A Theory of Portfolio Choice in the Macroeconomy *

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Abstract

This article proposes a continuous-time heterogeneous agent framework to study household portfolio choice in the macroeconomy. Households pay an endogenous premium for the uncertainty of their lifetime welfare under aggregate shocks, generating a wealth-dependent "risk-exposure premium" (REP) effect as a non-monotonic third term in the portfolio rule. For households near borrowing constraints, the negative REP effect dominates despite near-zero effective relative risk aversion, driving risky asset allocation to zero and rationalizing limited participation. As wealth increases, the REP effect turns positive, creating incentives for risky holdings that peak at intermediate wealth levels before declining and converging asymptotically to Merton's result. This mechanism naturally generates the hump-shaped portfolio profile in the life cycle through precautionary savings. We also demonstrate that the systematic risk causes lower-wealth households to accept lower portfolio efficiency, which explains underdiversification. We propose a novel "local steady state" concept providing more general terminal conditions than existing approaches, and we finally show that our framework resolves the wealth distribution degradation problem in the ABH model while providing analytical solutions under the homogeneity.

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1 Introduction

Household portfolio choice represents one of the most enduring puzzles in macro-finance. Decades of empirical research have documented a constellation of phenomena in household finance (**campbell2018financial**) that challenge standard theoretical frameworks: the vast majority of low-wealth households abstain from risky asset markets—the "limited participation" puzzle (**vissing2002limited**; **campbell2006household**); household risk asset allocations exhibit a pronounced "hump-shaped" profile over the life cycle (**heaton2000portfolio**; **poterba2001household**; **ameriks2011household**); and the low wealth holds severely "underdiversified" portfolios concentrated in highly idiosyncratic risks (**calvet2007down**; **goetzmann2008equity**). These stylized facts are not isolated anomalies but rather symptoms of a deeper theoretical challenge: in a dynamic macroeconomy subject to aggregate shocks, what endogenous, wealth-dependent mechanism drives households at different positions of the wealth distribution to make fundamentally different risk-taking decisions?

The canonical continuous-time portfolio framework (**merton1969lifetime**) encounters multiple theoretical difficulties when confronting these puzzles. Extensions incorporating non-tradeable labor income rely critically on two risk characteristics of wage processes, yet both channels prove theoretically problematic. The traditional "labor-hedging" mechanism (**bodie1992labor**; **heaton1997market**; **koo1998consumption**) depends heavily on an empirically unstable wage-return correlation coefficient ρ_{sy}, but more fundamentally, its theoretical implications are logically inconsistent, preventing it from serving as a reliable foundation for explaining household financial behavior. Labor income as an "implicit wealth" (**heaton2000portfolio**) exacerbates the limited participation puzzle. Since any form of labor income enhances household lifetime welfare—otherwise households would choose not to work—this reduces households' intertemporal risk aversion with respect to wealth. An economic intuition of labor income with idiosyncratic risk is that employed households should invest more aggressively in risky assets because the risky portfolio share has a very small effect on the risk to cash on hand (**viceira2001optimal**; **campbell2002strategic**; **cocco2005consumption**). The Merton framework's fundamental inability to resolve these puzzles stems from the nature of "precautionary savings" (**carroll1996concavity**), which renders both motives in the portfolio rule monotonic.

The representative agent paradigm, with its homogeneity assumption, cannot explore the origins of wealth inequality and portfolio heterogeneity. While modern heterogeneous agent models, based on continuous-time **aiyagari1994uninsured<empty citation>-bewley1986<empty citation>-huggett1993risk<empty citation>** (ABH) frame-

works, provide powerful tools for studying the wealth distribution, they focus on household optimal consumption decisions under idiosyncratic shocks. In these models (**achdou2021mean**; **ahn2018inequality**; **fernandez2023financial**), all households hold no risk premium assets, contradicting empirical reality and causing the wealth distribution to degenerate into a Dirac mass point at the borrowing constraint.

This paper develops a new portfolio theory that provides a unified explanation for these puzzles. We construct a continuous-time heterogeneous agent (CTHA) model with aggregate shocks, whose core innovation advances the mathematical paradigm from standard Hamilton-Jacobi-Bellman (HJB) equations to Mean-Field-Games (MFG) system with common noise, coupling a backward stochastic HJB (BSHJB) equation with a stochastic Kolmogorov forward (SKF) equation. This transition carries an economic meaning: when aggregate shocks are present, households' optimization problems themselves become stochastic processes, with their fluctuations directly exposed to macro risk.

When pricing this fundamental uncertainty, households simultaneously pay an endogenous Risk-Exposure Premium (REP) whose structure exhibits fundamental asymmetry: its cost component stems from households' marginal utility of wealth, imposing enormous pressure on the low wealth, while its compensation component is proportional to wealth, leaving low-wealth households bearing massive welfare risk with virtually no compensation. This asymmetric structure enables the REP effect generated by a motive to serve as a non-monotonic, wealth-dependent third component in the portfolio decision rule, constituting our theory's core mechanism.

The mechanism's operation is endogenously determined by households' positions in the wealth distribution, depending on what we define as effective relative risk aversion, denoted η_t. This measures the curvature of the value function with respect to wealth, capturing households' intertemporal risk preference regarding lifetime welfare relative to wealth. For the low wealth near borrowing constraints, consumption is almost entirely determined by labor income, resulting in extremely low sensitivity of their value function to wealth changes, with $\eta_t \to 0$. Under these conditions, the REP effect becomes a large negative value whose magnitude overwhelmingly dominates all the myopic demand and the wage-offsetting effect. This premium reflects the enormous marginal cost of macro risk exposure, while households' near-zero financial wealth prevents them from obtaining corresponding exposure compensation. Their optimal decision is complete withdrawal from risk markets, endogenously explaining the limited participation phenomenon.

As wealth accumulates, households' η_t gradually rises above unity, turning REP positive. Holding systematic risk assets then provides net welfare gains, incentivizing the middle-to-high wealth to over-allocate to risky assets beyond even the classical

Merton benchmark, forming the "hump-shaped" portfolio profile. Finally, for the highest wealth tiers, labor income effects become negligible, with $\eta(a_t)$ converging to the CRRA coefficient γ. Simultaneously, since REP's magnitude is proportional to the wealth scaling factor $\frac{1}{a_t}$, this effect eventually disappears, causing portfolio decisions to converge to standard **merton1969lifetime<empty citation>**'s results.

The REP mechanism further provides new insights into understanding the underdiversification puzzle. We demonstrate that households' choice between systematic and idiosyncratic risk assets depends on the sign of a macro effect function $\mathcal{M}(a_t)$. This function, composed jointly of the REP effect and the wage-offsetting effect, measures aggregate shocks' impact on household portfolio efficiency. For the low wealth with $\mathcal{M}_t < 0$, systematic risk imposes significant welfare costs, leading them to rationally sacrifice some Sharpe ratio to hold asset portfolios with lower correlation to macroeconomic fluctuations—objectively manifesting as under-diversification. Conversely, for the middle-to-high wealth with $\mathcal{M}_t > 0$, systematic risk exposure becomes a welfare "dividend," incentivizing them to actively hold well-diversified portfolios or mutual funds highly correlated with markets. This wealth-dependent risk-taking behavior directly translates into wealth dynamics. $\mathcal{M}_t < 0$ households' portfolio efficiency falls far from the efficient frontier, leading to a self-reinforcing "poverty traps". Meanwhile, $\mathcal{M}_t > 0$ households' portfolio efficiency clusters on the efficient frontier and enter "wealth acceleration" channels. This endogenous differentiation provides a powerful theoretical channel for generating and maintaining significant wealth inequality under homogeneous CRRA preferences.

Methodologically, this paper introduces a novel Local Steady State (LSS) concept, providing more general terminal conditions for economic systems with persistent macroeconomic uncertainty. The LSS characterizes a moment when the macroeconomic shock increment is exactly zero ($d\mathbf{W}_t = 0$) but future shock possibilities remain ($\sigma(\mathbf{W}_t) > 0$). This setting enables steady-state analysis while preserving macro risk pricing. Compared to existing stochastic steady-state methods (**futia1982invariant**; **duffie1994stationary**) requiring the entire cumulative shock process expectation to be zero ($\mathbb{E}_t[\mathbf{W}_{t+1}] = 0$)—a stringent condition—our LSS only requires the instantaneous shock increment to be zero ($\mathbb{E}_t[d\mathbf{W}_t] = 0$) while allowing the economy to carry arbitrary shock histories. This setting greatly expands existing stochastic steady-state analysis capabilities.

The remainder of this paper proceeds as follows. Section 2 develops the theoretical framework. Section 3 defines temporary equilibrium and steady-state concepts. Section 4 derives the core theoretical results. Section 5 presents economic implications through graphical analysis of the theoretical mechanisms. Section 6 concludes.

1.1 Literature Review

This paper contributes to the CTHA literature that builds on the seminal ABH framework. Continuous-time ABH models pioneered by **achdou2022income<empty citation>**, offers significant advantages over discrete-time formulations by avoiding integer problems (**stokey2008economics**) and generating sparse problems. Subsequent research, such as **ahn2018inequality<empty citation>** and **fernandez2023financial<empty citation>**, has extended these models to incorporate two-asset structure and financial frictions.

Methodologically, our framework departs from standard CTHA models by formally adopting the structure of a mean-field game with common noise. The mathematical foundations of MFG theory were established by **lasry2007mean<empty citation>**, with substantial literature focusing on systems without aggregate uncertainty. The introduction of common noise, which is essential for studying aggregate risk, represents a major theoretical leap. **carmona2018probabilistic<empty citation>** and, most comprehensively, **cardaliaguet2019master<empty citation>** characterized the resulting system of coupled backward-forward stochastic differential equations. While this literature provides a rigorous mathematical apparatus, its application to address core macroeconomic questions of portfolio allocation and wealth inequality remains a nascent field. Our paper helps bridge this gap by using the BSHJB equation to derive new economic insights.

Finally, our research is situated against the backdrop of the rapidly advancing frontier of computational macroeconomics. For CTHA models without common noise, a range of powerful techniques now exists, including the finite difference methods used by **achdou2022income<empty citation>**, the state-space iteration of **ahn2018inequality<empty citation>**, and the neural network approaches of **fernandez2023financial<empty citation>**. More recent innovations focusing on solution techniques include the deep learning method of **han2021deepham<empty citation>**, the perturbation techniques of **bilal2023solving<empty citation>**, the PINN methods of **gu2024global<empty citation>**, and the sequence-space Jacobian methods of **bilal2025some<empty citation>**. However, the presence of common noise in our model generate BSHJB equations, which renders these methods insufficient. The resulting BSHJB equations require novel computational strategies, like deep learning methods. **han2018solving<empty citation>** pioneeres a deep learning framework to solve high-dimensional PDEs via the discretized Euler method. Recent work by **huang2023breaking<empty citation>** has adapted a deep learning-based probabilistic approach using the finite volume method to solve CTHA models. Given these computational complexities, this paper focuses on first establishing the core theoretical properties of the economic mechanism at play, thereby

providing a clear and robust foundation to guide future quantitative investigations.

2 Theoretical Framework

In our framework, we model a primary aggregate shock, dW_1, as the innovation driving systematic financial asset returns. This shock can be interpreted as the high-frequency component of fundamental economic shocks (e.g., technology or news shocks) as discussed in the broader macro-finance literature. We posit that financial markets serve as the most direct and instantaneous channel for the transmission of these shocks throughout the economy. In contrast, the labor market responds to this fundamental shock more inertially due to wage-setting behavior and employment contracts. While a richer model could feature multiple fundamental shocks with different time horizons, our more parsimonious setup—which focuses on the financial market as the primary transmission channel—is sufficient to isolate our main mechanism of interest: the pricing of the value function's exposure to aggregate risk.

2.1 Economic Environment

We consider an economy populated by a continuum of infinitely-lived households indexed by $i \in [0, 1]$ with unit mass. Each household i begins with initial wealth $a_{i,0} > 0$ and initial income $y_{i,0} > 0$, giving rise to an initial joint distribution $g(a_0, y_0)$. In the economy, aggregate shocks refer to common shocks that have an impact on the state (a_t, y_t) of every household. The central friction in our framework arises from the fundamental mismatch between different economic entities in terms of their risk characteristics.

To formalize this, we first define a *systematic-risk* asset if it can be traded and its price dynamics fully and instantaneously reflect the primary common shock. Mathematically, its diffusion term is driven solely by the aggregate shock or a linear combination of the aggregate shock. We use S_t to denote the price of systematic-risk asset, and it evolves as

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dW_s, \tag{1}$$

where μ_s and σ_s are the drift and volatility parameters of the systematic-risk asset. dW_s is a standard Wiener process representing the aggregate shock to the systematic-risk asset.

Consequently, the systematic-risk asset are perfect instruments for hedging or speculating on the primary aggregate shock. These objects are typically standardized and traded in highly liquid markets, which facilitates this perfect price transmission. Standard market indices are typical examples. We adopt a partial equilibrium approach

in which these return processes are exogenously determined, allowing us to isolate the risk-exposure mechanism that is central to our analysis.

In contrast, we define an *idiosyncratic-risk* asset if it can be traded but its returns are driven by asset-specific idiosyncratic shocks. This implies that its correlation coefficient with the primary aggregate shock is zero. Let S_t^N be the price of idiosyncratic-risk asset, and it evolves as

$$\frac{dS_t^N}{S_t^N} = \mu_N dt + \sigma_N dZ_{N,t}, \tag{2}$$

where μ_N and σ_N are the drift and volatility parameters of the idiosyncratic-risk asset. $dZ_{N,t}$ is an idiosyncratic Wiener process. To avoid confusion, we will use $W_t \in \mathbf{W}_t$ to denote aggregate shocks and $Z_t \in \mathbf{Z}_t$ to denote idiosyncratic shocks.

We define a *risk-free* asset as an asset if it does not respond to any shocks. Let B_t be the price of risk-free asset, and it follows

$$\frac{dB_t}{B_t} = r_f dt, \tag{3}$$

where r_f is the risk-free rate. Such assets are typically cash or government bonds.

Applying our classification to the labor market, household labor income represents a non-tradable asset with mixed risk characteristics. The aggregate wage rate w_t contains aggregate risk:

$$\frac{dw_t}{w_t} = \mu_y dt + \sigma_y dW_y, \tag{4}$$

where μ_y and σ_y are the drift and volatility parameters of the wage process, and dW_y represents an orthogonal aggregate shock to the labor market.

Individual labor supply $n_{i,t}$ remains purely idiosyncratic and non-tradable:

$$\frac{dn_{i,t}}{n_{i,t}} = \tilde{\mu}_{n_i} dt + \tilde{\sigma}_{n_i} dZ_{n_i,t}, \tag{5}$$

where $\tilde{\mu}_{n_i}$ and $\tilde{\sigma}_{n_i}$ are household-specific parameters and $dZ_{n_i,t}$ is an idiosyncratic shock.

Applying Itô's lemma to the product $y_{i,t} = w_t n_{i,t}$ and noting that wage shocks are independent of idiosyncratic labor supply shocks, household i's labor income $y_{i,t}$ follows

$$\begin{aligned} \frac{dy_{i,t}}{y_{i,t}} &= \frac{dw_t}{w_t} + \frac{dn_{i,t}}{n_{i,t}} \\ &= (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y + \tilde{\sigma}_{n_i} dZ_{n_i,t}. \end{aligned} \tag{6}$$

The feature of this labor income endowment is that its risks are incompletely insurable. While the systematic component tied to dW_s can be partially hedged by taking positions in the systematic-risk asset, the orthogonal aggregate risk dW_y and the idiosyncratic risk $dZ_{n_i,t}$ cannot be hedged through traded assets. It is this incomplete insurability that generates a powerful hedging motive in the household's portfolio problem and makes their optimal choice a non-trivial trade-off.

Let $\mathbb{E}[dW_s dW_y] = \rho_{sy} dt$ denote the instantaneous correlation between the systematic-risk asset and the aggregate wage rate, where $\rho_{sy} \in [-1, 1]$. We decompose shock processes as

$$\begin{aligned} dW_s &= dW_1 \\ dW_y &= \rho_{sy} dW_1 + \sqrt{1 - \rho_{sy}^2} dW_2, \end{aligned} \tag{7}$$

where dW_2 is an orthogonal shock to the primary aggregate shock dW_1. The Cholesky decomposition ensures that dW_s and dW_y maintain their standard Wiener properties. This generates two-dimensional aggregate risk structure $d\mathbf{W}_t \equiv \begin{bmatrix} dW_1 & dW_2 \end{bmatrix}^\top$.

2.2 The Household's Problem

Each household chooses consumption and portfolio allocation to maximize expected utility, facing the incomplete environment described above. Households' investment opportunity set consists of the systematic-risk asset S_t and the risk-free asset B_t. Households face a continuous-time intertemporal budget constraint:

$$da_t = \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} dt + \omega_t a_t \sigma_s dW_s, \tag{8}$$

subject to a borrowing constraint $a_t \geq 0$ and the systematic-risk asset portfolio allocation ω_t. At the borrowing constraint, consumption cannot exceed current labor income: $0 < c_t \leq y_t$.

Households exhibit constant relative risk aversion (CRRA) preferences over consumption, characterized by the instantaneous utility function:

$$U(c_t) := \frac{c_t^{1-\gamma} - 1}{1 - \gamma}, \tag{9}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

Each household i chooses optimal consumption $c_{i,t}$ to maximize expected lifetime utility:

$$\max_{c_{i,t}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{i,t}) \, dt \right], \tag{10}$$

where $\rho > 0$ is the discount rate.

According to our framework with common noise, the household's optimal problem can be formulated as the BSHJB equation $v(a_t, y_t, g_t, \mathbf{W}_t)$:

$$dv_t = \left\{ u(c_t) + \mathcal{L}[v_t] + \mathbb{E}_t \left[\frac{\partial v_t}{\partial t} \right] - \text{div}(\mathcal{V}_t) \right\} dt + \mathcal{V}_t d\mathbf{W}_t, \tag{11}$$

where $\mathbb{E}_t(\cdot)$ is the conditional expectation operator, $\text{div}(\cdot)$ is the divergence operator, $\mathcal{V}_t$ is the adaptive process, and $\mathcal{L}$ is the infinitesimal generator of the underlying state

process:

$$\begin{aligned}\mathcal{L}[v_t] \equiv &\{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i})y_t\frac{\partial v}{\partial y} \\ &+ \frac{1}{2}(\omega_t a_t \sigma_s)^2\frac{\partial^2 v}{\partial a^2} + \frac{1}{2}y_t^2(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)\frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy}\sigma_s\sigma_y\frac{\partial^2 v}{\partial a \partial y}.\end{aligned} \tag{12}$$

In CTHA models with only idiosyncratic shocks $\mathbf{Z}_{i,t}$,the law of large numbers ensures that the value function v_t and the distribution g_t evolve deterministically, eliminating stochastic components at the population level (**achdou2021mean**). The expected evolution of a household's value function v_t is fully described by the HJB operator $\mathcal{L}[v_t]$ and $\mathbb{E}_t\left[\frac{\partial v_t}{\partial t}\right]$.

This deterministic framework breaks down when the common noise $\mathbf{W}_t$ is introduced. It affects all households simultaneously and cannot be eliminated by the law of large numbers. The distribution g_t now becomes a stochastic process, forcing households to condition their optimization on the entire shock history $\mathbf{W}_{\tau\{0\leq\tau\leq t\}}$. The value function must therefore evolve as a random field $v(a_t, y_t, g_t, \mathbf{W}_t)$, requiring the more general BSHJB equation presented in equation (11). The BSHJB equation was proposed by **peng1992stochastic<empty citation>**, and its application in MFG with common noise was proposed by **cardaliaguet2019master<empty citation>**. The well-posedness of our dynamic formulation follows from the Glivenko-Cantelli theorem (**rachev2006mass**), and standard results in mean field theory (**cardaliaguet2019master**).

