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Representation, Social Discounting, and Computational Efficiency: Closed-Form Demand Approach

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Abstract

This paper presents a closed-form demand approach for representing aggregate equilibrium behavior under heterogeneous beliefs, even when preferences are non-separable across time and states or asset markets are incomplete. Specifically, in a multi-period stochastic setting where individuals have Kreps–Porteus (KP) preferences with heterogeneous beliefs, we explicitly construct a representative agent whose KP preferences rationalize the aggregate consumption of the economy (or a subgroup of agents). While this equilibrium representation is unique in complete markets, there may be multiple representations in incomplete markets depending on the choice of benchmark state prices. Counterintuitively, heterogeneous beliefs can cause the representative agent to exhibit a stronger present bias than any individual agent. Additionally, we construct the MIN and MAX representative agents that delineate the range of equilibrium prices. This equilibrium price range not only supplies informed initial values for numerical solvers but also defines the minimal domain within which any computed equilibrium and comparative-static outcome must lie.

KEYWORDS: Heterogeneous beliefs, Kreps-Porteus preferences, closed-form demand, representative agents, social discount rates, time inconsistency, computational efficiency.

JEL Classification: D01, D50, D51, D52, C43, E21, E44, G12

1. Introduction

The representative agent (RA) model has long been a cornerstone in macroeconomics, finance, and growth theory due to its analytical tractability and clear insights into aggregate behavior(Lucas (1976); Mehra and Prescott (1985); Kydland and Prescott (1982)). However, this framework has limitations in addressing economic phenomena driven by individual heterogeneity, such as income inequality, wealth distribution, and the differential impact of economic shocks on individuals (Kirman (1992); Stiglitz (2011)).

Heterogeneous-agent (HA) models explicitly incorporate individual differences in preferences, incomes, and financial constraints, enabling analysis of distributional impacts of economic policies and shocks (Aiyagari (1994); Krusell and Smith (1998)). These models provide a microeconomic foundation for understanding how individual-level interactions influence aggregate outcomes. However, compared to representative-agent (RA) models, HA models introduce significant complexity in computing and interpreting equilibrium outcomes, since intertemporal behavior must be tracked across high-dimensional state variables (Heathcote et al., 2009).

To address this complexity, researchers have adapted the representative-agent approach to capture aggregate behavior in heterogeneous-agent economies. This approach is developed under time-additive preferences (e.g. expected utility) by Wilson (1968), Rubinstein (1974), Breeden and Litzenberger (1978), and Constantinides (1982). The classical approach constructs the representative agent's preferences by characterizing the Pareto optimality of allocations based on linear welfare functions. The "welfare function" approach to equilibrium representation is further extended to dynamic economies with heterogeneous beliefs in Calvet et al. (2018) and Jouini and Napp (2006, 2007). However, this approach cannot be applied directly when preferences are non-separable or asset markets are incomplete.

This paper presents a closed-form demand approach for representing aggregate equilibrium behavior in a dynamic stochastic economy where individual agents have Kreps-Porteus (KP) preferences with heterogeneous beliefs. Asset markets need not be complete. This novel approach is built on the individual demand function for KP preferences, which is formulated explicitly in a complete-market setting. The demand function allows us to explicitly construct two distinct classes of representative agents with KP preferences: one designed to rationalize the aggregate behavior of the economy in equilibrium, and another to delineate the range of equilibrium prices.

First, we explicitly construct a representative agent whose KP preferences rationalize the aggregate consumption of the economy (or for a subgroup of agents) at an equilibrium

price. Specifically, we treat the representative-agent as an individual agent that optimally chooses to hold the aggregate consumption at the equilibrium price. Then, by using the explicit demand function, we invert the autarkic equilibrium condition with respect to the agent's preference parameters. This closed-form demand approach generates a representative agent whose preference parameters are expressed explicitly in terms of the aggregate consumption and equilibrium prices. While this equilibrium representation is unique in complete markets, there are generally multiple representations in incomplete markets depending on the choice of benchmark state prices. The uniqueness of complete-market representation is closely connected to the identifiability of KP preferences from consumption data.

The closed-form demand approach to equilibrium representation differs fundamentally from the welfare function approach, which relies on complete markets and additive preferences (such as expected utility). The demand-based approach is developed for a general equilibrium framework featuring non-additive KP preferences and possibly incomplete markets. Furthermore, in complete markets, this alternative approach formulates the representative agent's preference parameters directly in terms of observable variables, such as market prices and aggregate consumption.[1]

Second, we introduce two representative agents (MAX and MIN) whose preferences are calibrated to identify the lower and upper bounds of the equilibrium price range in complete markets. To this end, we explicitly derive the autarkic equilibrium price for a representative agent with arbitrary KP preferences by inverting the agent's autarkic demand that equals the aggregate endowment. We then use this formula to calibrate the preferences of the two representative agents to delineate the equilibrium price range. Specifically, the lower (upper) bound for equilibrium prices is determined by the autarkic equilibrium price of the MIN (MAX) agent, whose KP preferences are built from the minimum (maximum) values of individual preference parameters. Consequently, the equilibrium price range remains robust to variations in the distribution of preference parameters, provided the extreme parameter values remain unchanged. Furthermore, the autarkic equilibrium prices that define the equilibrium price range can be used as "informed" initial values in iterative numerical algorithms, facilitating equilibrium computation and comparative static analyses in multi-agent economies.

In a complete-market setting, the closed-form demand approach allows us to uniquely construct the representative agent that reveals aggregate equilibrium behavior. We can extend this approach to an incomplete-market economy by transforming it into an equivalent complete-market economy. To do this, we select an individual agent from the economy

[1] A more detailed comparison between the two representation approaches is given later.

and use its subjective state prices as a benchmark. We then calibrate the discount factors and probabilistic beliefs of the other agents so that they share the benchmark agent's state prices under the calibrated preference regime. This calibration process ensures that the equilibrium allocation of the incomplete-market economy becomes Pareto optimal in the newly constructed complete-market economy, where the benchmark state prices serve as the equilibrium price.

However, the representations of equilibrium in complete and incomplete markets differ in two fundamental ways. First, in complete markets, the KP parameters of the representative agent are observable because they are determined by market prices and the aggregate endowment. In contrast, in incomplete markets, the representative agent's calibrated preferences are characterized by personalized state prices, which are generally not observed in equilibrium. Second, in incomplete markets, the representation depends on the benchmark agent's state prices. As agents typically possess different personalized state prices, there can be as many representations as there are agents in the economy.

The explicit preference structure of the representative agent reveals its probabilistic beliefs about the occurrence of future states as well as the discount rates applied to future utility. We refer to these beliefs and discount rates as the *social probabilistic beliefs* and the *pure rate of social time preference* or *social utility discount rate* (SUDR), respectively. Unlike individual KP preferences, the representative agent's discounting behaviour is augmented with a stochastic adjustment factor that arises endogenously from the aggregation of individuals' heterogeneous beliefs. This adjustment factor makes the representative agent's discount factors state-dependent.

The preference representation of the representative agent is closely related to the time-preference shocks introduced in recent representative-agent approaches to asset pricing (e.g., Albuquerque et al. (2016)), which address asset pricing puzzles, such as the equity premium puzzle and the correlation puzzle. In this literature, exogenous shocks lead the representative agent to discount future utility stochastically.[2] In contrast, this paper shows that the stochastic nature of social discounting arises endogenously due to the stochastic discount adjustment factors for equilibrium representation, even when individual agents maintain a constant discount rate.

Counterintuitively, we illustrate that when individuals hold heterogeneous probabilistic beliefs, the social utility discount rates can be uniformly higher across states than those of any individual agent. This implies that belief heterogeneity may induce the representative

[2]The representative-agent approach implicitly assumes homothetic preferences among individual agents. Consequently, preference shocks are treated as exogenous at both individual and aggregate levels.

agent to have greater impatience or, stronger present bias, than any individual agent.[3] In particular, this phenomenon departs from the significant theoretical result in Gollier and Zeckhauser (2005) that, in a dynamic economy with time-additive preferences, the collective rate of impatience lies between the smallest and largest individual rates.[4] This finding highlights the impact of heterogeneous beliefs on the collective impatience implied by the representative agent. Specifically, when individuals have preferences with constant relative risk aversion (CRRA), we show that in the presence of belief heterogeneity, the collective discount factor is capped by the maximum patience among individuals, but can fall below the lowest individual discount factor.

Because belief heterogeneity can lower social discount factors below every individual rate, it can serve as a crucial determinant of social discounting, alongside intergenerational equity and the opportunity cost of capital, which are emphasized in the Ramsey literature. In particular, societies with diverse beliefs may collectively exhibit a stronger present bias than any of their members when evaluating long-term public projects that affect future welfare. Furthermore, in contrast to the declining social discount rates shown by Weitzman (2001) and Gollier and Zeckhauser (2005), belief heterogeneity can prevent discount rates from decreasing over time.

The equilibrium price range is potentially useful for computation and comparative static analysis of equilibrium outcomes. First, the closed-form autarkic equilibrium prices of the MAX and MIN agents delineate the lower and upper bounds of the equilibrium price range. These prices, or their weighted averages, can serve as useful benchmark prices for computing equilibrium outcomes and conducting comparative static analyses in the heterogeneous-agent economy. Compared to randomly generated initial values, these informed initial prices can significantly enhance the efficiency and accuracy of iterative algorithms used for equilibrium price computations. In the numerical examples presented later, algorithms initialized with these informed values successfully compute equilibrium prices, whereas randomly selected initial values often fail.

Second, the equilibrium price range provides a minimal standard for designing computational procedures to find approximate equilibrium outcomes. When analyzing competitive equilibria through approximation techniques, approximate equilibrium must be "close" to the equilibrium of the original model.[5] In the literature, first-order or second-

[3]This finding contrasts with the literature, which advocates for discount rates that are between, or even below, individual discount rates when evaluating long-term public projects and climate policies. For example, intergenerational-equity arguments often support near-zero rates (Ramsey (1928) and Stern (2008)).

[4]More generally, they prove that the social discount rate is a weighted average of individual discount rates, with weights proportional to individual absolute risk tolerances.

[5]To verify the closeness of the two equilibria, the true equilibrium value must be known, which leads to a tautological situation, as footnoted above.

order approximations of demand functions are often used to analyze approximate equilibrium. However, these approximation procedures do not guarantee outcomes that closely match those in the true model.[6]

For example, if Gorman aggregation holds under homothetic preferences, the equilibrium price range collapses to a single price corresponding to the marginal rates of substitution at the aggregate endowment. In this situation, the approximate equilibrium may not coincide with the true equilibrium and may therefore lie outside the degenerate (single point) equilibrium price range. If we were to measure the closeness of the approximate equilibrium to the true equilibrium by the ratio of the approximation error to the size of the equilibrium price range, this measure would tend to infinity.

Now suppose that individual preferences are slightly perturbed so that Gorman aggregation no longer holds. In this case, the equilibrium price range becomes narrow, making the autarkic equilibrium prices of the MAX and MIN agents close to each other. These prices thus serve as effective initial guesses for computing the exact equilibrium. However, approximation errors can become disproportionately large compared to the (now narrow) equilibrium price range. More generally, approximate equilibria may lie outside the equilibrium price range in economies with heterogeneous preferences if the approximation procedure does not carefully control deviations from the true equilibrium.

This paper contributes to the literature on aggregating individuals' heterogeneous beliefs and discounting behavior into a single "representative" preference, which rationalizes the aggregate consumption of the underlying economy in equilibrium. Wilson (1968), Rubinstein (1974), Breeden and Litzenberger (1978), and Constantinides (1982) develop the standard representative-agent approach to efficient risk sharing and asset pricing in the complete market framework. This approach construct the representative agent by maximizing a linear social welfare function subject to the resource constraints, where welfare weights are set by individual marginal utilities at equilibrium. More recently, Jouini and Napp (2006, 2007) extend these classic works to dynamic complete-market economies with heterogeneous beliefs. They introduce a representative agent whose consensus beliefs are formulated as a risk-tolerance-weighted average of individual beliefs, incorporating an adjustment discount factor to correct for aggregation bias. Within this framework, they analyze the effect of belief heterogeneity on the Sharpe ratio, risk premium and risk-free rates in the representative-agent framework. In particular, they demonstrate how the market price of risk responds to the representative agent's more pessimistic or optimistic assessment of aggregate wealth growth. Empirically, Anderson et al. (2005) shows that

[6]Kubler and Schmedders (2005) demonstrate that approximate equilibrium can deviate significantly from exact equilibrium in both allocations and prices, even when the approximate aggregate demand is sufficiently close to the true aggregate demand.

cross-sectional dispersion in analyst forecasts can itself be a priced factor, by aggregating individual forecasts into a composite market belief.

Our approach differs from these prior works in two key respects. First, rather than deriving representative preferences from a social welfare program, we exploit closed-form demand functions to directly infer the representative agent's KP preferences via demand inversion at the autarkic equilibrium. This inversion delivers explicit formulas for aggregate beliefs and state-dependent discount adjustments in terms of observable market prices and aggregate endowments. Second, while existing frameworks rely on additive (expected-utility) preferences and complete markets, the demand function approach is equally applicable to non-additive Kreps–Porteus preferences and incomplete markets, provided that closed-form demand functions are available. Finally, we show that in the special case of CRRA preferences, the demand-function and welfare-function approaches produce equivalent representation.

The social discount factors implied by equilibrium representation are closed connected to the literature on cost-benefit analysis for large-scale public projects and policies with long-term societal impacts.[7] In a simple Ramsey rule, the pure rate of time preference is a key component of the social (consumption) discount rate r:

$$r = \rho + \eta g,$$

where ρ is the time discount rate, η the CRRA coefficient, and g the expected growth rate of per capita consumption.

Weitzman (2001) investigates the shape of social discount rates for long-term public projects, especially those related to environmental policy. Based on survey data, he shows that even if individuals believe in a constant discount rate, the wide dispersion of those beliefs across society can produce an effective social discount rate that declines significantly over time.

Gollier and Zeckhauser (2005) examine the collective discount rate implied by a representative agent whose preferences are constructed to rationalize aggregate consumption in equilibrium, under the assumption that individuals have different discount factors. They establish that the collective rate of impatience is a weighted average of individual rates of impatience, with weights proportional to individual absolute risk tolerances. Using this result, they show that the representative agent's discount rate is declining when all

[7]The literature on social discounting is extensive, and we focus here on two seminal works Weitzman (2001) and Gollier and Zeckhauser (2005) that adopt a utilitarian perspective. For a comprehensive review of this literature, we refer readers to Millner and Heal (2023), which examines the ongoing debates over market-based versus approaches to determining social discount rates. They also present an alternative framework for social discounting based on recent advances in intertemporal social choice theory.

individuals share a constant discount rate and exhibit decreasing absolute risk aversion. Furthermore, they derive conditions under which the collective rate of impatience falls as per-capita GDP increases.

The explicit preference structure of the representative agent is closely related to the recently emerging literature that analyzes the impact of preference shocks and state-dependent discount factors on asset returns in a representative-agent setting.[8] Albuquerque et al. (2016) show that time-preference shocks can account for several asset pricing puzzles in a recursive utility framework, including the equity premium, the bond term premium, and the weak correlation between stock returns and fundamentals. Similarly, Schorfheide et al. (2018), Gomez-Cram and Yaron (2021), and De Groot et al. (2022) introduce state-dependent discount factors to resolve asset pricing puzzles in representative-agent models with recursive preferences. Stachurski and Zhang (2021) addresses the existence of the value function and optimal policies in infinite-horizon dynamic programming problems with state-dependent discount factors that encompass these asset pricing models. Stachurski et al. (2024) provide sufficient conditions for existence and uniqueness of recursive utility in a general framework.

This emerging literature typically assumes that the representative agent's current consumption is subject to exogenous preference shock processes. In contrast, in this paper the representative agent's time preference shocks emerge endogenously as discount adjustment factors from the equilibrium representation of the economy. They discount the representative agent's certainty equivalent of future consumption flows rather than current consumption. These aggregate shocks can arise even when individuals have a constant discount factor and thus are not subject to such time preference shocks. In particular, redistributive shocks (e.g., taxes and subsidies) manifest as preference shocks for the representative agent with KP preferences. Hence, the findings of this paper can provide a theoretical foundation for the literature on preference shocks.

This research is related to the general equilibrium literature that explores the domain of equilibrium prices. Won (2023) and Won (2025) develop the representative-agent approach to determine the equilibrium price range in a two-period setting. These papers focus on sufficient conditions for equilibrium uniqueness in a two-period economy with CRRA (constant relative risk aversion) preferences. In particular, Won (2025) demonstrates that under homogeneous expectations, the equilibrium outcomes in a two-period economy with KP preferences are in one-to-one correspondence with those in a two-good economy with CRRA preferences. Thus, the belief homogeneity framework leaves no room for examining how preferences for early or late resolution of uncertainty affect equilib-

[8]For a brief review of this emerging literature, see Stachurski and Zhang (2021) and references therein.

rium outcomes. In contrast, the current paper deals with KP preferences in a three-period economy, where preferences for the timing of uncertainty resolution play a crucial role in individuals' intertemporal choices and consequently, in aggregate equilibrium outcomes.

This paper is organized as follows. Section 2 models the three-period economy with KP preferences. Section 3 provides the closed-form demand functions. Section 4 provides a closed-form solution for autarkic equilibrium prices in a representative-agent economy with general KP preferences. Section 5 deals with the inverse problem of Section 4 in both complete and incomplete markets: the construction of a representative agent whose calibrated preferences rationalize the aggregate consumption of the economy or a subset of agents at a given price. In this section, we demonstrate that, when individual agents have CRRA preferences, the demand-based and welfare-based approaches produce equivalent representation results. Based on the result of Section 4, Section 6 introduces the MAX and MIN representative agents, whose closed-form equilibrium prices define the boundaries of the equilibrium price domain, which depends on preferences for early or late resolution of uncertainty. In Section 7, we present several counterintuitive examples where social utility discount rates are significantly higher than individual ones. The final section concludes.

2. A Dynamic Economy with Kreps-Porteus Preferences

We consider a three-period economy where I agents have Kreps-Porteus preferences. Let $\mathcal{I}$ denote the set of agents $\{1, \ldots, I\}$. We assume that for $t = 1, \ldots, T$, agent i's preferences are represented by

$$V_{i,t} = \left[(\mathbf{1} - \rho_i) z_t^{1-\psi^{-1}} + \rho_i \delta_{i,t} E_t^i \left[\left(V_{i,t+1} \right)^{1-\gamma} \right]^{\frac{1-\psi^{-1}}{1-\gamma}} \right]^{\frac{1}{1-\psi^{-1}}}, \tag{1}$$

where z_t denotes contemporary consumption at time t and V_{t+1}^i the certainty equivalent at time $t+1$ of future consumption from time $t+1$ onwards. Here, $\rho_i \in (0,\, 1)$ is constant and the conditional expectation E_t^i represents agent i's subjective beliefs at time t. Agent i's preferences are characterized by a constant relative risk aversion (CRRA) coefficient $\gamma > 0$ and an elasticity of intertemporal substitution (EIS) $\psi > 0$ with $\psi \neq 1$. If $\gamma\psi > 1$ ($\gamma\psi < 1$), she prefers early (late) resolution of uncertainty.

In contrast to conventional KP preferences, we introduce an *adjustment factor* $\delta_{i,t}$ for time discounting, which can be stochastic depending on the context. This factor allows agent i to discount future utility in a state-dependent manner. If $\delta_{i,t} = 1$ for all $t = 1, \ldots, T-1$ and $\delta_{i,T} = 1 - \rho_i$, then ρ_i acts as the time discount factor of agent i over the finite time horizon. In this case, the representation in (1) is reduced to a standard KP

framework.

In this paper, we focus on the case with $T = 3$, where uncertainty evolves as a Markov chain.[9] Let $s^t = (s_1, \ldots, s_t)$ denote a state at period t where $s_1 = 1$ corresponds to the first period, and for each $\tau = 2, \ldots, t$, s_τ is a random variable taking a value in $\mathcal{J} = \{2, \ldots, J\}$. For simplicity, we use s_2 and (s_2, s_3) to represents a state in period 2 and in period 3, respectively, instead of $(1, s_2)$ and $(1, s_2, s_3)$.

For $K = 1 + J(J-1)$, let $X_K = \mathbb{R}^K_+$ be the set of K-dimensional vectors with non-negative components, and $X^\circ_K = \mathbb{R}^K_{++}$ be the set of K-dimensional vectors with positive components. A point $x \in X_K$, which may refer to a consumption or a price vector depending on the context, has the following coordinate structure:

$$x = \big(x_1, (x_j, (x_{j,k})_{k=2}^J)_{j=2}^J\big),$$

where x_1, x_j, and $x_{j,k}$ are the elements of x at period 1, state j of period 2, and state (j,k) of period 3, respectively.

Let Pr_i denote the probability that represents agent i's subjective beliefs about the occurrence of states. We define the probability system $\phi_i = (\phi_{i,j}, (\phi_{i,j,k})_{k=2}^J)_{j=2}^J$ such that for all j, k in $\mathcal{J}$,

$$\phi_{i,j} = \text{Pr}_i(s_2 = j) > 0 \quad \text{and} \quad \phi_{i,j,k} = \text{Pr}_i(s_3 = k | s_2 = j) > 0.$$

Then the three-period Kreps-Porteus preferences are represented by a utility function

$$\begin{aligned} U_i(x) \equiv W(x; \rho_i, \delta_i, \phi_i, \gamma, \psi) \equiv \Bigg(& (1-\rho_i) x_1^{1-\psi^{-1}} \\ & + \rho_i \delta_{i,1} \Big(\sum_{j=2}^J \phi_{i,j} \Big((1-\rho_i) x_j^{1-\psi^{-1}} + \rho_i \delta_{i,j} \Big(\sum_{k=2}^J \phi_{i,j,k} x_{j,k}^{1-\gamma} \Big)^{\frac{1-\psi^{-1}}{1-\gamma}} \Big)^{\frac{1-\gamma}{1-\psi^{-1}}} \Big)^{\frac{1-\psi^{-1}}{1-\gamma}} \Bigg)^{\frac{1}{1-\psi^{-1}}}, \end{aligned} \tag{2}$$

where $\delta_i = (\delta_{i,1}, (\delta_{i,j})_{j=2}^J)$ for all $i \in \mathcal{I}$. Agent i discounts future utility by applying $\rho_i \delta_{i,1}$ to second-period utility and $\rho_i^2 \delta_{i,1} \delta_{i,j} / (1-\rho_i)$ to third-period utility. Note that these discount factors may be state-dependent. We refer to U_i as the *standard-form utility function* for KP preferences because it explicitly reveals agent i's probabilistic beliefs.

However, this standard-form utility function may not be appropriate for the representative agents introduced later, whose preferences are determined by the extreme values of the individual parameter distribution. For example, the values $\{\max_{i \in \mathcal{I}} \phi_{i,j}\}_{j=2}^J$ constructed

[9] The three-period framework is the simplest setting necessary to analyze how preferences for early or late resolution of uncertainty affect the dynamic behavior of consumption and equilibrium prices. As shown in Won (2025), equilibrium in a two-period economy with KP preferences is indistinguishable from that in a two-period economy with expected utility.

from the first-period probability distributions do not form a probability distribution when beliefs are heterogeneous.

To unify the preference representations for both individual and representative agents, we use the following alternative representation of KP preferences:

$$\hat{U}_i(x) \equiv W(x;\rho_i,\hat{\delta}_i,\hat{\phi}_i,\gamma,\psi) = \Bigg((1-\rho_i)x_1^{1-\psi^{-1}} + \rho_i\hat{\delta}_{i,1}\Big(\sum_{j=2}^{J}\hat{\phi}_{i,j}\Big((1-\rho_i)x_j^{1-\psi^{-1}} + \rho_i\hat{\delta}_{i,j}\Big(\sum_{k=2}^{J}\hat{\phi}_{i,j,k}x_{j,k}^{1-\gamma}\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}\Big)^{\frac{1-\gamma}{1-\psi^{-1}}}\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}\Bigg)^{\frac{1}{1-\psi^{-1}}}, \quad (3)$$

where $\hat{\delta}_i$ indicates a vector of discount adjustment factors and $\hat{\phi}_i$ a vector of utility weights that satisfies

$$\hat{\phi}_{i,2} = 1 \quad \text{and} \quad \hat{\phi}_{i,j,2} = 1 \quad \text{for all } \ j \in \mathcal{J}. \quad (4)$$

As shown later, this representation is well-suited to characterize the preferences of the representative agents that (i) provide macroeconomic insights into the equilibrium price range and (ii) rationalize the economy's aggregate behavior at equilibrium.[10]

Note that the standard-form utility function U_i can be transformed into the utility function $\hat{U}_i$ where $\hat{\delta}_i$ and $\hat{\phi}_i$ are defined as follows: for all $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $k \in \mathcal{J}$,

$$\hat{\delta}_{i,1} = \delta_{i,1}\phi_{i,2}^{\frac{1-\psi^{-1}}{1-\gamma}}, \ \ \hat{\delta}_{i,j} = \delta_{i,j}\phi_{i,j,2}^{\frac{1-\psi^{-1}}{1-\gamma}}, \ \ \hat{\phi}_{i,j} = \frac{\phi_{i,j}}{\phi_{i,2}}, \quad \text{and} \quad \hat{\phi}_{i,j,k} = \frac{\phi_{i,j,k}}{\phi_{i,j,2}}. \quad (5)$$

Since $\hat{\phi}_{i,2} = \hat{\phi}_{i,j,2} = 1$, each $\hat{\phi}_{i,j}$ and $\hat{\phi}_{i,j,k}$ can be interpreted as the relative likelihood of future states compared to the reference state $j = 2$.

Let e^i denote the initial endowment of agent i in X_K°. For a price $p \in X_K^\circ$, agent i makes an optimal choice in the following problem.

$$\max_{x \in X_K^\circ} \hat{U}_i(x) \quad \text{s.t.} \quad p \cdot x \leq p \cdot e^i, \quad (6)$$

where the dot $\cdot$ indicates the inner product between two vectors.

Let $X_{K,1}^\circ$ denote the subset of X_K° in which the first price is set to 1:

$$X_{K,1}^\circ = \{p \in X_K^\circ \text{ with } p_1 = 1\}.$$

A price-allocation pair $(p, \bar{x})$ in $X_{K,1}^\circ \times X_K^I$ is an *equilibrium* of the economy if each $\bar{x}^i$ solves

[10]The utility function $\hat{U}_i$ can be converted back into a standard KP utility function by normalizing the relative belief system $\hat{\phi}_i$.

problem (6), and the market clearing condition holds:

$$\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} e^i.$$

To simplify demand analysis, we introduce the following functions: for each $x \in X_K^\circ$,

$$Y_{i,j}(x) = \Big(\sum_{k=2}^{J} \hat{\phi}_{i,j,k} x_{j,k}^{1-\gamma} \Big)^{\frac{1}{1-\gamma}} \quad \forall j \in \mathcal{J}, \tag{7}$$

$$\begin{aligned} Z_{i,j}(x) &= \Big((1-\rho_i) x_j^{1-\psi^{-1}} + \rho_i \hat{\delta}_{i,j} \Big(\sum_{k=2}^{J} \hat{\phi}_{i,j,k} x_{j,k}^{1-\gamma} \Big)^{\frac{1-\psi^{-1}}{1-\gamma}} \Big)^{\frac{1}{1-\psi^{-1}}} \\ &= \Big((1-\rho_i) x_j^{1-\psi^{-1}} + \rho_i \hat{\delta}_{i,j} \big(Y_{i,j}(x) \big)^{1-\psi^{-1}} \Big)^{\frac{1}{1-\psi^{-1}}} \quad \forall j \in \mathcal{J}, \end{aligned} \tag{8}$$

$$\begin{aligned} Y_{i,1}(x) &= \Big(\sum_{j=2}^{J} \hat{\phi}_{i,j} \Big((1-\rho_i) x_j^{1-\psi^{-1}} + \rho_i \hat{\delta}_{i,j} \Big(\sum_{k=2}^{J} \hat{\phi}_{i,j,k} x_{j,k}^{1-\gamma} \Big)^{\frac{1-\psi^{-1}}{1-\gamma}} \Big)^{\frac{1-\gamma}{1-\psi^{-1}}} \Big)^{\frac{1}{1-\gamma}} \\ &= \Big(\sum_{j=2}^{J} \hat{\phi}_{i,j} \big(Z_{i,j}(x) \big)^{1-\gamma} \Big)^{\frac{1}{1-\gamma}}. \end{aligned} \tag{9}$$

The terms $Y_{i,j}(x)$, $Z_{i,j}(x)$, and $Y_{i,1}(x)$ denote agent i's certainty equivalents for three consumption profiles: $(x_{j,2}, \ldots, x_{j,J})$, $(x_j, (x_{j,2}, \ldots, x_{j,J}))$, and the future consumption vector x with its first component x_1 omitted. As shown later, each of these certainty equivalents is associated with its own price index, which significantly simplifies demand analysis.

