Optimal Immigration and Long-Run Growth

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Abstract

We introduce immigration into a neoclassical growth model. A planner maximizes the utility of a representative native consumer in the steady state subject to providing each immigrant with a threshold level of utility. We derive a sufficient condition for the immigration rate to be positive in the planner's optimal steady state. We then implement the planner's optimal steady state in a decentralized competitive economy using a fiscal package consisting of an entry fee for immigrants to settle in the country, a consumption tax, and government bonds.

Immigration policy has become a major divisive issue in the United States, with passionate arguments put forth by people on all sides of questions relating to immigration. In this paper, we adopt a long-term macroeconomic perspective on the impact and desirability of immigration. We develop and analyze a simple neoclassical growth model of an economy that is an attractive destination for an unlimited supply of potential immigrants. We abstract from many of the rich set of considerations that arouse such passion in current debates and assume that all people, regardless of country of birth, have identical productivity when working in the destination country. We also assume that consumers, regardless of country of birth, have identical utility functions that depend only on their own consumption in each period of life. We ignore other considerations that might affect utility, such as an affinity for people of similar national origin or a taste for cultural diversity associated with a population consisting of people of various national origins. Our goal is to use neoclassical growth theory, extended to include immigration, to analyze the amount of immigration on that maximizes a well-defined social welfare function.

The social welfare function we adopt is the lifetime utility of a representative native consumer in the steady state. Notably, the utility of immigrants does not enter this social welfare function. Instead, the utility of immigrants is treated as a constraint–immigrants must receive a level of utility in the destination country that is greater than or equal to a threshold level of utility. We assume that children born in the destination country are treated as natives, regardless of whether their parents are natives or immigrants. (Had we adopted the alternative assumption that all descendents of immigrants are treated as immigrants, the steady state with immigrants would have a vanishingly small share of natives.) Finally, to address the concern that immigrants may consume public goods that are costly to produce, we assume that public goods are rival and non-excludable so that immigrants and natives consume identical amounts of costly public goods per person.

In a closed economy populated only by natives with constant productivity and a constant population growth rate, steady-state consumption per person is maximized by the capital-labor ratio for which the marginal product of capital, $f'(k)$, net of the depreciation rate of capital, δ, equals the population growth rate. Phelps (1961) dubbed such a steady state the

Golden Rule because it is the steady state chosen by a planner that treats every consumer in every generation equally—implementing the Biblical dictum to treat others as you would like to be treated. In this paper, the planner maximizes the utility of the representative native-born person in the steady state. In this case, the Golden Rule is narrow in scope, applying only to natives. As mentioned above, the consumption and utility of immigrants enters the planner's decision only as a constraint. The planner respects the threshold level of immigrants' utility but does not attempt to maximize their utility. Recognizing that the planner departs from the Biblical dictum, we will use the term Golden Rule in a narrow technical sense meaning that the planner maximizes the steady-state utility of given class of consumers—in this case, natives.

Our strategy in this paper is first to analyze the planner's problem to choose the amount of capital per worker, k, and the ratio of the number of immigrant workers to native workers, m, to maximize the social welfare function. Then we turn our attention to a competitive economy and analyze a package of taxes and government bonds that induces the steady state of the competitive economy to match the steady state chosen by the planner. In Section 2, we assume that all consumers live for only one period and give birth to an exogenous number, $1+\gamma \geq 1$, of children at the end of their lives. The planner chooses the levels of k and m that jointly maximize steady-state consumption per native. The optimal steady state is characterized by an extension of the Golden Rule: the marginal product of capital net of depreciation, $f'(k)-\delta$, equals the endogenous growth rate of population, $(1+\gamma)(1+m)-1$. In the special case in which $m=0$, the extended Golden Rule becomes $f'(k)-\delta=\gamma$, as in Phelps (1961) in the absence of immigration. We derive a sufficient condition for the optimal steady-state value of m to be positive: the consumption of a native in the Phelpsian Golden Rule in an economy without immigrants must exceed the sum of (1) the threshold level of consumption that a potential immigrant requires to willingly immigrate; and (2) the amount of capital needed to equip each of the immigrant's $1+\gamma$ children with capital equal to the per-worker capital-labor ratio in the Phelpsian Golden Rule.

A limitation of the model with one-period lifetimes in Section 2 is that consumers have no saving decision and hold no capital. As a step toward developing a competitive economy

with a meaningful saving decision by individual consumers, Section 3 extends lifetimes to two periods. In the first period of life, consumers work, consume, and use their saving to buy assets to carry into the second period of life. They also give birth to $1+\gamma$ young consumers at the end of the first period of life. In the second period of life, consumers do not work. With two-period lifetimes, the planner maximizes the present value of utility over the two periods of life of a representative native in the steady state. Also, with two periods, potential immigrants have a threshold level of utility (rather than simply consumption) required in order to willingly immigrate. The mapping from consumption to utility depends on the endogenous rate of return on capital, and the planner must take account of this endogeneity in maximizing the social welfare function. Despite these additional complications, the extended Golden Rule derived in Section 2 continues to hold without any change: $f'(k) - \delta = (1+\gamma)(1+m) - 1$.

Section 4 analyzes a version of the model with two-period lifetimes from Section 3, organized as a competitive economy. We show that if the optimal rate of immigration in the steady state is positive, the steady state in the competitive economy can be induced to match the planner's optimal steady state using three fiscal tools: (1) an entry fee charged to all young immigrants at the end of their first period of life; (2) a consumption tax levied at the same rate on all consumers regardless of national origin and regardless of age; and (3) government bonds. Because there is no risk in this model, the rate of return on government bonds equals the rate of return on capital. Hence, the extended Golden Rule implies that in the planner's optimal steady state, the rate of return on government bonds equals the economy's growth rate. Therefore, in the optimal steady state, the government earns zero seignorage. That is, bonds are not means of paying for government expenditures in the steady state. The role of government bonds is to induce the competitive economy to hold the optimal size of the capital stock by setting the amount of bonds equal to the amount of saving less the optimal capital stock.

1 Related Literature

This paper is related to two strands of literature: (1) the economic effects of immigration; and (2) analyses of the Golden Rule, which is the maximization of steady-state consumption (or utility) in a dynamic economy. In this section, we focus on the subsets of these literatures most relevant to the determination of the rate of immigration that maximizes steady-state welfare of natives in the host country.

Some of the early immigration literature emphasizes the impact of illegal, as opposed to legal, immigration. Indeed, the word "illegal" appears in the title of Ethier (1986), Hazari and Sgro (2003), and Moy and Yip (2006). The earliest of these three papers, discusses the impact of illegal immigration in a static economy without capital. Hazari and Sgro (2003) analyze the impact of illegal immigration in a dynamic economy with capital in which illegal immigrants as well as natives have infinite-horizons. Illegal immigrants receive a lower wage than equally productive natives. Hazari and Sgro (2003) finds that "if illegal migrants and domestic labour are perfect substitutes, then illegal migration necessarily lowers the long-run per capita consumption of domestic residents." (p 142) Using the model from Hazari and Sgro (2003), Moy and Yip (2006) provides an example in which native labor and immigrant labor are perfect substitutes in production, yet, contrary to Hazari and Sgro (2003), Moy and Yip (2006) finds than an increase in immigration can lead to a higher or lower level of consumption per native in the steady state. Moy and Yip (2006) attribute the difference in findings to "an indirect intertemporal effect that is overlooked by Hazari and Sgro." (p. 2473) More recently, Correani et al. (2015) in a framework similar to that used by Hazari and Sgro (2003) and Moy and Yip (2006), find that if native labor and immigrant labor are perfect substitutes in production, then, depending on the elasticity of output with respect to the capital stock in a Cobb-Douglas production function, an increase in immigration can either increase or decrease the steady-state level of consumption per capita.

While the studies described above focus on the impact on steady-state consumption per native, several other studies analyze the fiscal impact of immigration. Recognizing that the fiscal impact of an immigrant depends on the immigrant's age at the time of immigration,

as well as the skill of the immigrant, Storesletten (2000) calculates for the United States "the smallest increase in annual immigration required to balance the budget, given that the government is free to choose the distribution of age and skills of new immigrants, while restricted to keeping the current tax and spending policies unchanged." (p. 301). In a similar vein, though applied to Sweden, which has a much larger public sector than the United States, Storesletten (2003) computes the present value of the taxes paid by an immigrant minus the amount of government expenditure associated with that immigrant and finds that, for young working-age immigrants, this net present value is positive, though for the average immigrant, this net present value is negative.

Ben-Gad (2004) measures the impact of immigration on welfare as the sum of the impact on labor income and the impact on capital income. In this analysis, immigration reduces the capital-labor ratio, driving up the rate of return on capital, which benefits native owners of capital. However, if immigrants consume more publicly provided goods than they pay in taxes, the overall impact of immigration on welfare may be negative.

Within the set of papers that examine the fiscal impact of immigration, some papers focus on the role of public pensions or Social Security. Lee and Miller (2000), Naito (2014), and Schou (2006) present simulation results from models that include age-dependent public transfers. Razin and Sadka (1999, 2000), develop and apply an elegant overlapping generations model of immigration in a country with a pay-as-you-go Social Security system. Young workers are born with low skills and heterogeneous levels of innate ability that determine how much time must be diverted from employment to acquire a high level of skills. Low-ability individuals find it too costly to acquire the high level of skills and hence remain as workers with low skills and a low wage rate. High-ability individuals choose to devote some time to acquire a high level of skills and earn a high wage rate, but work for a smaller amount of time than low-skill workers. In both papers, workers contribute a fraction of their wage income when young to pay for a publicly-provided retirement benefit that is equal for all old consumers. In a small open economy, in which factor prices are exogenously given, these papers show that an increase in immigration is Pareto improving and domestic consumers would choose the immigration rate to be as high as possible. However, Razin and Sadka

(2000) shows, using numerical simulations, that if factor prices are endogenously determined, as would be the case in a closed economy, or an open economy with limited factor mobility, then it is possible that migration imposes a loss on all income groups in every generation, or imposes losses on some groups while yielding gains to other groups. In this case of endogenous factor prices, the optimal rate of immigration is not addressed.

