} & \text{if } \theta \geq \frac{\bar{b}-r^2}{1-\bar{b}}. \end{cases}$$

The expected equilibrium value of of U for bid space $\left[r, \bar{b}\right]$ is then

$$U(\bar{b}, r) = -\frac{1}{3}r^3 + \frac{(\bar{b}-r^2)^3}{3(1-\bar{b})^3} + r^2\left(-r + \frac{(\bar{b}-r^2)}{1-\bar{b}}\right) + \bar{b}\left(1 - \frac{(\bar{b}-r^2)^2}{(1-\bar{b})^2}\right) - \alpha\left(\frac{1-r^2}{1-\bar{b}}\right).$$

If $\mathcal{B} = \{\varnothing, 0\} \cup [r, \bar{b}]$, the symmetric equilibrium can take two different forms depending on whether $r < \bar{b}$ or $r = \bar{b}$. For $r < \bar{b}$, then bidders with type $\theta = 2r$ are indifferent between bidding 0 and r, and the equilibrium bid takes the form

$$b(\theta) = \begin{cases} 0 & \text{if } \theta < 2r \\ \frac{1}{2}\theta & \text{if } 2r \leq \theta < \frac{\bar{b}}{1-\bar{b}} \\ \bar{b} & \text{if } \theta \geq \frac{\bar{b}}{1-\bar{b}} \end{cases}.$$

If $r = \bar{b}$, then bidders with type $\theta = \frac{r}{1-r}$ are indifferent between bidding 0 and r, thus the equilibrium bid takes the form

$$b(\theta) = \begin{cases} 0 & \text{if } \theta < \frac{r}{1-r} \\ r & \text{if } \theta \geq \frac{r}{1-r} \end{cases}.$$

If $\mathcal{B} = \{\varnothing, 0\}$, then the value of the objective is 0.

The expected equilibrium value of V for bid space $\{0\} \cup [r, \bar{b}]$ is then

$$V(\bar{b}, r) = \begin{cases} -\frac{8}{3}r^3 + \frac{\bar{b}^3}{3(1-\bar{b})^3} + \bar{b}\left(1 - \frac{\bar{b}^2}{(1-\bar{b})^2}\right) - \alpha\left(\frac{1}{1-\bar{b}} - 2r\right) & \text{if } 0 < r < \bar{b} \\ r\left(1 - \left(\frac{r}{1-r}\right)^2\right) - \alpha & \text{if } 0 < r = \bar{b} \\ 0 & \text{if } 0 = r = \bar{b} \end{cases}.$$

To solve for the optimal mechanism, the designer hence needs to maximize $\max[U(\bar{b}, r), V(\bar{b}, r)]$ over reserve price r and bid cap $\bar{b}$. The solution for this maximization problem is:

$$\mathcal{B} = \begin{cases} \left[0.5, \frac{\frac{5}{4} - \sqrt{\alpha}}{2 - \sqrt{\alpha}}\right] & \text{if } \alpha < 0.25 \\ \{0\} \text{ or } \{0.5\} & \text{if } \alpha = 0.25 \\ \{0\} & \text{if } \alpha > 0.25 \end{cases}.$$

The detailed steps to find the analytical solution is in the appendix. Figure 1 illustrates the solution to this maximization problem for varying values of α. The solution thus shows that the reserve price does not depend on the relative weight α put on equity considerations vis-a-vis revenue maximization as long α (the weight put on equity) is low enough. The bid cap is decreasing in α over this range. For α above critical value $\alpha^* = 0.25$, the equity considerations become primary and optimal mechanism takes the form of a lottery with no price charged.

6.2 Type Distribution With an Atom

Atoms (mass points) in type distributions arise naturally in auction environments. For instance, under the US Affordable Care Act, firms with more than 50 employees face penalties if they do not offer health care insurance. In France, additional administrative obligations

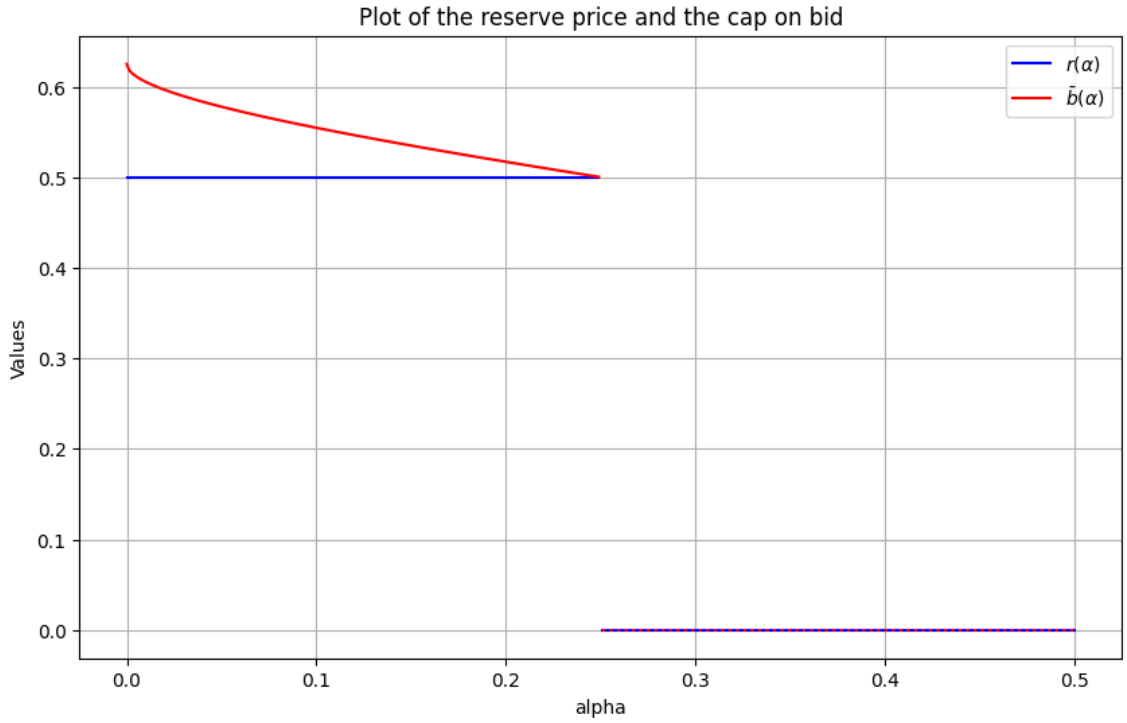


Figure 1: Example 1

apply once firms exceed 50 employees, creating bunching at the threshold, see Garicano, Lelarge, and Van Reenen (2016). Similarly, Jones and Lee (2018) documents that firms in Korea bunch in order to maintain small and medium sized business status. Harju, Matikka, and Rauhanen (2016) find analogous bunching behavior among Finnish firms. In auction contexts, bidders may also share a common outside option, leading to a positive probability mass at a particular point.

Such mass points break the smoothness assumptions underlying the traditional Myersonian approach. Our approach addresses this lacunae in mechanism design methodology: the first-price principle implies that properly designed first-price auction is optimal. We now illustrate the use of our tool results to construct optimal auctions in environments with atoms, where as the below example illustrates optimality may require introducing gaps in the bid space.

Suppose that there are two bidders. Each bidder's type is either 0.5 with a probability $p < 1$ or is drawn from a uniform distribution on $[0, 1]$ with the remaining probability $1 - p$, independently. The designer's goal is to maximize total surplus.[24] This objective can be

[24]Were revenue maximization the designer's goal, the optimal bid space would be $\mathcal{B} = [\frac{1}{2}, 1]$ for all atom probabilities p. The reason is that in this example the atom is at the optimal reserve price of the standard case with $p = 0$. If the atom is at value $a < .5$ then the optimal solution has bid space either $[\frac{1}{2}, 1]$ or $[a, 1]$ depending on the values of a and p. If the atom is at value $a > .5$ then the optimal bid space

achieved by a first-price auction with the bid space $\mathcal{B} = [0, \frac{1}{4}) \cup \left\{\frac{1+p}{4}\right\} \cup (\frac{3p+1}{4p+4}, \frac{1}{2}]$. In the first-price auction with the bid space $\mathcal{B}$, the symmetric equilibrium bidding strategy is

$$b(\theta) = \begin{cases} \frac{1}{2}\theta & \text{if } \theta < \frac{1}{2}, \\ \frac{1+p}{4} & \text{if } \theta = \frac{1}{2}, \\ \frac{p+(1-p)\theta^2}{2(p+(1-p)\theta)} & \text{if } \theta > \frac{1}{2}. \end{cases}$$

The verification that this profile of strategies in in equilibrium is in Appendix B.

6.3 Application: Credible Implementation

We now study that how the auctioneer can use a first-price auction to achieve his goal credibly in the sense of Akbarpour and Li (2020). Mapping their terminology to our study of auctions, we say that an **observation** of bidder i is (a_i, p_i), where, as above, p_i is the payment of bidder i and $a_i = 1$ iff bidder i wins, and it equals 0 otherwise. Given an allocation rule $(\mathbf{a}, \mathbf{p})$ and equilibrium $\boldsymbol{\sigma}$, another allocation rule $(\hat{\mathbf{a}}, \hat{\mathbf{p}})$ is **safe** if for every bidder i and every profile of types $(\theta_1, \ldots, \theta_n)$, the observation of i given the allocation $(\hat{\mathbf{a}}, \hat{\mathbf{p}})(\boldsymbol{\sigma}(\theta_1, \ldots, \theta_n))$ is the same as the observation of this bidder given allocation $(\mathbf{a}, \mathbf{p})\left(\boldsymbol{\sigma}\left(\theta_i, \theta'_{-i}\right)\right)$ for some profile of types θ'_{-i} of bidders other than i. A first-price auction is **credible** if no safe allocation rule may strictly improve the objective at some profile of types. Akbarpour and Li (2020) showed that first-price auctions are credible when the objective is a weighted average of revenue and welfare; using their approach, one can easily show a slightly more general result.

Lemma 5. *Suppose that $\Theta \subseteq \mathbb{R}_+$ and the bidders play a symmetric equilibrium. If the auctioneer maximizes a standard objective that is a function of expected revenue and expected total surplus, and that is weakly increasing in each argument, then any first-price auctions with $\mathcal{B}$ such that $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ is credible.*

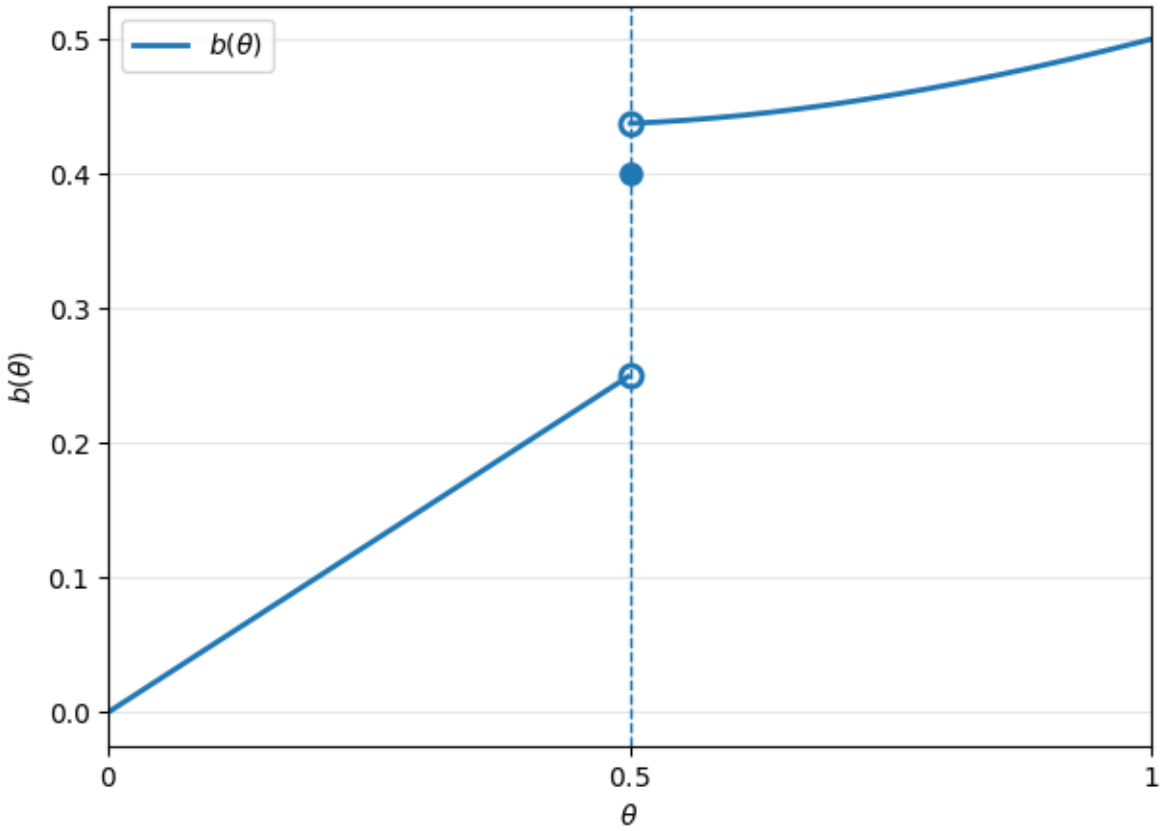
The restriction on the bid space $\mathcal{B} \cap \mathbb{R} \subseteq \mathbb{R}_+$ matters as in its absence there might be a safe deviation in which the item is not allocated when all bids are negative.

Proof. Denote the winner by i in a given first-price auction by following the rule. Suppose that the auctioneer safely deviate by picking a winner j, which is not i. If $b_j = b_i$, then the revenue does not change. Moreover, expected total surplus does not change either because the auctioneer cannot distinguish the types among the bidders who bid the same. If $b_j < b_i$, then the revenue decreases and the social welfare does not increase because any symmetric

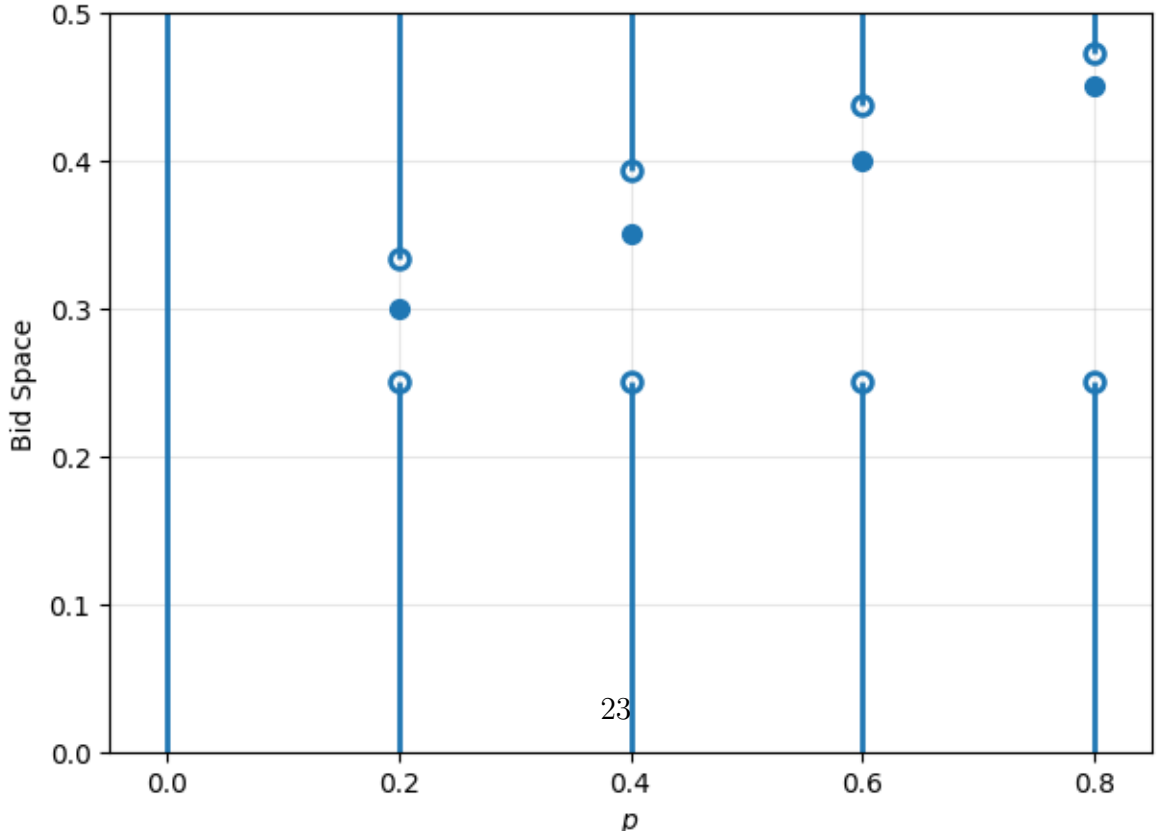
resembles the efficient bid space we construct below and, in general, take the form $[.5, \lim_{\theta \to a-} b(\theta)) \cup \{b(a)\} \cup (\lim_{\theta \to a+} b(\theta), b(1)]$.

Figure 2: Example 2

(a) The equilibrium bidding strategy when $p = 0.6$



(b) The efficient bid space for varying values of probability p of an atom at .5. Bids strictly above .5 are never used and could be removed from the bid space.



equilibrium is weakly monotonic. The safe deviation that does not sell the item does not increase the objective because we assume that the types and real bids are non-negative. □

This lemma and our Theorem 1 imply the following:

Theorem 6. *If $\Theta \subseteq \mathbb{R}_+$ and an auctioneer maximizes a standard objective that is a function of revenue and social welfare, and that is increasing in each argument, then this objective can be credibly achieved by a first-price auction.*

Proof. By Theorem 1, a first-price auction maximizes the objective. We can restrict our attention to the environment where $\Theta \in \mathbb{R}_+$and only positive bids are allowed without loss of generality, because if a bidder with a negative type wins the item and pays a negative bid, the auctioneer can increase the objective by setting $\min \mathcal{B} \cap \mathbb{R}$ to be non-negative. Then, the corollary is a direct implication of Lemma 5. □

While one may conjecture that the above results can extend to any standard objectives, the following example shows that some standard objectives cannot be credibly achieved by a first-price auction.

Example 3. There are two bidders and each bidder's type is drawn from $\Theta = \{0, 1\}$ uniformly at random. While the auctioneer prefers more revenue, he also has a distributional concern. Specifically, the objective of the auctioneer is $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)]$ if a bidder whose type is 1 wins the item, and $\mathbb{E}_\theta[t_1(\theta) + t_2(\theta)] + A$ if a bidder whose type is 0 wins the item. One can verify that, with $A \in [1, 2]$, the first-price auction with $\mathcal{B} = \{0, \frac{2}{3}\}$ maximizes the objective with the symmetric equilibrium $b(0) = 0$ and $b(1) = \frac{2}{3}$ and the resulting utility of the auctioneer is $\frac{1}{2} + \frac{1}{4}A$. Suppose that auctioneer deviates by allocating the item to a bidder that submitted 0 as bid when the other bidder submitted $\frac{2}{3}$ as bid. Note that this deviation is safe and the resulting utility is $\frac{1}{6} + \frac{3}{4}A$, which is strictly greater than $\frac{1}{2} + \frac{1}{4}A$ for $A > \frac{2}{3}$.

7 Discussion

We showed that the first price auctions are sufficient in optimal design. Might other auction pricing rules mechanisms be as universal? E.g. can we use second price auctions in lieu of first price auctions in Theorem 1? Can we implement the outcomes of first-price auctions in second-price auctions?

As the following example illustrates, this is in general not possible. In particular, the above proposition shows that with two types, an analogue of revenue equivalence obtains: same objectives can be achieved by first-price and second-price auctions. With three or more types, the revenue equivalence fails and the following example shows that revenue in a

first-price auction might be strictly higher the maximum revenue achievable in second-price auctions.

Example 4. There are 2 bidders, and each bidder i's value θ_i is drawn independently following

$$\theta_i = \begin{cases} 1 & \text{with probability } 0.6, \\ 11/4 & \text{with probability } 0.2, \\ 13/2 & \text{with probability } 0.2. \end{cases}$$

Suppose $\mathcal{B} = \{1, 2, 3\}$. Then,

$$\sigma(\theta_i) = \begin{cases} 1 & \text{if } \theta_i = 1, \\ 2 & \text{if } \theta_i = 11/4, \\ 3 & \text{if } \theta_i = 13/2. \end{cases}$$

is a symmetric equilibrium of the first-price auction with designed bids $\mathcal{B}$ and

$$Q(\theta) = \begin{cases} 0.6 & \text{if } \theta = 1, \\ 0.8 & \text{if } \theta = 11/4, \\ 1 & \text{if } \theta = 13/2. \end{cases}$$

The expected payment for each type is

$$T(1) = 0.3,\ T(11/4) = 1.4,\ T(13/2) = 2.7.$$

Now, consider a second-price auction to check if there exists a bids space such that the second-price auction achieves the same outcome with. Since the lowest type's utility in the equilibrium is 0, the equilibrium bid level for the lowest type is 1. The expected payment for the middle type when he places a bid level b_M is

$$b_L \times 0.6 + b_M \times \frac{1}{2} \times 0.2 = 0.6 + b_M \times \frac{1}{2} \times 0.2.$$

In order to match the expected payment under the second-price auction to the one in the first-price auction, b_M must solve the equation

$$0.6 + b_M \times \frac{1}{2} \times 0.2 = 1.4$$

and $b_M = 8$. We cannot separate the highest type from the middle type because a bidder with the value 13/2 will never place a bid above 8 in a symmetric equilibrium.

The idea behind of the example is as follows. In a second price auction with designed bids, the designer can make a bidder to place a bid above their own value. If the bidder pays the second highest bid with very high probability, the level of bid the bidder is willing to place can be arbitrarily large. In the example, the bid level that a middle type is willing to place is even above the value of the high type and it makes the high type deviates from the bid he was supposed to submit.

References

Akbarpour, Mohammad, Piotr Dworczak, and Scott Duke Kominers (2020). *Redistributive Allocation Mechanisms*. Working Paper.

Akbarpour, Mohammad and Shengwu Li (2020). "Credible Auctions: A Trilemma". In: *Econometrica* 88.2, pp. 425–467.

Bergemann, Dirk and Martin Pesendorfer (2007). "Information structures in optimal auctions". In: *Journal of Economic Theory* 137.1, pp. 580–609.

Bergson, Abram (1938). "A Reformulation of Certain Aspects of Welfare Economics". In: *The Quarterly Journal of Economics* 52.2, pp. 310–334.

Billingsley, Patrick (July 1999). *Convergence of probability measures*. en. Ed. by Patrick Billingsley. 2nd ed. Wiley Series in Probability and Statistics. Nashville, TN: John Wiley & Sons.

Border, Kim C. (1991). "Implementation of Reduced Form Auctions: A Geometric Approach". In: *Econometrica* 59.4, pp. 1175–1187.

— (2015). *Alternative Linear Inequalities*.

Carbajal, Juan Carlos and Jeffrey C. Ely (Jan. 2013). "Mechanism design without revenue equivalence". In: *Journal of Economic Theory* 148.1, pp. 104–133.

Carroll, Gabriel (2012). "When Are Local Incentive Constraints Sufficient?" In: *Econometrica* 80.2, pp. 661–686.

Chassang, Sylvain and Juan Ortner (2019). "Collusion in Auctions with Constrained Bids: Theory and Evidence from Public Procurement". In: *Journal of Political Economy* 127.5, pp. 2269–2300.

Chawla, Shuchi and Jason D. Hartline (2018). "Auctions with Unique Equilibria". In: *Proceedings of the Fourteenth ACM Conference on Electronic Commerce*. EC '13. Philadelphia, Pennsylvania, USA: Association for Computing Machinery, pp. 181–196.

Che, Yeon-Koo and Ian Gale (1998a). "Caps on Political Lobbying". In: *The American Economic Review* 88.3, pp. 643–651.

— (1998b). "Standard Auctions with Financially Constrained Bidders". In: *The Review of Economic Studies* 65.1, pp. 1–21.

— (2006). "Revenue comparisons for auctions when bidders have arbitrary types". In: *Theoretical Economics* 1.1, pp. 95–118.

Chung, Kim-Sau and Wojciech Olszewski (2007). "A non-differentiable approach to revenue equivalence". In: *Theoretical Economics* 2.4, pp. 469–487.

Condorelli, Daniele (2013). "Market and non-market mechanisms for the optimal allocation of scarce resources". In: *Games and Economic Behavior* 82, pp. 582–591.

Folland, G. B. (1984). *Real analysis : modern techniques and their applications / Gerald B. Folland.* eng. New York: Wiley.

Garicano, Luis, Claire Lelarge, and John Van Reenen (Nov. 2016). "Firm Size Distortions and the Productivity Distribution: Evidence from France". In: *American Economic Review* 106.11, pp. 3439–79.

Gershkov, Alex et al. (2013). "On the Equivalence of Bayesian and Dominant Strategy Implementation". In: *Econometrica* 81.1, pp. 197–220.

Harju, Jarkko, Tuomas Matikka, and Timo Rauhanen (Sept. 2016). *The Effects of Size-Based Regulation on Small Firms: Evidence from VAT Threshold.* Tech. rep. CESifo Working Paper No. 6115. Munich: CESifo Center for Economic Studies and Ifo Institute.

Heydenreich, Birgit et al. (2009). "Characterization of Revenue Equivalence". In: *Econometrica* 77.1, pp. 307–316.

Jones, Randall S. and Jae Wan Lee (Oct. 2018). *Enhancing dynamism in SMEs and entrepreneurship in Korea.* Tech. rep. No. 1510. Paris: OECD Publishing.

Kleiner, Andreas, Benny Moldovanu, and Philipp Strack (2021). "Extreme Points and Majorization: Economic Applications". In: *Econometrica* 89.4, pp. 1557–1593.

Lebrun, Bernard (2006). "Uniqueness of the equilibrium in first-price auctions". In: *Games and Economic Behavior* 55.1, pp. 131–151.

Lovejoy, William S. (2006). "Optimal Mechanisms with Finite Agent Types". In: *Management Science* 52.5, pp. 788–803.

Manelli, Alejandro M. and Daniel R. Vincent (2010). "Bayesian and Dominant-Strategy Implementation in the Independent Private-Values Model". In: *Econometrica* 78.6, pp. 1905–1938.

Maskin, Eric and J. Riley (2003). "Uniqueness of Equilibrium in Sealed High-Bid Auctions". In: *Games and Economic Behavior* 45. (issue in memory of Robert Rosenthal), pp. 395–409.

Maskin, Eric and John Riley (1984). "Optimal Auctions with Risk Averse Buyers". In: *Econometrica* 52.6, pp. 1473–1518.

Matthews, Steven A. (1984). "On the Implementability of Reduced Form Auctions". In: *Econometrica* 52.6, pp. 1519–1522.

May, Kenneth O. (1952). "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision". In: *Econometrica* 20.4, pp. 680–684.

McAfee, R.Preston and John McMillan (1988). "Multidimensional incentive compatibility and mechanism design". In: *Journal of Economic Theory* 46.2, pp. 335–354.

Ministry of Land, Infrastructure, Transport and Tourism (May 2005). *Responses to Low-Price Bids: Issues on Countermeasures for Dumping Bids*. Tech. rep. In Japanese. Government of Japan.

Mirrlees, J. A. (Apr. 1971). "An Exploration in the Theory of Optimum Income Taxation". In: *The Review of Economic Studies* 38.2, pp. 175–208.

Mishra, Debasis, Anup Pramanik, and Souvik Roy (2016). "Local incentive compatibility with transfers". In: *Games and Economic Behavior* 100, pp. 149–165.

Monteiro, Paulo Klinger (2009). "First-price auction symmetric equilibria with a general distribution". In: *Games and Economic Behavior* 65.1. Special Issue in Honor of Martin Shubik, pp. 256–269.