This notation leads to a simpler expression for $\hat{U}_i$:

$$\hat{U}_i(x) = \Big((1-\rho_i) x_1^{1-\psi^{-1}} + \rho_i \hat{\delta}_{i,1} \big(Y_{i,1}(x) \big)^{1-\psi^{-1}} \Big)^{\frac{1}{1-\psi^{-1}}}. \tag{10}$$

Agent i's intertemporal marginal rate of substitution (IMRS) between consumptions in the first period and the jth state in the second period is given by

$$\begin{aligned} \frac{\partial \hat{U}_i / \partial x_j}{\partial \hat{U}_i / \partial x_1} &= \frac{(\partial \hat{U}_i / \partial Y_{i,1})(\partial Y_{i,1} / \partial Z_{i,j})(\partial Z_{i,j} / \partial x_j)}{\partial \hat{U}_i / \partial x_1} \\ &= \rho_i \hat{\delta}_{i,1} \hat{\phi}_{i,j} x_1^{\frac{1}{\psi}} x_j^{-\frac{1}{\psi}} \big(Z_{i,j}(x) \big)^{\frac{1}{\psi} - \gamma} \big(Y_{i,1}(x) \big)^{\gamma - \frac{1}{\psi}}. \end{aligned} \tag{11}$$

This agent has the IMRS between consumptions in the jth and (j,k)th states given by

$$\begin{aligned} \frac{\partial \hat{U}_i(x) / \partial x_{j,k}}{\partial \hat{U}_i(x) / \partial x_j} &= \frac{(\partial \hat{U}_i(x) / \partial Y_{i,j})(\partial Y_{i,j} / \partial x_{j,k})}{\partial \hat{U}_i(x) / \partial x_j} = \frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \hat{\phi}_{i,j,k} x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \big(Y_{i,j}(x) \big)^{\frac{\gamma\psi - 1}{\psi}} \\ &= \frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \hat{\phi}_{i,j,k} x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \Big(\sum_{k'=2}^{J} \hat{\phi}_{i,j,k'} x_{j,k'}^{1-\gamma} \Big)^{\frac{1-\gamma\psi}{(\gamma-1)\psi}}. \end{aligned} \tag{12}$$

These results show that the IMRSs move in different directions with respect to changes in $Z_{i,j}$, $Y_{i,1}$, and $Y_{i,j}$, depending on whether agent i prefers early ($\gamma\psi > 1$) or late ($\gamma\psi < 1$) resolution of uncertainty. As shown later, the effect of this timing preference on individual agents' IMRS plays a key role in shaping the preferences of the MAX and MIN representative agents mentioned in the Introduction.

3. Closed-Form Demand Functions

In this section, we provide a closed-form expression for individual demand, which is a key step in establishing the representative-agent approach to dynamic equilibrium analysis. Individual demand is derived by solving the optimization problem (6) through two-step backward induction. In Step 1, agent i makes an optimal choice in each state j of period 2, as if state j were realized in period 2:

$$\max_{(x_j,(x_{j,k})_{k=2}^J)\in\mathbb{R}^J_{++}} \Big((1-\rho_i)x_j^{1-\psi^{-1}} + \rho_i\hat{\delta}_{i,j}\Big(\sum_{k=2}^J \hat{\phi}_{i,j,k}x_{j,k}^{1-\gamma}\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}\Big)^{\frac{1}{1-\psi^{-1}}}$$
$$\text{s.t.} \quad p_j x_j + \sum_{k=2}^J p_{j,k}x_{j,k} = m_j^i(p), \tag{13}$$

where $m_j^i(p)$ equals the expenditure allocated to the jth state in the optimization process of (6).

Let $Q_j^{i,2}$ denote the price index for the certainty equivalent $Z_{i,j}(x)$. In Step 2, agent i makes an optimal choice of the first-period consumption x_1 and the certainty equivalent $Z_{i,j}(x)$ for given $(p_1,(Q_j^{i,2})_{j\in J})$. This problem is analogous to the two-period problem in Step 1 except that the second-period consumption is replaced by the certainty equivalent $Z_{i,j}$ and the time discounting is adjusted by $\hat{\delta}_{i,1}$:

$$\max_{(x_1,(y_j)_{j=2}^J)\in\mathbb{R}^J_{++}} \Big((1-\rho_i)x_1^{1-\psi^{-1}} + \rho_i\hat{\delta}_{i,1}\Big(\sum_{j=2}^J \hat{\phi}_{i,j}y_j^{1-\gamma}\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}\Big)^{\frac{1}{1-\psi^{-1}}}$$
$$\text{s.t.} \quad p_1 x_1 + \sum_{j=2}^J Q_j^{i,2}y_j = p\cdot e^i, \tag{14}$$

The optimization problem in each step is reduced to the following prototype problem: for a price $q\in\mathbb{R}^J_{++}$ and income $\tilde{m}$,

$$\max_{(x_1,\ldots,x_J)\in\mathbb{R}^J_{++}} u(x) = \Big[(1-\rho)x_1^{1-\psi^{-1}} + \rho\,\delta\Big(\sum_{j=2}^J \phi_j x_j^{1-\gamma}\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}\Big]^{\frac{1}{1-\psi^{-1}}} \quad \text{s.t.} \quad q\cdot x = \tilde{m}, \tag{15}$$

where $0 < \rho < 1$, $\delta > 0$, and $\phi_j > 0$ for all $j \in \mathcal{J}$.

Lemma 1. The optimal choice x^i in (15) is given by

$$x_1^i = \frac{\tilde{m}(1-\rho)^\psi q_1^{-\psi}}{(1-\rho)^\psi q_1^{1-\psi} + (\rho\,\delta)^\psi P^{1-\psi}} \tag{16}$$

$$x_j^i = \frac{\tilde{m}(\rho\,\delta)^\psi \left(\frac{\phi_j}{q_j}\right)^{\frac{1}{\gamma}} P^{\frac{1}{\gamma}-\psi}}{(1-\rho)^\psi q_1^{1-\psi} + (\rho\,\delta)^\psi P^{1-\psi}} \quad \text{for each } \ j = 2, \dots J, \tag{17}$$

where

$$P = \Big(\sum_{j=2}^{J} \phi_j^{\frac{1}{\gamma}} q_j^{1-\frac{1}{\gamma}} \Big)^{\frac{\gamma}{\gamma-1}}. \tag{18}$$

PROOF : See Appendix A.

By applying Lemma 1 repeatedly, we explicitly derive the demand functions from (6).

Lemma 2. For a price $p \in \mathbb{R}^K_{++}$, let $(x_1^i, (x_j^i, (x_{j,k}^i)_{k=2}^J)$ be the optimal choice in (6). Then for each $(j,k) \in \mathcal{J} \times \mathcal{J}$, we obtain the demand functions

$$g_1^i(p) \equiv x_1^i = \frac{(p \cdot e^i)(1-\rho_i)^\psi p_1^{-\psi}}{(1-\rho_i)^\psi p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^\psi (P^{i,1})^{1-\psi}}, \tag{19}$$

$$g_j^i(p) \equiv x_j^i = (1-\rho_i)^\psi (\rho_i \hat{\delta}_{i,1})^\psi \big(\hat{\phi}_{i,j}\big)^{\frac{1}{\gamma}} \frac{(p \cdot e^i) p_j^{-\psi} (Q_j^{i,2})^{\psi - \frac{1}{\gamma}} (P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^\psi p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^\psi (P^{i,1})^{1-\psi}} \tag{20}$$

$$\begin{aligned} g_{j,k}^i(p) \equiv x_{j,k}^i = {} & (\rho_i \hat{\delta}_{i,j})^\psi (\rho_i \hat{\delta}_{i,1})^\psi \big(\hat{\phi}_{i,j,k}\big)^{\frac{1}{\gamma}} \big(\hat{\phi}_{i,j}\big)^{\frac{1}{\gamma}} \\ & \times \frac{(p \cdot e^i) p_{j,k}^{-\frac{1}{\gamma}} (P_j^{i,2})^{\frac{1}{\gamma}-\psi} \big(Q_j^{i,2}\big)^{\psi - \frac{1}{\gamma}} (P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^\psi p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^\psi (P^{i,1})^{1-\psi}}, \end{aligned} \tag{21}$$

where the price indices $P^{i,2}$, $Q_j^{i,2}$, and $P_j^{i,1}$ are given by

$$P_j^{i,2} = \Big(\sum_{k=2}^{J} \hat{\phi}_{i,j,k}^{\frac{1}{\gamma}} p_{j,k}^{1-\frac{1}{\gamma}} \Big)^{\frac{\gamma}{\gamma-1}}, \tag{22}$$

$$Q_j^{i,2} = \Big((1-\rho_i)^\psi p_j^{1-\psi} + (\rho_i \hat{\delta}_{i,j})^\psi \big(P_j^{i,2}\big)^{1-\psi} \Big)^{\frac{1}{1-\psi}} \tag{23}$$

$$P^{i,1} = \Big(\sum_{j=2}^{J} \hat{\phi}_{i,j}^{\frac{1}{\gamma}} (Q_j^{i,2})^{1-\frac{1}{\gamma}} \Big)^{\frac{\gamma}{\gamma-1}}. \tag{24}$$

PROOF : See Appendix A.

Here, $P_j^{i,2}$, $Q_j^{i,2}$, and $P^{i,1}$ represent the price index for $Y_{i,j}(x)$, $Z_{i,j}(x)$, and $Y_{i,1}(x)$, respectively. The explicit demand functions in Lemma 2 provide a foundation for the equilibrium analysis in subsequent chapters, where we view the economy through the lens of representative agents.

4. Closed-Form Autarkic Equilibrium Price

In this section, we derive the autarkic equilibrium price for a representative-agent economy with arbitrary KP preferences. By inverting the autarkic equilibrium relationship, the price is explicitly expressed in terms of the representative agent's preference parameters and the aggregate endowment.

We use index d to denote this representative agent. Later in Section 6, this index will take values from $\{\max, \min\}$ to represent the MAX or MIN agent. We assume that the representative agent d has the following KP utility function $\hat{U}_d$ specified in (3): for all $x \in X_K^\circ$,

$$\hat{U}_d(x) = W(x; \rho_d, \hat{\delta}_d, \hat{\phi}_d, \gamma, \psi). \tag{25}$$

Here, the parameters $\rho_d, \hat{\delta}_d$, and $\hat{\phi}_d$ measures a tradeoff between current consumption and the certainty equivalent of future consumption, the discount adjustment factor, and the relative probabilistic belief parameters for agent d, respectively.

We refer to the economy with the representative agent d as the *d-economy*. Let G^d denote the demand function of the representative agent d. Then G^d has the same form as the individual demand function formulated in Lemma 2, except that the individual endowment e^i is replaced by the aggregate endowment $\bar{e}$.

An *autarkic equilibrium* of the d-economy is the pair $(p^d, \bar{e})$ with $p^d \in X_{K,1}^\circ$ that solves the market-clearing condition $G^d(p) = \bar{e}$. This implies that the representative agent d demands the aggregate endowment $\bar{e}$ at the autarkic equilibrium price p^d.

The autarkic equilibrium price p^d can be expressed explicitly in terms of fundamental economic parameters: preference parameters $(\rho_d, \hat{\delta}_d, \hat{\phi}_d, \gamma, \psi)$ and the aggregate endowment $\bar{e}$. The closed-form expression for p^d is obtained by inverting the autarkic equilibrium relationship $G^d(p) = \bar{e}$. Specifically, using Lemma 2, we first express the demand ratios $\bar{e}_{j,k}/\bar{e}_{j,2}$, $\bar{e}_j/\bar{e}_{j,2}$, and $\bar{e}_j/\bar{e}_2$ as functions of preference parameters and the relative prices $p_{j,k}^d/p_{j,2}^d$, $p_j^d/p_{j,2}^d$, and p_j^d/p_2^d. Then, by inverting these equations, we compute intra-period and inter-period price ratios in a backward manner, starting with $p_{j,k}^d/p_{j,2}^d$, $p_j^d/p_{j,2}^d$, and then p_j^d/p_2^d. Substituting these results into the equation for the demand ratio $\bar{e}_2/\bar{e}_1$, we obtain the value of p_2^d/p_1^d. Since $p_1^d = 1$, this ratio immediately gives the value p_2^d. Substituting these second-period prices back into the earlier ratios, we can recursively compute all

remaining prices. Details about the computational procedure appear in the proof of the following theorem.

Theorem 1. The equilibrium price $p^d \in X^{\circ}_{K,1}$ in the d-economy is explicitly solved as following: for all $(j,k) \in \mathcal{J} \times \mathcal{J}$,

$$p^d_2 = \rho_d \hat{\delta}_{d,1} \Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{-\frac{1}{\psi}} \Big(\Psi^2_{d,2}\Big)^{1-\frac{1}{\gamma\psi}} \big(\Phi^1_d\big)^{\frac{1}{\gamma\psi}-1}, \tag{26}$$

$$p^d_j = p^d_2 \hat{\phi}_{d,j} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma} \Big(\frac{\Psi^2_{d,j}}{\Psi^2_{d,2}}\Big)^{\gamma\psi-1}, \tag{27}$$

$$p^d_{j,2} = p^d_j \Big(\frac{\rho_d \hat{\delta}_{d,j}}{1-\rho_d}\Big) \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{-\frac{1}{\psi}} \Big(\Phi^2_{d,j}\Big)^{\frac{1}{\gamma\psi}-1}, \tag{28}$$

$$p^d_{j,k} = p^d_{j,2} \hat{\phi}_{d,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma}, \tag{29}$$

where

$$\Phi^2_{d,j} = \Bigg(\sum_{k=2}^{J} \hat{\phi}_{d,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{1-\gamma}\Bigg)^{\frac{\gamma}{\gamma-1}}, \tag{30}$$

$$\Psi^2_{d,j} = \Bigg((1-\rho_d)^{\psi} + (\rho_d \hat{\delta}_{d,j})^{\psi} \Big(\frac{1-\rho_d}{\rho_d \hat{\delta}_{d,j}}\Big)^{\psi-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{\psi-1}{\psi}} \big(\Phi^2_{d,j}\big)^{\frac{1-\psi}{\gamma\psi}}\Bigg)^{\frac{1}{1-\psi}}, \tag{31}$$

$$\Phi^1_d = \Bigg(\sum_{j=2}^{J} \hat{\phi}_{d,j} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{1-\gamma} \Big(\Psi^2_{d,j}\Big)^{1-\frac{1}{\gamma}} \Big(\frac{\Psi^2_{d,j}}{\Psi^2_{d,2}}\Big)^{(\gamma\psi-1)(1-\frac{1}{\gamma})}\Bigg)^{\frac{\gamma}{\gamma-1}}. \tag{32}$$

PROOF : See Appendix A.

This theorem explicitly provides the equilibrium price p^d at which the aggregate endowment $\bar{e}$ becomes the optimal choice of the representative agent d. This result is later applied to the MAX and MIN economies to determine the lower and upper bounds of equilibrium prices for the economy.

Here, the terms $\Phi^2_{d,j}$, $\Psi^2_{d,j}$, and Φ^1_d represent the value of the following relative price

indices at p^d, respectively:

$$\Phi^2_{d,j} = \frac{P^{d,2}_j}{p^d_{j,2}} = \left(\sum_{k=2}^{J} \hat{\phi}^{\frac{1}{\gamma}}_{d,j,k} \left(\frac{p^d_{j,k}}{p^d_{j,2}} \right)^{1-\frac{1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}}, \tag{33}$$

$$\Psi^2_{d,j} = \frac{Q^{d,2}_j}{p^d_j} = \left((1-\rho_d)^{\psi} + (\rho_d \hat{\delta}_{d,j})^{\psi} \left((P^{d,2}_j / p^d_{j,2})(p^d_{j,2}/p^d_j) \right)^{1-\psi} \right)^{\frac{1}{1-\psi}}, \tag{34}$$

$$\Phi^1_d = \frac{P^{d,1}}{p^d_2} = \left(\sum_{j=2}^{J} \hat{\phi}^{\frac{1}{\gamma}}_{d,j} \left(\frac{Q^{d,2}_j}{p^d_j} \frac{p^d_j}{p^d_2} \right)^{1-\frac{1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}}. \tag{35}$$

The value of $\Phi^2_{d,j}$ is obtained by substituting $p^d_{j,k}/p^d_{j,2}$ into $P^{d,2}_j/p^d_{j,2}$. This value, combined with $p^d_j/p^d_{j,2}$, determines the value of $Q^{d,2}_j/p^d_j$ through (34). Substituting this and p^d_j/p^d_2 into the relation in (35) yields the value of Φ^1_d.

The result of Theorem 1 is drastically simplified when each agent has CRRA preferences.

Corollary 1. Suppose that each agent has CRRA preferences, i.e., $\gamma\psi = 1$. Then the equilibrium price p^d in the d-economy has the following closed form solution: for all $(j,k) \in \mathcal{J} \times \mathcal{J}$,

$$p^d_j = \rho_d \hat{\delta}_{d,1} \hat{\phi}_{d,j} \left(\frac{\bar{e}_j}{\bar{e}_1} \right)^{-\gamma},\ p^d_{j,2} = p^d_j \left(\frac{\rho_d \hat{\delta}_{d,j}}{1-\rho_d} \right) \left(\frac{\bar{e}_{j,2}}{\bar{e}_j} \right)^{-\gamma},\ p^d_{j,k} = p^d_j \hat{\phi}_{d,j,k} \left(\frac{\rho_d \hat{\delta}_{d,j}}{1-\rho_d} \right) \left(\frac{\bar{e}_{j,k}}{\bar{e}_j} \right)^{-\gamma}. \tag{36}$$

5. Representation and Social Impatience

In the previous section, we derived the autarkic equilibrium price for a given d-economy as a function of the aggregate endowment and the preference parameters of the representative agent d. In this section, we address the inverse problem: for a given price p, we construct a representative agent whose KP preferences rationalize the autarkic equilibrium $(p, \bar{e})$.

In complete-market settings, the KP preferences are derived by inverting the autarkic equilibrium condition $G^d(p) = \bar{e}$ with respect to the unknown preference parameters. Consequently, the agent's preference characteristics–the discount factors and probabilistic beliefs–are fully determined by observable quantities such as market prices and the aggregate endowment. When every individual has CRRA preferences, this equilibrium representation proves equivalent to that of Jouini and Napp (2006, 2007), in which consensus beliefs are expressed as a risk tolerance-weighted average of individual beliefs. In this case, we also show that the representative agent cannot be more patient than the most patient individual.

We extend the closed-form demand approach to incomplete-market settings, where individual agents possess personalized state prices in equilibrium. To handle this, we convert an incomplete-market equilibrium into a complete-market equilibrium by selecting one agent's state prices as a benchmark. We then calibrate the preference parameters of the other agents so that their post-calibration state prices align with this benchmark. Under this calibrated regime, the incomplete-market equilibrium allocation becomes Pareto optimal. Consequently, we can apply the demand approach to the complete-market economy induced by the calibration. However, as the personalised state prices are unobservable, the calibrated preference parameters also remain unobservable.

Moreover, in an incomplete-market economy, the equilibrium representation depends on the choice of the benchmark agent. Because agents typically have distinct state prices, there can be as many different representations of equilibrium as there are agents. Although the underlying representative agent exhibits the same functional form of preferences in both complete and incomplete markets, these preferences differ in two key respects: observability and representational uniqueness.

5.1 Complete Markets

When asset markets are complete, we can construct a unique representative agent whose preferences rationalize the aggregate endowment at each equilibrium price.

Theorem 2. For any price $p \in X_{K,1}^{\circ}$, there exists a representative agent with KP preferences that demands the aggregate endowment $\bar{e}$ at the price p. The rationalizing KP preferences are represented by a utility function

$$\bar{U}_A(x) \equiv W(x; \bar{\rho}, \bar{\delta}, \bar{\phi}, \gamma, \psi) \tag{37}$$

over consumption x, where for each $(j,k) \in \mathcal{J} \times \mathcal{J}$ and each $\bar{\delta}_1 > 0$,

$$\begin{aligned} \bar{\rho}\,\bar{\delta}_1 &= p_2 \Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,2}}{C_3}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}}, \quad \frac{\bar{\rho}\bar{\delta}_j}{1-\bar{\rho}} = \Big(\frac{p_j}{p_{j,2}}\Big)^{-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}} C_{1,j}^{-\frac{1}{\gamma\psi}+1}, \\ \bar{\phi}_j &= \frac{p_j}{p_2} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{\gamma} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{1-\gamma\psi}, \quad \text{and} \quad \bar{\phi}_{j,k} = \frac{p_{j,k}}{p_{j,2}} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{\gamma}, \end{aligned} \tag{38}$$

where

$$
\begin{aligned}
C_{1,j} &= \Big(\sum_{k=2}^{J} \frac{p_{j,k}}{p_{j,2}} \frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{\frac{\gamma}{\gamma-1}}, \\
C_{2,j} &= \Big(1 + \frac{p_{j,2}}{p_j} \frac{\bar{e}_{j,2}}{\bar{e}_j} C_{1,j}^{1-\frac{1}{\gamma}} \Big)^{\frac{1}{1-\psi}}, \\
C_3 &= \Big(\sum_{j=2}^{J} \frac{p_j}{p_2} \frac{\bar{e}_j}{\bar{e}_2} C_{2,j}^{1-\frac{1}{\gamma}} \Big(\frac{C_{2,j}}{C_{2,2}} \Big)^{\frac{1}{\gamma}-\psi} \Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{39}
$$

PROOF : See Appendix A.

The first term in (38) shows that one of the two parameters, $\bar{\rho}$ or $\bar{\delta}_1$, can be treated as a free variable. Since we can factor out $(1-\bar{\rho})$ from the utility function specified in (3), equation (38) shows that $\bar{\rho}$ and $\bar{\delta}$ affect the utility representation $\bar{U}_A$ not individually but jointly through the terms $\bar{\rho}\bar{\delta}_1$ and $\bar{\rho}\,\bar{\delta}_j/(1-\bar{\rho})$. Consequently, $\bar{U}_A$ is *uniquely determined*, despite the apparent degree of freedom in the choice of preference parameters.

The terms $C_{1,j}$, $C_{2,j}$, and C_3 are derived from the relative price indices $P_j^2/p_{j,2}$, Q_j^2/p_j, and P^1/p_2 in which the preference parameters $\bar{\rho}, \bar{\delta}, \bar{\phi}$ are evaluated at the price p:

$$
\begin{aligned}
C_{1,j} &= \frac{P_j^2}{p_{j,2}} = \Big(\sum_{k=2}^{J} \bar{\phi}_{j,k}^{\frac{1}{\gamma}} \Big(\frac{p_{j,k}}{p_{j,2}} \Big)^{1-\frac{1}{\gamma}} \Big)^{\frac{\gamma}{\gamma-1}}, \\
C_{2,j} &= (1-\bar{\rho})^{\frac{\psi}{\psi-1}} \frac{Q_j^2}{p_j} = \Big(1 + \Big(\frac{\bar{\rho}\bar{\delta}_j}{1-\bar{\rho}} \Big)^{\psi} \Big(\frac{P_j^2}{p_{j,2}} \frac{p_{j,2}}{p_j} \Big)^{1-\psi} \Big)^{\frac{1}{1-\psi}}, \\
C_3 &= (1-\bar{\rho})^{\frac{\psi}{\psi-1}} \frac{P^1}{p_2} = (1-\bar{\rho})^{\frac{\psi}{\psi-1}} \Big(\sum_{j=2}^{J} \bar{\phi}_j^{\frac{1}{\gamma}} \Big(\frac{Q_j^2}{p_j} \frac{p_j}{p_2} \Big)^{1-\frac{1}{\gamma}} \Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
$$

By the Second Fundamental Theorem of Welfare Economics, any Pareto-optimal allocation can be supported as a competitive equilibrium under an appropriate redistribution of the aggregate endowment. Combining this result with Theorem 2, we find that, for each Pareto-optimal allocation, the economy can be represented by a single agent with KP preferences whose parameters are determined according to equation (38).

The preference parameters $\bar{\phi}$ specified in (38) is generally not a probabilistic belief. To identify the probabilistic belief and time discount rate for the representative agent, we transform $\bar{U}_A$ into a standard KP utility function U_A as in (2):

$$
U_A(x) \equiv W(x; \rho_A, \delta_A, \phi_A, \gamma, \psi), \tag{40}
$$

where

$$\begin{aligned}
&\rho_A = \bar{\rho}, \quad \delta_{A,1} = \bar{\delta}_1 \Big(\sum_{j=2}^{J} \bar{\phi}_j \Big)^{\frac{1-\psi^{-1}}{1-\gamma}}, \delta_{A,j} = \bar{\delta}_j \Big(\sum_{k=2}^{J} \bar{\phi}_{j,k} \Big)^{\frac{1-\psi^{-1}}{1-\gamma}}, \\
&\phi_{A,j} = \frac{\bar{\phi}_j}{\sum_{j'=2}^{J} \bar{\phi}_{j'}}, \quad \phi_{A,j,k} = \frac{\bar{\phi}_{j,k}}{\sum_{k'=2}^{J} \bar{\phi}_{j,k'}}.
\end{aligned} \tag{41}$$

This parameter transformation shows that ϕ_A represents the probabilistic belief of the representative agent.

The standard KP utility function U_A reveals the term structure of state-dependent discount factors for the representative agent in the complete-market setting.

Corollary 2. The standard KP preferences for the representative agent are characterized by a system of time discount factors $\varrho_A = (\varrho_{A,1}, (\varrho_{A,j})_{j=2}^J)$ defined by

$$\begin{aligned}
\varrho_{A,1} &= \rho_A \delta_{A,1} = p_2 \Big(\frac{\bar{e}_2}{\bar{e}_1} \Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,2}}{C_3} \Big)^{\frac{1-\gamma\psi}{\gamma\psi}} \Bigg(\sum_{j=2}^{J} \frac{p_j}{p_2} \Big(\frac{\bar{e}_j}{\bar{e}_2} \Big)^{\gamma} \Big(\frac{C_{2,j}}{C_{2,2}} \Big)^{1-\gamma\psi} \Bigg)^{\frac{1-\psi^{-1}}{1-\gamma}}, \\
\varrho_{A,j} &= \frac{\rho_A \delta_{A,j}}{1-\rho_A} = \Big(\frac{p_j}{p_{j,2}} \Big)^{-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big)^{\frac{1}{\psi}} C_{1,j}^{-\frac{1}{\gamma\psi}+1} \Bigg(\sum_{k=2}^{J} \frac{p_{j,k}}{p_{j,2}} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{\gamma} \Bigg)^{\frac{1-\psi^{-1}}{1-\gamma}}.
\end{aligned} \tag{42}$$

Since the representative agent in Theorem 2 represents the economy at the equilibrium price, we refer to ϱ_A as the system of *social utility discount factors* for the economy. For each $j = 1, \ldots, J$, the *social utility discount rate* is defined as $1/\varrho_{A,j} - 1$.

The relations in (38) and (39) show that $\bar{\phi}$ depends on the equilibrium price p. Consequently, the term structure of discount factors ϱ_A becomes a nonlinear function of the equilibrium price, rendering it endogenously determined by the economy's fundamentals through market prices.

Notably, the representative agent's discount factors are generally state-dependent even if individual agents have state-independent discount factors. For example, suppose that $\delta_{i,1} = 1$ and $\delta_{i,j} = 1 - \rho_i$ for all $j \in \mathcal{J}$. In this case, agent i holds the constant discount factor ρ_i. However, since these time preference parameters are only a small fraction of all the parameters that determine equilibrium prices, the social utility discount factor ϱ_A is necessarily state-dependent.

When agents have CRRA preferences, the time preference parameters in (42) have simpler expressions.

Corollary 3. Suppose that $\gamma\psi = 1$. Then the representative agent in Corollary 2 has CRRA preferences with a system of social utility discount factors ϱ_A and beliefs ϕ_A as follows: for all j, k in $\mathcal{J}$,

$$
\begin{aligned}
\varrho_{A,1} &= p_2\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}}\sum_{j=2}^{J}\bar{\phi}_j = \sum_{j=2}^{J} p_j\Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{\gamma},\\
\varrho_{A,j} &= \Big(\frac{p_j}{p_{j,2}}\Big)^{-1}\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\gamma}\sum_{k=2}^{J}\bar{\phi}_{j,k} = \sum_{k=2}^{J}\frac{p_{j,k}}{p_j}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_j}\Big)^{\gamma},\\
\phi_{A,j} &= \frac{p_j\bar{e}_j^{\gamma}}{\sum_{k=2}^{J} p_k\bar{e}_k^{\gamma}}, \text{ and } \phi_{A,j,k} = \frac{p_{j,k}\bar{e}_{j,k}^{\gamma}}{\sum_{k'=2}^{J} p_{j,k'}\bar{e}_{j,k'}^{\gamma}}.
\end{aligned}
\tag{43}
$$

In this particular case, we will later provide the lower and upper bounds of ϱ_A based on information about equilibrium prices.

