When Lions meets Krugman: A Mean-Field Game framework for the New Economic Geography*

M. Bahlali† R. Boucekkine‡ Q. Petit§

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Abstract

We propose a mean-field game (MFG) set-up to study the dynamics of spatial agglomeration in a continuous space-time framework where trade across locations follows Krugman [1996]. Forward-looking intertemporal utility-maximizing agents work and migrate in a two-dimensional geography and face idiosyncratic shocks. Equilibrium wages depend on their common distribution and adjust statically according to the underlying trade model. We first prove existence and uniqueness of static trade equilibria. We also prove the existence of an MFG equilibrium. In the racetrack economy, we obtain analytical solutions for small sinusoidal perturbations around the flat-earth equilibrium, and we identify the sets of parameters that lead to agglomeration or dispersion after identifying four distinct driving underlying mechanisms.

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†Aix-Marseille University and CNRS, AMSE, France. E-mail: mohamed.bahlali@univ-amu.fr

‡Aix-Marseille University and CNRS, AMSE, France. E-mail: raouf.boucekkine@univ-amu.fr

§EDF, Paris, France. E-mail: quentin.petit@edf.fr

1 Introduction

Admittedly, the *New Economic Geography* (NEG) is one of the most important breakthroughs in the economic literature in the last 40 years. This breakthrough came on both the theoretical and empirical fronts and is still suggesting various major research questions. On the former, Fujita and Thisse [2002] have drawn a parallelism between NEG and growth theory: "*...in a world of globalization, agglomeration may well be the territorial counterpart of economic growth much in the same way as growth seems to foster inequality among individuals...*". Indeed, just like the convergence/divergence conceptual and empirical debates have been driving the economic growth area for decades, similar structuring debates have been taking place on agglomeration/dispersion in NEG since the 90s, in particular following the seminal work of Krugman [1991].

This said, being a quite young field, NEG research is still a highly dynamic field with plenty of interesting (some even crucial) open questions. Many of these questions have to do with the modelling of factor mobility across space, which is inherent in analysis of agglomeration vs dispersion forces. While the traditional two-country models and the much less metaphorical core/periphery models (following Krugman [1991]) have been intensively used to this end, comparatively very few papers have built on a continuous space modelling. The overwhelming majority of NEG models do not integrate saving behavior over time, therefore overlooking capital accumulation over time and capital mobility across space as mentioned by Desmet and Rossi-Hansberg [2010]. A few controlled spatiotemporal models encompassing capital dynamics have been more recently developed sharply expanding and enriching the capital spatial diffusion approach advocated by Isard and Liossatos [1979]. See in particular Brock and Xepapadeas [2008], Boucekkine et al. [2013] and Boucekkine et al. [2019]. All these works are however framed as controlled centralized problems, unfortunately not easily leading to economically relevant enough decentralized counterparts. The latter remains an interesting open question.

In contrast to capital mobility across space, labor mobility has been extensively studied in core-periphery models, and has been also the object of several studies in continuous space, see for example Krugman [1996] and more recently Mossay [2013] to which our work is more directly connected. In the NEG, it's usually assumed that skilled workers can migrate and choose to more to a different work location (see again Fujita and Thisse [2002]). When the decision to migrate is endogenous, it's very often assumed that this decision is based

on current returns. In sum, it's generally assumed that the decision is myopic, which denies any forward-looking behavior component in the migration process and the inherent spatial adjustments. It's only in 2001 that Baldwin [2001] introduced forward-looking migration decisions in the seminal 1991 Krugman core-periphery model, although ultimately only producing a numerical exploration. Ottaviano [2001] is the first work yielding some analytical results in a forward-looking core-periphery model of footloose entrepreneurs.

The literature is much scarcer in the continuous space forward-looking case. To our knowledge, Mossay [2013] is the most comprehensive contribution in this respect, also from the technical viewpoint. We concur with Mossay's view that the generalization of the approach taken in the core-periphery framework (with individual migration adjustment costs) to multi-regional settings is hardly envisageable, and advocate for a more exploration of continuous space models. We mentioned before the earlier attempt of Krugman [1996], to which we must add the pioneering work in continuous space by Fujita et al. [1999].[1] However, in both papers, individual spatial migrations are myopic systematically leading to agglomeration to dominate at equilibrium. We therefore focus from now on the 2013 Mossay contribution.

In this paper, Mossay solves a continuous space version of Krugman's trade model augmented with endogenous forward-looking *local* migrations (in line with Sonnenschein [1981]). Along with Mossay, we assume that consumption adjustment is costless, migration is costly, thereby leading to consumption goods prices adjusting instantaneously while migration involves an adjustment process. More importantly, we use essentially the same decision problem faced by individuals at any location. In particular, in the dynamic forward-looking migration problem, the instantaneous utility function is driven by utility from consumption and by a quadratic cost of mobility (here mobility is measured by velocity). We expand the analysis by making the individual maximization problem stochastic: the law of motion of any individual is assumed to follow a brownian motion subject to idiosyncratic noise. We also consider a two-dimensional spatial setting (a torus) to make the general problem more realistic. We use the racetrack economy (circle) in the last section of the paper for a closer comparison with the huge existing literature on agglomeration/dispersion mechanisms using this spatial setting (including Mossay [2013]).

Much more importantly, we depart radically from Mossay [2013] in the so-

[1] Mossay [2003] is another interesting contribution in the same vein.

lution methodology. We show that the individuals' problems can be solved as a **mean-field** game, and as such, we show that the induced NEG model with trade and migration can be solved using the tools of this currently flourishing branch of game theory. The very reason of why these individuals' problems constitute an MFG is that the (instantaneous) indirect utility from consumption of the varieties produced monopolistically in the continuous space (under instantaneous adjustment of consumption prices) is a function of the **real** wage earned at any location with the associated local price index depends on the distribution of nominal wages in the space. More precisely, the local real prices depend on an average of the distribution of nominal wages weighted by the inherent iceberg trade costs. This is the so-called MFG interaction term which makes this canonical NEG model solvable by the MFG methodology. Since the latter is systematic, we argue that it can be in principle applied to a much larger class of continuous space and time models of trade and (local) migration.

MFGs have been introduced in mathematics about 20 years ago (see (Huang et al. [2006], Lasry and Lions [2007]). They have been more recently applied to economics by a group of mathematicians and economists led by Lions (see e.g. Achdou et al. [2022]). They are typically applied in the case of large populations with certain characteristics' distributions: while no individual can affect the aggregate variables involved, the individual decisions do depend on some interaction terms typically capturing some key indicators/statistics of the distributions. In our case, local price indexes play the role of interaction terms. As the distributions move with all the individuals' decisions, a very fancy and appealing (functional) fixed-point problem arises, ultimately leading to concepts of MFG equilibrium we will touch later in this paper. A further complexity coming from economic applications is the adjonction of general equilibrium conditions.[2] In our NEG application, we have of course to add this layer to the standard MFG layers, that typically Hamilton-Jacobi-Bellman (HJB) equations for individual dynamic decisions, and Fokker-Planck (FP) equations for the associated evolution of the dynamics of distributions involved.

While Mossay [2013] was not contemporaneous with the MFG rise in economics, the author's alternative computations, not directly using HJB nor FP equations but guided by a strong internal mathematical consistency, turn out to be correct. This is our view an authentic *tour de force*. We argue however

[2]Of course, the overwhelming part of MFG applications in areas like industrial organisation are in partial equilibrium and do not have this additional layer. See for example Aydin et al. [2025].

that the NEG literature would benefit from the MFG approach for a systematic anayltical approach to continuous space NEG models. This approach is expanding in both its analytical and computational components, and could allow to go much beyond the current state of continuous space modelling in NEG (and more broadly in economic dynamics).

The paper is organized as follows. Section 2 presents the NEG model and studies both the static (or instantaneous in the terminology of Mossay [2013]) equilibrium and the MFG structure of the migration individual decisions. The (dynamic) MFG equilibrium is defined precisely and characterized. For both the static and dynamic equilibria, we provide existence proofs. To our knowledge, these existence proofs are novel in the NEG literature with continuous space. Section 3 is more applied: we use a recetrack version of the model to study its local dynamic properties by a standard local pertubation method. We single out four distinct mechanisms locally shaping agglomeration vs dispersion. Section 4 concludes.

2 Model

We consider a spatial economy extending along the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, populated by a continuum of agents who choose their location over time in order to maximize their welfare. The model is structured in two layers. At each point in time, a static equilibrium determines local wages and prices based on the spatial distribution of agents. This static layer is a two-dimensional extension of the New Economic Geography model from Krugman [1996], without the agricultural sector. On top of this, agents engage in forward-looking migration decisions under uncertainty. Their dynamics are described by a mean-field game, where individual incentives depend on future wages and the evolving spatial distribution of the population. The interaction between the static and dynamic layers is captured by the fact that the instantaneous utility function in the mean-field game is given by the real wage computed in static equilibrium.