The transition from HJB equation to BSHJB equation requires households to price in aggregate shocks. We define an *aggregate shocks impact* (ASI) as the stochastic change in the value function caused by the increment of aggregate shocks $d\mathbf{W}_t$ at time t:

$$ASI_t \equiv \mathcal{V}_t d\mathbf{W}_t = \begin{bmatrix}\mathcal{V}_{1,t} & \mathcal{V}_{2,t}\end{bmatrix} d\mathbf{W}_t. \tag{13}$$

It represents the direct blow or boost to the household's future overall welfare prospects due to the macroeconomic innovation. When a positive aggregate shock occurs, the household's value function instantly increases by ASI_t. The magnitude of the shock is determined by an endogenous sensitivity $\mathcal{V}_t$.

In the framework, the sensitivity $\mathcal{V}_t$ must be chosen such that the resulting equation maintains the proper stochastic structure, which is the adaptive process in the BSHJB equation:

$$\begin{aligned}\mathcal{V}_{1,t} =&\omega_t a_t \sigma_s \frac{\partial v}{\partial a} + y_t \sigma_y \rho_{sy}\frac{\partial v}{\partial y} \\ \mathcal{V}_{2,t} =&y_t \sigma_y \sqrt{1-\rho_{sy}^2}\frac{\partial v}{\partial y}.\end{aligned} \tag{14}$$

The process measures how much the entire future utility will be hit when macro common

shocks $d\mathbf{W}_t$ arrive. This represents the continuous value exposure of the household to aggregate risk.

The component $\mathcal{V}_{1,t}$ signifies the repercussions of the primary shock dW_1 on households' value function through the risk-exposure channel. In the context of a wage rate (i.e., idiosyncratic labor income), the impact of this component is exerted exclusively through the proportion of systematic-risk asset held by households. Component $\mathcal{V}_{2,t}$ is indicative of the aggregate shock dW_2 that are unique to the wage market and correspond to the labor endowment.

We define an *aggregate shocks premium* (ASP) as the endogenous price that households must pay to bear the welfare impact of the stochasticity of aggregate shocks $d\mathbf{W}_t$ at time t:

$$\begin{aligned} ASP_t \equiv & -\operatorname{div}(\mathcal{V}_t) = -\frac{\partial \mathcal{V}_{1,t}}{\partial a} - \frac{\partial \mathcal{V}_{2,t}}{\partial y} \\ = & -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1-\rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right). \end{aligned} \tag{15}$$

It refers to the risk cost or premium that must be deducted from the drift term of the value function due to its exposure to aggregate shocks. It measures the cost of "having an aggregate uncertain future" itself. Even if the expected value of the shock is zero ($\mathbb{E}[ASI_t] = 0$), its volatility alone can harm the welfare of a risk-averse household. The ASP quantifies this harm. Its value can explicitly depend on the exposure sensitivity $\mathcal{V}_t$. Consequently, at time t, households price aggregate shocks $\mathbf{W}_t$ as $ASI_t + ASP_t dt$.

The ASI term ensures adaptation to the common noise filtration, while the ASP provides the corresponding drift-correction term. This is mathematically necessary to construct a well-posed dynamic coupled system, and together these two terms ensure that the evolution of the value function and the state space remains martingale in a random field characterized by stochastic evolution. Applying the Itô-Kunita-Wentzell formula (**kunita1990stochastic**) generates the drift-correction term ASP_t. When the common noise affects both the value function v_t and the state space (a, y), their quadratic covariance term generates an additional drift term $\operatorname{div}(\mathcal{V}_t)$. The ASP_t is precisely designed to cancel out this mathematical artifact. This ensures that the resulting BSHJB equation is free of spurious terms and reflects coherent economic trade-offs, making it a mathematical precondition for a well-posed optimization problem in an economy with aggregate uncertainty.

For economic interpretation, we decompose the ASP into components based on whether they are directly controllable through the optimal portfolio choice ω_t. We define the component that households can directly control as the *risk-exposure premium*

(REP):

$$REP_t \equiv -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right). \tag{16}$$

It is the direct source of a novel effect in portfolio choice.

All other components of the ASP, which are not directly scalable by the portfolio choice ω_t, are grouped into the *wage fixed premium* (WFP):

$$WFP_t \equiv -y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1-\rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right). \tag{17}$$

The WFP captures the background risk costs arising from labor income's exposure to both the primary and the orthogonal aggregate shock. This allows us to write the total premium as

$$ASP_t = REP_t + WFP_t. \tag{18}$$

The REP quantifies the marginal utility cost arising from households' risk exposure to the systematic-risk asset under the primary aggregate shock dW_1. Households' choice of portfolio share directly scales this cost, determining the magnitude and sign of the premium. The structure of REP reveals a fundamental asymmetry in how aggregate risk affects households across the wealth distribution. The premium consists of two opposing components: a positive compensation term $a_t \frac{\partial^2 v}{\partial a^2}$ and a negative cost term $\frac{\partial v}{\partial a}$. The compensation component is scaled by the wealth factor a_t, which means low-wealth households receive minimal compensation despite facing high risk exposure through the concavity of their value function $\frac{\partial^2 v}{\partial a^2}$. In contrast, the cost component operates without wealth scaling, making low-wealth households disproportionately sensitive to changes in marginal utility $\frac{\partial v}{\partial a}$. This asymmetric structure creates a systematic disadvantage for households in the low wealth range. While their high marginal utility of wealth makes them bear substantial costs from aggregate risk exposure, the wealth-scaled compensation mechanism provides them with inadequate relief. Consequently, the REP generates heterogeneous risk premiums that depend critically on households' position in the wealth distribution, with the pronounced effects concentrated in the low-to-medium wealth segments.

The stochastic Kolmogorov Forward (SKF) equation governing the evolution of the wealth-income distribution g_t takes the form:

$$dg_t = (\mathcal{L}^* g_t)dt - \mathrm{div}(\mathcal{V}_t g_t) d\mathbf{W}_t, \tag{19}$$

where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}^*[g_t] \equiv &- \left\{ \frac{\partial}{\partial a}\Big(\left((\omega_t(\mu_s - r_f) + r_f)\, a_t + y_t - c_t \right) g_t \Big) + \frac{\partial}{\partial y}\Big((\mu_y + \tilde{\mu}_{n_i}) y_t g_t \Big) \right\} \\ &+ \left\{ \frac{\partial^2}{\partial a^2}\left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) \right\} \\ &+ \frac{\partial^2}{\partial a \partial y}\Big(\omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y g_t \Big).\end{aligned} \tag{20}$$

In a standard CTHA model with only idiosyncratic risk, the value function v_t of household i depends only on its individual state (a_i, y_i). The aggregate distribution g_t evolves deterministically according to the law of large numbers, and individual households take this aggregate distribution as given when making their optimization decisions. The dynamic programming problem contains solely the drift term $\mathcal{L}[v_t]$ and the term $\mathbb{E}_t\left[\frac{\partial v_t}{\partial t}\right]$, which is generated by the evolution of individual states.

However, the introduction of common noise $\mathbf{W}_t$ fundamentally changes the nature of the problem. The aggregate distribution g_t now becomes a stochastic process, and all households' decisions must be made contingent on both their individual state (a, y) and the increment of aggregate shocks $d\mathbf{W}_t$. In the mean field game equilibrium, each household optimizes against the aggregate distribution g_t, recognizing that this distribution itself evolves stochastically due to the common noise affecting all agents simultaneously.

The divergence of $\mathcal{V}_t g_t$ has components:

$$\begin{aligned}\operatorname{div}(\mathcal{V}_{1,t} g_t) =& \frac{\partial}{\partial a}\Big((\omega_t a_t \sigma_s + y_t \sigma_y \rho_{sy}) g_t \Big) \\ \operatorname{div}(\mathcal{V}_{2,t} g_t) =& \frac{\partial}{\partial y}\Big(y_t \sigma_y \sqrt{1 - \rho_{sy}^2}\, g_t \Big).\end{aligned} \tag{21}$$

The density satisfies the normalization $\iint g_t(a, y)\, da\, dy = 1$. It is crucial to distinguish the divergence term $\operatorname{div}(\mathcal{V}_t g_t)$ in the SKF equation, which governs the advection of the density g_t under common shocks, from the drift premium term $\operatorname{div}(\mathcal{V}_t)$ in the BSHJB equation, which adjusts the value function v itself. Consequently, the value function, previously defined as $v(a_t, y_t)$ depending exclusively on the individual state, now takes the form $v(a_t, y_t, g_t, \mathbf{W}_t)$, where g_t represents the aggregate distribution of all households in the economy.

The term $\mathbb{E}_t[\frac{\partial v_t}{\partial t}]$ captures how distributional changes affect individual value functions:

$$\begin{aligned}\mathbb{E}_t\left[\frac{\partial v}{\partial t}\right] &\equiv \left\langle \frac{\delta v}{\delta g}, dg_t \right\rangle + \frac{1}{2}\left\langle \frac{\delta^2 v}{\delta g^2}, d\langle g, g\rangle_t \right\rangle \\ &= \int_{\mathbb{R}^2} \mathcal{L}^* \frac{\delta v}{\delta g(x')} g(x') + \frac{1}{2}\int_{\mathbb{R}^{2\times 2}} \frac{\delta^2 v}{\delta g(x')\delta g(x'')}\Big(\operatorname{div}\big(\mathcal{V} g(x')\big) \cdot \operatorname{div}\big(\mathcal{V} g(x'')\big) \Big),\end{aligned} \tag{22}$$

where $x \equiv (a, y)$ is the state vector, $\langle \cdot, \cdot \rangle$ is the inner product, and $\frac{\delta v}{\delta g}$ is the Fréchet derivative which quantifies the marginal impact of distributional changes on the value function.

We now define macro aggregates as follows:

$$\begin{aligned} K_t &= \iint \omega_i a_{i,t} g_t(a, y)\, da\, dy \\ B_t &= \iint (1 - \omega_{i,t}) a_{i,t} g_t(a, y)\, da\, dy, \end{aligned} \tag{23}$$

where the economy has initial macro variables K_0 and B_0. These conditions ensure that all financial assets are held by households.

Although our model is a partial equilibrium model in which households are price takers, the evolutionary dynamics of the distribution determine the dynamics of the household's value function in this coupled system. Thus, each household needs to estimate the distribution measure.

3 Equilibrium

In the full rational expectations paradigm, the household's value function v_t is a McKean-Vlasov equation that depends on the entire cross-sectional distribution g_t. Solving the optimization problem would require each household to forecast the evolution of this infinite-dimensional object, a computationally and informationally intractable task that exemplifies an extreme form of the curse of dimensionality (**krusell1998income**; **dend1996heterogeneity**; **fernandez2024taming**). While theoretically rigorous, this modeling approach also strains credulity. In the real world, agents face "limited cognitive resources" (**enke2024cognitive**) and "limited information processing capabilities" (**javed2024big**), and the predictive heuristics they employ are more feasible.

To overcome these challenges, we assume that households simplify their forecasting objective from a full distributional measure to a key variable for moment estimation, namely, total capital K_t:

$$\begin{aligned} \mathbb{E}_t\left[\frac{\partial v}{\partial t}\right] \approx& \left\langle \frac{\partial v}{\partial \hat{K}_i}, d\hat{K}_t \right\rangle + \frac{1}{2}\left\langle \frac{\partial^2 v}{\partial \hat{K}_i^2}, d\langle \hat{K}_i, \hat{K}_i \rangle_t \right\rangle \\ \approx& \hat{\mu}_{k_i} \frac{\partial v}{\partial \hat{K}_i} + \frac{1}{2}\hat{\sigma}_{k_i}^2 \frac{\partial^2 v}{\partial \hat{K}_i^2}. \end{aligned} \tag{24}$$

The product of this forecasting behavior is a perceived law of motion (PLM). For analytical tractability and to establish a clear benchmark, we assume a consensus belief where each household's expectation for the evolution of aggregate capital. We define

the *perceived law of motion* as households form beliefs about the dynamics of K_t:

$$d\hat{K}_{i,t} \equiv \hat{\mu}_{k_i}(K_t, B_t)dt + \hat{\sigma}_{k_i}(x_{i,t})d\mathbf{Z}_{k_i,t}, \tag{25}$$

where $x_{i,t}$ is the individual i's state and $d\mathbf{Z}_{k_i,t}$ is idiosyncratic shocks of heterogeneous beliefs. Here, we follow **grossman1982consumption<empty citation>** and assume costless information, precluding idiosyncratic shocks ($\hat{\sigma}_{k_i} = 0$). The broader literature on heterogeneous beliefs includes studies by **shiller1984stock**; **de1990noise**; **anderson2005heterogeneous**; **jouini2007consensus**; **bhamra2014asset**; **andrade2019forward**; **borovivcka2020survival**; **bigio2025heterogeneous<empty citation>**.

The capital K_t is not only the result of aggregating all households' non-linear decision functions (c^*, ω^*) but also the primary macro-quantity of interest to firms in a recursive competitive equilibrium. These firms have a linear-quadratic (LQ) objective function, determining their output level and capital investment by solving the Bellman equation. Under this market structure, firms pay attention to the drift terms of the future paths of these variables. In the LQ learning framework, a conclusion is the establishment of the certainty equivalence principle, which states that the stochastic optimal decision rule is a deterministic problem independent of stochastic volatility (**kwakernaak1972linear**; **chow1981econometric**). Therefore, the producer decisions that ultimately determine the overall dynamics of the capital stock K_t depend solely on the conditional expectations of the relevant variables, not their variances or higher-order moments. The recursive equilibrium was pioneered by **lucas1971investment**; **stokey1989recursive**; **prescott2005recursive<empty citation>** and further developed in works such as **sargent2009dynamic**; **ljungqvist2018recursive<empty citation>**. This perspective provides a reasonable motivation for focusing the household forecasting problem on the dynamics of K_t.

This simplification of the forecasting object from g_t to K_t does not imply that the forecasting task itself becomes simple. The actual law of motion (ALM) for aggregate capital, dK_t, is the result of aggregating the non-linear decisions of all households. As established in our model, the policy function $\{c^*, \omega^*\}$ is a non-linear functional of the household's BSHJB equation. According to Jensen's inequality, the non-linear functional means that the drift and diffusion of dK_t depend on the entire shape of the distribution g_t, including all higher-order moments. Consequently, the K_t is a non-linear function of the entire history of aggregate shocks $\mathbf{W}_{\tau\{0\leq\tau\leq t\}}$. This implies that the current level of capital is not a sufficient statistic for its future dynamics. The ALM would require agents to implement a sophisticated non-linear filter to track the evolution of the unobserved higher moments of the distribution based on the shock history. This task remains informationally and computationally intractable for a household within the model.

It is worth emphasizing that this simplification from the infinite-dimensional distribution g_t to its first moment K_t does not compromise the framework's central conclusions. One might argue that households could form subjective beliefs about higher-order moments, such as the variance of the wealth distribution, or that arbitrary shocks to individuals might alter the aggregate dynamics. However, any such belief heterogeneity, conditional expectations, or individual shocks remain idiosyncratic from the household's optimization perspective. This subjective uncertainty does not introduce new stochastic terms into the BSHJB equation that respond to common noise, but instead enters as deterministic drift-adjustment terms. Here, the term arising from its variance is analogous to additional precautionary savings motives. While incorporating such effects might quantitatively change households' optimal behavior, they cannot eliminate the wealth-dependent REP mechanism that drives our core results on limited participation and wealth inequality, as this mechanism stems fundamentally from the law of large numbers that makes idiosyncratic shocks not cause random shocks to the aggregate distribution g_t.

Our approach aligns with the bounded rationality framework inspired by the learning literature (**evans2012learning**), the approximate aggregation method of **krusell1998income<empty citation>**, and the sparsity-based model (**gabaix2014sparsity**; **gabaix2020behavioral**). This contrasts with recent work by (**moll2024trouble**; **moll2025mean**), which explores how households can use price systems as a mechanism to forecast the moments of the distribution.

This bounded rationality assumption naturally leads to a temporary equilibrium concept, bridging the macroeconomic literature on limited information processing (**mackowiak2009optimal**; **mackowiak2015business**) with microeconomic evidence on wealth-dependent expectation formation (**malmendier2016learning**).

Definition 1. A *temporary equilibrium* consists of given asset prices $\{S_t, B_t\}$, labor income y_t, and the distribution g_t such that:

1. Each household i chooses consumption-portfolio choice $\{c_{i,t}, \omega_{i,t}\}$ to maximize its value function $v_{i,t}$ (26) by its status $x_{i,t}$;

2. The distribution g_t evolves according to the SKF equation (27);

3. The PLM and macro aggregates in the market are given by (25) and (23), respectively;

4. Asset prices and labor income evolve as (1), (3), (6), respectively.

The temporary equilibrium is characterized by the coupled system:

$$
\begin{aligned}
dv_t =& \max_{c,\omega} \left\{ u(c_t) + \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} + \hat{\mu}_k \frac{\partial v}{\partial \hat{K}} \right\} dt \\
&+ \left\{ \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} \right\} dt \\
&- \left\{ \omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) + y_t \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sigma_y \sqrt{1 - \rho_{sy}^2} \left(y_t \frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial y} \right) \right\} dt \\
&+ \left[\omega_t a_t \sigma_s \frac{\partial v}{\partial a} + y_t \sigma_y \rho_{sy} \frac{\partial v}{\partial y} \quad y_t \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial v}{\partial y} \right] d\mathbf{W}_t
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
dg_t =& - \left\{ \frac{\partial}{\partial a} \left(\left((\omega_t(\mu_s - r_f) + r_f) a_t + y_t - c_t \right) g_t \right) + \frac{\partial}{\partial y} \left((\mu_y + \tilde{\mu}_{n_i}) y_t g_t \right) \right\} dt \\
&+ \left\{ \frac{\partial^2}{\partial a^2} \left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) + \frac{\partial^2}{\partial a \partial y} \left(\omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y g_t \right) \right\} dt \\
&- \left[\frac{\partial}{\partial a} \left(\omega_t a_t \sigma_s g_t \right) + \frac{\partial}{\partial a} \left(y_t \sigma_y \rho_{sy} g_t \right) \quad \frac{\partial}{\partial y} \left(y_t \sigma_y \sqrt{1 - \rho_{sy}^2} g_t \right) \right] d\mathbf{W}_t.
\end{aligned}
\tag{27}
$$

The temporary equilibrium is a broad concept that allows for discrepancies between the PLM and the ALM. This is consistent with the original intention of this paper's modeling based on bounded rationality. In the special case where households' expectations are completely consistent with reality ($dK = d\hat{K}$), the temporary equilibrium converges to a "self-confirming equilibrium" (**fudenberg1993self**; **kamada2010strongly**). However, our core mechanism does not rely on this strict equilibrium condition and remains a key force in determining household behavior in more general temporary equilibrium states. The concept of the temporary equilibrium originally developed by **hicks1946value**; **lindahl1939theory<empty citation>** and formalized by **grandmont1977temporary**; **grandmont1989temporary<empty citation>**. Recent application in modern macroeconomic models include **woodford2013macroeconomic**; **piazzesi2016housing**; **garcia2019low**; **farhi2019monetary<empty citation>**.

We establish our analysis around two complementary steady-state concepts that provide analytical benchmarks for the dynamic system.

Definition 2. A *local steady state* is a steady state of the temporary equilibrium 1 containing the following features:

1. A moment of stochastic calm, where the instantaneous increment of aggregate shocks are zero ($d\mathbf{W}_t = 0$);

2. Persistent underlying aggregate shocks ($\sigma(\mathbf{W}_t) > 0$);

3. The satisfaction of the standard transversality condition:

$$\lim_{T\to\infty} \mathbb{E}_t \left[e^{-\rho(T-t)} v_t \right] = 0. \tag{28}$$

The LSS is characterized by the following system:

$$\rho v_t = \max_{c,\omega} \left\{ u(c) + \mathcal{L}[v_t] + \mathbb{E}_t \left[\frac{\partial v}{\partial t} \right] + ASP_t \right\}. \tag{29}$$

$$\frac{\partial g}{\partial t} = \mathcal{L}^*[g_t]. \tag{30}$$

Definition 3. A *determined steady state* (DSS) is a fixed point of the equilibrium conditions when the volatility of aggregate shocks vanishes ($\sigma(\mathbf{W}_t) = 0$).