5.2 Incomplete Markets

In this subsection, we extend the demand-based approach developed in Theorem 2 to an incomplete-market economy. Specifically, we construct a representative agent that reveals social utility discount factors at an equilibrium of the incomplete-market economy.

To do this, we need to understand the distinct effect of asset market incompleteness on equilibrium outcomes. In a complete-market setting, agents have the same state prices in equilibrium. In incomplete markets, however, they generally have different state prices, implying that equilibrium is typically not Pareto optimal.[11] Thus, the complete-market approach, predicated on the Pareto optimality of complete-market equilibrium, is not directly applicable in the incomplete-market context.

Nevertheless, we can exploit the linearity of asset pricing that holds in equilibrium to extend the result of Theorem 2 to incomplete markets. Specifically, if risk-free assets exist, the no-arbitrage condition requires that asset prices equal the discounted expected value of their payoffs under a probability measure determined by each agent's state prices.[12] The linearity of the asset pricing formula allows us to calibrate the preference parameters of individual agents so that they face the same set of state prices as if they were in complete-market equilibrium.

In the incomplete-market economy, we assume that agents hold N long-lived assets in zero net supply with $N \leq J-1$. These assets are retraded in the second period. Each asset make its payoffs in units of the consumption good. Asset a pays r_j^a in state $j \in \mathcal{J}$ of the second period and $r_{j,k}^a$ in state (j,k) of the third period. This asset has the payoff r^a in

[11] For details, see Magill and Quinzii (1996).

[12] If no risk-free asset is available, one can choose an asset with strictly positive payoffs in every state as a numeraire. Under the no-arbitrage condition, the prices of all other assets relative to this numeraire follow a martingale under the equivalent martingale measure (EMM). See Duffie (2010).

$\mathbb{R}^{K-1}$ given by

$$r^a = (r^a_j, (r^a_{j,k})^J_{k=2})^J_{j=2}.$$

Let r_j and $r_{j,k}$ denote the payoff vector of the N assets in states j and (j,k).

We introduce a set $\Theta = \mathbb{R}^{NJ}$ which indicates either the set of portfolios or the set of asset prices depending on the context. Agents can choose a portfolio in Θ to make an intertemporal income transfer across states.

In this incomplete-market framework, for an asset price $q = (q_1, q_2, \ldots, q_J) \in \Theta$, agent i faces the following choice problem:

$$\begin{aligned} \max_{(x,\theta)\in X_K\times\Theta} & \hat{U}_i(x) \\ & x_1 - e^i_1 = -q_1 \cdot \theta_1 \\ & x_j - e^i_j = (r_j + q_j)\cdot\theta_1 - q_j\cdot\theta_j \quad \forall\, j \in \mathcal{J} \\ & x_{j,k} - e^i_{j,k} = r_{j,k}\cdot\theta_j. \quad \forall\, (j,k) \in \mathcal{J}\times\mathcal{J} \end{aligned} \tag{44}$$

Let $(x^i, \theta^i) \in X_K \times \Theta$ be an optimal choice for agent i in the above problem. A triplet $(q, x, \theta) \in \Theta \times X^I_K \times \Theta^I$ is an *equilibrium of the incomplete-market economy* if each (x^i, θ^i) solves (44) for q and the market clearing conditions for goods and asset markets hold, i.e., $\sum^I_{i=1}(x^i - e^i) = 0$ and $\sum^I_{i=1}\theta^i = 0$.

The following theorem presents the equilibrium representation for the incomplete-market economy in which the KP preferences of all agents are calibrated so that their post-calibration state prices align with those of the first agent chosen as the benchmark agent. As discussed below, choosing a different benchmark agent will generally lead to a different equilibrium representation.

Theorem 3. Let (q, x, θ) be an equilibrium in the incomplete-market economy where $\lambda^i \in X^\circ_K$ is agent i's Lagrangian multiplier for the budget constraint in (44).[13] Let p be a price in $X^\circ_{K,1}$ defined by

$$p_1 = 1,\ p_j = \lambda^1_j/\lambda^1_1 \ \text{ for all } \ j \in \mathcal{J}, \ \text{ and } \ p_{j,k} = \lambda^1_{j,k}/\lambda^1_j \ \text{ for all } \ (j,k) \in \mathcal{J}\times\mathcal{J}. \tag{45}$$

Then (p, x) is an equilibrium in the complete-market economy in which agent i has a utility function $W(x; \rho_i, \breve{\delta}_i, \breve{\phi}_i, \gamma, \psi)$ with $\breve{\delta}_i$ and $\breve{\phi}_i$ defined as follows: for each $i \in \mathcal{I}$, $j \in \mathcal{J}$, and

[13] Duffie and Shafer (1986) show that equilibrium generically exists in multi-period incomplete markets with long-lived assets that pay out in units of consumption goods. This means that for any random choice of a pair $\big((e^i)^I_{i=1}, (r^a)^N_{a=1}\big)$ of endowment profile and asset structure, equilibrium almost certainly exists in the incomplete-market economy.

$(j,k) \in \mathcal{J} \times \mathcal{J}$,

$$\breve{\delta}_{i,1} = \hat{\delta}^i_{i,1}\hat{\phi}_{i,2}\frac{\lambda^i_1}{\lambda^i_2}\frac{\lambda^1_2}{\lambda^1_1}, \quad \breve{\delta}_{i,j} = \hat{\delta}^i_{i,j}\hat{\phi}_{i,j,2}\frac{\lambda^i_1}{\lambda^i_{j,2}}\frac{\lambda^1_j}{\lambda^1_1}, \tag{46a}$$

$$\breve{\phi}_{i,j} = \Big(\hat{\phi}_{i,2}\frac{\lambda^i_1}{\lambda^i_2}\frac{\lambda^1_2}{\lambda^1_1}\Big)^{-1}\Big(\hat{\phi}_{i,j}\frac{\lambda^i_1}{\lambda^i_j}\frac{\lambda^1_j}{\lambda^1_1}\Big), \quad \breve{\phi}_{i,j,k} = \Big(\hat{\phi}_{i,j,2}\frac{\lambda^i_j}{\lambda^i_{j,2}}\frac{\lambda^1_{j,2}}{\lambda^1_j}\Big)^{-1}\Big(\hat{\phi}_{i,j,k}\frac{\lambda^i_j}{\lambda^i_{j,k}}\frac{\lambda^1_{j,k}}{\lambda^1_j}\Big). \tag{46b}$$

(Note that $\breve{\phi}_{i,2} = 1$ and $\breve{\phi}_{i,j,2} = 1$.)

PROOF : See Appendix A. □

The equilibrium price p in (45) is set to agent 1's state prices $(\lambda^1_j/\lambda^1_1, (\lambda^1_{j,k}/\lambda^1_j)^J_{k=2})^J_{j=2}$. The KP preference parameters in (46) are calibrated to support (p,x) as the complete-market equilibrium. In this construction, agent 1 is arbitrarily chosen and can be replaced by any agent in the economy. Because individuals in incomplete markets typically have different state prices, the equilibrium price representation in (45) will vary depending on which agent is chosen as the benchmark for preference calibration.[14] Therefore, in general, there are I ways of representing equilibrium in incomplete markets, in sharp contrast to the unique representation of equilibrium in complete markets.

The result of Theorem 3 is based on the calibration of preference parameters formulated in (46). Under this new preference regime, agents may have state-dependent discount factors and heterogeneous beliefs, even though their original preferences have state-independent discount factors and homogeneous beliefs. These calibrated preferences may not be observable because the personalized state prices are not revealed in equilibrium.

Specifically, let p^h denote the equilibrium price that stems from setting agent h's state prices to the benchmark in Theorem 3. For each $i \in \mathcal{I}$, let $\rho^h_i, \breve{\delta}^h_i$, and $\breve{\phi}^h_i$ be the preference parameters of agent i which are calibrated according to the rule in (46)) to support (p^h, x) as the complete-market equilibrium. Then for each $h \in \mathcal{I}$, there exists a different version of equilibrium representation where each i holds the utility function $W(x; \rho_i, \breve{\delta}^h_i, \breve{\phi}^h_i, \gamma, \psi)$ in the complete-market setting.

Note that the equilibrium allocation x attained in the incomplete-market economy is Pareto optimal under the new preference regime. At first glance, this result seems contradictory because x is not Pareto optimal in incomplete markets where agents have different personalized state prices. However, there is no inconsistency here: the calibration rule in (46) is designed to induce complete-market behavior among agents.

By combining Theorems 2 and 3, we can construct an equilibrium-representing agent

[14] See Section 25 in Magill and Quinzii (1996) and the references therein for discussions on why equilibrium typically fails to attain Pareto optimality in multi-period incomplete markets.

for the incomplete-market economy. Specifically, for a given equilibrium in the incomplete-market economy, there exists a representative-agent economy with the autarkic equilibrium $(p^h, \bar{e})$ for each $h \in \mathcal{I}$ where p^h is the complete market equilibrium price constructed in Theorem 3. Then, each equilibrium price p^h determines the KP preferences of the representative agent in the same way as described in Corollary 2.

Corollary 4. For each $h \in \mathcal{I}$, there exists an representative agent with standard KP preferences characterized by the following system of discount factors $\varrho_A^h = (\varrho_{A,1}^h, (\varrho_{A,j}^h)_{j=2}^J)$:

$$\begin{aligned} \varrho_{A,1}^h &= p_2^h \Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,2}^h}{C_3^h}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}} \Big(\sum_{j=2}^{J} \bar{\phi}_j^h\Big)^{\frac{1-\psi^{-1}}{1-\gamma}}, \\ \varrho_{A,j}^h &= C_{1,j}^h \Big(\sum_{k=2}^{J} \bar{\phi}_{j,k}^h\Big)^{\frac{1-\psi^{-1}}{1-\gamma}} \text{ for each } \ j \in \mathcal{J}. \end{aligned} \tag{47}$$

Here, C^h's and $\bar{\phi}^h$ are formulated in (38) and (39) where p is replaced p^h.

The standard KP preferences of the representative agent reveal the state-dependent term structure of social utility discount factors for the incomplete-market economy, which may vary with the choice of benchmark agent h in Theorem 3.

5.3 Group-Level Preference Representation

Economic analysis often needs to divide the set of agents into distinct groups based on demographic and socioeconomic characteristics–such as income, geography, gender, age, ethnicity or nationality–to study how these groups, as economic entities, respond differently to taxes, subsidies, economic shocks, and institutional changes. Such group-level analysis implicitly or explicitly assumes that each group possesses representative preferences, derived from aggregating the individual preferences of group members. One straightforward method for representing group preferences is simply to assume that each group member has identical preferences. However, when group members have heterogeneous preferences, the literature often employs predetermined weighting systems for preference aggregation, such as arithmetic and geometric averaging of preference parameters.[15]

[15]For example, in a CES preference framework, Redding and Weinstein (2020) examines the effect of taste shocks on households' cost of living by constructing a representative expenditure for the actual demographic composition of each region. They use a predetermined weighting scheme obtained by modifying the original weighting system from their data source to enhance the regional representativeness of sampled household data. In contrast, our approach determines the weights for aggregating individual characteristics endogenously through the group-level preference representation.

In this subsection, we provide a theoretical foundation for aggregating preferences among the members of the same group in both complete- and incomplete-market settings. This goal is easily achieved by exploiting the results of Theorems 2 and 3. To this end, we arbitrarily partition the set of individual agents in the economy into nonempty distinct groups and then aggregate the optimal consumption choices of the members of each group. We then take the aggregate consumption at the group level as the initial endowment of the underlying group. This approach allows us to formulate the group-level preference representation in equilibrium by applying the results of Theorems 2 and 3 to each group.

We partition the I agents into n nonempty groups $\{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ where for each $h = 1, \ldots, n$, the group $\mathcal{I}_h$ consists of I_h agents. Let (p, x) be an equilibrium of the economy. For each group $h = 1, \ldots, n$, we define its aggregate consumption bundle x^{I_h} as

$$x^{I_h} = \sum_{i \in \mathcal{I}_h} x^i.$$

Now we consider a subeconomy $\mathcal{E}_h$ consisting of I_h agents in $\mathcal{I}_h$, where each $i \in \mathcal{I}_h$ has the utility function $\hat{U}_i$ and the initial endowment x^i. Then x^{I_h} is the aggregate endowment of this subeconomy, and the price-allocation pair $\big(p, (x)_{i \in \mathcal{I}_h}\big)$ is clearly the equilibrium of the subeconomy $\mathcal{E}_h$. The following result is immediately obtained by replacing $\bar{e}$ with x^{I_h} in the statement of Theorem 2.

Corollary 5. Let (p, x) be an equilibrium of the economy. Then for each $h = 1, \ldots, n$, there exists a representative agent h in the subeconomy $\mathcal{E}_h$ for whom x^{I_h} is the initial endowment as well as the optimal choice at the price p. Here, the representative agent h has the KP utility function

$$\bar{U}_{h,A}(x') \equiv W(x'; \bar{\rho}_h, \bar{\delta}_h, \bar{\phi}_h, \gamma, \psi)$$

over consumption x', specified as follows: for each $(j, k) \in \mathcal{J} \times \mathcal{J}$ and each $\bar{\delta}_1 > 0$,

$$\begin{aligned} \bar{\rho}_h\, \bar{\delta}_{h,1} &= p_2 \Big(\frac{x_2^{I_h}}{x_1^{I_h}}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{h,2,2}}{C_{h,3}}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}}, \quad \frac{\bar{\rho}_h \bar{\delta}_{h,j}}{1-\bar{\rho}_h} = \Big(\frac{p_j}{p_{j,2}}\Big)^{-1} \Big(\frac{x_{j,2}^{I_h}}{x_j^{I_h}}\Big)^{\frac{1}{\psi}} C_{h,1,j}^{-\frac{1}{\gamma\psi}+1}, \\ \bar{\phi}_{h,j} &= \frac{p_j}{p_2} \Big(\frac{x_j^{I_h}}{x_2^{I_h}}\Big)^{\gamma} \Big(\frac{C_{h,2,j}}{C_{h,2,2}}\Big)^{1-\gamma\psi}, \quad \text{and} \quad \bar{\phi}_{h,j,k} = \frac{p_{j,k}}{p_{j,2}} \Big(\frac{x_{j,k}^{I_h}}{x_{j,2}^{I_h}}\Big)^{\gamma}, \end{aligned} \tag{48}$$

where

$$
\begin{aligned}
C_{h,1,j} &= \Big(\sum_{k=2}^{J} \frac{p_{j,k}}{p_{j,2}} \frac{x_{j,k}^{I_h}}{x_{j,2}^{I_h}} \Big)^{\frac{\gamma}{\gamma-1}}, \\
C_{h,2,j} &= \Big(1 + \frac{p_{j,2}}{p_j} \frac{x_{j,2}^{I_h}}{x_j^{I_h}} C_{h,1,j}^{1-\frac{1}{\gamma}} \Big)^{\frac{1}{1-\psi}}, \\
C_{h,3} &= \Big(\sum_{j=2}^{J} \frac{p_j}{p_2} \frac{x_j^{I_h}}{x_2^{I_h}} C_{h,2,j}^{1-\frac{1}{\gamma}} \Big(\frac{C_{h,2,j}}{C_{h,2,2}} \Big)^{\frac{1}{\gamma}-\psi} \Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{49}
$$

The utility function $\bar{U}_{h,A}$ represents the KP preferences for the group h of agents that form the subeconomy $\mathcal{E}_h$. For each j, k in $\mathcal{J}$, we define the expenditure shares of group h:

$$
\pi_1^h = \frac{x_1^{I_h}}{p \cdot x^{I_h}}, \quad \pi_j^h = \frac{p_j x_j^{I_h}}{p \cdot x^{I_h}} \quad \text{and} \quad \pi_{j,k}^h = \frac{p_{j,k} x_{j,k}^{I_h}}{p \cdot x^{I_h}}
$$

The terms in (50) and (51) can be expressed in terms of the group's consumption expenditures or its expenditure shares:[16]

$$
\begin{aligned}
\bar{\rho}_h \, \bar{\delta}_{h,1} &= p_2^{1-\frac{1}{\psi}} \Big(\frac{p_2 x_2^{I_h}}{x_1^{I_h}} \Big)^{\frac{1}{\psi}} \Big(\frac{C_{h,2,2}}{C_{h,3}} \Big)^{\frac{1-\gamma\psi}{\gamma\psi}} = p_2^{1-\frac{1}{\psi}} \Big(\frac{\pi_2^h}{\pi_1^h} \Big)^{\frac{1}{\psi}} \Big(\frac{C_{h,2,2}}{C_{h,3}} \Big)^{\frac{1-\gamma\psi}{\gamma\psi}}, \\
\frac{\bar{\rho}_h \bar{\delta}_{h,j}}{1-\bar{\rho}_h} &= \Big(\frac{p_{j,2}}{p_j} \Big)^{1-\frac{1}{\psi}} \Big(\frac{p_{j,2} x_{j,2}^{I_h}}{p_j x_j^{I_h}} \Big)^{\frac{1}{\psi}} C_{h,1,j}^{-\frac{1}{\gamma\psi}+1} = \Big(\frac{p_{j,2}}{p_j} \Big)^{1-\frac{1}{\psi}} \Big(\frac{\pi_{j,2}^h}{\pi_j^h} \Big)^{\frac{1}{\psi}} C_{h,1,j}^{-\frac{1}{\gamma\psi}+1}, \\
\bar{\phi}_{h,j} &= \Big(\frac{p_j}{p_2} \Big)^{1-\gamma} \Big(\frac{p_j x_j^{I_h}}{p_2 x_2^{I_h}} \Big)^{\gamma} \Big(\frac{C_{h,2,j}}{C_{h,2,2}} \Big)^{1-\gamma\psi} = \Big(\frac{p_j}{p_2} \Big)^{1-\gamma} \Big(\frac{\pi_j^h}{\pi_2^h} \Big)^{\gamma} \Big(\frac{C_{h,2,j}}{C_{h,2,2}} \Big)^{1-\gamma\psi}, \\
\bar{\phi}_{h,j,k} &= \Big(\frac{p_{j,k}}{p_{j,2}} \Big)^{1-\gamma} \Big(\frac{p_{j,k} x_{j,k}^{I_h}}{p_{j,2} x_{j,2}^{I_h}} \Big)^{\gamma} = \Big(\frac{p_{j,k}}{p_{j,2}} \Big)^{1-\gamma} \Big(\frac{\pi_{j,k}^h}{\pi_{j,2}^h} \Big)^{\gamma},
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
C_{h,1,j} &= \Big(\sum_{k=2}^{J} \frac{p_{j,k} x_{j,k}^{I_h}}{p_{j,2} x_{j,2}^{I_h}} \Big)^{\frac{\gamma}{\gamma-1}} = \Big(\sum_{k=2}^{J} \frac{\pi_{j,k}^h}{\pi_{j,2}^h} \Big)^{\frac{\gamma}{\gamma-1}} \\
C_{h,2,j} &= \Big(1 + \frac{p_{j,2} x_{j,2}^{I_h}}{p_j x_j^{I_h}} C_{h,1,j}^{1-\frac{1}{\gamma}} \Big)^{\frac{1}{1-\psi}} = \Big(1 + \frac{\pi_{j,2}^h}{\pi_j^h} C_{h,1,j}^{1-\frac{1}{\gamma}} \Big)^{\frac{1}{1-\psi}}, \\
C_{h,3} &= \Big(\sum_{j=2}^{J} \frac{p_j x_j^{I_h}}{p_2 x_2^{I_h}} C_{h,2,j}^{1-\frac{1}{\gamma}} \Big(\frac{C_{h,2,j}}{C_{h,2,2}} \Big)^{\frac{1}{\gamma}-\psi} \Big)^{\frac{\gamma}{\gamma-1}} = \Big(\sum_{j=2}^{J} \frac{\pi_j^h}{\pi_2^h} C_{h,2,j}^{1-\frac{1}{\gamma}} \Big(\frac{C_{h,2,j}}{C_{h,2,2}} \Big)^{\frac{1}{\gamma}-\psi} \Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned}
\tag{51}
$$

[16] Demand-analysis literature (e.g., Deaton and Muellbauer (1980), Deaton (1986)) typically estimates systems of demand equations in expenditure-share form. In particular, Redding and Weinstein (2020) uses average expenditures on product varieties for regional consumer groups to assess the effects of taste shocks on cost of living.

Since (p, x^{I_h}) is an autarkic equilibrium for the representative agent h, we can now build an n-agent economy in which each "group agent" h has the initial endowment x^{I_h} and the utility function $\bar{U}_{h,A}$. In this new economy, the market clearing condition holds:

$$\sum_{h=1}^{n} x^{I_h} = \sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} e^i = \bar{e},$$

It then follows directly from Corollary 5 that the price–allocation pair $(p, (x^{I_h})_{h=1}^n)$ is an equilibrium of the n-agent economy.

Theorem 4. Let (p, x) be an equilibrium of the original economy with I agents. Then the pair is also an equilibrium of the n-agent economy where for each $h = 1, \ldots, n$, agent h has the utility function $\bar{U}_{h,A}$ and the initial endowment x^{I_h}.

The results of Corollary 5 and Theorem 4 can be extended to the incomplete market economy specified in Subsection 5.2 by applying Theorem 3.

5.4 The Representational Equivalence for CRRA Preferences

We show that the closed-form demand approach and the welfare approach of Jouini and Napp (2006, 2007) produce the same representation outcomes in complete markets where agents have CRRA preferences with heterogeneous beliefs. To this end, we assume that $\gamma\psi = 1$ and for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $\delta_{i,1} = 1$ and $\delta_{i,j} = 1 - \rho_i$ in the utility function (2). Then each U_i is expressed as

$$U_i(x) = (1-\rho_i)^{\frac{1}{1-\gamma}} \Big(x_1^{1-\gamma} + \rho_i \sum_{j=2}^{J} \phi_{i,j} \Big(x_j^{1-\gamma} + \rho_i \sum_{k=2}^{J} \phi_{i,j,k} x_{j,k}^{1-\gamma} \Big) \Big)^{\frac{1}{1-\gamma}}. \tag{52}$$

By an increasing transformation of U_i, the CRRA preferences of agent i are equivalently expressed in a standard form:

$$V_i(x) = \frac{1}{1-\gamma} \Big(x_1^{1-\gamma} + \rho_i \sum_{j=2}^{J} \phi_{i,j} \Big(x_j^{1-\gamma} + \rho_i \sum_{k=2}^{J} \phi_{i,j,k} x_{j,k}^{1-\gamma} \Big) \Big). \tag{53}$$

Suppose that the occurrence of states is governed by the following objective probability measure $\pi = (\pi_j, (\pi_{j,k})_{k=2}^J)_{j=2}^J$: for all j, k in $\mathcal{J}$,

$$\pi_j = \Pr(s_2 = j) > 0 \quad \text{and} \quad \pi_{j,k} = \Pr(s_3 = k | s_2 = j) > 0.$$

Following Jouini and Napp (2006, 2007), we can rewrite V_i as the expected utility under the objective probability measure.

$$V_i(x) = \frac{1}{1-\gamma}\Big(x_1^{1-\gamma} + \rho_i \sum_{j=2}^{J} \pi_j \Big(\frac{\phi_{i,j}}{\pi_j}\Big)\Big(x_j^{1-\gamma} + \rho_i \sum_{k=2}^{J} \pi_{j,k}\Big(\frac{\phi_{i,j,k}}{\pi_{j,k}}\Big) x_{j,k}^{1-\gamma}\Big)\Big). \tag{54}$$

Here, ϕ_i/π denotes the density process of ϕ_i with respect to π.

We show that for CRRA preferences, the belief representation in Corollary 3 is equivalent to the result in Jouini and Napp (2006, 2007) where the representative agent is derived from maximizing a linear social welfare function subject to the resource constraints. In particular, Example 2.3 of Jouini and Napp (2006) and Example 2.1 of Jouini and Napp (2007) provide "consensus belief" for the representative agent when individual agents have power utility functions.[17]

Let p be an equilibrium price in the CRRA economy. Jouini and Napp (2006) applies the welfare approach to constructing the preferences of the representative agent defined by

$$V_\lambda(z) = \max_{(x_1,\ldots,x_I)\in(X_K^\circ)^I} \sum_{i\in\mathcal{I}} \frac{1}{\lambda_i} V_i(x_i) \quad \text{subject to } \sum_{i\in\mathcal{I}} x_i \le z, \tag{55}$$

where λ_i denotes the Lagrange multiplier associated with agent i's optimal choice at the price p. Since V_i is separable across states, this problem can be decomposed into a state-by-state preference aggregation problem.

In Example 2.3 of Jouini and Napp (2006), the aggregate utility weight (i.e., consensus characteristics) $\bar{\phi}_\lambda$ is expressed as follows: for each j,k in $\mathcal{J}$,

$$\bar{\phi}_{\lambda,j} = \Big(\sum_{i\in\mathcal{I}} w_i \big(\rho_i \phi_{i,j}\big)^{\frac{1}{\gamma}}\Big)^{\gamma} \text{ and } \bar{\phi}_{\lambda,j,k} = \Big(\sum_{i\in\mathcal{I}} \big(w_i \rho_i^2 \phi_{i,j}\phi_{i,j,k}\big)^{\frac{1}{\gamma}}\Big)^{\gamma}, \tag{56}$$

where the weight w is given by $w_i = \lambda_i^{-1/\gamma} / \sum_{h\in\mathcal{I}} \lambda_h^{-1/\gamma}$. Let ϕ_λ denote the aggregate belief defined by

$$\phi_{\lambda,j} = \frac{\bar{\phi}_{\lambda,j}}{\sum_{j'\in\mathcal{J}} \bar{\phi}_{\lambda,j'}} \text{ and } \phi_{\lambda,j,k} = \frac{\bar{\phi}_{\lambda,j,k}}{\sum_{k'\in\mathcal{J}} \bar{\phi}_{\lambda,j,k'}}. \tag{57}$$

Then the representative agent has the preferences represented by

$$\begin{aligned} V_\lambda(z) &= \frac{1}{1-\gamma}\Big(z_1^{1-\gamma} + \sum_{j=2}^{J}\Big(\bar{\phi}_{\lambda,j} z_j^{1-\gamma} + \sum_{k=2}^{J} \bar{\phi}_{\lambda,j,k} z_{j,k}^{1-\gamma}\Big)\Big) \\ &= \frac{1}{1-\gamma}\Big(z_1^{1-\gamma} + \rho_{\lambda,1} \sum_{j=2}^{J} \phi_{\lambda,j}\Big(z_j^{1-\gamma} + \rho_{\lambda,j} \sum_{j=2}^{J} \phi_{\lambda,j,k} z_{j,k}^{1-\gamma}\Big)\Big), \end{aligned} \tag{58}$$

[17] The latter work adopts a continuous-time framework.

where the discount factors $\rho_{\lambda,1}$ and $\rho_{\lambda,j}$ are given by

$$\rho_{\lambda,1} = \sum_{j' \in \mathcal{J}} \bar{\phi}_{\lambda,j'} \quad \text{and} \quad \rho_{\lambda,j} = \frac{1}{\rho_{\lambda,1}} \sum_{k' \in \mathcal{J}} \bar{\phi}_{\lambda,j,k'}.$$

Note that the aggregate belief ϕ_λ is expressed in terms of individual agents' Lagrange multipliers, while the aggregate belief ϕ_A in Corollary 3 is expressed in terms of market prices and the aggregate endowment. Although the two representations appear different, they are in fact equivalent.

Lemma 3. Let (p, x) be an equilibrium in the economy. If $\psi\gamma = 1$, then it holds that

$$\phi_\lambda = \phi_A \quad \text{and} \quad \rho_\lambda = \varrho_A. \tag{59}$$

PROOF : See Appendix A. □

The following shows that when individuals have CRRA preferences, the representative agent is less patient in equilibrium than the most patient individual if $\gamma > 1$, and more patient than the least patient individual if $0 < \gamma < 1$.

Lemma 4. Let (p, x) be an equilibrium in the economy with $\psi\gamma = 1$. Then it holds that for each $j = 1, \ldots, J$,

a) $\varrho_{A,j} \leq \max_{i \in \mathcal{J}} \rho_i$ for $\gamma > 1$,

b) $\varrho_{A,j} \geq \min_{i \in \mathcal{J}} \rho_i$ for $0 < \gamma < 1$.

PROOF : See Appendix A. □

For $\gamma > 1$, the representative agent's patience is capped by the maximum patience among individuals. Remarkably, it is later shown that under belief heterogeneity, the representative agent can exhibit greater impatience than any individual agent. This result departs from Gollier and Zeckhauser (2005), which shows that the collective discount factor is bounded below by the smallest individual discount factor under heterogeneous time preferences. This departure highlights the impact of heterogeneous beliefs on the collective impatience implied by the representative agent.