Myerson, Roger B. (1981). "Optimal Auction Design". In: *Mathematics of Operations Research* 6.1, pp. 58–73.

Nikzad, Afshin (July 2022). "Constrained Majorization: Applications in Mechanism Design". In: *Proceedings of the 23rd ACM Conference on Economics and Computation*. ACM.

Olszewski, Wojciech, Philip J Reny, and Ron Siegel (2023). *Equilibrium existence in independent private-value first-price auctions*. In Progress.

Olszewski, Wojciech and Ron Siegel (2019). "Bid caps in large contests". In: *Games and Economic Behavior* 115, pp. 101–112.

Pai, Mallesh M. and Rakesh Vohra (2014). "Optimal auctions with financially constrained buyers". In: *Journal of Economic Theory* 150, pp. 383–425.

Pycia, Marek (2019). *Evaluating with Statistics: Which Outcome Measures Differentiate Among Matching Mechanisms?* Working Paper.

Reny, Philip J. (1999). "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games". In: *Econometrica* 67.5, pp. 1029–1056.

— (2011). "On the Existence of Monotone Pure-Strategy Equilibria in Bayesian Games". In: *Econometrica* 79.2, pp. 499–553.

Reny, Philip J. and Shmuel Zamir (2004). "On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions". In: *Econometrica* 72.4, pp. 1105–1125.

Riley, John G. and William F. Samuelson (1981). "Optimal Auctions". In: *The American Economic Review* 71.3, pp. 381–392.

Rochet, Jean-Charles (1987). "A necessary and sufficient condition for rationalizability in a quasi-linear context". In: *Journal of Mathematical Economics* 16.2, pp. 191–200.

Samuelson, Paul A. (1956). "Social Indifference Curves". In: *The Quarterly Journal of Economics* 70.1, pp. 1–22.

Sen, Amartya (1970). *Collective Choice and Social Welfare*. San Francisco: Holden Day.

Spence, Michael (1974). "Competitive and optimal responses to signals: An analysis of efficiency and distribution". In: *Journal of Economic Theory* 7.3, pp. 296–332.

Szech, Nora (2015). "Tie-breaks and bid-caps in all-pay auctions". In: *Games and Economic Behavior* 92, pp. 138–149.

Vohra, Rakesh V. (2005). *Advanced Mathematical Economics*. Advanced Texts in Economics Series. Routledge.

Woodward, Kyle (2016). "Strategic Ironing in Pay-as-Bid Auctions: Equilibrium Existence with Private Information". In: *Proceedings of the 2016 ACM Conference on Economics and Computation*. EC '16. Maastricht, The Netherlands: Association for Computing Machinery, p. 121.

A Other Results

A.1 Extreme Points of Reduced Form Direct Mechanisms with Finite Type Spaces

In this auxiliary section, we restrict attention to finite type spaces and study the set of incentive compatible symmetric social rules and show that each extreme point of the set takes a special form that can be generated by a first-price auction. A natural class of symmetric social rule allocates the object with equal probabilities to the bidders with the highest type. We generalize this class by allowing equal-probability allocation to bidders in a highest interval of types (which call the highest tier) as follows. We say that a social rule $\mathbf{y}$ is **monotonic hierarchical** if there exists a monotone partition $\Theta_1, \Theta_2, \ldots, \Theta_J$ of Θ such that for all i, one of the following two possibilities obtains:

$$q_i(\theta_1, \ldots \theta_n) = \begin{cases} \frac{1}{|\{l:\theta_l \in \Theta_j\}|}, & \text{if } \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^{J} \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise,} \end{cases}$$

or

$$q_i(\theta_1, \ldots \theta_n) = \begin{cases} \frac{1}{|\{n:\theta_n \in \Theta_j\}|}, & \text{if } j \geq 2, \ \ \theta_i \in \Theta_j \text{ and } \theta_{i'} \notin \cup_{l=j+1}^{J} \Theta_l \text{ for all } i' \in N, \\ 0, & \text{otherwise.} \end{cases}$$

The former possibility encompasses social rules that always allocate the object, while the latter possibility encompasses social rules in which the object is unallocated if no bidder's type is above a certain threshold. The popular implementation of the latter social rules is via a reserve price. More generally, in the appendix, we prove the following:

Lemma 6. *Any individually rational, incentive compatible, and monotonic hierarchical mechanism is a first-price auction with a subsidy.*

This lemma, together with the decomposition result we state next, allows us to prove the key results of Section 4.

Lemma 7. *Any symmetric, feasible, incentive compatible, and individually rational interim allocation is a convex combination of monotonic hierarchical interim allocations.*

In particular, this lemma implies that if an interim allocation Q is an extreme point of the space of symmetric, feasible, incentive compatible, and individually rational interim allocation, then Q is monotonic hierarchical. Lemma 7 extends an insight of Manelli and Vincent (2010), who studied continuous type spaces. We establish this theorem as immediate consequence of the following two lemmas, proven in the appendix. In proving the first lemma—but not the second—we are able to leverage the approach that Manelli and Vincent (2010) employed in their analysis.[25]

In order to prove Lemma 7, we study the space, denoted by $\mathcal{Q}$ of non-decreasing mappings from Θ to $[0, 1]$, where each mapping also satisfies the condition in 13. The first lemma shows that any extreme point of the space is an interim expected allocation of a monotonic hierarchical social rule.

Lemma 8. *The following three conditions are equivalent:*

(i) *Q is an extreme point of the set $\mathcal{Q}$ of non-decreasing interim single-agent allocations such that $(Q, ..., Q)$ is feasible.*

[25]Earlier developments of Manelli and Vincent's structure theorem, most notably Gershkov et al. (2013), analyzed discrete type spaces, but their approach was different from ours; in particular, they did not characterize incentive-compatible mechanisms in terms of convex combinations of simpler classes of rules, and neither Lemma 8 nor Lemma 9 has a counterpart in their analysis.

(ii) *There is a monotonic hierarchical rule* $\mathbf{y}$ *such that* $(Q, ..., Q) \in \mathcal{Q}^n$ *is an interim allocation rule of* $\mathbf{y}$.

(iii) *The coarsest monotone partition* $\{\Theta_1, \ldots, \Theta_J\}$ *of* Θ *that is adapted to* Q *satisfies*

$$Q(\Theta_j) = \frac{[\sum_{l=1}^{j} \pi(\Theta_l)]^n - [\sum_{l=1}^{j-1} \pi(\Theta_l)]^n}{n\pi(\Theta_j)}, \text{ for } j = 2, \ldots, J, \tag{1}$$

$$Q(\Theta_1) = 0 \text{ or } \frac{\pi(\Theta_1)^{n-1}}{n}. \tag{2}$$

This characterization of extreme points implies that, in our environment, any symmetric interim allocation rule is a convex combination of interim allocation rules of hierarchical social rules. To extend this insight to mechanisms, that is allocation rules and payment rules, we use Farkas Alternative to prove the following decomposition lemma. [26]

Lemma 9. *(Decomposition Lemma) Consider a mechanism* $(\mathbf{Q}, \mathbf{T}) = ((Q_1, T_1), \ldots, (Q_n, T_n))$ *that is incentive compatible and individually rational. Suppose that, for some functions* $Q_i^k : \Theta \to [0, 1]$ *and weights* $\alpha_k > 0$ *with* $\sum_{k=1}^{K} \alpha_k = 1$, *we have*

$$Q_i = \sum_{k=1}^{K} \alpha_k Q_i^k \quad \text{for all } i \in N.$$

Then the following are equivalent:

(i) For every i, $Q_i^1(\theta), \ldots Q_i^K(\theta)$ *are non-decreasing in* θ.

(ii) There exist transfers $T_i^1, \ldots, T_i^K$ *such that each pair* $\left(Q_i^k, T_i^k\right)$ *is incentive compatible, individually rational, and*

$$T_i(\theta) = \sum_{k=1}^{K} \alpha_k T_i^k(\theta) \quad \text{for all types } \theta \text{ and all } i.$$

A.2 Two-Type Equivalence Between First-price and Second-price Auctions

4 relied on there being three types. With two types only, as in Example 1, second-price auctions are also sufficient to implement many objectives. This is implied by the following

[26] This lemma has no direct counterpart in Manelli and Vincent (2010), who focus on allocations. As they observe, in the continuous environment they study, the payment functions can be reconstructed, up to a constant, from incentive compatibility conditions. In our discrete environment such a recovery is not possible. When $K = 1$, this lemma reduces to a finite-type version of the standard insight that monotonicity of allocation is necessary and sufficient for implementability, cf. Lemma 12. The decomposition part of this lemma, when $K > 1$, is new.

Proposition 1. *Any expected allocation and payment in a pure symmetric equilibrium in a first-price auction can be implemented by a second-price auctions if* $|\Theta| \leq 2$.

Proof. Suppose that $\Theta = \{\theta_L, \theta_H\}$ and $\theta_H > \theta_L$. Denote the probability that a bidder has a type θ_H by π. Given a symmetric equilibrium in the first-price auction, denote the expected allocation and the expected payment for a bidder with type θ by $Q(\theta)$ and $T(\theta)$. Note that $Q(\theta_H) \geq Q(\theta_L)$. If $Q(\theta_L) = Q(\theta_H)$=0, define $\mathcal{B} = \{b, \emptyset\}$, where b is any number strictly greater than θ_H. Then, the only equilibrium strategy in the second-price auction with $\mathcal{B}$ is to choose $\emptyset$. If $Q(\theta_L) = 0$ and $Q(\theta_H) > 0$, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that the bidders with the higher type choose b and the bidders with the lower type choose $\emptyset$. If $Q(\theta_H) = Q(\theta_L)$>0, define $\mathcal{B} = \{b, \emptyset\}$, where $b = \frac{T(\theta_H)}{Q(\theta_H)}$. Then, it is a symmetric equilibrium that all the bidders choose b. Suppose that $Q(\theta_H) > Q(\theta_L) > 0$. Define $\mathcal{B} = \{b_L, b_H, \emptyset\}$, where $b_L = \frac{T(\theta_L)}{Q(\theta_L)}$ and $b_H = \frac{T(\theta_H)-(n-1)T(\theta_L)}{Q(\theta_H)-(n-1)Q(\theta_L)}$ and consider a strategy where the bidders with the lower type chooses b_L and the bidders with the higher type choose b_H. We show that (i) b_H is well-defined, (ii)$b_H > b_L$, and (iii) it is an equilibrium that the bidders with the lower type choose b_L and the bidders with the higher type choose b_H.

(i) Note that $Q(\theta_H) = \sum_{m=0}^{n-1} \frac{1}{m+1}\binom{n-1}{m}\pi^m(1-\pi)^{n-m-1} = \sum_{m=1}^{n-1} \frac{1}{m+1}\binom{n-1}{m}\pi^m(1-\pi)^{n-m-1}+ (1-\pi)^{n-1}$ and $Q(\theta_L) = \frac{1}{n-1}(1-\pi)^{n-1}$. Therefore, $Q(\theta_H) > (n-1)Q(\theta_L)$.

(ii) Note that $[Q(\theta_H) - (n-1)Q(\theta_L)] * b_L + (n-1)T(\theta_L) = Q(\theta_H)T(\theta_L)/Q(\theta_L) < T(\theta_H)$, which implies $b_L < b_H$.

(iii) It is trivial that the expected payment of a low-type bidder in the given strategy in the second-price auction is $T(\theta_L)$. For a bidder with the type θ_H,he pays $b_L$only when all the other bidders have the lower type. Therefore, the expected payment of a bidder with the higher type is $(1-\pi)^{n-1}b_L + (Q(\theta_H) - (1-\pi)^{n-1})b_H = T(\theta_H)$. Because the expected winning probability and the expected payment for each type in the suggested strategy in the second-price auction coincide with the ones in the first-price auction, the suggested strategy is indeed an equilibrium strategy. □

B Proofs

B.1 Proofs of Results in Section 3

Proof of Lemma 1

Proof. It is straightforward that any linear combination of standard and convex objectives with nonnegative coefficients is standard and convex; similarly for any integral over such objectives (where convexity is needed to check condition (ii)).

It remains to show that the ratio of U_π and V_π is standard under the assumptions of the lemma. The ratio satisfies condition (i) as soon as when U_π and V_π do. Condition (ii) additionally relies on affinity of V_π. To check condition (iii) note that the convexity of U_π and affinity of V_π imply:

$$\frac{U_\pi\left(\lambda(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)(\mathbf{Q}_2,\mathbf{T}_2)\right)}{V_\pi\left(\lambda(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)(\mathbf{Q}_2,\mathbf{T}_2)\right)} \leq \frac{\lambda U_\pi(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)U_\pi(\mathbf{Q}_2,\mathbf{T}_2)}{\lambda V_\pi(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)V_\pi(\mathbf{Q}_2,\mathbf{T}_2)},$$

and by the mediant inequality,

$$\frac{\lambda U_\pi(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)U_\pi(\mathbf{Q}_2,\mathbf{T}_2)}{\lambda V_\pi(\mathbf{Q}_1,\mathbf{T}_1)+(1-\lambda)V_\pi(\mathbf{Q}_2,\mathbf{T}_2)} \leq \max\left\{\frac{U_\pi(\mathbf{Q}_1,\mathbf{T}_1)}{V_\pi(\mathbf{Q}_1,\mathbf{T}_1)}, \frac{U_\pi(\mathbf{Q}_2,\mathbf{T}_2)}{V_\pi(\mathbf{Q}_2,\mathbf{T}_2)}\right\}.$$

To check continuity condition (iv) suppose that each of U_π and V_π satisfy (iv) and $V_\pi > 0$. Then, for fixed π, the continuity in (Q,T) implies that locally V_π is bounded away from 0, and hence U_π/V_π is continuous in (Q,T). The continuity condition (iv) follows because each of U_π and V_π satisfy (iv) and V_π is bounded away from 0 □

B.2 Proofs of Results in Section A.1

We rely on two well-known results from linear programming.

Lemma 10. *(Farkas' Alternative)*[27] *Suppose that $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{l\times n}$, and $d \in \mathbb{R}^l$. Then exactly one of the following two statements is true:*

- There exists $x \in \mathbb{R}^n$ such that $Cx = d$ and $Ax \leq b$.
- There exist $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^m$ such that $zA + yC = 0$, $z \geq 0$, and $z \cdot b + y \cdot d < 0$.

Lemma 11. *(Fundamental Theorem of Linear Programming)*[28] *Suppose that $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Consider a set $P = \{x \in \mathbb{R}^n | Ax \leq b\}$. Then, $x^* \in P$ is an extreme point of P if and only if $Ax^* \leq b$ has n binding constraints that are linearly independent.*

We also rely on the following well-known results in mechanism design.

Lemma 12. *(Characterizations of Incentive Compatibility)*

- *(Spence-Mirrlees Condition)*[29] *For every interim allocation rule $\mathbf{Q}$, there exists an interim payment rule $\mathbf{T}$ such that the reduced-form mechanism $(\mathbf{Q},\mathbf{T})$ is incentive compatible if and only if $\mathbf{Q}$ is non-decreasing.*

[27] See Border (2015) for this version of Farkas' Alternative.

[28] See Vohra (2005).

[29] *For the continuous version of the condition, see Mirrlees (1971) and Spence (1974). For the finite type version, see Rochet (1987)*

- *(Local IC guarantees Global IC)*[30] *A mechanism* $(\mathbf{Q}, \mathbf{T})$ *is incentive compatible and individually rational if and only if for all* $i \in N$,

$$Q_i(\theta^t)\theta^t - T_i(\theta^t) \geq Q_i(\theta^{t-1})\theta^t - T_i(\theta^{t-1}), \text{ for all } t = 1, \ldots, T, \tag{3}$$
$$Q_i(\theta^{t-1})\theta^{t-1} - T_i(\theta^{t-1}) \geq Q_i(\theta^t)\theta^{t-1} - T(\theta^t), \text{ for all } t = 2, \ldots, T. \tag{4}$$

Because we assumed that $Q_i(\theta^0) = 0$ and $T_i(\theta^0) = 0$ (cf. 8), the condition (3) encompasses individual rationality for type θ^1.

Lemma 13. *(Maskin-Riley-Matthews-Border Condition)*[31] *A symmetric interim non-decreasing allocation rule* $\mathbf{Q} = (Q, ..., Q)$ *is feasible if and only if*

$$n \int_{\{\theta \in \Theta : Q(\theta) \geq \alpha\}} Q(\theta) d\pi(\theta) \leq 1 - (\pi(\{\theta \in \Theta : Q(\theta) < \alpha\}))^n, \text{ for all } \alpha \in [0, 1]. \tag{5}$$

Then, the above lemmas tell us that the set of symmetric allocations that are feasible and incentive compatible is

$$\mathcal{Q}^{\mathcal{S}} = \{(Q, ..., Q) : \Theta^n \to [0, 1]^n \mid Q \text{ satisfies (5) and non-decreasing}\}. \tag{6}$$

Lemma 14. *The set of symmetric, feasible, individually rational, and incentive compatible mechanisms is convex.*

Proof. Consider two arbitrary symmetric, feasible, individually rational, and incentive compatible mechanisms $(\mathbf{Q}, \mathbf{T}) = ((Q, \ldots, Q), (T, \ldots, T))$ and $(\mathbf{Q}', \mathbf{T}') = ((Q', \ldots, Q'), (T', \ldots, T'))$. Individual rationality, and incentive compatibility are preserved under convex combination because they are given by linear inequalities in the interim allocation and the interim payment. Furthermore, $\lambda(\mathbf{Q}, \mathbf{T}) + (1 - \lambda)(\mathbf{Q}', \mathbf{T}')$ is symmetric, where the interim allocation and the interim payment for a single bidder are $\lambda Q + (1 - \lambda)Q'$ and $\lambda T + (1 - \lambda)T'$, for any $\lambda \in [0, 1]$. It remains to show that $\lambda(\mathbf{Q}, \mathbf{T}) + (1 - \lambda)(\mathbf{Q}', \mathbf{T}')$ is feasible. By the definition of feasibility, there exists ex post allocation $q : \Theta^n \to [0, 1]^n$ and $q' : \Theta^n \to [0, 1]^n$such that $\sum_{\theta_{-i} \in \Theta_{-i}} q_i(\theta_i, \theta_{-i})\pi(\theta_i, \theta_{-i}) = Q(\theta)$ and $\sum_{\theta_{-i} \in \Theta_{-i}} q'_i(\theta_i, \theta_{-i})\pi(\theta_i, \theta_{-i}) = Q'(\theta)$. Hence, we

[30] *For more detailed discussions, see McAfee and McMillan (1988), Carroll (2012), and Mishra, Pramanik, and Roy (2016). The equivalence of local and global incentive compatibility conditions relies on the single crossing property, which is satisfied in our setting.*

[31] Maskin and Riley (1984) studied the condition when the type space is continuous and the allocation is a step function in their analysis of optimal auctions with risk-averse bidders. Matthews (1984) showed that the condition holds for any increasing allocation functions. Border (1991) proved the condition for general type spaces and allocation functions.

have

$$\lambda Q(\theta) + (1-\lambda)Q'(\theta) = \sum_{\theta_{-i}\in\Theta_{-i}} \left(\lambda q_i(\theta_i,\theta_{-i}) + (1-\lambda)q_i(\theta_i,\theta_{-i})\right)\pi(\theta_i,\theta_{-i}).$$

Then, $\lambda q + (1-\lambda)q'$ is an ex-post allocation rule, because

$$\sum_{i\in N}\left(\lambda q_i(\theta_i,\theta_{-i}) + (1-\lambda)q_i(\theta_i,\theta_{-i})\right) = \lambda\sum_{i\in N} q_i(\theta_i,\theta_{-i}) + (1-\lambda)\sum_{i\in N} q'_i(\theta_i,\theta_{-i}) \leq 1.$$

Therefore, $\lambda\left(\mathbf{Q},\mathbf{T}\right) + (1-\lambda)\left(\mathbf{Q}',\mathbf{T}'\right)$ feasible by definition. □

To prove Lemma 6, we rely on the following auxiliary lemma.

Lemma 15. *Let (Q,T) be an incentive compatible mechanism on $\Theta \subseteq \mathbb{R}$. Fix a scalar $s \in \mathbb{R}$. For any $\theta,\ \theta' \in \Theta$ such that $Q(\theta)\,\theta - T(\theta) - s > 0$, $\theta' > \theta$, and $Q(\theta) > 0$, the following equivalence holds:*

$$Q(\theta') > Q(\theta) \iff \frac{T(\theta')+s}{Q(\theta')} > \frac{T(\theta)+s}{Q(\theta)}.$$

Proof. Because the mechanism is incentive compatible, $Q(\cdot)$ is non-decreasing, thus we have (i) $Q(\theta') \geq Q(\theta)$, hence $\frac{T(\theta')+s}{Q(\theta')}$ is well defined. The incentive compatibility also implies that (ii) $Q(\theta') = Q(\theta) \iff T(\theta') = T(\theta)$. Sufficiency is a direct implication of (i) and (ii). To prove the necessity, notice that by subtracting s from the incentive compatibility constraint for type θ we have:

$$Q(\theta)\,\theta - T(\theta) - s \geq Q(\theta')\,\theta - T(\theta') - s,$$

which can be rearranged to

$$\frac{Q(\theta)}{Q(\theta')}\left(\theta - \frac{T(\theta)+s}{Q(\theta)}\right) \geq \theta - \frac{T(\theta')+s}{Q(\theta')}.$$

Because $\frac{Q(\theta)}{Q(\theta')} \in (0,1)$ and $\theta - \frac{T(\theta)+s}{Q(\theta)} > 0$ by the assumption, we have

$$\frac{T(\theta')+s}{Q(\theta')} > \frac{T(\theta)+s}{Q(\theta)}.$$

□

Proof of Lemma 6

Proof. Let (Q,T) be an individually rational, incentive compatible, and monotonic hierarchical mechanism on a finite type space Θ. For this proof, embed the symbols $\varnothing$ and r^- into

$\mathbb{R}$ so that $\varnothing < r^- < \inf\Theta$; this artificial ordering is used only to rank bids and select the winner. If the bid $b(r^-)$ wins, the bidder pays the reserve price $\min\left(\mathcal{B}_k - \{\varnothing, r^-\}\right)$. Assume every bidder follows a bidding strategy $b\left(\cdot\right)$ satisfying $Q\left(\theta'\right) > Q\left(\theta\right) \iff b\left(\theta'\right) > b\left(\theta\right)$ and $Q\left(\theta\right) = 0 \implies b\left(\theta\right) = \varnothing$. In a first-price auction, this strategy profile induces the interim allocation rule $Q\left(\cdot\right)$ because

(i) If θ and θ' lie in the same tier under Q, then $b\left(\theta\right) = b\left(\theta'\right)$; hence they lie in the same tier in the auction.

(ii) If θ' belongs to a higher tier than θ under Q, then $b\left(\theta'\right) > b\left(\theta\right)$; hence θ' belongs to a higher tier than θ in the auction.

(iii) If $Q\left(\theta\right) = 0$, then the bid $b\left(\theta\right) = \varnothing$, which never wins.

Define $s = \min_{\theta\in\Theta}\left\{Q(\theta)\theta - T(\theta)\right\}$. Because (Q, T) is individually rational and Θ if finite, the minimum exists and $s \geq 0$. Partition the type space Θ into

$$\Theta_0 = \left\{\theta \mid Q(\theta) = 0\right\} \quad \text{and} \quad \Theta_+ = \left\{\theta \mid Q(\theta) > 0\right\}.$$

If $\Theta_0 = \Theta$, then $Q(\theta) = 0$ for all $\theta \in \Theta$. By the incentive compatibility, $T(\theta)$ must be constant across all types in Θ, and the definition of s gives $T(\theta) = -s$. A first-price auction whose bid space is $\mathcal{B} = \{\varnothing\}$ and that provides a subsidy s implements (Q, T). Henceforth, we assume $\Theta_0 \neq \Theta$. Let us denote the lowest type in Θ_+ by θ_L. We construct the bidding strategy except for type θ_L by

$$b\left(\theta\right) = \frac{T\left(\theta\right) + s}{Q\left(\theta\right)}, \quad \text{for every } \theta \in \Theta_+ - \left\{\theta_L\right\},$$

and $b\left(\theta\right) = \varnothing$ for $\theta \in \Theta_0$. By construction, we have $Q\left(\theta\right)\theta - T\left(\theta\right) \geq s$ for every $\theta \in \Theta$, thus $\theta \geq b\left(\theta\right)$ for all $\theta \in \Theta_+ - \{\theta_L\}$. Furthermore, if $\theta = b\left(\theta\right)$ for some $\theta > \theta_L$, then the inequality becomes strict for θ because the incentive compatibility implies that

$$Q\left(\theta\right)\theta - T\left(\theta\right) \geq Q\left(\theta_L\right)\theta - T\left(\theta_L\right) > Q\left(\theta_L\right)\theta_L - T\left(\theta_L\right) \geq s.$$

By Lemma 15, we have

$$Q\left(\theta'\right) > Q\left(\theta\right) \iff b\left(\theta'\right) > b\left(\theta\right), \quad \text{for all } \theta,\ \theta' \in \Theta_+ - \left\{\theta_L\right\}.$$

Because $b\left(\theta\right) = \varnothing$ for every $\theta \in \Theta_0$, the same equivalence extends to

$$Q\left(\theta'\right) > Q\left(\theta\right) \iff b\left(\theta'\right) > b\left(\theta\right), \quad \text{for all } \theta,\ \theta' \in \Theta - \left\{\theta_L\right\}.$$

We have already ensured $Q(\theta) = 0 \implies b(\theta) = \varnothing$ and, because $\varnothing < b$ for every $b \neq \varnothing$, we need only choose $b(\theta_L)$ so that

$$Q(\theta') > Q(\theta) \iff b(\theta') > b(\theta) \quad \text{for all } \theta, \theta' \in \Theta_+$$

We handle three mutually exclusive cases.

Case 1. If $\frac{T(\theta_L)+s}{Q(\theta_L)} < b(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$, set $b(\theta_L) = \frac{T(\theta_L)+s}{Q(\theta_L)}$.

We first show that $Q(\theta_L) < Q(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$. Suppose, to the contrary, that there exists $\theta \in \Theta_+ - \{\theta_L\}$ such that $Q(\theta_L) = Q(\theta)$. By construction, we have $b(\theta) = \frac{T(\theta_L)+s}{Q(\theta_L)}$ contradicting $\frac{T(\theta_L)+s}{Q(\theta_L)} < b(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$. Therefore, we have $Q(\theta_L) < Q(\theta)$ and $b(\theta_L) < b(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$, which establishes the desired equivalence.

Case 2. If $\frac{T(\theta_L)+s}{Q(\theta_L)} = b\left(\hat{\theta}\right)$ and $Q(\theta_L) = Q\left(\hat{\theta}\right)$ for some $\hat{\theta} \in \Theta_+ - \{\theta_L\}$, set $b(\theta_L) = \frac{T(\theta_L)+s}{Q(\theta_L)}$.

Suppose that there exists $\theta \in \Theta_+ - \{\theta_L\}$ such that $Q(\theta) > Q(\theta_L)$ but $b(\theta) = b(\theta_L)$. Because $Q(\theta_L) = Q\left(\hat{\theta}\right)$ and $b(\theta_L) = b\left(\hat{\theta}\right)$, we have $Q(\theta) > Q\left(\hat{\theta}\right)$ and $b(\theta) = b\left(\hat{\theta}\right)$, contradicting Lemma 15. Conversely, $Q(\theta) = Q(\theta_L)$ implies $b(\theta) = b(\theta_L)$, so $b(\theta) > b(\theta_L)$ only if $Q(\theta) > Q(\theta_L)$. Therefore, we have the desired equivalence.

Case 3. If $\frac{T(\theta_L)+s}{Q(\theta_L)} = b\left(\hat{\theta}\right)$ for some $\hat{\theta} \in \Theta_+ - \{\theta_L\}$ and $Q(\theta_L) < Q(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$, set $b(\theta_L) = r^-$.