2.1 Static equilibrium

The static layer of our model follows the core structure of the New Economic Geography framework introduced by Krugman [1996], but is adapted to a continuous two-dimensional space and excludes the agricultural sector. We abstract from agriculture to streamline the analysis: since the agricultural good serves

mainly as an exogenous stabilizing force in the standard core-periphery model, removing it allows us to focus only on the endogenous wage differentials and spatial interactions within the manufacturing sector. For the sake of completeness and clarity, we provide a full presentation of the static equilibrium structure below, even though it is well known to readers familiar with the NEG literature.

Let time $t \in [0, \infty)$ be fixed. At each location $x \in \mathbb{T}^2$, the density of residents is denoted by $\mu_t(x)$, where μ_t is a probability measure on $\mathbb{T}^2$. Each location x produces $n_t(x) > 0$ differentiated varieties of a single manufactured good. A representative agent residing at x consumes over the full spectrum of varieties and has Dixit–Stiglitz preferences:

$$U_t(x) = \max_{q_t(\cdot,x,\cdot)} \left\{ \left(\int_{\mathbb{T}^2} \int_0^{n_t(y)} q_t(y,x,i)^{\frac{\sigma-1}{\sigma}} \, di \, dy \right)^{\frac{\sigma}{\sigma-1}} \right\}$$

subject to the budget constraint:

$$\int_{\mathbb{T}^2} \int_0^{n_t(y)} p_t(y,x,i) \, q_t(y,x,i) \, di \, dy = w_t(x)\mu_t(x),$$

where $q_t(y,x,i)$ and $p_t(y,x,i)$ denote the quantity and price of variety i produced at y and consumed at x, $w_t(x)$ is the nominal wage at x, and $\sigma > 1$ is the elasticity of substitution.

Solving the consumer's problem yields the demand function:

$$q_t(y,x,i) = p_t(y,x,i)^{-\sigma} \, P_t(x)^{\sigma-1} \, w_t(x)\mu_t(x), \tag{1}$$

where the local price index is:

$$P_t(x) = \left(\int_{\mathbb{T}^2} \int_0^{n_t(y)} p_t(y,x,i)^{1-\sigma} \, di \, dy \right)^{\frac{1}{1-\sigma}}. \tag{2}$$

The agent's indirect utility at location x is simply the real wage:

$$V(x, w_t) = \frac{w_t(x)}{P_t(x)}.$$

On the production side, we assume that there is a large number of manufacturing firms, each specializing in the production of a single variety. They are in monopolistic competition. Manufacturing firms operate under economies

of scale, requiring both a fixed cost and a constant marginal cost. The labor demand for producing a quantity $Q_t(x,i)$ of variety i at location x follows:

$$L_t(x,i) = \alpha + \beta Q_t(x,i)$$

where α represents the fixed labor requirement, and β is the marginal labor input per unit of output.

Given the isoelastic demand (1), firms set a price $p_t^0(y,i)$ at a constant markup over marginal cost, and the price received by the producer of variety i at y is:

$$p_t^0(y,i) = \frac{\sigma}{\sigma-1}\beta w_t(y) \tag{3}$$

Taking iceberg trade costs into account, the consumer price of a variety i produced at y and consumed at x is:

$$p_t(y,x,i) = p_t^0(y,i)\cdot\tau_d(y,x) = \frac{\sigma}{\sigma-1}\,\beta\, w_t(y)\,\tau_d(y,x),$$

where $\tau_d(x,y) = e^{d|x-y|}$ and $d \geq 0$ captures the intensity of trade frictions.

Zero-profit condition implies that firms produce a constant quantity:

$$Q_t(x,i) = \frac{\alpha(\sigma-1)}{\beta}. \tag{4}$$

This is for the offer side. Let us derive the demand of each variety. Because all varieties are produced at the same scale, the density $n(s,t)$ of manufactured varieties at any location is directly proportional to the local worker density:

$$\mu_t(y) = \int_0^{n_t(y)} L_t(y,i)\,di = \alpha\sigma n(s,t) \tag{5}$$

Injecting (5) and (3) into (2), we get that

$$P_t(x) = \frac{\beta\sigma}{\sigma-1}(\alpha\sigma)^{\frac{1}{\sigma-1}}\left(\int_{\mathbb{T}^2}\mu_t(y)w_t(y)^{1-\sigma}\tau_d(x,y)^{1-\sigma}dy\right)^{\frac{1}{1-\sigma}} \tag{6}$$

Integrating (1) and using (3) and (6), we obtain the demand for variety i produced at x as:

$$Q_t(x,i) = \int_{\mathbb{T}^2}\frac{(\frac{\sigma}{\sigma-1}\beta)^{-\sigma}w_t(x)^{-\sigma}\tau_d(x,y)^{1-\sigma}}{\frac{\beta\sigma}{\sigma-1}(\alpha\sigma)\left(\int_{\mathbb{T}^2}\mu_t(z)w_t(z)^{1-\sigma}\tau_d(y,z)^{1-\sigma}dy\right)}w_t(y)\mu_t(y)dy \tag{7}$$

Equalizing offer (4) and demand (7), we get the following equilibrium equation on wage, for all x in $\mathbb{T}^2$:

$$w_t(x)^\sigma = \left(\frac{\beta}{\alpha(\sigma-1)}\right)\left(\frac{\sigma-1}{\beta\sigma}\right)^\sigma \int_{\mathbb{T}^2}\left[\frac{\tau_d(y,x)^{1-\sigma}}{\int_{\mathbb{T}^2}\mu_t(z)(w_t(z)\tau_d(y,z))^{1-\sigma}dz}w_t(y)\mu_t(y)\right]dy.$$

Following Fujita et al. [1999], p. 55, it is possible to choose the units of measurement for output and the density of firms in order to have $\left(\frac{\beta}{\alpha(\sigma-1)}\right)\left(\frac{\sigma-1}{\beta\sigma}\right)^\sigma = 1$. We thus define a Krugman equilibrium wage as follows.

Definition 1. *Given any probability measure μ on $\mathbb{T}^2$, we say that $w : \mathbb{T}^2 \to \mathbb{R}_+$ is a Krugman equilibrium wage if it satisfies*

$$w(x)^\sigma = \int_{\mathbb{T}^2}\frac{\tau(y,x)^{1-\sigma}w(y)}{\int_{\mathbb{T}^2}(w(z)\tau(y,z))^{1-\sigma}d\mu(z)}d\mu(y), \quad \forall x \in \mathbb{T}^2, \tag{8}$$

and

$$\int_{\mathbb{T}^2} w(x)d\mu(x) = 1. \tag{9}$$

Condition (9) normalizes the average level of wages in the economy. It serves to close the static model by fixing a common unit for nominal variables. Since equation (8) is homogeneous in w, this constraint prevents equilibrium indeterminacy arising from the fact that any scalar multiple of w would otherwise be a valid solution.

Definition 1 provides a continuous-space formulation of the Krugman trade equilibrium with exogenous and immobile labor. While existence and uniqueness results are well established in the discrete case (see, for instance, Alvarez and Lucas [2007], Allen et al. [2020], and Kucheryavyy et al. [2023]) such results have not been established in a continuous setting. The following proposition addresses this gap by proving the existence of a Krugman equilibrium wage for any population distribution and its uniqueness.

Theorem 2. *Fix any μ of $\mathcal{P}(\mathbb{T}^2)$. If τ is symmetric, of class C^1, and there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \tau \leq C \text{ and } \|\tau'\|_\infty \leq C,$$

then there exists a Krugman equilibrium wage map w. Moreover, there exists a constant C' only depending on τ such that

$$\frac{1}{C'} \leq w \leq C' \text{ and } \|w'\|_\infty \leq C. \tag{10}$$

If, furthermore, supp $\mu = \mathbb{T}^2$*, then there is at most one equilibrium.*

The proof relies on a fixed-point argument for existence. The uniqueness is more challenging to obtain. Without condition (9) uniqueness is hopeless: if w_0 satisfies (8), then so does tw_0 for any $t > 0$. Therefore, we reformulate the problem to work in Ξ^0, a linear subspace of the space of continuous functions, in which uniqueness can be recovered. We introduce a function $G : \Xi^0 \to \Xi^0$ such that any Krugman equilibrium w_0 can be mapped to a zero of G, called u_0. We can show that $DG(u_0)$ is an isomorphism of Ξ^0 providing local uniqueness. In terms of global uniqueness, we observe that when $\tau \equiv 1$, there is a unique Krugman equilibrium. Fix any τ, and define $\tau_t = (1-t)+t\tau$. If $G : [0,1]\times\Xi^0 \to \Xi^0$ is such that the zeros of $G(t,\cdot)$ are exactly the Krugman equilibria associated with τ_t, then $G(0,\cdot)$ has only one zero. We can show that no other zeros of $G(t,\cdot)$ appear as t grows to 1. We use the Leray-Schauder degree theory to formalise this reasoning.

Beyond existence and uniqueness, Proposition 2 also provides a regularity result: the Krugman equilibrium wage is Lipschitz-continuous and bounded above and below by two positive constants that depend only on the trade cost function. Crucially, these bounds hold independently of the population distribution. To our knowledge, this is the first result proving such regularity for Krugman-type equilibria in a continuous setting. This property will play an important role in the analysis that follows, as it ensures that the wage function remains smooth enough to guarantee the existence of solutions to the dynamic mean-field game.