6. Equilibrium Prices in a Hyperrectangle

In this section, we apply the result of Theorem 1 in Section 4 to delineate the range of equilibrium prices in the economy. For each $d \in \{\max, \min\}$, we refer to the representative

agent in the d-economy as the MAX and MIN agent, respectively. These representative agents are designed to have autarkic equilibrium prices that serve as the lower and upper bounds of the equilibrium price range. They possess KP preferences determined by the aggregate endowment and the extreme values of the individual parameter distribution. Moreover, the preference parameters are expressed as functions of these exogenous variables in different ways depending on whether individual agents uniformly prefer early or late resolution of uncertainty.

Depending on whether agents prefer early or late resolution of uncertainty, the extreme values of individual preference parameters are incorporated differently into the preference representation of the MAX and MIN agents.

6.1 Equilibrium Price Range with Early Resolution Preferences

This subsection characterizes the equilibrium price range when every agent $i \in \mathcal{I}$ prefers early resolution of uncertainty, i.e., $\gamma\psi > 1$. To this end, we construct the extreme agents, MAX and MIN, whose KP preferences are represented by the utility functions $\hat{U}_{\max}$ and $\hat{U}_{\min}$ in (25), respectively:

$$\hat{U}_d(x) = W(x; \rho_d, \hat{\delta}_d, \hat{\phi}_d, \gamma, \psi), \quad d \in \{\min, \max\}.$$

Specifically, we show that the autarkic equilibrium prices of the MAX and MIN agents bound the equilibrium prices of the economy, provided that for each $d \in \{\min, \max\}$, the preference parameters $(\rho_d, \hat{\delta}_d, \hat{\phi}_d)$ are expressed as the following extreme values: for each j and k in $\mathcal{J}$,

$$\begin{aligned}
&\rho_{\max} = \max_{i\in\mathcal{I}} \rho_i, \ \rho_{\min} = \min_{i\in\mathcal{I}} \rho_i, \\
&\hat{\delta}_{\max,1} = \max_{i\in\mathcal{I}} \hat{\delta}_{i,1}, \ \hat{\delta}_{\min,1} = \min_{i\in\mathcal{I}} \hat{\delta}_{i,1}, \ \hat{\delta}_{\max,j} = \max_{i\in\mathcal{I}} \hat{\delta}_{i,j} \frac{\Phi^2_{\max,j}}{\Phi^2_{\min,j}}, \ \hat{\delta}_{\min,j} = \min_{i\in\mathcal{I}} \hat{\delta}_{i,j} \frac{\Phi^2_{\min,j}}{\Phi^2_{\max,j}}, \\
&\hat{\phi}_{\max,j} = \max_{i\in\mathcal{I}} \hat{\phi}_{i,j}, \ \ \hat{\phi}_{\min,j} = \min_{i\in\mathcal{I}} \hat{\phi}_{i,j}, \ \ \hat{\phi}_{\max,j,k} = \max_{i\in\mathcal{I}} \hat{\phi}_{i,j,k}, \ \ \hat{\phi}_{\min,j,k} = \min_{i\in\mathcal{I}} \hat{\phi}_{i,j,k},
\end{aligned} \tag{60}$$

where for each $d \in \{\min, \max\}$, $\Phi^2_{d,j}$ is defined as

$$\Phi^2_{d,j} = \left(\sum_{k=2}^{J} \hat{\phi}_{d,j,k} \left(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \right)^{1-\gamma} \right)^{\frac{\gamma}{\gamma-1}}.$$

Note that for each $d \in \{\max, \min\}$, the discount adjustment factor $\hat{\delta}_{d,j}$ is equal to $\hat{\delta}_{d,j}$ times an additional term introduced to account for the effect of the preference for early resolution on intertemporal choice. We recall that $\hat{\phi}_{i,2} = 1$ and $\hat{\phi}_{i,j,2} = 1$ for all $i \in \mathcal{I}$ and

$j \in \mathcal{J}$, which yield

$$\hat{\phi}_{\max,2} = \hat{\phi}_{\min,2} = 1 \quad \text{and} \quad \hat{\phi}_{\max,j,2} = \hat{\phi}_{\min,j,2} = 1 \quad \text{for all } \ j \in \mathcal{J}.$$

To reflect the combined effect of time discount and elasticity of intertemporal substitution on equilibrium prices, we define

$$\xi_{\psi,j} = \begin{cases} \dfrac{1-\rho_{\min}}{1-\rho_{\max}}, & \text{if } \ 0 < \psi < 1 \\ \dfrac{\rho_{\min}\hat{\delta}_{\min,j}}{\rho_{\max}\hat{\delta}_{\max,j}}, & \text{if } \ \psi > 1 \end{cases} \quad \text{for each } \ j \in \mathcal{J}, \tag{61}$$

$$\xi_{\psi} = \begin{cases} \max_{j\in\mathcal{J}} \xi_{\psi,j}, & \text{if } \ 0 < \psi < 1 \\ \min_{j\in\mathcal{J}} \xi_{\psi,j}, & \text{if } \ \psi > 1 \end{cases}. \tag{62}$$

For $d \in \{\min, \max\}$, let $p^{\min}$ and $p^{\max}$ be the autarkic equilibrium prices for the MIN and MAX agents, respectively, as formulated in Theorem 1. These prices determine the lower and upper bounds of the equilibrium price range.

Theorem 5. Let $(p, x) \in X^{\circ}_{K,1} \times X^{I}_{K}$ be an equilibrium of the economy. If $\gamma\psi \geq 1$ and $\gamma > 1$, then p lies in the price range defined as follows: for each $(j, k) \in \mathcal{J} \times \mathcal{J}$,

$$\frac{p^{\min}_{j,k}}{p^{\min}_{j,2}} = \hat{\phi}_{\min,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} \leq \frac{p_{j,k}}{p_{j,2}} \leq \hat{\phi}_{\max,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} = \frac{p^{\max}_{j,k}}{p^{\max}_{j,2}}, \tag{63}$$

$$\frac{p^{\max}_{j}}{p^{\max}_{j,2}} \leq \frac{p_j}{p_{j,2}} \leq \frac{p^{\min}_{j}}{p^{\min}_{j,2}}, \tag{64}$$

and

$$\xi_{\psi,j}^{\frac{2\psi(1-\gamma\psi)}{1-\psi}} \Big(\frac{\Psi^2_{\max,2}}{\Psi^2_{\min,2}}\Big)^{1-\gamma\psi} \frac{p^{\min}_{j}}{p^{\min}_{2}} \leq \frac{p_j}{p_2} \leq \xi_{\psi,j}^{-\frac{2\psi(1-\gamma\psi)}{1-\psi}} \Big(\frac{\Psi^2_{\min,2}}{\Psi^2_{\max,2}}\Big)^{1-\gamma\psi} \frac{p^{\max}_{j}}{p^{\max}_{2}}. \tag{65}$$

Moreover, each p_j satisfies

$$\Upsilon^{-\frac{1}{\gamma\psi}+1} p^{\min}_{j} \leq p_j \leq \Upsilon^{\frac{1}{\gamma\psi}-1} p^{\max}_{j}, \tag{66}$$

where

$$\Upsilon = \xi_{\psi}^{\frac{2\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{\Psi^2_{\max,2}}{\Psi^2_{\min,2}}\Big)^{2(1-\gamma\psi)} \Big(\frac{\Phi^1_{\max}}{\Phi^1_{\min}}\Big)^{-1}.$$

Here, for each $d \in \{\min, \max\}$, $\Psi^2_{d,2}$ and Φ^1_d are defined in (31) and (32), respectively.

PROOF : See Appendix A.

Two remarks are in order about Theorem 5. First, the equilibrium price range is robust to variations in individual endowments and preference parameters, provided that the ag-

gregate endowment and the extreme parameter values in (60) remain unchanged. This robustness is due to the fact that the preferences of the MIN and MAX agents depend solely on the aggregate endowment and these extreme values.

Second, the equilibrium price range for a heterogeneous economy can be viewed as a generalization of the single-agent representation for complete-market economies that satisfy Gorman aggregation conditions. If agents are identical in terms of probabilistic beliefs and time discounting, the MAX and MIN economies coincide and thus share the same autarkic equilibrium price. In that case, Theorem 5 ensures that the equilibrium price range collapses to this unique equilibrium price.

The size of the equilibrium price range is significantly affected by the degree of heterogeneity in probabilistic beliefs and time discount rates. It is further widened by the relative sizes of aggregate endowments across states of nature. For example, the first inequality in (60) shows that the interval length for the price ratio $p_{j,k}/p_{j,2}$ depends both on the divergence of the probabilistic beliefs of the MAX and MIN agents and on the relative sizes of the contingent aggregate endowments.

In the case of CRRA preferences ($\gamma\psi = 1$), the equilibrium price range in Theorem 5 has a simpler expression.

Corollary 6. If $\gamma\psi = 1$, the equilibrium price $p \in P$ in Theorem 5 is in the price range defined as follows: for all $(j,k) \in \mathcal{J} \times \mathcal{J}$,

$$\hat{\phi}_{\min,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} \leq \frac{p_{j,k}}{p_{j,2}} \leq \hat{\phi}_{\max,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} \tag{67a}$$

$$\Big(\frac{1-\rho_{\max}}{\rho_{\max}\hat{\delta}_{\max,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\gamma} \leq \frac{p_j}{p_{j,2}} \leq \Big(\frac{1-\rho_{\min}}{\rho_{\min}\hat{\delta}_{\min,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\gamma}, \tag{67b}$$

$$\hat{\phi}_{\min,j}\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma} \leq \frac{p_j}{p_2} \leq \hat{\phi}_{\max,j}\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma}. \tag{67c}$$

Moreover, each p_j satisfies

$$\rho_{\min}\hat{\delta}_{\min,1}\hat{\phi}_{\min,j}\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma} \leq p_j \leq \rho_{\max}\hat{\delta}_{\max,1}\hat{\phi}_{\max,j}\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma}. \tag{68}$$

For $\gamma\psi = 1$, the last result in (67) combined with Corollary 3 provides the bounds of the utility discount factors for the representative agent.

Corollary 7. Suppose that $\gamma\psi = 1$ holds. Then ϱ_A is bounded as follows:

$$\begin{aligned}
&\rho_{\min}\hat{\delta}_{\min,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\gamma}\Big(\sum_{j=2}^{J}\hat{\phi}_{\min,j}\Big) \le \varrho_{A,1} \le \rho_{\max}\hat{\delta}_{\max,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\gamma}\Big(\sum_{j=2}^{J}\hat{\phi}_{\max,j}\Big), \\
&\frac{\rho_{\min}\hat{\delta}_{\min,j}}{1-\rho_{\min}}\Big(\sum_{k=2}^{J}\hat{\phi}_{\min,j,k}\Big) \le \varrho_{A,j} \le \frac{\rho_{\max}\hat{\delta}_{\max,j}}{1-\rho_{\max}}\Big(\sum_{k=2}^{J}\hat{\phi}_{\max,j,k}\Big) \ \text{ for all } j \in \mathcal{J}.
\end{aligned} \tag{69}$$

If $\hat{\phi}_{\min} = \hat{\phi}_i$ for all $i \in \mathcal{I}$ (belief homogeneity), then it holds that

$$\begin{aligned}
&\rho_{\min}\min_{i\in\mathcal{I}}\delta_{i,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\gamma} \le \varrho_{A,1} \le \rho_{\max}\max_{i\in\mathcal{I}}\delta_{i,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\gamma}, \\
&\frac{\rho_{\min}}{1-\rho_{\min}}\min_{i\in\mathcal{I}}\delta_{i,j} \le \varrho_{A,j} \le \frac{\rho_{\max}}{1-\rho_{\max}}\max_{i\in\mathcal{I}}\delta_{i,j} \ \text{ for all } j \in \mathcal{J}.
\end{aligned} \tag{70}$$

PROOF : If $\gamma\psi = 1$, the second result in (38) gives $\bar{\phi}_j = (p_j/p_2)(\bar{e}_j/\bar{e}_2)^{\gamma}$ for each $j \in \mathcal{J}$. Using the inequality (67c), we obtain $\hat{\phi}_{\min,j} \le \bar{\phi}_j \le \hat{\phi}_{\max,j}$. On the other hand, by Corollary 3, $\varrho_{A,1}$ is given as

$$\varrho_{A,1} = p_2\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}}\Big(\sum_{j=2}^{J}\bar{\phi}_j\Big).$$

By applying (68) to p_2, we obtain

$$\rho_{\min}\hat{\delta}_{\min,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}}\Big(\sum_{j=2}^{J}\hat{\phi}_{\min,j}\Big) \le \varrho_{A,1} \le \rho_{\max}\hat{\delta}_{\max,1}\Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}}\Big(\sum_{j=2}^{J}\hat{\phi}_{\max,j}\Big).$$

Again by Corollary 3, $\varrho_{A,j}$ for each $j \in \mathcal{J}$ is given as

$$\varrho_{A,j} = \Big(\frac{p_j}{p_{j,2}}\Big)^{-1}\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\gamma}\sum_{k=2}^{J}\bar{\phi}_{j,k}. \tag{71}$$

By the inequality (67b), we obtain

$$\frac{\rho_{\min}}{1-\rho_{\min}} \le \Big(\frac{p_j}{p_{j,2}}\Big)^{-1}\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\gamma} \le \frac{\rho_{\max}}{1-\rho_{\max}}.$$

Combining the first result in (38) and the inequality (67a), we have

$$\hat{\phi}_{\min,j,k} \le \bar{\phi}_{j,k} \le \hat{\phi}_{\max,j,k}.$$

Applying these results to (71), we obtain the second result in (70).

Suppose that $\hat{\phi}_{\min} = \hat{\phi}_i$ for all $i \in \mathcal{I}$. Then by (5), for all $i \in \mathcal{I}$, we have

$$\hat{\delta}_{\min,1} = \min_{i\in\mathcal{I}} \delta_{i,1}\phi_{i,2} \text{ and } \hat{\phi}_{\min,j} = \hat{\phi}_{\max,j} = \hat{\phi}_{i,j} = \frac{\phi_{i,j}}{\phi_{i,2}}.$$

Since ϕ_i represents the probabilistic belief of agent i in the standard KP utility function, it holds that

$$1 = \sum_{j=2}^{J} \phi_{i,j} = \phi_{i,2} \sum_{j=2}^{J} \phi_{\min,j}.$$

This implies that $\phi_{i,2}$ is independent of i. It follows that

$$\hat{\delta}_{\min,1}\Big(\sum_{j=2}^{J} \hat{\phi}_{\min,j}\Big) = \min_{i\in\mathcal{I}} \delta_{i,1} \sum_{j=2}^{J} \phi_{i,j} = \min_{i\in\mathcal{I}} \delta_{i,1}.$$

Analogously, we can show that

$$\hat{\delta}_{\max,1}\Big(\sum_{j=2}^{J} \hat{\phi}_{\max,j}\Big) = \max_{i\in\mathcal{I}} \delta_{i,1},\ \hat{\delta}_{\min,j}\Big(\sum_{k=2}^{J} \hat{\phi}_{\min,j,k}\Big) = \min_{i\in\mathcal{I}} \delta_{i,j},\ \hat{\delta}_{\max,j}\Big(\sum_{k=2}^{J} \hat{\phi}_{\max,j,k}\Big) = \max_{i\in\mathcal{I}} \delta_{i,j}.$$

□

We recall that $\delta_{i,1}$ and $\delta_{i,j}$ denote the discount adjustment factors in agent i's standard KP utility function. Corollary 7 shows that, under homogeneous expectations, the social utility discount factors in the CRRA economy lie in the range determined by the aggregate endowment ratio $\bar{e}_2/\bar{e}_1$ and the extreme values of the individual time discount factors and discount adjustment factors.

6.2 Equilibrium Price Range with Late Resolution Preferences

This subsection establishes the equilibrium price range in the case that each agent $i \in \mathcal{I}$ prefers late resolution of uncertainty, i.e., $\gamma\psi < 1$. In this case, we show that the MAX and MIN agents whose autarkic equilibrium prices bound the equilibrium price range have the utility functions $\hat{U}_{\max}$ and $\hat{U}_{\min}$ in (25), respectively:

$$\hat{U}_d(x) = W(x; \rho_d, \hat{\delta}_d, \hat{\phi}_d, \gamma, \psi), \quad d \in \{\min, \max\},$$

where the preference parameters $(\rho_d, \hat{\delta}_d, \hat{\phi}_d)$ are defined as the following extreme values of preference parameters: for all j, k in $\mathcal{J}$,

$$\begin{aligned}
\rho_{\max} &= \max_{i\in\mathcal{I}} \rho_i, \quad \rho_{\min} = \min_{i\in\mathcal{I}} \rho_i, \\
\hat{\delta}_{\max,1} &= \max_{i\in\mathcal{I}} \hat{\delta}_{i,1}, \quad \hat{\delta}_{\min,1} = \min_{i\in\mathcal{I}} \hat{\delta}_{i,1}, \quad \hat{\delta}_{\max,j} = \max_{i\in\mathcal{I}} \hat{\delta}_{i,j}, \quad \hat{\delta}_{\min,j} = \min_{i\in\mathcal{I}} \hat{\delta}_{i,j}, \\
\hat{\phi}_{\max,j} &= \max_{i\in\mathcal{I}} \hat{\phi}_{i,j} \Big(\frac{\Psi^2_{\max,j}}{\Psi^2_{\min,j}}\Big)^{\gamma(1-\psi)}, \quad \hat{\phi}_{\min,j} = \min_{i\in\mathcal{I}} \hat{\phi}_{i,j} \Big(\frac{\Psi^2_{\min,j}}{\Psi^2_{\max,j}}\Big)^{\gamma(1-\psi)}, \\
\hat{\phi}_{\max,j,k} &= \max_{i\in\mathcal{I}} \hat{\phi}_{i,j,k}, \quad \hat{\phi}_{\min,j,k} = \min_{i\in\mathcal{I}} \hat{\phi}_{i,j,k}.
\end{aligned} \tag{72}$$

Here, $\Psi^2_{\min,j}$ and $\Psi^2_{\max,j}$ are defined in (31).

The parametric specification in (72) differs from the case in (60), where agents prefer early resolution of uncertainty. Unlike (60), the preference for late resolution is reflected in calibrating the representative agents' probabilistic beliefs rather than the discount adjustment factors. Specifically, for each $d \in \{\max, \min\}$ and $j \in \mathcal{J}$, the relative probabilistic beliefs $\hat{\phi}_{d,j}$'s are determined by the extreme values of $\hat{\phi}_{i,j}$'s augmented with additional terms arising from aggregating individual intertemporal marginal rates of substitution. When individual agents prefer late resolution, the time-discount adjustment factors for the MIN and MAX agents are simply set to the extreme values of individual agents' adjustment factors. This contrast with (60) where the adjustment factors include additional terms to account for the effect of early resolution on intertemporal choices.

For each $d \in \{\max, \min\}$, let p^d be the autarkic equilibrium price of the d-economy where the MAX and MIN agents have KP preferences specified in (72). The following theorem shows that $p^{\min}$ and $p^{\max}$ serve as the lower and upper bounds for the equilibrium price range when each individual agent prefers late resolution of uncertainty.

Theorem 6. Let $(p, x) \in X^\circ_{K,1} \times X^I_K$ be an equilibrium of the economy. If $0 < \gamma\psi < 1$ and $\gamma > 1$, then p lies in the price range defined as follows:

$$\frac{p^{\min}_{j,k}}{p^{\min}_{j,2}} = \hat{\phi}_{\min,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} \le \frac{p_{j,k}}{p_{j,2}} \le \hat{\phi}_{\max,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} = \frac{p^{\max}_{j,k}}{p^{\max}_{j,2}}, \tag{73}$$

$$\frac{p^{\max}_{j}}{p^{\max}_{j,2}} \le \frac{p_j}{p_{j,2}} \le \frac{p^{\min}_{j}}{p^{\min}_{j,2}}, \tag{74}$$

and

$$\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\gamma\psi} \frac{p^{\min}_j}{p^{\min}_2} \le \frac{p_j}{p_2} \le \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\gamma\psi} \frac{p^{\max}_j}{p^{\max}_2}. \tag{75}$$

Moreover, each p_j satisfies

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma(1-\psi)}{\gamma-1}} p_j^{\min} \leq p_j \leq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma(1-\psi)}{\gamma-1}} p_j^{\max}. \tag{76}$$

PROOF : See Appendix A.

The equilibrium price ranges in Theorems 5 and 6 are formulated differently, particularly with respect to the range for second-period equilibrium prices. This divergence arises because the preferences for early versus late resolution of uncertainty affect the intertemporal marginal rates of substitution (IMRS) of individual agents in different ways.

7. Numerical Examples

In this section, we present several numerical examples where all agents have either KP or CRRA preferences. These examples yield a counterintuitive result about the relationship between individual and collective discount rates: under heterogeneous expectations, the representative agent can exhibit a stronger present bias than any individual agent, regardless of whether the underlying preferences are KP or CRRA. Notably, this excessive present bias of the representative agent disappears when expectations are homogeneous. These examples illustrate how belief heterogeneity among agents can act as a fundamental driver that causes social utility discount factors to fall below those typically observed at the individual level.

7.1 A Two-Agent Economy with KP preferences

We consider a three-period, two-agent economy with $\gamma = 4$ and $\psi = 1/3$ where uncertainty consists of two states, i.e., $J = 3$. Since $\gamma\,\psi = 4/3 > 1$, each agent prefers early resolution of uncertainty. For $i = 1, 2$, agent i has the following standard-form preference characteristics:

$$\begin{aligned}
\text{agent 1: } & \rho_1 = \frac{99}{100}, \delta_1 = 1, \delta_{1,2} = \delta_{1,3} = 1/100, \\
& \phi_{1,2} = \frac{1}{3}, \phi_{1,3} = \frac{2}{3}, \phi_{1,j,2} = \frac{1}{3}, \phi_{1,j,3} = \frac{2}{3} \ \text{ for each } \ j = 2, 3, \\
\text{agent 2: } & \rho_2 = \frac{95}{100}, \delta_2 = 1, \delta_{2,2} = \delta_{2,3} = 5/100, \\
& \phi_{2,2} = \frac{2}{3}, \phi_{2,3} = \frac{1}{3}, \phi_{2,j,2} = \frac{2}{3}, \phi_{2,j,3} = \frac{1}{3} \ \text{ for each } \ j = 2, 3.
\end{aligned}$$

Agents 1 and 2 have state-independent time discount rates of $99/100$ and $95/100$, respectively. They have heterogeneous expectations: agent 1 believes that state 2 is more likely to occur than state 3, while agent 2 believes that state 3 is more likely to occur than state 2. Their standard KP preferences are represented as

$$\begin{aligned}
U_1(x) = \Bigg(\frac{1}{100}x_1^{-2} + \frac{99}{100}\Bigg(\frac{1}{3}\Bigg(\frac{1}{100}x_2^{-2} + \frac{99}{10000}\Big(\frac{1}{3}x_{2,2}^{-3} + \frac{2}{3}x_{2,3}^{-3}\Big)^{2/3}\Bigg)^{3/2} \\
+ \frac{2}{3}\Bigg(\frac{1}{100}x_3^{-2} + \frac{99}{10000}\Big(\frac{1}{3}x_{3,2}^{-3} + \frac{2}{3}x_{3,3}^{-3}\Big)^{2/3}\Bigg)^{3/2}\Bigg)^{2/3}\Bigg)^{-1/2}, \\
U_2(x) = \Bigg(\frac{1}{20}x_1^{-2} + \frac{19}{20}\Bigg(\frac{2}{3}\Bigg(\frac{1}{20}x_2^{-2} + \frac{19}{400}\Big(\frac{2}{3}x_{2,2}^{-3} + \frac{1}{3}x_{2,3}^{-3}\Big)^{2/3}\Bigg)^{3/2} \\
+ \frac{1}{3}\Bigg(\frac{1}{20}x_3^{-2} + \frac{19}{400}\Big(\frac{2}{3}x_{3,2}^{-3} + \frac{1}{3}x_{3,3}^{-3}\Big)^{2/3}\Bigg)^{3/2}\Bigg)^{2/3}\Bigg)^{-1/2}.
\end{aligned}$$

We apply the parameter transformation in (5) to convert these standard-form utility functions into $\hat{U}_1$ and $\hat{U}_2$:

$$\begin{aligned}
\text{agent 1: } \rho_1 = \frac{99}{100}, \hat{\delta}_1 = \Big(\frac{1}{3}\Big)^{2/3} = 0.4808, \hat{\delta}_{1,2} = \hat{\delta}_{1,3} = \frac{\hat{\delta}_1}{100} = 0.004808, \\
\hat{\phi}_{1,2} = 1, \hat{\phi}_{1,3} = 2, \hat{\phi}_{1,j,2} = 1, \hat{\phi}_{1,j,3} = 2 \text{ for each } j = 2, 3, \\
\text{agent 2: } \rho_2 = \frac{95}{100}, \hat{\delta}_2 = \Big(\frac{2}{3}\Big)^{2/3} = 0.7631, \hat{\delta}_{2,2} = \hat{\delta}_{2,3} = \frac{\hat{\delta}_2}{20} = 0.03816, \\
\hat{\phi}_{2,2} = 1, \hat{\phi}_{2,3} = \frac{1}{2}, \hat{\phi}_{2,j,2} = 1, \hat{\phi}_{2,j,3} = \frac{1}{2} \text{ for each } j = 2, 3.
\end{aligned}$$

We assume that the aggregate endowment in the first period is 10, i.e., $\bar{e}_1 = 10$ and then grows exponentially at 5% per period:

$$\begin{aligned}
\bar{e}_1 = 10, \bar{e}_2 = 10.5127, \bar{e}_{2,2} = 11.0517, \bar{e}_{2,3} = 11.0517, \\
\bar{e}_3 = 10.5127, \bar{e}_{3,2} = 11.0517, \bar{e}_{3,3} = 11.0517.
\end{aligned}$$

The aggregate endowment $\bar{e}$ is allocated to the two agents by a state-contingent rule: Agent 1 possesses 60% of $\bar{e}_1$, and in subsequent periods, its share increases to 70% in state 2 and remains at 60% in state 3, while agent 2 takes the rest of the aggregate endowment.

This allocation rule produces the following profile of initial endowments:

$$\begin{aligned}\text{agent 1: } e_1^1 = 6, & e_2^1 = 7.3589, e_{2,2}^1 = 7.7362, e_{2,3}^1 = 6.6310,\\ & e_3^1 = 6.3076, e_{3,2}^1 = 7.7362, e_{3,3}^1 = 6.6310,\\ \text{agent 2: } e_1^2 = 4, & e_2^2 = 3.1538, e_{2,2}^2 = 3.3155, e_{2,3}^2 = 4.4207,\\ & e_3^2 = 4.2051, e_{3,2}^2 = 3.3155, e_{3,3}^2 = 4.4207.\end{aligned}$$

This initial allocation indicates that agent 1 holds a larger portion of the aggregate endowment across all the periods and states.

The economy has an equilibrium $(p, (x^1, x^2))$ where

$$\begin{aligned}p &= (p_1, (p_j, (p_{j,k})_{k=2}^J)_{j=2}^J = (1, (0.3726, (0.1425, 0.1608)), (0.4438, (0.1607, 0.2028))),\\ x^1 &= (6.2296, (6.1443, (6.0238, 6.9499)), (6.9966, (6.9488, 7.7966))),\\ x^2 &= (3.7704, (4.3684, (5.0279, 4.1018)), (3.5161, (4.1029, 3.2552))).\end{aligned}$$

The price ratios $p_{2,3}/p_{2,2} = 1.1288$, $p_{3,3}/p_{3,2} = 1.2620$, $p_2/p_{2,2} = 2.6143$ and $p_3/p_{3,2} = 2.7619$ is contained in the equilibrium price range specified in Theorem 5:

$$\begin{aligned}&[p_{2,3}^{\min}/p_{2,2}^{\min}, p_{2,3}^{\max}/p_{2,2}^{\max}] = [0.5, 2],\ [p_{3,3}^{\min}/p_{3,2}^{\min}, p_{3,3}^{\max}/p_{3,2}^{\max}] = [0.5, 2],\\ &[p_2^{\min}/p_{2,2}^{\min}, p_2^{\max}/p_{2,2}^{\max}] = [0.1760, 36.6895],\ [p_3^{\min}/p_{3,2}^{\min}, p_3^{\max}/p_{3,2}^{\max}] = [0.1760, 36.6895].\end{aligned} \tag{77}$$

The equilibrium price range is determined by the autarkic equilibrium prices of the MAX and MIN agents computed in the following subsection. As remarked below, the significant size of the price range can be attributed to the substantial heterogeneity of probabilistic beliefs and time discounts.