Shea and Woodfield (1996) examines optimal immigration in a model without capital. Output is a function of skilled labor and unskilled labor, and unskilled labor can be transformed into skilled labor by using some skilled labor to teach unskilled labor. Interestingly, this paper uses the term "Golden Rule education policy" to describe the education policy that maximizes steady-state consumption per worker.

The literature on the Golden Rule capital-labor ratio begins with Phelps (1961), which shows that the level of the capital-labor ratio that maximizes consumption per unit of labor in the steady state of a closed economy is characterized by a marginal product of capital, net of depreciation, equal to the growth rate of the economy. A steady state in which the capital-labor ratio is higher than the Golden Rule capital-labor ratio has lower consumption per unit of labor than in the Golden Rule steady state. Such a steady state is said to have a dynamically inefficient overaccumulation of capital. Diamond (1965) develops an overlapping generations model with capital and demonstrates that it is possible for the competitive steady state to have a dynamically inefficient overaccumulation of capital. Cass (1972) develops a complete characterization of dynamic inefficiency for economies both outside the steady state and in the steady state. Abel et al. (1989) develops sufficient conditions to declare a stochastic economy either dynamically inefficient or dynamically efficient, though many stochastic economies cannot be declared either dynamically efficient or dynamically inefficent by these sufficient conditions.

In most of the Golden Rule literature, including the papers cited above, the growth rate of the labor force is exogenous. However, the introduction of a permanent stream of immigration increases the growth rate of the labor force. Some early papers on the Golden Rule have endogenized the growth rate of the labor force through channels other than immigration. Davis (1969) specifies the population growth rate to depend endogenously on

the level of income. Samuelson (1980) endogenizes the size of the labor force by endogenizing the labor force participation rate to be an increasing function of the wage rate. In both of these papers, the capital-labor ratio that maximizes consumption per person in the steady state is no longer characterized by Phelps's condition that the net marginal product of capital equals the growth rate. However, as we emphasize in the current paper, the introduction of a permanent flow of immigration, which effectively endogenizes the growth rate of population and labor force, does not change the Phelpsian Golden Rule prescription that the capital-labor ratio that maximizes steady-state consumption per person is attained when the net marginal product of capital equals the growth rate of the labor forcre. To reconcile these findings, we note that the labor force in both Davis (1969) and Samuelson (1980) is a function of an endogenous variable, but the growth rate of the labor force in our model is a function of a decision variable–namely the rate of immigration–chosen by the planner.

2 One-Period Lifetimes

We begin with a simple demographic structure in which each person lives for only one period. A generation t person is born at the beginning period t and gives birth to $1+\gamma \geq 1$ generation $t+1$ people at the end of period t, which is the beginning of period $t+1$. The country is a closed economy in the sense that it does not trade goods, services, or assets with other countries. However, in each period t, M_t people born outside the country immigrate into the country and join the N_t native-born people in the employed labor force,

$$L_t = N_t + M_t. \tag{1}$$

Like natives, each immigrant gives birth to $1+\gamma$ children, and since these children are born in the country of interest, they are natives. Since all people in the labor force in period t give birth to $1+\gamma$ native-born children in generation $t+1$, we have

$$N_{t+1} = (1+\gamma)(N_t + M_t) = (1+\gamma)(1+m_t)N_t, \tag{2}$$

where $m_t \equiv \frac{M_t}{N_t}$ is the ratio of immigrants to natives in generation t. Thus, $(1+\gamma)(1+m_t)$ is the gross growth rate of natives from generation t to generation $t+1$.

In period t, capital, K_t, and labor, L_t, are used to produce homogeneous output equal to $F(K_t, L_t)$, where $F(K_t, L_t)$ is strictly increasing, concave, and linearly homogeneous in K_t and L_t. Also, $F(0, L_t) = 0$, $F(K_t, 0) = 0$, $F_K(0, L_t) = \infty$, $F_L(K_t, 0) = \infty$, $F_K(\infty, L_t) = 0$, and $F_L(K_t, \infty) = 0$. Natives each consume c_t and immigrants each consume x_t so aggregate consumption in period t is $N_t c_t + M_t x_t$. Aggregate gross investment in period t is $K_{t+1} - (1-\delta) K_t$, where $\delta \in [0,1]$ is the depreciation rate of capital per period. The government buys an amount of output equal to $g_t > 0$ per person to provide rival, non-excludable goods, so total government purchases in period t equal $(N_t + M_t) g_t$. The sum of aggregate consumption, gross investment, and government purchases equals total output. Thus, the expenditure identity for GDP is

$$N_t c_t + M_t x_t + K_{t+1} - (1-\delta) K_t + (N_t + M_t) g_t = F(K_t, N_t + M_t). \tag{3}$$

Divide both sides of equation (3) by N_t and use $m_t \equiv \frac{M_t}{N_t}$ to obtain

$$c_t + m_t x_t + \frac{K_{t+1}}{K_t}\frac{K_t}{L_t}\frac{L_t}{N_t} - (1-\delta)\frac{K_t}{L_t}\frac{L_t}{N_t} + (1+m_t) g_t = \frac{L_t}{N_t}\frac{1}{L_t} F(K_t, L_t). \tag{4}$$

Define the capital-labor ratio in period t, $k_t \equiv \frac{K_t}{L_t}$, and the intensive form of the production function $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$, where $f'(k) > 0$, $f(k) - k f'(k) > 0$, and $f''(k) < 0$ for all $k > 0$, $f(0) = 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. These assumptions imply that[1]

$$\lim_{k \to 0} k f'(k) = 0. \tag{5}$$

Use the definitions $k_t \equiv \frac{K_t}{L_t}$ and $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$ to rewrite equation (4) as

$$c_t + m_t x_t + \frac{K_{t+1}}{K_t} k_t (1+m_t) - (1-\delta) k_t (1+m_t) + (1+m_t) g_t = (1+m_t) f(k_t). \tag{6}$$

[1] $k f'(k) > 0$ for $k > 0$ and $f(k) - k f'(k) > 0$ for $k > 0$ imply that $0 \leq \lim_{k \to 0} k f'(k) \leq \lim_{k \to 0} f(k) = 0$, so $\lim_{k \to 0} k f'(k) = 0$.

Henceforth, we confine attention to steady states, which are characterized by constant values of $\frac{M_t}{N_t} = m$ and $\frac{K_t}{L_t} = k$. The values of c_t, x_t, and g_t are constant in a steady state and the steady-state values of these variables are indicated simply by dropping the time subscript. Equation (2) implies that in a steady state, the gross growth rate of N_t is

$$\Gamma(m) \equiv (1+\gamma)(1+m). \tag{7}$$

The constancy of $\frac{M_t}{N_t}$ implies that the gross growth rate of M_t is also $\Gamma(m)$. Therefore, the gross growth rate of the labor force, $L_t = N_t + M_t$, is $\Gamma(m)$, and the constancy of the capital-labor ratio, $\frac{K_t}{L_t}$, implies that the gross growth rate of K_t is equal to $\Gamma(m)$.

In a steady state, the GDP expenditure identity in equation (6) can be written as

$$c = C(k,m) - mx, \tag{8}$$

where

$$C(k,m) \equiv (1+m)\left[f(k) + (1-\delta)k - \Gamma(m)k - g\right] \tag{9}$$

is the aggregate consumption of natives and immigrants, normalized by the number of natives.

We assume that there is an unlimited supply of potential immigrants who are willing to immigrate into the country provided that their consumption in the destination country, x, exceeds an exogenous threshhold $\xi > 0$. One can think of ξ as the amount of consumption the potential immigrant would have in their country of origin, plus the costs of travelling to the new country, getting adjusted, and myriad other considerations. It is sufficient for our purposes that such a threshhold value of x exists.

For given values of ξ and g, the planner chooses k, m, and x to maximize steady-state consumption per native, $c = C(k,m) - mx$ in equation (8). Because consumers live for only period, there is no need to introduce a utility function in this section. Provided that utility is strictly increasing in consumption, maximizing c is equivalent to maximizing the

utility derived from c; for immigrants, a threshold level of consumption ξ is equivalent to a threshold level of utility to be willing to immigrate into the country.[2]

Formally, the planner solves

$$\max_{k>0,m\geq 0,x} C(k,m) - mx \qquad \text{s.t. } x \geq \xi. \tag{10}$$

Because x enters the maximand in equation (10) negatively and linearly, the optimal value of x is ξ. Therefore, the planner's problem simplifies to

$$\max_{k>0,m\geq 0} C(k,m) - m\xi. \tag{11}$$

Partially differentiate $C(k,m) - m\xi$ with respect to k and set $\frac{\partial C(k,m)}{\partial k}$ equal to zero to obtain

$$1 + f'(k) - \delta = \Gamma(m). \tag{12}$$

Equation (12) implies that $r \equiv f'(k) - \delta$, which is the net rate of return on capital, equals the net growth rate of the economy, $\Gamma(m) - 1$. The equality of the net rate of return on capital and the net growth rate of the economy is the celebrated Golden Rule from Phelps (1961), extended to include endogenous growth resulting from immigration.