2.2 Dynamic equilibrium

So far, the framework has been static. We now extend it to a dynamic setting in which agents can gradually relocate across space over time, while anticipating the future evolution of the economy. We assume that agents are forward-looking: they form expectations about future wages and spatial distributions, and choose their trajectory accordingly in order to maximize expected utility over a finite time horizon.

To formalize this, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and fix a time horizon $T > 0$. Each agent controls their own location, which evolves as a stochastic process. More precisely, the position $X(t)$ of a represen-

tative agent follows

$$\begin{cases} dX(t) = \beta(t)\,dt + \sqrt{2\nu}\,dB(t), \quad 0 \le t \le T, \\ \mathcal{L}(X_0) = \mu_0, \end{cases} \tag{11}$$

where $\nu > 0$ is the diffusivity parameter, $(B(t))_{t\geq 0}$ is a standard Brownian motion, and $(\beta(t))_{t\geq 0}$ is a control process representing the agent's velocity.

At any time t, given a trajectory of spatial wage profiles $(w(t))_{t\geq 0}$, the agent chooses their velocity so as to maximize their expected discounted utility:

$$\max_{(\beta(t))_{t\geq 0}} \mathbb{E}\left[\int_0^T e^{-\rho t}\left[V(X(t), w(t)) - c(\beta(t))\right] dt\right] \tag{12}$$

where $V(x, w)$ is the real wage at location x, and $c(\beta(t)) = \frac{c_0}{2}|\beta(t)|^2$ is a quadratic adjustment cost of migration effort, with $c_0 > 0$. The quadratic form is standard in the literature (see, e.g., Baldwin and Venables [1994], Mossay [2013]), as it captures increasing marginal costs of movement and allows for analytical tractability. Moreover, we assume migration is local (that is, the agents adjust their position gradually over time) which is consistent with the continuous adjustment frameworks used in Sonnenschein [1981], Mossay [2006, 2013].

The function $V(x, w)$ describes the real wage available to a worker at location x, given a spatial wage profile w, and is defined by

$$V(x, w) = \frac{w(x)}{\left(\int_{\mathbb{T}^2} (w(z)\tau_d(x, z))^{1-\sigma} d\mu(z)\right)^{\frac{1}{1-\sigma}}},$$

where $\tau_d(x, z)$ is the trade cost function between locations x and z, and $\sigma > 1$ is the elasticity of substitution.

We now define a dynamic equilibrium as follows:

Definition 3. *We say that a flow of distributions $(\mu(t))_{t\in[0,T]} \in C([0, T], \mathcal{P}(\mathbb{T}^2))$ is a dynamic equilibrium if there exists a trajectory of wages $(w(t))_{t\in[0,T]}$, a stochastic process $(X(t))_{t\in[0,T]}$, and an optimal control $(\beta(t))_{t\in[0,T]}$, such that:*

1. *$X(t)$ satisfies the stochastic differential equation* (11)*;*
2. *The control $(\beta(t))$ maximizes the agent's expected utility* (12)*;*
3. *The law of $X(t)$ satisfies $\mathcal{L}(X(t)) = \mu(t)$ for all $t \in [0, T]$;*

4. *At each time t, $w(t)$ is the wage profile determined by the static Krugman equilibrium* (8)-(9) *given the distribution $\mu(t)$.*

This equilibrium describes a mean-field interaction: each agent takes the aggregate distribution μ as given, but collectively their actions determine its evolution. The wages $w(t)$ at each instant depend on $\mu(t)$, and thus influence agents' incentives via the real wage function $V(x, w(t))$.

To characterize optimal behavior, let

$$u(t,x) = \min_{(\beta(s))_{s\geq t}} \mathbb{E}\left\{ \int_t^T e^{-\rho(s-t)} \left[c_0 \frac{|\beta(s)|^2}{2} - V\left[X(s), w(s, X(s)), \mu(s)\right] \right] ds \middle| X(t) = x \right\} \tag{13}$$

be the value function of the problem. Then u solves a Hamilton–Jacobi–Bellman (HJB) equation reflecting the trade-off between current costs and expected future gains:

$$\begin{cases} -\partial_t u - \nu \Delta u + \rho u + H(x, \nabla u, w(t), \mu(t)) = 0, & (t,x) \in [0,T) \times \mathbb{T}^2 \\ u(T,x) = 0, & x \in \mathbb{T}^2 \end{cases}$$

where, for all $(x, p, w, \mu) \in \mathbb{T}^2 \times \mathbb{R}^2 \times C^1(\mathbb{T}^2) \times \mathcal{P}(\mathbb{T}^2)$,

$$H(x,p,w,\mu) = \max_{\beta \in \mathbb{R}^2} \left\{ -p \cdot \beta - c_0 |\beta|^2/2 + V(x,w,\mu) \right\}$$

is the Hamiltonian of the problem. The optimal control maximizes the Hamiltonian is

$$\beta^*(t,x) = \operatorname*{argmax}_{\beta \in \mathbb{R}^2} \left\{ -\nabla u(t,x) \cdot \beta - c_0 \frac{|\beta|^2}{2} \right\} = -\frac{1}{c_0} \nabla u(t,x)$$

This expression defines a closed-loop policy: the optimal migration speed at time t depends only on the agent's current location x and on the anticipated wage trajectory w trough u. Given this optimal policy, HJB writes

$$\begin{cases} -\partial_t u - \nu \Delta u - \frac{|\nabla u|^2}{2c_0} + \rho u = -V(x, w(t), \mu(t)), & (t,x) \in [0,T) \times \mathbb{T}^2 \\ u(T,x) = 0, & x \in \mathbb{T}^2. \end{cases} \tag{14}$$

Fixing u and letting free μ, the aggregate distribution evolves according to a Fokker–Planck (FP) equation, which describes how the mass of agents flows

across space under optimal migration:

$$\begin{cases} \partial_t \mu - \nu \Delta \mu + \frac{1}{c_0} \text{div}\,(\mu \nabla u) = 0, & (t,x) \in (0,T] \times \mathbb{T}^2, \\ \mu(0,x) = \mu_0, & x \in \mathbb{T}^2. \end{cases}$$

We now reformulate the notion of dynamic equilibrium accordingly.

Definition 4. *We say that a flow of distributions $\mu \in C^2([0,T] \times \mathbb{T}^2)$ is a mean-field game equilibrium if there exists a value function $u \in C^2([0,T] \times \mathbb{T}^2)$ and a trajectory of wages $w \in C([0,T] \times \mathbb{T}^2)$ such that the following equations are satisfied in the classical sense:*

$$-\partial_t u - \nu \Delta u - \frac{|\nabla u|^2}{2c_0} + \rho u = -V(x, w(t), \mu(t)), \quad (15)$$

$$\partial_t \mu - \nu \Delta \mu + \frac{1}{c_0} \text{div}\,(\mu \nabla u) = 0, \quad (16)$$

$$\int_{\mathbb{T}^2} \frac{\tau(y,x)^{1-\sigma} w(t,y)}{\int_{\mathbb{T}^2} (w(t,z)\tau(y,z))^{1-\sigma} \mu(t,z) dz} \mu(t,y) dy = w(t,x)^{\sigma}, \quad (17)$$

completed with $\mu(0) = \mu_0$, $u(T) = 0$, and such that for any t

$$\int \mu(t,x) dx = 1 \text{ and } \int d\mu(t,x) dx = 1. \quad (18)$$

It is remarkable that Mossay [2013] cleverly derived equations (15)-(16) (without noise and in the racetrack economy) without relying on the mean-field game formalism. He obtained the HJB equation from an Euler-Lagrange equation (Mossay [2013], equation 17) and the FP equation as a conservation law (Mossay [2013], equation 19). His approach demonstrated great intuition. As we will see, the mean-field game formalism allows us to obtain more advanced results (existence of equilibria in the general case, consideration of uncertainty, more refined analysis of agglomeration and dispersion effects).

The following theorem establishes the existence of dynamic equilibria.

Theorem 5. *There exists at least one mean-field game equilibrium, in the sense of Definition 4.*

A fixed-point strategy is used to establish the existence of equilibria. What differs in the MFG system (15)-(17) with the classical MFG system, is the additional equation (17) (see [Achdou et al. (2023), Diogo A. Gomes et al. (2018)] for similar settings). The Krugman equilibrium $w(t,\cdot)$ associated with

$\mu(t,\cdot)$ is uniquely determinated providing the stability of w with respect to μ. This property is needed to show the continuity necessary to apply Schauder's fixed-point theorem leading to the existence of equilibria.

Idea of proof. The key steps to prove the existence of a dynamic equilibirum is by using the Schauder fixed-point theorem. We define a map Λ by the following steps:

1. Fix μ a distribution trajectory.
2. Associate w the trajectory of Krugman equilibria.
3. Associate u a solution of the HJB equation.
4. Define $\Lambda[\mu]$ the solution of the Fokker-Planck equation.