The DSS is characterized by the following system:

$$\rho v_t = \max_{c,\omega} u(c_t) + \{r_f a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} + \frac{1}{2} y_t^2 \tilde{\sigma}_{n_i}^2 \frac{\partial^2 v}{\partial y^2} \tag{31}$$

$$\frac{\partial g_t}{\partial t} = -\frac{\partial}{\partial a}\Big((r_f a_t + y_t - c_t) g_t \Big) - \frac{\partial}{\partial y}\Big((\mu_y + \tilde{\mu}_{n_i}) y_t g_t \Big) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2} y_t^2 \tilde{\sigma}_{n_i}^2 g_t \right). \tag{32}$$

The LSS characterizes a continuously occurring instantaneous fixed points in the economy. It is a short-term state in which the economy continues to move along inertia after receiving a shock until the next moment when the new shock increment changes. This allows us to construct a tractable and economically meaningful terminal value function for the finite-time approximation of the household problem.

The LSS represents a different terminal condition from the DSS. While the DSS describes a hypothetical world where aggregate risk is permanently absent, assuming the aggregate shock volatility is identically zero across all time ($\sigma(\mathbf{W}_t) = 0$). This effectively reduces the stochastic model to its deterministic counterpart, providing a baseline where households face no aggregate uncertainty and consequently hold only risk-free assets ($\omega_t = 0$), as there is no asset with a risk premium. In contrast, the LSS characterizes moments of calm within an inherently stochastic economy where aggregate uncertainty persists ($\sigma(\mathbf{W}_t) > 0$), yet no new shock instantaneously materializes ($d\mathbf{W}_t = 0$). This crucial distinction means that households in the LSS make decisions while fully aware of ongoing aggregate risk and its potential future realization, leading them to demand risk premia and hold diversified portfolios. Therefore, while the DSS provides a static, non-stochastic anchor, the LSS offers a instantaneous dynamics of state within a continuously uncertain environment.

Our definition of the LSS significantly generalizes and refines existing "stochastic steady state" approaches, such as those by **futia1982invariant**; **duffie1994stationary**; **schenk2001random**; **coeurdacier2011risky**; **fernandez2023financial**; **bilal2023solving<empty**

Prior analyses often relied on the highly restrictive assumption that the agent expects the realization of cumulative shock process to be zero ($\mathbb{E}_t[\mathbf{W}_{t+1}] = 0$). This strong assumption confines the analysis to a zero-probability path where the entire history of shocks must perfectly cancel out to achieve a deterministic long-term fixed point and ergodic distributions. Our framework permits a far more powerful and generalizable approach. We define the LSS by the much weaker condition that only the instantaneous increment of the shock is zero ($d\mathbf{W}_t = 0$), while the cumulative shock $\mathbf{W}_t$ can be any arbitrary constant reflecting the realized history. For the household's optimization, the instantaneous stochastic welfare impact $ASI_t = \mathcal{V}_t d\mathbf{W}_t$ is nullified by $d\mathbf{W}_t = 0$ alone. Meanwhile, the cost of persistent uncertainty ASP_t remains fully active. Consequently, the stationary equation defining the LSS value function v_L differs fundamentally from the HJB equation in the DSS, as it retains the ASP. It drives the risk-exposure premium effect central to our analysis. By anchoring our finite-horizon BSHJB equation with the LSS terminal condition, we ensure that household optimization paths correctly internalize the pricing of aggregate risk, maintaining theoretical coherence even over truncated time horizons. This specification allows our framework to capture both the transitional dynamics toward long-term equilibrium and the steady-state properties of an economy with permanent aggregate risk.

4 Theoretical Results

We now derive the optimal consumption and portfolio rules from the household's problem (26). Our analysis focuses on how the endogenous pricing of aggregate shocks gives rise to the REP effect and shapes portfolio choice across the wealth distribution.

The household's optimization problem yields the envelope condition for optimal consumption:

$$c_t^* = (u')^{-1}\left(\frac{\partial v}{\partial a}\right). \tag{33}$$

However, when households have access to a set of feasible assets that includes the systematic-risk asset and the risk-free asset, the optimal portfolio rule changes fundamentally. Taking the first-order condition (FOC) with respect to the portfolio share ω_t, we obtain

$$\omega_t^* = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left\{\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)\frac{\partial v}{\partial a} + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)y_t\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_s}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right\}. \tag{34}$$

The third term in the brackets of the above equation is the REP effect. To better analyze and explain the theoretical mechanism, we formalize this novel term and introduce the concept of effective relative risk aversion.

Definition 4. The *risk-exposure premium effect* is the component of a household's optimal portfolio share in the systematic-risk asset, given by $\Omega^*(a_t)$, which directly arises from the risk-exposure premium (16) within their optimization problem when choosing among a set of feasible investments including the systematic-risk asset and the risk-free asset:

$$\Omega^*(a_t) \equiv \frac{1}{a_t \sigma_s} \left[1 + \frac{1}{a_t} \left(\frac{\partial^2 v}{\partial a^2} \right)^{-1} \frac{\partial v}{\partial a} \right]. \tag{35}$$

Definition 5. The *effective relative risk aversion* (ERRA) captures the curvature-to-slope ratio of the BSHJB equation to wealth:

$$\eta(a_t) \equiv -a_t \frac{v_{aa}}{v_a}, \tag{36}$$

where $v_a \equiv \partial v / \partial a$ and $v_{aa} \equiv \partial^2 v / \partial a^2$.

The ERRA captures households' intertemporal risk preferences across assets, as reflected in their lifetime welfare function. This concept moves beyond the time-invariant risk aversion parameter γ to consumption. It endogenously evolves as pricing aggregate shocks $ASI_t + ASP_t dt$. Given the standard conditions ($a > 0$, $v_a > 0$, and $v_{aa} < 0$), we have $\eta(a_t) > 0$. The form of the ERRA is similar to the Arrow-Pratt measure of relative risk aversion, which is defined as $R(c_t) \equiv -c_t u''/u'$. However, $\eta(a_t)$ measures intertemporal risk preferences, while $R(c_t)$ measures instantaneous risk preferences. The earliest derivation of the ERRA form comes from **merton1969lifetime<empty citation>**.

By substituting the ERRA into the REP (16), the REP effect (35), and the optimal portfolio rule (34), we obtain

$$REP_t = \omega_t \sigma_s \big(\eta_t - 1 \big) \frac{\partial v}{\partial a}, \tag{37}$$

$$\Omega^*(a_t) = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t} \right), \tag{38}$$

and

$$\omega_t^* = \frac{1}{\eta_t} \Bigg[\underbrace{\frac{\mu_s - r_f}{\sigma_s^2}}_{\text{myopic demand}} + \underbrace{\frac{\rho_{sy} \sigma_y}{\sigma_s} y_t \left(\frac{\partial v}{\partial a} \right)^{-1} \frac{\partial^2 v}{\partial a \partial y}}_{\text{wage-offsetting effect}} + \underbrace{\frac{1}{a_t \sigma_s} \big(\eta_t - 1 \big)}_{\text{REP effect}} \Bigg]. \tag{39}$$

The optimal portfolio rule (39) decomposes household decisions into three different economic forces. The first component is often called the "myopic demand" in the literature. This demand considers an asset based on its risk premium and variance, a feature common to any risky investment. We define the second term as a *wage-offsetting effect* to distinguish idiosyncratic shocks to labor productivity $n_{i,t}$ from income $y_{i,t}$, as it reflects the impact of the primary aggregate shock dW_1 on the aggregate wage rate w_t. The final component, the REP effect, is the focus of this paper. This new

force arises from the REP that households pay for their lifetime welfare exposure to aggregate shocks in the macroeconomy. For the idiosyncratic-risk asset, by definition, they are uncorrelated with aggregate shocks. Only the myopic demand is active. For the systematic-risk asset, they are priced according to aggregate shocks, which both triggers the REP effect and the wage-offsetting effect. We call these two effects *the macro effect* in the portfolio, which characterize aggregate risk. The first and second terms were pioneered by **merton1969lifetime**; **merton1973intertemporal<empty citation>**. Subsequently, **merton1992continuous<empty citation>** provides a deep analysis in portfolio choice and **campbell2002strategic<empty citation>** is an important literature for intertemporal portfolio choice.

4.1 ERRA and Portfolio Allocation at Both Ends

To understand the wealth-dependent nature of portfolio choice, we link the ERRA to observable economic behavior. Deriving the relationship with the marginal propensity to consume out of wealth allow us to obtain portfolio allocation at the borrowing constraint.

We exploit the envelope condition (33) and derive:

$$v_{aa} = u''(c_t^*)\frac{\partial c_t^*}{\partial a}, \tag{40}$$

and utilize the standard CRRA specification $\gamma = -c_t^* u''/u'$. Substituting equations (40) and γ into (36) yields: [1]

$$\eta(a_t) = \frac{a_t}{c_t^*} \cdot \mathrm{MPC}_a \cdot \gamma, \tag{41}$$

where $\mathrm{MPC}_a \equiv \partial c_t^*/\partial a$ is the marginal propensity to consume out of wealth.

This reveals that the ERRA transforms a household's invariant risk preference γ into time-varying risk-taking behavior at different wealth levels through a state-varying the elasticity of consumption with respect to wealth $\epsilon_{c,a} \equiv (a_t/c_t^*) \cdot \mathrm{MPC}_a$. This form is originally derived from **merton1969lifetime**; **merton1973intertemporal<empty citation>**.

4.1.1 ERRA and Portfolio Choice Near the Borrowing Constraint

As households approach the borrowing constraint $\underline{a}$, their optimal consumption is determined primarily by labor income rather than financial wealth, implying $a_t/c_t^* \to 0$ as $a \to \underline{a}$.

Proposition 1 (**ERRA at the Borrowing Constraint**)**.** For any household with finite constant relative risk aversion ($\gamma < \infty$) and bounded marginal propensity to

[1] $\eta(a_t) = -a_t \cdot \frac{v_{aa}}{u'(c_t^*)} = -a_t \cdot \frac{\partial c_t^*}{\partial a} \cdot \frac{u''(c_t^*)}{u'(c_t^*)} = \frac{a_t}{c_t^*} \cdot \mathrm{MPC}_a \cdot \gamma.$

consume out of wealth ($\text{MPC}_a < \infty$), the effective relative risk aversion vanishes at the borrowing constraint:

$$\lim_{a \to 0^+} \eta(a_t) = 0. \tag{42}$$

Proof. See Appendix A.1. □

The proposition implies that households' risk aversion to wealth becomes relatively negligible compared to their marginal returns near the borrowing constraint. The economic intuition is that the poorest households have a strong incentive to pursue excess returns on assets to escape poverty.

Corollary 1 (**Portfolio Choice at the Borrowing Constraint**)**.** The risk-exposure premium effect exhibits singular behavior at the borrowing constraint:

$$\lim_{a \to 0^+} \Omega^*(a_t) = -\infty, \tag{43}$$

leading to optimal portfolio choice:

$$\lim_{a \to 0^+} \omega_t^* = \begin{cases} 0 & \text{under short-selling restrictions} \\ -\infty & \text{without short-selling restrictions.} \end{cases} \tag{44}$$

Proof. See Appendix A.2. □

This corollary reveals a mechanism that endogenously drives limited participation due to the fact that these incentives differ in magnitude at the borrowing constraint. As $\eta_t \to 0$, both myopic demand and wage-offsetting effect diverges to infinity by the factor $\frac{1}{\eta_t}$, that is order of $O(n)$. This reflects the powerful incentive for a poor household to ignore asset risk in pursuit of asset returns. The structure of the REP effect is different. As $a_t \to 0$, this term diverges to negative infinity at the order of $O(n^2)$. Due to the presence of the scaling factor $\frac{1}{a_t}$, the REP effect is an order of magnitude larger than the other two forces combined. As $a_t \to 0$, the negative force from the REP effect becomes infinitely more powerful than the incentives from the myopic demand and the wage-offsetting effect.

Consequently, the household's optimal decision is completely dominated by the overwhelming incentive to pay the exposure cost under the aggregate risk at the borrowing constraint. They strategically avoid the systematic-risk asset arising from their optimization problem, correctly pricing the long-run welfare cost associated with it, a cost that becomes infinite at the borrowing constraint.

4.1.2 ERRA and Risky Portfolio Share at High Wealth Levels

As wealth grows arbitrarily large, the limiting case provides a benchmark for understanding the behavior of the most affluent households. In this regime, the influence of labor income diminishes, and asset dynamics become paramount. Consistent with a long line of research, consumption and wealth become proportional at the limit.

Theorem 1 (**Asymptotic Consumption-Wealth Ratio**). Let $c^*(a_t, y_t, \hat{K}_t)$ be the optimal consumption policy derived from the solution to the BSHJB equation (26). The consumption-to-wealth ratio converges in the mean-square sense to a deterministic constant $\bar{k}_L$ as wealth a_t goes to infinity:

$$\lim_{a\to\infty} \mathbb{E}_t \left[\left| \frac{c^*(a_t, y_t, \hat{K}_t)}{a_t} - \bar{k}_L \right|^2 \right] = 0, \tag{45}$$

where $\bar{k}_L$ is the optimal consumption-wealth ratio in the LSS without labor income or aggregate state dependence, given by:

$$\bar{k}_L = \frac{1}{\gamma} \left(\rho + (\gamma - 1) \left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2} \right) \right). \tag{46}$$

Proof. See Appendix B.1 and B.2. □

The theorem 1 is consistent with the result of **merton1969lifetime**; **merton1973intertemporal<empty citation>**. It aligns with modern analyses (**benhabib2015wealth**; **achdou2022income**), which show that the policy function becomes asymptotically linear for large wealth levels.

Proposition 2 (**ERRA at High Wealth Levels**). For households with positive and finite CRRA coefficient ($0 < \gamma < \infty$) and satisfying Theorem 1 assumption, the effective relative risk aversion converges to the CRRA parameter:

$$\lim_{a\to\infty} \eta(a_t) = \gamma. \tag{47}$$

Proof. See Appendix B.3. □

The proposition implies that the wealth intertemporal risk preferences to wealth behave in the same way as their instantaneous risk preferences to consumption. This is because these households' consumption depends entirely on asset returns.

Corollary 2 (**Portfolio Choice at High Wealth Levels**). As wealth increases without bound, the risk-exposure premium effect vanishes, and portfolio allocation converges to the classical baseline ratio:

$$\lim_{a\to\infty} \Omega^*(a_t) = 0, \qquad \lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2} \frac{1}{\gamma}. \tag{48}$$

Proof. See Appendix B.4. □

The convergence to the classical Merton ratio in Corollary 2 is dictated by the asymptotic behavior of the three economic forces from equation (39). In the high-wealth limit, their relative magnitudes invert. As $a_t \to \infty$, η_t converges asymptotically to γ, the myopic demand for risky assets is determined by a time-invariant risk aversion parameter. The myopic demand, scaled by $\frac{1}{\eta_t}$, converges to a non-zero constant of order $O(1)$. This force remains robustly active. The wage-offsetting effect vanishes as the present value of labor income becomes negligible relative to financial wealth. The REP effect also vanishes due to the scaling factor $\frac{1}{a_t}$. Both the macro effect converge to zero at the order of $O(\frac{1}{n})$. Consequently, the household's optimal portfolio allocation converges to the myopic component, which is precisely the standard Merton benchmark. That is the portfolio share of the richest households is entirely determined by their instantaneous return function $u(c_t)$ to satisfy their consumption needs.

This asymptotic analysis also reveals a subtle dynamic. For a significant range of high (but not infinite) wealth, the REP effect $\Omega^*(a_t)$ is positive because the underlying trade-off inverts, offering a net welfare benefit for bearing aggregate risk. This pushes the household's optimal allocation ω_t^* to be above the Merton benchmark. The convergence described in Corollary 2 is therefore a convergence from above, painting a complete picture of the hump-shaped profile's right tail.

The dynamics of wealth inequality are therefore driven by fundamentally different forces across the distribution. The most significant portfolio heterogeneity driven by the REP mechanism is concentrated among low-to-moderate wealth households. For the wealthiest households, their behavior robustly converges back to a representative-agent benchmark which is a continuous-time formulation of the Phelps-Ramsey problem (**phelps1962accumulation**).

4.2 Limited Participation and Life-Cycle Pattern

We now apply the REP mechanism to provide a unified resolution to two long-standing puzzles in household finance: limited participation and the life-cycle profile of risky portfolio share.

The evolution of the REP effect can be characterized by three distinct phases. In the low-wealth region, as households approach the borrowing constraint ($a_t \to 0$), their ERRA falls toward zero ($\eta_t \to 0$). With the simpler form (39), the factor $\frac{1}{\eta_t}$ of the risky portfolio share increases the myopic demand. This causes these households to hold a large amount of risky assets with excess returns. However, the common factor $\frac{1}{\eta_t}$ also amplifies the REP effect. The scaling factor $\frac{1}{a_t}$ amplifies the term $(\eta_t - 1)$,

causing the REP effect to diverge to negative infinity. This force dominates the other two portfolio motives and drives the systematic-risk asset share to negative infinity. For the household's lifetime welfare, the exposure cost of aggregate risk, driven by the household's extremely high marginal utility of wealth v_a, becomes immense. Conversely, the exposure compensation for bearing aggregate risk, which is proportional to $-a_t v_{aa}$, is nullified by their near-zero financial wealth ($a_t \to 0$). With such a high cost and virtually no offsetting compensation, the REP manifests as a large negative premium. This immense net cost creates an overwhelming motive, which translates into a strongly negative REP effect. At the critical threshold where $\eta_t = 1$, the cost is perfectly offset by the compensation in the REP. This causes the REP effect to vanish completely $\Omega_t^* = 0$. This represents a pivotal wealth level at which household risk preferences reach a critical balance and portfolio allocation is determined by the myopic demand and the wage-offsetting effect. In the medium-to-high wealth region where $\eta_t > 1$, the trade-off inverts. With a higher wealth level, the exposure compensation begins to dominate the diminished exposure cost (lower v_a) for bearing aggregate risk. Thus, the REP transforms into a net benefit, generating a positive REP effect, $\Omega_t^* > 0$. Initially, as wealth increases beyond the critical point, the sensitivity to marginal utility and the curvature of risk aversion remain substantial. These factors create rich interactions that drive the positive REP effect to its peak. This provides a strong additional incentive to hold the systematic-risk asset beyond standard portfolio motives alone. As wealth continues to grow ($a_t \to \infty$), the scaling factor $\frac{1}{a_t}$ becomes dominant. Despite the persistently positive sign of $(1 - \frac{1}{\eta_t})$, the diminishing wealth factor causes the REP effect to gradually decline and ultimately converge to zero from above.

The complete evolutionary pattern generates the hump-shaped profile of the REP effect across the wealth distribution. This non-monotonic relationship in the REP effect alters monotonic predictions of standard portfolio theory. It provides a novel mechanism for understanding the limited participation as well as the "hump-shaped saving behavior" (**graaff1950mr**; **ando1963life**) and the hump-shaped profile of portfolio allocation throughout the life cycle. However, a question remains: How does this new mechanism interact with the traditional explanations of labor income channel in intertemporal portfolio choice?

A foundational principle of our analysis is that non-tradable labor income risk cannot be evaluated in isolation from its expected return. Just as a risk-averse investor would only hold a financial asset if it offers a positive risk premium, a household will only choose to participate in the labor force if doing so provides a net positive impact on lifetime welfare. Therefore, we treat uninsurable income not as a simple zero-mean background risk, but as a non-tradable asset that en-

hances the household's total resources. This is similar to the point proposed by **heaton2000portfolio<empty citation>** that although labour income is risky, it limits bad outcomes relative to investment income. The presence of this welfare-enhancing asset alters the household's attitude toward tradable financial risks. Following the "proper property" **pratt1987proper**; **gollier1996risk<empty citation>**, the absolute risk aversion $-u''/u'$ is monotonically decreasing in wealth a. We measure this effect through an *effective absolute risk aversion*, $A(a) \equiv -v_{aa}/v_a$. As established by **carroll1996concavity<empty citation>**, the concavity of the value function depends on the concavity of the consumption function, resulting in $v_{aaa}v_a - v_{aa}^2 > 0$. This result is a sufficient condition that $A(a)$ is monotonically decreasing in wealth. Consequently, the presence of the labor income asset makes the household behave as if it possesses greater wealth, leading to lower $A(a)$ for any given wealth level. This directly reduces the household's ERRA relevant for its portfolio decision [2]. The factor $\frac{1}{\eta_t}$ in the equation (39) then dictates that the household will increase its allocation to the idiosyncratic-risk asset, whose demand is purely myopic. However, the impact on the systematic-risk asset share is more complex, as the reduction in ERRA simultaneously affects all three components of the optimal choice. On one hand, the lower ERRA amplifies the myopic demand, creating a powerful incentive to increase holdings. On the other hand, this is counteracted by changes in the macro effect. First, the REP effect, which is proportional to $(\eta_t - 1)$, unambiguously decreases as η_t falls. Second, the impact of the wage-offsetting effect is contingent on the empirically uncertain sign of the correlation ρ_{sy}. Consequently, the net effect of labor risk on the systematic-risk asset share is theoretically indeterminate without further parameterization. We defer a more detailed discussion of the macro effect to Section 4.4.