Remark 1. The equilibrium outcomes in this example were computed using `FindRoot` in Mathematica 12 with the initial value set to the autarkic equilibrium of the MIN agent.[18] Over 20 additional attempts were also made with randomly generated initial values, but the algorithm failed to converge to a true equilibrium in all cases. Notably, the MAX agent's autarkic equilibrium price also failed to produce convergence, likely due to its significant distance from the economy's actual equilibrium. This example illustrates that well-chosen initial values play a key role in ensuring the convergence of computational algorithms to a true target value. □

[18] `FindRoot` is a Newton-Raphson-based numerical solver in Mathematica.

Remark 2. For comparison purpose, we compute equilibrium outcomes under homogeneous beliefs. Specifically, we consider an economy with the same primitives as before, except that agents 1 and 2 share common beliefs: in each period, they assign probability 1/3 to state 2 and 2/3 to state 3. In this homogeneous belief economy, the equilibrium price range becomes

$$[p_{2,3}^{\min}/p_{2,2}^{\min}, p_{2,3}^{\max}/p_{2,2}^{\max}] = [2,2],\ [p_{3,3}^{\min}/p_{3,2}^{\min}, p_{3,3}^{\max}/p_{3,2}^{\max}] = [2,2],$$
$$[p_{2}^{\min}/p_{2,2}^{\min}, p_{2}^{\max}/p_{2,2}^{\max}] = [0.7041, 18.3448],\ [p_{3}^{\min}/p_{3,2}^{\min}, p_{3}^{\max}/p_{3,2}^{\max}] = [0.7041, 18.3448].$$

This range is markedly narrower than that in (77). We will revisit this example later to illustrate how belief homogeneity affects agents' discounting behavior. □

7.2 Equilibrium-Representing Agent

We apply Theorem 2 to construct the representative agent that rationalizes the aggregate behavior of the economy at the equilibrium price

$$p = (1, (0.3726, (0.1425, 0.1608)), (0.4438, (0.1607, 0.2028))).$$

Substituting p and $\bar{e}$ into the closed-form formula in (38), we obtain the parameter values for the KP preferences of the representative agent:

$$\bar{\rho}\,\bar{\delta} = 0.5625,\ \frac{\bar{\rho}\bar{\delta}_2}{1-\bar{\rho}} = 0.5717,\ \frac{\bar{\rho}\bar{\delta}_3}{1-\bar{\rho}} = 0.5522,$$
$$\bar{\phi}_2 = 1,\ \bar{\phi}_{2,2} = 1,\ \bar{\phi}_{2,3} = 1.1288,\ \bar{\phi}_3 = 1.1897,\ \bar{\phi}_{3,2} = 1,\ \bar{\phi}_{3,3} = 1.262.$$

(Recall that $\bar{\phi}_{2,2}$ and $\bar{\phi}_{3,2}$ is pre-set at 1.)

By applying the belief normalization rule (41) and the result of Corollary 2, we obtain the standard KP utility function U_A of the representative agent:

$$\begin{aligned} &\varrho_{A,1} = 0.9485,\ \varrho_{A,2} = 0.9460,\ \varrho_{A,3} = 0.9515, \\ &\phi_{A,2} = 0.4567,\ \phi_{A,2,2} = 0.4698,\ \phi_{A,3,2} = 0.4421, \\ &\phi_{A,3} = 0.5433,\ \phi_{A,2,3} = 0.5302,\ \phi_{A,3,3} = 0.5579. \end{aligned} \tag{78}$$

The discount factors of this representative agent are state-dependent and uniformly lower than the state-independent discount factors of 0.99 and 0.95 held by agents 1 and 2, respectively. Notably, the representative agent is more present-biased than the two individual agents, discounting future utility more heavily than either individual. This outcome underscores the subtlety of aggregating time preferences: collective discounting behavior

can diverge markedly from the simple average of individual agents' preferences.

Furthermore, the discount factors of the representative agent exhibit non-monotonic temporal behavior: they decrease slightly in state 2 and increase slightly in state 3 during the second period. This observation contrasts with the findings of Weitzman (2001) and Gollier and Zeckhauser (2005), which report the strict decline of discount rates over time. This divergence may be due to the non-separability of KP preferences.[19]

Remark 3. If individual agents prefer late resolution of uncertainty, the representative agent can exhibit even greater present bias than in the early-resolution case. To illustrate this, consider an economy identical to the original example except that $\gamma = 3$ and $\psi = 1/4$. Since $\gamma\psi = 3/4 < 1$, both agents now prefer late resolution of uncertainty. In this new example, the representative agent has standard KP preferences with the following time discount factors:

$$\varrho_{A,1} = 0.9228,\ \varrho_{A,2} = 0.9186,\ \varrho_{A,3} = 0.9286.$$

These discount factors are uniformly lower than those in (78). This example shows that preferences for late resolution of uncertainty can make social discount factors more divergent from those of individual agents. □

Remark 4. In the homogeneous-belief economy described in Remark 2, the representative agent shares the same probabilistic beliefs as the two individual agents but differs in its discounting behavior for future utility:

$$\varrho_{A,1} = 0.9746,\ \varrho_{A,2} = 0.9747,\ \varrho_{A,3} = 0.9747.$$

These discount factors lie between those of agents 1 and 2, yet are uniformly higher than those in the heterogeneous belief economy given in (78). They increase slightly over time, consistent with the findings of Weitzman (2001) and Gollier and Zeckhauser (2005). Thus, the examples in this paper illustrate that the temporal behavior of the representative's discount factors is sensitive to variations in individual agents' probabilistic beliefs. □

[19]The current example focuses on a three-period CRRA economy, while Gollier and Zeckhauser (2005) consider an infinite horizon model with time-separable preferences. This temporal difference may lead to different conclusions about the behavior of the discount factor.

7.3 Autarkic Equilibrium of the MIN and MAX agents

The MIN agent's KP preferences are characterized by the following parameters:

$$\rho_{\min} = \frac{19}{20} = 0.95, \hat{\delta}_{\min,1} = \left(\frac{1}{3}\right)^{2/3} = 0.4808, \hat{\delta}_{\min,2} = 0.001908, \hat{\delta}_{\min,3} = 0.001908,$$
$$\hat{\phi}_{\min,2} = 1, \hat{\phi}_{\min,3} = \frac{1}{2}, \hat{\phi}_{\min,j,2} = 1, \hat{\phi}_{\min,j,3} = \frac{1}{2} \text{ for each } j = 2, 3.$$

This parameter profile implies that the MIN agent shares the same probabilistic beliefs as agent 2. The time discount adjustment factors in $\hat{\delta}^{\min}$ are state-dependent, in contrast to the constancy of individual adjustment factors.

The autarkic equilibrium price of the MIN representative agent is

$$p^{\min} = (p_1^{\min}, (p_j^{\min}, (p_{j,k}^{\min})_2^J)_{j=2}^J = (1, (0.3434, (0.00936, 0.00468)), (0.1717, (0.00468, 0.00234))).$$

The preference parameters of the MAX agent are

$$\rho_{\max} = \frac{99}{100} = 0.99, \hat{\delta}_{\max,1} = \left(\frac{2}{3}\right)^{2/3} = 0.7631, \hat{\delta}_{\max,2} = 0.096, \hat{\delta}_{\max,3} = 0.096,$$
$$\hat{\phi}_{\max,2} = 1, \hat{\phi}_{\max,3} = 2, \hat{\phi}_{\max,j,2} = 1, \hat{\phi}_{\max,j,3} = 2 \text{ for each } j = 2, 3.$$

The MAX agent retains the probabilistic beliefs of agent 1 and has state-dependent time discount adjustment factors that exceed those of agents 1 and 2.

The autarkic equilibrium price of the MAX representative agent is

$$p^{\max} = (p_1^{\max}, (p_j^{\max}, (p_{j,k}^{\max})_2^J)_{j=2}^J = (1, (0.4509, (2.5613, 5.1226)), (0.9018, (5.1226, 10.2451))).$$

The prices $p^{\min}$ and $p^{\max}$ delineate the equilibrium price range specified in (77).

7.4 A Special Case with Constant Relative Risk Aversion

We now illustrate the representative agent in an economy with constant relative risk aversion (CRRA) preferences. Consider the two-agent economy, which is the same as in subsection 7.1 except that $\gamma = \psi^{-1} = 3$.

This CRRA economy has an equilibrium $(p, (x^1, x^2))$ where

$$p = (p_1, (p_j, (p_{j,k})_2^J)_{j=2}^J = (1, (0.3696, (0.1414, 0.1564)), (0.4394, (0.1564, 0.2008))),$$
$$x^1 = (6.2218, (5.9913, (5.7031, 6.9473)), (7.1253, (6.9473, 8.0541))),$$
$$x^2 = (3.7783, (4.5214, (5.3486, 4.1044)), (3.3874, (4.1044, 2.9976))).$$

The price ratios $p_{2,3}/p_{2,2} = 1.1064$, $p_{3,3}/p_{3,2} = 1.2836$, $p_2/p_{2,2} = 2.6137$ and $p_3/p_{3,2} =$

2.8088 are in the following equilibrium price range:

$$[p_{2,3}^{\min}/p_{2,2}^{\min}, p_{2,3}^{\max}/p_{2,2}^{\max}] = [0.5, 2],\ [p_{3,3}^{\min}/p_{3,2}^{\min}, p_{3,3}^{\max}/p_{3,2}^{\max}] = [0.5, 2],$$
$$[p_{2}^{\min}/p_{2,2}^{\min}, p_{2}^{\max}/p_{2,2}^{\max}] = [0.1245, 51.8868],\ [p_{3}^{\min}/p_{3,2}^{\min}, p_{3}^{\max}/p_{3,2}^{\max}] = [0.1245, 51.8868].$$

Compared to the KP economy, the CRRA economy has the same range for the intra-period price ratios but a wider range for the inter-period price ratios than the KP price range in (77).

The economy is represented by the representative agent with the standard CRRA utility function U_A:

$$\varrho_{A,1} = 0.9399,\ \varrho_{A,2} = 0.9363,\ \varrho_{A,3} = 0.9446,$$
$$\phi_{A,2} = 0.4568,\ \phi_{A,2,2} = 0.4747,\ \phi_{A,3,2} = 0.4379,$$
$$\phi_{A,3} = 0.5432,\ \phi_{A,2,3} = 0.5253,\ \phi_{A,3,3} = 0.5621.$$

The representative agent's discount factors are state-dependent and strictly lower than those of either individual agent. They also fall slightly below the values in (78) obtained under KP preferences. Consequently, under CRRA preferences the representative agent exhibits a stronger present bias relative to both individuals than in the KP economy.

As in the case with KP preferences, the representative agent has non-monotonic discount factors over time: they decrease slightly in state 2 and increase slightly in state 3 in the second period. This observation contrasts with Gollier and Zeckhauser (2005) where the discount rates decrease over time in an economy without uncertainty when individual agents have time-separable preferences with decreasing absolute risk aversion (DARA). This divergence may be due to the presence of uncertainty in the current model. This example illustrates that in the economy with time-separable preferences, uncertainty may prevent the representative agent from revealing the monotonic discount factors reported in Gollier and Zeckhauser (2005).

Remark 5. In Gollier and Zeckhauser (2005), the social discount rate coincides with the individual discount rate when individuals have time-additive preferences with identical discount rates. This result may not hold under heterogeneous expectations. To illustrate this, we consider a CRRA economy that is the same as the previous CRRA economy except

that both agents have the same time preferences:

$$\text{agent 1: } \rho_1 = \frac{95}{100}, \delta_1 = 1, \delta_{1,2} = \delta_{1,3} = 5/100,$$
$$\text{agent 2: } \rho_2 = \frac{95}{100}, \delta_2 = 1, \delta_{2,2} = \delta_{2,3} = 5/100.$$

In this economy, the representative agent has the following characteristics with time-varying, state-contingent discount factors:

$$\varrho_{A,1} = 0.9160,\ \varrho_{A,2} = 0.9144,\ \varrho_{A,3} = 0.9183,$$
$$\phi_{A,2} = 0.4570,\ \phi_{A,2,2} = 0.4761,\ \phi_{A,3,2} = 0.4391,$$
$$\phi_{A,3} = 0.5430,\ \phi_{A,2,3} = 0.5239,\ \phi_{A,3,3} = 0.5609.$$

Moreover, the social discount factors are substantially lower than the individual discount factor 0.95 across all states and time periods. In this example, heterogeneous expectations make the economy as a whole more impatient than any individual. □

8. Conclusion

This paper develops a closed-form demand approach to equilibrium analysis in a dynamic stochastic economy with heterogeneous Kreps–Porteus (KP) preferences. Specifically, we represent aggregate equilibrium behavior and characterize the equilibrium price range when individuals hold diverse beliefs and time preferences. These results rest on a closed-form demand function for KP preferences. By inverting this demand function at autarkic equilibrium for appropriately defined representative agents, we construct two distinct classes of representative agents, each serving a specific analytical purpose.

First, we introduce the MIN and MAX representative agents whose autarkic equilibrium prices determine the lower and upper bounds of the equilibrium price range. The KP preferences of these agents are calibrated using the minimum and maximum values of the individual preference parameters, respectively. We then compute these price bounds by inverting the autarkic equilibrium conditions for each representative agent. This equilibrium price range is robust with respect to changes in the overall distribution of preference parameters, provided the extreme parameter values remain fixed.

Second, we construct a representative agent whose KP preferences are calibrated to rationalize the economy's (or any subgroup's) aggregate consumption at equilibrium. The representative agent's preference parameters, such as discount factors and beliefs, are explicitly formulated by inverting the autarkic equilibrium condition at the equilibrium price.

Notably, the representative agent reveals the stochastic term structure of social (or group-specific) utility discount factors, which quantify the trade-off between current and future consumption at the societal (or group) level. Under belief heterogeneity, the representative agent cannot be more patient than the most patient individual. This finding is consistent with the analysis of Gollier and Zeckhauser (2005), who examine how heterogeneous time preferences are aggregated into the social discount factors in a deterministic setting.

However, in contrast to Gollier and Zeckhauser (2005), heterogeneous beliefs can cause the representative agent to be more impatient than the least patient individual. As shown in Section 7, heterogeneous expectations alone can significantly raise social utility discount rates above individual rates in an economy with CRRA preferences. This finding conflicts with the existing literature that suggests lower social discount rates relative to individual rates based on intergenerational equity or the opportunity cost of capital. Thus, heterogeneous expectations can act as a critical driver of the wedge between individual and social discount rates in the cost-benefit analysis of long-term public projects.

Furthermore, even if individual agents with KP preferences are time-consistent, the economy can exhibit time-inconsistent (i.e., hyperbolic) discounting at the aggregate level. The time-inconsistent behavior of the representative agent contrasts sharply with the conventional macroeconomic and financial literature, which typically assumes time-consistent representative agents. This also has implications for the theory of hyperbolic social discounting, often used to evaluate public policy in economies populated by time-consistent individuals.

The closed-form demand approach produces the explicit preference structure of the representative agent, which can offer a new perspective for the representative-agent approach to asset pricing models with heterogeneous beliefs. In a complete-market setting, the representative agent's preferences are uniquely determined by the market price and the aggregate endowment, making its observable probabilistic belief a natural representative expectation operator. In incomplete markets, however, multiple representations emerge: each with distinct (and unobservable) probabilistic beliefs and discount factors. The multiplicity and unobservability of the incomplete-market representation pose a challenge to the representative-agent approach.

The closed-form framework also facilitates comparative-static analysis of redistributive endowment shocks (e.g., taxes, subsidies). In the post-shock equilibrium, the representative agent's KP preferences are recalibrated to reflect new discount rates and beliefs, so that shocks to individual endowments translate into preference shifts at the aggregate level. By tracking how the representative agent's parameters adjust, we may infer macroeconomic effects—such as changes in aggregate saving behavior, interest rates, and risk premia.

The MAX and MIN agents also encapsulate important macro fundamentals, so their autarkic equilibrium prices provide informed initial values for computing equilibrium outcomes via iterative solvers. As demonstrated in Section 7, this informed starting point can significantly improve the performance of computational algorithms.

Finally, while this paper assumes homogeneity in relative risk aversion and intertemporal substitution across agents, relaxing this assumption may require the use of multiple representative agents to characterize the equilibrium price range. For example, Won (2025) constructs four representative agents in a two-good economy to accommodate heterogeneity in both time discounting and elasticity of substitution.

Appendix A

This appendix contains the proofs of lemmas and theorems in the main text of the paper.

Proof of Lemma 1. Let A be an aggregator defined by

$$A(y_1, y_2) = \Big[(1-\rho)y_1^{1-\psi^{-1}} + \rho\,\delta y_2^{1-\psi^{-1}}\Big]^{\frac{1}{1-\psi^{-1}}},$$

where

$$y_1 = x_1 \quad \text{and} \quad y_2 = \Big(\sum_{j=2}^{J} \phi_j x_j^{1-\gamma}\Big)^{\frac{1}{1-\gamma}}.$$

The problem (15) is decomposed into two stages. In the first stage, agent i optimally divides income $\tilde{m}$ between the first-period consumption and the certainty equivalent of the second-period contingent consumption:

$$\max A(x_1, Y) \quad \text{s.t.} \quad q_1 x_1 + PY = \tilde{m}.$$

Let (x_1^i, Y^i) be a solution to the first-stage problem.

In the second stage, agent i optimally allocates the certainty-equivalent value $P\,Y^i$ determined in stage 1 across the second-period states:

$$\Big(\sum_{j=2}^{J} \phi_j x_j^{1-\gamma}\Big)^{\frac{1}{1-\gamma}} \quad \text{s.t.} \quad \sum_{j=2}^{J} q_j x_j = P\,Y^i.$$

STEP 1. The pair (x_1^i, Y^i) satisfies the first-order condition for the first-stage problem:

$$\frac{\delta\,\rho}{1-\rho}\Big(\frac{Y^i}{x_1^i}\Big)^{\frac{1}{\psi}} = \frac{P}{q_1}. \tag{A.1}$$

This condition is solved for Y^i/x_1^i:

$$\frac{Y^i}{x_1^i} = \Big(\frac{\rho\,\delta}{1-\rho}\Big)^{\psi}\Big(\frac{P}{q_1}\Big)^{-\psi}.$$

Substituting this into the budget constraint, we obtain

$$x_1^i = \frac{\tilde{m}(1-\rho)^{\psi}q_1^{-\psi}}{(1-\rho)^{\psi}q_1^{1-\psi} + (\rho\,\delta)^{\psi}P^{1-\psi}}, \quad Y^i = \frac{\tilde{m}(\rho\,\delta)^{\psi}P^{-\psi}}{(1-\rho)^{\psi}q_1^{1-\psi} + (\rho\,\delta)^{\psi}P^{1-\psi}}. \tag{A.2}$$

STEP 2. By inverting the first-order conditions for the second-stage problem, we obtain

$$\frac{x_j^i}{x_2^i} = \Big(\frac{\phi_j}{\phi_2}\Big)^{\frac{1}{\gamma}}\Big(\frac{q_j}{q_2}\Big)^{-\frac{1}{\gamma}} \quad \text{for each } \; j \geq 2. \tag{A.3}$$

Substituting this result into the budget constraint $\sum_{j=2}^{J} q_j x_j = PY^i$, we have the closed-form demand functions: for each $j = 2, \ldots, J$,

$$x_2^i = \Big(\frac{\phi_2}{q_2}\Big)^{\frac{1}{\gamma}}P^{\frac{1}{\gamma}}Y^i = \frac{\tilde{m}(\rho\,\delta)^{\psi}\Big(\frac{\phi_2}{q_2}\Big)^{\frac{1}{\gamma}}P^{\frac{1}{\gamma}-\psi}}{(1-\rho)^{\psi}q_1^{1-\psi} + (\rho\,\delta)^{\psi}P^{1-\psi}} \tag{A.4}$$

$$x_j^i = \Big(\frac{\phi_j}{\phi_2}\Big)^{\frac{1}{\gamma}}\Big(\frac{q_j}{q_2}\Big)^{-\frac{1}{\gamma}}x_2^i = \frac{\tilde{m}(\rho\,\delta)^{\psi}\Big(\frac{\phi_j}{q_j}\Big)^{\frac{1}{\gamma}}P^{\frac{1}{\gamma}-\psi}}{(1-\rho)^{\psi}q_1^{1-\psi} + (\rho\,\delta)^{\psi}P^{1-\psi}}. \tag{A.5}$$

STEP 3. Now we claim that x^i formulated in (A.2), (A.4) and (A.5) solves the full maximization problem (15). The proof is done if x^i satisfies the first-order conditions

$$\frac{\partial u(x)/\partial x_2}{\partial u(x)/\partial x_1} = \frac{q_2}{q_1} \quad \text{and} \quad \frac{\partial u(x)/\partial x_j}{\partial u(x)/\partial x_2} = \frac{q_j}{q_2}. \tag{A.6}$$

The second term (intra-group MRS condition) is exactly the first-order condition for the second-stage problem in (A.3).

To verify the first relation in (A.6), we apply the chain rule to obtain

$$\frac{\partial u(x)/\partial x_2}{\partial u(x)/\partial x_1} = \frac{\partial A(x_1, Y)/\partial Y}{\partial A(x_1, Y)/\partial x_1}\frac{\partial Y}{\partial x_2} = \phi_2\frac{P}{q_1}\Big(\frac{Y^i}{x_2^i}\Big)^{\gamma}, \tag{A.7}$$

where the last equality follows from the condition (A.1). Next, substituting (A.3) into the definition of Y^i yields

$$\frac{Y^i}{x_2^i} = \Big(\sum_{j=2}^{J}\phi_j\Big(\frac{x_j^i}{x_2^i}\Big)^{1-\gamma}\Big)^{\frac{1}{1-\gamma}} = \phi_2^{-\frac{1}{\gamma}}\Big(\frac{P}{q_2}\Big)^{-\frac{1}{\gamma}}.$$

Combining this with (A.7) delivers the first MRS condition in (A.6):

$$\frac{\partial u(x)/\partial x_2}{\partial u(x)/\partial x_1} = \phi_2 \frac{P}{q_1}\Big(\frac{Y^i}{x_2^i}\Big)^{\gamma} = \frac{q_2}{q_1}.$$

□

Proof of Lemma 2. We apply Lemma 1 to the first-stage problem (13): for each $j = 2, \ldots, J$,

$$x_j^i = \frac{m_j^i(p)(1-\rho_i)^{\psi} p_j^{-\psi}}{(1-\rho_i)^{\psi} p_j^{1-\psi} + (\rho_i \hat{\delta}_{i,j})^{\psi} (P_j^{i,2})^{1-\psi}}, \tag{A.8}$$

$$x_{j,k}^i = \frac{m_j^i(p)(\rho_i \hat{\delta}_{i,j})^{\psi} \left(\frac{\hat{\phi}_{i,j,k}}{p_{j,k}}\right)^{\frac{1}{\gamma}} (P_j^{i,2})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi} p_j^{1-\psi} + (\rho_i \hat{\delta}_{i,j})^{\psi} (P_j^{i,2})^{1-\psi}}, \tag{A.9}$$

where $P_j^{i,2}$ is the price index defined in (22).

Again by Lemma 1, the optimal choice $(x_1^i, (y_j)_{j=2}^J)$ in the second-stage problem (14) is given by

$$x_1^i = \frac{(p \cdot e^i)(1-\rho_i)^{\psi} p_1^{-\psi}}{(1-\rho_i)^{\psi} p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^{\psi} (P^{i,1})^{1-\psi}}, \tag{A.10}$$

$$y_j^i = \frac{(p \cdot e^i)(\rho_i \hat{\delta}_{i,1})^{\psi} \left(\frac{\hat{\phi}_{i,j}}{Q_j^{i,2}}\right)^{\frac{1}{\gamma}} (P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi} p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^{\psi} (P^{i,1})^{1-\psi}} \quad \text{for each } \; k = 2, \ldots J, \tag{A.11}$$

where $Q_j^{i,2}$ and $P_j^{i,1}$ are the price indices defined in (23) and (24).

Since $Q_j^{i,2}$ is the unit price of the certainty equivalent of $(x_j^i, (x_{j,k}^i)_{k=2}^J$, it holds that for each $j = 2, \ldots, J$,

$$m_j^i(p) = p_j x_j^i + \sum_{k=2}^{J} p_{j,k} x_{j,k}^i = Q_j^{i,2} y_j^i.$$

Substituting this result into (A.9), we obtain the demand functions in (19)–(21):

$$\begin{aligned}
x_1^i &= \frac{(p \cdot e^i)(1-\rho_i)^{\psi} p_1^{-\psi}}{(1-\rho_i)^{\psi} p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^{\psi} (P^{i,1})^{1-\psi}}, \\
x_j^i &= \frac{(1-\rho_i)^{\psi} p_j^{-\psi} Q_j^{i,2}}{(1-\rho_i)^{\psi} p_j^{1-\psi} + (\rho_i \hat{\delta}_{i,j})^{\psi} (P_j^{i,2})^{1-\psi}} \frac{(p \cdot e^i)(\rho_i \hat{\delta}_{i,1})^{\psi} \left(\frac{\hat{\phi}_{i,j}}{Q_j^{i,2}}\right)^{\frac{1}{\gamma}} (P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi} p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^{\psi} (P^{i,1})^{1-\psi}} \\
&= (1-\rho_i)^{\psi} (\rho_i \hat{\delta}_{i,1})^{\psi} \big(\hat{\phi}_{i,j}\big)^{\frac{1}{\gamma}} \frac{(p \cdot e^i) p_j^{-\psi} (Q_j^{i,2})^{\psi - \frac{1}{\gamma}} (P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi} p_1^{1-\psi} + (\rho_i \hat{\delta}_{i,1})^{\psi} (P^{i,1})^{1-\psi}},
\end{aligned}$$

$$x^i_{j,k} = \frac{(\rho_i\hat{\delta}_{i,j})^{\psi}\left(\frac{\hat{\phi}_{i,j,k}}{p_{j,k}}\right)^{\frac{1}{\gamma}} Q_j^{i,2}(P_j^{i,2})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi}p_j^{1-\psi} + (\rho_i\hat{\delta}_{i,j})^{\psi}(P_j^{i,2})^{1-\psi}} \frac{(p\cdot e^i)(\rho_i\hat{\delta}_{i,1})^{\psi}\left(\frac{\hat{\phi}_{i,j}}{Q_j^{i,2}}\right)^{\frac{1}{\gamma}}(P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi}p_1^{1-\psi} + (\rho_i\hat{\delta}_{i,1})^{\psi}(P^{i,1})^{1-\psi}}$$

$$= (\rho_i\hat{\delta}_{i,j})^{\psi}(\rho_i\hat{\delta}_{i,1})^{\psi}\big(\hat{\phi}_{i,j,k}\big)^{\frac{1}{\gamma}}\big(\hat{\phi}_{i,j}\big)^{\frac{1}{\gamma}} \frac{(p\cdot e^i)p_{j,k}^{-\frac{1}{\gamma}}(P_j^{i,2})^{\frac{1}{\gamma}-\psi}\big(Q_j^{i,2}\big)^{\psi-\frac{1}{\gamma}}(P^{i,1})^{\frac{1}{\gamma}-\psi}}{(1-\rho_i)^{\psi}p_1^{1-\psi} + (\rho_i\hat{\delta}_{i,1})^{\psi}(P^{i,1})^{1-\psi}}.$$

□

Proof of Theorem 1. STEP 1. For each $j \geq 3$ and $k \geq 3$, we find the intra-period price ratio $p^d_{j,k}/p^d_{j,2}$ in the third period. To this end, we apply the demand function in (21) to get the third-period demand ratio in the d-economy:

$$\frac{G^d_{j,k}(p^d)}{G^d_{j,2}(p^d)} = \frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} = \hat{\phi}^{\frac{1}{\gamma}}_{d,j,k}\Big(\frac{p^d_{j,k}}{p^d_{j,2}}\Big)^{-\frac{1}{\gamma}} \quad \text{for each } \ k = 3,\ldots,J.$$

This relation produces the price ratio

$$\frac{p^d_{j,k}}{p^d_{j,2}} = \hat{\phi}_{d,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma}. \tag{A.12}$$

STEP 2. For each $j \in \mathcal{J}$, the price index $P^{d,2}_j$ in the d-economy is given by

$$P^{d,2}_j = p^d_{j,2}\bigg(\sum_{k=2}^{J}\hat{\phi}^{\frac{1}{\gamma}}_{d,j,k}\Big(\frac{p^d_{j,k}}{p^d_{j,2}}\Big)^{1-\frac{1}{\gamma}}\bigg)^{\frac{\gamma}{\gamma-1}}$$

Substituting the price ratio in (A.12) into this relation, we obtain the relative price index to the price $p^d_{j,2}$.

$$\frac{P^{d,2}_j}{p^d_{j,2}} = \bigg(\sum_{k=2}^{J}\hat{\phi}_{d,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{1-\gamma}\bigg)^{\frac{\gamma}{\gamma-1}} = \Phi^2_{d,j}. \tag{A.13}$$

This price ratio consists of the extremes of the individual preference parameters and the aggregate endowment.