Finding a fixed-point of Λ is equivalent to find a dynamic equilibrium. To apply the Schauder fixed point theorem we need to prove that there exists a non empty, compact and convex set K such that $\Lambda(K) \subset K$, and that it is continuous. The first observation is that μ belongs to $C([0,T],\mathcal{P}(\mathbb{T}^2))$ which is a compact set for the right topology. It is possible to show that $\Lambda(C([0,T],\mathcal{P}(\mathbb{T}^2))) \subset C([0,T],\mathcal{P}(\mathbb{T}^2))$. Concerning the continuity of Λ, thanks to the uniqueness of Krugman equilibria and their regularity in space, it is possible to show that if μ_n converges to μ, then w_n converges to w. Using classical arguments on the stability of the solution of the HJB equation, we deduce that u_n converges to u. Similarly, we establish that $\Lambda[\mu_n]$ converges to $\Lambda[\mu]$. □

3 The racetrack economy

We now consider a one-dimensional application of the model. The geography, known as the racetrack economy, is represented by the circle $\mathbb{T}^1_R$ of radius $R > 0$. This framework, widely used in spatial economics (see, e.g., Krugman [1996] Fujita et al., 1999), provides an analytically convenient benchmark to study the formation of spatial patterns.

Within this setting, the mean-field game structure proves particularly fruitful to analyze the agglomeration and dispersion forces in a dynamic and uncertain environment. Indeed, HJB and Fokker-Planck equations explicitly capture the forward-looking behavior of agents and their exposure to idiosyncratic shocks. This allows us to formalize and quantify how expectations and uncertainty affect the evolution of population distributions in space.

The remainder of this section introduces a linearized version of the model around a spatially uniform steady state. This simplified framework admits closed-form sinusoidal solutions, which allow us to identify and interpret the main forces shaping the dynamics of spatial concentration: myopic adjustment, forward-looking expectations, and the management of uncertainty.

3.1 Equilibrium

We begin by defining a dynamic equilibrium in the racetrack economy, following the same definition as in the two-dimensional case.

Definition 6. *In the racetrack economy, $\mu \in C^{1,2}([0,T] \times \mathbb{T}^1_R)$ is a dynamic equilibrium if there exists a trajectory of Krugman wages $w \in C([0,T] \times \mathbb{T}^1_R)$ and $u \in C^{1,2}([0,T] \times \mathbb{T}^1_R)$ such that:*

$$\begin{aligned} -\partial_t u - \nu \Delta u - \frac{|\nabla u|^2}{2c_0} + \rho u &= -V(x, w(t)), \\ \partial_t \mu - \nu \Delta \mu + \frac{1}{c_0} \operatorname{div}(\mu \nabla u) &= 0, \\ \int_{\mathbb{T}^2} \frac{\tau(y,x)^{1-\sigma} w(t,y)}{\int_{\mathbb{T}^2} (w(t,z)\tau(y,z))^{1-\sigma} \mu(t,z) dz} \mu(t,y) dy &= w(t,x)^{\sigma}, \end{aligned}$$

completed with $\mu(0,\cdot) = \mu_0$, $u(T,\cdot) = 0$, and such that for any $t \geq 0$:

$$\int_{\mathbb{T}^1_R} \mu(t,x) dx = 2\pi R \text{ and } \int_{\mathbb{T}^1_R} w(t,x)\mu(t,x) dx = 2\pi R.$$

Under the assumptions of Theorem 5, such a dynamic equilibrium exists for any initial condition $\mu_0 \in \mathcal{P}(\mathbb{T}^1_R)$. In particular, the system admits a unique stationary uniform solution given by

$$\bar{\mu} = 1, \ \bar{u} = -\left[2\rho R \int_0^{\pi} e^{-d(\sigma-1)x} dx\right]^{-1}, \ \bar{w} = 1, \ \bar{V} = \left[2R \int_0^{\pi} e^{-d(\sigma-1)x} dx\right]^{-1}. \tag{19}$$

To analyze the mechanisms of agglomeration and dispersion, we consider a small deviation from this stationary equilibrium. Specifically, following Krugman [1996], we introduce a sinusoidal perturbation of the initial population distribution:

$$\tilde{\mu}_0(x) = \delta_\mu \cos(kx) \tag{20}$$

with $k \in \frac{1}{2\pi R}\mathbb{N}^*$ and $\delta_\mu > 0$ small. This perturbation affects the initial wage

equilibrium through the static Krugman equation, generating a corresponding perturbation $\tilde{w}_{\tilde{\mu}_0} \in L^2(\mathbb{T}^1_R)$ on the nominal wage, and $\tilde{V}_{\tilde{\mu}_0} \in L^2(\mathbb{T}^1_R)$ on the real wage. Following Fujita et al. [1999], these perturbations can be characterized analytically using a first-order approximation.

Proposition 7. *For all $x \in \mathbb{T}^1_R$, we have, at first-order approximation:*

$$\tilde{V}_{\tilde{\mu}_0}(x) = \delta_V \cos(kx), \tag{21}$$

where

$$\frac{\delta_V}{\delta_\mu} = G^{-1} \frac{1-\rho^2}{\rho} \frac{1-Z}{1-Z(1-\rho)-\rho Z^2}$$

$$G = \left[R \int_0^\pi e^{-d(\sigma-1)z} dz \right]^{1/1-\sigma}$$

$$Z = RG^{\sigma-1} \int_{-\pi}^{\pi} \cos(kz) e^{-d(\sigma-1)|z|} dz.$$

and $\rho := (\sigma-1)^{-1}$. The ratio (δ_V/δ_μ) is positive when R is large.

Proposition 7 states that, at first order, the sinusoidal perturbation (20) of the initial population distribution induces a corresponding sinusoidal perturbation (21) of the initial utility. When the radius R is large, the ratio δ_V/δ_μ is positive, indicating that utility increases at locations where the local population density rises. This property stems from the fact that our static model is a particular case of Krugman [1996], where only the manufacturing sector is active. In our setting, there are no centrifugal forces arising from immobile agriculture, but only from love of variety. Consequently, the "no-black-hole" condition [Fujita et al., 1999, p. 59] does not hold here[3] and, with myopic agents, any small perturbation of population density would evolve toward agglomeration.

Thus, the Krugman model without the agricultural sector leads to agglomeration as migration decisions are not forward-looking. This result is well-known. However, our model is dynamic and stochastic: agents form rational expectations and face uncertainty. Additional forces related to these features therefore come into play. We will see that the mean-field game equations make it possible to identify and quantify these forces. But before doing so, to facilitate the analysis, let us linearize the Fokker-Planck-HJB system around the stationary equilibrium.

[3]Specifically, since the share of expenditure on manufactured goods m is equal to one, we have $\sigma(1-m) = 0$, whereas the no-black-hole condition requires $\sigma(1-m) > 1$.

Definition 8. *In the racetrack economy, we say that $\tilde{\mu} \in C^{1,2}([0,T] \times \mathbb{T}^1_R)$ is a solution to the linearized perturbation problem if there exists a trajectory of value function perturbations $\tilde{u} \in C^{1,2}([0,T] \times \mathbb{T}^1_R)$ such that:*

$$-\partial_t \tilde{u} - \nu \Delta \tilde{u} + \rho \tilde{u} = -\tilde{V}_{\tilde{\mu}_t}(x, w(t)), \tag{22}$$

$$\partial_t \tilde{\mu} - \nu \Delta \tilde{\mu} + \frac{1}{c_0} \Delta \tilde{u} = 0, \tag{23}$$

completed with $\tilde{\mu}(0,x) = \delta_\mu \cos(kx)$, $u(T,x) = 0$ for all $x \in \mathbb{T}^1_R$.

This type of linearization has proven useful in economic applications of mean-field games, particularly for analyzing shocks in heterogeneous-agent macroeconomic models (see Bilal [2023], Alvarez et al. [2023]). In our case, it is justified by the fact that the initial perturbation $\tilde{\mu}_0$ is small.

Our linearized HJB-FP system (22)-(23) may admit multiple solutions. Following Fujita et al. [1999], we focus on sinusoidal solutions that oscillate at the same frequency as the initial perturbation. We characterize this class of solutions in closed form in the following proposition.

Proposition 9. *The linearized system of Definition 8 admits a solution of the form*

$$\tilde{\mu}(t,x) = A(t)\cos(kx), \tag{24}$$

$$\tilde{u}(t,x) = B(t)\cos(kx), \tag{25}$$

if and only if

$$\frac{\delta_V}{\delta_\mu} < \frac{c_0 \rho^2}{4k^2} + \frac{c_0 \nu^2 k^2}{2} + c_0 \rho \nu.$$

The amplitudes $A(t)$ and $B(t)$ are linear combinations of exponentials:

$$A(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},$$
$$B(t) = (C_1/\beta_1) e^{\lambda_1 t} + (C_2/\beta_2) e^{\lambda_2 t},$$

with

$$\lambda_{1,2} = \frac{\rho \pm \sqrt{\rho^2 + 4\left(\frac{\nu^2 k^4}{2} + \rho \nu k^2 - \frac{\delta_V k^2}{\delta_\mu c_0}\right)}}{2}.$$

$$\beta_{1,2} = -\frac{\nu k^2 + \rho - \lambda_{1,2}}{(\delta_V/\delta_\mu)}$$

$$C_1 = \frac{\delta_\mu \beta_2 e^{\lambda_2 T}\left(1 - \frac{\delta_V}{\delta_\mu}\beta_1\right)}{\Delta}, \quad C_2 = \frac{-\delta_\mu \beta_1 e^{\lambda_1 T}\left(1 - \frac{\delta_V}{\delta_\mu}\beta_2\right)}{\Delta},$$

and $\Delta = \beta_1\beta_2(e^{\lambda_1 T} - e^{\lambda_2 T})\frac{\delta_V}{\delta_\mu} + \beta_2 e^{\lambda_2 T} - \beta_1 e^{\lambda_1 T}$.