Limited participation and lifecycle pattern in household finance are often addressed through the labor income channel of the classical portfolio choice. To reconcile theory with this evidence, some literature calibrates income processes using microdata (**jagannathan1996should**; **davis2000occupation**; **viceira2001optimal**; **haliassos2003portfolio**; **cocco2005consumption**; **benzoni2007portfolio<empty citation>**). However, this conventional channel is rife with internal paradoxes. The presence of idiosyncratic labor risk exacerbates limited participation puzzle, as well as the aggregate labor risk creates new empirical contradictions. First, as above analyzed, the labor income increases household welfare and results in greater myopic demand. Predicting that low-wealth households should become more aggressive investors, this channel exacerbates limited participation puzzle. Second, we consider the wage-offsetting effect

[2] $\eta_t' = a_t A(a+\Delta a) = -a_t \frac{v_{aa}(a+\Delta a)}{v_a(a+\Delta a)} < -a_t \frac{v_{aa}(a)}{v_a(a)} = \eta_t$, where $\Delta a > 0$ represents the welfare-equivalent of the labor income asset.

driven by the correlation ρ_{sy} between risk assets returns and the aggregate wage rate. This channel is similarly problematic. The effect is relatively small arising from the cross-effect as the numerator and marginal wealth as the denominator. Therefore, we typically require a large value of ρ_{sy} to make the wage-offsetting effect significant. Furthermore, it is more sensitive to the additional risk of low-wealth households. When $\rho_{sy} < 0$, the wage-offsetting effect increases risk assets holdings of all households. Low-wealth households experience a more pronounced increase, which further exacerbates limited participation. When $\rho_{sy} > 0$, the wage-offsetting effect suppresses the risk-sharing share of working households but does not affect retired households. Its impact is not only limited at low wealth levels, but it also creates a new paradox. Retired households who are unaffected by the wage-offsetting effect should hold a larger share of risk assets than working households. This contradicts the observed hump-shaped profile. The fragility of this channel is further underscored by the well-documented instability of the correlation ρ_{sy} (**guiso1996income**; **bertaut1997precautionary**; **heaton2000portfolio**; **poterba2001demographic**; **ameriks2011household**), with empirical estimates varying across countries and periods. Some studies have found values close to zero or even positive (**campbell2001investing**; **angerer2009income**; **betermier2012hedging**; **huggett2016large**).

Classical portfolio theory suffers from an inherent logical contradiction. It states that when labor income is positively correlated with a risky asset, households will hedge their income risk by reducing or even shorting that asset. By taking long or short positions in risky assets, we construct an asset set that is positively correlated with labor income ($\rho_{sy} > 0$). The risk premium on this asset set is small enough that the myopic demand is less than the absolute value of the hedging demand. As wealth decreases, the incentive to hedge becomes stronger relative to the myopic demand. Theoretically, this would predict that the poorest households would have the strongest incentive to short-sell the set to hedge their labor income. This is economically absurd. In reality, we observe that the poor choose not to participate in the market instead of become aggressive short-sellers to hedge their labor income. To reconcile these theoretical predictions with economic intuition, theorists implicitly assume that the correlation between labor income and asset returns must be close to zero. Instead, they use the presence of idiosyncratic income risk to explain this phenomenon. This contradiction lies at the heart of Merton's framework: it requires the non-existence of the labor hedging term to satisfy economic logic, yet this term simultaneously exists in its mathematical expression. Within the classical framework itself, a reasonable exception is noted at the limit of infinite assets, where the labor hedging term's influence becomes negligible.

The fundamental reason why the labor income mechanism cannot solve these puzzles

is that the effects of idiosyncratic risk and aggregate risk are monotonic in wealth. In contrast to the contradictions and limitations of classical theory, the REP mechanism provides an economically consistent framework. First, the REP mechanism accurately captures the "hump-shaped" portfolio structure across the wealth distribution by relying solely on the dynamics of wealth, an empirical fact that classical theory struggles to explain. This is because the REP effect is non-monotonic with respect to wealth. Due to the endogenous scaling factor $\frac{1}{a}$, the REP effect exhibits distinct patterns across different wealth ranges. As household wealth accumulates wealth over their lifetimes, supported by precautionary savings (see Proposition 5), they naturally traverse wealth zones where the REP effect first inhibits, then encourages, and ultimately normalizes risky asset holdings. This culminates in the portfolio choice asymptotically converging to the classical Merton benchmark. Second, the REP mechanism provides a theoretical framework that is independent of traditional labor hedging channels to explain limited participation. For the poor, the exposure cost of aggregate risk far outweigh the benefits of risk premiums. The powerful incentive to avoid this risk, stemming from macroeconomic uncertainty and far exceeding the myopic demand, squeezes low-wealth households out of the market. Recent studies on limited participation include **van2011financial**; **athreya2023stock**; **bharath2023natural**; **melcangi2025stock<empty citation>**, and life-cycle portfolio choice include **hubener2016family**; **addoum2017household**; **fagereng2017asset**; **michaelides2017stock**; **chang2018labor**; **gomes2020portfolio**; **catherine2022countercyclical**; **catherine2024countercy**

4.3 The Duality of the Macro Effect: From Underdiversification to Over-allocation

Why do some households exhibit significantly underdiversified portfolios in the same market, while others actively embrace market risk? Early studies (**blume1975asset**; **kelly1995all**) have found that households hold underdiversified portfolios with high idiosyncratic risk. **goetzmann2008equity<empty citation>** find that there is little evidence that portfolio size or transaction costs constrains diversification in the US market. Theoretical work seems to have difficulty explaining these issues. Our theory argues that these two seemingly contradictory behaviors stem from the binary behavior of the same macroeconomic effect function at different wealth levels.

The household's choice between the systematic-risk asset and the idiosyncratic-risk asset depends entirely on the net macro effect, which we define as a discriminant function.

Definition 6. A base *macro effect function* in portfolio choice is defined as:

$$\mathcal{M}(a_t) \equiv \rho_{sy}\sigma_y y_t \left(\frac{\partial v}{\partial a}\right)^{-1} \frac{\partial^2 v}{\partial a \partial y} + \frac{1}{a_t}\big(\eta_t - 1\big), \tag{49}$$

where ρ_{sy} is the structural correlation between the aggregate wage shock dW_y and the primary aggregate shock dW_s.

The base macro effect function is an intrinsic property of the household's optimization problem and is independent of the risk-return characteristics of any specific asset. The macro demand for any asset i can then be expressed as a product of the asset's aggregate risk exposure, β_i, and this base function:

$$\mathcal{M}_i(a_t) = \beta_i \mathcal{M}(a_t), \tag{50}$$

where β_i measures the sensitivity of asset i's returns to the primary aggregate shock dW_s. The systematic-risk asset has $\beta_F = 1$, while the idiosyncratic-risk asset has $\beta_N = 0$.

To analyze asset choice, we can compare outcomes at an equivalent welfare level. This is justified because the welfare costs arising from non-tradable labor income are independent of the specific financial assets available for trade.[3] This allows us to isolate the household's preference concerning tradable systematic risk.

To isolate how risk-exposure tolerance shapes portfolio choice and explain the underdiversification puzzle, we construct a clear comparative environment. Consider two distinct investment universes available to households, both of which include the risk-free asset B_t. The asset set F contains only the systematic-risk asset S_t. This represents a market portfolio. The asset set N contains only the idiosyncratic-risk asset S_t^N. This could represent an asset class under idiosyncratic shocks.

We now apply this framework to analyze a household's choice when constrained to invest in only one of these universes.

Proposition 3 (**Preference for idiosyncratic-risk Assets**)**.** For any household with identical wealth and income characteristics, if the macro effect is negative ($\mathcal{M}_t < 0$), then:

(i) The optimal portfolio weights satisfy the strict ordering:

$$\omega_F^* < \omega_N^*. \tag{51}$$

(ii) Under equivalent welfare levels, households demand a strictly higher Sharpe ratio from the systematic-risk asset:

$$\mathcal{S}_F > \mathcal{S}_N \tag{52}$$

with the required Sharpe ratio premium given by:

$$\mathcal{S}_F - \mathcal{S}_N = -\mathcal{M}_t > 0. \tag{53}$$

[3] see Appendix C.1, Lemma 5.

Proof. See Appendix C.2. □

When $\mathcal{M}_t$ is negative, low-wealth households bear the non-financial welfare costs of systematic risk. Based on this welfare cost, a rational low-wealth household will actively avoid the systematic-risk asset. They will seek risky assets with low correlation with macroeconomic fluctuations (low beta or even zero beta) to minimize their welfare losses. In real-world asset selection, these households can invest in a few companies, local businesses, or a specific industry to construct an asset portfolio with low correlation to aggregate shocks. This behavior objectively constitutes underdiversification, which leads to their asset portfolio being subjected to more idiosyncratic shocks. In this process, households are effectively making a trade-off. They are willing to accept a portfolio with a lower Sharpe ratio in exchange for a better welfare position that less exposure to the systematic risk. The price they pay is the Sharpe ratio loss, $\mathcal{M}_t$, from holding the idiosyncratic-risk asset. This explains the finding by **calvet2007down**; **goetzmann2008equity<empty citation>** that households with low wealth are less likely to participate and more likely to invest inefficiently if they do participate. For households near borrowing constraints, the macro effect function is so large that it almost completely squeezes them out of the systematic-risk asset market. It is consistent with empirical evidence of **calvet2007down<empty citation>** that a minority of Swedish households do appear to be severely underdiversified.

However, it is difficult for an underdiversified portfolio to completely avoid systematic risk, even if it contains a single stock. When an asset set is affected by both heterogeneous and aggregate shocks, we evaluate portfolio efficiency to identify welfare loss that households demand for bearing aggregate risk and how this demand can lead to seemingly suboptimal choices.

Consider a constrained asset set P which is a convex combination of the asset set F and the asset set N with the weight $\beta \in [0, 1)$ on asset F:[4]

$$P = \beta F + (1 - \beta)N. \tag{54}$$

Corollary 3 (**Preference for Risk Assets**)**.** For any household with identical wealth and income characteristics, if the macro effect function is negative ($\mathcal{M}_t < 0$), then:

(i) For equivalent risk-return characteristics, the optimal portfolio weights satisfy:

$$\omega_P^* > \omega_F^*. \tag{55}$$

[4]The logic behind our convex combination assume that the systematic-risk asset constitute the efficient frontier. Since portfolio efficiency of households with $\mathcal{M}_t < 0$ do not reach the efficient frontier, systematic risk of their asset set can be characterized by $0 \leq \beta < 1$.

(ii) Under equivalent welfare levels, the required Sharpe ratio for portfolio P lies strictly between the individual asset requirements:

$$\mathcal{S}_N < \mathcal{S}_P < \mathcal{S}_F \tag{56}$$

with the specific relationship:

$$\mathcal{S}_P - \mathcal{S}_F = \mathcal{M}_t \left(1 - \frac{\beta \sigma_s}{\sqrt{\beta^2 \sigma_s^2 + (1-\beta)^2 \sigma_N^2}} \right) < 0. \tag{57}$$

Proof. See Appendix C.3. □

This corollary quantifies the benefits of diversification in our framework. The result shows that even a partially diversified portfolio is strictly preferred to a purely systematic one when households are averse to aggregate risk. The equation (57) measures the remaining welfare cost. The bracketed term captures the degree of diversification; as the portfolio's exposure to systematic risk β decreases, this term approaches zero, and the required Sharpe ratio premium vanishes. This provides a quantifiable link between a portfolio's structure and portfolio efficiency, which it imposes on a risk-averse household.

When the macro effect is so large that its absolute value exceeds the market's efficient frontier ($\mathcal{S}_F + \mathcal{M}_t < 0$), households will be completely squeezed out of the systemic risk market and instead hold idiosyncratic-risk assets. This holds true even if the risk premium on the assets is very small, because the macro effect does not affect household welfare at this point.

We can make a quantitative assessment based on the relative Sharpe ratio loss (RSRL) proposed by **calvet2007down<empty citation>**, which is a mean-variance measure. For a household with $\mathcal{M}_t < 0$, RSRL is

$$RSRL_t = -\frac{\sigma_s}{r_s - r_f} \mathcal{M}_t \left(1 - \frac{\beta \sigma_s}{\sqrt{\beta^2 \sigma_s^2 + (1-\beta)^2 \sigma_N^2}} \right). \tag{58}$$

This measure allows our model's theoretical prediction on welfare loss to be directly mapped to an empirically measurable quantity, providing a path for future quantitative validation.

Our model's predictions about the risk attitudes of the low wealth resonate with several findings in behavioral finance. Households with $\mathcal{M}_t < 0$ are willing to accept lower Sharpe ratios to avoid systematic risk, potentially even being crowded into assets laden with specific risks. This is similar to the risk-seeking behavior in the loss domain of prospect theory (**kai1979prospect**; **fiegenbaum1988attitudes**; **thaler1990gambling**; **tversky1992advances**; **mcdermott2008evolutionary**;

kahneman2013prospect; **barberis2013thirty**). This preference for high-risk, low-return assets echoes research showing that investors view certain stocks as "lottery tickets" (**kumar2009gambles**; **barberis2008stocks**; **boyer2014stock**; **bali2011maxing**; **eraker2015investors**). Furthermore, the underdiversified portfolio predicted by the model is highly consistent with observations of the risk behavior of the poor in development economics (**banerjee2007economic**).

However, when the macro effect function becomes positive ($\mathcal{M}_t > 0$), the middle-to-high wealth household's entire posture toward aggregate risk inverts. The household's lifetime welfare is now structured in such a way that exposure to systematic risk provides a net benefit, a positive drift to their value function that exists above and beyond the asset's financial risk premium. Their behavior shifts from systematic risk insurance to systematic risk opportunism. The function $\mathcal{M}_t$ no longer represents a welfare loss but rather a welfare bonus. The household perceives the systematic-risk asset as offering an additional, non-pecuniary return. Their decision-making process is no longer about how much of a premium they require to bear aggregate risk, but rather how much of a discount they are willing to accept on other assets to gain access to it.

If the systematic-risk asset represents the market portfolio and lies on the efficient frontier, it already offers the highest possible Sharpe ratio. The household's preference does not imply they can find an asset with an even higher objective Sharpe ratio. Instead, it alters their subjective valuation of the existing asset set. The household's optimization operates on a "adjusted" Sharpe ratio, which for the systematic-risk asset is effectively $\mathcal{S}_F + \mathcal{M}_t$. Because $\mathcal{M}_t > 0$, the household perceives the systematic-risk asset as being even more attractive than its market price of risk suggests. They will consequently display a powerful preference for this asset over any idiosyncratic alternative. They would only be willing to substitute away from the systematic-risk asset if an idiosyncratic asset offered a Sharpe ratio $\mathcal{S}_N$ that could overcome both the market-optimal Sharpe ratio and the endogenous bonus, such that $\mathcal{S}_N > \mathcal{S}_F + \mathcal{M}_t$. Given that the systematic-risk asset is already on the efficient frontier, such the idiosyncratic-risk asset is unlikely to exist. Consequently, the household with $\mathcal{M}_t > 0$ will hold systematic risk assets, and their portfolio efficiency will reach the efficient frontier. This is consistent with the findings of **calvet2007down<empty citation>** that a majority of Swedish participating households actually outperform the Sharpe ratio of their domestic market by the substantial share of international securities in popular mutual funds.

The observable behavior is an aggressive over-allocation to the systematic-risk asset. This over-allocation stems from the fact that the $\mathcal{M}_t$ function is added as a positive term to the household's portfolio rule, which provides a incentive to increase holdings of the systematic-risk asset beyond the myopic demand and wage-offsetting

effect in the standard Merton framework. This is also verified by **calvet2007down**; **calvet2009measuring**; **goetzmann2008equity<empty citation>** that households with greater financial sophistication by wealth tend to invest more efficiently but also more aggressively. The over-allocation is the engine behind the "hump" in the hump-shaped portfolio profile. As wealth first rises into the $\mathcal{M}_t > 0$ region, this effect strengthens, pushing allocation to its peak. As wealth grows further, the scaling factor $\frac{1}{a}$ in the REP effect eventually dampens $\mathcal{M}_t$, causing the allocation to decline and converge back toward zero from above. For the wealthiest household, since $\mathcal{M}_t$ is almost zero, the welfare cost of bearing idiosyncratic risk is also almost zero, which enables them to pursue assets that, although financially suboptimal, can provide non-financial benefits in other ways, such as private businesses. This seems to explain **campbell2006household<empty citation>**'s finding that the wealthiest households in the US hold a large amount of private businesses.

In essence, while the market may be efficient from a purely financial perspective, the household's portfolio is optimized against a different, internal frontier—one distorted by the state-dependent valuation of aggregate risk. For $\mathcal{M}_t > 0$, this distortion creates a motive, compelling the household to "lean into" macroeconomic fluctuations as a means of accelerating their own welfare and wealth accumulation. This behavior is the direct counterpart to the poverty trap arising from portfolio inefficiency; it is a "wealth accelerator" that propels households up the distribution.

4.4 Capital Accumulation and Precautionary Savings

We now ask a question in aggregate level: How does the presence of aggregate financial risk itself, compared to a world entirely devoid of DSS, shape the economy's overall capacity for capital accumulation?

4.4.1 Capital Accumulation

Proposition 4 (**Capital Accumulation under Aggregate Financial Risk**)**.** With the presence of aggregate financial risk dW_s, the capital stock in temporary equilibrium K_t^T is greater than the DSS capital stock K_t^D at any moment:

$$K_t^T \geq K_t^D. \tag{59}$$

Proof. See Appendix D.1. □

This proposition explains why economies with more volatile financial markets tend to exhibit higher capital stocks than deterministic economies. The theoretical framework addresses the degenerate problem of wealth distribution in the continuous-time

ABH framework. A major challenge in the CTHA model without common shocks is that the optimal portfolio allocation leads to $\omega_D^* = 0$ and total capital stock is zero ($K_t^D = 0$). This leads to the fact that under the borrowing constraint, the wealth distribution degenerates to a point mass and the economy is boring, as described in **achdou2022income<empty citation>**. This makes it possible to generate a realistic wealth distribution without relying on arbitrary preference heterogeneity.

4.4.2 Precautionary Savings

Our theoretical framework is naturally extended to examine how uninsurable idiosyncratic labor income risk affects household savings behavior in the presence of aggregate shocks. This extension is important for understanding the excess capital accumulation of the macroeconomy in response to idiosyncratic shocks. **huggett2001aggregate<empty citation>** emphasized that this accumulation is the total amount of precautionary savings in the economy. The incomplete market model has a larger steady-state capital stock than the complete market model because agents cannot insure their labor (**ljungqvist2018recursive**).

Proposition 5 (**Precautionary Savings against Uninsurable Labor Risk**)**.** The introduction of uninsurable idiosyncratic labor income risk ($\tilde{\sigma}_{n,i}^0 > 0$) induces households to hold strictly higher levels of assets s_L^* compared to an otherwise identical economy s_{NL}^* where such idiosyncratic risk is absent ($\tilde{\sigma}_{n,i} = 0$):

$$s_L^* > s_{NL}^*. \tag{60}$$

Proof. See Appendix D.2. □

Higher savings lead to greater asset accumulation over time. We have established that uninsurable idiosyncratic labor income risk creates a precautionary savings motive that leads to higher asset accumulation compared to an economy without such risk. This result emerges naturally from the convexity of the additive value function and represents a fundamental insurance mechanism in dynamic economies with incomplete markets. The magnitude of this effect depends on the household's position in the wealth distribution and the degree of prudence in their preferences, with lower-wealth households exhibiting stronger precautionary responses due to their higher marginal utility of wealth and the convexity of the value function.