By construction, $Q(\theta_L) < Q(\theta)$ and $b(\theta_L) < b(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$, so the desired equivalence follows. Moreover, two additional properties of this case will be needed later; $\frac{T(\theta_L)+s}{Q(\theta_L)} = \theta_L$ and there is no type θ such that $r^- < b(\theta) < \theta_L$. By construction, $Q(\theta_L)\theta_L - T(\theta_L) \geq s$ hence $\theta_L \geq \frac{T(\theta_L)+s}{Q(\theta_L)}$. If the inequality is strict, then Lemma 15 gives $b(\theta_L) < b(\theta)$ for all $\theta \in \Theta_+ - \{\theta_L\}$, which is a contradiction. Hence the equality must hold: $\frac{T(\theta_L)+s}{Q(\theta_L)} = \theta_L$ and in particular $b\left(\hat{\theta}\right) = \theta_L$. With the identity in hand, suppose that there exists θ such that $r^- < b(\theta) < \theta_L$. Because $b(\theta) > r^-$, we must have $Q(\theta) > 0$. Moreover, $b(\theta) < \theta_L$ gives

$$\frac{T(\theta)+s}{Q(\theta)} = b(\theta) < \theta_L = \frac{T(\theta_L)+s}{Q(\theta_L)}.$$

Lemma 15 then implies $Q(\theta) < Q(\theta_L)$; together with $Q(\theta) > 0$, this means $0 < Q(\theta) < Q(\theta_L)$, contradicting the definition of θ_L as the smallest type in Θ_+.

Having already shown that the bidding strategy $b(\cdot)$ induces the interim allocation $Q(\cdot)$ in a first-price auction, we now verify that it also induces the interim transfer $T(\cdot)$ when the subsidy s is given to all bidders and the bid space is $\mathcal{B} = b(\Theta)$. Let $T_{FPA}(\theta)$ denote the expected payment of type θ in this first-price auction.

For $\theta \in \Theta_0$, because $Q(\theta) = 0$, by construction we have $T(\theta) = s$. In the auction, these

bidders receive the subsidy s and pay nothing, so $T_{FPA}(\theta) = s$. Hence, $T(\theta) = T_{FPA}(\theta) = s$ for every $\theta \in \Theta_0$.

For θ such that $b(\theta) = \frac{T(\theta)+s}{Q(\theta)}$, the expected payment in the auction equals to the winning probability multiplied by the bid minus the subsidy;

$$T_{FPA}(\theta) = Q(\theta) b(\theta) - s = T(\theta).$$

For θ such that $b(\theta) = r^-$, we necessarily have $\theta = \theta_L$ by construction. Because there is no type θ such that $r^- < b(\theta) < \theta_L$, the reserve price in the bid space $\mathcal{B} = b(\Theta)$ equals θ_L; consequently bidders with type θ_L pay θ_L whenever they win. We have already established that $\theta_L = \frac{T(\theta_L)+s}{Q(\theta_L)}$ if $b(\theta_L) = r^-$. Hence, the expected payment for θ_L in the auction is

$$T_{FPA}(\theta_L) = Q(\theta_L)\theta_L - s = Q(\theta_L)\frac{T(\theta_L)+s}{Q(\theta_L)} - s = T(\theta_L).$$

Therefore, we have $T(\theta) = T_{FPA}(\theta)$ for every $\theta \in \Theta$. When all bidders follow $b(\cdot)$, the first price auction with subsidy s and the bid space $b(\Theta)$ thus reproduces both the interim allocation $Q(\cdot)$ and the interim transfer $T(\cdot)$. It remains to verify that $b(\cdot)$ is a symmetric equilibrium strategy. Indeed, because (Q, T) is incentive compatible, no bidder can profitably deviate to a bid used by some other type; and because every permissible bid in $\mathcal{B}$ is used with positive probability, there are no unused bids to which a profitable deviation could be made. □

Proof of Lemma 7

Proof. By Lemma 14, the set of non-decreasing and feasible interim allocation is convex. Lemma 8, together with Carathéodory's theorem, completes the proof. □

Proof of Lemma 8

Proof. Let $\Theta = \{\theta^1 < \cdots < \theta^T\}$ be the finite type set and let π (the prior belief) be the probability mass function such that $\pi(\theta^t) > 0$. Define discrete CDF

$$F(\theta^t) = \sum_{x<\theta^t} \pi(x), \qquad F(\theta^1) = 0.$$

The set $\mathcal{Q}$ can be written as the set of all non-decreasing $Q : \Theta \to [0,1]$ satisfying, for each $t = 1, \ldots, T$,

$$n \sum_{s=t}^{T} Q(\theta^s)\pi(\theta^s) \leq 1 - F(\theta^t)^n.$$

In particular, and for $t = 1, \ldots, T$,

$$Q(\theta^t) \geq Q(\theta^{t-1}),$$

where $Q(\theta^0)$ is defined to be 0. For each $t = 1, \ldots, T$, the first inequality is the t-th feasibility constraint and the second inequality is the t-th monotonicity constraint.

Proof of (i) $\iff$ (iii). Because $\mathcal{Q} \subseteq \mathbb{R}^T$ is a polyhedron, any extreme point Q admits T linearly independent binding constraints. Suppose that Q is an extreme point of $\mathcal{Q}$, and let $\{\Theta_1, \ldots, \Theta_J\}$ be the coarsest monotonic partition adapted to Q (so Q is constant on each block and strictly increases across blocks), hence $T - J$ monotonicity constraints are binding if $Q(\Theta_1) > 0$ or $T - J + 1$ monotonicity constraints are binding if $Q(\Theta_1) = 0$.

We claim that for every $j \in \{2, \ldots, J\}$ there exists $\theta^{t_j} \in \Theta_j$ such that the t_j-th feasibility constraints is binding. We will address the case $j = 1$ later. By way of contradiction, suppose that none of the feasibility constraints are binding on block Θ_j for some j. Define Q' and Q'' by shifting the value of Q on the block Θ_j up and down by a small $\varepsilon > 0$:

$$Q'(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) + \varepsilon & \text{if } \theta \in \Theta_j, \end{cases} \qquad Q''(\theta) = \begin{cases} Q(\theta) & \text{if } \theta \notin \Theta_j, \\ Q(\theta) - \varepsilon & \text{if } \theta \in \Theta_j. \end{cases}$$

We choose ε small enough that all feasibility constraints that were strict remain satisfied (they are strict on Θ_j by assumption), and across-block monotonicity is preserved, hence both Q' and Q'' are in $\mathcal{Q}$. This contradicts the extremality of Q because $\frac{1}{2}Q' + \frac{1}{2}Q'' = Q$.

We next show that the binding feasibility in Θ_j occurs at its minimal type for every $j \in \{2, \ldots, J\}$ by induction. A useful identity is the finite geometric sum:

$$\frac{x^n - y^n}{x - y} = \sum_{s=0}^{n-1} x^{n-1-s} y^s.$$

Consider $j = J$. If the feasibility constraints binds at some $\theta^{t_J} \in \Theta_J$, then, we have

$$nQ(\Theta_J)(1 - F(\theta^{t_J})) = 1 - F(\theta^{t_J})^n,$$

and

$$nQ(\Theta_J) = \frac{1 - F(\theta^{t_J})^n}{1 - F(\theta^{t_J})} = \sum_{s=0}^{n-1} F(\theta^{t_J})^s.$$

We claim that $\theta^{t_J} = \min \Theta_J$. By way of contradiction, suppose that there exists $\theta \in \Theta_J$ such that $\theta < \theta^{t_J}$. Evaluating feasibility at θ gives

$$nQ(\Theta_J) \leq \frac{1 - F(\theta)^n}{1 - F(\theta)} = \sum_{s=0}^{n-1} F(\theta)^s,$$

which is a contradiction because $F(\theta) < F(\theta^{t_J})$ makes the right-hand side strictly smaller. Hence $\theta_{t_J} = \min \Theta_J$.

Now, assume that the feasibility constraint binds at $\theta^{t_{j+1}} = \min \Theta_{j+1}$ as an inductive step. Consider Θ_j and θ^{t_j} at which the feasibility constraint is binding. We claim that $\theta^{t_j} = \min \Theta_j$. By way of contradiction, suppose that there exist $\theta \in \Theta_j$ such that $\theta < \theta^{t_j}$. The two binding t_j-th and t_{j+1}-th feasibility constrains give

$$nQ(\Theta_j)(F(\theta^{t_{j+1}}) - F(\theta^{t_j})) + n \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) = 1 - F(\theta^{t_j})^n,$$

and

$$n \sum_{x \geq \theta^{t_{j+1}}} Q(x)\pi(x) = 1 - F(\theta^{t_{j+1}})^N$$

Subtracting gives

$$nQ(\Theta_j) = \frac{F(\theta^{t_{j+1}})^n - F(\theta^{t_j})^n}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \sum_{s=0}^{n-1} F(\theta^{t_{j+1}})^{n-1-s} F(\theta^{t_j})^s.$$

Evaluating the feasibility constraint at θ yields the analogous inequality

$$nQ(\Theta_j) \leq \sum_{s=0}^{n-1} F(\theta^{t_{j+1}})^{n-1-s} F(\theta)^s,$$

which is a contradiction because $F(\theta^{t_j}) > F(\theta)$. By induction, the binding feasibility constraint in each block occurs at its minimum. The above step also shows that

$$\begin{aligned} nQ(\Theta_j) =& \frac{F(\theta^{t_{j+1}})^n - F(\theta^{t_j})^n}{F(\theta^{t_{j+1}}) - F(\theta^{t_j})} = \frac{F(\theta^{t_{j+1}})^n - F(\theta^{t_j})^n}{\pi(\Theta_j)}, \\ =& \frac{(\sum_{k=1}^{j} \pi(\Theta_k))^n - (\sum_{k=1}^{j-1} \pi(\Theta_k))^n}{\pi(\Theta_j)}, \text{ for } j > 1. \end{aligned}$$

Now, we address the case $j = 1$. Because Q must satisfy T binding constraints to be an extreme point, either a feasibility constraint or a monotonicity constraint must bind on Θ_1. If a feasibility constraint is binding, then by the same induction argument above, the binding occurs at the minimum θ^1. In this case, we have

$$nQ(\Theta_1) = \frac{F(\theta^{t_2})^n - F(\theta^1)^n}{F(\theta^{t_2}) - F(\theta^1)} = \frac{F(\theta^{t_2})^n}{F(\theta^{t_2})} = \frac{\pi(\Theta_1)^n}{\pi(\Theta_1)} = \pi(\Theta_1)^{n-1}.$$

If the monotonicity constraint is binding, then we have $Q(\Theta_1) = 0$.

To prove the converse, assume Q satisfies (iii) with coarsest monotonic partition $\{\Theta_1, \ldots, \Theta_J\}$ adapted to Q. If $Q(\Theta_1) = 0$, then we have $J-1$ binding feasibility constraints and $T-J+1$ binding monotonicity constraints. If $Q(\Theta_1) > 0$, then we have J binding feasibility constraints and $T - J$ binding monotonicity constraints. In either case, T constraints are bind. It remains to show that these binding constraints are linearly independent. This follows from the observation that all values of Q are uniquely determined by the binding constraints. Indeed, the feasibility constraint for the top block Θ_J pins down

$$Q(\Theta_J) = \frac{1 - F(\theta^{t_J})^n}{n(1 - F(\theta^{t_J}))}.$$

Furthermore, the feasibility constraint for block Θ_j involves only $Q(\Theta_r)$ terms with $r \geq j$. Thus, once $Q(\Theta_{j+1}), \ldots, Q(\Theta_J)$ are known, the binding feasibility constraint at Θ_j uniquely determines $Q(\Theta_j)$. Hence, by induction, this pins down all $Q(\Theta_j)$ for $j \in \{2, \ldots, J\}$. For the block Θ_1, if the feasibility constraint is binding, then we obtain $Q(\Theta_1) = \frac{\pi(\Theta_1)^{n-1}}{n}$. If instead the monotonicity constraint at θ^1 binds, then we have $Q(\Theta_1) = 0$. Therefore, the system of binding constraints determines Q uniquely, which implies that the constraints are linearly independent. This completes the proof of (i) $\iff$ (iii).

Proof of (ii) $\iff$ (iii): Suppose that Q is monotonic hierarchical with a coarsest monotonic partition $\{\Theta_1, \ldots, \Theta_J\}$ adapted to Q. The winning probability of a bidder with type in Θ_j can be expressed as

$$Q(\Theta_j) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \left(\pi(\Theta_j)^{k-1} \Big[\sum_{l=1}^{j-1} \pi(\Theta_l) \Big]^{n-k} \right).$$

Using $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$, this becomes

$$Q(\Theta_j) = \frac{1}{n} \sum_{k=1}^{n} \binom{n}{k} \pi(\Theta_j)^{k-1} \Big[\sum_{l=1}^{j-1} \pi(\Theta_l) \Big]^{n-k} = \frac{1}{n\pi(\Theta_j)} \sum_{k=1}^{n} \binom{n}{k} \left(\pi(\Theta_j)^{k} \Big[\sum_{l=1}^{j-1} \pi(\Theta_l) \Big]^{n-k} \right).$$

By applying the binomial theorem, we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(\pi\left(\Theta_j\right)^{k-1}\left[\sum_{l=1}^{j-1}\pi(\Theta_l)\right]^{n-k}\right)=\left(\left[\sum_{l=1}^{j-1}\pi(\Theta_l)\right]+\pi\left(\Theta_j\right)\right)^{n}=\left[\sum_{l=1}^{j}\pi(\Theta_l)\right]^{n},$$

hence we have

$$Q(\Theta_j)=\frac{1}{n\pi\left(\Theta_j\right)}\left[\left[\sum_{l=1}^{j}\pi\left(\Theta_l\right)\right]^{n}-\left[\sum_{l=1}^{j-1}\pi\left(\Theta_l\right)\right]^{n}\right].$$

For $j=1$, this expression gives $Q(\Theta_1)=\frac{\pi(\Theta_1)^{n-1}}{\pi(\Theta_1)}$. If the designer keeps the object when all bidders lie in Θ_1, then we have $Q(\Theta_1)=0$ instead. This is exactly the statement in (iii).

To prove the converse, assume Q satisfies (iii). Because the formula in (iii) is expressed with a monotonic partition, we can define a monotonic hierarchical rule $\tilde{Q}$ associated with the monotonic partition $\{\Theta_1,\ldots,\Theta_J\}$. By the same counting argument as above, this rule yields

$$\tilde{Q}\left(\Theta_j\right)=\frac{1}{n\pi\left(\Theta_j\right)}\left[\left[\sum_{l=1}^{j}\pi\left(\Theta_l\right)\right]^{n}-\left[\sum_{l=1}^{j-1}\pi\left(\Theta_l\right)\right]^{n}\right],\quad\text{for } j\geq 2,$$

and for the lowest block, depending on the treatment of Θ_1, we have

$$\tilde{Q}\left(\Theta_1\right)=0 \text{ or } \frac{\pi\left(\Theta_1\right)^{n-1}}{n}.$$

Because Q and $\tilde{Q}$ coincide on all blocks, it follows that Q is monotonic hierarchical. □

Proof of Lemma 9

Proof. If $\left(Q_i^k,T_i^k\right)$ is incentive compatible, then Q_i^k must be non-decreasing by Lemma 12, hence we have (ii) $\Longrightarrow$ (i). In order to prove (i) $\Longrightarrow$ (ii), let the type set be $\left\{\theta^1,\ldots,\theta^T\right\}$ with $\theta^t<\theta^{t+1}$ for all $t\in\{1,\ldots,T-1\}$. Suppose that (Q_i,T_i) is incentive compatible and individually rational for bidder i, and that there exist non-decreasing functions $Q_i^k:\Theta\to[0,1]$ and weights $\alpha_k>0$ such that

$$Q_i=\sum_{k=1}^{K}\alpha_k Q_i^k.$$

For each $k\in\{1,\ldots,K\}$ and $t\in\{2,\ldots,T\}$, define the increments

$$\Delta Q_i^k\left(\theta^t\right)=Q_i^k\left(\theta^t\right)-Q_i^k\left(\theta^{t-1}\right).$$

By monotonicity of Q_i^k, we have $\Delta Q_i^k(\theta^t) \geq 0$ for all $k \in \{1, \ldots, K\}$ and $t \in \{2, \ldots, T\}$. We now seek $T_i^k(\theta^t)$ that satisfy, for each k:

– Downward local incentive compatibility for $t = 2, \ldots, T$:

$$Q_i^k(\theta^t)\theta^t - T_i^k(\theta^t) \geq Q_i^k(\theta^{t-1})\theta^t - T_i^k(\theta^{t-1}) \iff T_i^k\left(\theta^t\right) - T_i^k\left(\theta^{t-1}\right) \leq \Delta Q_i^k\left(\theta^t\right)\theta^t.$$

– Upward local incentive compatibility for $t = 2, \ldots, T$:

$$Q_i^k(\theta^{t-1})\theta^{t-1} - T_i^k(\theta^{t-1}) \geq Q_i^k(\theta^t)\theta^{t-1} - T_i^k(\theta^t) \iff T_i^k\left(\theta^t\right) - T_i^k\left(\theta^{t-1}\right) \geq \Delta Q_i^k\left(\theta^t\right)\theta^{t-1}.$$

– Individual rationality at the lowest type:

$$Q_i^k\left(\theta^1\right)\theta^1 - T_i^k\left(\theta_1\right) \geq 0 \iff Q_i^k\left(\theta^1\right)\theta^1 \geq T_i^k\left(\theta^1\right).$$

– Mixture identity for $t = 1, \ldots, T$:

$$\sum_{k=1}^{K} \alpha_k T_i^k\left(\theta^t\right) = T_i\left(\theta^t\right).$$

For convenience, we set $Q_i^k(\theta^t) = T_i^k(\theta^t) = 0$ for $t = 0$. Then the individual rationality at the lowest type can be treated as a special case of downward local incentive compatibility:

$$\Delta Q_i^k\left(\theta^1\right)\theta^1 \geq T_i^k\left(\theta^1\right) - T_i^k\left(\theta^0\right).$$

The feasibility of the above linear system in the variables $T_i^k(\theta^t)$ ensures the existence of transfers that make each pair $\left(Q_i^k, T_i^k\right)$ incentive compatible and individually rational. To establish the feasibility, we apply Lemma 10. For this, let $z_{t,t}^k$ denote the dual variables corresponding to the local downward incentive compatibility constraints, $z_{t,t-1}^k$ the dual variables corresponding to the local upward incentive compatibility constraints, respectively and y_t the dual variable corresponding to the mixture identities. By Lemma 10, the original linear system is feasible if and only if the following dual system is infeasible:

$$-z_{t+1,t+1}^k + z_{t+1,t}^k + z_{t,t}^k - z_{t,t-1} + \alpha_k y_t = 0, \text{ for all } t \text{ and } k, \tag{7}$$

with the convention $z_{T+1,T}^k = z_{T+1,T+1}^k = 0$, and

$$\sum_{t=1}^{T}\sum_{k=1}^{K}[z_{t,t}^k \Delta Q_i^k(\theta^t)\theta_t - z_{t,t-1}^k \Delta Q_i^k(\theta^t)\theta^{t-1}] + \sum_{t=1}^{T} y_t T_i(\theta^t) < 0 \tag{8}$$

in variables $z^k_{t,t} \geq 0$, $z^k_{t,t-1} \geq 0$, and unrestricted y_t. To prove infeasibility of the dual system (and hence feasibility of the original system), suppose instead that the dual system is feasible. Define the cumulative sum $Y_t = \sum_{s=t}^{T} y_s$ (with $Y_{T+1} = 0$). Summing the equality (7) from s to T gives:

$$z^k_{t,t} - z^k_{t,t-1} = -\alpha_k Y_t. \tag{9}$$

Because $y_t = Y_t - Y_{t+1}$, we have

$$\sum_{t=1}^{T} y_t T_i\left(\theta^t\right) = \sum_{t=1}^{T} Y_t T_i\left(\theta^t\right) - \sum_{t=1}^{T} Y_{t+1} T_i\left(\theta^t\right).$$

By shifting the index in the second sum, we obtain

$$\sum_{t=1}^{T} Y_t T_i\left(\theta^t\right) - \sum_{t=1}^{T} Y_{t+1} T_i\left(\theta^t\right) = \sum_{t=1}^{T} Y_t T_i\left(\theta^t\right) - \sum_{t=2}^{T+1} Y_t T_i\left(\theta^{t-1}\right).$$

Because we set $Y_{T+1} = 0$ and $T_i\left(\theta^0\right) = 0$, this can be simplified to

$$\sum_{t=1}^{T} y_t T_i\left(\theta^t\right) = \sum_{t=1}^{T} Y_t \left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right)\right).$$

By substituting these into the inequality (8), we obtain

$$\sum_{t=1}^{T} \left[\sum_{k=1}^{K} \Delta Q^k_i\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right) z^k_{t,t-1} + Y_t\left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^t\right) \right] < 0,$$

where $\Delta Q_i\left(\theta^t\right) = \sum_{k=1}^{K} \alpha_k \Delta Q^k_i\left(\theta^t\right)$. We analyze the sign of the left hand-side term by term. For each t, the first part of the summand is

$$\sum_{k=1}^{K} \Delta Q^k_i\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right) z^k_{t,t-1}.$$

This is non-negative because $\Delta Q^k_i\left(\theta^t\right) \geq 0$, $\theta^t > \theta^{t-1}$, and $z^k_{t,t-1} \geq 0$. Hence, for the inequality to hold strictly, the second term must be negative for some t. The second part is

$$Y_t\left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^t\right).$$

must be negative. To determine the sign of the second term, recall that

$$\Delta Q_i\left(\theta^t\right)\theta^{t-1} \leq T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) \leq \Delta Q_i\left(\theta^t\right)\theta^t$$

because (Q_i, T_i) is incentive compatible and individually rational. Therefore, we have

$$\left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^t\right) \leq 0.$$

For the second term to be strictly negative, we must have $Y_t > 0$ for some t. Suppose that $Y_t > 0$ for some fixed t. From the equality (9), we have

$$z^k_{t,t-1} = z^k_{t,t} + \alpha_k Y_t \geq \alpha_k Y_t,$$

using $z^k_{t,t} \geq 0$. Thus, the first term admits the lower bound

$$\sum_{k=1}^{K} \Delta Q_i^k\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right) z^k_{t,t-1} \geq Y_t \sum_{k=1}^{K} \alpha_k \Delta Q_i^k\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right),$$

which is simplified to

$$\sum_{k=1}^{K} \Delta Q_i^k\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right) z^k_{t,t-1} \geq Y_t \Delta Q_i\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right).$$

Hence, the total contribution to the summation at t is bounded below by

$$Y_t\left(\Delta Q_i\left(\theta^t\right)\left(\theta^t - \theta^{t-1}\right) + T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^t\right),$$

which is simplified to

$$Y_t\left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^{t-1}\right).$$

Because $T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) \geq \Delta Q_i\left(\theta^t\right)\theta^{t-1}$ by the incentive compatibility of (Q_i, T_i), we have

$$Y_t\left(T_i\left(\theta^t\right) - T_i\left(\theta^{t-1}\right) - \Delta Q_i\left(\theta^t\right)\theta^{t-1}\right) \geq 0.$$

Therefore, for every t, the summand is non-negative. This contradicts the feasibility of the dual system, completing the proof. □

B.3 Proofs of Results in Section 4

Proof of Lemma 3 and Lemma 4

Proof. Recall that only anonymity and convexity of maximand set are needed to establish Lemma 2. Hence, we can focus on symmetric mechanisms. Lemma 3 follows directly from Lemma 4. To establish Lemma 4, let (Q, T) be an arbitrary symmetric, individually ra-

tional, incentive compatible, and feasible mechanism. By Lemma 7, there exist monotonic hierarchical allocations $\left(Q^k\right)_{k=1}^K$ and coefficients $\alpha_k \geq 0$ such that $Q = \sum_{k=1}^K \alpha_k Q^k$. By Lemma 9, there exists $\left(T^k\right)_{k=1}^K$ such that $\left(Q^k, T^k\right)$ is incentive compatible for all k and $T = \sum_{k=1}^K \alpha_k T^k$. By Lemma 6, each $\left(Q^k, T^k\right)$ is a first-price auction with subsidy, and this completes the proof. □

The following two lemmas are straightforward; we provide their proofs because we use the constructions introduced therein in the proofs of Theorem 1 and Theorem 2.

Lemma 16. *Let π be a Borel probability measure on a compact subset Θ in $\mathbb{R}$. Then there exists a sequence of Borel probability measures $(\pi_k)_{k\in\mathbb{N}}$ such that each π_k has finite support, π_k converges to π weakly, and $\pi_k(I) \to \pi(I)$ for each Borel set I with at most countable boundary ∂I.*

Proof. Denote the set of atoms $A = \{x \in \Theta \mid \pi(\{x\}) > 0\}$. Notice that A is at most countable. Decompose $\pi = \pi_a + \pi_c$, where $\pi_a = \sum_{x\in A} \pi(\{x\})\, \delta_x$ and π_c is the remaining part of π after removing the atoms. We approximate π_a and π_c separately with $\pi_{a,k}$ and $\pi_{c,k}$ as follows. Enumerate $A = \{a_1, a_2, \ldots\}$ and define the scalars $\alpha_k = \sum_{i\leq k} \pi(\{a_i\})$ and $\beta_k = \sum_{i>k} \pi(\{a_i\})$ and note that $\alpha_k + \beta_k = \pi_a(\Theta)$. Because α_k and β_k are bounded, we have the convergence

$$\alpha_k \to \pi_a(\Theta), \quad \beta_k \to 0 \quad \text{as} \quad k \to \infty$$

We define $\pi_{a,k}$ to pick up the first k atoms in μ and redistribute β_k proportionally over the first k atoms:

$$\pi_{a,k} = \sum_{i\leq k} \left(\frac{\alpha_k + \beta_k}{\alpha_k}\right) \pi(\{a_i\})\, \delta_{a_i}.$$

Then $\pi_{a,k}$ converges to π_a weakly. Indeed, for any bounded continuous function f, we have

$$\begin{aligned}\left|\int f d\pi_{a,k} - \int f d\pi_{a,k}\right| &= \left|\left(\frac{\alpha_k + \beta_k}{\alpha_k}\right) \sum_{i\leq k} \pi(\{a_i\})\, f(a_i) - \sum_{i\geq 1} \pi(\{a_i\})\, f(a_i)\right| \\ &= \left|\left(\frac{\beta_k}{\alpha_k}\right) \sum_{i\leq k} \pi(\{a_i\})\, f(a_i) - \sum_{i>k} \pi(\{a_i\})\, f(a_i)\right|,\end{aligned}$$

and the right-hand side converges to 0 as $k \to \infty$. The convergence of $\pi_{a,k}(I)$ to $\pi_a(I)$ for any Borel set I is followed by setting $f = 1_{\{a_i \in I\}}$.