Definition 1 *Define $k(m)$ as the unique value of k that satisfies the Golden Rule condition in equation (12) for a given value of m, and define $k_G \equiv k(0)$ as the Phelpsian Golden Rule value of k in the absence of immigration.*

Equation (12) implies that

$$\lim_{m\to\infty} k(m) = 0. \tag{13}$$

The definition of $k(m)$ together with equation (12) implies that $1-\delta-\Gamma(m) = -f'(k(m))$,

[2]In Section 3, all consumers live for two-periods, and we introduce utility functions and a threshold level of utility for potential immigrants.

so the normalized aggregate consumption in equation (9) can be expressed as a function of only m

$$C\left(k\left(m\right),m\right)=\left(1+m\right)\left[f\left(k\left(m\right)\right)-k\left(m\right)f'\left(k\left(m\right)\right)-g\right]. \tag{14}$$

Lemma 1 $\frac{dC(k(m),m)}{dm}=f\left(k\left(m\right)\right)-k\left(m\right)f'\left(k\left(m\right)\right)-g-\Gamma\left(m\right)k\left(m\right)$.

Having defined the function $k\left(m\right)$, the Golden Rule capital-labor ratio for a given value of m, the planner's problem can be written as the choice of a single scalar, m, as $\max_{m\geq 0}\Omega\left(m\right)$, where

$$\Omega\left(m\right)\equiv C\left(k\left(m\right),m\right)-m\xi. \tag{15}$$

Differentiate $\Omega\left(m\right)$ with respect to m, and use Lemma 1 to obtain

$$\Omega'\left(m\right)=f\left(k\left(m\right)\right)-k\left(m\right)f'\left(k\left(m\right)\right)-g-\Gamma\left(m\right)k\left(m\right)-\xi. \tag{16}$$

Definition 2 $\overline{\xi}\equiv f\left(k_G\right)-k_Gf'\left(k_G\right)-\left(1+\gamma\right)k_G-g$.

To ensure that $\overline{\xi}$ is positive, we maintain the following two-part assumption throughout the paper.[3]

Assumption 1 *Assume that*

1. $\alpha_G\equiv\frac{k_Gf'(k_G)}{f(k_G)}<\frac{\gamma+\delta}{2(\gamma+\delta)+1-\delta}\leq\frac{1}{2}$
2. $0\leq g<f\left(k_G\right)-k_Gf'\left(k_G\right)-\left(1+\gamma\right)k_G$

[3]The upper bound on g in Statement 2 of Assumption 1 is $f\left(k_G\right)-k_Gf'\left(k_G\right)-\left(1+\gamma\right)k_G=f\left(k_G\right)-k_Gf'\left(k_G\right)-\left(1+f'\left(k_G\right)-\delta\right)k_G=f\left(k_G\right)-2k_Gf'\left(k_G\right)-\left(1-\delta\right)k_G=f\left(k_G\right)\left\{1-2\frac{k_Gf'(k_G)}{f(k_G)}-\frac{1-\delta}{\gamma+\delta}\frac{k_Gf'(k_G)}{f(k_G)}\right\}$ $=f\left(k_G\right)\left\{1-\left[2+\frac{1-\delta}{\gamma+\delta}\right]\alpha_G\right\}$, where $\alpha_G\equiv\frac{k_Gf'(k_G)}{f(k_G)}$. Therefore, the upper bound will be positive if and only if $1-\left[2+\frac{1-\delta}{\gamma+\delta}\right]\alpha_G>0$, or equivalently, $\alpha_G<\left[\frac{2(\gamma+\delta)+1-\delta}{\gamma+\delta}\right]^{-1}=\frac{\gamma+\delta}{2(\gamma+\delta)+1-\delta}\leq\frac{1}{2}$.

We have not shown that $\arg\max_{m\geq 0}\Omega(m)$ is unique. To avoid any ambiguity, we define the optimal steady-state value of m to be

$$m^* \equiv \min\left\{\arg\max_{m\geq 0}\Omega(m)\right\}. \tag{17}$$

Proposition 1 *If* $\xi < \overline{\xi} \equiv f(k_G) - k_G f'(k_G) - (1+\gamma)k_G - g$, *then*

1. $\Omega'(0) = \overline{\xi} - \xi > 0$
2. $\lim_{m\to\infty}\Omega'(m) = -g - \xi < 0$
3. $0 < m^* < \infty$
4. $0 < k^* = k(m^*) < k_G$

Since the proof of Proposition 1 is both simple and instructive, we include it in the text.

Proof of Proposition 1. Assume that $\xi < \overline{\xi}$. Evaluate $\Omega'(m)$ in equation (16) at $m = 0$, using $k(0) = k_G$, to obtain $\Omega'(0) = f(k_G) - f'(k_G)k_G - (1+\gamma)k_G - g - \xi = \overline{\xi} - \xi > 0$, which is Statement 1. Use $\lim_{m\to\infty} k(m) = 0$ from equation (13) along with the expression for $\Omega'(m)$ from equation (16) and the fact, from equation 5, that $\lim_{k\to 0} kf'(k) = 0$ to obtain $\lim_{m\to\infty}\Omega'(m) = -g - \xi < 0$, which is Statement 2. Statement 1 implies that m^* cannot equal 0, and Statement 2 implies that m^* cannot be infinite. Therefore, m^* is positive and finite, which is Statement 3. Finally, equation (12) implies that $1 + f'(k^*) - \delta = \Gamma(m^*) > \Gamma(0) = 1 + f'(k_G) - \delta$. Therefore, since $\Gamma(m^*)$ is finite, $f'(k^*)$ is finite and hence $k^* > 0$. Also, since $f'(k^*) > f'(k_G)$ and $f''(k) < 0$, we have $k^* = k(m^*) < k_G$, which completes the proof of Statement 4. ■

We now interpret the assumption $\xi < \overline{\xi}$ in Proposition 1 and show its implications. Use the definition of $\overline{\xi}$ and rearrange $\xi < \overline{\xi}$ to obtain

$$\underbrace{f(k_G) - f'(k_G)k_G}_{\text{marginal product of labor}} > \underbrace{\xi + g}_{\text{output to immigrant}} + \underbrace{(1+\gamma)k_G}_{\text{new capital needed}}. \tag{18}$$

Imagine that the economy is in a Golden Rule steady state with zero immigration so that $k = k_G$. Now consider the impact of an immigrant. The left hand side of equation

(18), which is the marginal product of labor, shows the additional output produced by the immigrant. The right hand side of equation (18) shows the resources that must be diverted from the consumption of natives in the form of: (1) ξ units of consumption for the immigrant; (2) g additional units of government purchases; and (3) the cost of providing each of the immigrant's $1+\gamma$ children with k_G units of capital. If the immigrant's contribution to output on the left hand side of equation (18) exceeds the costs to the natives on the right hand side, then the optimal immigration rate is positive.

It is well-known[4] that in the steady state of a closed economy without immigration, the Golden Rule is characterized by capital income equal to capital investment, or equivalently, non-capital income, in the form of wages, equal to non-investment expenditure in the form of private consumption, $C(k_G, 0)$, and government expenditure, g. Therefore, we can replace labor income on the left hand side of equation (18) by $C(k_G, 0) + g$, and then subtract g from both sides of the resulting equation, to obtain

$$\underbrace{C(k_G, 0)}_{\text{consumption of native}} > \underbrace{\xi}_{\text{consumption of immigrant}} + \underbrace{(1+\gamma) k_G}_{\text{new capital needed}} . \tag{19}$$

The condition in equation (19) is stronger than simply "the consumption of an immigrant, ξ, is less than the consumption of a native, $C(k_G, 0)$." Equation (19) requires that the consumption of an immigrant is less than the consumption of a native by at least the cost of providing each of the immigrant's $1+\gamma$ children with k_G units of capital.

3 Two-Period Lifetimes

In this section, we extend the lifetimes of individual consumers from one period to two periods, so that in a competitive economy, individual consumers have a meaningful consumption/saving decision and hold capital. Extending lifetimes to two periods introduces some additional considerations. Consumers who live for two periods obtain utility from consumption in both periods. We assume that natives and immigrants have identical util-

[4]For example, Phelps (1961) and Abel et al. (1989).

ity functions. The planner maximizes the utility of a representative native in the steady state. In addition, potential immigrants evaluate their prospective utility from consumption in their new country and compare it to an exogenous threshold level of utility. This comparison is complicated by the fact that the lifetime utility from consumption in their new country depends on the rate of return in the new country, which is an endogenous variable. Fortunately, much of the analytic apparatus developed for the case of one-period lifetimes continues to be useful when we extend lifetimes to two periods.

We begin by extending the national income accounting framework to reflect the assumption that all consumers live for two periods. As in the case of one-period lifetimes, each consumer, whether a native or an immigrant, gives birth to $1+\gamma$ children at the end of the first period of their life. The generations overlap so that in each period, there are young and old natives as well as young and old immigrants. All young consumers work and all old consumers are retired, so total employed labor in period t, is $L_t \equiv N_t + M_t$, where, as in the version with one-period lifetimes, N_t natives are born at the beginning of period t and M_t foreign-born young consumers immigrate into the country at the beginning of period t. In period t, each of the N_t young natives consumes c_t^y and each of the M_t young immigrants consumes x_t^y. Also in period t, the N_{t-1} old natives each consume c_t^o and the M_{t-1} old immigrants each consume x_t^o, so aggregate consumption in period t is $N_t c_t^y + N_{t-1} c_t^o + M_t x_t^y + M_{t-1} x_t^o$. As in Section 2, aggregate gross investment in period t is $K_{t+1} - (1-\delta) K_t$. For simplicity, we scale government purchases of goods to the number of young consumers, so that, as in Section 2, government purchases of goods equal $(N_t + M_t) g$. Therefore, the GDP expenditure identity is

$$N_t c_t^y + N_{t-1} c_t^o + M_t x_t^y + M_{t-1} x_t^o + K_{t+1} - (1-\delta) K_t + (N_t + M_t) g = F(K_t, L_t). \quad (20)$$