3.2 Analysis of agglomeration and dispersion forces

The linearized dynamics derived in Proposition 9 provide a tractable way to study the forces that drive agglomeration or dispersion. The amplitudes of $\tilde{u}$ and $\tilde{\mu}$ take the form of a sum of two exponentials. Several cases arise depending on the values taken by the parameters.

If

$$\frac{\delta_V}{\delta_\mu} < \frac{c_0 k^2 \nu^2}{2} + c_0 \rho \nu,$$

one exponential, $e^{\lambda_1 t}$, diverges because $\lambda_1 > 0$, while the other, $e^{\lambda_2 t}$, vanishes because $\lambda_2 < 0$. Besides, over an infinite horizon,

$$C_1 \underset{T\to\infty}{\longrightarrow} 0, \quad C_2 \underset{T\to\infty}{\longrightarrow} \delta_\mu.$$

This means that when the time horizon is large, the divergent exponential disappears from the solution. In this case, the perturbation $\tilde{\mu}$ dampens over time, and eventually vanishes. There is dispersion.

Else, if

$$\frac{c_0 k^2 \nu^2}{2} + c_0 \rho \nu < \frac{\delta_V}{\delta_\mu} < \frac{c_0 \rho^2}{4k^2} + \frac{c_0 k^2 \nu^2}{2} + c_0 \rho \nu,$$

both exponentials $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ diverge, because $\lambda_1, \lambda_2 > 0$. Over an infinite horizon,

$$C_1 \underset{T\to\infty}{\longrightarrow} 0, \quad C_2 \underset{T\to\infty}{\longrightarrow} \delta_\mu.$$

In this case, the perturbation $\tilde{\mu}$ grows over time, leading to agglomeration.

We summarize these results for large horizon in the following corollary.

Corollary 10. *Assume that* $\frac{\delta_V}{\delta_\mu} < \frac{c_0\rho^2}{4k^2} + \frac{c_0 k^2 \nu^2}{2} + c_0\rho\nu$. *When the time horizon* T *is large, there exists a solution to the linearized problem such that for all* $0 \le t \le T$, $x \in \mathbb{T}^1_R$,

$$\tilde{\mu}(t,x) = \delta_\mu e^{\lambda t}\cos(kx), \tag{26}$$

with $\lambda = \frac{1}{2}\left[\rho - \sqrt{\rho^2 + 4\left(\frac{k^4\nu^2}{2} + \rho k^2 \nu - \frac{\delta_V k^2}{\delta_\mu c_0}\right)}\right]$.

We now want to go more in detail and quantify the extent to which strategic and dynamic behaviors (rational expectations, consideration of uncertainty) contribute to agglomeration and dispersion effects. To quantify them, we use

the Herfindahl–Hirschman Index (HHI), a classical measure of concentration. For any probability density function f defined on $\mathbb{T}^1_R$, we define the HHI of f as

$$H[f] = \int_{\mathbb{T}^1_R} f(x)^2 \, dx.$$

The next proposition illustrates how the HHI captures the degree of spatial concentration:

Proposition 11. *The following statements hold:*

- *The uniform distribution minimizes the HHI over $\mathcal{P}(\mathbb{T}^1_R)$.*
- *Any sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{T}^1_R)$ that converges (in the sense of distributions) to a Dirac mass δ_{x_0} at some point $x_0 \in \mathbb{T}^1_R$ satisfies:*

$$\lim_{k\to\infty} H[f_k] = +\infty.$$

The HHI thus provides a continuous measure of spatial concentration: it reaches its minimum for a uniform distribution and diverges as mass becomes increasingly concentrated at a point. We will use this metric to evaluate how our racetrack economy evolves from an initial perturbation and to identify the forces that either reinforce or counteract agglomeration.

Given the perturbation setup introduced earlier, the HHI at time t takes the form

$$H[\mu](t) = \int_{\mathbb{T}^1_R} (\bar{\mu} + \tilde{\mu}(t,x))^2 \, dx = \int_{\mathbb{T}^1_R} (1 + \tilde{\mu}(t,x))^2 \, dx.$$

Using the fact that $\int_{\mathbb{T}^1_R} \tilde{\mu}(t,x)\, dx = 0$, a second-order expansion yields:

$$H[\mu](t) = \int_{\mathbb{T}^1_R} \tilde{\mu}(t,x)^2 \, dx.$$

To assess whether the system tends toward agglomeration or dispersion, we consider the time derivative of the HHI:

$$\frac{1}{2} H[\mu]'(t) = \int_{\mathbb{T}^1_R} \partial_t \tilde{\mu}(t,x)\, \tilde{\mu}(t,x)\, dx.$$

A positive value indicates increasing spatial concentration (agglomeration), whereas a negative value indicates movement toward uniformity (dispersion). Using the linearized Fokker–Planck equation (23), this derivative decomposes

as:

$$\frac{1}{2}H[\mu]'(t) = \underbrace{\nu \int \Delta\tilde{\mu}(t,x)\tilde{\mu}(t,x)\,dx}_{\text{Idiosyncratic shocks}} + \underbrace{\frac{1}{c_0}\int \Delta\tilde{u}(t,x)\tilde{\mu}(t,x)\,dx}_{\text{Controlled migrations}} \tag{27}$$

To clarify the interpretation of this equation, note the following: the value function (13) is negative, as it results from the minimization of a quantity that itself takes negative values. Although this does not affect the optimal control (which remains unchanged), the agent's intertemporal utility at the optimum is actually the negative of this value function. From now on, we define $u^+ := -u$ (and $\tilde{u}^+ = -\tilde{u}$) and interpret it as the agent's actual intertemporal utility. Thus:

$$\frac{1}{2}H[\mu]'(t) = \underbrace{\nu \int \Delta\tilde{\mu}(t,x)\tilde{\mu}(t,x)\,dx}_{\text{Idiosyncratic shocks}} - \underbrace{\frac{1}{c_0}\int \Delta\tilde{u}^+(t,x)\tilde{\mu}(t,x)\,dx}_{\text{Controlled migrations}} \tag{28}$$

The interpretation of equation (28) is as follows. Agglomeration and dispersion dynamics are driven by agent movement, which can be either involuntary due to exogenous shocks or controlled by the agents through migrations.

The involuntary term related to idiosyncratic shocks, $\nu \int \tilde{\mu}(t,x)\Delta\tilde{\mu}(t,x)\,dx$, is always negative[4] and therefore systematically contributes to dispersion. This reflects the fact that idiosyncratic shocks introduce random, uncoordinated movements across space. As agents are hit by independent Brownian shocks, their trajectories diverge over time, leading to a smoothing of the population density. This diffusive mechanism is purely entropic: it pushes the system toward a more uniform distribution, regardless of economic incentives or spatial preferences.

The controlled migration term depends on the interplay between the value function and the population distribution. If positive deviations of $\tilde{\mu}$ tend to coincide with local minima of $\tilde{u}^+$ (that is, if $\int \Delta\tilde{u}^+\,\tilde{\mu} > 0$) then agents are more concentrated in locations where the value function is relatively low. In such cases, we can expect agents to migrate toward less crowded areas offering higher value, which contributes to dispersion (i.e., $H[\mu]' < 0$). Conversely, if $\tilde{\mu}$ and $\Delta\tilde{u}^+$ are negatively correlated, migration flows reinforce existing concentrations, leading to agglomeration.

By performing a double integration by parts on the controlled migration term, and then using the linearized HJB equation (22), this term can be decomposed into several interpretable components. This is shown in the following

[4]This follows from an integration by parts, using periodic boundary conditions.

proposition.

Proposition 12. *Let $\tilde{\mu}$ and $\tilde{u}$ be solutions to the linearized problem as in Definition 8. For all $0 \leq t \leq T$:*

$$H[\mu]'(t) = \underbrace{\nu \int \Delta\tilde{\mu}(t,x)\tilde{\mu}(t,x)\,dx}_{\textit{Idiosyncratic shocks}} \underbrace{- \frac{1}{\rho c_0} \int \tilde{V}_{\tilde{\mu}_t}(x)\Delta\tilde{\mu}(t,x)\,dx}_{\textit{Instantaneous utility}}$$
$$\underbrace{+ \frac{\nu}{\rho c_0} \int \Delta\tilde{u}(t,x)\Delta\tilde{\mu}(t,x)\,dx}_{\textit{Uncertainty}} + \underbrace{\frac{1}{\rho c_0} \int \partial_t\tilde{u}(t,x)\Delta\tilde{\mu}(t,x)\,dx}_{\textit{Rational expectations}}.$$

Proposition 12 identifies four distinct forces that shape the evolution of agglomeration and dispersion. The first term originates from the Fokker–Planck equation and captures the contribution of idiosyncratic shocks. As discussed previously, it is always negative and thus acts as a force of dispersion.