In order to approximate π_c, we use the usual quantile approximation. Let F_c be the cumulative distribution function of π_c. As π_c has no atoms, for each positive integer k we can partition the support π_c into k intervals of equal probability mass. Let $x_{k,j}$ be an

arbitrary point in the interior of the j-th interval of the partition:

$$x_{k,j} \in F_c^{-1}\left(\left(\frac{j-1}{k}\pi_c(\Theta), \frac{j}{k}\pi_c(\Theta)\right)\right), \qquad j = 1, \ldots, k.$$

We define $\pi_{c,k}$ by putting mass $\pi_c(\Theta)/k$ at each $x_{k,j}$, that is:

$$\pi_{c,k} = \frac{\pi_c(\Theta)}{k} \sum_{i=1}^{k} \delta_{x_{k,j}}.$$

By construction, $\|F_{c,k} - F_c\|_\infty \leq \frac{\pi_c(\Theta)}{k}$, thus $\pi_{c,k}$ converges to π_c weakly. Let I be a Borel set with the countable boundary. Because the probability measure π_c is atomless, $\pi_c(\partial I) = 0$. By the Portmanteau theorem, $\pi_{c,k}(I)$ converges to $\pi_c(I)$. By setting $\pi_k = \pi_{a,k} + \pi_{c,k}$, we establish the convergence of $\pi_k \to \pi$ and $\pi_k(I) \to \pi(I)$ for any Borel I with at most countable boundary. □

In the next lemma we used the following convergence concept: $(I_k)_{k\in\mathbb{N}}$ converges to I almost surely in measure π, if $\pi((I_k - I) \cup (I - I_k)) \to 0$ as $k \to \infty$. Equivalently $\mathbf{1}_{I_k} \to \mathbf{1}_I$ almost surely in measure π.[32]

Lemma 17. *Let π be a Borel probability measure on a compact subset Θ in $\mathbb{R}$, and $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of Borel probability measures as constructed in the proof of Lemma 16. Let $(I_k)_{k\in\mathbb{N}}$ be a sequence of intervals converging to an interval I almost surely in measure π. Then $\pi_k(I_k)$ converges to $\pi(I)$ as $k \to \infty$.*

Proof. Let us denote the symmetric difference of I and I_k by $D_k = (I_k - I) \cup (I - I_k)$. Notice that

$$|\pi_k(I_k) - \pi(I)| \leq |\pi_k(I_k) - \pi_k(I)| + |\pi_k(I) - \pi(I)|.$$

Because I has at most two boundary points, Lemma 16 implies that $\pi_k(I)$ converges to $\pi(I)$, so the second term vanishes. It remains to show that $|\pi_k(I_k) - \pi_k(I)|$ converges to 0. Because $\pi_k(I_k) - \pi_k(I) = \pi_k(I_k - I) - \pi_k(I - I_k)$, by definition of D_k, we have

$$|\pi_k(I_k) - \pi_k(I)| \leq \pi_k(I_k - I) + \pi_k(I - I_k) = \pi_k(D_k).$$

Let $F_k(x) = \pi_k((-\infty, x] \cap \Theta)$ and $F(x) = \pi((-\infty, x] \cap \Theta)$. The construction of π_k in Lemma 16 gives

- atomic part: $\|F_{a,k} - F_a\|_\infty \leq \beta_k$,

[32] Also equivalently: there exists a measurable set E with $\mu(E) = 0$ such that for every $\theta \notin E$, there is an integer $k(\theta)$ for which $\theta \in I_k \iff \theta \in I$ for all $k \geq k(\theta)$.

- continuous part: $\|F_{c,k} - F_c\|_\infty \leq \frac{\pi_c(\Theta)}{k}$,

hence,

$$d_k := \sup_{x\in\Theta} |F_k(x) - F(x)| \leq \|F_{a,k} - F_a\|_\infty + \|F_{c,k} - F_c\|_\infty \to 0.$$

D_k is the union of at most two disjoint intervals, whose four endpoints each contribute at most d_k; consequently

$$|\pi_k(D_k) - \pi(D_k)| \leq 4d_k \to 0.$$

Furthermore, because I_k converges to I almost surely, by dominated convergence theorem, we infer

$$\pi(D_k) = \int |\mathbf{1}_{I_k} - \mathbf{1}_I| \, d\pi \to 0,$$

hence, $\pi_k(D_k)$ converges to 0. Therefore, we conclude that $|\pi_k(I_k) - \pi_k(I)|$ converges to 0, and this completes the proof. □

Proof of Theorem 1

Proof. For finite type spaces, the result follows directly from Lemma 7, Lemma 3, quasi-convexity of the objective, and Carathéodory's theorem. Let us consider the general case of Θ being a compact subset of $\mathbb{R}$ and π be a Borel probability measure on Θ. Let $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of Borel probability measures from the proof of Lemma 16. By the discrete version of Theorem 1, for each π_k, there exists a first-price auction with subsidy s_k, a symmetric equilibrium bidding strategy b_k, and a bid space $\mathcal{B}_k$ such that the induced mechanism (Q_k, T_k) maximizes the objective under π_k. Moreover, we choose $b_k(\cdot)$ and s_k as in the proof of Lemma 6. By construction, there is at most one type $\theta \in \Theta_k$ such that $b_k(\theta) = r^-$, we denote this type by $\theta_{r^-,k}$ onward. If such a type exists, then by construction $\theta_{r^-,k} \in \mathcal{B}_k$ and $Q_k(\theta_{r^-,k})\,\theta_{r^-,k} = T_k(\theta_{r^-,k}) + s_k$, thus $\theta_{r^-,k}$ serves as the reserve price, i.e. $\theta_{r^-,k} = \min(b_k(\Theta_k) - \{\varnothing, r^-\})$. For the sake of this proof, we may assume $\mathcal{B}_k \subset \mathbb{R}$ by embedding $\varnothing$ and r^- in $\mathbb{R}$ so that $\varnothing < r^- < \inf\Theta$.[33] We extend the functions Q_k, T_k, and b_k to all $\theta \in \Theta$ by selecting

$$x(\theta) \in \arg\max_{x\in\Theta_k} (Q_k(x)\,\theta - T_k(x))$$

so that $x(\theta) = \theta$ for $\theta \in \Theta_k$. For $\theta \in \Theta$ such that $Q_k(x(\theta))\,\theta - T_k(x(\theta)) \geq s_k$, we define

$$Q_k(\theta) = Q_k(x(\theta)), \quad T_k(\theta) = T_k(x(\theta)), \quad b_k(\theta) = b_k(x(\theta)).$$

[33] Of course, a bidder winning with bid $\varnothing$ is not given an object and pays nothing, while a bidder winning with bid r^- pays the reserve price.

For $\theta \in \Theta$ such that $Q_k(x(\theta))\,\theta - T_k(x(\theta)) < s_k$, we define

$$Q_k(\theta) = 0, \quad T_k(\theta) = -s_k, \quad b_k(\theta) = \varnothing.$$

The extended (Q_k, T_k) remains incentive compatible on Θ because $x(\theta)$ maximizes $Q_k(x)\,\theta - T_k(x)$ and it remains individually rational because $Q_k(\theta)\,\theta - T_k(\theta) \geq s_k$, for all $\theta \in \Theta$. Because the support of types remains Θ_k, and b_k was a symmetric equilibrium in the first-price auction with subsidy s_k on Θ_k, we infer that the strategy profile in which every player of any type in Θ (including outside of support Θ_k) plays the extended strategy $b_k(\cdot)$ is a symmetric equilibrium. In this equilibrium, every type $\theta \in \Theta$ receives the interim outcome $(Q_k(x(\theta)), T_k(x(\theta)))$. Furthermore, the incentive compatibility of the extended (Q_k, T_k) implies that the extended b_k is weakly increasing in θ.

Furthermore, in the extended strategy there can be at most one $\theta_{r^-,k}$ such that $b_k(\theta_{r^-,k}) = r^-$. Indeed, for bidders with type $\theta < \theta_{r^-,k}$, bidding r^- is strictly dominated by staying out of the auction, and for bidders with type $\theta > \theta_{r^-,k}$, bidding the reserve price $\theta_{r^-,k}$ strictly dominates bidding r^-.

Because b_k is weakly increasing in type, Helly's selection theorem together with the standard diagonal argument guarantee that the sequence $(b_k, s_k)_{k\in\mathbb{N}}$ admits a pointwise convergent subsequence. Without loss of generality, in the following, we assume that the sequence $(b_k, s_k)_{k\in\mathbb{N}}$ itself is this convergent subsequence.

Let (b_∞, s) be the limit of this convergent sequence; note that b_∞ is then weakly increasing and $b_\infty(\theta) \leq \theta$ for all $\theta \in \Theta$. Our goal is to show that $b_\infty(\cdot)$ constitutes a symmetric equilibrium bidding strategy in a first-price auction with bid space $\mathcal{B}_\infty = b_\infty(\Theta)$, and that it maximizes the objective in the limit. For the rest of the proof, we use the following notation for any $k \in \mathbb{N} \cup \{\infty\}$:

$$I_k(\theta) = \{x \in \Theta \mid b_k(x) = b_k(\theta)\}, \qquad J_k(\theta) = \{x \in \Theta \mid b_k(x) < b_k(\theta)\}.$$

For each $k \in \mathbb{N}$, because $Q_k(\cdot)$ is the interim winning probability implemented by the first-price auction with the equilibrium bidding strategy $b_k(\cdot)$, we have

$$Q_k(\theta) = \begin{cases} \sum_{j=1}^{n} \frac{1}{j}\binom{n-1}{j-1} \left(\pi_k(J_k(\theta))\right)^{n-j} \left(\pi_k(I_k(\theta))\right)^{j-1} & \text{if } b_k(\theta) > \varnothing, \\ 0 & \text{if } b_k(\theta) = \varnothing. \end{cases}$$

For the expected payment, we have

$$T_k(\theta) = \begin{cases} b_k(\theta) Q_k(\theta) - s_k & \text{if } b_k(\theta) > r^-, \\ \theta Q_k(\theta) - s_k & \text{if } b_k(\theta) = r^-, \\ -s_k & \text{if } b_k(\theta) = \varnothing. \end{cases}$$

Similarly, we denote by $Q_\infty(\cdot)$ the interim winning probability implemented by the first-price auction with the bidding strategy $b_\infty(\cdot)$ in the limit, thus we have

$$Q_\infty(\theta) = \begin{cases} \sum_{j=1}^{n} \frac{1}{j} \binom{n-1}{j-1} \left(\pi\left(J_\infty(\theta)\right)\right)^{n-j} \left(\pi\left(I_\infty(\theta)\right)\right)^{j-1} & \text{if } b_\infty(\theta) > \varnothing, \\ 0 & \text{if } b_\infty(\theta) = \varnothing. \end{cases}$$

Claim 1. *For all $\theta \in \Theta$, if $I_k(\theta) \to I_\infty(\theta)$ almost surely in measure π, then $J_k(\theta) \to J_\infty(\theta)$ almost surely in measure π.*

Proof of Claim 1. First, suppose that x is such that $b_\infty(x) \neq b_\infty(\theta)$. Because b_k converges to b_∞ pointwise, we infer that $b_k(x) \neq b_k(\theta)$ for all sufficiently large k, and, hence, $\mathbf{1}_{J_k(\theta)}(x) \to \mathbf{1}_{J_\infty(\theta)}(x)$. In particular,

$$X^\theta \subseteq I_\infty(\theta),$$

for $X^\theta = \left\{x \mid \mathbf{1}_{J_k(\theta)}(x) \not\to \mathbf{1}_{J_\infty(\theta)}(x)\right\}$ (recall that the level set $I_\infty(\theta)$ of x is the set of x such that $b_\infty(x) = b_\infty(\theta)$). If the level set $I_\infty(\theta)$ has π measure 0, then the Claim obtains.

By way of contradiction, suppose $J_k(\theta)$ does not converge to $J_\infty(\theta)$ almost surely in measure π that is $\pi\left(X^\theta\right) > 0$. By the above displayed inclusion, for all $x \in X^\theta$ we have $b_\infty(x) = b_\infty(\theta)$ and hence $\mathbf{1}_{J_\infty(\theta)}(x) = 0$. Thus,

$$X^\theta = \left\{x \mid \mathbf{1}_{J_k(\theta)}(x) \not\to 0\right\}.$$

and, for $x \in X^\theta$ we have $b_k(x) < b_k(\theta)$ infinitely often. Hence, $\mathbf{1}_{I_k(\theta)}(x) \not\to 1$ for all $x \in X^\theta$. In particular,

$$X^\theta \subseteq \left\{x \mid \mathbf{1}_{I_k(\theta)}(x) \not\to 1\right\} \cap I_\infty(\theta)$$

By assumption, $I_k(\theta) \to I_\infty(\theta)$ π-almost surely, and hence for almost all $x \in I_\infty(\theta)$, we have $\mathbf{1}_{I_k(\theta)}(x) \to 1$. Thus, X^θ has π measure 0, a contradiction that completes the proof. □

Claim 2. *If $b_\infty(\theta) = \varnothing$, then there exists K such that $b_k(\theta) = \varnothing$ for all $k > K$. Likewise, if $b_\infty(\theta) = r^-$, then there exists K such that $b_k(\theta) = r^-$ for all $k > K$. Moreover, there is at most one type θ such that $b_\infty(\theta) = r^-$. If such a type θ exists, then*

$\theta = \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$.

Proof of Claim 2: Because $\varnothing$ and r^- are isolated points, convergence of the sequence $b_k(\theta)$ to $b_\infty(\theta)$ can only occur through eventual equality. Hence, there exists a finite index K beyond which $b_k(\theta)$ coincides with the limit value $b_\infty(\theta)$. In order to see the uniqueness of a type that bids r^- in the limit, suppose instead that $b_\infty(\theta) = b_\infty(\theta') = r^-$ for two distinct types $\theta, \theta' \in \Theta$. Then there exist indices K_θ and $K_{\theta'}$ such that $b_k(\theta) = r^-$ for all $k > K_\theta$ and $b_k(\theta') = r^-$ for all $k > K_{\theta'}$. Setting $K = \max\{K_\theta, K_{\theta'}\}$ yields $b_k(\theta) = b_k(\theta') = r^-$ for every $k > K$, contradicting the fact that there is at most one type to bid r^- for each finite k. Denote by θ_{r^-} the type that bids r^- in the limit if such a type exists.

It remains to show that $\theta_{r^-} = \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$. Because there exists $K_{\theta_{r^-}}$ such that $b_k\left(\theta_{r^-}\right) = r^-$ for all $k > K_{\theta_{r^-}}$ and there cannot be more than one type bidding r^- for each k, we have $\theta_{r^-,k} = \theta_{r^-}$ for all $k > K_{\theta_{r^-}}$. First, we show that $\theta_{r^-} \leq \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$. To the contrary, suppose that $\theta_{r^-} > \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$, thus there exists θ' such that $\theta_{r^-} > b_\infty\left(\theta'\right) > r^-$. Because $b_k\left(\theta'\right)$ converges to $b_\infty\left(\theta'\right)$, there exists $K_{\theta'}$ such that $r^- < b_k\left(\theta'\right) < \theta_{r^-}$ for all $k > K_{\theta'}$. Therefore, for all $k > \max\left\{K_{\theta_{r^-}}, K_{\theta'}\right\}$, we have $r^- < b_k\left(\theta'\right) < \theta_{r^-,k}$, which is a contradiction because $b_k\left(\theta'\right) = \varnothing$ for every $\theta' < \theta_{r^-,k}$. In order to show that $\theta_{r^-} \geq \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$, to the contrary, suppose that $\theta_{r^-} < \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$, thus there exists $\theta' \in \Theta$ such that $\theta_{r^-} < \theta' < \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$. Because $\theta' > \theta_{r^-}$ and there exists at most one type that bids r^- in the limit, we have $b_\infty(\theta') > r^-$, hence $b_\infty\left(\theta'\right) \in b_\infty\left(\Theta\right) - \{\varnothing, r^-\}$. By combining $\theta' < \inf\left(b_\infty\left(\Theta\right) - \{\varnothing, r^-\}\right)$ and $b_\infty\left(\theta'\right) \in b_\infty\left(\Theta\right) - \{\varnothing, r^-\}$, we have $\theta' < b_\infty\left(\theta'\right)$, which is a contradiction to that $b_\infty\left(\theta\right) \leq \theta$ for all $\theta \in \Theta$. □

With Claim 2, we can define the interim payment in the first-price auction when every bidder follows b_∞ as following:

$$T_\infty\left(\theta\right) = \begin{cases} b_\infty\left(\theta\right) Q_\infty\left(\theta\right) - s & \text{if } b_\infty\left(\theta\right) > r^-, \\ \theta Q_\infty\left(\theta\right) - s & \text{if } b_\infty\left(\theta\right) = r^-, \\ -s & \text{if } b_\infty\left(\theta\right) = \varnothing. \end{cases}$$

Claim 3. *If $b_\infty\left(\theta\right) = \varnothing$, then $Q_k\left(\theta\right) \to Q_\infty\left(\theta\right)$.*

Proof of Claim 3. If $b_\infty(\theta) = \varnothing$, then there exists K such that for all $k > K$, we have $b_k(\theta) = \varnothing$ by Claim 1. Consequently, $Q_k(\theta) = 0$ for all $k > K$, implying $Q_k(\theta) \to Q_\infty(\theta)$, as required. □

Claim 4. *For any fixed θ, if $\pi\left(I_\infty\left(\theta\right) - \{\theta\}\right) = 0$ then $Q_k\left(\theta\right) \to Q_\infty\left(\theta\right)$.*

Proof of Claim 4. Because the case $b_\infty\left(\theta\right) = \varnothing$ is covered by Claim 3, we may assume that $b_\infty\left(\theta\right) \neq \varnothing$ and $\pi\left(I_\infty\left(\theta\right) - \{\theta\}\right) = 0$. Take any $x \in \Theta - \{\theta\}$ and analyze two cases:

If $b_\infty\left(x\right) > b_\infty\left(\theta\right)$, then there exists K such that $b_k\left(x\right) > b_k\left(\theta\right)$ for all $k > K$. Denoting

by $\mathbf{1}_J$ the indicator function of set J, we can express this as:

$$\mathbf{1}_{J_k(\theta)}(x) = \mathbf{1}_{J_\infty(\theta)}(x) = 0, \quad \mathbf{1}_{I_k(\theta)}(x) = \mathbf{1}_{I_\infty(\theta)}(x) = 0 \qquad \text{for all } k > K.$$

If $b_\infty(x) < b_\infty(\theta)$, then there exists K such that $b_k(x) < b_k(\theta)$ for all $k > K$. Hence,

$$\mathbf{1}_{J_k(\theta)}(x) = \mathbf{1}_{J_\infty(\theta)}(x) = 1, \quad \mathbf{1}_{I_k(\theta)}(x) = \mathbf{1}_{I_\infty(\theta)}(x) = 0 \qquad \text{for all } k > K.$$

Combining two cases, as $k \to \infty$ we have

$$\mathbf{1}_{J_k(\theta)}(x) \longrightarrow \mathbf{1}_{J_\infty(\theta)}(x), \quad \mathbf{1}_{I_k(\theta)}(x) \longrightarrow \mathbf{1}_{I_\infty(\theta)}(x) \qquad \text{for every } x \notin I_\infty(\theta) - \{\theta\}.$$

Because $\pi(I_\infty(\theta) - \{\theta\}) = 0$ by the assumption of the claim, the convergences hold almost surely in measure π. Furthermore, as b_∞ is non-decreasing, $I_\infty(\theta)$ and $J_\infty(\theta)$ are (possibly one-point) intervals in Θ, thus their boundaries $\partial I(\theta)$ and $\partial J(\theta)$ are finite. Therefore, by Lemma 17, we have $\pi_k(I_k(\theta)) \to \pi(I_\infty(\theta))$ and $\pi_k(J_k(\theta)) \to \pi(J_\infty(\theta))$. Finally, because $Q_k(\theta)$ depends continuously on $(\pi_k(I_k(\theta)), \pi_k(J_k(\theta)))$, and similarly, $Q_\infty(\theta)$ depends continuously on $(\pi(I_\infty(\theta)), \pi(J_\infty(\theta)))$, the convergence $Q_k(\theta) \to Q_\infty(\theta)$ follows. □

Claim 5. *For any $\theta \in \Theta$, if $Q_\infty(\theta) = 0$ then $Q_k(\theta) \to Q_\infty(\theta)$.*

Proof of Claim 5. The case $b_\infty(\theta) = \varnothing$ is covered by Claim 3. If $b_\infty(\theta) > \varnothing$ and $Q_\infty(\theta) = 0$, then we have $\pi(J_\infty(\theta)) = 0$ and $\pi(I_\infty(\theta)) = 0$. By Claim 4, we have $Q_k(\theta) \to Q_\infty(\theta)$. □

Claim 6. *If $b_\infty(\theta) = \theta$ for some $\theta \in \Theta$, then $b_\infty(\theta') = \varnothing$ for all $\theta' \in \{x \in \Theta \mid x < \theta\}$.*

Proof of Claim 6. For each k, the incentive compatibility for type θ implies

$$Q_k(\theta)(\theta - b_k(\theta)) \geq Q_k(\theta')(\theta - b_k(\theta')), \text{ for all } \theta' \in \{x \in \Theta \mid x < \theta\}.$$

The left-hand side converges to 0, whereas $\theta - b_k(\theta')$ is bounded away from zero for every k; hence we have $Q_k(\theta') \to 0$. For $Q_k(\theta')$ to converge 0, one of the following must occur:

(i) $b_k(\theta') = \varnothing$ for all sufficiently large k; or

(ii) both $\pi_k(J_k(\theta'))$ and $\pi_k(I_k(\theta'))$ converge to 0.

Assume, to the contrary, that there exists $\theta' \in \{x \in \Theta \mid x < \theta\}$ such that $b_k(\theta') > \varnothing$. Because $\{x \in \Theta \mid x \leq \theta'\} \subseteq J_k(\theta') \cup I_k(\theta')$ for every k, case (ii) forces $\pi_k(\{x \in \Theta \mid x \leq \theta'\}) \to 0$. Lemma 16 then gives $\pi(\{x \in \Theta \mid x \leq \theta'\}) = 0$. Because Θ is the support of π, this can hold only if $\theta' = \min(\Theta)$. Take any θ'' with $\theta' < \theta'' < \theta$. Because θ' is the minimum of the support, the interval $[\theta', \theta'')$ must have positive mass, hence $\pi(\{x \in \Theta \mid x \leq \theta''\}) > 0$. Hence, neither alternative (i) or (ii) can be true for θ'', which is a contradiction. □

Claim 6 therefore guarantees that there is at most one type θ with $b_\infty(\theta) = \theta$, which we denote θ_L when it exists. Claim 6 also implies that no two distinct types θ, θ' can satisfy $b_\infty(\theta) = r^-$ and $b_\infty(\theta') = \theta'$. For expositional compactness in the claim below $I_\infty(\theta_L)$ is an empty set if θ_L is undefined (that is if no type satisfies $b_\infty(\theta) = \theta$).

Claim 7. For a fixed $\theta \in \Theta - I_\infty(\theta_L)$, if $\pi(I_\infty(\theta) - \{\theta\}) > 0$, then for every $x, y \in I_\infty(\theta)$, there exists $k(x,y)$ such that $b_k(x) = b_k(y)$ for all $k > k(x,y)$. Furthermore, *for every* $\theta \in \Theta - I_\infty(\theta_L)$*, we have* $Q_k(\theta) \to Q_\infty(\theta)$.

Proof of Claim 7. To prove the first part, consider $\theta \in \Theta - I_\infty(\theta_L)$ such that $\pi(I_\infty(\theta) - \{\theta\}) > 0$. Pick arbitrary $x, y \in I_\infty(\theta)$. If $b_\infty(\theta) \in \{\varnothing, r^-\}$, then Claim 2 guarantees that $b_k(x) = b_k(y)$ for all sufficiently large k. Henceforth, assume $b_\infty(\theta) > r^-$. Without loss of generality, suppose that $x < y$. To derive a contradiction, assume $b_k(x) < b_k(y)$ infinitely often. Because $b_k(\cdot)$ is weakly monotonic and $\pi(I_\infty(\theta) - \{\theta\}) > 0$, we can find $x', y' \in I_\infty(\theta)$ such that $b_k(x') < b_k(y')$ infinitely often and $\pi([x', y'] \cap \Theta) > 0$. The incentive compatibility for the type x' gives:

$$Q_k(x')[x' - b_k(x')] \geq Q_k(y')[x' - b_k(y')],$$

which rearranges to

$$[Q_k(y') - Q_k(x')](x' - b_k(x')) \leq Q_k(y')(b_k(y') - b_k(x')).$$

Because $b_k(x') < b_k(y')$ infinitely often, there exists a subsequence along which this inequality holds. Hence, accounting for tie-breaking, we have $Q_k(y') - Q_k(x') \geq \frac{1}{N}\pi_k([x', y'] \cap \Theta)$ along the subsequence. By Lemma 17, we have $\pi_k([x', y'] \cap \Theta) \to \pi([x', y'] \cap \Theta) > 0$. Therefore, along the subsequence,

$$\lim_{k\to\infty} [Q_k(y') - Q_k(x')](x' - b_k(x')) \geq \frac{1}{n}\pi([x', y'] \cap \Theta)(x' - b_\infty(x')).$$

Because $x > \theta_L$, we have $x' > b_\infty(x')$, and thus the right-hand side is strictly positive. This gives:

$$\lim_{k\to\infty} [Q_k(y') - Q_k(x')](x' - b_k(x')) > 0.$$

However, because $\lim_{k\to\infty} b_k(x') = \lim_{k\to\infty} b_k(y')$, we also have

$$\lim_{k\to\infty} Q_k(y')(b_k(y') - b_k(x')) = 0,$$

which contradicts the earlier inequality, thus proving the first part of the claim. To prove the second part of the claim, consider a type $\theta \in \Theta - I_\infty(\theta_L)$:

If $\theta < \theta_L$, then Claim 3 and Claim 6 prove the convergence $Q_k(\theta) \to Q_\infty(\theta)$.

If $\theta > \theta_L$ and $\pi(I_\infty(\theta) - \{\theta\}) = 0$, then Claim 4 proves the convergence.

If $\theta > \theta_L$ and $\pi(I_\infty(\theta) - \{\theta\}) > 0$, then for all $x \in I_\infty(\theta)$, the first part of the claim implies that $b_k(x) = b_k(\theta)$ for all sufficiently large k, hence $\mathbf{1}_{I_k(\theta)}(x) \to \mathbf{1}_{I_\infty(\theta)}(x) = 1$. For all $x \notin I_\infty(\theta)$, the pointwise convergence of $b_k(\cdot)$ implies that $b_k(x) \neq b_k(\theta)$ for all sufficiently large k, hence $\mathbf{1}_{I_k(\theta)}(x) \to \mathbf{1}_{I_\infty(\theta)}(x) = 0$. Therefore, we have the convergence of $I_k(\theta)$ to $I_\infty(\theta)$. The result then follows from Lemma 17 and Claim 1, which together complete the proof. □

Claim 8. *Suppose there exists $b_\infty(\theta_L) = \theta_L$. If there exists $\theta \in I_\infty(\theta_L) - \{\theta_L\}$ such that $\pi(I_\infty(\theta_L) - \{\theta_L, \theta\}) > 0$, then for every $x, y \in I_\infty(\theta_L) - \{\theta_L\}$, there exists $k(x,y)$ such that $b_k(x) = b_k(y)$ for all $k > k(x,y)$.*

Proof of Claim 8. Pick arbitrary $x, y \in I_\infty(\theta_L) - \{\theta_L\}$. To derive a contradiction, suppose that $b_k(x) < b_k(y)$ infinitely often. Because $b_k(\cdot)$ is weakly monotonic and $\pi(I_\infty(\theta_L) - \{\theta_L, \theta\}) > 0$, we can find $x', y' \in I_\infty(\theta_L) - \{\theta_L\}$ such that $b_k(x') < b_k(y')$ infinitely often and $\pi([x', y'] \cap \Theta) > 0$. The incentive compatibility for the type x' gives:

$$Q_k(x')[x' - b_k(x')] \geq Q_k(y')[x' - b_k(y')],$$

which we rearrange as

$$[Q_k(y') - Q_k(x')](x' - b_k(x')) \leq Q_k(y')(b_k(y') - b_k(x')).$$

Because $b_k(x') < b_k(y')$ infinitely often, there exists a subsequence along which this inequality holds. Hence, the tie-breaking implies that $Q_k(y') - Q_k(x') \geq \frac{1}{n}\pi_k([x', y'] \cap \Theta)$ along the subsequence. By 17, we have $\pi_k([x', y'] \cap \Theta) \to \pi([x', y'] \cap \Theta) > 0$. Therefore, along the subsequence,

$$\lim_{k\to\infty} [Q_k(y') - Q_k(x')](x' - b_k(x')) \geq \frac{1}{n}\pi([x', y') \cap \Theta)(x' - b_\infty(x')).$$

Because $x > \theta_L$, we have $x' > b_\infty(x')$, and thus the right-hand side is strictly positive. This gives:

$$\lim_{k\to\infty} [Q_k(y') - Q_k(x')](x' - b_k(x')) > 0.$$

However, because $\lim_{k\to\infty} b_k(x') = \lim_{k\to\infty} b_k(y')$, we also have

$$\lim_{k\to\infty} Q_k(y')(b_k(y') - b_k(x')) = 0,$$

which contradicts the earlier inequality. □

Claim 7 establishes that $Q_k(\theta) \to Q_\infty(\theta)$ for all $\theta \in \Theta$ if there does not exist a type θ_L such that $b_\infty(\theta_L) = \theta_L$. If such a type θ_L exists then Claim 7 tells us that $Q_k(\theta) \to Q_\infty(\theta)$ provided $\theta \in \Theta - I_\infty(\theta_L)$. To prove that $Q_k(\theta) \to Q_\infty(\theta)$ in the remaining case—in which θ_L exists and $\theta \in I_\infty(\theta_L)$, we consider the following three exhaustive and mutually exclusive cases:

Case (i): $I_\infty(\theta_L) - \{\theta_L\} = \emptyset$.