Divide both sides of equation (20) by N_t to obtain

$$c_t^y + \frac{N_{t-1}}{N_t} c_t^o + \frac{M_t}{N_t} x_t^y + \frac{M_{t-1}}{N_{t-1}} \frac{N_{t-1}}{N_t} x_t^o + \frac{K_{t+1}}{K_t} \frac{K_t}{L_t} \frac{L_t}{N_t} - (1-\delta) \frac{K_t}{L_t} \frac{L_t}{N_t} + \left(1 + \frac{M_t}{N_t}\right) g = \frac{L_t}{N_t} \frac{1}{L_t} F(K_t, L_t). \quad (21)$$

Recall that $m_t \equiv \frac{M_t}{N_t}$, $k_t \equiv \frac{K_t}{L_t}$, and $f(k_t) \equiv \frac{1}{L_t} F(K_t, L_t)$ to rewrite equation (21) as

$$\begin{array}{r} c_t^y + \frac{N_{t-1}}{N_t} c_t^o + m_t x_t^y + m_{t-1} \frac{N_{t-1}}{N_t} x_t^o \\ + \frac{K_{t+1}}{K_t} k_t (1 + m_t) - (1 - \delta) k_t (1 + m_t) + (1 + m_t) g \end{array} = (1 + m_t) f(k_t). \tag{22}$$

In a steady state, $k_t = k$, $m_t = m$, $c_t^y = c^y$, $c_t^o = c^o$, $x_t^y = x^y$ and $x_t^o = x^o$ are all constant. As in Section 2, the gross growth rates of K_t, L_t, M_t, and N_t are all equal to $\Gamma(m) \equiv (1 + m)(1 + \gamma)$ in a steady state. Therefore, in a steady state, equation (22) can be written, after isolating the consumption of natives on the left hand side, as

$$c^y + \frac{1}{\Gamma(m)} c^o = C(k, m) - m \left[x^y + \frac{1}{\Gamma(m)} x^o \right], \tag{23}$$

where, from equation (9), $C(k, m) \equiv (1 + m)[f(k) + (1 - \delta) k - \Gamma(m) k - g]$ is the aggregate consumption of natives and immigrants normalized by the number of young natives.

The planner chooses c^y, c^o, x^y, x^o, k, and m to maximize the utility of a representative native in the steady state, $u(c^y, c^o)$, subject to the constraint in equation (23) and the constraint that the utility of a representative immigrant in the steady state, $u(x^y, x^o)$, is at least as high as a threshold value, $\overline{u}$,

$$u(x^y, x^o) \geq \overline{u}, \tag{24}$$

which is the analog of the constraint $x \geq \xi$ in Section 2. As in Section 2, this constraint will hold as an equality and leads to the optimal values of x^y and x^o. To obtain expressions for x^y and x^o, we need to introduce a specific utility function. For simplicity, we assume that the utility function is logarithmic and write the binding version of the constraint in equation (24) as

$$(1 - \beta) \ln x^y + \beta \ln x^o = \overline{u}, \tag{25}$$

where $0 \leq \beta < 1$.

For a given value of m, the planner chooses x^y and x^o to minimize the amount of con-

sumption devoted to immigrants in each period in the steady state subject to the constraint in equation (25). This minimized value of consumption devoted to immigrants is $mX(m)$, where

$$X(m) \equiv \min_{x^y, x^o} x^y + \frac{1}{\Gamma(m)} x^o \quad \text{s.t.} \quad (1-\beta)\ln x^y + \beta \ln x^o = \overline{u}. \tag{26}$$

We show in equations (A.8) and (A.9) in Appendix A that

$$X(m) = \xi (1+m)^{-\beta} \leq \xi, \tag{27}$$

where

$$\xi \equiv \exp(\overline{u})(1-\beta)^{-(1-\beta)} \beta^{-\beta} (1+\gamma)^{-\beta} > 0. \tag{28}$$

The definition of ξ in equation (28) is equivalent to ξ in the case of one-period lifetimes in Section 2. In both cases, ξ is a measure of the threshold value of the present value of consumption required by an immigrant. With one-period lifetimes, this present value is invariant to the rate of return, but with two-period lifetimes, the present value depends on the rate of return, $1 + f'(k) - \delta = \Gamma(m) = (1+\gamma)(1+m)$, which accounts for the factor $(1+m)^{-\beta}$ in equation (27).

Now substitute $X(m)$ for $x^y + \frac{1}{\Gamma(m)} x^o$ in equation (23). Since natives have the same utility function that immigrants have, the planner's problem can be written as

$$\max_{c^y, c^o} (1-\beta)\ln c^y + \beta \ln c^o \quad \text{s.t.} \quad c^y + \frac{1}{\Gamma(m)} c^o = C(k,m) - mX(m). \tag{29}$$

With the logarithmic utility function in equation (29), the optimal values of consumption

by natives are[5]

$$c^y = (1-\beta)\left[C\left(k,m\right) - mX\left(m\right)\right] \tag{30}$$

and

$$c^o = \Gamma\left(m\right)\beta\left[C\left(k,m\right) - mX\left(m\right)\right]. \tag{31}$$

Evaluating the logarithmic utility function of natives at the optimal values of c^y and c^o in equations (30) and (31), respectively, yields

$$\begin{aligned} u\left(c^y, c^o\right) &= \ln\left[C\left(k,m\right) - mX\left(m\right)\right] + \beta \ln \Gamma\left(m\right) \\ &\quad + (1-\beta)\ln\left(1-\beta\right) + \beta \ln \beta. \end{aligned} \tag{32}$$

Partially differentiate the right hand side of equation (32) with respect to k and set the partial derivative equal to zero obtain $\frac{\partial C(k,m)}{\partial k} = 0$, just as in the case with one-period lifetimes in Section 2. Therefore, the expression for the Golden Rule in equation (12) in Section 2 continues to hold with two-period lifetimes. Thus, the function relating the Golden Rule capital stock to m, $k\left(m\right)$, is unchanged from Section 2, though, of course, the optimal values of m and k may differ from their values in that section.

The planner's problem can be formulated as the choice of a single scalar, m, by substituting $k\left(m\right)$ for k and $\left(1+\gamma\right)\left(1+m\right)$ for $\Gamma\left(m\right)$ in equation (32) and ignoring additive constants to obtain $\max_{m\geq 0} V\left(m\right)$, where

$$V\left(m\right) \equiv \begin{cases} \ln\left[C\left(k\left(m\right),m\right) - mX\left(m\right)\right] + \beta\ln\left(1+m\right), & \text{if } C\left(k\left(m\right),m\right) - mX\left(m\right) > 0 \\ -\infty, & \text{if } C\left(k\left(m\right),m\right) - mX\left(m\right) \leq 0. \end{cases} \tag{33}$$

[5]See equations (A.4) and (A.5) in Appendix A, with c^y replacing z^y, c^o replacing z^o, and $C\left(k,m\right) - mX\left(m\right)$ replacing ϕ.

The following proposition states that $V(m)$ attains a maximum at a finite value of m.

Proposition 2 $0 \leq \arg\max_{m \geq 0} V(m) < \infty$.

We have not shown that $\arg\max_{m \geq 0} V(m)$ is unique. Hence, as in Section 2, to avoid any ambiguity, we define the optimal steady-state value of m to be

$$m^{**} \equiv \min \left\{ \arg\max_{m \geq 0} V(m) \right\}. \tag{34}$$

Therefore, if $m^{**} > 0$, we know that $V(m^{**}) > V(0)$ and the optimal rate of immigration in the steady state is unambiguously positive. The two asterisks in m^{**} indicate that we are considering two-period lifetimes, in contrast to m^{*}, which refers to one-period lifetimes.

Definition 3 $\overline{\overline{\xi}} \equiv (1+\beta)\,\overline{\xi} + \beta\,(1+\gamma)\,k_G > \overline{\xi} > 0$.

Lemma 2 $V'(0) = \frac{1}{C(k_G,0)} \left(\overline{\overline{\xi}} - \xi \right)$.

Lemma 2 is useful in proving the following proposition.

Proposition 3 *If* $\xi < \overline{\overline{\xi}}$, *then*

1. $0 < m^{**} < \infty$
2. $0 < k^{**} < k_G$

Definition 4 $\mathcal{C}(m) \equiv c^y + \frac{1}{\Gamma(m)} c^o$ *in a steady state with immigration ratio* m *and* $k = k(m)$.

Use the definition of $\mathcal{C}(m)$ to rewrite the constraint in equation (29) as

$$\mathcal{C}(m) + mX(m) = C(k, m). \tag{35}$$

Now divide both sides of equation (35) by $1+m$, substitute $k(m)$ for k, and use the expression for $C(k(m), m)$ from equation (14) to obtain

$$\frac{1}{1+m} \mathcal{C}(m) + \frac{m}{1+m} X(m) = f(k(m)) - k(m) f'(k(m)) - g. \tag{36}$$

Equation (36) shows that for $m > 0$, $f(k(m)) - k(m) f'(k(m)) - g$ lies between $\mathcal{C}(m)$ and $X(m)$. The next step is to determine whether $\mathcal{C}(m^{**})$ is larger than or smaller than $X(m^{**})$.

Proposition 4 *If* $\xi < \overline{\overline{\xi}}$ *and* $\beta \leq \overline{\beta} \equiv \frac{1+\gamma}{\delta+\gamma} \frac{\alpha_G}{1-\alpha_G - \frac{g}{f(k_G)}}$, *then* $\mathcal{C}(m^{**}) > X(m^{**})$.

We interpret $\overline{\beta}$ at the end of Section 4.1.

4 Implementation of the Planner's Optimal Steady State in a Competitive Economy

Up to this point, we have focused on the planner's problem. Now we consider competitive economies in which consumers live for two periods. In the competitive steady state, young consumers, both native and immigrant, earn pre-tax wage income

$$w(k) \equiv f(k) - k f'(k). \tag{37}$$

Our goal is to derive a package of taxes and government bonds that leads the steady state of a competitive economy with two-period lifetimes to have $k = k^{**}$ and $m = m^{**}$, which are the planner's optimal values of k and m, respectively, derived in Section 3.