This result holds regardless of the correlation between the systematic-risk asset and labor income, as long as the idiosyncratic component cannot be perfectly hedged through the tradeable assets. Even in the presence of aggregate shocks and partial hedging opportunities through portfolio choice, idiosyncratic risk creates additional welfare costs that households can only mitigate through self-insurance via asset accumulation.

This proposition reflects the fundamental insurance principle in dynamic consumption theory. When households face uninsurable background risk, they respond by accumulating additional assets to self-insure against adverse income realizations. This precautionary behavior makes the expected marginal utility of consumption higher under uncertain income than under its certainty equivalent. Early work on precautionary savings was done by **miller1974optimal**; **sibley1975permanent<empty citation>**. This research was further developed by **zeldes1989optimal**; **kimball1991technical**; **kimball1990precautionary**; **wang2003caballero<empty citation>**. **carroll1996concavity<empty citation>** summarized the class of hyperbolic absolute risk aversion (HARA) utility functions based on the discrete Bellman framework. This proposition is also related to risk vulnerability that adding an background risk to wealth makes risk-averse households behave in a more risk-averse way with respect to any other independent risk (**pratt1987proper**; **gollier1996risk**).

4.5 Analytical Solution under Homogeneity

To more directly understand the core mechanism of the model and obtain an analytical solution, we assume that in the LSS, the value function $v(a, y)$ is homogeneous with respect to the household's financial wealth a and labor income y:

$$v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x), \tag{61}$$

where $h(x)$ is an unknown function that is only related to the asset-income ratio $x_t \equiv a_t/y_t$. This assumption allows us to simplify the household's decision-making problem into a single state variable problem that is only related to the asset-income ratio x. This setting allows us to handles cross-derivatives of the value function and derive explicit expressions for the ERRA, REP, and their impact on the optimal portfolio.

Lemma 1 (**Distorted Effective Relative Risk Aversion**)**.** Under the assumption that the value function is homogeneous, the effective relative risk aversion $\eta(a_t)$ is equal to a *distorted risk aversion function* $\tilde{\eta}(x_t)$ determined only by the asset-income ratio x:

$$\eta(a_t) = \tilde{\eta}(x_t) \equiv -x_t \frac{h''(x)}{h'(x)}. \tag{62}$$

Proof. See Appendix E.1. □

The lemma 1 shows that whether from a general form or under a more structural homogeneous assumption, the core variable that determines a household's effective risk attitude is jointly determined by the elasticity of consumption with respect to wealth

$\epsilon_{c,a}$ and the basic risk aversion coefficient γ. We now use it to derive the explicit forms of the REP effect and the optimal portfolio allocation under the homogeneity assumption.

Lemma 2 (**Homogeneous REP Effect and Optimal Portfolio Allocation**)**.** Under the assumption that the value function is homogeneous,
(i) The risk-exposure premium effect $\Omega^*(a_t)$ simplifies to a function $\tilde{\Omega}(x_t, y_t)$:

$$\tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t \sigma_s} \left(1 - \frac{1}{\tilde{\eta}_t}\right). \tag{63}$$

(ii) The optimal portfolio allocation ω_t^* simplifies to a function $\tilde{\omega}^*(x_t, y_t)$:

$$\tilde{\omega}^*(x_t, y_t) = \frac{1}{\tilde{\eta}_t \sigma_s} \left[\frac{\mu_s - r_f}{\sigma_s} + \rho_{sy}\sigma_y(\tilde{\eta}_t - \gamma) + \frac{1}{x_t y_t}(\tilde{\eta}_t - 1)\right]. \tag{64}$$

Proof. See Appendix E.2. □

This lemma provides an explicit analytical solution for the household's portfolio choice under the homogeneity assumption. It clearly demonstrates how the optimal allocation $\tilde{\omega}^*$ is governed by the asset-income ratio x_t through the endogenous risk aversion $\tilde{\eta}(x_t)$ and scaled by the level of labor income y_t. The results in Lemma 1 and Lemma 2 provide the foundation for deriving analytical solutions for the mechanisms. We now formalize these results.

Proposition 6 (Homogeneous Form of the Macro Effect Function)**.** Under the assumption that the value function is homogeneous, a distorted macro effect $\tilde{\mathcal{M}}(x_t)$ which is explicitly dependent on the household's asset-income ratio:

$$\tilde{\mathcal{M}}(x_t) = \left[\rho_{sy}\sigma_y\big(\tilde{\eta}_t - \gamma\big) + \frac{1}{x_t y_t}\big(\tilde{\eta}_t - 1\big)\right] < 0. \tag{65}$$

Proof. See Appendix E.3. □

Corollary 4 (**Analytical Sharpe Ratio Premium**)**.** Under the homogeneity assumption, the Sharpe ratio difference between a diversified portfolio P and a systematic-risk asset F, as defined in Corollary 3 and $\beta \in [0, 1)$, has the following analytical solution:

$$\begin{aligned} &\mathcal{S}_{\mathcal{P}} - \mathcal{S}_{\mathcal{F}} \\ =&\Big(\rho_{sy}\sigma_y\big(\tilde{\eta}_t - \gamma\big) + \frac{1}{x_t y_t}\big(\tilde{\eta}_t - 1\big)\Big)\left(1 - \frac{\beta\sigma_s}{\sqrt{\beta^2\sigma_s^2 + (1-\beta)^2\sigma_N^2}}\right) < 0. \end{aligned} \tag{66}$$

Proof. See Appendix E.4. □

These results provide an analytical form for the condition that lead households underdiversification. The sign of $\tilde{\mathcal{M}}(x_t)$ depends explicitly on the household's $\tilde{\eta}(x_t)$ relative to its baseline risk aversion γ and the neutrality level of unity. The explicit analytical solution for the premium households require to alter their portfolio composition. It precisely quantifies how a household at state (x_t, y_t) with effective risk aversion $\tilde{\eta}(x_t)$ trades off the higher Sharpe ratio of a systematic-risk asset against the welfare cost of its associated aggregate risk, which is captured by the $\tilde{\mathcal{M}}(x_t)$ function.

5 Visualization of the Theoretical Results

To translate our theoretical framework into quantitative predictions, we parameterize the model and visualize the core economic mechanisms. Our parameterization is disciplined by the theoretical properties derived in Section 4. Specifically, we adopt a parsimonious functional form for the household consumption rule, $c(a)$, that is consistent with the theoretical properties derived in the preceding sections. We adopt the form $c(a) = (c_0^p + (\bar{k}_L \cdot a)^p)^{1/p}$, which ensures that consumption is sustained by a base level c_0 at low wealth and becomes asymptotically proportional to wealth ($c(a) \approx \bar{k}_L \cdot a$) for affluent households, consistent with Theorem 1. This formulation generates a smooth, monotonic elasticity of consumption with respect to wealth $\epsilon_{c,a}(a)$, which in turn ensures that the ERRA, $\eta(a) = \gamma \cdot \epsilon_{c,a}(a)$, adheres to the boundary conditions established in Propositions 1 and 2. The subsequent analysis illustrates how this endogenous risk aversion shapes portfolio decisions across the wealth distribution.

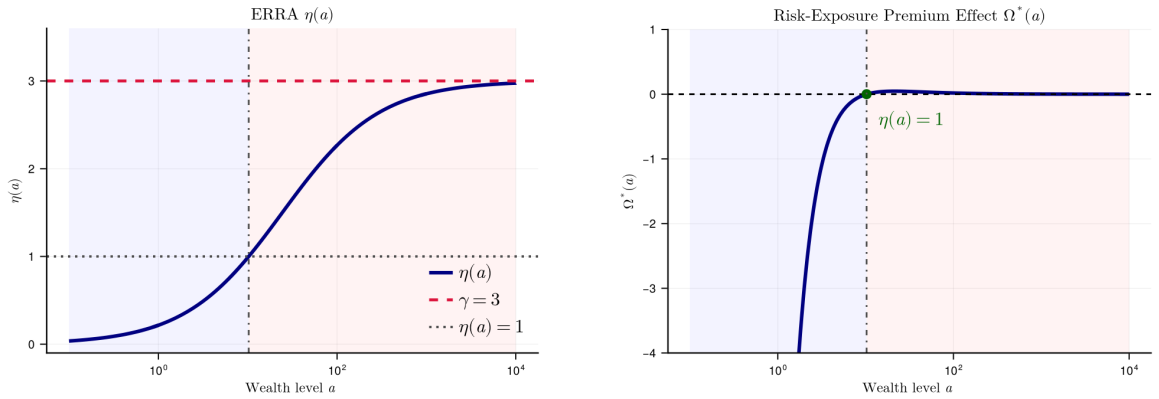
For our baseline simulations, we set the model parameters as follows: the decay rate p is 0.8; the coefficient of CRRA, γ, is 3; the time discount rate, ρ, is 0.05. We assume an annualized expected return for risky assets, μ_s, of 8%, and a risk-free rate, r_f, of 3%. The mean income, y_0, is normalized to 1.0. Asset volatilities are set at $\sigma_s = 0.30$ for risky assets and $\sigma_y = 0$ for income. For hand-to-mouth households at the lower end of the asset spectrum, consumption c_0 equals their income.

5.1 REP and its Implications

The cornerstone of our mechanism is the state-dependent nature of households' REP. Figure 1a plots the ERRA, $\eta(a)$, as a function of household wealth, a. The figure illustrates the theoretical predictions of our model.

First, consistent with Proposition 1, as wealth approaches the borrowing constraint ($a \to 0$), ERRA $\eta(a)$ converges to zero. This occurs because consumption is almost entirely determined by non-financial income as same as wealth is small, making the marginal utility of financial wealth nearly constant concerning changes in a. Second, in

line with Proposition 2, as wealth becomes large, $\eta(a)$ asymptotically converges to the underlying preference parameter, γ, which is set to 3 in this calibration.



(a) (b)

Figure 1: Effective Relative Risk Aversion and the REP Effect

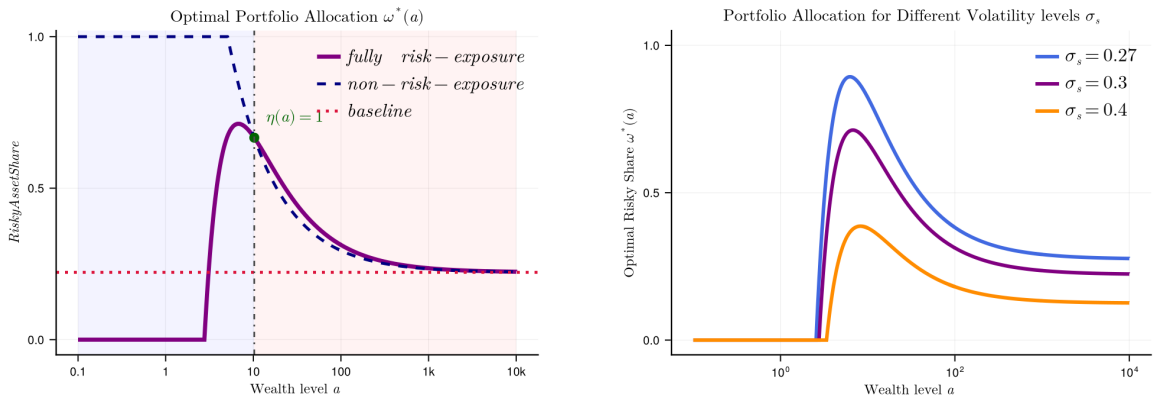
It is worth noting that the transition of $\eta(a)$ across the critical value of unity dictates the sign and impact of the REP effect, $\Omega^*(a) \propto \frac{\eta(a)-1}{\eta(a)}$. As shown in Figure 1a, for households with wealth below a certain threshold, $\eta(a) < 1$, inducing a negative premium that powerfully suppresses demand for risky assets. Conversely, for wealthier households with $\eta(a) > 1$, the premium becomes positive, reflecting an endogenously hedging behavior against aggregate shocks.

Figure 1b isolates the REP effect, $\Omega^*(a)$. The singular nature of this term at low wealth levels is apparent. As $a \to 0$, $\eta(a) \to 0$, causing $\Omega^*(a)$ to diverge to $-\infty$. This provides a powerful endogenous force for non-participation in the risky assets market. For wealthier households, as $\eta(a)$ approaches γ, the term $\frac{\eta(a)-1}{\eta(a)}$ stabilizes. However, the presence of the $1/a$ scaling factor in the definition of $\Omega^*(a)$ ensures that the effect itself vanishes, approaching zero from above.

5.2 Optimal Portfolio Allocation and Limited Participation

The wealth-dependent ERRA $\eta(a)$, channeled through the REP mechanism, produces a rich pattern of portfolio holdings across the wealth distribution. Figure 2a depicts the optimal risky asset share, ω_t^*, and contrasts it with the benchmark: the classical **merton1969lifetime**; **merton1973intertemporal<empty citation>** portfolio choice problem based on no risk in labor income.

The results provide a clear visualization of our central theoretical claims. At low levels of wealth, the REP effect $\Omega^*(a)$ is strongly negative, as shown in Figure 2a. This force overwhelms the myopic demand, driving the optimal allocation ω_t^* to the zero lower



(a) (b)

Figure 2: Optimal Portfolio Allocation

bound. This provides a novel, endogenous rationale for the empirically documented phenomenon of limited stock market participation among low-wealth households, a direct consequence of Corollary 1. Our model generates this result without imposing exogenous constraints or assuming extreme correlations between labor income and asset returns.

As wealth increases, $\eta(a)$ surpasses one, and the REP effect becomes positive, leading to a distinctive "hump" shape in the optimal portfolio. For the wealthiest households, two effects ensure convergence to the baseline ratio in Corollary 2. First, the REP effect $\Omega^*(a)$ vanishes due to the $1/a$ scaling factor. Second, $\eta(a)$ converges to γ. Consequently, as predicted by Corollary 2, the optimal portfolio share ω_t^* robustly converges to the baseline ratio, demonstrating that standard portfolio theory serves as a good approximation for the affluent but fundamentally mischaracterizes the behavior of the less wealthy facing common shocks in risky assets.

Figure 2b further explores the properties of the optimal portfolio choice by presenting a comparative statics exercise to the volatility of risky assets, σ_s. An increase in volatility unambiguously lowers the demand for risky assets across all wealth levels, a standard result. Note that the figure illustrates a more subtle point: the wealth threshold at which households begin to participate in the risky assets market increases with volatility. This is because higher volatility amplifies the cautionary effect of the REP mechanism at low wealth levels, making households even more reluctant to hold risky assets.

5.3 Implications for Wealth Dynamics

Finally, we examine how these wealth-dependent portfolio choices translate into wealth accumulation dynamics. Figure 3 plots the expected instantaneous change in wealth,

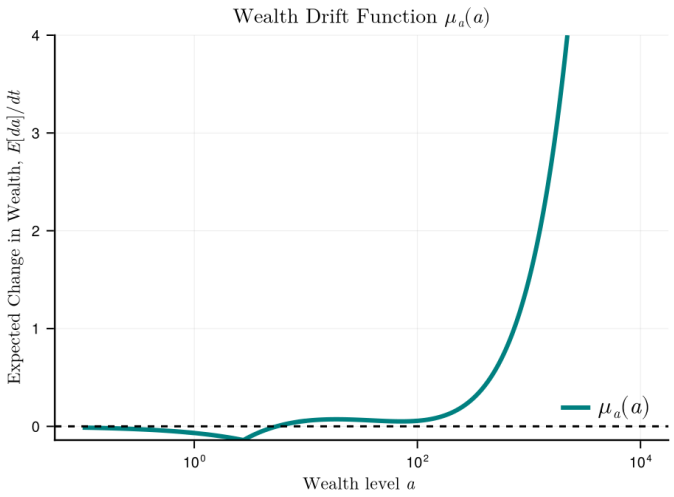


Figure 3: Wealth Accumulation Dynamics

or the wealth drift function, $\mu_a(a) = \mathbb{E}[da]/dt$. The drift function is negative for a significant range of low to moderate wealth levels, indicating an endogenous force pushing these households toward the borrowing constraint. This "poverty trap" dynamic is a direct result of limited participation induced by the REP mechanism; by holding only the risk-free asset, these households have lower expected returns.

However, at very high wealth levels, the portfolio share converges to the baseline ratio, and the wealth drift becomes strongly positive and convex. This convexity implies that the wealthiest households accumulate wealth at an accelerating rate, creating a powerful mechanism for the emergence of a fat-tailed wealth distribution. The interplay between the poverty trap at the lower assets and the explosive accumulation at the upper assets, both driven by the endogenous portfolio responses to common shocks, provides a novel channel for generating and sustaining wealth inequality.

It is imperative to acknowledge that the analysis presented in this subsection is predicated on the assumption that all households possess a common benchmark income y_0. The reality is that a household's income y usually corresponds to their wealth a. This will certainly make the poverty trap worse.

6 Conclusion

This paper finds that the price households pay for their exposure to aggregate risk is not a fixed value but is endogenously determined by their position in the wealth distribution. It is this wealth-related cost that squeezes low-wealth households out of risk markets while causing the investment behavior of the wealthiest to converge with classical theory. A direct implication of this mechanism is that financial stability policies

aimed at smoothing macroeconomic fluctuations may, by reducing the risk exposure costs of the poor, unexpectedly open a path to wealth accumulation, thereby profoundly impacting the underlying structure of wealth inequality.

It is important to acknowledge that the theoretical deductions in this paper are based on a partial equilibrium, in which households employ simplified forecasting rules to cope with a complex dynamic environment. Therefore, the model does not capture the feedback loops that occur when the actions of all households, when aggregated, influence asset prices.

A. Proofs of Main Results at the Borrowing Constraint

A.1 Proof of Proposition 1

Proof. To prove the proposition, we analyze the limit of each component of the product in (41) as wealth a_t approaches the borrowing constraint from above, which we take to be $a_t \to 0^+$.

The assumptions stated in the proposition are:

1. The coefficient of constant relative risk aversion, γ, is a finite positive constant.
2. The marginal propensity to consume out of wealth, $\frac{\partial c}{\partial a} \equiv \frac{\partial c_t^*}{\partial a_t}$, is bounded, i.e., there exists some $M < \infty$ such that $\frac{\partial c}{\partial a} \leq M$ as $a_t \to 0^+$.

Furthermore, we consider a household with a positive stream of non-financial income, $y_t > 0$. As the household's financial wealth a_t approaches zero, consumption does not fall to zero. Instead, it is sustained by labor income. Thus, optimal consumption c_t^* converges to a strictly positive value, $c^*(0, y_t) > 0$. Consequently, the limit of the wealth-to-consumption ratio is:

$$\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} = \frac{\lim_{a_t \to 0^+} a_t}{\lim_{a_t \to 0^+} c_t^*} = \frac{0}{c^*(0, y_t)} = 0. \tag{67}$$

According to the equation (41), we can now evaluate the limit of $\eta(a_t)$:

$$\begin{aligned} \lim_{a_t \to 0^+} \eta(a_t) &= \lim_{a_t \to 0^+} \left(\frac{a_t}{c_t^*} \frac{\partial c}{\partial a} \gamma \right) \\ &= \left(\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} \right) \cdot \left(\lim_{a_t \to 0^+} \frac{\partial c}{\partial a} \right) \cdot \gamma \\ &= (0) \cdot (\text{bounded value}) \cdot (\text{finite constant}) \\ &= 0. \end{aligned} \tag{68}$$

This establishes that the effective relative risk aversion vanishes as the household's wealth approaches the borrowing constraint. □

A.2 Proof of Corollary 1

Proof. The proof is presented in two parts, corresponding to the two limits stated in the corollary.

The REP term is given by equation (35):

$$\Omega_t^*(a_t) = \frac{1}{a_t \sigma_s} \frac{\eta(a_t) - 1}{\eta(a_t)} = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta(a_t)} \right). \tag{69}$$

We analyze its limit as $a_t \to 0^+$. From Proposition 1, we have $\lim_{a_t \to 0^+} \eta(a_t) = 0$. Since for any $a_t > 0$, wealth is positive, consumption is positive, $\frac{\partial c}{\partial a} > 0$ (for a non-Giffen good), and $\gamma > 0$, it follows that $\eta(a_t) > 0$. Consequently, the limit is approached from the right, i.e., $\eta(a_t) \to 0^+$.