Case (ii): There exists $\theta \in I_\infty(\theta_L) - \{\theta_L\}$ such that $b_k(\theta_L) = b_k(\theta)$ for all sufficiently large k.

Case (iii): $I_\infty(\theta_L) - \{\theta_L\}$ is not empty, and $b_k(\theta_L) < b_k(\theta)$ infinitely often for all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$.

Claim 9. *In cases (i) and (ii), we have $Q_k(\theta) \to Q_\infty(\theta)$ for all $\theta \in \Theta$.*

Proof of Claim 9. For $\theta \in \Theta - I_\infty(\theta_L)$, Claim 6 establishes the convergence $Q_k(\theta) \to Q_\infty(\theta)$. Henceforth, we consider $\theta \in I_\infty(\theta_L)$.

In case (i), because θ_L is the only type in $I_\infty(\theta_L)$, it suffices to prove the convergence of $Q_k(\theta_L)$ to $Q_\infty(\theta_L)$. Claim 4 establishes the convergence of $Q_k(\theta_L)$ to $Q_\infty(\theta_L)$ because we have $\pi(I_\infty(\theta_L) - \{\theta_L\}) = \pi(\emptyset) = 0$.

In case (ii), let us denote by θ_f the type in $I_\infty(\theta_L) - \{\theta_L\}$ such that $b_k(\theta_L) = b_k(\theta_f)$ for all sufficiently large k. We consider two different sub-cases:

(a) There exists $\theta' \in I_\infty(\theta_L) - \{\theta_L\}$ such that $\pi(I_\infty(\theta_L) - \{\theta_L, \theta'\}) > 0$.

(b) For all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$, we have $\pi(I_\infty(\theta_L) - \{\theta_L, \theta\}) = 0$.

In case (a), Claim 8 established that, for any $x, y \in I_\infty(\theta_L) - \{\theta_L\}$, we have $b_k(x) = b_k(y)$ for all sufficiently large k. By setting $x = \theta_f$ and y to be an arbitrary type θ in $I_\infty(\theta_L) - \{\theta_L\}$, we have $b_k(\theta_L) = b_k(\theta_f)$ and $b_k(\theta_f) = b_k(\theta)$ for all sufficiently large k. In case (b), because Θ is the support of π, we have $I_\infty(\theta_L) - \{\theta_L\} = \{\theta\}$ for some θ. Moreover, by the assumption of the case (ii), we have $I_\infty(\theta_L) - \{\theta_L\} = \{\theta_f\}$ and $b_k(\theta_L) = b_k(\theta_f)$ for all sufficiently large k. Combining two sub-cases, we established that for all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$, there exists $k(\theta)$ such that $b_k(\theta_L) = b_k(\theta)$ for all $k \geq k(\theta)$. Now, we want to show that $I_k(\theta)$ converges to $I_\infty(\theta_L)$ for all $\theta \in I_\infty(\theta_L)$. Suppose that $x \in I_\infty(\theta_L)$, thus $\mathbf{1}_{I_\infty(\theta_L)}(x) = 1$. We already have shown that, for all sufficiently large k, $b_k(x) = b_k(\theta_L)$, hence $\mathbf{1}_{I_k(\theta)}(x) = 1$. Suppose that $x \notin I_\infty(\theta_L)$, thus $\mathbf{1}_{I_\infty(\theta_L)}(x) = 0$. Because b_k converges to b_∞ pointwise, we have $b_k(x) \neq b_k(\theta_L)$ for all sufficiently large k. However, we already have shown that $b_k(\theta) = b_k(\theta_L)$ for all sufficiently large k, therefore, we have $\mathbf{1}_{I_k(\theta)}(x) = 0$ for all sufficiently large k. Therefore, we have the convergence of $I_k(\theta)$ to $I_\infty(\theta_L)$ for all $\theta \in I_\infty(\theta_L)$. Because $I_\infty(\theta) = I_\infty(\theta_L)$ for all $\theta \in I_\infty(\theta_L)$, Claim 1 and Lemma 17 prove the convergence of $Q_k(\theta)$ to $Q_\infty(\theta)$. $\square$

Except for case (iii) above, we have now proven the convergence $(Q_k, b_k, s_k) \to (Q_\infty, b_\infty, s)$.

This convergence implies that also $T_k(\cdot)$ converges to $T_\infty(\cdot)$ pointwise on Θ. Therefore (and except for case (iii) above) we conclude that $(Q_k, T_k, b_k, s_k) \to (Q_\infty, T_\infty, b_\infty, s)$, and that (b_∞, s) implements (Q_∞, T_∞) in a symmetric equilibrium of the first-price auction.

Consider now case (iii). In this case, we prove the convergence $Q_k(\theta) \to Q_\infty(\theta)$ for a subsequence of (b_k) defined as follows:

- If $\inf(I_\infty(\theta_L) - \{\theta_L\}) \in I_\infty(\theta_L) - \{\theta_L\}$ (that is the infimum is the minimum), we choose a subsequence b_{k_m} such that $b_{k_m}(\theta_L) < b_{k_m}(\inf(I_\infty(\theta_L) - \{\theta_L\}))$ for all $m \in \mathbb{N}$. (Recall that in case (iii), there are infinitely many k_m that satisfy this property).

- If $\inf(I_\infty(\theta_L) - \{\theta_L\}) \notin I_\infty(\theta_L) - \{\theta_L\}$, then we denote by $r_1, r_2, \ldots$ a strictly decreasing sequence of elements of $I_\infty(\theta_L) - \{\theta_L\}$ that converges to $\inf(I_\infty(\theta_L) - \{\theta_L\})$. For each $m \in \mathbb{N}$, we choose a subsequence b_{k_m} such that $b_{k_m}(\theta_L) < b_{k_m}(r_m)$. Because $b_k(\cdot)$ is non-decreasing, this implies $b_{k_m}(\theta_L) < b_{k_m}(x)$ for all $x \geq r_m$.

To simplify notation, and without loss of generality, we assume that the original sequence $(b_k)_{k\in\mathbb{N}}$ itself is the convergent subsequence $(b_{k_m})_{m\in\mathbb{N}}$. Then, for all $x \in I_\infty(\theta_L) - \{\theta_L\}$, there exists $K(x)$ such that $b_k(\theta_L) < b_k(x)$ for all $k \geq K(x)$.

Now consider the following modification of the limiting bidding function b_∞:

$$b_\infty^-(\theta) = \begin{cases} b_\infty(\theta) & \text{if } \theta \neq \theta_L, \\ r^- & \text{if } \theta = \theta_L. \end{cases}$$

Our goal is to show that (b_∞^-, s) implements $\lim_{k\to\infty}(Q_k, T_k)$ in case (iii). Notice that $b_\infty^-(\cdot)$ is non-decreasing by Claim 6. Let $(Q_\infty^-(\cdot), T_\infty^-(\cdot))$ denote the interim allocation and the interim payment when every bidder follows the bidding strategy $b_\infty^-(\cdot)$ in the limit. Because we modify the bid only at type θ_L, replacing it with r^-, Claim 6 implies that $Q_\infty(\theta) = Q_\infty^-(\theta)$ for all $\theta < \theta_L$. Furthermore, for any $\theta > \theta_L$, if θ_L was not tied with θ under $b_\infty(\cdot)$, then the modification does not change the winning probability. Hence, we have $Q_\infty(\theta) = Q_\infty^-(\theta)$ and $T_\infty(\theta) = T_\infty^-(\theta)$ for all $\theta \in \theta \in \Theta - I_\infty(\theta_L)$. For types in $I_\infty(\theta_L)$, the interim allocation under $b_\infty^-(\cdot)$ is given by:

For $\theta = \theta_L$,

$$Q_\infty^-(\theta_L) = \sum_{j=1}^{n} \frac{1}{j} \binom{n-1}{j-1} \left(\pi(\{x \in \Theta \mid x < \theta_L\})\right)^{n-j} \left(\pi(\{\theta_L\})\right)^{j-1}.$$

For all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$,

$$Q_\infty^-(\theta) = \sum_{j=1}^{n} \frac{1}{j} \binom{n-1}{j-1} \left(\pi(\{x \in \Theta \mid x \leq \theta_L\})\right)^{n-j} \left(\pi(I_\infty(\theta) - \{\theta_L\})\right)^{j-1}.$$

Claim 10. *In case (iii), $Q_k(\theta) \to Q_\infty^-(\theta)$ for all $\theta \in \Theta$.*

Proof of Claim 10. For all $\theta \in \Theta - I_\infty(\theta_L)$, we have $Q_\infty^-(\theta) = Q_\infty(\theta)$, and the convergence follows from previous analysis. To prove the convergence for $\theta \in I_\infty(\theta_L)$, by Lemma 17, it suffices to show the following almost sure convergences in measure π.

1. $I_k(\theta_L) \to \{\theta_L\}$
2. $I_k(\theta) \to I_\infty(\theta) - \{\theta_L\}$ for all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$
3. $J_k(\theta_L) \to \{x \in \Theta \mid x < \theta_L\}$
4. $J_k(\theta) \to \{x \in \Theta \mid x \leq \theta_L\}$ for all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$

To prove the convergence of $I_k(\theta_L)$ to $\{\theta_L\}$, consider x such that $\mathbf{1}_{\{\theta_L\}}(x) = 1$, thus $x = \theta_L$. Because $\theta_L \in I_k(\theta_L)$ for all k, we have $\mathbf{1}_{I_k(\theta_L)}(x) = 1$ for all k, hence the convergence follows. Now consider x such that $\mathbf{1}_{\{\theta_L\}}(x) = 0$, thus either $x < \theta_L$ or $x > \theta_L$. If $x < \theta_L$, then $b_\infty(x) = \varnothing$ by Claim 6, and by Claim 2, $b_k(x) = \varnothing$ for all sufficiently large k, so $\mathbf{1}_{I_k(\theta_L)}(x) = 0$. If $x > \theta_L$, then by the construction of the subsequence, there exists $K(x)$ such that $b_k(\theta_L) < b_k(x)$ for all $k \geq K(x)$, implying $\mathbf{1}_{I_k(\theta_L)}(x) = 0$ for all $k \geq K(x)$. Therefore, $\mathbf{1}_{I_k(\theta_L)}(x)$ converges to $\mathbf{1}_{\{\theta_L\}}(x)$ for all x.

To prove the convergence of $I_k(\theta)$ to $I_\infty(\theta) - \{\theta_L\}$ for $\theta \in I_\infty(\theta_L) - \{\theta_L\}$, consider x such that $\mathbf{1}_{I_\infty(\theta_L)-\{\theta_L\}}(x) = 1$, thus $b_\infty(x) = b_\infty(\theta_L)$. Claim 8 implies that for all sufficiently large k, we have $b_k(x) = b_k(\theta)$, so $\mathbf{1}_{I_k(\theta)}(x) = 1$. Now consider x such that $\mathbf{1}_{I_\infty(\theta_L)-\{\theta_L\}}(x) = 0$. Then, either $x = \theta_L$ or $x \notin I_\infty(\theta_L)$. If $x = \theta_L$, because $b_k(x) < b_k(\theta)$ for all sufficiently large k, hence $\mathbf{1}_{I_k(\theta)}(x) = 0$. If $x \notin I_\infty(\theta_L)$, then $b_\infty(x) \neq b_\infty(\theta_L)$ and $b_\infty(\theta_L) = b_\infty(\theta)$, thus we have $b_\infty(x) \neq b_\infty(\theta)$. The pointwise convergence of b_k to b_∞ implies that $b_k(x) \neq b_k(\theta)$ for all sufficiently large k, hence $\mathbf{1}_{I_k(\theta)}(x) = 0$. Therefore, $I_k(\theta)$ converges to $I_\infty(\theta) - \{\theta_L\}$ for $\theta \in I_\infty(\theta_L) - \{\theta_L\}$.

To prove the convergence of $J_k(\theta_L)$ to $\{x \in \Theta \mid x < \theta_L\}$, consider $x < \theta_L$, thus $b_\infty(x) = \varnothing$ and $\mathbf{1}_{\{x<\theta_L\}}(x) = 1$. By Claim 2, $b_k(x) = \varnothing$ for all sufficiently large k, while $b_k(\theta_L)$ converges to θ_L. Therefore, $\mathbf{1}_{J_k(\theta_L)}(x) = 1$ for all sufficiently large k, and the convergence follows. Now consider $x \geq \theta_L$, thus $\mathbf{1}_{\{x<\theta_L\}}(x) = 0$. The monotonicity of b_k implies that $b_k(x) \geq b_k(\theta)$ for all k, hence we have $\mathbf{1}_{J_k(\theta_L)}(x) = 0$ for all k. Therefore, $J_k(\theta_L)$ converges to $\{x \in \Theta \mid x < \theta_L\}$.

To prove the convergence of $J_k(\theta)$ to $\{x \in \Theta \mid x \leq \theta_L\}$ for all $\theta \in I_\infty(\theta_L) - \{\theta_L\}$, consider $x \leq \theta_L$, thus $\mathbf{1}_{\{x\leq\theta_L\}}(x) = 1$. If $x < \theta_L$, then $b_\infty(x) = \varnothing$ and $b_k(x) = \varnothing$ for all sufficiently large k. Hence, $\mathbf{1}_{J_k(\theta)}(x)$ converges to 1. If $x = \theta_L$, because of how we construct the subsequence, we have $b_k(\theta_L) < b_k(\theta)$ for all sufficiently large k. Hence, $\mathbf{1}_{J_k(\theta)}(\theta_L) = 1$ and the desired convergence follows. □

Claim 10 implies that $(Q_k, T_k, s_k) \to (Q_\infty^-, T_\infty^-, s)$, on the selected subsequence, also in case (iii). Indeed, notice that $T_k(\theta) = b_k(\theta) Q_k(\theta) - s_k$ converges to $b_\infty(\theta) Q_\infty^-(\theta) - s$. Because $b_\infty(\theta) = b_\infty^-(\theta)$ for all $\theta \neq \theta_L$, we obtain the convergence of $T_k(\theta)$ to $b_\infty^-(\theta) Q_\infty^-(\theta) - s$,

which corresponds to the interim payment for type θ when all bidders follow the bidding strategy $b_\infty^-(\cdot)$ in the limit. Now, consider the type $\theta = \theta_L$. Because $b_\infty(\theta_L) = \theta_L$, the limit of $T_k(\theta)$ is $\theta_L Q_\infty^-(\theta) - s$. Because $b_\infty^-(\theta_L) = r^-$, type θ_L pays θ_L in the event of the winning. Therefore, the interim payment for type θ_L is $T_\infty^-(\theta_L) = \theta_L Q_\infty^-(\theta_L) - s$. Hence, we have pointwise convergence of T_k to T_∞^- on Θ.

From now let us denote the limit of (Q_k, T_k) be (Q, T). It remains to prove that (Q, T) maximizes the objective under the measure π.[34] Let $M(\mu)$ denote the set of feasible, symmetric, individually rational, and incentive compatible mechanisms when the commonly known type distribution is given by probability measure μ. The interim allocation and payment (Q, T) belongs to $M(\pi)$ because they are the interim allocation and payment in a symmetric equilibrium in a first-price auction.

To prove the optimality of (Q, T) we need to show that there is no other $\left(\tilde{Q}, \tilde{T}\right) \in M(\pi)$ such that $U_\pi\left(\left(\tilde{Q}, \tilde{T}\right)\right) > U_\pi((Q, T))$. We prove it by way of contradiction. Suppose that there exists $\left(\tilde{Q}, \tilde{T}\right) \in M(\pi)$ such that $U_\pi\left(\left(\tilde{Q}, \tilde{T}\right)\right) > U_\pi((Q, T))$. For each k, we define

$$\left(\tilde{Q}_k, \tilde{T}_k\right) = \left(c_k \tilde{Q}, c_k \tilde{T}\right)$$

where c_k is defined using the auxiliary concepts of γ_k, in turn defined in terms of L_k and R_k as follows:

$$c_k = \begin{cases} \min\left\{1, \frac{1}{\gamma_k}\right\} & \text{if } \gamma_k > 0, \\ 1 & \text{if } \gamma_k = 0, \end{cases}$$

$$\gamma_k = \sup\left\{\frac{L_k(t)}{R_k(t)} \mid t \in [0,1],\ R_k(t) > 0\right\},$$

$$L_k(t) = n \int_{\left\{\tilde{Q}(\theta) \geq t\right\}} \tilde{Q}(x)\, d\pi_k(x),$$

$$R_k(t) = 1 - \left(\pi_k\left(\left\{\tilde{Q}(\theta) < t\right\}\right)\right)^n.$$

Here, γ_k is well-defined because the set $\{t \in [0,1] : R_k(t) > 0\}$ is non-empty. By Lemma 13, $\left(\tilde{Q}_k, \tilde{T}_k\right)$ is feasible if and only if it satisfies Maskin-Riley-Matthews-Border inequalities. The scaling factor c_k ensures that the inequalities are satisfied and hence that $\left(\tilde{Q}_k, \tilde{T}_k\right)$ is feasible. Indeed, Maskin-Riley-Matthews-Border inequalities take the form $c_k L_k\left(\frac{t}{c_k}\right) \leq R_k\left(\frac{t}{c_k}\right)$ for all $t \in [0, c_k]$, and this inequality is guaranteed by the construction of γ_k and c_k. Furthermore, $\left(\tilde{Q}_k, \tilde{T}_k\right)$ is incentive compatible because $\left(\tilde{Q}, \tilde{T}\right)$ is incentive compatible and c_k is positive

[34] Recall that (Q_k, T_k) maximizes the objective U_{π_k} for each k.

and does not depend on bidder's type; $\left(\tilde{Q}_k, \tilde{T}_k\right)$ is individually rational because $\left(\tilde{Q}, \tilde{T}\right)$ is individually rational and c_k is positive. As $\left(\tilde{Q}_k, \tilde{T}_k\right)$ is also symmetric, we conclude that $\left(\tilde{Q}_k, \tilde{T}_k\right) \in M(\pi)$.

We now show that $\left(\tilde{Q}_k, \tilde{T}_k\right) \to \left(\tilde{Q}, \tilde{T}\right)$ pointwise, which reduces to showing that c_k converges to 1. To prove $c_k \to 1$, it is enough to show that $\limsup_k \gamma_k \leq 1$. By Lemma 16, $R_k(t)$ converges to $R(t) := 1 - \left(\pi\left(\left\{\tilde{Q}(\theta) < t\right\}\right)\right)^n$. In order to show that $L_k(t)$ converges to $L(t) := n\int_{\left\{\tilde{Q}(\theta)\geq t\right\}} \tilde{Q}(x)\, d\pi(x)$, we re-express L_k via the change of variables as

$$L_k(t) = n\int_{\left\{\tilde{Q}(\theta)\geq t\right\}} \tilde{Q}(x)\, d\pi_k(x) = n\int_t^1 \pi_k\left(\left\{\tilde{Q}(\theta) \geq s\right\}\right) ds$$

and analogously for $L(t)$. Lemma 16 gives $\pi_k\left(\left\{\tilde{Q}(\theta) \geq s\right\}\right) \to \pi\left(\left\{\tilde{Q}(\theta) \geq s\right\}\right)$, and we apply the dominated convergence theorem to conclude $L_k(t) = n\int_t^1 \pi_k\left(\left\{\tilde{Q}(\theta) \geq s\right\}\right) ds \to n\int_t^1 \pi\left(\left\{\tilde{Q}(\theta) \geq s\right\}\right) ds = L(t)$. Now, by way of contradiction (set within the overall by contradiction argument), suppose that $\limsup_k \gamma_k > 1 + \varepsilon$, for some $\varepsilon > 0$. We can take a subsequence (still indexed by k) such that $R_k(t_k) > 0$ and $\frac{L_k(t_k)}{R_k(t_k)} > 1 + \varepsilon$ for all k. Because $t_k \in [0,1]$, we can select a further subsequence such that t_k converges to some limit $t^* \in [0,1]$. If $R(t^*) > 0$, then the pointwise convergence of L_k and R_k leads to $\frac{L(t^*)}{R(t^*)} > 1 + \varepsilon$, which contradicts the feasibility of $\tilde{Q}$. If $R(t^*) = 0$, then $\pi_k\left(\left\{\tilde{Q}(\theta) \geq t_k\right\}\right)$, which we denote by p_k, converges to 0. Because $0 \leq \tilde{Q} \leq 1$, we have $L_k(t_k) \leq np_k$ and $R_k(t_k) = 1 - (1-p_k)^n \geq np_k - \frac{n(n-1)}{2}p_k^2$, where the inequality follows from the binomial theorem, $(1-p_k)^n \geq 1 - np_k + \frac{n(n-1)}{2}p_k^2$. Therefore, we have

$$\frac{L_k(t_k)}{R_k(t_k)} \leq \frac{np_k}{np_k - \frac{n(n-1)}{2}p_k^2} = \frac{1}{1 - \frac{n-1}{2}p_k} \to 1$$

as $p_k \to 0$, contradicting that $\frac{L_k(t_k)}{R_k(t_k)} > 1 + \varepsilon$ for all k. Thus, we have $\limsup_k \gamma_k \leq 1$, and hence $c_k \to 1$ and $\left(\tilde{Q}_k, \tilde{T}_k\right) \to \left(\tilde{Q}, \tilde{T}\right)$ pointwise.

By the continuity assumption on the objective, we have $U_{\pi_k}\left(\tilde{Q}_k, \tilde{T}_k\right) \to U_\pi\left(\tilde{Q}, \tilde{T}\right)$ and $U_{\pi_k}(Q_k, T_k) \to U_\pi(Q, T)$. Because we assumed $U_\pi\left(\tilde{Q}, \tilde{T}\right) > U_\pi(Q, T)$, it follows that $U_{\pi_k}\left(\tilde{Q}_k, \tilde{T}_k\right) > U_{\pi_k}(Q_k, T_k)$ for all sufficiently large k, contradicting that (Q_k, T_k) maximizes U_{π_k} for all k.

The final step of the proof is to show that there exists an optimal mechanism with $s = 0$ when the objective function is weakly increasing in revenue. Because the subsidy s

is given uniformly to all bidders as a lump sum, reducing s to zero for any given optimal pair (b, s) does not affect the interim allocation Q, but it increases the interim payment T. Furthermore, reducing subsidy s to zero does not violate individual rationality: if doing so were to violate individual rationality for some type θ, then we have $Q(\theta)\theta - (T(\theta) + s) < 0$, which implies $Q(\theta)\theta - T(\theta) < s$. In that case, bidders with type θ would have obtained a strictly higher payoff of s, by bidding $\varnothing$ that is choosing to stay out of the auction and merely collect the subsidy s. They would then achieve the payoff of $Q(\theta)\theta - T(\theta) = s$, and hence $b(\cdot)$ would not have been an equilibrium bidding strategy. This contradiction shows that reducing the subsidy to 0 preserves individual rationality of all bidder types. Therefore, there exists an optimal solution with $s = 0$. □

B.4 Proofs of Results in Section 5

Proof of Theorem 2

Because the subsidy is given universally, its amount does not affect the equilibrium. Hence, we focus on bidding strategies. If the bid space is finite, the existence of a pure-strategy symmetric equilibrium is implied by Reny (2011). Because $\mathcal{B}$ is compact, it can be approximated by finite subsets $\mathcal{B}_k$ so that for any $b \in \mathcal{B}$, $(b - \frac{1}{k}, b + \frac{1}{k}) \cap \mathcal{B}_k \neq \varnothing$ for any $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, there exists a symmetric and monotonic pure-strategy equilibrium of the first-price auction with bid space $\mathcal{B}_k$. Let us select one such equilibrium strategy and denote it $b_k(\cdot)$. Because $\mathcal{B}$ is compact, we can ensure that each $\mathcal{B}_k$ includes $\min(\mathcal{B})$ and $\max(\mathcal{B})$. We can further assume that $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$. We then have a sequence of monotonic functions $b_k(\cdot)$ which admits a point-wise convergent subsequence by Helly's selection theorem. For notational convenience, we can assume that $(b_k)_{k\in\mathbb{N}}$ itself is this convergent subsequence. Let us denote its limit by $b(\cdot)$. We claim that the limit constitutes a symmetric equilibrium in the first-price auction under $\mathcal{B}$. For each $k \in \mathbb{N}$, the probability of winning for type θ only depends on the $\pi(\{x|b_k(x) < b_k(\theta)\})$ and $\pi(\{x|b_k(x) = b_k(\theta)\})$, and it can be expressed as

$$\left(\pi(\{x|b_k(x) < b_k(\theta)\})\right)^{N-1} + \sum_{n=1}^{N-1} \frac{1}{n+1} \binom{N-1}{n} \left(\pi(\{x|b_k(x) < b_k(\theta)\})\right)^{N-1-n} \left(\pi(\{x|b_k(x) = b_k(\theta)\})\right)^{n}.$$

Let us denote $\{x|b_k(x) = b_k(\theta)\}$ by $I_k(\theta)$ and $\{x|b(x) = b(\theta)\}$ by $I(\theta)$. We use the following claims to prove that $b(\cdot)$ is in equilibrium.

Claim 1. $\limsup_k I_k(\theta) \subseteq I(\theta)$.

Proof of Claim 1. If $x \in \limsup_k I_k(\theta)$, then $b_k(x) = b_k(\theta)$ infinitely often. Because $b_k(\cdot)$

converges to $b(\cdot)$, $b(x) = b(\theta)$, thus $x \in I(\theta)$. $\square$

Claim 2. *If $\mu(I(\theta)) > 0$, then:*
either $I(\theta) \subseteq \liminf_k I_k(\theta)$,
or $b(x) = x$ for some $x \in I(\theta)$ and $I(\theta) - \{x\} \subseteq \liminf_k I_k(\theta)$.

Proof of Claim 2. Suppose that there exists $\theta' \in I(\theta) - \liminf_k I_k(\theta)$. Notice that $\theta' \neq \theta$ because $\liminf_k I_k(\theta)$ includes θ. For any K, there exists $k > K$ such that $b_k(\theta') \neq b_k(\theta)$. The monotonicity of b_k allows us to pick $x, y \in I(\theta)$ such that for any K, there exists $k > K$ such that $b_k(x) < b_k(y)$ and $\mu([x, y]) > 0$. Because $b_k(\cdot)$ is in a symmetric equilibrium, the following must hold:

$$[P_k(b_k(y)) - P_k(b_k(x))](x - b_k(x)) \leq P_k(b_k(y))(b_k(y) - b_k(x)),$$

where $P_k(b)$ represents the winning probability from placing a bid b assuming all the other bidders follow the bidding strategy $b_k(\cdot)$.