We consider a package of taxes and government bonds that pays for government purchases g and implements the planner's optimal steady-state values of c^y, c^o, x^y, and x^o, together with $m = m^{**}$, $k = k^{**} = k(m^{**})$, and $w = w^{**} = w(k(m^{**}))$. Specifically, we consider (1) an entry fee, τ_M, paid by a young immigrant at the end of the period in which they immigrate (after earning wages in the destination country); (2) a consumption tax at rate $\tau_C > -1$, levied on the consumption purchases of all consumers–natives and immigrants– in both periods of life; and (3) one-period government bonds that pay an interest rate $r = f'(k) - \delta = \Gamma(m) - 1$.

Neither of the taxes creates a distortion. The entry tax is a tax on the act of immigrating into the destination country, but it does not distort the decision to immigrate because the

combination of tax rates is chosen to allow an immigrant to purchase goods with present value $X(m^{**})$, which is sufficient to allow potential immigrants to achieve their threshold level of utility in the destination country. The consumption tax does not introduce intertemporal distortions because the tax rate is the same during both periods of a consumer's life; and since all consumers face the same consumption tax rate, there is no distortion between natives and immigrants. The role of government bonds is simply to adjust the portion of consumers' saving allocated to capital, so that the steady state of the competitive economy has the same value of k as in the planner's optimal steady state. Since the interest rate, r, equals the net growth rate of the economy, $\Gamma(m) - 1$, bonds have no impact on the government's budget constraint in the steady state. That is, with $r = \Gamma(m) - 1$ in the planner's optimal steady state, bond seignorage equals zero.[6]

The budget constraint of a young immigrant in the planner's optimal steady state is

$$(1 + \tau_C) x^y + \frac{1}{1+r} (1 + \tau_C) x^o = w^{**} - \tau_M. \tag{38}$$

In the planner's optimal steady state, $X(m^{**}) = x^y + \frac{1}{\Gamma(m^{**})} x^o$ and $1 + r = \Gamma(m^{**})$, so the budget constraint of an immigrant in equation (38) can be written as

$$(1 + \tau_C) X(m^{**}) = w^{**} - \tau_M. \tag{39}$$

The optimal consumption expenditure (including the consumption tax) of a young immigrant is $(1 + \tau_C) x^y = (1 - \beta)(w^{**} - \tau_M)$, so the saving of a young immigrant, after paying the entry fee, τ_M, and spending $(1 + \tau_C) x^y$ on consumption is

$$s_M = (w^{**} - \tau_M) - (1 + \tau_C) x^y = \beta (w^{**} - \tau_M). \tag{40}$$

[6] A non-distortionary capital income tax, as described in Abel and Panageas (2024), also would have no impact on the government budget constraint when $r = \Gamma(m) - 1$, because, as shown in that paper, the tax revenue collected by such a tax is proportional to $r - (\Gamma(m) - 1)$.

The budget constraint of a young native in the planner's optimal steady state is

$$(1+\tau_C)\,c^y + \frac{1}{1+r}\,(1+\tau_C)\,c^o = w^{**}. \tag{41}$$

Since $1+r = \Gamma(m^{**})$ in the planner's optimal steady state, the budget constraint of a native in the planner's optimal steady state can be written, using $\mathcal{C}(m) \equiv c^y + \frac{1}{\Gamma(m)}c^o$ from Definition 4, as

$$(1+\tau_C)\,\mathcal{C}(m^{**}) = w^{**}. \tag{42}$$

The optimal consumption expenditure of a young native in the planner's optimal steady state is $(1+\tau_C)\,c^y = (1-\beta)\,w^{**}$, so the saving of this native is

$$s_N = w^{**} - (1+\tau_C)\,c^y = \beta w^{**}. \tag{43}$$

In period t in the steady state, the aggregate saving of the N_t young natives, who each save $s_N = \beta w^{**}$, and the M_t young immigrants, who each save $s_M = \beta\,(w^{**} - \tau_M)$, is

$$S_t = \beta\left[N_t w^{**} + M_t\,(w^{**} - \tau_M)\right]. \tag{44}$$

In the aggregate, young consumers use their saving at the end of period t to purchase capital, K_{t+1}, and one-period government bonds, B_{t+1}. Therefore, $K_{t+1} + B_{t+1} = S_t$, so equation (44) implies that

$$K_{t+1} + B_{t+1} = \beta\left[N_t w^{**} + M_t\,(w^{**} - \tau_M)\right]. \tag{45}$$

Dividing both sides of equation (45) by N_t implies[7] that in the planner's optimal steady state,

$$(1+m^{**})\,\Gamma(m^{**})\,k^{**} + (1+m^{**})\,\Gamma(m^{**})\,b = \beta\left[w^{**} + m^{**}\,(w^{**} - \tau_M)\right], \tag{46}$$

[7]Recall that $\frac{1}{N_t} = \frac{L_t}{N_t}\frac{L_{t+1}}{L_t}\frac{1}{L_{t+1}} = (1+m)\,\Gamma(m)\,\frac{1}{L_{t+1}}$.

where $b_{t+1} \equiv \frac{B_{t+1}}{L_{t+1}} = b$ in the steady state.

In period t in the steady state, the government's budget constraint is

$$M_t \tau_M + \tau_C \left(N_t c^y + M_t x^y + N_{t-1} c^o + M_{t-1} x^o \right) = \left(N_t + M_t \right) g. \tag{47}$$

Dividing both sides of the government budget constraint in equation (47) by N_t implies that in the planner's optimal steady state,

$$m^{**} \tau_M + \tau_C \left[\mathcal{C}\left(m^{**} \right) + m^{**} X\left(m^{**} \right) \right] = \left(1 + m^{**} \right) g. \tag{48}$$

Equations (39), (42), (46), and (48) form a four-equation linear system in τ_M, τ_C, and b. However equation (48) is a linear combination of equations (39) and (42),[8] so we ignore equation (48). The other three equations form a linear system that we write in matrix form (with the equations ordered as (42), (39), and (46)) as

$$\Lambda \times \begin{bmatrix} 1 + \tau_C \\ \tau_M \\ b \end{bmatrix} = \begin{bmatrix} w^{**} \\ w^{**} \\ \beta \left(1 + m^{**} \right) w^{**} - \left(1 + m^{**} \right) \Gamma\left(m^{**} \right) k^{**} \end{bmatrix}, \tag{49}$$

where

$$\Lambda \equiv \begin{bmatrix} \mathcal{C}\left(m^{**} \right) & 0 & 0 \\ X\left(m^{**} \right) & 1 & 0 \\ 0 & m^{**} \beta & \left(1 + m^{**} \right) \Gamma\left(m^{**} \right) \end{bmatrix}. \tag{50}$$

Since $\det \Lambda = \left(1 + m^{**} \right) \Gamma\left(m^{**} \right) \mathcal{C}\left(m^{**} \right) > 0$, equations (42), (39), and (46) are independent and there is a unique solution for the fiscal parameters τ_C, τ_M, and b. Also, since Λ is

[8] Multiply equation (39) by m^{**}, then add equation (42) minus equation (48) to the resulting equation to obtain $\mathcal{C}\left(m^{**} \right) + m^{**} X\left(m^{**} \right) = \left(1 + m^{**} \right) w^{**} - \left(1 + m^{**} \right) g$. Divide both sides by $1 + m^{**}$ to obtain $w^{**} - g = \frac{1}{1+m^{**}} \mathcal{C}\left(m^{**} \right) + \frac{m^{**}}{1+m^{**}} X\left(m^{**} \right) = f\left(m^{**} \right) - k\left(m^{**} \right) f'\left(k\left(m^{**} \right) \right) - g$, where the second equality follows from equation (36). Therefore, $w^{**} = f\left(m^{**} \right) - k\left(m^{**} \right) f'\left(k\left(m^{**} \right) \right)$, which holds identically in m^{**}, since $w^{**} = w\left(k^{**} \right) = f\left(m^{**} \right) - k\left(m^{**} \right) f'\left(k\left(m^{**} \right) \right)$. Hence, equations (39), (42), and (48) are linearly dependent.

triangular, the solution to the matrix equation in (49) particularly easy to obtain as

$$\begin{bmatrix} 1+\tau_C \\ \tau_M \\ b \end{bmatrix} = \begin{bmatrix} \frac{w^{**}}{\mathcal{C}(m^{**})} \\ \left(1-\frac{X(m^{**})}{\mathcal{C}(m^{**})}\right) w^{**} \\ \beta\left[1+m^{**}\frac{X(m^{**})}{\mathcal{C}(m^{**})}\right]\frac{w^{**}}{(1+m^{**})\Gamma(m^{**})}-k^{**} \end{bmatrix}. \tag{51}$$

Proposition 5 *Under the assumptions of Proposition 4, optimal* $\tau_M > 0$.

One of the assumptions of Proposition 4, namely $\xi < \bar{\bar{\xi}}$, implies that $m^{**} > 0$. Proposition 5 states that under the assumptions of Proposition 4, the fiscal package that implements the optimal steady state includes a positive entry fee for immigrants. This entry fee is one channel by which resources are extracted from immigrants to increase the welfare of natives.

To interpret τ_C, observe that the budget constraint in equation (48) implies

$$\tau_C = \frac{(1+m^{**})\, g - m^{**}\tau_M}{\mathcal{C}\,(m^{**}) + m^{**}X\,(m^{**})}, \tag{52}$$

where the denominator, $\mathcal{C}\,(m^{**}) + m^{**}X\,(m^{**})$, is positive. If the government purchases an amount of output, $(1+m^{**})\, g$, greater than the revenue collected from the entry fee paid by immigrants, $m^{**}\tau_M$, then the tax rate on consumption, τ_C, is positive. However, if government purchases are smaller than the revenue collected from entry fees, the tax rate on consumption is negative.