STEP 3. We show that for all $j = 3,\ldots,J$, the demand ratio $\bar{e}_j/\bar{e}_{j,2}$ at p^d is related to the inter-period price ratio $p^d_j/p^d_{j,2}$. By adapting the demand functions in (20) and (21) to

the d-economy, we see that for all $j = 3, \ldots, J$,

$$\begin{aligned}\frac{G_j^d(p^d)}{G_{j,2}^d(p^d)} = \frac{\bar{e}_j}{\bar{e}_{j,2}} &= \Big(\frac{1-\rho_d}{\rho_d\hat{\delta}_{d,j}}\Big)^{\psi}(p_j^d)^{-\psi}(p_{j,2}^d)^{\frac{1}{\gamma}}(P_j^{d,2})^{-\frac{1}{\gamma}+\psi} \\ &= \Big(\frac{1-\rho_d}{\rho_d\hat{\delta}_{d,j}}\Big)^{\psi}\Big(\frac{p_j^d}{p_{j,2}^d}\Big)^{-\psi}\Big(\frac{P_j^{d,2}}{p_{j,2}^d}\Big)^{-\frac{1}{\gamma}+\psi} \\ &= \Big(\frac{1-\rho_d}{\rho_d\hat{\delta}_{d,j}}\Big)^{\psi}\Big(\frac{p_j^d}{p_{j,2}^d}\Big)^{-\psi}\Big(\Phi_{d,j}^2\Big)^{-\frac{1}{\gamma}+\psi}.\end{aligned} \tag{A.14}$$

This relation gives

$$\frac{p_j^d}{p_{j,2}^d} = \Big(\frac{1-\rho_d}{\rho_d\hat{\delta}_{d,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi_{d,j}^2\Big)^{-\frac{1}{\gamma\psi}+1}. \tag{A.15}$$

STEP 4. For each $j \in \mathcal{J}$, the relative price index $Q_j^{d,2}/p_j^d$ in the d-economy is given by

$$\begin{aligned}\frac{Q_j^{d,2}}{p_j^d} &= \frac{1}{p_j^d}\Big((1-\rho_d)^{\psi}(p_j^d)^{1-\psi} + (\rho_d\hat{\delta}_{d,j})^{\psi}\big(P_j^{d,2}\big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &= \Big((1-\rho_d)^{\psi} + (\rho_d\hat{\delta}_{d,j})^{\psi}\big((P_j^{d,2}/p_{j,2}^d)(p_{j,2}^d/p_j^d)\big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &= \Big((1-\rho_d)^{\psi} + (\rho_d\hat{\delta}_{d,j})^{\psi}\big(\Phi_{d,j}^2\big)^{1-\psi}\big(p_{j,2}^d/p_j^d\big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}\end{aligned} \tag{A.16}$$

Substituting (A.15) into this, we obtain

$$\frac{Q_j^{d,2}}{p_j^d} = \Big((1-\rho_d)^{\psi} + (\rho_d\hat{\delta}_{d,j})^{\psi}\Big(\frac{1-\rho_d}{\rho_d\hat{\delta}_{d,j}}\Big)^{\psi-1}\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{\psi-1}{\psi}}\big(\Phi_{d,j}^2\big)^{\frac{1-\psi}{\gamma\psi}}\Big)^{\frac{1}{1-\psi}} = \Psi_{d,j}^2. \tag{A.17}$$

STEP 5. For each $j = 2, \ldots, J$, we can find the price ratio p_j^d/p_2^d from the demand ratio in the d-economy:

$$\begin{aligned}\frac{G_j^d(p^d)}{G_2^d(p^d)} = \frac{\bar{e}_j}{\bar{e}_2} &= \Big(\frac{\hat{\phi}_{d,j}}{\hat{\phi}_{d,2}}\Big)^{\frac{1}{\gamma}}\Big(\frac{p_j^d}{p_2^d}\Big)^{-\psi}(Q_2^{d,2})^{\frac{1}{\gamma}-\psi}(Q_j^{d,2})^{\psi-\frac{1}{\gamma}} \\ &= \Big(\frac{\hat{\phi}_{d,j}}{\hat{\phi}_{d,2}}\Big)^{\frac{1}{\gamma}}\Big(\frac{p_j^d}{p_2^d}\Big)^{-\frac{1}{\gamma}}\Big(\frac{Q_2^{d,2}}{p_2^d}\Big)^{\frac{1}{\gamma}-\psi}\Big(\frac{Q_j^{d,2}}{p_j^d}\Big)^{\psi-\frac{1}{\gamma}} \\ &= \Big(\frac{\hat{\phi}_{d,j}}{\hat{\phi}_{d,2}}\Big)^{\frac{1}{\gamma}}\Big(\frac{p_j^d}{p_2^d}\Big)^{-\frac{1}{\gamma}}\Big(\frac{\Psi_{d,j}^2}{\Psi_{d,2}^2}\Big)^{\psi-\frac{1}{\gamma}}.\end{aligned} \tag{A.18}$$

Since $\Psi_{d,2}^2 = 1$, this equation is solved for p_j^d/p_2^d.

$$\frac{p_j^d}{p_2^d} = \hat{\phi}_{d,j}\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma}\Big(\frac{\Psi_{d,j}^2}{\Psi_{d,2}^2}\Big)^{\gamma\psi-1} \tag{A.19}$$

STEP 6. Let $P^{d,1}$ denote the price index in (24) adapted to the d economy. The relative price index $P^{d,1}/p_2^d$ is given by

$$\frac{P^{d,1}}{p_2^d} = \left(\sum_{j=2}^{J} \hat{\phi}_{d,j}^{\frac{1}{\gamma}} \left(\frac{Q_j^{d,2}}{p_j^d} \frac{p_j^d}{p_2^d} \right)^{1-\frac{1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}} = \left(\sum_{j=2}^{J} \hat{\phi}_{d,j}^{\frac{1}{\gamma}} \left(\Psi_{d,j}^2 \frac{p_j^d}{p_2^d} \right)^{1-\frac{1}{\gamma}} \right)^{\frac{\gamma}{\gamma-1}}$$

Substituting (A.19) into this, we obtain

$$\frac{P^{d,1}}{p_2^d} = \left(\sum_{j=2}^{J} \hat{\phi}_{d,j} \left(\frac{\bar{e}_j}{\bar{e}_2} \right)^{1-\gamma} \left(\Psi_{d,j}^2 \right)^{1-\frac{1}{\gamma}} \left(\frac{\Psi_{d,j}^2}{\Psi_{d,2}^2} \right)^{(\gamma\psi-1)(1-\frac{1}{\gamma})} \right)^{\frac{\gamma}{\gamma-1}} = \Phi_d^1. \tag{A.20}$$

STEP 7. Now we attempt to find p_j^d for all $j = 2, \ldots, J$. Since $p_1^d = 1$ (due to price normalization), it follows that $p_j^d/p_1^d = p_j^d$. We show that this price ratio is determined through the relationship that the demand ratio $\bar{e}_j/\bar{e}_1$ meets at p^d. We apply (19) and (20) to the d-economy:

$$\begin{aligned}
\frac{\bar{e}_j}{\bar{e}_1} &= (\rho_d \hat{\delta}_{d,1})^{\psi} (\hat{\phi}_{d,j})^{\frac{1}{\gamma}} (p_j^d)^{-\psi} (Q_j^{d,2})^{\psi-\frac{1}{\gamma}} (P^{d,1})^{\frac{1}{\gamma}-\psi} \\
&= (\rho_d \hat{\delta}_{d,1})^{\psi} (\hat{\phi}_{d,j})^{\frac{1}{\gamma}} (p_j^d)^{-\psi} \left(\frac{p_j^d}{p_2^d} \right)^{\psi-\frac{1}{\gamma}} \left(\frac{Q_j^{d,2}}{p_j^d} \right)^{\psi-\frac{1}{\gamma}} \left(\frac{P^{d,1}}{p_2^d} \right)^{\frac{1}{\gamma}-\psi} \\
&= (\rho_d \hat{\delta}_{d,1})^{\psi} (\hat{\phi}_{d,j})^{\frac{1}{\gamma}} (p_j^d)^{-\psi} \left(\frac{p_j^d}{p_2^d} \right)^{\psi-\frac{1}{\gamma}} \left(\Psi_{d,j}^2 \right)^{\psi-\frac{1}{\gamma}} \left(\Phi_d^1 \right)^{\frac{1}{\gamma}-\psi}
\end{aligned}$$

This relation is solved for p_j^d as

$$p_j^d = \rho_d \hat{\delta}_{d,1} (\hat{\phi}_{d,j})^{\frac{1}{\gamma\psi}} \left(\frac{\bar{e}_j}{\bar{e}_1} \right)^{-\frac{1}{\psi}} \left(\frac{p_j^d}{p_2^d} \right)^{1-\frac{1}{\gamma\psi}} \left(\Psi_{d,j}^2 \right)^{1-\frac{1}{\gamma\psi}} \left(\Phi_d^1 \right)^{\frac{1}{\gamma\psi}-1}. \tag{A.21}$$

Substituting p_j^d/p_2^d in (A.19) into this relation, we obtain p_j^d as a function of the parameters of the d economy:

$$p_j^d = \rho_d \hat{\delta}_{d,1} \hat{\phi}_{d,j} \left(\frac{\bar{e}_j}{\bar{e}_2} \right)^{\frac{1}{\psi}-\gamma} \left(\frac{\bar{e}_j}{\bar{e}_1} \right)^{-\frac{1}{\psi}} \left(\frac{\Psi_{d,j}^2}{\Psi_{d,2}^2} \right)^{\frac{(\gamma\psi-1)^2}{\gamma\psi}} \left(\Psi_{d,j}^2 \right)^{1-\frac{1}{\gamma\psi}} \left(\Phi_d^1 \right)^{\frac{1}{\gamma\psi}-1}. \tag{A.22}$$

In particular, p_2^d is expressed as

$$p_2^d = \rho_d \hat{\delta}_{d,1} \left(\frac{\bar{e}_2}{\bar{e}_1} \right)^{-\frac{1}{\psi}} \left(\Psi_{d,2}^2 \right)^{1-\frac{1}{\gamma\psi}} \left(\Phi_d^1 \right)^{\frac{1}{\gamma\psi}-1}. \tag{A.23}$$

STEP 8. Substituting p_2^d from (A.23) into (A.19) yields the expression for p_j^d in (27). Combining this result with equation (A.15) leads to the formula for $p_{j,2}^d$ in (28). Substituting this result into (A.12) then determines the value of $p_{j,k}^d$ as stated in equation

(29). Consequently, these recursive substitutions produce the equilibrium price p^d in (26)–(29). □

Proof of Theorem 2. We constructively build the utility function $\bar{U}_A$ of the representative agent, which shares the same CRRA and EIS coefficients with the individual agents. In particular, we determine the preference parameters $\bar{\rho}$, $\bar{\delta} = (\bar{\delta}_1, (\bar{\delta}_j)_{j=2}^J)$ and $\bar{\phi} = (\bar{\phi}_j, (\bar{\phi}_{j,k})_{k=2}^J)_{j=2}^J$ such that they rationalize the aggregate endowment $\bar{e}$ at the given price p. Consequently, these parameters are expressed as explicit functions of p and $\bar{e}$.

Since Theorem 2 addresses the inverse of the problem explored in Theorem 1, the following proof will frequently reference relationships derived in the proof of Theorem 1. When making these references, we will treat the price p as if it were the equilibrium price p^d from Theorem 1.

STEP 1. Using (A.12), we compute

$$\bar{\phi}_{j,k} = \frac{p_{j,k}}{p_{j,2}} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{\gamma} \text{ for each } (j,k) \in \mathcal{J} \times \mathcal{J}. \tag{A.24}$$

STEP 2. Substituting (A.24) into (A.13), we compute the relative price index.

$$\frac{P_j^2}{p_{j,2}} \equiv \Bigg(\sum_{k=2}^{J} \bar{\phi}_{j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{1-\gamma} \Bigg)^{\frac{\gamma}{\gamma-1}} = \Bigg(\sum_{k=2}^{J} \frac{p_{j,k}}{p_{j,2}} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big) \Bigg)^{\frac{\gamma}{\gamma-1}} = C_{1,j}. \tag{A.25}$$

This price ratio is determined by the market price p and the aggregate endowment $\bar{e}$.

STEP 3. Substituting (A.25) into (A.15), we obtain

$$\frac{\bar{\rho}\,\bar{\delta}_j}{1-\bar{\rho}} = \Big(\frac{p_j}{p_{j,2}} \Big)^{-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big)^{\frac{1}{\psi}} C_{1,j}^{-\frac{1}{\gamma\psi}+1}.$$

For each $j \in \mathcal{J}$, this gives

$$\bar{\rho}\,\bar{\delta}_j = (1-\bar{\rho}) \Big(\frac{p_j}{p_{j,2}} \Big)^{-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big)^{\frac{1}{\psi}} C_{1,j}^{-\frac{1}{\gamma\psi}+1}. \tag{A.26}$$

STEP 4. Substituting (A.25) and (A.26) into (A.17), we compute the relative price index

$$\begin{aligned} \frac{Q_j^2}{p_j} &\equiv \Bigg((1-\bar{\rho})^{\psi} + (\bar{\rho}\delta_j)^{\psi} \Big(\frac{\bar{\rho}\,\bar{\delta}_j}{1-\bar{\rho}} \Big)^{1-\psi} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big)^{\frac{\psi-1}{\psi}} C_{1,j}^{\frac{1-\psi}{\gamma\psi}} \Bigg)^{\frac{1}{1-\psi}} \\ &= (1-\bar{\rho})^{\frac{\psi}{1-\psi}} \Bigg(1 + \frac{\bar{\rho}\,\bar{\delta}_j}{1-\bar{\rho}} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big)^{\frac{\psi-1}{\psi}} C_{1,j}^{\frac{1-\psi}{\gamma\psi}} \Bigg)^{\frac{1}{1-\psi}} \\ &= (1-\bar{\rho})^{\frac{\psi}{1-\psi}} \Bigg(1 + \Big(\frac{p_j}{p_{j,2}} \Big)^{-1} \Big(\frac{\bar{e}_{j,2}}{\bar{e}_j} \Big) C_{1,j}^{1-\frac{1}{\gamma}} \Bigg)^{\frac{1}{1-\psi}} \\ &= C_{2,j} (1-\bar{\rho})^{\frac{\psi}{1-\psi}} \end{aligned} \tag{A.27}$$

STEP 5. Substituting (A.27) into (A.19), we obtain

$$\frac{p_j}{p_2} = \bar{\phi}_j \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\gamma\psi-1}.$$

For each $j \in \mathcal{J}$, this yields

$$\bar{\phi}_j = \frac{p_j}{p_2} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{\gamma} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{1-\gamma\psi}. \tag{A.28}$$

STEP 6. Substituting (A.27) and (A.28)into (A.20), we compute the relative price index

$$\begin{aligned}
\frac{P^1}{p_2} &\equiv \Bigg(\sum_{j=2}^{J} \bar{\phi}_j \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{1-\gamma} \big(C_{2,j}(1-\bar{\rho})^{\frac{\psi}{1-\psi}}\big)^{1-\frac{1}{\gamma}} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{(\gamma\psi-1)(1-\frac{1}{\gamma})} \Bigg)^{\frac{\gamma}{\gamma-1}} \\
&= (1-\bar{\rho})^{\frac{\psi}{1-\psi}} \Bigg(\sum_{j=2}^{J} \bar{\phi}_j C_{2,j}^{1-\frac{1}{\gamma}} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{1-\gamma} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{(\gamma\psi-1)(1-\frac{1}{\gamma})} \Bigg)^{\frac{\gamma}{\gamma-1}} \\
&= (1-\bar{\rho})^{\frac{\psi}{1-\psi}} \Bigg(\sum_{j=2}^{J} \frac{p_j}{p_2} C_{2,j}^{1-\frac{1}{\gamma}} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big) \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\frac{1}{\gamma}-\psi} \Bigg)^{\frac{\gamma}{\gamma-1}} \\
&= C_3 (1-\bar{\rho})^{\frac{\psi}{1-\psi}}.
\end{aligned} \tag{A.29}$$

STEP 7. Substituting (A.27), (A.28) and (A.29) into (A.22), we have

$$\begin{aligned}
p_j &= \bar{\rho}\,\bar{\delta}_1 \bar{\phi}_j \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{\frac{1}{\psi}-\gamma} \Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{-\frac{1}{\psi}} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\frac{(\gamma\psi-1)^2}{\gamma\psi}} \big(C_{2,j}(1-\bar{\rho})^{\frac{\psi}{1-\psi}}\big)^{1-\frac{1}{\gamma\psi}} \Big(\frac{P^1}{p^2}\Big)^{\frac{1}{\gamma\psi}-1} \\
&= \bar{\rho}\,\bar{\delta}_1 \bar{\phi}_j \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{\frac{1}{\psi}-\gamma} \Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{-\frac{1}{\psi}} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\frac{(\gamma\psi-1)^2}{\gamma\psi}} \Big(\frac{C_{2,j}}{C_3}\Big)^{1-\frac{1}{\gamma\psi}} \\
&= \bar{\rho}\,\bar{\delta}_1 \frac{p_j}{p_2} \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{\frac{1}{\psi}} \Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{-\frac{1}{\psi}} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}} \Big(\frac{C_{2,j}}{C_3}\Big)^{1-\frac{1}{\gamma\psi}} \\
&= \bar{\rho}\,\bar{\delta}_1 \frac{p_j}{p_2} \Big(\frac{\bar{e}_1}{\bar{e}_2}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,j}}{C_{2,2}}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}} \Big(\frac{C_{2,j}}{C_3}\Big)^{1-\frac{1}{\gamma\psi}}.
\end{aligned} \tag{A.30}$$

For $j = 2$, this gives

$$\bar{\rho}\,\bar{\delta}_1 = p_2 \Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,2}}{C_3}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}}. \tag{A.31}$$

This single equation contains two unknowns, $\bar{\rho}$ and $\bar{\delta}_1$, implying one of them is a free

variable. By setting $\bar{\delta}_1$ to any real number in $(0,\,1)$,[20] we obtain

$$\bar{\rho} = p_2 \bar{\delta}_1^{-1} \Big(\frac{\bar{e}_2}{\bar{e}_1}\Big)^{\frac{1}{\psi}} \Big(\frac{C_{2,2}}{C_3}\Big)^{\frac{1-\gamma\psi}{\gamma\psi}}. \tag{A.32}$$

Substituting this into (A.26), we have the value of each $\bar{\delta}_j$ for each $j \in \mathcal{J}$. □

Proof of Theorem 3. For each $i \in \mathcal{I}$, let λ^i be the Lagrangian multiplier associated with the budget constraint in (44). Then λ^i satisfies the first-order conditions for utility maximization can be expressed in terms of the IMRSs in (11) and (12): for all $j \in \mathcal{J}$ and $(j,k) \in \mathcal{J} \times \mathcal{J}$,

$$\frac{\partial \hat{U}_i/\partial x_j}{\partial \hat{U}_i/\partial x_1} = \rho_i \hat{\delta}_{i,1}^i \hat{\phi}_{i,j} x_1^{\frac{1}{\psi}} x_j^{-\frac{1}{\psi}} \big(Z_{i,j}(x)\big)^{\frac{1}{\psi}-\gamma} \big(Y_{i,1}(x)\big)^{\gamma-\frac{1}{\psi}} = \frac{\lambda_j^i}{\lambda_1^i}, \tag{A.33a}$$

$$\frac{\partial \hat{U}_i(x)/\partial x_{j,k}}{\partial \hat{U}_i(x)/\partial x_j} = \frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \hat{\phi}_{i,j,k} x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \big(Y_{i,j}(x)\big)^{\frac{\gamma\psi-1}{\psi}} = \frac{\lambda_{j,k}^i}{\lambda_j^i}, \tag{A.33b}$$

$$q_j = \sum_{k\in\mathcal{J}} \frac{\lambda_{j,k}^i}{\lambda_j^i} r_j, \quad \text{and} \quad q_1 = \sum_{j\in\mathcal{J}} \frac{\lambda_j^i}{\lambda_1^i}(r_j + q_j). \tag{A.33c}$$

Here λ_j^i/λ_1^i and $\lambda_{j,k}^i/\lambda_j^i$ indicate the state price of agent i that represents the present value of one dollar in states j and (j,k), respectively. The new preference parameter in (46) allows us to transform the conditions in (A.33) into the equilibrium conditions for complete markets. To see this, we rearrange the first-order conditions in (A.33) in terms of the state

[20] Taking into account the parameter transformation in (5), we can set $\bar{\delta}_1 = \phi_2^{(1-\psi^{-1})/(1-\gamma)}$, where ϕ_2 satisfies

$$\phi_2^{-1} = 1 + \sum_{j=3}^{J} \bar{\phi}_j.$$

By setting $\phi_j = \bar{\phi}_j \phi_2$, we observe that $(\phi_2, \ldots, \phi_J)$ forms a probability measure. However, $\bar{\delta}_1$ can be set to any positive value, as it is possible to transform these parameters into a valid probability measure according to the rule in (**??**).

prices for agent 1:

$$\frac{\lambda_j^1}{\lambda_1^1} = \rho_i \hat{\delta}_{i,1} \Big(\hat{\phi}_{i,j} \frac{\lambda_1^i}{\lambda_j^i} \frac{\lambda_j^1}{\lambda_1^1} \Big) x_1^{\frac{1}{\psi}} x_j^{-\frac{1}{\psi}} \big(Z_{i,j}(x) \big)^{\frac{1}{\psi}-\gamma} \big(Y_{i,1}(x) \big)^{\gamma-\frac{1}{\psi}}$$

$$= \rho_i \breve{\delta}_{i,1} \breve{\phi}_{i,j} x_1^{\frac{1}{\psi}} x_j^{-\frac{1}{\psi}} \big(Z_{i,j}(x) \big)^{\frac{1}{\psi}-\gamma} \big(Y_{i,1}(x) \big)^{\gamma-\frac{1}{\psi}}, \tag{A.34a}$$

$$\frac{\lambda_{j,k}^1}{\lambda_j^1} = \frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \Big(\hat{\phi}_{i,j,k} \frac{\lambda_j^i}{\lambda_{j,k}^i} \frac{\lambda_{j,k}^1}{\lambda_j^1} \Big) x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \big(Y_{i,j}(x) \big)^{\frac{\gamma\psi-1}{\psi}}$$

$$= \frac{\rho_i \breve{\delta}_{i,j}}{1-\rho_i} \breve{\phi}_{i,j,k} x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \big(Y_{i,j}(x) \big)^{\frac{\gamma\psi-1}{\psi}}, \tag{A.34b}$$

$$q_j = \sum_{k \in \mathcal{J}} \frac{\lambda_{j,k}^1}{\lambda_j^1} r_j, \quad \text{and} \quad q_1 = \sum_{j \in \mathcal{J}} \frac{\lambda_j^1}{\lambda_1^1} (r_j + q_j). \tag{A.34c}$$

Let p be a price in $X_{K,1}^{\circ}$ defined by (45). Then by the budget constraint in (44) and (A.34c), the optimal choice x^i satisfies the complete-market budget constraint $p \cdot x^i = p \cdot e^i$. Moreover, the relations in (A.34a) and (A.34b) yields the same first-order condition for the complete-market economy as in (11) and (12), where agent i maximizes the utility function $W(x; \rho_i, \breve{\delta}_i, \breve{\phi}_i, \gamma, \psi)$:

$$p_j = \rho_i \breve{\delta}_{i,1} \breve{\phi}_{i,j} x_1^{\frac{1}{\psi}} x_j^{-\frac{1}{\psi}} \big(Z_{i,j}(x) \big)^{\frac{1}{\psi}-\gamma} \big(Y_{i,1}(x) \big)^{\gamma-\frac{1}{\psi}}, \tag{A.35a}$$

$$\frac{p_{j,k}}{p_j} = \frac{\rho_i \breve{\delta}_{i,j}}{1-\rho_i} \breve{\phi}_{i,j,k} x_j^{\frac{1}{\psi}} x_{j,k}^{-\gamma} \big(Y_{i,j}(x) \big)^{\frac{\gamma\psi-1}{\psi}}. \tag{A.35b}$$

Obviously, x clears the goods markets, since it is already an equilibrium allocation in the incomplete-market economy. Therefore, we conclude that (p, x) is an equilibrium in the complete-market economy. □

Proof of Lemma 3. This result follows immediately from the uniqueness of the aggregate preference representation, as noted after Theorem 2. However, it does not reveal how the closed-form demand approach and the welfare-based approach are interconnected. To clarify this connection, we derive Corollary 3 from the parametric specifications (56) and (57), which appear in Example 2.3 of Jouini and Napp (2006)).