Notice that the right hand side of the inequality is arbitrarily close to 0, but the limit of left hand side is bounded below by $\frac{1}{N}\mu([x, y])^{N-1}(x - b(x))$ because the gain in the winning probability by changing a bid from $b_k(x)$ to $b_k(y)$ is at least $\frac{1}{N}\mu([x, y])^{N-1}$. Therefore, if $b(x) < x$, it leads to a contradiction because the type $x$would deviate to $b_k(y)$ for some k. $\square$

Claim 3. $\mu(I_k(\theta)) \to \mu(I(\theta))$.

Proof of Claim 3. For $I(\theta)$ such that $\mu(I(\theta)) > 0$, we established $\limsup_k I_k(\theta) \subseteq I(\theta) \subseteq \liminf_k I_k(\theta)$ with an exception that might arise for a single type $x \in I(\theta)$ such that $x = b(\theta)$, which can be ignored by the atom-less assumption. Therefore, we establish that $\lim_k \mu(I_k(\theta)) = \mu(I(\theta))$ by applying Fatou's lemma. For $I(\theta)$ such that $\mu(I(\theta)) = 0$, we still have $\limsup_k I_k(\theta) \subseteq I(\theta)$. Because $\mu(I_k(\theta)) \geq 0$ for all k and $\mu(I(\theta)) = 0$, $\lim_k \mu(I_k(\theta)) = 0$. $\square$

Claim 4. *For any $\theta \in \Theta$, $\mu(\{x|b_k(x) < b_k(\theta)\}) \to \mu(\{x|b(x) < b(\theta)\})$.*

Proof of Claim 4. We want to show that $\limsup_k \{x|b_k(x) < b_k(\theta)\} \subseteq \{x|b(x) < b(\theta)\}$ and $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ μ-almost everywhere. To show the first, suppose that there exists $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$ but $b(\theta') \geq b(\theta)$. Because $b_k(\cdot)$ converges to $b(\cdot)$, it must be $b(\theta') = b(\theta)$. By claim 2, either $\mu(I(\theta) = 0$ or $\theta' \in \liminf_k I_k(\theta)$. If $\mu(I(\theta)) = 0$, then $\limsup_k \{x|b_k(x) < b_k(\theta)\} \subseteq \{x|b(x) < b(\theta)\}$ μ-almost everywhere. If $\theta' \in \liminf_k I_k(\theta)$, then it leads to a contradiction to the assumption that $\theta' \in \limsup_k \{x|b_k(x) < b_k(\theta)\}$. It remains to show that $\{x|b(x) < b(\theta)\} \subseteq \liminf_k \{x|b_k(x) < b_k(\theta)\}$ μ-almost everywhere. If $b(\theta') < b(\theta)$, then there exists K such that $b_k(\theta') < b_k(\theta)$ for all $k \geq K$, thus $\theta' \in \liminf_k \{x|b_k(x) < b_k(\theta)\}$. Therefore, $\mu(\{x|b_k(x) < b_k(\theta)\}) \to \mu(\{x|b(x) < b(\theta)\})$. $\square$

Claim 5. *If $b \in \mathcal{B} - b(\Theta)$ then $\mu(\{x|b_k(x) < b\}) \to \mu(\{x|b(x) < b\})$.*

Proof of Claim 5. Pick an arbitrary $\theta \in \limsup_k \{x|b_k(x) < b\}$. Then, $b_k(\theta) < b$ infinitely often and $b_k(\theta) \to b(\theta) \neq b$. This implies that $b(\theta) < b$, thus, $\limsup_k \{x|b_k(x) < b\} \subseteq \{x|b(x) < b\}$. To see $\{x|b(x) < b\} \subseteq \liminf_k \{x|b_k(x) < b\}$, pick an arbitrary θ such that $b(\theta) < b$. Because $b_k(\theta)$ converges to $b(\theta)$, for any $\varepsilon > 0$, there exists K such that $b_k(\theta) < b - \varepsilon$ for all $k > K$. Therefore, $\theta \in \liminf_k \{x|b_k(x) < b\}$. Applying Fatou's lemma completes the proof. $\square$

Claim 3 and 4 establish that there is no profitable deviation to a bid $b \in b(\Theta)$. Consider a deviation to a bid $b \in \mathcal{B} - b(\Theta)$. Because $P_k(b)$ converges to $P(b)$ by Claim 5, if b were a profitable deviation in the limit and belonged to some $\mathcal{B}_\ell$ then the monotonicity of $\mathcal{B}_k$ ensures that it would also be a profitable deviation at some step $k \geq \ell$. It remains to consider the case in which $b \notin \cup_{k\in\mathbb{N}}\mathcal{B}_k$. If b were a profitable deviation and $\theta - b \leq 0$, so were deviating to $b - \varepsilon$ for any $\varepsilon \geq 0$. Because $\cup_{k\in\mathbb{N}}\mathcal{B}_k$ is dense in $\mathcal{B}$, there exists $\varepsilon \geq 0$ such that $b - \varepsilon \in \cup_{k\in\mathbb{N}}\mathcal{B}_k$, and there would be a profitable deviation to $b - \varepsilon$ at some step k. If b were a profitable deviation and $\theta - b > 0$, then there exists small $\varepsilon > 0$ such that deviating to $b + \varepsilon'$ is also a profitable deviation for any $\varepsilon' \in [0, \varepsilon]$. Because $\cup_{k\in\mathbb{N}}\mathcal{B}_k$ is dense in $\mathcal{B}$,there exists $b' \in \cup_{k\in\mathbb{N}}\mathcal{B}_k$ such that $b' \in [b, b + \varepsilon]$ and there would be a profitable deviation to b' at some step k.

Proof of Theorem 3

Proof. Given a symmetric profile of bidding strategies $b(\cdot)$, regardless of the restrictions on the bid space, we can express the winning probability of a bidder by placing a bid $b \in \mathbb{R}$ as

$$P(b) = (\mu(\{\theta|b > b(\theta)\}))^{N-1} + \sum_{n=1}^{N-1} \frac{1}{n+1} \binom{N-1}{n} (\mu(\{\theta|b > b(\theta)\}))^{N-1-n} (\mu(\{\theta|b = b(\theta)\}))^{n},$$

where we use two conventions: (i) $b(\theta) = -\infty$ if type θ drops and does not submit a bid, and (ii) we resolve $(\mu(\{\theta|b > b(\theta)\}))^{0}$ to equal 1 when $\mu(\{\theta|b > b(\theta)\}) = 0$. The win probability $P(\cdot)$ is monotone in b and, for $b > \inf(\mathcal{B})$, it is discontinuous at b if and only if $\mu(\{\theta|b = b(\theta)\}) > 0$. Suppose that $b(\cdot)$ is in equilibrium under $\mathcal{B}$, but not under $cl(\mathcal{B})$. Then, there exists a type $\theta \in \Theta$ and $b \in cl(\mathcal{B}) - \mathcal{B}$ such that $P(b)(\theta - b) > P(b(\theta))(\theta - b(\theta))$. Since $b \in cl(\mathcal{B}) - \mathcal{B}$, there exists a sequence $(b_k)_{k\in\mathbb{N}}$ that converges to b with each $b_k \in \mathcal{B}$ for all k. Consider a case where $b > \inf(\mathcal{B})$. Because $\mu(\{\theta|b = b(\theta)\}) = 0$ as b is not a valid bid under $\mathcal{B}$, thus, $P(\cdot)$ is continuous at b, there exists k such that $P(b_k)(\theta - b_k) > P(b(\theta))(\theta - b(\theta))$, which is a contradiction because $b(\cdot)$ is an equilibrium under $\mathcal{B}$. In the remaining case, $b = \inf(\mathcal{B})$; notice that $b_k \geq b$ for all k in the convergent sequence (b_k) with $b_k \in \mathcal{B}$. Thus, $P(b_k) \geq P(b)$ for all k, and b_k can be arbitrarily close to b, thus,

$P(b_k)(\theta - b_k) > P(b(\theta))\,(\theta - b(\theta))$ for some k, which leads to the same contradiction. $\square$

Lemma Lemma 18 and its Proof

Lemma 18. *Suppose bidders type are drawn from an atomless distribution with interval support Θ. Let $\mathcal{B}$ and $\hat{\mathcal{B}}$ be two bid spaces, let $b(\cdot)$ be a symmetric equilibrium bid strategy for bid space $\mathcal{B}$ and let $\hat{b}(\cdot)$ be a symmetric equilibrium bid strategy for bid space $\hat{\mathcal{B}}$. Let $[\theta_1, \theta_2] \subseteq \Theta$ and suppose that (i) $b(\cdot)$ is strictly increasing on $[\theta_1, \theta_2]$, and (ii) $\hat{b}(\cdot)$ is constant on $[\theta_1, \theta_2]$. The following then holds:*

1. If F is convex on $[\theta_1, \theta_2]$ and no type below θ_1 submits the bid $\hat{b}(\theta_1) = \hat{b}(\theta_2)$, then the difference in expected utility, $u(\theta) - u(\theta_1)$, under the equilibrium with $b(\cdot)$ is less than or equal to the difference, $\hat{u}(\theta) - \hat{u}(\theta_1)$, under the equilibrium with $\hat{b}(\cdot)$ for all $\theta \in [\theta_1, \theta_2]$

2. If F is concave on $[\theta_1, \theta_2]$ and no type above θ_2 submits the bid $\hat{b}(\theta_1) = \hat{b}(\theta_2)$, then the difference in expected utility, $u(\theta_2) - u(\theta_1)$, under the equilibrium with $b(\cdot)$ is greater than or equal to the difference, $\hat{u}(\theta_2) - \hat{u}(\theta_1)$, under the equilibrium with $\hat{b}(\cdot)$.

Proof. Let us define an auxiliary function $\phi(x, y) = \frac{x^n - y^n}{x - y}$ for $x > y \geq 0$. For any integer $n > 1$, the function ϕ is strictly increasing in x and y because $t \mapsto t^n$ is strictly convex and $\phi(x, y)$ is the secant slope of t^n between x and y. Let F be the cumulative distribution function of a bidder's type (recall that bidders draw their types iid). Because $b(\cdot)$ is strictly increasing on $[\theta_1, \theta_2]$, the standard envelope calculation gives

$$u(\theta) - u(\theta_1) = \int_{\theta_1}^{\theta} Q(x)\, dx = \int_{\theta_1}^{\theta} F(x)^{n-1}\, dx.$$

Under $\hat{b}$, the set of types submitting the pooled bid $\hat{b}(\theta_1) = \hat{b}(\theta_2)$ is the interval $[\theta^-, \theta^+]$ for some $\theta^+ \geq \theta_2$ and $\theta^- \leq \theta_1$.[35] For any type in $[\theta^-, \theta^+]$, the winning probability, denoted by $\hat{Q}$, is expressed as:

$$\hat{Q} = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \left[F(\theta^+) - F(\theta^-)\right]^{k-1} \left[F(\theta^-)\right]^{n-k}.$$

[35] The set of types submitting the pooled bid is an interval, but may not be a closed interval. For any different types of interval, the same proof applies.

Using $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$, we can simplify $\hat{Q}$ as follows:

$$\begin{aligned}\hat{Q} &= \frac{1}{n\left[F(\theta^+) - F(\theta_1)\right]} \sum_{k=1}^{n} \binom{n}{k} \left[F(\theta^+) - F(\theta^-)\right]^k \left[F(\theta^-)\right]^{n-k} \\ &= \frac{F(\theta^+)^n - F(\theta^-)^n}{n\left[F(\theta^+) - F(\theta^-)\right]} \\ &= \frac{1}{F(\theta^+) - F(\theta^-)} \int_{\theta^-}^{\theta^+} F(x)^{n-1} dF(x),\end{aligned}$$

where the final identity is from Lebesgue-Stieltjes change of variables. Because the winning probability is constant on $[\theta^-, \theta^+]$, for every $x, y \in [\theta^-, \theta^+]$, we have:

$$\hat{u}(x) - \hat{u}(y) = \hat{Q} \cdot (x - y) = \frac{(\theta - \theta^-)}{F(\theta^+) - F(\theta^-)} \int_{\theta^-}^{\theta^+} F(x)^{n-1} dF(x).$$

Now, we consider the first part of the lemma, where F is convex on on $[\theta_1, \theta_2]$ and $\theta^- = \theta_1$ by assumption. Because $F(x)^{n-1}$ is increasing $F(\cdot)$ is convex on $[\theta_1, \theta_2]$, the following inequality holds for all $\theta \in [\theta_1, \theta_2]$ by Chebyshev integral inequality:

$$\frac{1}{F(\theta) - F(\theta_1)} \int_{\theta_1}^{\theta} F(x)^{n-1} dF(x) \geq \frac{1}{\theta - \theta_1} \int_{\theta_1}^{\theta} F(x)^{n-1} dx.$$

Because $\frac{1}{F(\theta)-F(\theta_1)} \int_{\theta_1}^{\theta} F(x)^{n-1} dF(x) = \frac{F(\theta)^n - F(\theta_1)^n}{n[F(\theta)-F(\theta_1)]} = \phi\left(F(\theta), F(\theta_1)\right)$ and $\phi(x, y)$ is strictly increasing in x, we have for all $\theta \in [\theta_1, \theta_2]$,

$$\frac{1}{F(\theta^+) - F(\theta_1)} \int_{\theta_1}^{\theta^+} F(x)^{n-1} dF(x) \geq \frac{1}{\theta - \theta_1} \int_{\theta_1}^{\theta} F(x)^{n-1} dx,$$

which is equivalent to

$$\frac{\theta - \theta_1}{F(\theta^+) - F(\theta_1)} \int_{\theta_1}^{\theta^+} F(x)^{n-1} dF(x) \geq \int_{\theta_1}^{\theta} F(x)^{n-1} dx.$$

Because $\hat{u}(\theta) - \hat{u}(\theta_1) = \frac{\theta-\theta_1}{F(\theta^+)-F(\theta_1)} \int_{\theta_1}^{\theta^+} F(x)^{n-1} dF(x)$ and $u(\theta) - u(\theta_1) = \int_{\theta_1}^{\theta} F(x)^{n-1} dx$, we have the desired inequality $\hat{u}(\theta) - \hat{u}(\theta_1) \geq u(\theta) - u(\theta_1)$ for all $\theta \in [\theta_1, \theta_2]$.

Now, we consider the second part of the lemma, where F is concave on $[\theta_1, \theta_2]$ and $\theta^+ = \theta_2$. Because F is concave, Chebyshev integral inequality gives for all $\theta \in [\theta_1, \theta_2]$

$$\frac{1}{F(\theta) - F(\theta_1)} \int_{\theta_1}^{\theta} F(x)^{n-1} dF(x) \leq \frac{1}{\theta - \theta_1} \int_{\theta_1}^{\theta} F(x)^{n-1} dx.$$

Especially, for $\theta = \theta_2$,we have

$$\frac{1}{F(\theta_2) - F(\theta_1)} \int_{\theta_1}^{\theta_2} F(x)^{n-1} dF(x) \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} F(x)^{n-1} dx.$$

Because $\frac{1}{F(\theta_2)-F(\theta_1)} \int_{\theta_1}^{\theta_2} F(x)^{n-1} dF(x) = \frac{F(\theta_2)^n - F(\theta_1)^n}{n[F(\theta_2)-F(\theta_1)]} = \phi\left(F(\theta_2), F(\theta_1)\right)$ and $F(\theta_1) \geq F(\theta^-)$,

$$\frac{1}{F(\theta_2) - F(\theta^-)} \int_{\theta^-}^{\theta_2} F(x)^{n-1} dF(x) \leq \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} F(x)^{n-1} dx,$$

which is equivalent to

$$\frac{\theta_2 - \theta_1}{F(\theta_2) - F(\theta^-)} \int_{\theta^-}^{\theta_2} F(x)^{n-1} dF(x) \leq \int_{\theta_1}^{\theta_2} F(x)^{n-1} dx.$$

Because $\frac{\theta_2-\theta_1}{F(\theta_2)-F(\theta^-)} \int_{\theta^-}^{\theta_2} F(x)^{n-1} dF(x) = \hat{u}(\theta_2) - \hat{u}(\theta_1)$ and $\int_{\theta_1}^{\theta_2} F(x)^{n-1} dx = u(\theta_2) - u(\theta_1)$, we have the desired inequality $\hat{u}\left(\theta_2\right) - \hat{u}\left(\theta_1\right) \leq u(\theta_2) - u(\theta_1)$. □

Proof of Theorem 4

To prove the theorem, we first show that any mixed strategy equilibrium is equivalent to a pure strategy equilibrium, following the similar steps as in Lemma 3.9 and Lemma 3.10 of Chawla and Hartline (2018). Although their model does not include a bid cap, the same reasoning applies to rule out mixed strategy equilibria in our setting. For expositional simplicity, we let $b_i(\theta)$ denote the support of the mixed strategy employed by bidder i of type θ. If $b_i\left(\theta\right)$ is a singleton, then bidder i of type θ employs a pure strategy. For the purpose of our proof, it suffices to focus on the support of the mixing rather than the precise distribution.

Claim 1. If $b_i\left(\cdot\right)$ is a best response to other bidders' strategy and $\theta < \theta'$, then for any $b \in b_i\left(\theta\right)$ and $b' \in b_i\left(\theta'\right)$, we have $b \leq b'$.

Proof. By way of contradiction, suppose that $\theta < \theta'$ but there are $b \in b_i\left(\theta\right)$ and $b' \in b_i\left(\theta'\right)$ with $b > b'$. We express the payoff of bidder i with type θ by bidding b and b' as

$$G_i\left(b\right)\left(\theta' - b\right) = G_i\left(b\right)\left(\theta - b\right) + \left(\theta' - \theta\right) G_i\left(b\right)$$

and

$$G_i\left(b'\right)\left(\theta' - b'\right) = G_i\left(b'\right)\left(\theta - b'\right) + \left(\theta' - \theta\right) G_i\left(b'\right),$$

where $G_i\left(x\right)$ denotes the probability of winning of bidder i by bidding x. Because b' is an

optimal bid for bidder i with type θ', we have

$$G_i(b)(\theta - b) + (\theta' - \theta) G_i(b) \leq G_i(b')(\theta - b') + (\theta' - \theta) G_i(b'),$$

which can be rearranged to

$$G_i(b)(\theta - b) - G_i(b')(\theta - b') \leq (\theta' - \theta)(G_i(b') - G_i(b)).$$

Because b is an optimal bid for bidder i with type θ, the left-hand side of the inequality is non-negative. Hence, we have $G_i(b') \geq G_i(b)$ as $\theta' > \theta$. Because we assume that $b > b'$, it must be $G_i(b') = G_i(b)$, hence bidding b' and bidding b give the same winning probability. This contradicts that b' is an optimal bid for bidder i with type θ' because bidding b would have given the same winning probability with a smaller payment. □

Claim 2. Let $(b_i(\cdot))_{i \in N}$ be a profile of bidding strategy in an equilibrium. Then, there exists an equilibrium formed by $(b'_i(\cdot))_{i \in N}$, where $b'_i = b_i$ almost everywhere and b'_i is a pure strategy for all i.

Proof. For each θ and i, set $b'_i(\theta) = \min\{b \in b_i(\theta)\}$, which is a pure strategy. Because $(b_i(\cdot))_{\in N}$ is in equilibrium, $b'_i(\cdot)$ is a best response against $(b_j(\cdot))_{j \in N - \{i\}}$ for all $i \in N$. We show that $b'_i(\cdot)$ is a best response against $\left(b'_j(\cdot)\right)_{j \in N - \{i\}}$ for all i, which implies that $(b'_i(\cdot))_{i \in N}$ forms an equilibrium. Let X_i denote the bid distribution over bids induced by $b_i(\cdot)$ and the type distribution π; similarly, let X'_i denote the distribution over bids induced by $b'_i(\cdot)$ and π. For $b \in [\underline{b}, \bar{b}]$, define

$$\theta(b) = \sup\{v \in \Theta \mid \exists b' \in b_i(v) \text{ with } b' \leq b\}.$$

(If the set is empty, take $\theta(b) = \underline{\theta}$, so that $F(\theta(b)) = 0$. If $b = \bar{b}$, then $\theta(b) = \bar{\theta}$ and $F\left(\theta\left(\bar{b}\right)\right) = 1$.) For all $x < \theta(b)$ and $b' \in b_i(x)$, by Claim 1 and the definition of $\theta(b)$, we have $b' \leq b$. Hence, we have

$$Pr(X_i \leq b) \geq \pi(\{x \in \Theta \mid x < \theta(b)\}).$$

Because $b'_i(x) = \min b_i(x)$, we have $b'_i(x) \leq b$ for all $x < \theta(b)$; hence

$$Pr(X'_i \leq b) \geq \pi(\{x \in \Theta \mid x < \theta(b)\}).$$

For all $x > \theta(b)$, by the definition of $\theta(b)$ there is no $b' \in b_i(x)$ with $b' \leq b$; hence

$$Pr(X_i \leq b) \leq \pi(\{x \in \Theta \mid x \leq \theta(b)\}).$$

Because $b_i'(x) = \min b_i(x)$ and $b' > b$ for all $b' \in b_i(x)$, we have $b_i'(x) > b$; hence

$$Pr(X_i' \leq b) \leq \pi(\{x \in \Theta \mid x \leq \theta(b)\}).$$

Because π is atomless,

$$\pi(\{x \in \Theta \mid x < \theta(b)\}) = \pi(\{x \in \Theta \mid x \leq \theta(b)\}) = F(\theta(b))$$

hence we have, for all i and b,

$$Pr(X_i' \leq b) = Pr(X_i \leq b).$$

Therefore, $b_i(\cdot)$ and $b_i'(\cdot)$ generate the same distribution of bids, hence $b_i'(\cdot)$ is a best response against $\left(b_j'(\cdot)\right)_{j \in N - \{i\}}$, so $(b_i'(\cdot))_{i \in N}$ forms an equilibrium. Finally, Claim 1 implies that b_i' and b_i can be different at most countable number of types, where the bidding functions jump. Because we assume that the type distribution is atomless, we have $b_i'(\theta) = b_i(\theta)$ almost everywhere for all i. □

Now, we restrict our attention to equilibria in pure strategies by starting with establishing the following claims to show that all bid functions in any pure strategy equilibrium share a common cutoff type for bidding $\bar{b}$.

Claim 3. In any pure strategy equilibrium, there is no tie-breaking at any bid level other than $\bar{b}$.

Proof. Assume for contradiction that some bidder i's bid function $b_i(\cdot)$ is constant on an interval and we denote the value of $b_i(\cdot)$ on the interval by b_i. In any equilibrium involving $b_i(\cdot)$, no other bidder j can choose b_i because bidding $b_i + \varepsilon$ would yield a strictly higher payoff for a small $\varepsilon > 0$. If no other bidders select b_i, the original $b_i(\cdot)$ cannot be a best response for bidder i, since reducing it to $b_i - \varepsilon$ would improve bidder i's payoff. □

With the following claim we show that bid function for each bidder jumps at the exactly the same type in any pure strategy-equilibrium. Let us use θ_i to denote the lowest type at which bidder i bids $\bar{b}$ and b_i to denote the the left-sided of $b_i(x)$ as x approaches θ_i for each i.

Claim 4. In any pure strategy equilibrium, $\theta_i = \theta_j$ for all i and j.

Proof. We can encode the distribution of the maximum bid among the other $n-2$ bidder s into a single distribution function G, such that $G(b)$ represents the probability the highest bid among $n-2$ bidders is strictly less than b. By Claim 1, $G(\cdot)$ has no atoms except possibly at $\bar{b}$. Let $Q_i(b)$ and $Q_j(b)$ denote the winning probabilities for bidder i and j, respectively. Let $F_i(b)$ and $F_j(b)$ represent the probabilities that they bid strictly less than b. Then, for $b < \bar{b}$, $Q_i(b)$ equals $G(b)F_j(b)$. To address the potential tie-breaking at $\bar{b}$, we define $P(k|b)$ as the probability that exactly $k-1$ bidders among the other $n-2$ select b and $n-k-1$ choose lower bids. Thus, $Q_i(\bar{b})$ is given by:

$$Q_i(\bar{b}) = \sum \frac{1}{k} P(k|b) F_j(\bar{b}) + \sum_{k=1}^{n-1} \frac{1}{k+1} P(k|b)(1 - F_j(\bar{b})).$$

Assume for contradiction that $\theta_i < \theta_j$ for some i and j in equilibrium. We analyze two cases separately: $b_i > b_j$ and $b_i \leq b_j$. Let us consider the case $b_i > b_j$ (see Figure 3). We show that if bidder i with type θ_i weakly prefers b_i to b_j, which should be the case if bidder i plays the equilibrium, then bidder j with type θ_j strictly prefers b_i to b_j, which is a contradiction. We can express the expected payoff of bidder i with type θ_i when following the equilibrium as follows:

$$u_i(\theta_i, b_i) = Q_i(b_i)(\theta_i - b_i) = G(b_i)F_j(b_i)\left(\theta_i - b_i\right).$$

By noticing that bidder j does not play any bid between b_i and b_j:

$$u_i(\theta_i, b_i) = G(b_i)F_j(b_j)\left(\theta_i - b_i\right).$$

If bidder i deviates to b_j:

$$u_i(\theta_i, b_i) = Q_i(b_j)(\theta_i - b_j) = G(b_j)F_j(b_j)\left(\theta_i - b_j\right).$$

Since bidder i with type θ_i prefers b_i:

$$G(b_i)F_j(b_i)\left(\theta_i - b_i\right) \geq G(b_j)F_j(b_j)\left(\theta_i - b_j\right),$$

thus

$$G(b_i)(\theta_i - b_i) \geq G(b_j)(\theta_i - b_j).$$

Because $\theta_j > \theta_i$ and $G(b_i) \geq G(b_j)$, $G(b_i)(\theta_j - b_i) > G(b_j)(\theta_j - b_j)$. Since $F_i(b_i) > F_i(b_j)$, multiplying $F_i(b_i)$ on the left hand side and $F_i(b_j)$ on the right hand side leads to:

$$G(b_i)F_i(b_i)(\theta_j - b_i) > G(b_j)F_i(b_j)(\theta_j - b_j).$$

Notice that the left hand side is $u_j(\theta_j, b_i)$ and the right hand side is $u_j(\theta_j, b_j)$, thus $u_j(\theta_j, b_i) > u_j(\theta_j, b_j)$, which is a contradiction. It remains to analyze the case $b_i \leq b_j$ (see Figure 4). For this case, we use the indifferent condition of bidder j at the cutoff type θ_j to show that bidder i with type θ_j would deviate in the equilibrium. The indifferent condition for bidder j at the cutoff type θ_j can be written as:

$$\left[\sum \frac{1}{k} P(k|b) F_i(\bar{b}) + \sum_{k=1}^{n-1} \frac{1}{k+1} P(k|b)(1 - F_i(\bar{b}))\right] \left(\theta_j - \bar{b}\right) = G(b_j)F_i(b_j)\left(\theta_j - b_j\right).$$

Notice that $F_i(b_j) = F_i(\bar{b})$ because bidder i does not choose a bid in $\left(b_i, \bar{b}\right)$, which includes the interval $\left(b_j, \bar{b}\right)$ in the equilibrium. Dividing both sides of the equality by $F_i(\bar{b})$ leads to:

$$\left[\sum \frac{1}{k} P(k|b) + \sum_{k=1}^{n-1} \frac{1}{k+1} P(k|b) \left(\frac{1 - F_i(\bar{b})}{F_i(\bar{b})}\right)\right] \left(\theta_j - \bar{b}\right) = G(b_j)\left(\theta_j - b_j\right).$$

Because we assume that $\theta_i < \theta_j$, $F_i(\bar{b}) < F_j(\bar{b})$, thus $\left(\frac{1-F_j(\bar{b})}{F_j(\bar{b})}\right) < \left(\frac{1-F_i(\bar{b})}{F_i(\bar{b})}\right)$. By substituting $\left(\frac{1-F_i(\bar{b})}{F_i(\bar{b})}\right)$ with $\left(\frac{1-F_j(\bar{b})}{F_j(\bar{b})}\right)$:

$$\left[\sum \frac{1}{k} P(k|b) + \sum_{k=1}^{n-1} \frac{1}{k+1} P(k|b) \left(\frac{1 - F_j(\bar{b})}{F_j(\bar{b})}\right)\right] \left(\theta_j - \bar{b}\right) < G(b_j)\left(\theta_j - b_j\right),$$

and

$$\left[\sum \frac{1}{k} P(k|b) \left(F_j(\bar{b})\right) + \sum_{k=1}^{n-1} \frac{1}{k+1} P(k|b) \left(1 - F_j(\bar{b})\right)\right] \left(\theta_j - \bar{b}\right) < G(b_j)F_j(\bar{b})\left(\theta_j - b_j\right).$$

The left hand side of inequality is $u_i(\theta_j, \bar{b})$ and the right hand side is $u_i(\theta_j, b_j)$ by noticing that $F_j(\bar{b}) = F_j(b_j)$. This is a contradiction because $u_i(\theta_j, \bar{b})$ is the equilibrium payoff and $u_i(\theta_j, b_j)$ is a profitable deviation. □

Proof. We now turn to the proof of Theorem 4. By Claim 1 and 2, there exists a common cutoff type that the bidding strategy jumps to $\bar{b}$. Let us denote the cutoff type by $\hat{\theta}$ and consider a first-price auction with the type space $\left[\underline{\theta}, \hat{\theta}\right]$ with the type distribution defined by the conditional distribution on $\left[\underline{\theta}, \hat{\theta}\right]$. Suppose that a profile of bid functions $(b_i(\cdot))_{i \in N}$

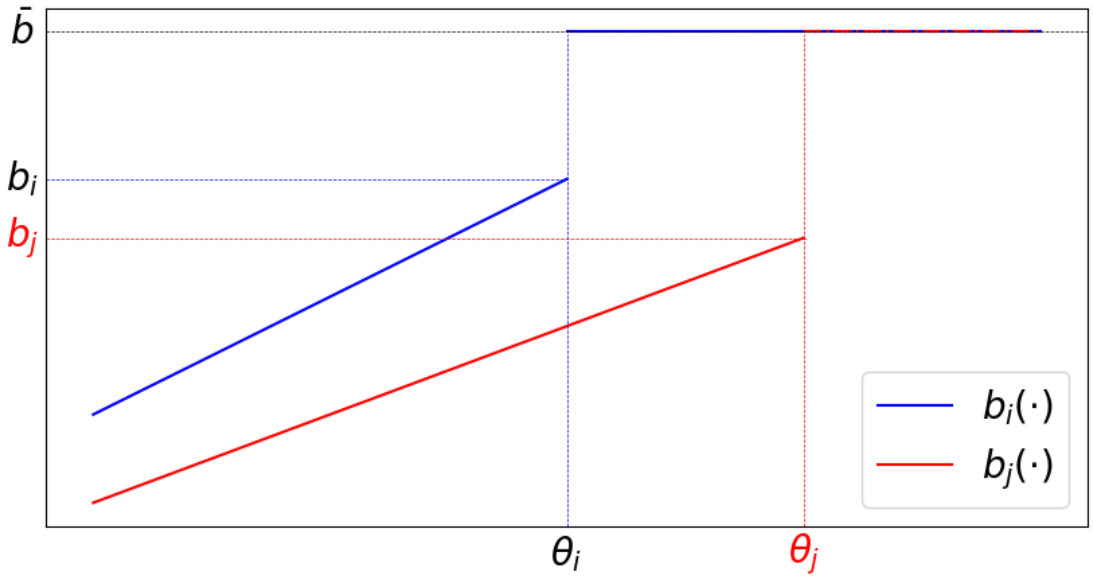


Figure 3: If bidder i with type θ_i prefers b_i to b_j, bidder j with type θ_j would strictly prefer b_j to b_i.