To interpret the optimal value of b in equation (51), multiply both sides of $b = \beta\left[1+m^{**}\frac{X(m^{**})}{\mathcal{C}(m^{**})}\right] \times \frac{w^{**}}{(1+m^{**})\Gamma(m^{**})} - k^{**}$ by $N_t\,(1+m^{**})\,\Gamma\,(m^{**}) = L_t\Gamma\,(m^{**}) = L_{t+1}$, so that in any period t when the economy is in the optimal steady state

$$bL_{t+1} = \beta\left[N_t\mathcal{C}\,(m^{**}) + M_tX\,(m^{**})\right]\frac{w^{**}}{\mathcal{C}\,(m^{**})} - k^{**}L_{t+1}. \tag{53}$$

Now use $1+\tau_C = \frac{w^{**}}{\mathcal{C}(m^{**})}$ from equation (51) along with $b = \frac{B_{t+1}}{L_{t+1}}$ and $k^{**} = \frac{K_{t+1}}{L_{t+1}}$ to rewrite equation (53) as

$$B_{t+1} = \beta\,(1+\tau_C)\left[N_{\,t}\mathcal{C}\,(m^{**}) + M_tX\,(m^{**})\right] - K_{t+1}. \tag{54}$$

Equation (54) states that when the optimal steady state prevails, the amount of government bonds held by young consumers at the end of period t, B_{t+1}, equals $\beta(1+\tau_C)[N_t\mathcal{C}(m^{**}) + M_t X(m^{**})]$, which is the aggregate saving by these consumers during t, minus K_{t+1}, which is the amount of capital that these consumers hold at the end of period t. The optimal value of b can be positive, zero, or negative. To illustrate the role of the saving propensity β in determining the sign of the optimal steady-state value of government bonds, we consider the optimal steady state of an economy without immigration in the following subsection.

4.1 Implementation of Planner's Optimal Steady State in the Absence of Immigration

In this subsection we examine the implementation of the planner's optimal steady state in which the optimal rate of immigration, m^{**}, equals zero. We begin with the following proposition, which presents sufficient conditions for $m^{**} = 0$.

Proposition 6 *With two-period lifetimes, if* $\xi > (1+\widehat{m})^{\beta}[f(k_G) - k_G f'(k_G) - g]$, *then* $m^{**} = 0$, *where* $\widehat{m}$ *is defined in Lemma 3 in the Appendix.*

If it turns out that $m^{**} = 0$, then there are no immigrants in the steady state, so equation (39) and the entry fee, τ_M, are no longer relevant. The optimal steady-state values of the consumption tax rate, τ_C, and the amount of government bonds per young person, b, are determined by equations (42) and (46).

Proposition 7 *If* $m^{**} = 0$, *then*

1. $k^{**} = k(0) = k_G$
2. *optimal* $\tau_C = \frac{g}{w(k_G) - g}$
3. *optimal* $b = \left(\beta \frac{1-\alpha_G}{\alpha_G} \frac{\delta+\gamma}{1+\gamma} - 1\right) k_G$, *where* $\alpha_G \equiv \frac{k_G f'(k_G)}{f(k_G)}$.

In a planner's optimal steady state with $m^{**} = 0$, we have $k^{**} = k(0) = k_G$, which is the Phelpsian Golden Rule capital-labor ratio. In the Phelpsian Golden Rule, total non-investment spending, specifically, the sum of consumption and government purchases, equals

total wage income. Thus, consumption normalized by the number of young consumers equals $w(k_G) - g$, so that consumption tax revenue collected by the government is $\tau_C (w(k_G) - g)$. This tax revenue equals the amount of government purchases per capita, g, so $\tau_C = \frac{g}{w(k_G)-g}$, which is Statement 2 of Proposition 7.

To illustrate Statement 3 of Proposition 7, consider the steady state of a competitive economy with $m^{**} = 0$. Government purchases are paid for by the consumption tax as described above. Setting $m^{**} = 0$, which implies $\Gamma(m^{**}) = 1 + \gamma$ and $k^{**} = k_G$, equation (46) yields

$$(1+\gamma) k_G + (1+\gamma) b = \beta w(k_G). \tag{55}$$

In a steady state with $k = k_G$, equation (55) equates the asset accumulation by young consumers at the end of a period, $(1+\gamma)(k_G + b)$, with their saving out of wage income during that period, $\beta w(k_G)$.

Now define β_G to be the value of β for which equation (55) holds with $b = 0$, that is,

$$(1+\gamma) k_G = \beta_G w(k_G), \tag{56}$$

so that when $k = k_G$, the saving of young consumers, $\beta_G w(k_G)$, is equal to the capital held in the Golden Rule steady state with zero immigration. Dividing each side of equation (55) by the corresponding side of equation (56) yields

$$\frac{b}{k_G} = \frac{\beta - \beta_G}{\beta_G}. \tag{57}$$

Equation (57) implies that the optimal amount of government bonds will be positive if and only if the saving propensity, β, is greater than β_G, which is the value of the saving propensity for which the level of saving by young consumers equals the Golden Rule amount of capital. Thus, if $\beta > \beta_G$, saving by young consumers would be excessive, in the sense that economy would have a dynamically inefficient overaccumulation of capital. The role of government bonds in this situation is to absorb saving in excess of the Golden Rule capital stock.

To relate the value of b in equation (57) to the value of b in Statement 3 of Proposition 7. Use $k_G = \frac{\alpha_G f(k_G)}{f'(k_G)} = \frac{\alpha_G f(k_G)}{\delta+\gamma}$ to substitute for k_G on the left hand side of equation (56) and $w(k_G) = f(k_G) - k_G f'(k_G) = (1-\alpha_G) f(k_G)$ to substitute for $w(k_G)$ on the right hand side of equation (56) to obtain

$$\beta_G = \frac{1+\gamma}{\delta+\gamma} \frac{\alpha_G}{1-\alpha_G}. \tag{58}$$

Using the expression for β_G from equation (58), Statement 3 of Proposition 7 can be written as $b = \left(\frac{\beta}{\beta_G} - 1\right) k_G$, which equivalent to equation (57).

To interpret $\overline{\beta} \equiv \frac{1+\gamma}{\delta+\gamma} \frac{\alpha_G}{1-\alpha_G - \frac{g}{f(k_G)}}$ in Proposition 4, suppose that instead of a consumption tax, the government pays for its purchases by levying a lump-sum tax equal to g on each young consumer. In this case, the saving of a young consumer is $\beta(w(k) - g)$. Therefore, if $\beta = \overline{\beta} = \frac{1+\gamma}{\delta+\gamma} \frac{\alpha_G f(k_G)}{w(k_G)-g}$, then the saving of young consumers when $k = k_G$ equals $\overline{\beta}(w(k_G) - g)$ $= \frac{1+\gamma}{\delta+\gamma} \alpha_G f(k_G) = \frac{1+\gamma}{\delta+\gamma} k_G f'(k_G) = (1+\gamma) k_G$. That is, if g is financed by lump-sum taxes on the young, the nontrivial steady state of the economy without immigration and without government bonds satisfies the Phelpsian Golden Rule with $k = k_G$ if $\beta = \overline{\beta}$.

5 Conclusion

We have developed and analyzed a neoclassical growth model extended to allow for the possibility of immigration into a country. Except for possible immigration into the country, the country is a closed economy that does not trade goods, services, or assets with other countries. Our goal is to answer two important questions related to immigration. First, is the optimal rate of immigration positive in the steady state? To address this question, we need to specify a social welfare function. Motivated by Phelps's seminal work on the Golden Rule of capital accumulation, we specify the social welfare function to be the utility of a representative consumer in the long run–in particular, the utility of a representative native-born consumer in the long run. We consider the optimization problem solved by a planner who allocates output to consumption of natives, consumption of immigrants, and capital to be carried into the following period. We derive sufficient conditions for the optimal amount

of immigration to be positive. Second, if the optimal rate of immigration of positive, what package of taxes and government bonds will lead the steady state of a competitive economy to be the same as the planner's optimal steady state? We show that the planner's optimal steady state with positive immigration can be implemented by a combination of (1) an entry fee imposed on immigrants, (2) a consumption tax levied at the same rate (which could be positive, negative, or zero) on all consumers, regardless of age or country of birth, and (3) government bonds.

To distill the first question to its essential elements, in Section 2 we consider a non-overlapping generations model, in which each person lives for only one period and gives birth to a fixed number, $1 + \gamma$, of children at the end of that period. Because individual lifetimes are only one period in length, there is no need to specify a utility function for consumers. Provided that utility is strictly increasing in consumption, the maximum level of utility per native in the steady state is attained by the maximum level of consumption per native in the steady state. Thus, the social welfare function is the level of consumption of a representative native in the steady state, in the spirit of Phelps's Golden Rule. If the optimal level of immigration is positive in the long run, the steady state that attains the highest level of consumption by a representative native satisfies a simple extension of the Golden Rule in which the net rate of return on capital, r, satisfies $1 + r = (1 + m)(1 + \gamma)$, which is the gross growth rate of population; in the absence of immigration, $m = 0$ so the extended Golden Rule becomes $r = \gamma$, which is Phelps's Golden Rule.

The level of the capital-labor ratio for which $r = \gamma$ has a special role in our analysis and we denote that value of the capital-labor ratio as k_G. We assume that there is an unlimited supply of potential immigrants provided that they receive a level of consumption in the destination country at least as large as an exogenous threshold. We prove that a sufficient condition for the rate of immigration to be positive in the optimal steady state is that the consumption of a representative native when $k = k_G$ exceeds the sum of the threshold level of consumption of an immigrant plus the cost of devoting $(1 + \gamma) k_G$ units of output to provide capital to an immigrant's $1 + \gamma$ children. While it is true that immigrants receive lower consumption per person than natives receive, they receive the threshold level of consumption

at which they are willing to immigrate. Notably, if the social welfare function were to treat natives and immigrants equally, the planner would allocate equal consumption to natives and immigrants. The steady state that would maximize this alternative social welfare function would have $k = k_G$, which is attained only when $m = 0$. Potential immigrants are not helped at all by this alternative social welfare function.