The remaining three terms arise from the linearized Hamilton–Jacobi–Bellman equation and reflect how agents adjust their location based on different economic motives.

The instantaneous utility term originates from the source term $\tilde{V}_{\tilde{\mu}_t}(x)$ in the HJB equation and captures myopic migration behavior. Agents react to current spatial differences in utility, relocating toward areas with higher real wages.

The uncertainty term comes from the Laplacian $\nu\Delta\tilde{u}$ in the HJB equation. It reflects the anticipation of random shocks.

Finally, the rational expectation term comes from the time derivative $\partial_t\tilde{u}$ in the HJB equation. It captures the forward-looking component of agents' behavior: individuals form expectations about how the value of each location will evolve, and adjust their migration decisions accordingly.

These forces are not always aligned. Some promote agglomeration, others drive dispersion. To understand their respective roles more precisely, we restrict attention to the class of sinusoidal solutions introduced in Proposition 9 and Corollary 10.

Proposition 13. *Let $\tilde{\mu}$ be the sinusoidal perturbation solution to the linearized*

problem given in Corollary 10. We have, for all $0 \leq t \leq T$*:*

$$H[\mu]'(t) = e^{-2\lambda t} \left[\begin{array}{ccc} \underbrace{-\nu\sigma_0^2 k^2 \delta_\mu^2}_{\textit{Idiosyncratic shocks}} & \underbrace{+\frac{\delta_V}{\delta_\mu}\frac{k^2}{\rho c_0}\delta_\mu^2}_{\textit{Instantaneous reward}} & \underbrace{+\nu k^2 \delta_\mu^2 \frac{k^2}{\beta_2 \rho c_0}}_{\textit{Uncertainty management}} \\ \underbrace{+\frac{k^2\lambda}{\rho c_0}\frac{\delta_\mu^2}{\beta_2}}_{\textit{Rational expectations}} & & \end{array} \right]$$

In this form, the sign of each term is explicit. As expected, the contribution of idiosyncratic shocks is negative, confirming that such random shocks induce dispersion. This reflects the diffusive effect discussed earlier, whereby uncoordinated movements tend to flatten the spatial distribution.

The term related to instantaneous utility has the same sign as the ratio δ_V/δ_μ. According to Proposition 7, this ratio is positive in our model. This means that locations with higher population density also offer higher instantaneous utility. As a result, agents are attracted to already dense regions, reinforcing concentration. This mechanism thus promotes agglomeration, and the corresponding contribution to $H[\mu]'(t)$ is positive.

Conversely, the term related to uncertainty has the opposite sign of δ_V/δ_μ and is therefore negative. The intuition is as follows. Noise naturally tends to spread the population by reducing density in crowded areas and increasing it elsewhere. Since δ_V/δ_μ is positive, utility is currently higher in densely populated locations. However, agents anticipate that stochastic fluctuations will gradually redistribute the population, making less crowded areas more attractive in the future. Expecting these zones to gain in utility over time, agents choose to move toward them preemptively. This behavior, driven by uncertainty, reinforces spatial dispersion, and the corresponding contribution to $H[\mu]'(t)$ is negative.

Finally, the term associated with rational expectations is always positive. The intuition behind this is the following: since the ratio δ_V/δ_μ is positive, being in a densely populated area increases instantaneous utility. Agents therefore anticipate that others will also be drawn to these crowded locations, causing future utility to rise in those places. This expectation creates an incentive for agents to join the crowd immediately, reinforcing agglomeration dynamics.

This decomposition shows that, in a dynamic and uncertain environment, agglomeration and dispersion cannot be understood solely through static trade-offs such as love of variety versus congestion. They also reflect how agents form

expectations, respond to uncertainty, and adjust to the evolving distribution of others. The mean-field game framework allows these effects to be modeled and quantified explicitly.

4 Conclusion

In this paper, building on Mossay [2013], we have presented why and how the rising MFG methodology can be used to overcome the technical difficulties arising in NEG modelling in continuous space. By providing such a systematic solution strategy, this quickly developing methodology is likely to help handling more intricate NEG models in continuous space. As shown in this paper, it may also open the door to a deeper and more efficient theoretical approach possibly delivering existence proofs of equilibria in various circumstances.

Several extensions can already be envisaged regarding the state of art in MFG. In first place, introducing exogenous heterogeneity traits across players (here e.g. differences in productivity across locations as in the original Armington [1969]) would not disable the method obviously although equilibrium existence may be more challenging. Second, there is also room to introduce policy and even interactions between the policy maker (typically referred to as dominating player) and the economic agents (or market players) as evidenced in Bensoussan et al. [2016]. Third, one can make the stochastic component more complex if needed for realism. For example, one can add common noise to idiosyncratic noises (see Carmona et al. [2015]). Last but not least, the MFG computational literature is booming and in principle, it should be possible to obtain the transitional dynamics inherent in MFG games with increasingly efficient computational methods (see Bayraktar and Zhang [2023] for instance).

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Appendix: Proofs

Proof of Proposition 2

Existence

We first establish the existence of a wage function w satisfying (8) and (9). Let us introduce

$$\Xi = \left\{ g \in C^1(\mathbb{T}^2) \ : \ \frac{1}{C_1} \leq g(x) \leq C_1, \ \|g'\|_\infty \leq C_2 \text{ and } \int g(x) d\mu(x) = 1 \right\},$$

for some constants C_1 and C_2 to fix later. It is a convex and compact set of $C(\mathbb{T}^2, \mathbb{R})$ for the uniform topology. Let us now define the application $\Lambda : \Xi \to \Xi$ as follows: for any w in Ξ,

1. associate $f[w](x) = \int_{\mathbb{T}^2} \frac{\tau(x,y)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z)} d\mu(y)$, for any $x \in \mathbb{T}^2$,
2. then set $\Lambda[w](x) = f[w](x)^{\frac{1}{\sigma}} / \left(\int_{\mathbb{T}^2} f[w](y)^{\frac{1}{\sigma}} d\mu(y) \right)$.

Step 1. We start by showing that Λ is well-defined. Fix any w in Ξ. Notice that by construction,

$$\int_{\mathbb{T}^2} \Lambda[w](x) d\mu(x) = 1.$$

To show estimates on $\Lambda[w]$, we first establish estimates on $f[w]$. Using the fact that τ is bounded by below and above, there exists $C_\tau > 0$ such that

$$f[w](x) \leq C_\tau \frac{\int_{\mathbb{T}^2} w(y) d\mu(y)}{\int_{\mathbb{T}^2} w(z)^{1-\sigma} d\mu(z)}.$$

From the convexity of $\omega \mapsto \omega^{1-\sigma}$ on $\mathbb{R}_+^*$, Jensen's inequality yields

$$f[w](x) \leq C_\tau \frac{\int_{\mathbb{T}^2} w(y) d\mu(y)}{\left(\int_{\mathbb{T}^2} w(z) d\mu(z) \right)^{1-\sigma}}$$

Using that $\int_{\mathbb{T}^2} w(y) d\mu(y) = 1$, we conclude that $f(x) \leq C_\tau$.

On the other hand, we have for any z, $w(z)^{1-\sigma} \leq C_1^{\sigma-1}$. Therefore,

$$\left(\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z) \right)^{-1} \geq C_1^{1-\sigma} \left(\int_{\mathbb{T}^2} (\tau(y,z))^{1-\sigma} d\mu(z) \right)^{-1}.$$

Thus, there exists C'_τ such that $f(x) \geq C'_\tau C_1^{1-\sigma}$.

Coming back to $\Lambda[w]$, from its definition and the estimates on f, we deduce that

$$\left(\frac{C'_\tau}{C_\tau}\right)^{\frac{1}{\sigma}} C_1^{\frac{1}{\sigma}-1} \leq \Lambda[w](x) \leq \left(\frac{C_\tau}{C'_\tau}\right)^{\frac{1}{\sigma}} C_1^{1-\frac{1}{\sigma}}.$$

Therefore, we fix C_1 large enough to ensure that

$$\frac{1}{C_1} \leq \left(\frac{C'_\tau}{C_\tau}\right)^{\frac{1}{\sigma}} C_1^{\frac{1}{\sigma}-1} \text{ and } \left(\frac{C_\tau}{C'_\tau}\right)^{\frac{1}{\sigma}} C_1^{1-\frac{1}{\sigma}} \leq C_1.$$

In others words, we set C_1 bigger than C_τ/C'_τ. To highlight the fact that C_1 may only depends on τ, we choose $C_1 = C_\tau/C'_\tau$.