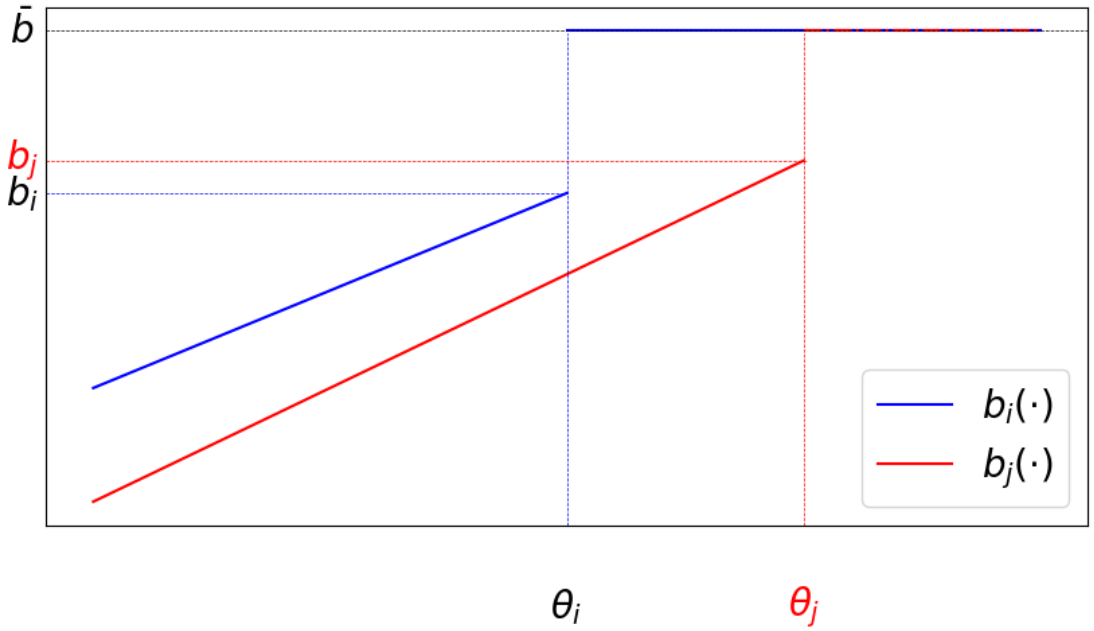


Figure 4: If bidder j with type θ_j is indifferent between $\bar{b}$ and b_j, bidder i with type θ_j would strictly prefer b_j to $\bar{b}$.

is in an equilibrium in the original game with type space $\left[\underline{\theta}, \overline{\theta}\right]$. Consider a profile of bid functions, where each bid function is defined by:

$$b_i\big|_{\left[\underline{\theta},\hat{\theta}\right]}(x) = \begin{cases} b_i(x) & \text{if } x \in \left[\underline{\theta}, \hat{\theta}\right) \\ \lim_{\varepsilon \to 0^+} b_i(x-\varepsilon) & \text{if } x = \hat{\theta}. \end{cases},$$

Notice that $\left(b_i\big|_{\left[\underline{\theta},\hat{\theta}\right]}(\cdot)\right)_{i \in N}$ is also an equilibrium of the first-price auction with the type space $\left[\underline{\theta}, \hat{\theta}\right]$ and the bid space $[\underline{b}, \infty)$, which is a bid space without a restriction other than a reserve price. For a symmetric environment where only a reserve price is allowed, Chawla and Hartline (2018) show that the equilibrium is unique and it is symmetric.[36] Because the cutoff type is common among bidders in equilibrium, we can rule out any asymmetric equilibria in the original auction with type space $\left[\underline{\theta}, \overline{\theta}\right]$. The characterization of the equilibrium bidding strategy on the left side of the cutoff type $\hat{\theta}$ follows the result from Monteiro (2009)

$$b(\theta) = \begin{cases} \varnothing & \text{if } \theta < \underline{b}, \\ \theta - \frac{\int_{\underline{b}}^{\theta} F(x)^{n-1} dx}{F(\theta)^{n-1}} & \text{if } \underline{b} \leq \theta < \hat{\theta}. \end{cases}$$

If $\bar{\theta} - \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1} dx > \bar{b}$, then the bidding cap $\bar{b}$ is not binding, hence we have the equilibrium

$$b(\theta) = \begin{cases} \varnothing & \text{if } \theta < \underline{b}, \\ \theta - \frac{\int_{\underline{b}}^{\theta} F(x)^{n-1} dx}{F(\theta)^{n-1}} & \text{if } \theta \geq \underline{b}, \end{cases}$$

proving the second part of the theorem. To prove the first part of the theorem, where the bidding cap is binding, notice that the cutoff type $\hat{\theta}$ is determined by the indifferent condition:

$$\left[\hat{\theta} - \left(\hat{\theta} - \frac{\int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx}{F(\hat{\theta})^{n-1}}\right)\right] F\left(\hat{\theta}\right)^{n-1} = \left[\hat{\theta} - \bar{b}\right] Pr(\bar{b}),$$

where $Pr(\bar{b})$ is the probability of winning by bidding $\bar{b}$ when other bidders with type θ bid $\bar{b}$ if and only if $\theta \geq \hat{\theta}$. Accounting for tie-breaking,

$$Pr(\bar{b}) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \left[1 - F(\hat{\theta})\right]^{k-1} \left[F(\hat{\theta})\right]^{n-k}.$$

[36] The uniqueness was established under stronger assumptions (Lebesgue absolute continuity of F) by e.g. Maskin and Riley (2003) and Lebrun (2006).

Using $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$, we can simplify $Pr(\bar{b})$ as follows:

$$\begin{aligned} Pr(\bar{b}) &= \frac{1}{n\left[1-F(\hat{\theta})\right]} \sum_{k=1}^{n} \binom{n}{k} \left[1-F(\hat{\theta})\right]^{k} \left[F(\hat{\theta})\right]^{n-k} \\ &= \frac{1-F(\hat{\theta})^{n}}{n\left[1-F(\hat{\theta})\right]} \\ &= \frac{1}{1-F(\hat{\theta})} \int_{\hat{\theta}}^{\bar{\theta}} F(x)^{n-1} dF(x), \end{aligned}$$

where the final identity is from Lebesgue-Stieltjes change of variables. By substituting $\frac{1}{1-F(\hat{\theta})} \int_{\hat{\theta}}^{\bar{\theta}} F(x)^{n-1} dF(x)$ into the indifferent condition, we have

$$\left[\hat{\theta} - \bar{b}\right] \int_{\hat{\theta}}^{\bar{\theta}} F(x)^{n-1} dF(x) = \left[1-F(\hat{\theta})\right] \int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx.$$

The next step is to show that the indifferent condition gives a unique solution in $[\underline{b}, \bar{\theta})$ if Using $\frac{1}{1-F(\hat{\theta})} \int_{\hat{\theta}}^{\bar{\theta}} F(x)^{n-1} dF(x) = \frac{1-F(\hat{\theta})^{n}}{n\left[1-F(\hat{\theta})\right]}$, we can express the indifference condition as

$$(\hat{\theta} - \bar{b}) \frac{1-F(\hat{\theta})^{n}}{\left[1-F(\hat{\theta})\right]} = n \int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx.$$

Because $\frac{1-F(\hat{\theta})^{n}}{\left[1-F(\hat{\theta})\right]} = \sum_{k=1}^{n-1} F(\hat{\theta})^{k}$, we have

$$(\hat{\theta} - \bar{b}) \sum_{k=1}^{n-1} F(\hat{\theta})^{k} = n \int_{\underline{b}}^{\hat{\theta}} F(x)^{n-1} dx.$$

Define $L(\theta) = (\theta - \bar{b}) \sum_{k=0}^{n-1} F(\theta)^{k}$, $R(\theta) = n \int_{\underline{b}}^{\theta} F(x)^{n-1} dx$, and $D(\theta) = L(\theta) - R(\theta)$. We show that $D(\theta) = 0$ has a unique solution in $[\underline{b}, \bar{\theta})$ by establishing a series of claims.

Claim 1. $D(\theta)$ is strictly increasing on $[\underline{b}, \bar{b}]$.

Proof of Claim 1. Fix $s > t \geq \bar{b}$. Because F is nondecreasing, we have

$$L(s) - L(t) = (s-t) \sum_{k=0}^{n-1} F(s)^{k} + (t - \bar{b}) \Big(\sum_{k=0}^{n-1} F(s)^{k} - \sum_{k=0}^{n-1} F(t)^{k} \Big) \geq (s-t) \sum_{k=0}^{n-1} F(s)^{k}.$$

Also,

$$R(s) - R(t) = n \int_{t}^{s} F(x)^{n-1} dx \leq n(s-t) F(s)^{n-1}.$$

Therefore

$$D(s) - D(t) \geq (s-t)\Big(\sum_{k=0}^{n-1} F(s)^k - nF(s)^{n-1}\Big).$$

For any $y \in [0,1)$, because $y^{n-1} \leq y^k$ for $k = 0, \ldots, n-1$ with strict inequality for some k when $y < 1$, we have $\sum_{k=0}^{n-1} y^k > ny^{n-1}$. As long as $s < \bar{\theta}$ we have $F(s) < 1$, hence $\Big(\sum_{k=0}^{n-1} F(s)^k - nF(s)^{n-1}\Big)$ is strictly positive and thus $D(s) > D(t)$. So D is strictly increasing on $[\bar{b}, \bar{\theta})$.

Claim 2. $D(\bar{b}) \leq 0$ and $D(\bar{b}) > 0$.

Proof of Claim 2. At $\theta = \bar{b}$, we have $L(\bar{b}) = (\bar{b} - \bar{b}) \sum_{k=1}^{n-1} F(\underline{b})^k = 0$ and $R(\underline{b}) = n \int_{\underline{b}}^{\bar{b}} F(x)^{n-1} dx \geq 0$, hence $D(\bar{b}) \leq 0$. As $x \to \hat{\theta}$ from below, $F(x) \to 1$ and $\sum_{k=1}^{n-1} F(\hat{\theta})^k \to n$, while $R(x) \to n \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1} dx$. Hence,

$$\lim_{x \to \bar{\theta}-} D(x) = n(\bar{\theta} - \bar{b}) - n \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1} dx = n \left[(\bar{\theta} - \bar{b}) - \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1} dx \right].$$

By assumption that $\underline{\theta} < \bar{b} < \bar{\theta} - \int_{\underline{b}}^{\bar{\theta}} F(x)^{n-1} dx$, we have $\lim_{x \to \bar{\theta}-} D(x) > 0$.

We established that D is strictly increasing, that $D(\underline{b}) \leq 0$, and that $D(x) > 0$ for x sufficiently close to $\bar{\theta}$. Furthermore, D is continuous by the atomless assumption. Therefore, there exists a unique $\hat{\theta} \in [\underline{b}, \hat{\theta})$ such that $D(\hat{\theta}) = 0$. Hence, we establish the uniqueness of the pure strategy equilibrium with the profile of bidding strategies given by

$$b_i(\theta) = \begin{cases} \varnothing & \text{if } \theta < \underline{b} \\ \theta - \frac{\int_{\underline{b}}^{\theta} F(x)^{n-1} dx}{F(\theta)^{n-1}} & \text{if } \underline{b} \leq \theta < \hat{\theta}, \\ \bar{b} & \text{if } \theta \geq \hat{\theta}. \end{cases}$$

Because bidders on the cutoff types are indifferent, there also exist other, essentially equivalent, equilibria in which those types choose different bids between two indifferent choices. □

Proof of Theorem 5

Proof. Let $u(\cdot)$ and $\hat{u}(\cdot)$ denote the interim utilities in the equilibria $b(\cdot)$ and $\hat{b}(\cdot)$, respectively. Recall that, by 4, $\hat{b}(\theta)$ is strictly increasing on $[r, \hat{\theta}]$ and equals $\bar{b}$ on $\theta \geq \hat{\theta}$. Define $\tilde{\theta} = \inf\{x \mid b(x) = \bar{b}\}$ and $\Delta(\theta) = u(\theta) - \hat{u}(\theta)$.

For $\theta \leq r$, because r is the minimum allowed bid in $\hat{\mathcal{B}}$, either bidders with type θ opt out of the auction or bid their own value in the equilibrium in $\hat{b}(\theta)$. The analogous analysis holds for $b(\theta)$ because $r = \inf\{\theta \mid b(\theta) > \varnothing\}$. Therefore, we have $\Delta(\theta) = 0$ and $u(\theta) = \hat{u}(\theta)$

for all $\theta \leq r$.

For $\theta \in (r, \min\left\{\hat{\theta}, \tilde{\theta}\right\}]$, by Theorem 4, $\hat{b}(\theta)$ is strictly increasing. On an interval where $b(\theta)$ is strictly increasing, $\Delta(\theta)$ is constant by the payoff equivalence theorem. On an interval where $b(\theta)$ is flat, $\Delta(\theta)$ is non-decreasing by Lemma 18. Therefore, we have $u(\theta) \geq \hat{u}(\theta)$ for all $r < \theta \leq \min\left\{\hat{\theta}, \tilde{\theta}\right\}$.

Now, we prove that $\tilde{\theta} \geq \hat{\theta}$. By way of contradiction, suppose that $\tilde{\theta} < \hat{\theta}$. Because $\tilde{\theta} < \hat{\theta}$, we have $\min\left\{\hat{\theta}, \tilde{\theta}\right\} = \tilde{\theta}$ and there is a positive mass bidding $\bar{b}$ in the equilibrium $b(\theta)$. Hence, there is a discontinuity in $b(\cdot)$ at type $\tilde{\theta}$ and bidders with type $\tilde{\theta}$ are indifferent between bidding $\bar{b}$ and $\lim_{x \to \tilde{\theta}-} b(x)$. By the previous analysis for $\theta \in (r, \min\left\{\hat{\theta}, \tilde{\theta}\right\}]$, we have $u\left(\tilde{\theta}\right) \geq \hat{u}\left(\tilde{\theta}\right)$. Consider bidders with type $\tilde{\theta}$ in the equilibrium in $\hat{b}(\theta)$. Because $\tilde{\theta} < \hat{\theta}$, bidding $\bar{b}$ yields a strictly higher winning probability in $\hat{b}(\theta)$ than in $b(\theta)$. Hence, if a bidder with type $\tilde{\theta}$ were to bid $\bar{b}$ when everyone else follows $\hat{b}(\theta)$, then the payoff would strictly exceed $u\left(\tilde{\theta}\right)$. Because $u\left(\tilde{\theta}\right) \geq \hat{u}\left(\tilde{\theta}\right)$, such a deviation would be profitable for type $\tilde{\theta}$ contradicting that $\hat{b}(\cdot)$ forms an equilibrium. Hence, we must have $\tilde{\theta} \geq \hat{\theta}$.

We now show $u(\theta) \geq \hat{u}(\theta)$ for all $\theta \in [\hat{\theta}, \bar{\theta}]$.

– If $\theta \in (\tilde{\theta}, \bar{\theta}]$, then both equilibria prescribe bid $\bar{b}$. The set of types bidding $\bar{b}$ is weakly smaller under $b(\cdot)$ than under $\hat{b}(\cdot)$ (because $\tilde{\theta} \geq \hat{\theta}$), so the winning probability (and hence utility) from bidding $\bar{b}$ is weakly higher under $b(\cdot)$.

– If $\theta \in (\hat{\theta}, \tilde{\theta}]$, then under $\hat{b}$ the equilibrium bid is $\bar{b}$. In the $b(\cdot)$-equilibrium, $\bar{b}$ is available; let $q_b^{\bar{b}}$ be the winning probability from bidding $\bar{b}$ against opponents who follow b. Because $b(\cdot)$ is in the equilibrium, we have

$$u(\theta) \geq (\theta - \bar{b}) q_b^{\bar{b}}.$$

Because fewer types bid $\bar{b}$ under $b(\cdot)$ than under $\hat{b}(\cdot)$, we have $q_b^{\bar{b}} \geq q_{\hat{b}}^{\bar{b}}$, where $q_{\hat{b}}^{\bar{b}}$ is the winning probability under $\hat{b}(\cdot)$ when bidding $\bar{b}$. But $\hat{u}(\theta) = (\theta - \bar{b}) q_{\hat{b}}^{\bar{b}}$. Hence $u(\theta) \geq \hat{u}(\theta)$.

Combining the above analyses, we have $u(\theta) \geq \hat{u}(\theta)$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. We compare the expected revenue by analyzing two different regions; $\theta \leq \hat{\theta}$ and $\theta > \hat{\theta}$.

– For $\theta \leq \hat{\theta}$, recall that $\hat{b}(\theta)$ is strictly increasing on $[r, \hat{\theta}]$ and $\hat{b}(\theta) = \varnothing$ for $\theta < r$, while $b(\theta)$ may be flat on some region in $[r, \hat{\theta}]$ and $b(\theta) = \varnothing$ for $\theta < r$. Hence, the total expected surplus conditional on $\theta \leq \hat{\theta}$ is weakly higher in $\hat{b}(\theta)$ than in $b(\theta)$. On the other hand, the expected payoff for each type θ is weakly lower in $\hat{b}(\theta)$ than in $b(\theta)$. Hence, the expected revenue conditional on $\theta \leq \hat{\theta}$ is weakly higher in $\hat{b}(\theta)$.

– For $\theta \geq \hat{\theta}$, notice that bidders with type θ above $\hat{\theta}$ submit $\bar{b}$ in the equilibrium in $\hat{b}(\theta)$, while bidders with type between $\hat{\theta}$ and $\tilde{\theta}$ may bid lower than $\bar{b}$ in the equilibrium in $b(\theta)$. Hence, the expected revenue conditional on $\theta \geq \hat{\theta}$. Therefore, the expected revenue is weakly

higher in $\hat{b}(\theta)$ than in $b(\theta)$. $\square$

C Omitted Derivation of the Optimal Solution in Section 6.1

Instead of r and $\bar{b}$, we can transform the choice variables in the maximization problem into r and θ^*, where θ^* is the cutoff type that chooses $\bar{b}$. Then, we can rewrite the objective functions $U(\bar{b}, r)$ by setting $\bar{b} = \frac{\theta^*+r^2}{1+\theta^*}$ as:

$$U(\theta^*, r) = -\frac{4}{3}r^3 + \frac{1}{3}\theta^{*3} + \theta^* + r^2 - \theta^{*2} - \alpha(1+\theta^*)$$

and $V(\bar{b}, r)$ by setting $\bar{b} = \frac{\theta^*}{1+\theta^*}$ as:

$$V(\theta^*, r) = \begin{cases} V_1(\theta^*, r) = -\frac{8}{3}r^3 + \frac{1}{3}\theta^{*3} + \theta^* - \theta^{*2} - \alpha(1+\theta^*-2r) & \text{if } 0 < r < \frac{\theta^*}{1+\theta^*} \\ V_2(\theta^*) = \theta^*(1-\theta^*) - \alpha & \text{if } 0 < r = \frac{\theta^*}{1+\theta^*} \\ V_3(\theta^*) = 0 & \text{if } 0 = r = \frac{\theta^*}{1+\theta^*} \end{cases}.$$

By taking the first order condition for $U(\theta^*, r)$ with respect to θ^*:

$$\frac{\partial U}{\partial \theta^*}(\theta^*, r) = \theta^{*2} - 2\theta^* + 1 - \alpha = 0 \implies \theta^* = 1 - \sqrt{\alpha} \text{ and } \bar{b} = \frac{1 + r^2 - \sqrt{\alpha}}{2 - \sqrt{\alpha}}.$$

By taking the first order condition for $U(\theta^*, r)$ with respect to r:

$$\frac{\partial U}{\partial r}(\theta^*, r) = -4r^2 + 2r = 0 \implies r = 0.5.$$

The maximized value $U(\theta^*, r)$ as a function of α is:

$$U(\alpha) = -2\alpha + \frac{2}{3}\alpha^{1.5} + \frac{5}{12}.$$

By taking the first order condition for $V_1(\theta^*, r)$ with respect to θ^*:

$$\frac{\partial V_1}{\partial \theta^*}(\theta^*, r) = \theta^{*2} - 2\theta^* + 1 - \alpha = 0 \implies \theta^* = 1 - \sqrt{\alpha} \text{ and } \bar{b} = \frac{1 - \sqrt{\alpha}}{2 - \sqrt{\alpha}}$$

By taking the first order condition for $V_1(\theta^*, r)$ with respect to r:

$$\frac{\partial V_1}{\partial r}(\theta^*, r) = -8r^2 + 2\alpha = 0 \implies r = \frac{\sqrt{\alpha}}{2}.$$

Notice that the condition on α for $r < \frac{\theta^*}{1+\theta^*}$ is $\alpha < 6 - 4\sqrt{2}$ and the maximized value of V_1 as a function of α is:

$$V_1(\alpha) = \frac{4}{3}\alpha^{3/2} - 2\alpha + \frac{1}{3}.$$

By taking the first order condition for $V_2(\theta^*)$ with respect to θ^*:

$$\frac{\partial V_2}{\partial \theta^*}(\theta^*) = 1 - 2\theta^* = 0 \implies \theta^* = \frac{1}{2}.$$

The maximized value of V_2 as a function of α is:

$$V_2(\alpha) = \frac{1}{4} - \alpha.$$

One can verify that $\max\{U(\alpha), V_1(\alpha), V_2(\alpha), 0\}$ is $U(\alpha)$ if $\alpha \leq 0.25$ and 0 if $\alpha \geq 0.25$. Therefore, the optimal bid spaces for varying α is

$$\mathcal{B} = \begin{cases} [0.5, \frac{\frac{5}{4} - \sqrt{\alpha}}{2 - \sqrt{\alpha}}] & \text{if } \alpha < 0.25 \\ \{0\} \text{ or } \{0.5\} & \text{if } \alpha = 0.25 \\ \{0\} & \text{if } \alpha > 0.25 \end{cases}.$$

Omitted Verification of the Symmetric Equilibrium in Section 6.2

To verify this is indeed an equilibrium, consider bidder i of type θ. The expected payoff of bidder i of type θ by bidding b given that the other bidder follows the strategy $b(\theta)$ i

$$u(\theta, b) = \begin{cases} 2(1-p)b(\theta - b) & \text{if } b \in [0, \frac{1}{4}), \\ \frac{1}{2}(\theta - b) & \text{if } b = \frac{1+p}{4}, \\ x(b)(\theta - b) & \text{if } b \in (\frac{3p+1}{4p+4}, \frac{1}{2}], \end{cases}$$

where

$$x(b) = p + (1-p)\left(b + \sqrt{b^2 - \frac{p(1-2b)}{1-p}}\right).$$

We compute the maximizer and the resulting payoff on $[0, \frac{1}{4})$, $\{\frac{1+p}{4}\}$, and $(\frac{3p+1}{4p+4}, \frac{1}{2}]$ separately.