Although the model with one-period lifetimes provides a simple answer to the first question–whether the optimal immigration rate is positive–consumers in that framework do not make any intertemporal decisions. In particular, those consumers do not hold capital from one period to the next. To introduce a capital investment decision for individual consumers, in Section 3 we modify the model to have overlapping generations of consumers who each live for two periods, as in the classic Diamond (1965) model. We must introduce a utility function, which, for simplicity, we assume to be logarithmic in the consumption of each period of life. Instead of the social welfare function being the level of *consumption* of a representative native in the steady state, we assume that the social welfare function is the level of *utility* of a representative native in the steady state. Similarly, there is a threshold level of utility rather consumption, that makes potential immigrants willing to immigrate. Despite these changes, the extended Golden Rule, $1+r = (1+m)(1+\gamma)$, is unchanged from the one-period lifetime version of the model in Section 2, though specific values of r and m in the planner's optimal steady state can differ from those in the one-period lifetime version of the model. We derive sufficient conditions for rate of immigration in the planner's optimal steady state to be positive; these conditions are similar in spirit, though more complicated in detail, than the sufficient condition in Section 2.

Having characterized the planner's optimal steady state, we turn, in Section 4, to the implementation question. Specifically, we take as given the planner's optimal steady-state allocations of consumption to young natives, young immigrants, old natives, and old immigrants, as well as the optimal immigration rate, m, and optimal capital-labor ratio (and resulting wage rate and rate of return on capital). We then derive a fiscal package consisting of an entry fee for immigrants, a uniform consumption tax on all consumers, and government bonds, that lead a competitive economy to attain these optimal allocations. The

consumption tax collects enough revenue to pay for any excess of government purchases over revenue from the entry fee. Because the extended Golden Rule implies that rates of return on capital and on government bonds equal the population growth rate in the optimal steady state, seignorage from bonds is zero in that steady state. The role of government bonds in the steady state is to induce the amount of capital held in consumers' portfolios to match the capital-labor ratio dictated by the extended Golden Rule. A positive amount of government bonds absorbs saving in excess of the Golden Rule capital stock, thereby avoiding a dynamically inefficient overaccumulation of capital.

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Appendix

A Two-Period Consumption Problem with Log Utility

Consider a person who lives for two periods and has logarithmic utility

$$u\left(z^y, z^o\right) = (1-\beta)\ln z^y + \beta \ln z^o \tag{A.1}$$

where z^y is consumption when young and z^o is consumption when old. The person chooses z^y and z^o to

$$\max_{z^y, z^o} (1-\beta)\ln z^y + \beta \ln z^o \quad \text{s.t.} \quad z^y + \frac{1}{1+r} z^o = \phi, \tag{A.2}$$

which can be written as

$$\max_{z^y} (1-\beta)\ln z^y + \beta \ln\left[(1+r)\left(\phi - z^y\right)\right]. \tag{A.3}$$

The first-order condition associated with the maximization in equation (A.3) is $\frac{1-\beta}{z^y} = \frac{\beta}{\phi - z^y}$, which implies

$$z^y = (1-\beta)\,\phi \tag{A.4}$$

and hence

$$z^o = (1+r)\,\beta\phi. \tag{A.5}$$

Evaluating the utility function $u\left(z^y, z^o\right)$ from equation (A.1) at the optimal values of z^y and z^o in equations (A.4) and (A.5), respectively, yields

$$u\left(z^y, z^o\right) = \ln\phi + \beta\ln(1+r) + (1-\beta)\ln(1-\beta) + \beta\ln\beta. \tag{A.6}$$

A.1 Planner's Optimal Steady State

In the planner's optimal steady state, equation (12) implies that $1+r = 1+f'(k)-\delta = \Gamma(m) = (1+m)(1+\gamma)$. The value of ϕ for which the maximized value of $u(z^y, z^o)$ equals $\overline{u}$ is

$$\phi = \exp(\overline{u})(1-\beta)^{-(1-\beta)}\beta^{-\beta}(1+\gamma)^{-\beta}(1+m)^{-\beta}. \tag{A.7}$$

Since $\phi = z^y + \frac{1}{1+r}z^o = z^y + \frac{1}{\Gamma(m)}z^o$ in the maximization problem in equation (A.2), it is the value of $X(m)$ in the minimization problem in equation (26), which is the dual to the maximization problem (with z^y replacing x^y and z^o replacing x^o) in equation (A.2). Therefore,

$$X(m) = \xi(1+m)^{-\beta}, \tag{A.8}$$

where

$$\xi \equiv \exp(\overline{u})(1-\beta)^{-(1-\beta)}\beta^{-\beta}(1+\gamma)^{-\beta} > 0. \tag{A.9}$$

B Selected Proofs and Lemmas

Proof of Lemma 1. Differentiate $1+f'(k(m))-\delta = \Gamma(m)$ from equation (12) with respect to m to obtain $f''(k(m))k'(m) = 1+\gamma$, which implies $k'(m) = \frac{1+\gamma}{f''(k(m))}$. Differentiate $C(k(m),m)$ in equation (14) with respect m to obtain $\frac{dC(k(m),m)}{dm} = \frac{C(k(m),m)}{1+m} - (1+m)k(m)f''(k(m))k'(m)$. Now use $k'(m) = \frac{1+\gamma}{f''(k(m))}$ to obtain $\frac{dC(k(m),m)}{dm} = \frac{C(k(m),m)}{1+m} - \Gamma(m)k(m) = f(k(m)) - k(m)f'(k(m)) - g - \Gamma(m)k(m)$. ■

Lemma 3 *There exists a finite $\widehat{m} > 0$ such that $C(k(m),m) - mX(m) \leq 0$ for all $m \geq \widehat{m}$.*

Proof of Lemma 3. Observe that equation (12) implies $1+f'(k(m))-\delta = \Gamma(m) = (1+\gamma)(1+m)$, which implies that (1) $\lim_{m\to\infty} f'(k(m)) = \infty$ and hence $\lim_{m\to\infty} k(m) = 0$ and (2) $\lim_{m\to\infty}\Gamma(m)k(m) = \lim_{m\to\infty}[1+f'(k(m))-\delta]k(m) = \lim_{m\to\infty} f'(k(m))k(m)$

$= 0$, where the final equality follows from equation (5). From equation (14), $C(k(m), m) = (1+m)[f(k(m)) - k(m) f'(k(m)) - g]$, so $C(k(m), m) - mX(m) = (1+m) A(m)$, where $A(m) \equiv f(k(m)) - k(m) f'(k(m)) - g - \frac{m}{1+m} X(m)$. Observe that $\lim_{m\to\infty} A(m) = \lim_{m\to\infty} \left[f(k(m)) - k(m) f'(k(m)) - g - \frac{m}{1+m} X(m)\right] =$ $\lim_{m\to\infty} \left[-g - \frac{m}{1+m} X(m)\right] \leq -g < 0$. Therefore, there exists a finite $\widehat{m}$ such that $A(m) \leq 0$ and hence $C(k(m), m) - mX(m) \leq 0$ for all $m \geq \widehat{m}$. ■

Proof of Proposition 2. Define $\widehat{V}(m) \equiv \max\{V(m), V(0) - \varepsilon\}$ for small $\varepsilon > 0$. Lemma 3 implies that if $m \geq \widehat{m}$, where $\widehat{m}$ is finite, then $C(k(m), m) - mX(m) \leq 0$, so $V(m) = -\infty$, and hence $\widehat{V}(m) = V(0) - \varepsilon < V(0) = \widehat{V}(0)$. Therefore, if $\widehat{V}(m)$ attains a maximum, it must be attained for $0 \leq m \leq \widehat{m}$. $\widehat{V}(m)$ is continuous on the closed and bounded domain $\{m : 0 \leq m \leq \widehat{m}\}$, so it attains a maximum on this set. That is, $\arg\max_{m\geq 0} \widehat{V}(m) \in \{m : 0 \leq m \leq \widehat{m}\}$. Observe that $\max_{0\leq m\leq \widehat{m}} \widehat{V}(m) = \max_{0\leq m\leq \widehat{m}} V(m)$, so $\arg\max_{0\leq m\leq \widehat{m}} V(m) = \arg\max_{0\leq m\leq \widehat{m}} \widehat{V}(m) \in \{m : 0 \leq m \leq \widehat{m}\}$. ■

Proof of Lemma 2. Recall $\widehat{V}(m) \equiv \max\{V(m), V(0) - \varepsilon\}$ for small $\varepsilon > 0$, from the proof of Proposition 2. Assume that $V(m_0) > V(0) - \varepsilon$ for small $\varepsilon > 0$, which implies that $\widehat{V}(m) = V(m) = \ln[C(k(m), m) - mX(m)] + \beta \ln(1+m)$ for $m \geq 0$ in a neighborhood of m_0. Therefore, for $m \geq 0$ in this neighborhood, $\widehat{V}'(m) = V'(m) =$ $\frac{1}{C(k(m),m)-mX(m)} \left\{\frac{dC(k(m),m)}{dm} - X(m) - mX'(m)\right\} + \frac{\beta}{1+m}$. Differentiate $X(m) = \xi(1+m)^{-\beta}$ from equation (27) with respect to m to obtain $X'(m) = -\beta \frac{X(m)}{1+m}$. Use this expression for $X'(m)$ and rearrange the resulting equation to obtain $V'(m) = \frac{1}{C(k(m),m)-mX(m)} \times$ $\left\{\frac{dC(k(m),m)}{dm} + \beta \frac{C(k(m),m)}{1+m} - X(m)\right\}$. Use $\frac{dC(k(m),m)}{dm} = \frac{C(k(m),m)}{1+m} - \Gamma(m) k(m)$ from Lemma 1 to substitute $\frac{dC(k(m),m)}{dm} + \Gamma(m) k(m)$ for $\frac{C(k(m),m)}{1+m}$ to obtain $V'(m) = \frac{1}{C(k(m),m)-mX(m)}$ $\times \left\{(1+\beta) \frac{dC(k(m),m)}{dm} + \beta\Gamma(m) k(m) - X(m)\right\}$.