It remains to show that $\Lambda[w]$ is of class C^1 and its derivative is uniformly bounded. It is easy to see that $\Lambda[w]$ is differentiable, and

$$\Lambda[w]'(x) = \frac{f[w](x)^{\frac{1}{\sigma}-1}}{\int_{\mathbb{T}^2} f[w](y)^{\frac{1}{\sigma}} d\mu(y)} f[w]'(x),$$

where

$$f[w]'(x) = \int_{\mathbb{T}^2} \frac{\tau(x,y)^{-\sigma} w(y)}{\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z)} \frac{\partial \tau}{\partial x}(x,y) d\mu(y).$$

From the continuity of $f[w]$ and $f[w]'$, we deduce that $\Lambda[w]$ is C^1. Then, using the bounds on w, τ, $\frac{\partial \tau}{\partial x}$, and $f[w]$, we can deduce that there exists a constant C''_τ which only depends on τ (since all the estimates only depend on τ), such that

$$\|\Lambda[w]'\|_\infty \leq C''_\tau.$$

By setting $C_2 = C''_\tau$, we have proved that $\Lambda[w]$ belongs to Ξ.

Step 2. Let us show that finding a Krugman equilibrium is equivalent of finding a fixed-point of Λ. The implication is trivial. Conversly, let us assume that w is a fixed-point of Λ. Then, the definition of Λ yields

$$C\Lambda[w](x)^\sigma = \int_{\mathbb{T}^2} \frac{\tau(x,y)^{1-\sigma} w(y)}{\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z)} d\mu(y), \tag{29}$$

where

$$C = \left(\int_{\mathbb{T}^2} f[w](y)^{\frac{1}{\sigma}} d\mu(y)\right)^\sigma.$$

Note that the result follows the fact that $C = 1$. Multiplying equation (29) by

$w(x)^{1-\sigma}$ and integrating it against μ, we obtain

$$C\int_{\mathbb{T}^2}\Lambda[w](x)^{\sigma}w(x)^{1-\sigma}d\mu(x)=\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\frac{\tau(x,y)^{1-\sigma}w(y)}{\int_{\mathbb{T}^2}(w(z)\tau(y,z))^{1-\sigma}d\mu(z)}d\mu(y)w(x)^{1-\sigma}d\mu(x).$$

Using Tonelli's theorem and rearranging the terms, we obtain

$$C\int_{\mathbb{T}^2}\Lambda[w](x)^{\sigma}w(x)^{1-\sigma}d\mu(x)=\int_{\mathbb{T}^2}\frac{\int_{\mathbb{T}^2}(w(x)\tau(x,y)^{1-\sigma}d\mu(x)}{\int_{\mathbb{T}^2}(w(z)\tau(y,z))^{1-\sigma}d\mu(z)}w(y)d\mu(y).$$

We conclude that $C=1$, by using the symmetry of τ, the fact that $\Lambda[w]=w$, and that $\int w d\mu=1$. Therefore, w satisfies (8) and (9).

Step 3. We show that Λ is continuous for the uniform topology. Let (w_n) be a sequence of Ξ converging to w. The uniform bounds on (w_n) and its convergence to w yield that

$$\int_{\mathbb{T}^2}(w_n(z)\tau(\cdot,z))^{1-\sigma}d\mu(z)\to\int_{\mathbb{T}^2}(w(z)\tau(\cdot,z))^{1-\sigma}d\mu(z)$$

uniformly. From the previous convergence, the uniform bounds on (w_n) and its convergence to w, we deduce that $f[w_n]$ converges to $f[w]$ uniformly. The same kind of arguments allows us to conclude that $\Lambda[w_n]$ converges to $\Lambda[w]$ uniformly.

Step 4. From Step 1 and 3, Λ is a continuous map from Ξ into itself. Moreover, Ξ is a compact, convex and non empty subset of $C(\mathbb{T}^2,\mathbb{R})$ for C_τ chosen large enough. Therefore, we use Schauder's fixed-point theorem to conclude that there exists w in Ξ such that $w=\Lambda[w]$.

Local Uniqueness

The purpose of this paragraph is to use the Inverse Function Theorem to deduce the local uniqueness of a Krugman equilibrium. Let us fix w^0 a Krugman equilibrium, introduce

$$B=\left\{g\in C^0(\mathbb{T}^2,\mathbb{R})\ :\ \frac{1}{C_1}\leq g\leq C_1 \text{ and } \int g(x)d\mu(x)=1\right\}$$

and define $F:B\to C^0(\mathbb{T}^2,\mathbb{R})$ by

$$F(w)=f[w]^{\frac{1}{\sigma}}=\left(\int_{\mathbb{T}^2}\frac{\tau(x,y)^{1-\sigma}w(y)}{\int_{\mathbb{T}^2}(w(z)\tau(y,z))^{1-\sigma}d\mu(z)}d\mu(y)\right)^{\frac{1}{\sigma}}.$$

Step 1. Let us show that 1 is a eigenvalue of $DF(w^0)$, and that if $DF(w^0)(h) = h$, then $h \in \text{span}\{w^0\}$.

First, it is easy to show that F is Gateau-Differentiable and that DF is continuous, therefore F is of class C^1. Moreover,

$$DF(w)(h) = \sigma f[w]^{\frac{1-\sigma}{\sigma}} D_w f[w](h).$$

Therefore,

$$DF(w)(h) = \sigma f[w]^{\frac{1-\sigma}{\sigma}} \left[\int_{\mathbb{T}^2} \frac{\tau(x,y)^{1-\sigma} h(y)}{\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z)} d\mu(y) \right.$$
$$\left. + (\sigma - 1) \int_{\mathbb{T}^2} \frac{\tau(x,y)^{1-\sigma} w(y) \int_{T^2} (w(z)\tau(y,z))^{1-\sigma} \frac{h(z)}{w(z)} d\mu(z)}{\left(\int_{\mathbb{T}^2} (w(z)\tau(y,z))^{1-\sigma} d\mu(z)\right)^2} d\mu(y) \right].$$

We claim that $DF(w_0)$ is a linear, compact and strongly positive operator of C^0.

Since the linearity is trivial, let us only check that it is compact and strongly positive. Let $\mathcal{B}$ be a bounded set of $(C^0(\mathbb{T}^2), \|.\|_\infty)$. There exists $b > 0$ such that $h \in \mathcal{B}$ implies $\|h\|_\infty \leq b$. The regularity of τ and w, and the bound on h implies that $\|DF(w)(h)\|_\infty$ and $\|DF(w)(h)'\|_\infty$ are bounded by above by a constant that only depends on τ, w and b. Then Arzelà-Ascoli's theorem ensures that $DF(w_0)(\mathcal{B})$ is compact in $(C^0(\mathbb{T}^2), \|.\|_\infty)$.

Now let us take a nonnegative function h, not equals to 0. The kernel functions multiplying h in the integrals are bounded by below by positive constants. Moreover, supp $\mu = \mathbb{T}^2$, then $DF(w_0)(h)$ is strongly positive on $\mathbb{T}^2$.

Since for any $t \in \mathbb{R}$, tw_0 is solution of $w = F(w)$, then $DF(w_0)(w_0) = w_0$. In order to conclude that if $h \in C^0(\mathbb{T}^2, \mathbb{R})$ satisfies $DF(w_0)(h) = h$, then $h \in \text{span}\{w_0\}$, we use the following result:

Theorem 14. *Let X be a Banach space, $K \subset X$ a solid cone, $T : X \to X$ a compact linear operator which is strongly positive. Then*

1. *the spectral radius $r(T)$ is a simple eigenvalue with an eigenvector $v \in \mathring{K}$ and there is no other eigenvalue with a positive eigenvector.*
2. *$|\lambda| < r(T)$ for all eigenvalue $\lambda \neq r(T)$.*

Proof. Add reference. □

Step 2. Let us introduce for $i = 0, 1$,

$$\Theta^i = \left\{ g \in C^0(\mathbb{T}^2, \mathbb{R}) \ : \ \int g d\mu = i \right\}.$$

Let $T : x^0 \to x^1$ be defined by

$$T(u) = u + 1.$$

It is a bijection. Note that $(\Theta^0, \|.\|_\infty)$ is a Banach space. Let us define $G : T^{-1}(B) \subset \Theta^0 \to \Theta^0$ by

$$G(u) = T(u) - \Lambda(T(u)).$$

The function G is differentiable in Θ^0 and

$$DG(u)(h) = h - \frac{DF(T(u))(h)}{\int_{\mathbb{T}^2} F(T(u))d\mu} + \frac{F(T(u))}{\left(\int_{\mathbb{T}^2} F(T(u))d\mu\right)^2} \int DF(T(u))(h)d\mu$$

Fix u_0 such that $T(u_0) = w_0$. We want to use the Fredholm alternative to show that $DG(u_0)$ is an isomorphism. We already know that $DF(T(u_0))$ is a compact operator. Moreover, let us establish that $\int DF(T(u_0))(h)d\mu = 0$ for any h. Let us notice that for any w,

$$\int F(w)^\sigma w^{1-\sigma} d\mu = \int w d\mu = 1.$$

Differentiating this equation with respect to w in the direction $h \in \Theta^0$, we get

$$\sigma \int F(w)^{\sigma-1} DF(w)(h) w^{1-\sigma} = (\sigma - 1) \int F(w)^\sigma w^{-\sigma} h d\mu.$$

Setting $w = w_0$ and using $F(w_0) = w_0$, we end up with

$$\sigma \int DF(w_0)(h)d\mu = (\sigma - 1) \int h d\mu = 0.$$

It is the wanted equality. With the fact that $\int F(T(u_0))d\mu = 1$, it allows us to write

$$DG(u_0)(h) = h - DF(T(u_0))(h)$$

Note that $DG(u_0)$ is a Fredholm operator on Θ^0. Let us verify that $\mathrm{Ker} DG(u_0)$ is reduced to $\{0\}$. Fix $h \in \mathrm{Ker} DG(u_0)$, then by step 1, $h = tw_0$ but the

condition $\int h = 0$ ensures that $t = 0$ meaning that $h = 0$. Therefore, $DG(u_0)$ is an isomorphism in Θ^0 and we can apply the Inverse Function Theorem to deduce the local uniqueness of u_0 in Θ^0.