– For region $b \in [0, \frac{1}{4})$, the first order condition is

$$\frac{\partial u}{\partial b} = 2(1-p)(\theta - 2b) = 0 \implies b(\theta) = \frac{1}{2}\theta.$$

Furthermore, u is strictly concave on the region because $\frac{\partial^2 u}{\partial b^2} = -4(1-p) < 0$, hence $b^* = \frac{1}{2}\theta$ is the unique maximizer. This lies in $[0, \frac{1}{4})$ if and only if $\theta < \frac{1}{2}$. Hence, for $\theta < \frac{1}{2}$, the maximizer within this region is $\frac{1}{2}\theta$, yielding

$$u\left(\theta, \frac{1}{2}\theta\right) = 2(1-p) \cdot \frac{\theta}{2}(\theta - \frac{\theta}{2}) = \frac{1-p}{2}\theta^2.$$

If $\theta \geq \frac{1}{2}$, the payoff $u(\theta, b)$ increases throughout, so the optimal within this region is the right-end supremum with value

$$\sup_{b<\frac{1}{4}} u(\theta, b) = 2(1-p) \cdot \frac{1}{4}(\theta - \frac{1}{4}) = \frac{1-p}{2}\left(\theta - \frac{1}{4}\right).$$

– For $b = \frac{1+p}{4}$, because the allowed bid is singleton, the maximized payoff is

$$u(\theta, \frac{1+p}{4}) = \frac{1}{2}\left(\theta - \frac{1+p}{4}\right).$$

– For the region $b \in (\frac{3p+1}{4p+4}, \frac{1}{2}]$, we define two auxiliary functions:

$$I(b) = b^2 - \frac{p(1-2b)}{1-p}, \qquad s(b) = \sqrt{I(b)}.$$

Note that

$$s'(b) = \frac{b + \frac{p}{1-p}}{s(b)}, \qquad x'(b) = (1-p) + \frac{(1-p)b + p}{s(b)}.$$

From the second equality, we have

$$x'(b)s(b) = p + (1-p)\left(s(b) + b\right) = x(b).$$

The first order condition on $u(\theta, b) = x(b)(\theta - b)$ gives

$$\frac{\partial u}{\partial b} = x'(b)(\theta - b) - x(b) = 0.$$

By substituting $x'(b)s(b) = x(b)$ into the first order condition, we have

$$x'(b)(\theta - b) - x'(b)s(b) = 0 \implies \theta - b - s(b) = 0.$$

For $\theta > \frac{1}{2}$, using $s(b) = \sqrt{I(b)}$ gives a closed from for the maximizing bid

$$b(\theta) = \frac{p + (1-p)\theta^2}{2\left[p + (1-p)\theta\right]}.$$

The second order condition gives

$$\frac{\partial^2 u}{\partial b^2} = x''(b)(\theta - b) - 2x'(b) < 0$$

because

$$x''(b) = \frac{(1-p)s(b) - ((1-p)b + b)s'(b)}{s(b)^2} = -\frac{p}{(1-p)s(b)^3} < 0.$$

Hence, $b(\theta) = \frac{p+(1-p)\theta^2}{2[p+(1-p)\theta]}$ is indeed the maximum on the region $b \in (\frac{3p+1}{4p+4}, \frac{1}{2}]$. Recall that

$$\frac{\partial u}{\partial b} = x'(b)(\theta - (b + s(b))).$$

Because $b + s(b)$ increasing in b and $b + s(b)$ is $\frac{1}{2}$ for $b = \frac{3p+1}{4p+4}$, we have $\frac{\partial u}{\partial b} < 0$ for all $\theta < \frac{1}{2}$ and $\frac{\partial u}{\partial b} = 0$ for $\theta = \frac{1}{2}$. Therefore, for bidders of type $\theta < \frac{1}{2}$, the optimal bidding is to bid as close as possible to $b = \frac{3p+1}{4p+4}$.

We now compare the best values attainable in each part of the bid space for each type.

(i) Consider bidders of type $\theta < \frac{1}{2}$.

– On the lower interval $b \in [0, \frac{1}{4})$, bidding $b(\theta) = \frac{1}{2}\theta$, which is the maximizer in the lower interval $[0, \frac{1}{4})$ gives the payoff $u(\theta, \frac{1}{2}\theta) = \frac{1-p}{2}\theta^2$.

– Bidding $b(\theta) = \frac{1+p}{4}$ gives the payoff $u(\theta, \frac{1+p}{4}) = \frac{1}{2}\left(\theta - \frac{1+p}{4}\right)$ and $\frac{1}{2}\left(\theta - \frac{1+p}{4}\right)$.

– On the upper interval $b \in (\frac{3p+1}{4p+4}, \frac{1}{2}]$, the supremum of the payoff $\frac{1+p}{2}\left(\theta - \frac{3p+1}{4p+4}\right)$ at its excluded lower endpoint $b = \frac{3p+1}{4p+4}$.

Because $\frac{1-p}{2}\theta^2 > \frac{1}{2}\left(\theta - \frac{1+p}{4}\right)$ and $\frac{1-p}{2}\theta^2 > \frac{1+p}{2}\left(\theta - \frac{3p+1}{4p+4}\right)$ for all $\theta \in [0, \frac{1}{2})$, the global maximizer for $\theta < \frac{1}{2}$ is $b(\theta) = \frac{1}{2}\theta$.

(ii) Consider bidders of type $\theta = \frac{1}{2}$.

– On the lower interval $b \in [0, \frac{1}{4})$, the supremum of the payoff is $\frac{1-p}{8}$ at the excluded endpoint $b = \frac{1}{4}$.

– Bidding $b(\theta) = \frac{1+p}{4}$ gives the payoff $u(\frac{1}{2}, \frac{1+p}{4}) = \frac{1-p}{8}$.

– On the upper interval $b \in (\frac{3p+1}{4p+4}, \frac{1}{2}]$, the supremum of the payoff is also $\frac{1-p}{8}$ at its

excluded endpoint $b = \frac{3p+1}{4p+4}$.

Hence, the global maximizer for $\theta = \frac{1}{2}$ is $b(\theta) = \frac{1+p}{4}$

(iii) Consider bidders of type $\theta > \frac{1}{2}$.

– On the lower interval $b \in [0, \frac{1}{4})$, the supremum of the payoff is $\frac{1-p}{2}\left(\theta - \frac{1}{4}\right)$ at the excluded end point $b = \frac{1}{4}$.

– Bidding $b(\theta) = \frac{1+p}{4}$ gives the payoff $u(\theta, \frac{1+p}{4}) = \frac{1}{2}\left(\theta - \frac{1+p}{4}\right)$.

– On the upper interval $b \in (\frac{3p+1}{4p+4}, \frac{1}{2}]$, bidding $b(\theta) = \frac{p+(1-p)\theta^2}{2(p+(1-p)\theta)}$, which is the maximizer in the upper interval, gives the payoff $u(\theta, \frac{p+(1-p)\theta^2}{2(p+(1-p)\theta)}) = p(\theta - \frac{1}{2}) + \frac{1-p}{2}\theta^2$.

Because $p(\theta - \frac{1}{2}) + \frac{1-p}{2}\theta^2 > \frac{1-p}{2}\left(\theta - \frac{1}{4}\right)$ and $p(\theta - \frac{1}{2}) + \frac{1-p}{2}\theta^2 > \frac{1}{2}\left(\theta - \frac{1+p}{4}\right)$ for all $\theta \in (\frac{1}{2}, 1]$, the global maximizer for $\theta > \frac{1}{2}$ is $b(\theta) = \frac{p+(1-p)\theta^2}{2(p+(1-p)\theta)}$.

D Asymmetric Environment

In this section, we present results for asymmetric environment. Throughout the analysis, we assume that each bidder i has a type distribution π_i, with finite support Θ_i.[37]

Consider bidder specific monotone partitions $\left\{\Theta_i^j \mid j = 1, \dots J_i\right\}$ for all $i \in N$ and a strict order $\prec$ on $\left\{\{i\} \times \left\{\Theta_i^j \mid j = 1, \dots J_i\right\} \mid i \in N\right\} \cup \{Keep\}$ such that

$$\left(\{i\}, \Theta_i^j\right) \prec \left(\{i\}, \Theta_i^{j+1}\right), \quad \text{for all } j = 1, \dots, J-1$$

and a mapping $g_i : \Theta_i \to \left\{\Theta_i^j \mid j = 1, \dots J_i\right\}$ defined by $\theta_i \in g_i(\theta_i)$.

Ex-post allocation $(q_i)_{i \in N}$ is **asymmetric hierarchical allocation**: $q_i(\theta_1, \dots, \theta_N) = 1$ if $(\{i\}, g_i(\theta_i)) \succ \max\{Keep, (\{j\}, g_j(\theta_j))\}$ for all $i \neq j$ and $q_i(\theta_1, \dots, \theta_N) = 0$ otherwise.

Let

$$\mathcal{Q} = \left\{(Q_i)_{i \in N} \mid Q_i : \Theta \to [0,1],\ Q_i \text{ is non-decerasing and feasible.}\right\}$$

The set $\mathcal{Q} \subseteq \mathbb{R}^{\sum_{i=1}^N T_i}$ is a polyhedron characterized by the following inequalities:

$$\sum_{i \in N} \sum_{s=t_i}^{T_i} Q_i(\theta_i^s)\, \pi_i(\theta_i^s) \leq 1 - \prod_{i \in N} \left[\sum_{s=1}^{t_i - 1} \pi_i(\theta_i^s)\right], \text{ for all } i \in N \text{ and } t_i = 1, \dots, T_i,$$

and

$$Q_i\left(\theta_i^t\right) \geq Q_i\left(\theta_i^{t-1}\right), \text{ for all } i \in N \text{ and } t = 1, \dots, T_i,$$

where $Q_i(\theta_i^0)$ is defined to be 0.

[37] The extension to general distributions follows the same steps as in the symmetric case.

Lemma 19. *An interim allocation $(Q_i)_{i\in N}$ is an extreme point of $\mathcal{Q}$ if and only if it is a reduced form of an asymmetric hierarchical allocation.*

For convenience, let us introduce notations to be used throughout the section. Fix an interim allocation $(Q_i)_{i\in N} \in \mathcal{Q}$. For each bidder $i \in N$, let

$$\Theta_i = \Theta_i^1 \cup \cdots \cup \Theta_i^{J_i}$$

be the coarsest monotone partition adapted to Q_i: Q_i is constant on each block and strictly increases across adjacent blocks. Write

$$Q_i(\theta) = \alpha_{i,j} \quad \text{for all } \theta \in \Theta_i^j.$$

Let $\mathcal{T} = \prod_{i\in N} \{1, \ldots, J_i\}$ be the lattice for block-index with the usual coordinate-wise order. For each $t \in \mathcal{T}$, define block-level tails:

$$p_i(t_i) = \sum_{\theta_i \in \cup_{r\geq t_i} \Theta_i^r} \pi_i(\theta_i), \quad U_i(t_i) = \sum_{\theta_i \in \cup_{r\geq t_i} \Theta_i^r} Q_i(\theta_i)\, \pi_i(\theta_i),$$

and

$$H(t) = \sum_{i\in N} U_i(t_i)$$

Let $U_i(t_i) = \sum_{s\geq t_i} Q_i(\theta_i^s)\, \pi_i(\theta_i^s)$ be the weighted upper tail of Q_i and $p_i(t_i) = \sum_{s\geq t_i} \pi_i(\theta_i^s)$ be the tail probability. Border's feasibility condition is then:

$$\sum_{i\in N} U_i(t_i) \leq 1 - \prod_{i\in N} (1 - p_i(t_i))$$

for every vector $t = (t_1, \ldots, t_N) \in \prod_{i\in N} \{1, \ldots, T_i\}$. To further simplify the notation, throughout the next the proof of the theorem, we use

$$F(t) = 1 - \prod_{i\in N} (1 - p_i(t_i)), \; H(t) = \sum_{i\in N} U_i(t_i).$$

Then, feasibility constraint at vector t is given by

$$H(t) \leq F(t).$$

The proof of Lemma 19 is based on the following two lemmas.

Lemma 20. *Let $(Q_i)_{i\in N} \in \mathcal{Q}$. Then $\mathcal{S} = \{t \in \mathcal{T} \mid H(t) = F(t)\}$ is closed under coordinate-*

wise meet and join: if $t, s \in \mathcal{S}$, then $t \wedge s, t \vee s \in \mathcal{S}$.

Proof. If $\mathcal{S}$ is empty, then the claim holds automatically. Suppose that $\mathcal{S}$ is not empty, hence we pick arbitrary $t, s \in \mathcal{S}$. Because $H(t)$ is additive across coordinates, we have

$$H(t) + H(s) = H(t \wedge s) + H(t \vee s).$$

Because $F(t)$ is non-decreasing and submodular, we have

$$F(t) + F(s) \geq F(t \wedge s) + F(t \vee s).$$

Because $t, s \in \mathcal{S}$, we have $H(t) + H(s) = F(t) + F(s)$, hence

$$H(t \wedge s) + H(t \vee s) \geq F(t \wedge s) + F(t \vee s).$$

By the feasibility condition of $(Q_i)_{i \in N}$ at $t \wedge s$ and $t \vee s$, we have

$$H(t \wedge s) + H(t \vee s) \leq F(t \wedge s) + F(t \vee s),$$

hence

$$H(t \wedge s) + H(t \vee s) = F(t \wedge s) + F(t \vee s).$$

The equality holds if and only if $H(t \wedge s) = F(t \wedge s)$ and $H(t \vee s) = F(t \vee s)$, therefore $t \wedge s, t \vee s \in \mathcal{S}$. □

Lemma 21. *Let $(Q_i)_{i \in N}$ be an extreme point of $\mathcal{Q}$. If $\mathcal{S} = \{t \mid H(t) = F(t)\}$ is non-empty, then there exists a finite sequence $\left(t^{(\ell)}\right)_{\ell=1}^{L}$ in $\mathcal{S}$ such that*

(i) $t^{(0)} = \min(\mathcal{S})$ and $t^{(L)} = \max(\mathcal{S})$ and $t^{(0)} \prec \cdots \prec t^{(L)}$ (if $\min(\mathcal{S}) \neq \max(\mathcal{S})$),[38]

(ii) for all $\ell = 1, \ldots, L$, no element of $\mathcal{S}$ lies strictly between $t^{(\ell-1)}$ and $t^{(\ell)}$,

(iii) for all $\ell = 1, \ldots, L$, we have $t^{(\ell)} = t^{(\ell-1)} + e_{i_\ell}$ for some $i_\ell \in N$,

(iv) for all $\ell = 1, \ldots, L$, we have $Q_i(\theta_i^{t^{(\ell)}}) = \prod_{j \neq i} \left(1 - p_j\left(t_j^{(\ell)}\right)\right)$.

Proof. By Lemma 20, $\mathcal{S}$ is finite lattice. Because $\mathcal{S}$ is non-empty and finite lattice, we have $t^{(0)} \in \mathcal{S}$ and $t^{(L)} \in \mathcal{S}$, hence establishing (i). If $t^{(0)} = t^{(L)}$, the proof is done. Suppose that $t^{(0)} \prec t^{(L)}$. Because $\mathcal{S}$ is finite and $\left\{u \in \mathcal{S} \mid t^{(\ell-1)} \prec u\right\}$ is non-empty for $\ell = 1, \ldots, L$, it has $\preceq$-minimal elements; we can pick any one of them and call it $t^{(\ell)}$. This produces a strictly increasing finite sequence $t^{(0)} \prec \cdots \prec t^{(L)}$ such that no element of $\mathcal{S}$ lies strictly between $t^{(\ell-1)}$ and $t^{(\ell)}$ for any ℓ, and this proves (ii). To prove (iii), for convenience, let us introduce

[38] The minimum and maximum are well-defined because $\mathcal{S}$ is a finite lattice by Lemma 20. The relation $s \prec t$ if and only if $s_i \leq t_i$ for all i and $s_i < t_i$ for some i.

$B(t) = \{(i,k) \mid k \geq t_i\}$. If $M = B(t^{(\ell)}) - B(t^{(\ell-1)})$ is a singletone then the proof is complete. Assume for a contradiction that $|M| \geq 2$. Let $I = \left\{ i \mid t_i^{(\ell)} > t_i^{(\ell-1)} \right\}$ be the set of coordinates that move from $t^{(\ell-1)}$ to $t^{(\ell)}$.

Case 1. $|I| \geq 2$, or $|I| = 1$ and Q_i is not constant on $\left\{ t_i^{(\ell-1)}, \ldots, t_i^{(\ell)} - 1 \right\}$.

In this case, M meets at least two flat segments. Pick two flat segments B^+ and B^- contained in M (possibly on different bidders). Define, for a small $\eta > 0$,

$$\Delta Q_{i,k} = \begin{cases} \eta & \text{if } (i,k) \in B^+, \\ -\eta \frac{\sum_{(i,k)\in B^+} \pi_i\left(\theta_i^k\right)}{\sum_{(i,k)\in B^-} \pi_i\left(\theta_i^k\right)} & \text{if } (i,k) \in B^-, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_i^+ = Q_i + \Delta Q_{i,k}, \quad Q_i^- = Q_i - \Delta Q_{i,k}.$$

To check the feasibility of Q_i^+ and Q_i^-, we partition the set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$; for $t \in \mathcal{T}_1$, the feasibility constraint is binding and for $t \in \mathcal{T}_2$, the feasibility constraint has slack. For $t \in \mathcal{T}_2$, because there is slack, we can set $\eta > 0$ small enough to satisfy the feasibility constraint due to the finiteness of $\mathcal{T}$. For $t \in \mathcal{T}_1$, if $M \subseteq B(t)$, then the feasibility constraints hold as equality at t after the perturbation because $\sum_M \Delta Q_{i,k} \pi_i\left(\theta_i^k\right) = 0$. For $t \in \mathcal{T}_1$, if $M \cap B(t) = \emptyset$, then the perturbation never shows up in the feasibility constraints, hence the constraints hold as equality. For $t \in \mathcal{T}_1$, if $M \cap B(t)$ is a proper subset of M, then there exists indices i with $t_i < t_i^{(\ell)}$ and j with $t_j > t_j^{(\ell-1)}$. Hence $w = \left(t^{(\ell-1)} \vee t\right) \wedge t^{(\ell)}$ is in $\mathcal{S}$ by Lemma 20 and $t^{(\ell-1)} \prec w \prec t^{(\ell)}$, which is a contradiction to (ii). Therefore, both Q_i^+ and Q_i^- are feasible. Because we perturb the flat segments by the same amount, monotonicity constraints are satisfied by setting η small enough. Notice that $\frac{1}{2}\left(Q_i^+\right)_{i\in N} + \frac{1}{2}\left(Q_i^-\right)_{i\in N} = (Q_i)_{i\in N}$ and this is a contradiction to the assumption that $(Q_i)_{i\in N}$ is an extreme point.

Case 2. M lies in a single flat segment over multiple types of a single bidder.

Here, $I = \{i\}$ and $Q_i\left(\theta_i^{t^{(\ell-1)}}\right) = \cdots = Q_i\left(\theta_i^{t^{(\ell)}}\right) = c$ for some $c \geq 0$. Notice that

$$H(t^{(\ell-1)}) - H(t^{(\ell)}) = c \sum_{k=t_i^{(\ell-1)}}^{t_i^{(\ell)}} \pi_i\left(\theta_i^k\right)$$

and

$$F(t^{(\ell-1)}) - F(t^{(\ell)}) = \left(\prod_{j\neq i} \left(1 - p_j\left(t_j^{(\ell-1)}\right)\right) \right) \sum_{k=t_i^{(\ell-1)}}^{t_i^{(\ell)}} \pi_i\left(\theta_i^k\right).$$

Because $H(t^{(\ell-1)}) = F(t^{(\ell-1)})$ and $H(t^{(\ell)}) = F(t^{(\ell)})$, we obtain

$$c = \prod_{j \neq i} \left(1 - p_j\left(t_j^{(\ell-1)}\right)\right).$$

But then for the first unit step $t^{(\ell-1)} \to t^{(\ell-1)} + e_i$, we have

$$H(t^{(\ell-1)}) - H(t^{(\ell-1)} + e_i) = c\pi_i\left(\theta_i^{t_i^{(\ell-1)}}\right)$$

and

$$F(t^{(\ell-1)}) - F(t^{(\ell-1)} + e_i) = \prod_{j \neq i} \left(1 - p_j\left(t_j^{(\ell-1)}\right)\right) \pi_i\left(\theta_i^{t_i^{(\ell-1)}}\right) = c\pi_i\left(\theta_i^{t_i^{(\ell-1)}}\right).$$

Because $H(t^{(\ell-1)}) - H(t^{(\ell-1)} + e_i) = F(t^{(\ell-1)}) - F(t^{(\ell-1)} + e_i)$ and $H(t^{(\ell-1)}) = F(t^{(\ell-1)})$, the feasibility constraint is binding at $t^{(\ell-1)} + e_i$, which is a contradiction to (ii) because $t^{(\ell-1)} \prec t^{(\ell-1)} + e_i \prec t^{(\ell)}$. Therefore, it must be $|M| = 1$, hence $t^{(\ell)} = t^{(\ell-1)} + e_i$ and this proves both (iii) and (iv). □

Proof of Lemma 19. Let $(Q_i)_{i \in N}$ be an extreme point of $\mathcal{Q}$. For each i, let $\Theta_i = \Theta_i^1 \cup \cdots \cup \Theta_i^{J_i}$ be the coarsest monotone partition adapted to Q_i and denote the common value of Q_i on Θ_i^j by $\alpha_{i,j}$. Let $[a_i(j), b_i(j)] \subseteq \{1, \ldots, T_i\}$ be the index interval of Θ_i^j.

Because $\mathcal{Q}$ is a polytope in $\mathbb{R}^{\sum_{i \in N} T_i}$, at least $\sum_{i \in N} T_i$ constraints must be binding. If $\mathcal{S} = \emptyset$, then for $(Q_i)_{i \in N}$ to be extremal, every monotonicity constraint must bind to have $\sum_{i \in N} T_i$ binding constraints; hence $Q_i(\theta_i) = 0$ for all i and θ_i. This is an asymmetric hierarchical allocation, where *Keep* is at the top. Henceforth, we assume $\mathcal{S} \neq \emptyset$. By Lemma 21 (ii), we pick a sequence of vectors $\min(\mathcal{S}) = t^{(0)} \prec \cdots \prec t^{(L)} = \max(\mathcal{S})$ such that $t^{(\ell)} \in \mathcal{S}$ and $t^{(\ell)} = t^{(\ell-1)} + e_{i_\ell}$ for some i_ℓ for all ℓ. By Lemma 21 (iv), for all ℓ, we have

$$Q_i(\theta_i^{t^{(\ell)}}) = \prod_{h \neq i} \left(1 - p_h\left(t_h^{(\ell)}\right)\right).$$

We implement this with an asymmetric hierarchical allocation as follows:

- Follow the chain $t^{(0)} \prec \cdots \prec t^{(L)}$ from $t^{(0)}$ to $t^{(L)}$.

- Along the chain, whenever k_ℓ is the first index of some partition block $\Theta_{i_\ell}^j$ (i.e., $k_\ell = a_{i_\ell}(j)$), record the partition label $(i_\ell, \Theta_{i_\ell}^j)$ after all previously recorded labels. (Thus each partition label is recorded exactly once, at the chain's first entry into that block.)

- Place *Keep* below every recorded label with $\alpha_{i,j} > 0$; any blocks with $\alpha_{i,j} = 0$ is placed below *Keep*.

This produces a strict order on $\left\{\left(i,\Theta_i^j\right)\right\}\cup\{Keep\}$ that is monotone within each bidder: because the chain raises type indices by one, it visits all indices $a_i\left(j\right),\ldots,b_i\left(j\right)$ before it can visit $a_i\left(j+1\right)$, thus we have $\left(i,\Theta_i^j\right)\prec\left(i,\Theta_i^{j+1}\right)$. Hence, this is an asymmetric hierarchical allocation. We show that this allocation rule generates the interim allocation given by

$$Q_i(\theta_i^{t_i^{(\ell)}})=\prod_{h\neq i}\left(1-p_h\left(t_h^{(\ell)}\right)\right).$$

Fix a partition label (i,Θ_i^j). Let r be the chain vector just before the chain first enters $a_i\left(j\right)$. Under the allocation rule, bidder i with $\theta_i\in\Theta_j$ with $(i,\Theta_j)\succ Keep$ wins the auction if and only iff every competitor j is assigned a partition label ranked below $\left(i,\Theta_i^j\right)$. By construction, the probability of that event is

$$\prod_{h\neq i}\left(1-p_h\left(r_h\right)\right).$$

By applying $Q_i(\theta_i^{t_i^{(\ell)}})=\prod_{j\neq i}\left(1-p_j\left(t_j^{(\ell)}\right)\right)$ at $r\to r+e_i$ with $r_i=a_i\left(j\right)$, we have

$$\prod_{h\neq i}\left(1-p_h\left(r_h\right)\right)=Q_i\left(\theta_i^{a_i(j)}\right).$$

Hence, $(Q_i)_{i\in N}$ is a reduced form of the asymmetric hierarchical allocation rule.

To show the converse, fix bidder-wise partitions $\left\{\Theta_i^j\right\}$ and strict order on $\left\{\left(i,\Theta_i^j\right)\right\}\cup\{Keep\}$. Let the induced interim rule be $(Q_i)_{i\in N}$ so $Q_i\left(\theta_i^j\right)=\alpha_{i,j}$ on Θ_i^j.

Starting from the minimum threshold $t^{(0)}$, sweep the partition in order from bottom to top. Whenever the sweep passes $\left(i,\Theta_i^j\right)$ above $Keep$, perform the consecutive type-level steps

$$t\to t+e_i\to t+2e_i+\cdots$$

$b_i(j)-a_i(j)+1$ times. This yields a type-level unit-step sequence $t^{(0)}\prec\cdots\prec t^{(L)}$. Consider an edge $r\to s=r+e_i$ inside block Θ_i^j (so $r_i=k\in\left[a_i\left(j\right),b_i\left(j\right)\right]$. The probability that all others are below (i,Θ_i^j) is $\prod_{h\neq i}(1-p_h(r_h))$, and by construction we do not change r_j while moving inside the block. Hence for each such step,

$$H(r)-H(s)=\pi_i(\theta_i^k)\alpha_{i,j}=\pi_i(\theta_i^k)\prod_{h\neq i}(1-p_h(r_h))=F(r)-F(s).$$

Inducting from the start gives $H(t^{(\ell)})=F(t^{(\ell)})$, hence the feasibility constraints on the whole sequence are binding. Hence, along the sequence, $\left(Q_i(\theta_i^k)\right)_{i,k}$ are uniquely pinned down by

a combination of monotonicity and feasibility constraints, hence $(Q_i)_{i\in N}$ is extremal.

Proposition 2. *If $(Q_i)_{i\in N}$ is a reduced form of a asymmetric hierarchical allocation and (Q_i,T_i) is incentive compatible and individually rational, then it is a first-price auction with subsidy.*

Proof. Set $s_i = \min\{Q_i(x)x - T_i(x) \mid x \in \Theta_i\}$ and $b_i(\theta) = \frac{T_i(\theta)\theta + s_i}{Q_i(\theta)}$ for θ such that $Q_i(\theta) > 0$ and $b_i(\theta) = \varnothing$ for θ such that $Q_i(\theta) = 0$. Then, by Lemma 15, $b_i(\theta') > b_i(\theta)$ if and only if $Q_i(\theta') > Q_i(\theta)$ for all $\theta' > \theta$ except when $b_i(\theta) = \theta$. By incentive compatibility, there exists at most one type θ such that $b_i(\theta) = \theta$. If $b_i(\theta) = \theta$ and $b_i(\theta') = \theta$ for some $\theta' > \theta$, we modify $b_i(\theta) = r^-$. Otherwise, we do not need to include r^- in the bid space. We construct individual specific bid space $\mathcal{B}_i = b_i(\Theta_i) \cup \{\varnothing\}$ for each i and bid interpretation mapping

$$\Phi_i : \mathcal{B}_i \to \mathbb{R} \cup \{\varnothing\}$$

such that $\Phi_i(\varnothing) = \varnothing$ for all i and

$$\Phi_i(b_i(\theta_i)) < \Phi_j(b_j(\theta_j)) \iff (\{i\}, g_i(\theta_i)) \prec (\{j\}, g_j(\theta_j)).$$

Then, the auction, where bidder with the highest $\Phi_i(b_i)$ wins the auction and pays $b_i(\theta_i)$, implement (Q_i, T_i) with bid specific subsidy s_i. □

Theorem 7. *Any individually rational and incentive compatible mechanism is a lottery over first-price auctions with subsidy.*

Proof. By Lemma 19 and Lemma 9, any extreme point of the set of individually rational and incentive compatible mechanisms is an asymmetric hierarchical. 2 completes the proof. □

Theorem 8. *If the objective is quasi-convex and continuous in $(Q_i, T_i)_{i\in N}$, then there exists a first-price auction with subsidy that maximizes the objective.*

Proof. By the continuity and the quasi-convexity, there exists an optimal on an extreme point. Theorem 7 and Carathéodory's theorem completes the proof. □