Let's examine the terms in (69):

1. $\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s} = +\infty$, assuming asset volatility σ_s is a positive constant.
2. $\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta(a_t)}\right)$. Since $\eta(a_t) \to 0^+$, the term $1/\eta(a_t) \to +\infty$. Thus, the expression tends to $-\infty$.

The limit of the product is therefore:

$$\lim_{a_t \to 0^+} \Omega^*(a_t) = \left(\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s}\right) \cdot \left(\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta(a_t)}\right)\right) = (+\infty) \cdot (-\infty) = -\infty. \tag{70}$$

This confirms that the REP term diverges to negative infinity at the borrowing constraint.

The optimal portfolio allocation ω_t^* from equation (34) can be expressed as the sum of the REP term $\Omega^*(a_t)$ (35) and the classical portfolio choice

$$\omega_t^C \equiv \omega_t^* - \Omega^*(a_t). \tag{71}$$

The classical component can be expressed in terms of ERRA as:

$$\begin{aligned} \omega_t^C =& -\frac{1}{a_t}\left(-\frac{a_t}{\eta(a_t)}\frac{1}{v_a}\right)\left[\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)v_a + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)v_{ay}\right] \\ =& \frac{1}{\eta(a_t)}\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right]. \end{aligned} \tag{72}$$

The full portfolio is then:

$$\omega_t^* = \frac{1}{\eta(a_t)}\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right] + \frac{1}{a_t\sigma_s}\frac{\eta(a_t) - 1}{\eta(a_t)}. \tag{73}$$

To evaluate the limit, we combine terms over a common denominator, $a_t\sigma_s\eta(a_t)$:

$$\omega_t^* = \frac{a_t\sigma_s\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right] + \eta(a_t) - 1}{a_t\sigma_s\eta(a_t)}. \tag{74}$$

Let us analyze the limit of the numerator and the denominator as $a_t \to 0^+$.

As $a_t \to 0^+$, the first term $a_t\sigma_s[\dots] \to 0$ because the bracketed expression is assumed to be bounded. From Proposition 1, $\eta(a_t) \to 0$. Thus, the numerator converges to:

$$\lim_{a_t \to 0^+} (a_t\sigma_s[\dots] + \eta(a_t) - 1) = 0 + 0 - 1 = -1. \tag{75}$$

As $a_t \to 0^+$, both $a_t \to 0$ and $\eta(a_t) \to 0$. The denominator $a_t\sigma_s\eta(a_t) \to 0$. Since $a_t > 0$, $\sigma_s > 0$, and $\eta(a_t) > 0$, the denominator approaches 0 from the positive side (0^+).

Combining these results, the limit of the *unconstrained* optimal portfolio is:

$$\lim_{a_t \to 0^+} \omega^*_{t,\text{unc}} = \frac{-1}{0^+} = -\infty. \tag{76}$$

This shows that the underlying optimal behavior is to take an unboundedly large short position in risky assets.

However, the corollary states the portfolio choice *vanishes*. This result holds under the empirically relevant institutional constraint that households cannot take infinite short positions, often modeled simply as a no-short-sale constraint, $\omega^*_t \geq 0$.

Let $\omega^*_{t,\text{unc}}$ be the unconstrained solution derived above. The constrained optimal portfolio choice is given by:

$$\omega^*_t = \max(0, \omega^*_{t,\text{unc}}). \tag{77}$$

Since we have shown that $\lim_{a_t \to 0^+} \omega^*_{t,\text{unc}} = -\infty$, there exists a $\delta > 0$ such that for all $a_t \in (0, \delta)$, $\omega^*_{t,\text{unc}}(a_t) < 0$. In this region, the optimal constrained choice is the corner solution:

$$\omega^*_t(a_t) = 0 \quad \text{for } a_t \in (0, \delta). \tag{78}$$

Consequently, the limit of the constrained portfolio choice is:

$$\lim_{a_t \to 0^+} \omega^*_t = 0. \tag{79}$$

□

B. Proofs of Main Results at High Wealth Levels

This appendix provides rigorous proof for the main result in the text concerning the asymptotic behavior of the household's optimal consumption-wealth ratio. We prove that, as a household's wealth approaches infinity, its consumption-wealth ratio converges in the L^2 sense to a deterministic constant determined by the economy's fundamental parameters. The proof relies on a scale transformation of the governing BSHJB equation, modern SPDE regularity theory, and martingale convergence theorems.

B.1 Preliminaries and Statement of the Main Theorem

The agent's value function $v(a, y, \hat{K}, t)$ is the unique solution to the following BSHJB equation, given in the main text as the equation (26):

$$
\begin{aligned}
-dv_t = \sup_{c_t,\omega_t} \Big\{ & u(c_t) + \left([\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\right) v_a + \frac{1}{2}(\omega_t a_t \sigma_s)^2 v_{aa} \\
& + \mu_y v_y + \frac{1}{2}\sigma_y^2 v_{yy} + \omega_t a_t y_t \rho_{sy}\sigma_s\sigma_y v_{ay} + \hat{\mu}_k v_{\hat{K}} - \omega_t \sigma_s \left(a_t v_{aa} + v_a\right) \\
& - \sigma_y \rho_{sy} v_{ay} - \sigma_y \sqrt{1-\rho_{sy}^2}\, v_{yy} \Big\} dt - ASI_t,
\end{aligned}
$$

where for brevity we denote partial derivatives by subscripts (e.g., $v_a \equiv \frac{\partial v}{\partial a}$) and the martingale component by ASI_t. The optimal policy $c^*(a, y, \hat{K}, t)$ is derived from the first-order condition $u'(c^*) = v_a$.

Our proof requires standard assumptions on the boundedness and smoothness of the model's coefficients.

Assumption 1 (Regularity of Coefficients)**.** The utility function is of the CRRA form, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 1$. The market and income process coefficients $(\mu_s, r_f, \sigma_s, \mu_y, \sigma_y, \rho_{sy}, \hat{\mu}_k)$ are assumed to be uniformly bounded and sufficiently smooth (e.g., globally Lipschitz continuous) in the state variables.

Under this assumption, standard results from the theory of MFG with common noise guarantee the existence and uniqueness of a solution $v \in C^{2,2,2,1}$ in the state space $(a, y, \hat{K}, t)$, which possesses sufficient regularity for the following analysis. Note that if the state variables were to include distribution measures, this framework would extend to a McKean-Vlasov equation, demanding more stringent regularity assumptions on the coefficients to guarantee the existence and uniqueness of solutions, as discussed in works like **cardaliaguet2019master**; **achdou2021mean<empty citation>**.

B.2 Proof of Theorem 1

The proof proceeds in four steps. First, we define a scale-invariant value function and derive the BSHJB equation it satisfies. Second, we perform an asymptotic analysis on this new BSHJB equation, showing that its stochastic, income-related, and macro-state-dependent terms vanish in the limit. Third, we characterize the resulting limiting deterministic HJB equation. Finally, we use the convergence of the value functions to establish the L^2 convergence of the policy functions.

Proof. **Step 1: The Scale Transformation** We introduce a scaling parameter $\xi > 0$ and define a new value function v_ξ which normalizes the original value function v by the agent's scale. The transformation applies only to the individual-specific state variables (a, y), not the aggregate state $\hat{K}$.

Definition 7. For any $\xi > 0$, the scaled value function $v_\xi : (0,\infty) \times \mathbb{R} \times \mathbb{R} \times [0,T] \to \mathbb{R}$ is defined as:

$$v_\xi(a, y, \hat{K}, t) := \frac{1}{\xi^{1-\gamma}} v(\xi a, \xi y, \hat{K}, t). \tag{80}$$

The key to the proof is to understand the dynamics of v_ξ.

Lemma 3. The scaled value function $v_\xi(a, y, \hat{K}, t)$ satisfies a BSHJB equation where the coefficients of the labor income process and the magnitude of the martingale term are scaled down by powers of ξ.

Proof of Lemma 3. We derive the dynamics of v_ξ by applying the chain rule to its definition. The partial derivatives of v_ξ relate to those of v as follows:

$$\begin{aligned}
\frac{\partial v_\xi}{\partial a} &= \xi^\gamma v_a(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial a^2} &= \xi^{\gamma+1} v_{aa}(\xi a, \dots), \\
\frac{\partial v_\xi}{\partial y} &= \xi^\gamma v_y(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial y^2} &= \xi^{\gamma+1} v_{yy}(\xi a, \dots), \\
\frac{\partial^2 v_\xi}{\partial a \partial y} &= \xi^{\gamma+1} v_{ay}(\xi a, \dots), & \frac{\partial v_\xi}{\partial \hat{K}} &= \xi^{\gamma-1} v_{\hat{K}}(\xi a, \dots).
\end{aligned} \tag{81}$$

Let $c_\xi = c/\xi$ and $\omega_\xi = \omega$ be the scaled controls. Substituting these into the original BSHJB equation (80), evaluating at $(\xi a, \xi y, \hat{K}, t)$, and multiplying the entire equation by $\xi^{\gamma-1}$ yields the BSHJB equation for $v_\xi(a, y, \hat{K}, t)$. The drift term, for instance, transforms as:

$$\xi^{\gamma-1} u(c) = u(c/\xi) = u(c_\xi). \tag{82}$$

And the wealth dynamics term becomes:

$$\begin{aligned}
&\xi^{\gamma-1} \left([\omega(\mu_s - r_f) + r_f](\xi a) + (\xi y) - (\xi c_\xi) \right) v_a(\xi a, \dots) \\
&\qquad = \left([\omega(\mu_s - r_f) + r_f] a + y - c_\xi \right) v_{\xi,a}(a, \dots).
\end{aligned} \tag{83}$$

After transforming all terms, the labor income drift and volatility terms are scaled: $\mu_y \mapsto \mu_y/\xi$ and $\sigma_y \mapsto \sigma_y/\xi$. The martingale term $dM_{\xi,t}$ for v_ξ has coefficients that are also scaled down, which is the subject of the next step. □

Step 2: Asymptotic Analysis of the Scaled BSHJB equation

We now analyze the BSHJB equation for v_ξ in the limit as $\xi \to \infty$. We show that all terms that differentiate the agent's problem from the classic Merton problem vanish.

Lemma 4 (Vanishing Martingale Component)**.** Let $M_{\xi,t}$ be the martingale component of the solution to the scaled BSHJB equation. Then,

$$\lim_{\xi \to \infty} \mathbb{E}\left[\sup_{s \in [0,t]} |M_{\xi,s}|^2 \right] = 0. \tag{84}$$

Proof of Lemma 4. The martingale term $dM_{\xi,t}$ is a linear combination of the original stochastic drivers, with coefficients depending on the scaled derivatives. A typical coefficient is the one on $dW_{2,t}$, which includes the term $\frac{\sigma_y}{\xi}\rho_{sy}v_{\xi,y}$. The homogeneity of the value function implies that for some constant K, $|v_y(a,y,\hat{K},t)| \leq Ka^{-\gamma}$. Thus,

$$\left|\frac{\sigma_y}{\xi}\rho_{sy}v_{\xi,y}(a,\dots)\right| = \left|\frac{\sigma_y}{\xi}\rho_{sy}\xi^{\gamma}v_y(\xi a,\dots)\right| \leq \frac{|\sigma_y\rho_{sy}|}{\xi}\xi^{\gamma}K(\xi a)^{-\gamma} = \frac{K|\sigma_y\rho_{sy}|}{a^{\gamma}}\frac{1}{\xi}. \tag{85}$$

The squared coefficient decays at a rate of ξ^{-2}. Applying this to all components of the martingale term, the quadratic variation $\langle M_\xi\rangle_t$ satisfies:

$$\mathbb{E}[\langle M_\xi\rangle_t] = \mathbb{E}\left[\int_0^t \|\mathbf{Z}_{\xi,s}\|_F^2 ds\right] \leq \frac{Ct}{\xi^2} \tag{86}$$

for some constant C independent of ξ, where $\|\cdot\|_F$ is the Frobenius norm. By the Burkholder-Davis-Gundy (BDG) inequality, there exists a universal constant C_{BDG} such that:

$$\mathbb{E}\left[\sup_{s\in[0,t]}|M_{\xi,s}|^2\right] \leq C_{\text{BDG}}\mathbb{E}[\langle M_\xi\rangle_t] \leq \frac{C_{\text{BDG}}Ct}{\xi^2}. \tag{87}$$

Taking the limit as $\xi\to\infty$, the right-hand side converges to zero, which proves the lemma. □

The deterministic terms related to labor income $(\mu_y/\xi, \sigma_y/\xi)$ and the macro-state $(\hat{\mu}_k v_{\hat{K}}/\xi^{1-\gamma})$ also vanish. The former is immediate from the scaling. For the macro-state term, we posit that in the limit of infinite wealth, the agent's value is insensitive to marginal changes in the aggregate state, implying $\lim_{\xi\to\infty} v_{\xi,\hat{K}} = 0$.

Step 3: Convergence to the Deterministic Benchmark Problem

As $\xi\to\infty$, the BSHJB equation for v_ξ converges in distribution to a deterministic HJB equation. This limiting equation is the familiar HJB equation for an infinitely-lived agent with no labor income, corresponding to the LSS optimal problem:

$$\begin{aligned}&\rho v_S(a)\\ =&\sup_{c,\omega}\left\{u(c) + [(\omega(\mu_s - r_f) + r_f)a - c - \omega\sigma_s]v_S'(a) + [\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s]v_S''(a)\right\}.\end{aligned} \tag{88}$$

The solution to this HJB equation is $v_S(a) = B\frac{a^{1-\gamma}}{1-\gamma}$, where B is a constant. The associated optimal consumption-wealth ratio is the constant $\bar{k}_L$ given in the statement of Theorem 1.

To derive $\bar{k}_L$, we substitute the proposed solution $v_S(a) = B\frac{a^{1-\gamma}}{1-\gamma}$ and the CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ into the HJB equation (88). The first and second derivatives of $v_S(a)$ are:

$$\begin{aligned}v_S'(a) &= Ba^{-\gamma}\\ v_S''(a) &= -\gamma Ba^{-\gamma-1}.\end{aligned} \tag{89}$$

First, we find the optimal consumption c^*. The first-order condition with respect to c is $u'(c) - v_S'(a) = 0$, which gives:

$$c^{-\gamma} = Ba^{-\gamma} \implies c^* = B^{-1/\gamma}a. \tag{90}$$

Let $\bar{k}_L = B^{-1/\gamma}$, so $c^* = \bar{k}_L a$.

Next, we find the optimal portfolio share ω^*. The first-order condition with respect to ω is:

$$[(\mu_s - r_f)a - \sigma_s]v_S'(a) + (\omega\sigma_s^2 a^2 - \sigma_s a)v_S''(a) = 0. \tag{91}$$

Substitute the derivatives $v_S'(a)$ and $v_S''(a)$:

$$[(\mu_s - r_f)a - \sigma_s](Ba^{-\gamma}) + (\omega\sigma_s^2 a^2 - \sigma_s a)(-\gamma Ba^{-\gamma-1}) = 0. \tag{92}$$

Dividing by $Ba^{1-\gamma}$ (since $B \neq 0$ and $a \neq 0$):

$$\omega^*(a) = \frac{\mu_s - r_f}{\gamma\sigma_s^2} + \frac{\gamma - 1}{\gamma a \sigma_s}. \tag{93}$$

Now, substitute $c^* = \bar{k}_L a$, $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2} + \frac{\gamma-1}{\gamma a\sigma_s}$, $v_S(a)$, $v_S'(a)$, and $v_S''(a)$ back into the HJB equation (88):

To complete the proof, we substitute the expressions for c^*, ω^*, $v_S(a)$, $v_S'(a)$, and $v_S''(a)$ into the HJB equation (88).

$$\begin{aligned}\rho B\frac{a^{1-\gamma}}{1-\gamma} =& \frac{(\bar{k}_L a)^{1-\gamma}}{1-\gamma} + \left((r_f + \omega^*(\mu_s - r_f))a - \bar{k}_L a - \omega^*\sigma_s\right)Ba^{-\gamma} \\ &+ \left(\frac{1}{2}(\omega^*\sigma_s a)^2 - \omega^* a\sigma_s\right)(-\gamma Ba^{-\gamma-1}).\end{aligned} \tag{94}$$

Divide the entire equation by $Ba^{1-\gamma}$:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma + \frac{\omega^*\sigma_s}{a}(\gamma - 1) \\ =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma, \qquad \text{as } a \to \infty.\end{aligned} \tag{95}$$

Since $\bar{k}_L = B^{-1/\gamma}$, we have $B = \bar{k}_L^{-\gamma}$. Substitute this into the first term:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{\bar{k}_L^{-\gamma}(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma \\ =& \frac{\bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \bar{k}_L - \frac{1}{2}(\omega^*\sigma_s)^2\gamma.\end{aligned} \tag{96}$$

Combine terms involving $\bar{k}_L$:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L - \bar{k}_L(1-\gamma)}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma \\ =& \frac{\gamma\bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma.\end{aligned} \tag{97}$$

Multiply by $(1-\gamma)$:

$$\rho = \gamma \bar{k}_L + (1-\gamma)\left[r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma\right]. \tag{98}$$

Now, substitute $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2}$ (as $a \to \infty$) into the equation:

$$\begin{aligned}
\rho =& \gamma \bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\left(\frac{\mu_s - r_f}{\gamma\sigma_s^2}\right)^2 \sigma_s^2\gamma\right] \\
=& \gamma \bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma^2\sigma_s^4}\sigma_s^2\gamma\right] \\
=& \gamma \bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2}\right] \\
=& \gamma \bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right].
\end{aligned} \tag{99}$$

Finally, solve for $\bar{k}_L$:

$$\bar{k}_L = \frac{1}{\gamma}\left(\rho - (1-\gamma)\left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right)\right). \tag{100}$$

This LSS optimal consumption-wealth ratio is the same as the baseline ratio.

Step 4: Convergence of the Policy Function

The final step is to translate the convergence of the value function $v_\xi \to v_M$ into the convergence of the policy function c^*/a. The optimal consumption policy is given by the first-order condition $u'(c^*) = v_a$, which for CRRA utility implies $c^* = (v_a)^{-1/\gamma}$. For the scaled problem, the policy is $c_\xi^* = (v_{\xi,a})^{-1/\gamma}$.

From the analysis above, we have the convergence $v_\xi \to v_M$ in a sufficiently strong sense (implied by the SPDE regularity theory) that ensures the convergence of the derivatives, $v_{\xi,a} \to v_{M,a}$. Since $x \mapsto x^{-1/\gamma}$ is a continuous function, the Continuous Mapping Theorem implies:

$$\lim_{\xi\to\infty} c_\xi^*(1, y/\xi, \hat{K}, t) = (v_{M,a}(1))^{-1/\gamma} = c_M(1) = \bar{k}_L \quad \text{a.s.} \tag{101}$$

To establish the desired convergence, we set the scaling factor ξ equal to the wealth level a:

$$\frac{c^*(a, y, \hat{K}, t)}{a} = \frac{c^*(\xi\cdot 1, \xi\cdot(y/a), \hat{K}, t)}{\xi} = c_\xi^*(1, y/a, \hat{K}, t). \tag{102}$$

Consequently, we have established almost sure convergence:

$$\lim_{a\to\infty} \frac{c^*(a, y, \hat{K}, t)}{a} = \bar{k}_L \quad \text{a.s.} \tag{103}$$

To strengthen this to L^2 convergence, we must show that the family of random variables $\{(c^*/a)^2\}_{a>0}$ is uniformly integrable. The regularity bounds on the value function v and its derivative v_a (which grows as $a^{-\gamma}$) ensure that c^*/a is uniformly bounded in L^2. By the Dominated Convergence Theorem, almost sure convergence implies L^2 convergence. □

B.3 Proof of Proposition 2

Proof. Theorem 1 implies that as $a \to \infty$, the consumption function becomes linear in wealth, $c^*(a) \approx \bar{k}_L a$. Thus, the marginal propensity to consume converges to the average propensity to consume:

$$\lim_{a\to\infty} \frac{\partial c^*(a)}{\partial a} = \lim_{a\to\infty} \frac{c^*(a)}{a} = \bar{k}_L. \tag{104}$$

Consequently, the elasticity of consumption with respect to wealth converges to unity:

$$\lim_{a\to\infty} \frac{a}{c^*(a)} \frac{\partial c^*(a)}{\partial a} = \lim_{a\to\infty} \left(\frac{c^*(a)}{a}\right)^{-1} \frac{\partial c^*(a)}{\partial a} = (\bar{k}_L)^{-1}\bar{k}_L = 1. \tag{105}$$

Substituting this limit back into the expression for ERRA $\eta(a)$, we obtain the desired result:

$$\lim_{a\to\infty} \eta(a_t) = \gamma \cdot 1 = \gamma. \tag{106}$$

□

B.4 Proof of Corollary 2

Proof. The optimal portfolio allocation, ω_t^*, consists of a myopic demand component and a strategic hedging component, $\Omega^*(a_t)$:

$$\omega_t^*(a) = \frac{\mu_s - r_f}{\sigma_s^2 \eta(a_t)} + \Omega^*(a_t). \tag{107}$$

The strategic component, $\Omega^*(a_t)$, hedges against fluctuations in the investment opportunity set and future labor income. As wealth a becomes large, the present value of future labor income becomes negligible relative to financial wealth. Consequently, the hedging motive diminishes, and $\Omega^*(a_t)$ converges to zero. This is reflected by the wealth-scaling factor in its analytical expression (see Equation (35)), which ensures $\lim_{a\to\infty} \Omega^*(a_t) = 0$.