Global Uniqueness

To deduce the global uniqueness of a solution of $\Lambda(w) = w$ in Θ^1, we will use the fact that when $\tau \equiv 1$ there is a unique Krugman equilibrium. In particular, we will show that the value of the Leray-Schauder degree is 1. Then, using the properties of the degree, we deduce that a change of τ does not modify its value. Therefore, from the degree formula and the previous steps, we deduce that there is at most one Krugman equilibrium.

Fix $\tau \equiv 1$. In this case, it is trivial to see that the unique Krugman equilibrium is $w_0 \equiv 1/\mu(\mathbb{T}^2)$. Therefore, the unique solution of $G_1 = 0$ (the dependency of G with respect to τ has been made clear) is $u_0 = T^{-1}(w_0)$.
It allows us to deduce the value of the Leray-Schauder degree as follows:

$$\deg_{LS}(G_1, T^{-1}(B), 0) = \sum_{a \in G_1^{-1}(0)} (-1)^{\sigma(a)},$$

where $\sigma(a)$ is the aglebric multiplicities of the eigenvalues of $D\Lambda(T(a))$ contained in $(1, +\infty)$. From the above, the sum is reduced to one element u_0, and from step 1 of the Local Uniqueness paragraph, we know that $D\Lambda(T(u_0))$ viewed as an operator of $C(\mathbb{T}^2, \mathbb{R})$ has 1 as simple eigenvalue and that the others have a norm strictly smaller than 1, hence $\sigma(u_0) = 0$, leading to

$$\deg_{LS}(G_1, T^{-1}(B), 0) = 1.$$

Now, fix any τ satisfying the conditions of the Theorem and introduce $\tau_t = (1 - t) + t\tau$. By homotopy invariance of the degree, for any $t \in [0, 1]$

$$\deg_{LS}(G_{\tau_t}, T^{-1}(B), 0) = 1.$$

Using the same argument developed above, we conclude that there is at most one Krugman equilibrium.

Proof of Proposition 7

Up to multiplicative constants, the proof closely follows Fujita et al. [1999]. Let

$$\begin{aligned} Y_0(x) &= \mu_0(x) w_0(x) \\ G_0(x) &= \left[R \int_{-\pi}^{\pi} w_0(y)^{1-\sigma} e^{-d(\sigma-1)|x-y|} \mu_0(y) dy \right]^{1/(1-\sigma)} \\ w_0(x)^{\sigma} &= \left[R \int_{-\pi}^{\pi} Y_0(y) G_0(y)^{\sigma-1} e^{-d(\sigma-1)|x-y|} dy \right] \\ V_0(x) &= w_0(x)/G_0(x). \end{aligned} \tag{30}$$

At the flat-earth equilibrium, we have $\bar{\mu} = 1$, $\bar{w} = 1$, $\bar{Y} = 1$, and $\bar{G} = \left[R \int_0^{\pi} e^{-d(\sigma-1)x} dx \right]^{1/1-\sigma}$. Totally differentiating (30) around the flat earth, we obtain:

$$\begin{aligned} \tilde{Y}_0(x) &= \tilde{\mu}_0(x) + \tilde{w}_0(x) \\ \frac{\tilde{G}_0(x)}{\bar{G}} &= \bar{G}^{\sigma-1} R \int_{-\pi}^{\pi} \tilde{w}_0(x+y) e^{-d(\sigma-1)|y|} dy \\ &\quad - \frac{\bar{G}^{\sigma-1}}{\sigma-1} R \int_{-\pi}^{\pi} \tilde{\mu}_0(x+y) e^{-d(\sigma-1)|y|} dy \\ \tilde{w}_0(x) &= \frac{\bar{G}^{\sigma-1}}{\sigma} R \int_{-\pi}^{\pi} \tilde{Y}_0(x+y) e^{-d(\sigma-1)|y|} dy \\ &\quad + \frac{\sigma-1}{\sigma} \bar{G}^{\sigma-1} R \int_{-\pi}^{\pi} (\tilde{G}_0(x+y)/\bar{G}) e^{-d(\sigma-1)|y|} dy \\ \tilde{V}_0(x) &= \left[\tilde{w}_0(x) - \tilde{G}_0(x)/\bar{G} \right] \bar{G}^{-1}. \end{aligned} \tag{31}$$

Consider a sinusoidal solution to the system (31) of the form:

$$\begin{aligned} \tilde{Y}_0(x) &= \delta_Y \cos(kx), \\ \tilde{G}_0(x)/\bar{G} &= \delta_G \cos(kx), \\ \tilde{w}_0(x) &= \delta_w \cos(kx), \\ \tilde{V}_0(x) &= \delta_V \cos(kx). \end{aligned} \tag{32}$$

where the constants δ_Y, δ_G, δ_w and δ_V have to be determined. Introducing

$$Z = R \bar{G}^{\sigma-1} \int_{-\pi}^{\pi} \cos(kx) e^{-d(\sigma-1)|x|} dx,$$

the system of equations (31) becomes:

$$\begin{aligned}
\delta_Y &= \delta_\mu + \delta_w, \\
\delta_G &= \frac{1}{1-\sigma} Z\delta_\mu + Z\delta_w, \\
\sigma\delta_w &= Z\delta_Y + (\sigma-1)Z\delta_G, \\
\delta_V &= \bar{G}^{-1}\left(\delta_w - \delta_G\right).
\end{aligned} \tag{33}$$

which gives:

$$\frac{\delta_V}{\delta_\mu} = \bar{G}^{-1}\frac{1-\rho^2}{\rho}\frac{1-Z}{1-Z(1-\rho)-\rho Z^2}$$

where $\rho = (\sigma-1)^{-1}$.

Proof of Proposition 9

Define

$$X(t) = \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}, \quad M = \begin{pmatrix} -\nu k^2 & -\frac{k^2}{c_0} \\ \frac{\delta_V}{\delta_\mu} & \nu k^2 + \rho \end{pmatrix}.$$

Then $X'(t) = M\,X(t)$, whose general solution is

$$X(t) = e^{Mt}X(0).$$

Since $A(0) = \delta_\mu$, write

$$X(0) = \begin{pmatrix} \delta_\mu \\ B(0) \end{pmatrix}, \quad e^{MT} = \left(e_{ij}\right)_{1\le i,j\le 2}$$

Hence

$$X(T) = \begin{pmatrix} e_{11}\,\delta_\mu + e_{12}\,B(0) \\ e_{21}\,\delta_\mu + e_{22}\,B(0) \end{pmatrix},$$

and the terminal condition $B(T) = \frac{\delta_V}{\delta_\mu}A(T)$ yields

$$e_{21}\,\delta_\mu + e_{22}\,B(0) = \frac{\delta_V}{\delta_\mu}\left(e_{11}\,\delta_\mu + e_{12}\,B(0)\right).$$

Rearranging gives the linear equation

$$\left(e_{22} - \tfrac{\delta_V}{\delta_\mu}e_{12}\right)B(0) = \left(\tfrac{\delta_V}{\delta_\mu}e_{11} - e_{21}\right)\delta_\mu.$$

Thus, if

$$e_{22} - \frac{\delta_V}{\delta_\mu}\, e_{12} \neq 0,$$

then there is a unique solution

$$B(0) = \frac{\left(\frac{\delta_V}{\delta_\mu} e_{11} - e_{21}\right) \delta_\mu}{e_{22} - \frac{\delta_V}{\delta_\mu} e_{12}},$$

and hence a unique trajectory

$$X(t) = e^{Mt} \begin{pmatrix} \delta_\mu \\ B(0) \end{pmatrix},$$

solving the mixed boundary-value problem. One can verify that this solution is the one given in the proposition. The non-degeneracy condition

$$e_{22} - \frac{\delta_V}{\delta_\mu}\, e_{12} \neq 0$$

is equivalent to $\Delta \neq 0$.