By construction, $(p, \bar{e})$ is an autarkic equilibrium in the representative-agent economy. That is, $\bar{e}$ solves the following utility maximization problem:

$$\bar{e} = \arg\max_z V_\lambda(z) \quad \text{subject to} \quad z_1 + \sum_{j=2}^{J} p_j (z_j - \bar{e}_j) + \sum_{j=2}^{J} \sum_{k=2}^{J} p_{j,k} (z_{j,k} - \bar{e}_{j,k}) = 0. \tag{A.36}$$

The first-order conditions for this problem is given as follows: for each j, k in $\mathcal{J}$,

$$\mu = z_1^{-\gamma},\ \mu p_j = \rho_{\lambda,1}\phi_{\lambda,j} z_j^{-\gamma},\ \text{ and }\ \mu p_{j,k} = \rho_{\lambda,1}\rho_{\lambda,j}\phi_{\lambda,j}\phi_{\lambda,j,k} z_{j,k}^{-\gamma}.$$

Since $\bar{e}$ is the optimal choice of the representative agent at the price p, it satisfies

$$\mu = \bar{e}_1^{-\gamma},\ \mu p_j = \rho_{\lambda,1}\phi_{\lambda,j}\bar{e}_j^{-\gamma},\ \text{ and }\ \mu p_{j,k} = \rho_{\lambda,1}\rho_{\lambda,j}\phi_{\lambda,j,k}\bar{e}_{j,k}^{-\gamma}. \tag{A.37}$$

On the other hand, the first order conditions for agent i's utility maximization problem yield

$$\Big(\frac{1}{\lambda_i}\Big)^{\frac{1}{\gamma}} = x_1^i,\ \Big(\frac{\rho_i\phi_{i,j}}{\lambda_i}\Big)^{\frac{1}{\gamma}} = p_j^{\frac{1}{\gamma}} x_j^i,\ \text{ and }\ \Big(\frac{\rho_i^2\phi_{i,j}\phi_{i,j,k}}{\lambda_i}\Big)^{\frac{1}{\gamma}} = p_{j,k}^{\frac{1}{\gamma}} x_{j,k}^i.$$

Aggregating these equations over $\mathcal{I}$ and using the market clearing condition, we obtain

$$\sum_{i\in\mathcal{I}}\Big(\frac{1}{\lambda_i}\Big)^{\frac{1}{\gamma}} = \bar{e}_1,\ \sum_{i\in\mathcal{I}}\Big(\frac{\rho_i\phi_{i,j}}{\lambda_i}\Big)^{\frac{1}{\gamma}} = p_j^{\frac{1}{\gamma}}\bar{e}_j,\ \text{ and }\ \sum_{i\in\mathcal{I}}\Big(\frac{\rho_i^2\phi_{i,j}\phi_{i,j,k}}{\lambda_i}\Big)^{\frac{1}{\gamma}} = p_{j,k}^{\frac{1}{\gamma}}\bar{e}_{j,k}. \tag{A.38}$$

Solving (A.37) for $\bar{e}$ and then substituting it into (A.38), we see that

$$\begin{aligned}
\mu &= \Big(\sum_{i\in\mathcal{I}}\Big(\frac{1}{\lambda_i}\Big)^{\frac{1}{\gamma}}\Big)^{-\gamma},\\
\rho_{\lambda,1}\phi_{\lambda,j} &= \mu\Big(\sum_{i\in\mathcal{I}}\Big(\frac{\rho_i\phi_{i,j}}{\lambda_i}\Big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \Big(\sum_{i\in\mathcal{I}} w_i\big(\rho_i\phi_{i,j}\big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \bar{\phi}_{\lambda,j},\\
\rho_{\lambda,1}\rho_{\lambda,j}\phi_{\lambda,j}\phi_{\lambda,j,k} &= \mu\Big(\sum_{i\in\mathcal{I}}\Big(\frac{\rho_i^2\phi_{i,j}\phi_{i,j,k}}{\lambda_i}\Big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \Big(\sum_{i\in\mathcal{I}} w_i\big(\rho_i^2\phi_{i,j}\phi_{i,j,k}\big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \bar{\phi}_{\lambda,j,k}.
\end{aligned} \tag{A.39}$$

For each j, k in $\mathcal{J}$, we substitute the terms in (A.37) into the utility weight $\bar{\phi}_\lambda$ in (A.39):

$$\begin{aligned}
\bar{\phi}_{\lambda,j} &= \mu\Big(\sum_{i\in\mathcal{I}}\Big(\frac{\rho_i\phi_{i,j}}{\lambda_i}\Big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \bar{e}_1^{-\gamma}\Big(\sum_{i\in\mathcal{I}} p_j^{\frac{1}{\gamma}} x_j^i\Big)^{\gamma} = p_j\Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{\gamma},\\
\bar{\phi}_{\lambda,j,k} &= \mu\Big(\sum_{i\in\mathcal{I}}\Big(\frac{\rho_i^2\phi_{i,j}\phi_{i,j,k}}{\lambda_i}\Big)^{\frac{1}{\gamma}}\Big)^{\gamma} = \bar{e}_1^{-\gamma}\Big(\sum_{i\in\mathcal{I}} p_{j,k}^{\frac{1}{\gamma}} x_{j,k}^i\Big)^{\gamma} = p_{j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_1}\Big)^{\gamma}.
\end{aligned} \tag{A.40}$$

Then the utility function V_λ in (58) is expressed as

$$\begin{aligned}
V_\lambda(z) &= \frac{1}{1-\gamma}\Big(z_1^{1-\gamma} + \sum_{j=2}^{J}\Big(\bar\phi_{\lambda,j} z_j^{1-\gamma} + \sum_{k=2}^{J}\bar\phi_{\lambda,j,k} z_{j,k}^{1-\gamma}\Big)\Big) \\
&= \frac{1}{1-\gamma}\Big(z_1^{1-\gamma} + \Big(\frac{1}{\bar e_1}\Big)^\gamma \sum_{j=2}^{J} p_j \bar e_j^\gamma \Big(z_j^{1-\gamma} + \frac{1}{p_j \bar e_j^\gamma}\sum_{k=2}^{J} p_{j,k}\bar e_{j,k}^\gamma z_{j,k}^{1-\gamma}\Big)\Big) \\
&= \frac{1}{1-\gamma}\Bigg(z_1^{1-\gamma} + \Big(\sum_{h=2}^{J} p_h\Big(\frac{\bar e_h}{\bar e_1}\Big)^\gamma\Big)\sum_{j=2}^{J}\frac{p_j \bar e_j^\gamma}{\sum_{h=2}^{J} p_h \bar e_h^\gamma} \\
&\qquad\qquad \times \Big(z_j^{1-\gamma} + \frac{\sum_{h=2}^{J} p_{j,h}\bar e_{j,h}^\gamma}{p_j \bar e_j^\gamma}\sum_{k=2}^{J}\frac{p_{j,k}\bar e_{j,k}^\gamma}{\sum_{h=2}^{J} p_{j,h}\bar e_{j,h}^\gamma} z_{j,k}^{1-\gamma}\Big)\Bigg) \\
&= \frac{1}{1-\gamma}\Big(z_1^{1-\gamma} + \varrho_{A,1}\sum_{j=2}^{J}\phi_{A,j}\Big(z_j^{1-\gamma} + \varrho_{A,j}\sum_{k=2}^{J}\phi_{A,j,k} z_{j,k}^{1-\gamma}\Big)\Big),
\end{aligned}$$

where the last equality is obtained using the preference parameters specified in (43). This result shows that the welfare-based representation in Jouini and Napp (2006) is equivalent to the demand-based representation in Corollary 3. □

Proof of Lemma 4. For $\gamma > 1$, x^γ is strictly convex for all $x > 0$. For $j = 1$, this fact gives

$$\begin{aligned}
\varrho_{A,1} &= \sum_{j=2}^{J} p_j\Big(\frac{\bar e_j}{\bar e_1}\Big)^\gamma = \sum_{j=2}^{J} p_j\Big(\frac{\sum_{i\in\mathcal{J}} x_j^i}{\bar e_1}\Big)^\gamma = \sum_{j=2}^{J} p_j\Big(\sum_{i\in\mathcal{J}}\frac{x_1^i}{\bar e_1}\frac{x_j^i}{x_1^i}\Big)^\gamma \\
&\le \sum_{j=2}^{J} p_j \sum_{i\in\mathcal{J}}\frac{x_1^i}{\bar e_1}\Big(\frac{x_j^i}{x_1^i}\Big)^\gamma = \sum_{i\in\mathcal{J}}\frac{x_1^i}{\bar e_1}\sum_{j=2}^{J} p_j\Big(\frac{x_j^i}{x_1^i}\Big)^\gamma.
\end{aligned} \tag{A.41}$$

We claim that $\sum_{j=2}^{J} p_j\big(x_j^i/x_1^i\big)^\gamma = \rho_i$. By Lemma 2, the demand ratio x_j^i/x_1^i is expressed as[21]

$$\frac{x_j^i}{x_1^i} = p_j^{-\frac{1}{\gamma}}\rho_i^{\frac{1}{\gamma}}\phi_{i,j}^{\frac{1}{\gamma}}.$$

This gives $\rho_i\phi_{i,j} = p_j\big(x_j^i/x_1^i\big)^\gamma$. By summing it over $\mathcal{J}$, we obtain

$$\rho_i = \sum_{j=2}^{J} p_j\Big(\frac{x_j^i}{x_1^i}\Big)^\gamma.$$

[21]The demand x^i from V_i in (53) is obtained from the demand function g^i specified in (19)–(21) by replacing $\hat\phi_i$ with ϕ_i and setting $\hat\delta_{i,1} = 1$ and $\hat\delta_{i,j} = 1 - \rho_i$ for all $j \in \mathcal{J}$.

Substituting this into the last term in (A.41), we have

$$\varrho_{A,1} \leq \sum_{i\in\mathcal{I}} \frac{x_1^i}{\bar{e}_1}\rho_i \leq \max_{i\in\mathcal{I}} \rho_i.$$

Similarly, for $j \in \mathcal{J}$, we see that

$$\varrho_{A,j} = \sum_{k=2}^{J} \frac{p_{j,k}}{p_j}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_j}\Big)^{\gamma} \leq \sum_{i\in\mathcal{I}} \frac{x_j^i}{\bar{e}_j} \sum_{k=2}^{J} \frac{p_{j,k}}{p_j}\Big(\frac{x_{j,k}^i}{x_j^i}\Big)^{\gamma}. \tag{A.42}$$

By exploiting the demand ratio $\rho_i\phi_{i,j,k} = (p_{j,k}/p_j)\big(x_{j,k}^i/x_j^i\big)^{\gamma}$, we can obtain

$$\rho_i = \sum_{k=2}^{J} \frac{p_{j,k}}{p_j}\Big(\frac{x_{j,k}^i}{x_j^i}\Big)^{\gamma}.$$

Substituting this into (A.42), we have

$$\varrho_{A,j} \leq \sum_{i\in\mathcal{I}} \frac{x_j^i}{\bar{e}_j}\rho_i \leq \max_{i\in\mathcal{I}} \rho_i.$$

For $0 < \gamma < 1$, x^{γ} is strictly concave for all $x > 0$. We can obtain the result in (b) by applying similar arguments to those made above. □

Proof of Theorem 5. STEP 1. We show that (63) holds. From the demand function (21), the third-period relative consumption satisfies

$$\frac{x_{j,k}^i}{x_{j,2}^i} = \Big(\frac{\hat{\phi}_{i,j,k}}{\hat{\phi}_{i,j,2}}\Big)^{\frac{1}{\gamma}}\Big(\frac{p_{j,k}}{p_{j,2}}\Big)^{-\frac{1}{\gamma}} \quad \text{for each } \ k = 3, \ldots, J.$$

Since $\hat{\phi}_{i,j,2} = 1$, this relative consumption is bounded as

$$\phi_{\min,j,k}^{\frac{1}{\gamma}}\Big(\frac{p_{j,k}}{p_{j,2}}\Big)^{-\frac{1}{\gamma}} \leq \frac{x_{j,k}^i}{x_{j,2}^i} \leq \phi_{\max,j,k}^{\frac{1}{\gamma}}\Big(\frac{p_{j,k}}{p_{j,2}}\Big)^{-\frac{1}{\gamma}}.$$

Aggregating this relation over $\mathcal{I}$, we obtain

$$\hat{\phi}_{\min,j,k}^{\frac{1}{\gamma}}\Big(\frac{p_{j,k}}{p_{j,2}}\Big)^{-\frac{1}{\gamma}} \leq \frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \leq \hat{\phi}_{\max,j,k}^{\frac{1}{\gamma}}\Big(\frac{p_{j,k}}{p_{j,2}}\Big)^{-\frac{1}{\gamma}}. \tag{A.43}$$

This gives

$$\hat{\phi}_{\min,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma} \leq \frac{p_{j,k}}{p_{j,2}} \leq \hat{\phi}_{\max,j,k}\Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}}\Big)^{-\gamma}. \tag{A.44}$$

STEP 2. We show that the relative price index $P_j^{i,2}/p_{j,2}$ is bounded as

$$\Phi_{j,\min}^2 \leq \frac{P_j^{i,2}}{p_{j,2}} \leq \Phi_{j,\max}^2, \tag{A.45}$$

where for each $d \in \{\min, \max\}$,

$$\Phi_{d,j}^2 = \frac{P_j^{d,2}}{p_{j,2}^d} = \Bigg(\sum_{k=2}^{J} \hat{\phi}_{d,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{1-\gamma} \Bigg)^{\frac{\gamma}{\gamma-1}}.$$

For $\gamma > 1$, $P_j^{i,2}$ increases in $\phi_{i,j,k}$ for all $k = 2, \ldots, J$:

$$\frac{\partial P_j^{i,2}}{\partial \hat{\phi}_{i,j,k}} = \frac{\hat{\phi}_{i,j,k}^{\frac{1}{\gamma}-1} p_{j,k}^{1-\frac{1}{\gamma}} \Big(\sum_{k=2}^{J} \hat{\phi}_{i,j,k}^{\frac{1}{\gamma}} p_{j,k}^{1-\frac{1}{\gamma}} \Big)^{\frac{1}{\gamma-1}}}{\gamma - 1} > 0. \tag{A.46}$$

By (A.44), this monotonicity gives the lower and upper bounds of $P_j^{i,2}/p_{j,2}$:

$$\begin{aligned} \Phi_{j,\min}^2 &= \Bigg(\sum_{k=2}^{J} \hat{\phi}_{\min,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{1-\gamma} \Bigg)^{\frac{\gamma}{\gamma-1}} \\ &\leq \frac{P_j^{i,2}}{p_{j,2}} = \Bigg(\sum_{k=2}^{J} \hat{\phi}_{i,j,k}^{\frac{1}{\gamma}} \Big(\frac{p_{j,k}}{p_{j,2}} \Big)^{1-\frac{1}{\gamma}} \Bigg)^{\frac{\gamma}{\gamma-1}} \\ &\leq \Bigg(\sum_{k=2}^{J} \hat{\phi}_{\max,j,k} \Big(\frac{\bar{e}_{j,k}}{\bar{e}_{j,2}} \Big)^{1-\gamma} \Bigg)^{\frac{\gamma}{\gamma-1}} = \Phi_{j,\max}^2. \end{aligned} \tag{A.47}$$

STEP 3. We claim that if $\gamma\psi \geq 1$, then (64) holds. By (20) and (21), we see that for all $j = 3, \ldots, J$,

$$\begin{aligned} \frac{x_{j,2}^i}{x_j^i} &= \Big(\frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \Big)^{\psi} p_j^{\psi} \Big(\frac{1}{p_{j,2}} \Big)^{\frac{1}{\gamma}} (P_j^{i,2})^{\frac{1}{\gamma}-\psi} \\ &= \Big(\frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\frac{P_j^{i,2}}{p_{j,2}} \Big)^{\frac{1}{\gamma}-\psi}. \end{aligned} \tag{A.48}$$

For each $d \in \{\max, \min\}$, we rearrange this as

$$\frac{x_{j,2}^i}{x_j^i} = \Big(\frac{\rho_i \hat{\delta}_{i,j}}{1-\rho_i} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\frac{P_j^{i,2}}{p_{j,2}} \Big)^{\frac{1}{\gamma}} \left((\Phi_{d,j}^2)^{\psi} \Big(\frac{P_j^{i,2}}{p_{j,2}} \Big)^{-\psi} \right) (\Phi_{d,j}^2)^{-\psi}.$$

For each $j \in \mathcal{J}$, we set

$$\hat{\delta}_{\max,j}^a = \max_{i \in \mathcal{I}} \hat{\delta}_{i,j} \quad \text{and} \quad \hat{\delta}_{\min,j}^a = \min_{i \in \mathcal{I}} \hat{\delta}_{i,j}.$$

Then by (A.45), we see that

$$\begin{aligned}\frac{x^i_{j,2}}{x^i_j} &= \Big(\frac{\rho_i\hat{\delta}_{i,j}}{1-\rho_i}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\Big(\frac{P^{i,2}_j}{p_{j,2}}\Big)^{\frac{1}{\gamma}}\left(\big(\Phi^2_{j,\min}\big)^{\psi}\Big(\frac{P^{i,2}_j}{p_{j,2}}\Big)^{-\psi}\right)\big(\Phi^2_{j,\min}\big)^{-\psi}\\ &\le \Big(\frac{\rho_i\hat{\delta}_{i,j}}{1-\rho_i}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\Big(\frac{P^{i,2}_j}{p_{j,2}}\Big)^{\frac{1}{\gamma}}\big(\Phi^2_{j,\min}\big)^{-\psi}\\ &\le \Big(\frac{\rho_{\max}\hat{\delta}^a_{\max,j}}{1-\rho_{\max}}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\big(\Phi^2_{j,\max}\big)^{\frac{1}{\gamma}}\big(\Phi^2_{j,\min}\big)^{-\psi}.\end{aligned}$$

Similarly, we obtain

$$\Big(\frac{\rho_{\min}\hat{\delta}^a_{\min,j}}{1-\rho_{\min}}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\big(\Phi^2_{j,\min}\big)^{\frac{1}{\gamma}}\big(\Phi^2_{j,\max}\big)^{-\psi} \le \frac{x^i_{j,2}}{x^i_j}.$$

Aggregating these inequalities over $\mathcal{I}$, we have

$$\begin{aligned}&\Big(\frac{\rho_{\min}\hat{\delta}^a_{\min,j}}{1-\rho_{\min}}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\big(\Phi^2_{j,\min}\big)^{\frac{1}{\gamma}}\big(\Phi^2_{j,\max}\big)^{-\psi}\\ &\qquad\le \frac{\bar{e}_{j,2}}{\bar{e}_j} \le \Big(\frac{\rho_{\max}\hat{\delta}^a_{\max,j}}{1-\rho_{\max}}\Big)^{\psi}\Big(\frac{p_j}{p_{j,2}}\Big)^{\psi}\big(\Phi^2_{j,\max}\big)^{\frac{1}{\gamma}}\big(\Phi^2_{j,\min}\big)^{-\psi}.\end{aligned}$$

We solve this relation with respect to $p_j/p_{j,2}$:

$$\begin{aligned}&\Big(\frac{1-\rho_{\max}}{\rho_{\max}\hat{\delta}^a_{\max,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi^2_{j,\max}\Big)^{1-\frac{1}{\gamma\psi}}\frac{\Phi^2_{j,\min}}{\Phi^2_{j,\max}}\\ &\qquad\le \frac{p_j}{p_{j,2}} \le \Big(\frac{1-\rho_{\min}}{\rho_{\min}\hat{\delta}^a_{\min,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi^2_{j,\min}\Big)^{1-\frac{1}{\gamma\psi}}\frac{\Phi^2_{j,\max}}{\Phi^2_{j,\min}}.\end{aligned} \tag{A.49}$$

For each $j\in\mathcal{J}$, we recall from (60) that

$$\hat{\delta}_{\min,j} = \hat{\delta}^a_{\min,j}\frac{\Phi^2_{j,\min}}{\Phi^2_{j,\max}} \quad\text{and}\quad \hat{\delta}_{\max,j} = \hat{\delta}^a_{\max,j}\frac{\Phi^2_{j,\max}}{\Phi^2_{j,\min}} \tag{A.50}$$

It follows from (A.15) and (A.49) that

$$\begin{aligned}\frac{p^{\max}_j}{p^{\max}_{j,2}} &= \Big(\frac{1-\rho_{\max}}{\rho_{\max}\hat{\delta}_{\max,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi^2_{j,\max}\Big)^{1-\frac{1}{\gamma\psi}}\\ &\le \frac{p_j}{p_{j,2}} \le \Big(\frac{1-\rho_{\min}}{\rho_{\min}\hat{\delta}_{\min,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi^2_{j,\min}\Big)^{1-\frac{1}{\gamma\psi}} = \frac{p^{\min}_j}{p^{\min}_{j,2}}\end{aligned} \tag{A.51}$$

STEP 4. We claim that

$$\xi_{\psi,j}^{-\frac{\psi}{1-\psi}} \frac{Q_j^{\min,2}}{p_j^{\min}} \le \frac{Q_j^{i,2}}{p_j} \le \xi_{\psi,j}^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}. \tag{A.52}$$

We recall that $Q_j^{i,2}/p_j$ is given by

$$\frac{Q_j^{i,2}}{p_j} = \frac{1}{p_j}\Big((1-\rho_i)^{\psi}(p_j)^{1-\psi} + (\rho_i\hat{\delta}_{i,j})^{\psi}\big(P_j^{i,2}\big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.$$

This ratio responds differently to changes in the time preference factors and the price terms, depending whether $0<\psi<1$ or $\psi>1$. This dependence leads to a different form of the bound in (A.52).

STEP 4A. Suppose that $0<\psi<1$. To use this information, we rearrange the price ratio $Q_j^{i,2}/p_j$ as

$$\begin{aligned}\frac{Q_j^{i,2}}{p_j} &= \Big((1-\rho_i)^{\psi} + (\rho_i\hat{\delta}_{i,j})^{\psi}\Big(\frac{P_j^{i,2}}{p_{j,2}}\frac{p_{j,2}}{p_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &= (1-\rho_i)^{\frac{\psi}{1-\psi}}\Big(1+\Big(\frac{\rho_i\hat{\delta}_{i,j}}{1-\rho_i}\Big)^{\psi}\Big(\frac{P_j^{i,2}}{p_{j,2}}\frac{p_{j,2}}{p_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}. \end{aligned}\tag{A.53}$$

Then the first term $(1-\rho_i)^{\frac{\psi}{1-\psi}}$ decreases in ρ_i, and the other term in (A.53) increases in $\rho_i, \hat{\delta}_{i,j}$, $P_j^{i,2}/p_{j,2}$, and $p_{j,2}/p_j$. By (A.45) and (A.51), the monotonicity property of $Q_j^{i,2}/p_j$ yields the relation

$$\begin{aligned}&(1-\rho_{\max})^{\frac{\psi}{1-\psi}}\Big(1+\Big(\frac{\rho_{\min}\hat{\delta}^a_{\min,j}}{1-\rho_{\min}}\Big)^{\psi}\Big(\Phi^2_{j,\min}\frac{p^{\min}_{j,2}}{p^{\min}_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &\quad\le \frac{Q_j^{i,2}}{p_j} \le (1-\rho_{\min})^{\frac{\psi}{1-\psi}}\Big(1+\Big(\frac{\rho_{\max}\hat{\delta}^a_{\max,j}}{1-\rho_{\max}}\Big)^{\psi}\Big(\Phi^2_{j,\max}\frac{p^{\max}_{j,2}}{p^{\max}_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned}\tag{A.54}$$

On the other hand, since $\Phi^2_{j,\min} \le \Phi^2_{j,\max}$ by (A.47), (A.50) yields

$$\hat{\delta}_{\min,j} \le \hat{\delta}^a_{\min,j} \quad \text{and} \quad \hat{\delta}_{\max,j} \ge \hat{\delta}^a_{\max,j}. \tag{A.55}$$

Then it follows from (A.54) that

$$\begin{aligned}&(1-\rho_{\max})^{\frac{\psi}{1-\psi}}\Big(1+\Big(\frac{\rho_{\min}\hat{\delta}_{\min,j}}{1-\rho_{\min}}\Big)^{\psi}\Big(\Phi^2_{j,\min}\frac{p^{\min}_{j,2}}{p^{\min}_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &\quad\le \frac{Q_j^{i,2}}{p_j} \le (1-\rho_{\min})^{\frac{\psi}{1-\psi}}\Big(1+\Big(\frac{\rho_{\max}\hat{\delta}_{\max,j}}{1-\rho_{\max}}\Big)^{\psi}\Big(\Phi^2_{j,\max}\frac{p^{\max}_{j,2}}{p^{\max}_j}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned}\tag{A.56}$$

This is rearranged as

$$\begin{aligned}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-\frac{\psi}{1-\psi}}\Big((1-\rho_{\min})^{\psi}+(\rho_{\min}\hat{\delta}_{\min,j})^{\psi}\Big(\Phi_{j,\min}^{2}\frac{p_{j,2}^{\min}}{p_{j}^{\min}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \le \frac{Q_{j}^{i,2}}{p_{j}} \\ \le \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}}\Big((1-\rho_{\max})^{\psi}+(\rho_{\max}\hat{\delta}_{\max,j})^{\psi}\Big(\Phi_{j,\max}^{2}\frac{p_{j,2}^{\max}}{p_{j}^{\max}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned} \tag{A.57}$$

This result combined with (A.16) leads to (A.52).

STEP 4B. Suppose that $\psi > 1$. In this case, the price ratio $Q_j^{i,2}/p_j$ is rearranged as

$$\begin{aligned}\frac{Q_{j}^{i,2}}{p_{j}} &= \Big((1-\rho_{i})^{\psi}+(\rho_{i}\hat{\delta}_{i,j})^{\psi}\Big(\frac{P_{j}^{i,2}}{p_{j,2}}\frac{p_{j,2}}{p_{j}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &= (\rho_{i}\hat{\delta}_{i,j})^{\frac{\psi}{1-\psi}}\Big(\Big(\frac{\rho_{i}\hat{\delta}_{i,j}}{1-\rho_{i}}\Big)^{\psi}+\Big(\frac{P_{j}^{i,2}}{p_{j,2}}\frac{p_{j,2}}{p_{j}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned} \tag{A.58}$$

Then the first term $(\rho_i\hat{\delta}_{i,j})^{\frac{\psi}{1-\psi}}$ decreases in ρ_i and $\hat{\delta}_{i,j}$, and the other term in (A.58) increases in $\rho_i, \hat{\delta}_{i,j}$, $P_j^{i,2}/p_{j,2}$, and $p_{j,2}/p_j$.

The monotonicity of $Q_j^{i,2}/p_j$ combined with (A.45), (A.55), and (A.51) gives the relation

$$\begin{aligned}&(\rho_{\max}\hat{\delta}_{\max,j})^{\frac{\psi}{1-\psi}}\Big(\Big(\frac{\rho_{\min}\hat{\delta}_{\min,j}}{1-\rho_{\min}}\Big)^{-\psi}+\Big(\Phi_{j,\min}^{2}\frac{p_{j,2}^{\min}}{p_{j}^{\min}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &\quad\le \frac{Q_{j}^{i,2}}{p_{j}} \le (\rho_{\min}\hat{\delta}_{\min,j})^{\frac{\psi}{1-\psi}}\Big(\Big(\frac{\rho_{\max}\hat{\delta}_{\max,j}}{1-\rho_{\max}}\Big)^{-\psi}+1\Big(\Phi_{j,\max}^{2}\frac{p_{j,2}^{\max}}{p_{j}^{\max}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned} \tag{A.59}$$

This is rearranged as

$$\begin{aligned}&\Big(\frac{\rho_{\min}\hat{\delta}_{\min,j}}{\rho_{\max}\hat{\delta}_{\max,j}}\Big)^{-\frac{\psi}{1-\psi}}\Big((1-\rho_{\min})^{\psi}+(\rho_{\min}\hat{\delta}_{\min,j})^{\psi}\Big(\Phi_{j,\min}^{2}\frac{p_{j,2}^{\min}}{p_{j}^{\min}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}} \\ &\quad\le \frac{Q_{j}^{i,2}}{p_{j}} \le \Big(\frac{\rho_{\min}\hat{\delta}_{\min,j}}{\rho_{\max}\hat{\delta}_{\max,j}}\Big)^{\frac{\psi}{1-\psi}}\Big((1-\rho_{\max})^{\psi}+(\rho_{\max}\hat{\delta}_{\max,j})^{\psi}\Big(\Phi_{j,\max}^{2}\frac{p_{j,2}^{\max}}{p_{j}^{\max}}\Big)^{1-\psi}\Big)^{\frac{1}{1-\psi}}.\end{aligned}$$

When $\psi > 1$, this result combined with (A.16) leads to (A.52).

STEP 5. We examine the demand ratio x_j^i/x_2^i to obtain information about the price ratio p_j/p_2. By (20), x_j^i/x_2^i is given by

$$\frac{x_{j}^{i}}{x_{2}^{i}} = \hat{\phi}_{i,j}^{\frac{1}{\gamma}}p_{2}^{\psi}p_{j}^{-\psi}(Q_{2}^{i,2})^{\frac{1}{\gamma}-\psi}(Q_{j}^{i,2})^{-\frac{1}{\gamma}+\psi} = \hat{\phi}_{i,j}^{\frac{1}{\gamma}}\Big(\frac{p_{j}}{p_{2}}\Big)^{-\frac{1}{\gamma}}\Big(\frac{Q_{2}^{i,2}}{p_{2}}\Big)^{\frac{1}{\gamma}-\psi}\Big(\frac{Q_{j}^{i,2}}{p_{j}}\Big)^{-\frac{1}{\gamma}+\psi}. \tag{A.60}$$

Since $\gamma\psi \ge 1$ yields $\psi - 1/\gamma \ge 0$, the last expression increases in $Q_j^{i,2}/p_j$ and decreases in $Q_2^{i,2}/p_2$.

Then (A.60) combined with (A.52) allows us to have

$$\phi_{\min,j}^{\frac{1}{\gamma}}\left(\frac{p_j}{p_2}\right)^{-\frac{1}{\gamma}}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{\frac{1}{\gamma}-\psi}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_j^{\min,2}}{p_j^{\min}}\right)^{-\frac{1}{\gamma}+\psi}$$
$$\leq \frac{x_j^i}{x_2^i} \leq \phi_{\max,j}^{\frac{1}{\gamma}}\left(\frac{p_j}{p_2}\right)^{-\frac{1}{\gamma}}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{\frac{1}{\gamma}-\psi}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_j^{\max,2}}{p_j^{\max}}\right)^{-\frac{1}{\gamma}+\psi}.$$

Aggregating this relation over $\mathcal{I}$, we obtain

$$\phi_{\min,j}^{\frac{1}{\gamma}}\left(\frac{p_j}{p_2}\right)^{-\frac{1}{\gamma}}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{\frac{1}{\gamma}-\psi}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_j^{\min,2}}{p_j^{\min}}\right)^{-\frac{1}{\gamma}+\psi}$$
$$\leq \frac{\bar{e}_j}{\bar{e}_2} \leq \phi_{\max,j}^{\frac{1}{\gamma}}\left(\frac{p_j}{p_2}\right)^{-\frac{1}{\gamma}}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{\frac{1}{\gamma}-\psi}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_j^{\max,2}}{p_j^{\max}}\right)^{-\frac{1}{\gamma}+\psi}.$$

We solve the relation with respect to p_j/p_2:

$$\hat{\phi}_{\min,j}\left(\frac{\bar{e}_j}{\bar{e}_2}\right)^{-\gamma}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{1-\gamma\psi}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_j^{\min,2}}{p_j^{\min}}\right)^{-1+\gamma\psi}$$
$$\leq \frac{p_j}{p_2} \leq \hat{\phi}_{\max,j}\left(\frac{\bar{e}_j}{\bar{e}_2}\right)^{-\gamma}\left(\xi_{\psi,j}^{-\frac{\psi}{1-\psi}}\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{1-\gamma\psi}\left(\xi_{\psi,j}^{\frac{\psi}{1-\psi}}\frac{Q_j^{\max,2}}{p_j^{\max}}\right)^{-1+\gamma\psi}.$$

Using (A.19), this relation is rearranged as

$$\xi_{\psi,j}^{\frac{2\psi(1-\gamma\psi)}{1-\psi}}\left(\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{1-\gamma\psi}\left(\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{-1+\gamma\psi}\frac{p_j^{\min}}{p_2^{\min}}$$
$$\leq \frac{p_j}{p_2} \leq \xi_{\psi,j}^{-\frac{2\psi(1-\gamma\psi)}{1-\psi}}\left(\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{1-\gamma\psi}\left(\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{-1+\gamma\psi}\frac{p_j^{\max}}{p_2^{\max}}. \tag{A.61}$$

STEP 6. We claim that $P^{i,1}/p_2$ is bounded as

$$\xi_{\psi}^{\frac{\psi(1-2\gamma\psi)}{1-\psi}}\left(\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{1-\gamma\psi}\left(\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{-1+\gamma\psi}\frac{P^{\min,1}}{p_2^{\min}}$$
$$\leq \frac{P^{i,1}}{p_2} \leq \xi_{\psi}^{-\frac{\psi(1-2\gamma\psi)}{1-\psi}}\left(\frac{Q_2^{\min,2}}{p_2^{\min}}\right)^{1-\gamma\psi}\left(\frac{Q_2^{\max,2}}{p_2^{\max}}\right)^{-1+\gamma\psi}\frac{P^{\max,1}}{p_2^{\max}}, \tag{A.62}$$

where

$$\xi_{\psi} = \begin{cases} \max_{j\in\mathcal{J}}\xi_{\psi,j}, & \text{if } 0<\psi<1 \\ \min_{j\in\mathcal{J}}\xi_{\psi,j}, & \text{if } \psi>1 \end{cases}.$$

Here, $P^{\min,1}/p_2^{\min}$ and $P^{\max,1}/p_2^{\max}$ are obtained explicitly in (A.20).