To evaluate $V'(m)$ at $m = 0$, note that $m_0 = 0$ satisfies the assumption $V(m_0) > V(0) - \varepsilon$ for small $\varepsilon > 0$, and use $k(0) = k_G$, $\Gamma(0) = 1 + \gamma$, and $X(0) = \xi$ to obtain $\widehat{V}'(0) = V'(0) =$ $\frac{1}{C(k_G,0)} \left\{(1+\beta) \frac{dC(k(m),m)}{dm}|_{m=0} + \beta(1+\gamma) k_G - \xi\right\}$. Observe from $\Omega(m) \equiv C(k(m), m) - m\xi$ in equation (15) that $\frac{dC(k(m),m)}{dm} = \Omega'(m) + \xi$, so Statement 1 of Proposition 1 implies $\frac{dC(k(m),m)}{dm}|_{m=0} = \Omega'(0) + \xi = \overline{\xi}$, and hence $V'(0) = \frac{1}{C(k_G,0)} \left\{(1+\beta)\overline{\xi} + \beta(1+\gamma) k_G - \xi\right\}$, which proves this lemma. ■

Proof of Proposition 3. Assume that $\xi < \overline{\overline{\xi}}$. From Lemma 2, $V'(0) = \frac{1}{C(k_G,0)}\left(\overline{\overline{\xi}} - \xi\right) > 0$, so $m^{**} > 0$. Also, the definition of m^{**} in equation (34) and Lemma 3, which defines and characterizes $\widehat{m}$, imply that $m^{**} \leq \widehat{m} < \infty$, which completes the proof of Statement 1. Statement 2 follows from $k^{**} \equiv k(m^{**})$ and the facts that $k(m)$ is strictly decreasing, $k(0) = k_G$, and $\lim_{m\to\infty} k(m) = 0$. ■

Proof of Proposition 4. This proof proceeds in two parts. Part I: Prove that $\mathcal{C}(0) > X(0)$. The definition $\overline{\xi} \equiv f(k_G) - k_G f'(k_G) - (1+\gamma)k_G - g$ along with $\mathcal{C}(0) = f(k_G) - k_G f'(k_G) - g$ imply that $\overline{\xi} = \mathcal{C}(0) - (1+\gamma)k_G$. Therefore, $\overline{\overline{\xi}} \equiv (1+\beta)\overline{\xi} + \beta(1+\gamma)k_G = (1+\beta)[\mathcal{C}(0) - (1+\gamma)k_G] + \beta(1+\gamma)k_G = \mathcal{C}(0) + \beta\mathcal{C}(0) - (1+\gamma)k_G$, so $\overline{\overline{\xi}} \leq \mathcal{C}(0)$ if and only if $\beta \leq \frac{(1+\gamma)k_G}{\mathcal{C}(0)} = \frac{(1+\gamma)k_G}{f(k_G)-k_G f'(k_G)-g}$. Use $\alpha_G f(k_G) = k_G f'(k_G) = (\gamma+\delta)k_G$, to obtain $\frac{(1+\gamma)k_G}{f(k_G)-k_G f'(k_G)-g} = \frac{(1+\gamma)\frac{\alpha_G f(k_G)}{\gamma+\delta}}{f(k_G)-k_G f'(k_G)-g} = \frac{1+\gamma}{\gamma+\delta}\frac{\alpha_G}{1-\alpha_G-\frac{g}{f(k_G)}} = \overline{\beta}$. Therefore, $\overline{\overline{\xi}} \leq \mathcal{C}(0)$ if and only if $\beta \leq \overline{\beta}$ and hence $X(0) = \xi < \overline{\overline{\xi}} \leq \mathcal{C}(0)$ since $\beta \leq \overline{\beta}$.

Part II: Prove that if $m^{**} > 0$, then $\mathcal{C}(m^{**}) > X(m^{**})$. Define $U_M(m) = \ln X(m) + \beta\ln(1+f'(k(m))-\delta) + (1-\beta)\ln(1-\beta) + \beta\ln\beta$ and $U_N(m) = \ln\mathcal{C}(m) + \beta\ln(1+f'(k(m))-\delta) + (1-\beta)\ln(1-\beta) + \beta\ln\beta$ as the planner's optimal values of utility of immigrants and natives, respectively, in a steady state with immigration ratio m and $k = k(m)$. Observe that $U_N(m) - U_M(m) = \ln\mathcal{C}(m) - \ln X(m)$, which implies $U_N(0) - U_M(0) = \ln\mathcal{C}(0) - \ln X(0) > 0$. Since $m^{**} \equiv \min\{\arg\max_{m\geq 0} V(m)\}$ and $m^{**} > 0$, we have $U_N(m^{**}) > U_N(0) > U_M(0) = \overline{u} = U_M(m^{**})$. Therefore, $\ln\mathcal{C}(m^{**}) - \ln X(m^{**}) = U_N(m^{**}) - U_M(m^{**}) > 0$, so $\mathcal{C}(m^{**}) > X(m^{**})$. ■

Proof of Proposition 5. Proposition 4 states that $\mathcal{C}(m^{**}) > X(m^{**})$. Therefore, the solution for τ_M in equation (51) implies that $\tau_M = \left(1 - \frac{X(m^{**})}{\mathcal{C}(m^{**})}\right)w^{**} > 0$. ■

Proof of Proposition 6. Assume that $\xi > (1+\widehat{m})^{\beta}[f(k_G) - k_G f'(k_G) - g]$, where $\widehat{m}$ is defined in Lemma 3. Also, assume, contrary to what is to be proved, that $m^{**} > 0$. From Lemma 3, $m^{**} \leq \widehat{m}$, so equation (26), $X(m) = \xi(1+m)^{-\beta}$, implies $X(m^{**}) \geq X(\widehat{m}) = \xi(1+\widehat{m})^{-\beta} > f(k_G) - k_G f'(k_G) - g \geq f(k(m^{**})) - k(m^{**})f'(k(m^{**})) - g$, where the final inequality follows from the fact that k_G attains the maximum over k of $f(k) - kf'(k) - g$. Evaluate equation (36) at $m = m^{**}$ to obtain $\frac{1}{1+m^{**}}\mathcal{C}(m^{**}) + \frac{m^{**}}{1+m^{**}}X(m^{**}) = f(k(m^{**})) - k(m^{**})f'(k(m^{**})) - g < X(m^{**})$, which implies $\mathcal{C}(m^{**}) < X(m^{**})$ and hence $\mathcal{C}(m^{**}) <$

$f\left(k\left(m^{**}\right)\right)-k\left(m^{**}\right)f'\left(k\left(m^{**}\right)\right)-g$. Use equation (12), $1+f'\left(k\right)-\delta=\Gamma\left(m\right)$, to substitute $1+f'\left(k\right)-\delta$ for $\Gamma\left(m\right)$ in equation (9) to obtain $C\left(k,m\right)=\left(1+m\right)\left[f\left(k\right)-f'\left(k\right)k-g\right]$ so $C\left(k\left(m^{**}\right),0\right)=f\left(k\left(m^{**}\right)\right)-f'\left(k\left(m^{**}\right)\right)k\left(m^{**}\right)-g>\mathcal{C}\left(m^{**}\right)$. A native consumer would prefer a steady state with $k=k\left(m^{**}\right)$ and $m=0$ to a steady state with $k=k\left(m^{**}\right)$ and $m=m^{**}>0$ because the present value of the lifetime consumption in the former steady state is $C\left(k\left(m^{**}\right),0\right)$, which is greater than the present value of lifetime consumption, $\mathcal{C}\left(m^{**}\right)$, in the latter steady state and the rate of return on capital, $1+f'\left(k\left(m^{**}\right)\right)-\delta$, is the same in both steady states. Therefore, the optimal value of m^{**} cannot be positive. ■

Proof of Proposition 7. Assume that $m^{**}=0$. Then $k^{**}=k\left(m^{**}\right)=k\left(0\right)=k_G$, which proves Statement 1. The aggregate consumption of natives, normalized by the number of young natives is $c^y+\frac{1}{\Gamma(0)}c^o=C\left(k_G,0\right)=f\left(k_G\right)+\left(1-\delta\right)k_G-\Gamma\left(0\right)k_G-g=f\left(k_G\right)-k_Gf'\left(k_G\right)-g=w\left(k_G\right)-g$. Therefore, equation (42), which states $\left(1+\tau_C\right)\left(c^y+\frac{1}{\Gamma(m^{**})}c^o\right)=w\left(k\left(m^{**}\right)\right)$, implies that $\left(1+\tau_C\right)=\frac{w(k_G)}{w(k_G)-g}$ and hence $\tau_C=\frac{g}{w(k_G)-g}$, which is Statement 2. If $m^{**}=0$, then equation (46) implies $b=\frac{\beta}{\Gamma(0)}w\left(k_G\right)-k_G$. Since $w\left(k_G\right)=f\left(k_G\right)-k_Gf'\left(k_G\right)=\left(1-\alpha\right)f\left(k_G\right)=\left(1-\alpha\right)\frac{k_Gf'k_G}{\alpha}=\frac{1-\alpha}{\alpha}\left(\delta+\gamma\right)k_G$ and $\Gamma\left(0\right)=1+\gamma$, we have $b=\left(\beta\frac{1-\alpha}{\alpha}\frac{\delta+\gamma}{1+\gamma}-1\right)k_G$, which proves Statement 3. ■