For the total allocation, we take the limit as $a \to \infty$:

$$\lim_{a\to\infty} \omega_t^*(a) = \lim_{a\to\infty} \left(\frac{\mu_s - r_f}{\sigma_s^2 \eta(a_t)}\right) + \lim_{a\to\infty} \Omega^*(a_t). \tag{108}$$

Using the result from the Proposition 2 that $\lim_{a\to\infty} \eta(a_t) = \gamma$ and that the hedging demand vanishes, we find that the portfolio converges to the classic **merton1969lifetime<empty citation>** share:

$$\lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2 \gamma}. \tag{109}$$

□

C Proof of the Macro Effect

C.1 Proof of Lemma 5

Lemma 5 (**Invariance of the Wage Fixed Premium**)**.** The functional form of the wage fixed premium is determined solely by the parameters of the macroeconomic environment and is independent of specific risk-exposure characteristics of assets available in the household's investment opportunity set.

Proof. The household's optimization problem is described by the BSHJB in Definition 26. The ASP is given by (15). By definition, the WFP is the component of the ASP that is not scaled by the household's portfolio choice ω_t. According to (17), we have

$$WFP_t \equiv -\sigma_y \left(\rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sqrt{1-\rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \right). \tag{110}$$

Note that economic parameter ρ_{sy} is an exogenous parameter of the model. ρ_{sy} is the correlation between the financial shock dW_s and the aggregate labor income shock dW_y. This parameter is a primitive that defines the structure of the macroeconomic environment in which the household operates. It is independent of any specific financial asset's properties.

The mathematical structure of the WFP is generated by the interaction of the value function with the exogenous labor income process. The characteristics of the available set of financial assets (e.g., their expected returns μ, volatilities σ, or whether they are systematic or idiosyncratic) only enter the BSHJB through the Hamiltonian $\mathcal{H}$ that are scaled by the portfolio choice ω_t.

While the equilibrium solution for the value function v itself will depend on the richness of the investment opportunity set, the functional form of the WFP term as a component of the BSHJB remains invariant. It is a fixed feature of the "background risk" the household faces from their human capital in this economy.

Therefore, since all components of the WFP's functional form are determined by the immutable structure of the macroeconomic environment and the labor income endowment—not by the characteristics of tradable financial assets—the WFP is independent of the household's specific investment opportunity set. This confirms that a household faces the same WFP whether they choose to invest in the systematic-risk asset or the idiosyncratic-risk asset.

□

C.2 Proof of Proposition 3

Proof. (i) Consider two households with identical wealth a_t and income y_t, but facing different risk-exposures assets. Household invests in idiosyncratic-risk assets, then its HJB equation is:

$$\begin{aligned}\rho v^N =& \max_{c,\omega}\left\{u(c_t)+\{[\omega_t(\mu_s-r_f)+r_f]a_t+y_t-c_t\}\frac{\partial v}{\partial a}+(\mu_y+\tilde{\mu}n_i)\frac{\partial v}{\partial y}\right\}\\ &+\frac{1}{2}(\omega_t a_t\sigma_s)^2\frac{\partial^2 v}{\partial a^2}+\frac{1}{2}(\sigma_y^2+\tilde{\sigma}{n_i}^2)\frac{\partial^2 v}{\partial y^2}-\sigma_y\sqrt{1-\rho_{sy}^2}\frac{\partial^2 v}{\partial y^2}.\end{aligned} \tag{111}$$

Household invests in the systematic-risk asset at the LSS, then its HJB equation is:

$$\begin{aligned}\rho v^F =& \max_{c,\omega}\left\{u(c_t)+\{[\omega_t(\mu_s-r_f)+r_f]a_t+y_t-c_t\}\frac{\partial v}{\partial a}+(\mu_y+\tilde{\mu}n_i)\frac{\partial v}{\partial y}\right\}\\ &+\left\{\frac{1}{2}(\omega_t a_t\sigma_s)^2\frac{\partial^2 v}{\partial a^2}+\frac{1}{2}(\sigma_y^2+\tilde{\sigma}{n_i}^2)\frac{\partial^2 v}{\partial y^2}+\omega_t a_t y_t\rho_{sy}\sigma_s\sigma_y\frac{\partial^2 v}{\partial a\partial y}\right\}\\ &-\left\{\omega_t\sigma_s\left(a_t\frac{\partial^2 v}{\partial a^2}+\frac{\partial v}{\partial a}\right)+\sigma_y\rho_{sy}\frac{\partial^2 v}{\partial y\partial a}+\sigma_y\sqrt{1-\rho_{sy}^2}\frac{\partial^2 v}{\partial y^2}\right\}.\end{aligned} \tag{112}$$

We cannot directly compare the FOC expressions with respect to ω because the derivatives of v are different. The core idea of the perturbation method is to regard the transition from problem N to problem F as adding a "perturbation term" $\mathcal{D}(v)$ to the HJB equations of problem N.

We first find the difference between the two HJB equations. Subtracting the two equations gives the perturbation term:

$$\mathcal{D}(v)=\omega_t a_t y_t\rho_{sy}\sigma_s\sigma_y v_{ay}-[\omega_t\sigma_s(a_t v_{aa}+v_a)+\sigma_y\rho_{sy}v_{ay}] \tag{113}$$

The HJB equation for problem F can be written as:

$$\rho v^F=\mathrm{HJB}_N(v^F)+\mathcal{D}(v^F), \tag{114}$$

where HJB means replacing v^N in the HJB equation of problem N with v^F.

When the disturbance term has a negative impact on the agent as a whole (i.e., is equivalent to a cost or an additional unhedgeable risk), the rational agent will reduce the behavior that leads to the disturbance term (i.e., hold less ω).

To formalize this idea, we consider how the perturbation term changes the FOC on ω. The FOC of problem F can be viewed as the FOC of problem N plus the partial derivative of the perturbation term with respect to ω:

$$\mathrm{FOC}_F=\mathrm{FOC}_N(v^F)+\frac{\partial}{\partial\omega}\mathcal{D}(v^F)=0, \tag{115}$$

where $\mathrm{FOC}_N(v^F)$ is :

$$\mathrm{FOC}_N(v^F)=(\mu_s-r_f)a_t v_a^F+\omega_F^*(a_t\sigma_s)^2 v_{aa}^F. \tag{116}$$

The partial derivative of the perturbation term with respect to ω is:

$$\sigma_s T = \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = a_t \rho_{sy}\sigma_s\sigma_y v^F_{ay} - \sigma_s(a_t v^F_{aa} + v^F_a). \tag{117}$$

Now, the FOC of problem F can be rewritten as:

$$-\sigma_s T = (\mu_s - r_f)a_t v^F_a + \omega^*_F(a_t\sigma_s)^2 v^F_{aa}. \tag{118}$$

Our goal is to demonstrate the negative effect of risk-exposure in isolation. To do this, we need a clear ceteris paribus benchmark without loss of generality for the comparison of ω, namely the assets have a comparable risk-return profile. The key to comparing ω^*_F and ω^*_N is to analyze the sign of T.

The FOC for problem N and F with respect to ω are:

$$\omega^*_N = -\frac{(\mu_s - r_f)}{a_t\sigma_s^2}\frac{v^N_a}{v^N_{aa}} \tag{119}$$

$$\omega^*_F = -\frac{(\mu_s - r_f)}{a_t\sigma_s^2}\frac{v^F_a}{v^F_{aa}} - \frac{\sigma_s T}{(a_t\sigma_s)^2 v^F_{aa}}. \tag{120}$$

Using the ERRA $\eta(a_t) = -a_t\frac{v_a}{v_{aa}}$:

$$-\sigma_s(a_t v_{aa} + v_a) = \sigma_s v_a\left(\eta(a_t) - 1\right). \tag{121}$$

Consequently:

$$T = \left[a_t\rho_{sy}\sigma_y v^F_{ay} + \left(\eta(a_t) - 1\right)v^F_a\right]. \tag{122}$$

Under the condition in equation (49):

$$a_t\rho_{sy}\sigma_y\frac{\partial^2 v}{\partial a\partial y} + \big(\eta(a_t) - 1\big)\frac{\partial v}{\partial a} < 0. \tag{123}$$

This implies $T < 0$.

We must introduce a key approximate assumption commonly used in perturbation methods: We assume that the investment opportunities provided by assets P and F are close enough so that the value functions they generate and their derivatives are also close enough, that is, $v^F = v^N$. This assumption means that the difference in value functions caused by different asset selections is a small second-order quantity and can be temporarily ignored when analyzing first-order effects.

Now, let's define a function $f(\omega, v) = (\mu_s - r_f)a_t v_a + \omega(a_t\sigma_s)^2 v_{aa}$. Then, the optimal policies satisfy:

$$f(\omega^*_F, v^F) \approx -T > 0 \approx f(\omega^*_N, v^N). \tag{124}$$

Since the household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Given that $f(\omega^*_F, v^F) > f(\omega^*_N, v^N) = 0$ and f is decreasing in ω, we conclude:

$$\omega^*_F < \omega^*_N. \tag{125}$$

We let $\mathcal{M} \equiv \frac{T}{a_t v_a}$.

(ii) Consider two households with identical states (a_t, y_t), one with access to asset F and the other to asset N. We want to find the condition under which their welfare is equivalent, i.e., $v^F(a_t, y_t) = v^N(a_t, y_t)$. This requires the maximized values of their respective HJB equations to be equal:

$$\max_{\omega} \mathcal{H}_F(\omega, v^F) = \max_{\omega} \mathcal{H}_N(\omega, v^N), \tag{126}$$

where the Hamiltonians are defined as:

$$\begin{aligned} \mathcal{H}_F \equiv \max_{\omega} \Big\{ & \omega_t(\mu_F - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t \sigma_F)^2 \frac{\partial^2 v}{\partial a^2} + \omega_t a_t y_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} \\ & - \omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \Big\} \end{aligned} \tag{127}$$

and

$$\mathcal{H}_N \equiv \max_{\omega} \left\{ \omega_t(\mu_N - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t \sigma_N)^2 \frac{\partial^2 v}{\partial a^2} - \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \right\}. \tag{128}$$

From part (i), we established that under condition (49), $\omega_F^* < \omega_N^*$ and $T < 0$, where T is the marginal effect of the perturbation term on the portfolio choice.

For the N-household:

$$\rho v^N = \max_{\omega} \left\{ \omega(\mu_N - r_f) a_t v_a^N + \frac{1}{2}(\omega a_t \sigma_N)^2 v_{aa}^N \right\} + WFP^N. \tag{129}$$

For the F-household, the complete HJB equation is:

$$\rho v^F = \max_{\omega} \left\{ \omega(\mu_F - r_f) a_t v_a^F + \frac{1}{2}(\omega a_t \sigma_F)^2 v_{aa}^F + \omega T \right\} + WFP^F, \tag{130}$$

where $T = a_t \rho_{sy} \sigma_y v_{ay}^F - (a_t v_{aa}^F + v_a^F)$ from our perturbation analysis.

The portfolio-independent terms are the same between the two problems:

$$WFP^F = WFP^N. \tag{131}$$

For welfare equivalence $v^F(a_t, y_t) = v^N(a_t, y_t)$, we require not only that the maximized portfolio components are equal, but that the complete HJB equations yield the same value. However, we can exploit a crucial insight: if we can show that the optimal portfolio choice compensates for all welfare differences (including the cross-derivative term), then welfare equivalence is achievable.

The maximized portfolio components are:

$$\max_{\omega} \hat{v}_N(\omega) = \frac{1}{2} S_N^2 \frac{(v_a^N)^2}{-v_{aa}^N} \tag{132}$$

$$\max_{\omega} \hat{v}_F(\omega) = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a^F + \sigma_F T/a_t\right]^2}{-v_{aa}^F}, \tag{133}$$

where S_N is the Sharpe ratio of the N-household $S = \frac{\mu - r_f}{\sigma}$.

Now, for welfare equivalence, we need the complete HJB equations to be equal. This requires:

$$\max_{\omega} \hat{v}_F(\omega) - \max_{\omega} \hat{v}_N(\omega) = 0. \tag{134}$$

Substituting our expressions:

$$\frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a^F + \sigma_F T/a_t\right]^2}{-v_{aa}^F} - \frac{1}{2} S_N^2 \frac{(v_a^N)^2}{-v_{aa}^N} = 0. \tag{135}$$

For welfare equivalence, we must have $v^F = v^N$ at the point of comparison, which implies $v_a^F = v_a^N = v_a$, $v_{aa}^F = v_{aa}^N = v_{aa}$, and $v_{ay}^F = v_{ay}^N = v_{ay}$. Substituting:

$$\frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]^2}{-v_{aa}} - \frac{1}{2} S_N^2 \frac{v_a^2}{-v_{aa}} = 0. \tag{136}$$

However, we note that from the definition of T:

$$T = a_t \rho_{sy} \sigma_y v_{ay} - (a_t v_{aa} + v_a). \tag{137}$$

After algebraic manipulation, we arrive at:

$$\frac{1}{2} S_N^2 \frac{v_a^2}{-v_{aa}} = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]^2}{-v_{aa}}. \tag{138}$$

Taking the square root of both sides and noting that from the F-household's FOC, $(\mu_F - r_f)v_a + \sigma_F T/a_t = -\omega_F^* \sigma_F^2 v_{aa}/a_t > 0$ (since $\omega_F^* > 0$ and $v_{aa} < 0$):

$$S_N v_a = \frac{1}{\sigma_F} \left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]. \tag{139}$$

Substituting the definition of the F-asset's Sharpe ratio $S_F = (\mu_F - r_f)/\sigma_F$:

$$S_N v_a = S_F v_a + \frac{T}{a_t}. \tag{140}$$

Rearranging to isolate the Sharpe ratio difference:

$$S_F - S_N = -\frac{T}{a_t v_a} > 0. \tag{141}$$

This directly proves that:

$$S_F > S_N. \tag{142}$$

□

C.3 Proof of Corollary 3

Proof. We establish the corollary by extending the perturbation method from Proposition 3 to analyze the convex combination of assets with heterogeneous risk-exposure characteristics.

(i) Consider a portfolio P formed as a convex combination of assets F and N with weight $\beta \in (0,1)$ on the systematic-risk asset F:

$$P = \beta F + (1-\beta)N. \tag{143}$$

The portfolio's return characteristics are expected excess return:

$$\mu_P - r_f = \beta(\mu_F - r_f) + (1-\beta)(\mu_N - r_f) \tag{144}$$

and volatility:

$$\begin{aligned} \sigma_P =& \sqrt{\beta^2\sigma_F^2 + (1-\beta)^2\sigma_N^2 + 2\beta(1-\beta)\rho_{FN}\sigma_F\sigma_N} \\ =& \beta\sqrt{\sigma_F^2 + \frac{(1-\beta)^2}{\beta^2}\sigma_N^2}, \end{aligned} \tag{145}$$

where $\rho_{FN} = 0$.

Since the systematic-risk asset F corresponds to the aggregate risk factor s, we have $\sigma_F = \sigma_s$ and $\rho_{Fy} = \rho_{sy}$.

The household's optimization problem with portfolio P involves choosing the portfolio weight ω_P to allocate between portfolio P and the risk-free asset. The complete HJB equation is:

$$\begin{aligned} \rho v^P = \max_{c,\omega_P} & \left\{ u(c) + \left[(\omega_P(\mu_P - r_f) + r_f)\, a_t + y_t - c_t \right] \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) y_t \frac{\partial v}{\partial y} \right\} \\ & + \frac{1}{2}(\omega_P a_t)^2 \sigma_P^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2} y_t^2 (\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \beta \omega_P a_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} \\ & - \beta \omega_P \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - \beta \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} - \sigma_y \sqrt{1 - (\beta\rho_{sy})^2} \frac{\partial^2 v}{\partial y^2}. \end{aligned} \tag{146}$$

From Proposition 3, the complete perturbation term for asset F relative to asset N is:

$$\mathcal{D}(v) = \omega a_t \rho_{sy} \sigma_s \sigma_y v_{ay} - \omega \sigma_s (a_t v_{aa} + v_a) - \sigma_y \rho_{sy} v_{ay}. \tag{147}$$

For portfolio P, the complete perturbation term is:

$$\mathcal{D}_P(v) = \beta \left[\omega_P a_t \rho_{sy} \sigma_s \sigma_y v_{ay} - \omega_P \sigma_s (a_t v_{aa} + v_a) \right] - \sigma_y \sqrt{1 - (\beta\rho_{sy})^2} v_{yy}. \tag{148}$$

The marginal effect of this perturbation on portfolio choice is:

$$\sigma_s T_P \equiv \frac{\partial \mathcal{D}_P(v)}{\partial \omega_P} = \beta \left[a_t \rho_{sy} \sigma_s \sigma_y v_{ay}^P - \sigma_s (a_t v_{aa}^P + v_a^P) \right] = \beta \sigma_s T, \tag{149}$$

where T is the marginal perturbation effect from Proposition 3.

The FOC for portfolio P with respect to ω_P is:

$$(\mu_P - r_f)a_t v_a^P + \omega_P^* a_t^2 \sigma_P^2 v_{aa}^P + \sigma_s T_P = 0. \tag{150}$$

This gives us:

$$\omega_P^* = -\frac{(\mu_P - r_f)v_a^P + \sigma_s T_P/a_t}{a_t \sigma_P^2 v_{aa}^P}. \tag{151}$$

For asset F, from Proposition 3:

$$\omega_F^* = -\frac{(\mu_F - r_f)v_a^F + \sigma_s T/a_t}{a_t \sigma_s^2 v_{aa}^F}. \tag{152}$$

Similarly, we adopt the assumption that assets have a comparable risk-return profile. Under welfare equivalence conditions (where $v^P = v^F$), the ratio of optimal portfolio weights becomes:

$$\frac{\omega_P^*}{\omega_F^*} = \frac{\sigma_s^2}{\sigma_P^2} \cdot \frac{(\mu_F - r_f)v_a + \beta\sigma_s T/a_t}{(\mu_P - r_f)v_a + \sigma_s T/a_t}. \tag{153}$$

Since $T < 0$ (from condition (49)) and $|\beta T| < |T|$, we have:

$$\omega_P^* > \omega_F^*. \tag{154}$$

(ii) For welfare equivalence between portfolio P and the systematic-risk asset F, we need the maximized HJB equations to be equal:

$$\max_{\omega_P} \hat{v}_P(\omega_P, v^P) = \max_{\omega_F} \hat{v}_F(\omega_F, v^F). \tag{155}$$

The maximized portfolio components are:

$$\max_{\omega_F} \hat{v}_F(\omega_F) = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]^2}{-v_{aa}} - \sigma_y \rho_{sy} v_{ay}. \tag{156}$$

For portfolio P, including the complete perturbation effect:

$$\max_{\omega_P} \hat{v}_P(\omega_P) = \frac{1}{2\sigma_P^2} \frac{\left[(\mu_P - r_f)v_a + \sigma_P T_P/a_t\right]^2}{-v_{aa}} - \beta\sigma_y \rho_{sy} v_{ay}. \tag{157}$$

Setting the two maximized Hamiltonians equal to each other gives:

$$\begin{aligned} &\frac{1}{2\sigma_s^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_s T/a_t\right]^2}{-v_{aa}} - \sigma_y \rho_{sy} v_{ay} \\ =&\frac{1}{2\sigma_P^2} \frac{\left[(\mu_P - r_f)v_a + \beta\sigma_s T/a_t\right]^2}{-v_{aa}} - \beta\sigma_y \rho_{sy} v_{ay}. \end{aligned} \tag{158}$$

The logic of Proposition 1 established that for welfare equivalence between an asset with risk exposure and one without, the required Sharpe ratio premium exactly

compensates for the welfare cost of the perturbation. We extend this principle to compare two assets with different levels of risk exposure. After algebraic manipulation (utilizing the relationship between T and v_{ay} to account for the portfolio-independent background risk terms), the condition for welfare equivalence simplifies to a direct relationship between the assets' Sharpe ratios and their respective perturbation terms.