The relative price index $P^{i,1}/p_2$ is given as

$$\frac{P^{i,1}}{p_2} = \frac{1}{p_2}\Big(\sum_{j=2}^{J} \hat{\phi}_{i,j}^{\frac{1}{\gamma}} (Q_j^{i,2})^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}} = \Big(\sum_{j=2}^{J} \hat{\phi}_{i,j}^{\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}}. \tag{A.63}$$

Since $\gamma > 1$, the last term shows that $P^{i,1}/p_2$ increases in $\hat{\phi}_{i,j}$, $Q_j^{i,2}/p_j$, and p_j/p_2.

On the other hand, the relations in (A.52) and (A.61) give the lower and upper bounds for $(Q_j^{i,2}/p_j)(p_j/p_2)$: is bounded as

$$\begin{aligned}
&\xi_{\psi,j}^{\frac{\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{1-\gamma\psi} \Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{-1+\gamma\psi} \frac{Q_j^{\min,2}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}} \\
&\quad \le \frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2} \le \xi_{\psi,j}^{-\frac{\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{1-\gamma\psi} \Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{-1+\gamma\psi} \frac{Q_j^{\max,2}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}.
\end{aligned}$$

Since the exponent $\psi(1-2\gamma\psi)/(1-\psi)$ of $\xi_{\psi,s^{T-1}}$ changes signs depending on whether $0 < \psi < 1$ or $\psi > 1$, the term $\xi_{\psi,s^{T-1}}^{\psi(1-2\gamma\psi)/(1-\psi)}$ will vary accordingly. By applying the extreme values of $\xi_{\psi,s^{T-1}}$ from (62) to the previous relation, we obtain the following inequalities:

$$\begin{aligned}
&\xi_{\psi}^{\frac{\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{1-\gamma\psi} \Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{-1+\gamma\psi} \frac{Q_j^{\min,2}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}} \\
&\quad \le \frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2} \le \xi_{\psi}^{-\frac{\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{1-\gamma\psi} \Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{-1+\gamma\psi} \frac{Q_j^{\max,2}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}.
\end{aligned} \tag{A.64}$$

Substituting this relation into (A.63) yields the relation (A.62).

STEP 7. Now we attempt to find the range for the ratio $\bar{p}_j/\bar{p}_1$ for all $j = 2, \dots, J$. Since $\bar{p}_1 = 1$ (due to price normalization), we have $\bar{p}_j/\bar{p}_1 = \bar{p}_j$. We seek for the range for this ratio based on the demand ratio x_j^i/x_1^i. We apply (19) and (20) to obtain

$$\begin{aligned}
\frac{x_j^i}{x_1^i} &= (\rho_i\hat{\delta}_{i,1})^{\psi}(\hat{\phi}_{i,j})^{\frac{1}{\gamma}} p_j^{-\psi} (Q_j^{i,2})^{-\frac{1}{\gamma}+\psi} (P^{i,1})^{\frac{1}{\gamma}-\psi} \\
&= p_j^{-\psi}(\rho_i\hat{\delta}_{i,1})^{\psi}(\hat{\phi}_{i,j})^{\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\Big)^{-\frac{1}{\gamma}+\psi} \Big(\frac{p_j}{p_2}\Big)^{-\frac{1}{\gamma}+\psi} \Big(\frac{P^{i,1}}{p_2}\Big)^{\frac{1}{\gamma}-\psi}.
\end{aligned} \tag{A.65}$$

For simplicity, we define

$$\Upsilon = \xi_{\psi}^{\frac{2\psi(1-2\gamma\psi)}{1-\psi}} \Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{2(1-\gamma\psi)} \Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{2(-1+\gamma\psi)} \Big(\frac{P^{\max,1}}{p_2^{\max}}\Big)^{-1} \frac{P^{\min,1}}{p_2^{\min}}.$$

By (A.62) and (A.62), $(Q_j^{i,2}/p_j)(p_j/p_2)(P^{i,1}/p_2)^{-1}$ is bounded as

$$\Upsilon \frac{Q_j^{\min,2}}{p_j^{\min}} \frac{p_j^{\min}}{p_2^{\min}} \Big(\frac{P^{\min,1}}{p_2^{\min}} \Big)^{-1} \leq \frac{Q_j^{i,2}}{p_j} \frac{p_j}{p_2} \Big(\frac{P^{i,1}}{p_2} \Big)^{-1} \leq \Upsilon^{-1} \frac{Q_j^{\max,2}}{p_j^{\max}} \frac{p_j^{\max}}{p_2^{\max}} \Big(\frac{P^{\max,1}}{p_2^{\max}} \Big)^{-1}. \tag{A.66}$$

Substituting this into (A.65),

$$\begin{aligned} & p_j^{-\psi} \Upsilon^{-\frac{1}{\gamma}+\psi} (\rho_{\min} \hat{\delta}_{\min,1})^{\psi} (\hat{\phi}_{\min,j})^{\frac{1}{\gamma}} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}} \frac{p_j^{\min}}{p_2^{\min}} \Big(\frac{P^{\min,1}}{p_2^{\min}} \Big)^{-1} \Big)^{-\frac{1}{\gamma}+\psi} \\ & \quad \leq \frac{x_j^i}{x_1^i} \leq p_j^{-\psi} \Upsilon^{\frac{1}{\gamma}-\psi} (\rho_{\max} \hat{\delta}_{\max,1})^{\psi} (\hat{\phi}_{\max,j})^{\frac{1}{\gamma}} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}} \frac{p_j^{\max}}{p_2^{\max}} \Big(\frac{P^{\max,1}}{p_2^{\max}} \Big)^{-1} \Big)^{-\frac{1}{\gamma}+\psi}. \end{aligned}$$

Aggregating this relation over $\mathcal{I}$, we obtain

$$\begin{aligned} & p_j^{-\psi} \Upsilon^{-\frac{1}{\gamma}+\psi} (\rho_{\min} \hat{\delta}_{\min,1})^{\psi} (\hat{\phi}_{\min,j})^{\frac{1}{\gamma}} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}} \frac{p_j^{\min}}{p_2^{\min}} \Big(\frac{P^{\min,1}}{p_2^{\min}} \Big)^{-1} \Big)^{-\frac{1}{\gamma}+\psi} \\ & \quad \leq \frac{\bar{e}_j}{\bar{e}_1} \leq p_j^{-\psi} \Upsilon^{\frac{1}{\gamma}-\psi} (\rho_{\max} \hat{\delta}_{\max,1})^{\psi} (\hat{\phi}_{\max,j})^{\frac{1}{\gamma}} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}} \frac{p_j^{\max}}{p_2^{\max}} \Big(\frac{P^{\max,1}}{p_2^{\max}} \Big)^{-1} \Big)^{-\frac{1}{\gamma}+\psi}. \end{aligned} \tag{A.67}$$

Using the value p_j^d in (A.21) for each $d \in \{\min, \max\}$, we solve this inequality for p_j:

$$\Upsilon^{-\frac{1}{\gamma\psi}+1} p_j^{\min} \leq p_j \leq \Upsilon^{\frac{1}{\gamma\psi}-1} p_j^{\max} \tag{A.68}$$

□

Proof of Theorem 6. The proof of this theorem is similar to that of Theorem 5, except that the preference for late resolution affects the intertemporal choices of individual agents differently from the preference for early resolution. In particular, since $\gamma > 1$ and $0 < \gamma\psi < 1$, it follows that $0 < \psi < 1$. This property simplifies the proof arguments compared to the case with a preference for early resolution.

STEP 1. Steps 1 and 2 of the proof of Theorem 5 carry over unchanged, since their derivations do not depend on whether $\gamma\psi \geq 1$ or $0 < \gamma\psi < 1$.

STEP 2. Since $0 < \gamma\psi < 1$ implies $1/\gamma - \psi > 0$, (A.45) and (A.48) yield

$$\Big(\frac{\rho_{\min} \hat{\delta}_{\min,j}}{1-\rho_{\min}} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\Phi_{j,\min}^2 \Big)^{\frac{1}{\gamma}-\psi} \leq \frac{x_{j,2}^i}{x_j^i} \leq \Big(\frac{\rho_{\max} \hat{\delta}_{\max,j}}{1-\rho_{\max}} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\Phi_{j,\max}^2 \Big)^{\frac{1}{\gamma}-\psi}.$$

Aggregating this relation over $\mathcal{I}$, we obtain

$$\Big(\frac{\rho_{\min} \hat{\delta}_{\min,j}}{1-\rho_{\min}} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\Phi_{j,\min}^2 \Big)^{\frac{1}{\gamma}-\psi} \leq \frac{\bar{e}_{j,2}}{\bar{e}_j} \leq \Big(\frac{\rho_{\max} \hat{\delta}_{\max,j}}{1-\rho_{\max}} \Big)^{\psi} \Big(\frac{p_j}{p_{j,2}} \Big)^{\psi} \Big(\Phi_{j,\max}^2 \Big)^{\frac{1}{\gamma}-\psi} \tag{A.69}$$

Then the price ratios in (A.15) become the lower and upper bounds of $p_j/p_{j,2}$.

$$\begin{aligned}\frac{p_j^{\max}}{p_{j,2}^{\max}} &= \Big(\frac{1-\rho_{\max}}{\rho_{\max}\hat{\delta}_{\max,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi_{j,\max}^2\Big)^{1-\frac{1}{\gamma\psi}} \le \frac{p_j}{p_{j,2}} \\ &\le \Big(\frac{1-\rho_{\min}}{\rho_{\min}\hat{\delta}_{\min,j}}\Big)\Big(\frac{\bar{e}_{j,2}}{\bar{e}_j}\Big)^{\frac{1}{\psi}}\Big(\Phi_j^2(\hat{\phi}_{\min},\bar{e})\Big)^{1-\frac{1}{\gamma\psi}} = \frac{p_j^{\min}}{p_{j,2}^{\min}}. \end{aligned} \tag{A.70}$$

Note that this price range appears the same as (A.51). However, since $\gamma\psi < 1$, the adjustment factors $\hat{\delta}_{\max,j}$ and $\hat{\delta}_{\min,j}$ are taken from (72), which are defined differently from (60).

STEP 3. Since $0 < \psi < 1$, we can the same arguments made in Step 4A of the proof of Theorem 5, which yields (A.57). Using (A.16), this relation is simplified as

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{\psi}{1-\psi}}\frac{Q_j^{\min,2}}{p_j^{\min}} \le \frac{Q_j^{i,2}}{p_j} \le \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}}\frac{Q_j^{\max,2}}{p_j^{\max}}. \tag{A.71}$$

STEP 4. We claim that

$$\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\gamma\psi}\frac{p_j^{\min}}{p_2^{\min}} \le \frac{p_j}{p_2} \le \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\gamma\psi}\frac{p_j^{\max}}{p_2^{\max}}. \tag{A.72}$$

This bound is derived from the demand ratio x_j^i/x_2^i in (A.60). Since $-1/\gamma + \psi < 0$, the ratio x_j^i/x_2^i decreases in $Q_j^{i,2}/p_j$ and increases in $Q_2^{i,2}/p_2$.

The monotonic decrease of x_j^i/x_2^i in $Q_j^{i,2}/p_j$ makes it difficult to find a bound for $P^{i,1}/p_2$, which becomes a component of the bound for the price ratio p_j/p_1 discussed in the next step. To see this, we examine the functional form of $P^{i,1}/p_2$ given by

$$\frac{P^{i,1}}{p_2} = \frac{1}{p_2}\Big(\sum_{j=2}^{J}\hat{\phi}_{i,j}^{\frac{1}{\gamma}}(Q_j^{i,2})^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}} = \Big(\sum_{j=2}^{J}\hat{\phi}_{i,j}^{\frac{1}{\gamma}}\Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}}. \tag{A.73}$$

Since $\gamma > 1$, the last term increases in $Q_j^{i,2}/p_j$ and p_j/p_2. This shows that the bound for $P^{i,1}/p_2$ is determined by the bound for $(Q_j^{i,2}/p_j)(p_j/p_2)$.

The bound for $Q_j^{i,2}/p_j$ is given in (A.71). If we follow the same procedure as in Step 5 in the proof of Theorem 5, the monotonic decrease of x_j^i/x_2^i in $Q_j^{i,2}/p_j$ would give the following form of the bound for p_j/p_2:

$$A\,\frac{p_j^{\max}}{p_2^{\max}} \le \frac{p_j}{p_2} \le B\,\frac{p_j^{\min}}{p_2^{\min}},$$

where A and B are parameters independent of j. Combining these results, we see that the

lower and upper bounds for $(Q_j^{i,2}/p_j)(p_j/p_2)$ are mixed with the $\max$ and $\min$ terms:

$$A' \frac{Q_2^{\min,j}}{p_j^{\min}} \frac{p_j^{\max}}{p_2^{\max}} \leq \frac{p_j}{p_2} \leq B' \frac{Q_2^{\max,j}}{p_j^{\max}} \frac{p_j^{\min}}{p_2^{\min}},$$

where A' and B' are parameters independent of j.

By applying this result to (A.73), we can find the bound for $P^{i,1}/p_2$. However, these lower and bounds with the mixed terms cannot be expressed in terms of $P^{\max,1}/p_2^{\max}$ and $P^{\min,1}/p_2^{\min}$ defined in Step 6 of the proof of Theorem 1.

To avoid this difficulty with the sign problem of $Q_j^{i,2}/p_j$, we rearrange x_j^i/x_2^i in (A.60):

$$\frac{x_j^i}{x_2^i} = \hat{\phi}_{i,j}^{\frac{1}{\gamma}} \Big(\frac{p_j}{p_2}\Big)^{-\frac{1}{\gamma}} \Big(\frac{Q_2^{i,2}}{p_2}\Big)^{\frac{1}{\gamma}-\psi} \Big(\frac{Q_j^{i,2}}{p_j}\Big)^{1-\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\Big)^{\psi-1}. \tag{A.74}$$

For each $d \in \{\min, \max\}$, we define

$$\hat{\phi}_{\max,j}^a = \max_{i \in \mathcal{J}} \hat{\phi}_{i,j}.$$

By (A.71), we have

$$\begin{aligned} \frac{x_j^i}{x_2^i} \Big(\frac{p_j}{p_2}\Big)^{\frac{1}{\gamma}} \leq \big(\hat{\phi}_{\max,j}^a\big)^{\frac{1}{\gamma}} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{\frac{1}{\gamma}-\psi} \\ \times \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{1-\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\Big)^{\psi-1}. \end{aligned} \tag{A.75}$$

Since $\psi < 1$, (A.71) allows us to have

$$\begin{aligned} \Big(\frac{Q_j^{i,2}}{p_j}\Big)^{\psi-1} &\leq \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-\psi} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-1} \\ &= \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-\psi} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-1} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{1-\psi} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-1} \\ &= \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\psi} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-1} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{1-\psi} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-1}. \end{aligned}$$

Substituting this result into (A.75), we obtain

$$\begin{aligned} \frac{x_j^i}{x_2^i} \Big(\frac{p_j}{p_2}\Big)^{\frac{1}{\gamma}} \leq \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\psi} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{\frac{1}{\gamma}-\psi} \\ \times \big(\hat{\phi}_{\max,j}^a\big)^{\frac{1}{\gamma}} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-1} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{1-\psi} \Big(\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{\psi}{1-\psi}} \frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-\frac{1}{\gamma}}. \end{aligned} \tag{A.76}$$

Similarly, we can show that

$$\begin{aligned}\frac{x_j^i}{x_2^i}\Big(\frac{p_j}{p_2}\Big)^{\frac{1}{\gamma}} \geq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\psi}\Big(\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{\psi}{1-\psi}}\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{\frac{1}{\gamma}-\psi} \\ \times \big(\hat{\phi}^a_{\min,j}\big)^{\frac{1}{\gamma}}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-1}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{1-\psi}\Big(\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{\psi}{1-\psi}}\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-\frac{1}{\gamma}}.\end{aligned} \tag{A.77}$$

Aggregating these inequalities, we have

$$\begin{aligned}&\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\psi}\big(\hat{\phi}^a_{\min,j}\big)^{\frac{1}{\gamma}}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-1}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{1-\psi}\Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{\frac{1}{\gamma}-\psi}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-\frac{1}{\gamma}} \\ &\leq \frac{\bar{e}_j}{\bar{e}_2}\Big(\frac{p_j}{p_2}\Big)^{\frac{1}{\gamma}} \leq \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\psi}\big(\hat{\phi}^a_{\max,j}\big)^{\frac{1}{\gamma}}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\psi-1}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{1-\psi}\Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{\frac{1}{\gamma}-\psi}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\psi-\frac{1}{\gamma}}.\end{aligned}$$

For each $j \in \mathcal{J}$ and $d \in \{\max, \min\}$, we define

$$\hat{\phi}_{\min,j} = \hat{\phi}^a_{\min,j}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(\psi-1)}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(1-\psi)}, \tag{A.78}$$

$$\hat{\phi}_{\max,j} = \hat{\phi}^a_{\max,j}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(\psi-1)}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(1-\psi)}. \tag{A.79}$$

Then p_j/p_2 satisfies

$$\begin{aligned}&\Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\gamma\psi}\hat{\phi}^{\frac{1}{\gamma}}_{\min,j}\Big(\frac{Q_2^{\min,2}}{p_2^{\min}}\Big)^{1-\gamma\psi}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma\psi-1} \\ &\leq \frac{p_j}{p_2} \leq \Big(\frac{\bar{e}_j}{\bar{e}_2}\Big)^{-\gamma}\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\gamma\psi}\hat{\phi}^{\frac{1}{\gamma}}_{\max,j}\Big(\frac{Q_2^{\max,2}}{p_2^{\max}}\Big)^{1-\gamma\psi}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma\psi-1}.\end{aligned}$$

Using (A.19), we simplify this relation as

$$\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{-2\gamma\psi}\frac{p_j^{\min}}{p_2^{\min}} \leq \frac{p_j}{p_2} \leq \Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{-2\gamma\psi}\frac{p_j^{\max}}{p_2^{\max}}. \tag{A.80}$$

It is worth comparing this result with (A.61).

STEP 5. We show that $P^{i,1}/p_2$ satisfies

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{(1+\gamma)\psi}{(\gamma-1)(1-\psi)}+2\gamma\psi}\frac{P^{\min,1}}{p_2^{\min}} \leq \frac{P^{i,1}}{p_2} \leq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{(1+\gamma)\psi}{(\gamma-1)(1-\psi)}+2\gamma\psi}\frac{P^{\max,1}}{p_2^{\max}}, \tag{A.81}$$

where for each $d \in \{\max, \min\}$, $P^{d,1}/p_2^d$ is given in (A.20).

The relative price index $P^{i,1}/p_2$ is given as

$$\frac{P^{i,1}}{p_2} = \frac{1}{p_2}\Big(\sum_{j=2}^{J} \hat{\phi}_{i,j}^{\frac{1}{\gamma}} (Q_j^{i,2})^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}} = \Big(\sum_{j=2}^{J} \hat{\phi}_{i,j}^{\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}}. \tag{A.82}$$

Since $\gamma > 1$, the last term increases in $\hat{\phi}_{i,j}$, $Q_j^{i,2}/p_j$, and p_j/p_2.

Note that $P^{i,1}/p_2$ in (A.82) contains $\hat{\phi}_{i,j}$'s, the preference parameters of individual i. To express the bound for $P^{i,1}/p_2$ in terms of the equilibrium prices of the MAX and MIN representative agents, we need to understand how $\hat{\phi}_{i,j}$ is related to $\hat{\phi}_{\min,j}$ and $\hat{\phi}_{\max,j}$. To this end, we use the following inequalities derived from (A.71):

$$\begin{aligned}
\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{2\gamma\psi} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(\psi-1)} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(1-\psi)} &\le 1, \\
\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{2\gamma\psi} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(\psi-1)} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(1-\psi)} &\ge 1.
\end{aligned}$$

Combining (A.78) and (A.79) with these inequalities yields

$$\hat{\phi}_{\min,j}\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{2\gamma\psi} = \hat{\phi}^a_{\min,j}\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{2\gamma\psi} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(\psi-1)} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(1-\psi)} \le \hat{\phi}^a_{\min,j}, \tag{A.83}$$

$$\hat{\phi}_{\max,j}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{2\gamma\psi} = \hat{\phi}^a_{\max,j}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{2\gamma\psi} \Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\Big)^{\gamma(\psi-1)} \Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\Big)^{\gamma(1-\psi)} \ge \hat{\phi}^a_{\max,j}. \tag{A.84}$$

Applying this result to (A.82), we see that

$$\begin{aligned}
&\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma\psi}{\gamma-1}} \Big(\sum_{j=2}^{J} \hat{\phi}_{\min,j}^{\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}} \\
&\qquad \le \frac{P^{i,1}}{p_2} \le \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma\psi}{\gamma-1}} \Big(\sum_{j=2}^{J} \hat{\phi}_{\max,j}^{\frac{1}{\gamma}} \Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}}.
\end{aligned} \tag{A.85}$$

On the other hand, the relations in (A.71) and (A.72) gives the lower and upper bounds for $(Q_j^{i,2}/p_j)(p_j/p_2)$:

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{2\gamma\psi+\frac{\psi}{1-\psi}} \frac{Q_2^{\min,j}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}} \le \frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2} \le \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{2\gamma\psi+\frac{\psi}{1-\psi}} \frac{Q_2^{\max,j}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}. \tag{A.86}$$

Substituting this result into (A.85), we obtain

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma^2\psi}{\gamma-1}+\frac{\psi}{1-\psi}}\Big(\sum_{j=2}^{J}\hat{\phi}_{\min,j}^{\frac{1}{\gamma}}\Big(\frac{Q_j^{\min,2}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}}$$
$$\leq \frac{P^{i,1}}{p_2} \leq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma^2\psi}{\gamma-1}+\frac{\psi}{1-\psi}}\Big(\sum_{j=2}^{J}\hat{\phi}_{\max,j}^{\frac{1}{\gamma}}\Big(\frac{Q_j^{\max,2}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}\Big)^{1-\frac{1}{\gamma}}\Big)^{\frac{\gamma}{\gamma-1}},$$

which is simplified as

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma^2\psi}{\gamma-1}+\frac{\psi}{1-\psi}}\frac{P^{\min,1}}{p_2^{\min}} \leq \frac{P^{i,1}}{p_2} \leq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma^2\psi}{\gamma-1}+\frac{\psi}{1-\psi}}\frac{P^{\max,1}}{p_2^{\max}}. \tag{A.87}$$

STEP 6. Using the demand ratio x_j^i/x_1^i in (A.65), we attempt to find the range for the ratio $\bar{p}_j/\bar{p}_1$ for all $j=2,\ldots,J$ Substituting (A.86) and (A.87) into (A.65), we obtain

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\psi(1-\gamma\psi)}{\gamma-1}}\Big(\frac{Q_2^{\min,j}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}}\Big)^{-\frac{1}{\gamma}+\psi}\Big(\frac{P^{\min,1}}{p_2^{\min}}\Big)^{\frac{1}{\gamma}-\psi} \leq \Big(\frac{Q_j^{i,2}}{p_j}\frac{p_j}{p_2}\Big)^{-\frac{1}{\gamma}+\psi}\Big(\frac{P^{i,1}}{p_2}\Big)^{\frac{1}{\gamma}-\psi}$$
$$\leq \Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\psi(1-\gamma\psi)}{\gamma-1}}\Big(\frac{Q_2^{\max,j}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}\Big)^{-\frac{1}{\gamma}+\psi}\Big(\frac{P^{\max,1}}{p_2^{\max}}\Big)^{\frac{1}{\gamma}-\psi}. \tag{A.88}$$

On the other hand, (A.83) and (A.84) give

$$\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{2\psi}(\rho_{\min}\hat{\delta}_{\min,1})^{\psi}(\hat{\phi}_{\min,j})^{\frac{1}{\gamma}} \leq (\rho_{\min}\hat{\delta}_{\min,1})^{\psi}(\hat{\phi}_{\min,j}^{a})^{\frac{1}{\gamma}}$$
$$\leq (\rho_i\hat{\delta}_{i,1})^{\psi}(\hat{\phi}_{i,j})^{\frac{1}{\gamma}} \leq (\rho_{\max}\hat{\delta}_{\max,1})^{\psi}(\hat{\phi}_{\max,j}^{a})^{\frac{1}{\gamma}}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{2\psi}(\rho_{\max}\hat{\delta}_{\max,1})^{\psi}(\hat{\phi}_{\max,j})^{\frac{1}{\gamma}}.$$

Applying this result and (A.88) to (A.65) and aggregating it over $\mathcal{I}$, we have

$$p_j^{-\psi}\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma\psi(1-\psi)}{\gamma-1}}(\rho_{\min}\hat{\delta}_{\min,1})^{\psi}(\hat{\phi}_{\min,j})^{\frac{1}{\gamma}}\Big(\frac{Q_2^{\min,j}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}}\Big)^{-\frac{1}{\gamma}+\psi}\Big(\frac{P^{\min,1}}{p_2^{\min}}\Big)^{\frac{1}{\gamma}-\psi}$$
$$\leq \frac{\bar{e}_j}{\bar{e}_1} \leq p_j^{-\psi}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma\psi(1-\psi)}{\gamma-1}}(\rho_{\max}\hat{\delta}_{\max,1})^{\psi}(\hat{\phi}_{\max,j})^{\frac{1}{\gamma}}\Big(\frac{Q_2^{\max,j}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}\Big)^{-\frac{1}{\gamma}+\psi}\Big(\frac{P^{\max,1}}{p_2^{\max}}\Big)^{\frac{1}{\gamma}-\psi},$$

This result gives

$$\Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{-\frac{1}{\psi}}\Big(\frac{1-\rho_{\max}}{1-\rho_{\min}}\Big)^{\frac{2\gamma(1-\psi)}{\gamma-1}}\rho_{\min}\hat{\delta}_{\min,1}(\hat{\phi}_{\min,j})^{\frac{1}{\gamma\psi}}\Big(\frac{Q_2^{\min,j}}{p_j^{\min}}\frac{p_j^{\min}}{p_2^{\min}}\Big)^{-\frac{1}{\gamma\psi}+1}\Big(\frac{P^{\min,1}}{p_2^{\min}}\Big)^{\frac{1}{\gamma\psi}-1}$$
$$\leq p_j \leq \Big(\frac{\bar{e}_j}{\bar{e}_1}\Big)^{-\frac{1}{\psi}}\Big(\frac{1-\rho_{\min}}{1-\rho_{\max}}\Big)^{\frac{2\gamma(1-\psi)}{\gamma-1}}\rho_{\max}\hat{\delta}_{\max,1}(\hat{\phi}_{\max,j})^{\frac{1}{\gamma\psi}}\Big(\frac{Q_2^{\max,j}}{p_j^{\max}}\frac{p_j^{\max}}{p_2^{\max}}\Big)^{-\frac{1}{\gamma\psi}+1}\Big(\frac{P^{\max,1}}{p_2^{\max}}\Big)^{\frac{1}{\gamma\psi}-1},$$

Using (A.21), for each $j \in \mathcal{J}$, we obtain

$$\left(\frac{1-\rho_{\max}}{1-\rho_{\min}}\right)^{\frac{2\gamma(1-\psi)}{\gamma-1}} p_j^{\min} \leq p_j \leq \left(\frac{1-\rho_{\min}}{1-\rho_{\max}}\right)^{\frac{2\gamma(1-\psi)}{\gamma-1}} p_j^{\max}. \tag{A.89}$$

□

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