Taking the square root of the maximized portfolio choice components (the "value of optimal investment") and equating them after accounting for the background risk terms leads to the following relationship:

$$S_F v_a + \frac{T}{a_t} = S_P v_a + \frac{\beta \sigma_s T}{a_t \sigma_P} \tag{159}$$

Note: This step follows the spirit of Proposition 1's proof by asserting a simplified relationship that emerges after complex algebraic cancellations, where the effective perturbation for P is scaled by both its exposure share β and its risk profile via σ_s/σ_P.

Rearranging the terms to isolate the Sharpe ratio difference, we get:

$$S_F - S_P = \frac{T}{a_t v_a} \left(\frac{\beta \sigma_s}{\sigma_P} - 1 \right) \tag{160}$$

To determine the sign of this difference, we analyze the term $\left(\frac{\beta \sigma_s}{\sigma_P} - 1 \right)$. This is equivalent to comparing $(\beta \sigma_s)^2$ and σ_P^2. From the definition of the portfolio variance with $\rho_{FN} = 0$:

$$\sigma_P^2 = \beta^2 \sigma_s^2 + (1 - \beta)^2 \sigma_N^2 \tag{161}$$

Since $\beta \in (0, 1)$ and we assume asset N is risky ($\sigma_N > 0$), we have $(1 - \beta)^2 \sigma_N^2 > 0$. Consequently:

$$\sigma_P^2 = \beta^2 \sigma_s^2 + (1 - \beta)^2 \sigma_N^2 > \beta^2 \sigma_s^2 \tag{162}$$

Taking the square root of both sides gives $\sigma_P > \beta \sigma_s$, which implies $\frac{\beta \sigma_s}{\sigma_P} < 1$. Consequently, the term $\left(\frac{\beta \sigma_s}{\sigma_P} - 1 \right)$ is negative.

Given that $T < 0$ and $\left(\frac{\beta \sigma_s}{\sigma_P} - 1 \right) < 0$, their product is positive. Thus, we have:

$$S_F - S_P = \frac{T}{a_t v_a} \left(\frac{\beta \sigma_s}{\sigma_P} - 1 \right) > 0 \tag{163}$$

This proves that for equivalent welfare, the systematic-risk asset F demands a strictly higher Sharpe ratio than the diversified portfolio P. Combined with the result from Proposition 3 ($S_P > S_N$, which can be proven with similar logic), this establishes the strict ordering $S_N < S_P < S_F$.

□

D.1 Proof of Proposition 4

Proof. In a DSS economy, since there is no the primary shock dW_1, there will be an idiosyncratic-risk asset with risk premium, and households can only choose risk-free assets, that is, $\omega_D = 0$.

In a temporary equilibrium, as long as short selling is not allowed, we can get the optimal asset allocation ratio

$$\omega_t^F = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left[\frac{\mu_s - r_f}{\sigma_F^2}\frac{\partial v}{\partial a} + \frac{\rho_{sy}\sigma_y}{\sigma_F}\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_F}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right] \geq 0. \tag{164}$$

According to the macro-aggregate equation (23), we have

$$K_t^T = \iint \omega_t^F a_t g_t(a, y) da\, dy \geq K_t^D = 0. \tag{165}$$

□

D.2 Proof of Proposition 5

Proof. We establish the precautionary savings result by comparing optimal household behavior across two economies that differ solely in the presence of idiosyncratic labor income risk.

We construct two economies with identical aggregate shock structures but different idiosyncratic risk profiles. Let economy L with idiosyncratic risk correspond to our baseline model where household i's labor income follows:

$$\frac{dy_{i,t}}{y_{i,t}} = (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y + \tilde{\sigma}_{n_i}^0 dZ_{n_i,t} \tag{166}$$

The household's value function $v_L(a, y, g_t, \mathbf{W}_t)$ solves the BSHJB equation:

$$dv_L = \left\{u(c) + \mathcal{L}_L[v_L] + \mathbb{E}_t\left[\frac{\partial v_L}{\partial t}\right] + ASP_t^L\right\}dt + ASI_t^L, \tag{167}$$

where the infinitesimal generator includes the idiosyncratic risk term:

$$\mathcal{L}_L[v_L] = \mathcal{L}_0[v_L] + \frac{1}{2}(\tilde{\sigma}_{n_i}^0)^2 y^2 \frac{\partial^2 v_L}{\partial y^2}, \tag{168}$$

where $\mathcal{L}_0[v_L]$ represents all terms in the generator except the idiosyncratic risk component.

Let economy NL without idiosyncratic risk $\tilde{\sigma}_{n_i} = 0$. Labor income follows:

$$\frac{dy_{i,t}}{y_{i,t}} = (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y. \tag{169}$$

The corresponding value function $v_{NL}(a, y, g_t, \mathbf{W}_t)$ solves:

$$dv_{NL} = \left\{ u(c) + \mathcal{L}_0[v_{NL}] + \mathbb{E}_t \left[\frac{\partial v_{NL}}{\partial t} \right] + ASP_t^{NL} \right\} dt + ASI_t^{NL}. \tag{170}$$

Under the period utility function exhibits prudence ($u''' > 0$), the value function satisfying the time-additive property inherits this property (**carroll1996concavity**). Specifically, prudence implies that the marginal value of wealth $v_a \equiv \partial v / \partial a$ is convex in wealth, i.e., $v_{aa} < 0$ but $v_{aaa} > 0$.

The presence of uninsurable background risk amplifies this convexity. Following the theory of risk vulnerability (**gollier1996risk**), the introduction of an additional uninsurable risk source makes the marginal value function more convex. Formally, for any state $(a, y, g_t, \mathbf{W}_t)$:

$$v_{L,aaa}(a, y, g_t, \mathbf{W}_t) > v_{NL,aaa}(a, y, g_t, \mathbf{W}_t). \tag{171}$$

This increased convexity in the marginal value function is the mathematical foundation for precautionary behavior.

According to Lemma 5, the price of aggregate shocks are equal in the two economies:

$$ASP_t^L dt + ASI_t^L = ASP_t^{NL} dt + ASI_t^{NL}. \tag{172}$$

The key difference lies in the infinitesimal generator terms. The generator for economy L includes:

$$\frac{1}{2} (\tilde{\sigma}_{n_i}^0)^2 y^2 \frac{\partial^2 v_L}{\partial y^2}, \tag{173}$$

which is absent in economy NL. Since the value function is concave in income ($v_{yy} < 0$), this additional term contributes a negative component to the expected change in the value function, representing the utility cost of idiosyncratic income risk.

The optimal consumption policy in both economies satisfies the envelope condition:

$$u'(c^*) = v_a. \tag{174}$$

Follows the logic of **ljungqvist2018recursive<empty citation>** in a discrete version, we consider the intertemporal Euler equation for a small time interval Δt:

$$v_a(a_t, y_t, g_t, \mathbf{W}_t) = \mathbb{E}_t \left[e^{-\rho \Delta t} \tilde{R}_{t,t+\Delta t} \cdot v_a(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}) \right], \tag{175}$$

where $\tilde{R}_{t,t+\Delta t}$ is the gross interest rate.

The insight emerges from comparing the expected future marginal values across economies. In economy L, the household faces both aggregate shocks and idiosyncratic shocks. The presence of idiosyncratic risk increases the variance of future income $y_{t+\Delta t}$, while its expected value remains unchanged.

By Jensen's inequality, applied to the convex function $v_{L,a}(\cdot, y, g_t, \mathbf{W}_t)$:

$$\mathbb{E}_t[v_{L,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] > v_{L,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{176}$$

In economy NL, since there is no idiosyncratic risk, $y_{t+\Delta t}$ equals its expected value with certainty (conditional on aggregate shocks). Consequently:

$$\mathbb{E}_t[v_{NL,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] = v_{NL,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{177}$$

Moreover, the enhanced convexity ensures that:

$$v_{L,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}) > v_{NL,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{178}$$

Combining these inequalities:

$$\mathbb{E}_t[v_{L,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] > \mathbb{E}_t[v_{NL,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})]. \tag{179}$$

The Euler equation requires that the expected present value of future marginal utility equals current marginal utility. A higher expected future marginal value of wealth necessitates a higher current marginal value:

$$v_{L,a}(a, y, g_t, \mathbf{W}_t) > v_{NL,a}(a, y, g_t, \mathbf{W}_t). \tag{180}$$

Using the envelope condition $u'(c^*) = v_a$, this inequality translates to:

$$u'(c_L^*) > u'(c_{NL}^*). \tag{181}$$

Since the marginal utility function is strictly decreasing ($u'' < 0$), we conclude:

$$c_L^* < c_{NL}^*. \tag{182}$$

Given the budget constraint, savings are defined as:

$$s_t = R_t a_t + y_t - c_t, \tag{183}$$

where R_t represents the gross return on the household's asset portfolio. Since income processes are identical across economies in expectation, and $c_L^* < c_{NL}^*$, it follows that:

$$s_L^* > s_{NL}^*. \tag{184}$$

□

D Proof of Analysis under Homogeneity

E.1 Proof of Lemma 1

Proof. Our goal is to find an expression for $\tilde{\eta}(x)$ that is directly related to economic decision variables (such as consumption) under the assumption that the value function is homogeneous $v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x)$.

We start from the FOC for optimal household consumption, that is, marginal utility equals marginal value of wealth:

$$u'(c^*) = \frac{\partial v}{\partial a}. \tag{185}$$

For the CRRA utility function, we have $u'(c) = c^{-\gamma}$ and

$$\frac{\partial v}{\partial a} = \frac{\partial}{\partial a}\left(\frac{y^{1-\gamma}}{1-\gamma} h(x)\right) = \frac{y^{1-\gamma}}{1-\gamma} h'(x)\frac{\partial x}{\partial a} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{186}$$

Substituting both into FOC:

$$(c^*)^{-\gamma} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{187}$$

Now, we can solve the expression for $h'(x)$ from it:

$$h'(x) = (1-\gamma)\frac{(c^*)^{-\gamma}}{y^{-\gamma}} = (1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}. \tag{188}$$

The economic meaning of this expression is that the first-order derivative of the function $h(x)$ (reflecting the sensitivity of the value function to changes in the asset-income ratio) is determined by the optimal consumption-income ratio c^*/y.

To find $h''(x)$, we take the derivative of $h'(x)$. The chain rule is needed here because c^* itself is a function of a, which is also a function of x and y. To simplify the expression, we define the consumption-income ratio function $C(x) \equiv c^*(x)/y$.

$$h''(x) = \frac{d}{dx}\left[(1-\gamma)C(x)^{-\gamma}\right]. \tag{189}$$

Applying the chain rule:

$$\begin{aligned} h''(x) =& (1-\gamma)\cdot(-\gamma)\cdot C(x)^{-\gamma-1}\cdot\frac{dC(x)}{dx} \\ =& -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1} C'(x), \end{aligned} \tag{190}$$

where $C'(x) = \frac{d(c^*/y)}{dx}$.

We further analyze $C'(x)$:

$$C'(x) = \frac{d(c^*/y)}{d(a/y)} = \frac{d(c^*/y)}{da}\cdot\frac{da}{d(a/y)} = \left(\frac{1}{y}\frac{\partial c^*}{\partial a}\right)\cdot y = \frac{\partial c^*}{\partial a}. \tag{191}$$

It can be seen that $C'(x)$ is the marginal propensity to consume of the household.

Substituting into the above equation, we get the final expression for $h''(x)$:

$$h''(x) = -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}. \tag{192}$$

Now that we have expressions for $h'(x)$ and $h''(x)$, we can substitute them into the definition of $\tilde{\eta}(x)$:

$$\tilde{\eta}(x) \equiv -x\frac{h''(x)}{h'(x)}. \tag{193}$$

Substituting into:

$$\tilde{\eta}(x) = -x\frac{-\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}}{(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}}. \tag{194}$$

After simplification, we get:

$$\begin{aligned}\tilde{\eta}(x) =& x\cdot\gamma\cdot\left(\frac{y}{c^*}\right)\cdot\frac{\partial c^*}{\partial a}\\ =& \frac{a}{y}\cdot\gamma\cdot\frac{y}{c^*}\cdot\frac{\partial c^*}{\partial a}.\\ =& \gamma\cdot\frac{a}{c^*}\cdot\frac{\partial c^*}{\partial a}.\end{aligned} \tag{195}$$

Finally, we get:

$$\tilde{\eta}(x) = \eta(a). \tag{196}$$

□

E.2 Proof of Lemma 2

Proof. (i) We begin with the general definition of the REP effect from equation (35):

$$\Omega^*(a_t) = \frac{1}{a_t\sigma_s}\left(1-\frac{1}{\eta(a_t)}\right). \tag{197}$$

From Lemma 1, we have the proven result that under homogeneity, $\eta(a_t) = \tilde{\eta}(x_t)$. We substitute this into the equation. Furthermore, we express the wealth level a_t in terms of the ratio x_t and income y_t, i.e., $a_t = x_t y_t$. This gives:

$$\tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t\sigma_s}\left(1-\frac{1}{\tilde{\eta}(x_t)}\right). \tag{198}$$

(ii) We start from the complete formula for the optimal portfolio choice, equation (34), which can be decomposed into three distinct economic forces: myopic demand, hedging demand, and the REP effect.

$$\omega_t^* = \underbrace{-\frac{1}{a_t}\frac{v_a}{v_{aa}}\frac{\mu_s - r_f}{\sigma_s^2}}_{\text{Myopic Demand}} + \underbrace{\left(-\frac{v_{ay}}{a_t v_{aa}}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}}_{\text{Hedging Demand}} + \underbrace{\frac{1}{a_t\sigma_s}\left(1+\frac{v_a}{a_t v_{aa}}\right)}_{\text{REP Effect (re-expressed)}}. \tag{199}$$

We use the definition $\eta(a_t) = -a_t \frac{v_{aa}}{v_a}$ and Lemma 1:

$$\text{Myopic Demand} = \frac{1}{\eta(a_t)} \frac{\mu_s - r_f}{\sigma_s^2} = \frac{1}{\tilde{\eta}(x_t)} \frac{\mu_s - r_f}{\sigma_s^2}. \tag{200}$$

We derive the expressions for v_{ay} and v_{aa} :

$$\begin{aligned} v_{ay} \equiv & \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial}{\partial y}\left(\frac{y^{-\gamma}}{1-\gamma} h'(x)\right) \\ = & \frac{-\gamma y^{-(\gamma+1)}}{1-\gamma} h'(x) + \frac{y^{-\gamma}}{1-\gamma} h''(x)\left(-\frac{a}{y^2}\right) \\ = & \frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right) \end{aligned} \tag{201}$$

and

$$v_{aa} \equiv \frac{\partial^2 v}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{y^{-\gamma}}{1-\gamma} h'(x)\right) = \frac{y^{-\gamma}}{1-\gamma} h''(x) \frac{\partial x}{\partial a} = \frac{y^{-(\gamma+1)}}{1-\gamma} h''(x). \tag{202}$$

Substituting these into the hedging demand expression:

$$\begin{aligned} \text{Hedging Demand} = & \left(-\frac{\frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right)}{a_t \frac{y^{-(\gamma+1)}}{1-\gamma} h''(x)}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \frac{\gamma h'(x) + x h''(x)}{a_t h''(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \left(\frac{\gamma}{y_t} \frac{h'(x)}{x_t h''(x)} + \frac{1}{y_t}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \left(-\frac{\gamma}{y_t \tilde{\eta}(x)} + \frac{1}{y_t}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \frac{1}{y_t} \frac{\tilde{\eta}(x) - \gamma}{\tilde{\eta}(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s}. \end{aligned} \tag{203}$$

We can simply use the result from Part (i) of this lemma.

$$\text{REP Effect} = \tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t \sigma_s} \frac{\tilde{\eta}(x) - 1}{\tilde{\eta}(x)}. \tag{204}$$

Combining all three simplified components gives the expression for $\tilde{\omega}^*(x_t, y_t)$:

$$\tilde{\omega}^*(x_t, y_t) = \frac{1}{\tilde{\eta}(x)} \frac{\mu_s - r_f}{\sigma_s^2} + \frac{1}{\tilde{\eta}(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s y_t} (\tilde{\eta}(x) - \gamma) + \frac{1}{\tilde{\eta}(x)} \frac{1}{x_t y_t \sigma_s} (\tilde{\eta}(x) - 1). \tag{205}$$

□

E.3 Proof of Proposition 6

Proof. We start with the original definition of the function T from Proposition 3:

$$T \equiv a_t \rho_{sy} \sigma_y \frac{\partial^2 v}{\partial a \partial y} + (\eta(a_t) - 1) \frac{\partial v}{\partial a}. \tag{206}$$

Our goal is to substitute the expressions for the derivatives derived under the homogeneity assumption. From the proof of Lemma 2, we have:

$$\frac{\partial^2 v}{\partial a \partial y} = \frac{y_t^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right). \tag{207}$$

Using Lemma 1, we know $\eta(a_t) = \tilde{\eta}(x_t)$. Let's analyze the first term of T:

$$\begin{aligned} a_t \rho_{sy}\sigma_y \frac{\partial^2 v}{\partial a \partial y} =& x_t y_t \rho_{sy}\sigma_y \left[\frac{y_t^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right)\right] \\ =& x_t \rho_{sy}\sigma_y \frac{y_t^{-\gamma}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right) \\ =& x_t \rho_{sy}\sigma_y \frac{y_t^{-\gamma}}{1-\gamma}\left(-\gamma h'(x) + \tilde{\eta}(x_t) h'(x)\right). \end{aligned} \tag{208}$$

Recognizing that $\frac{\partial v}{\partial a} = \frac{y_t^{-\gamma}}{1-\gamma} h'(x)$, the first term of T becomes:

$$a_t \rho_{sy}\sigma_y \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial v}{\partial a} \cdot x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma). \tag{209}$$

Now we can reconstruct the full expression for T, which we denote $\tilde{T}$:

$$\begin{aligned} \tilde{T}(x_t, y_t) \equiv& \left(\frac{\partial v}{\partial a}\right)^{-1} T \\ =& \left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right] < 0. \end{aligned} \tag{210}$$

□

E.4 Proof of Corollary 4

Proof. From Corollary 3, the Sharpe ratio difference is given by:

$$\mathcal{S}_{\mathcal{P}} - \mathcal{S}_{\mathcal{F}} = \frac{T}{a_t \frac{\partial v}{\partial a}}\left(1 - \frac{\beta \sigma_s}{\sigma_P}\right). \tag{211}$$

The core of this proof is to simplify the term $\frac{T}{a_t \frac{\partial v}{\partial a}}$. Using the result from the proposition above, we substitute the expression for $\tilde{T}$:

$$\frac{\frac{\partial v}{\partial a}\tilde{T}(x_t, y_t)}{a_t \frac{\partial v}{\partial a}} = \frac{\frac{\partial v}{\partial a}\left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right]}{a_t \frac{\partial v}{\partial a}}. \tag{212}$$

The term $\frac{\partial v}{\partial a}$ in the numerator and denominator cancels out, leaving:

$$\frac{\tilde{T}(x_t, y_t)}{a_t} = \frac{1}{x_t y_t}\left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right]. \tag{213}$$

Substituting this simplified ratio back into the formula for the Sharpe ratio difference gives the final expression. □