The Risk-Exposure Premium: A Theory of Portfolio Choice and the Wealth Distribution *

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Abstract

We develop a continuous-time heterogeneous agent model with aggregate risk to study portfolio choice and wealth dynamics. In our model, households face endogenous costs for hedging the stochasticity of their continuation value under aggregate shocks, creating a novel "risk-exposure premium" (REP). The REP generates a wealth-dependent intertemporal effect that drives heterogeneous risk-taking behavior. For households near borrowing constraints, the negative REP effect dominates despite near-zero relative risk aversion, driving systematic-risk asset allocation to zero and rationalizing limited participation in risky asset markets. As wealth increases, the REP effect turns positive, creating strong incentives for risky holdings that peak at intermediate wealth levels before declining as wealth scaling dominates. This mechanism traces the complete hump-shaped lifecycle portfolio trajectory observed empirically. We prove the existence of a "poverty trap region" where households rationally choose lower return-risk idiosyncratic assets to avoid aggregate risk, causing wealth stagnation and generating realistic wealth distribution. We propose a novel "local steady state" concept as a short-term fixed point that provides a more general terminal condition compared to existing approaches. The model preserves precautionary saving motives while providing analytical solutions under homogeneity assumptions that enable explicit comparison of the three economic forces governing portfolio choice.

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1 Introduction

Empirical research in household finance has documented two persistent and puzzling phenomena that challenge conventional economic theory. First, low-wealth households exhibit "limited participation" in high-return risky asset markets, with households approaching borrowing constraints choosing not to hold any risky assets. Second, households' risky asset allocations display a pronounced "hump-shaped" lifecycle pattern, with middle-aged households holding significantly higher portfolio shares than both young and elderly households. These facts stand in stark contrast to the predictions of classical portfolio theory, the continuous-time framework (Merton, 1969), which suggests that households with minimal wealth should exhibit near-zero relative risk aversion and thus display enormous myopic demand for risky assets.

The theoretical literature has extensively explored the role of labor income in reconciling these empirical patterns with economic intuition. However, this research faces fundamental conceptual challenges. On one hand, the introduction of labor income typically indicates that households should be more willing to bear financial risk, thereby reinforcing the theoretical prediction that low-wealth households should aggressively pursue risky investments. On the other hand, resolving the limited participation puzzle requires a positive correlation between asset returns and labor income, yet empirical estimates of this correlation coefficient remain highly controversial in the literature, with substantial disagreement regarding both its magnitude and stability. Even assuming such correlation exists, the resulting hedging demand tends to uniformly depress risky asset holdings across all working-age households relative to retirees, contradicting the observed hump-shaped lifecycle pattern.

This paper argues that the key to resolving these interconnected puzzles lies in shifting the analytical perspective from isolated individual decision-making to the dynamic interaction between individual choices and macroeconomic fluctuations. We construct a continuous-time heterogeneous agent (CTHA) model incorporating aggregate risk, transmitted through a tradable systematic-risk asset, forces households to choose their optimal exposure. To model this, we employ the mathematical framework of Mean Field Games (MFG) with common noise. This theoretical approach transforms the household problem from the traditional Hamilton-Jacobi-Bellman analytical framework into a stochastic backward partial differential equation framework. This theoretical structure reveals a previously overlooked economic force that we term the risk-exposure premium (REP). The REP represents the endogenous price that households must pay to hedge against the inherent stochasticity of their continuation value itself. In the face of aggregate shocks, household lifetime utility and distributional evolution form

coupled stochastic fields, with households actively managing their risk exposure to handle systematic risk for maximizing their welfare.

The REP mechanism provides a unified and powerful explanation for the aforementioned empirical puzzles. Our central theoretical finding is that the REP creates a wealth-dependent intertemporal effect of remarkable strength. We first establish that for households approaching borrowing constraints, their effective relative risk aversion approaches zero; however, the REP-dominated intertemporal effect exerts an overwhelmingly negative impact on their portfolios, driving their optimal systematic-risk asset allocation to zero and endogenously explaining limited participation. As wealth increases beyond this initial region, the REP effect turns positive, providing a strong additional incentive to hold the systematic-risk asset and pushing allocation to its peak. As wealth continues to grow, this positive effect is eventually dampened by the wealth scaling factor, causing the allocation to decline. Finally, we demonstrate that the wealthiest households exhibit asymptotically linear consumption behavior; their effective risk aversion converges to the constant relative risk aversion coefficient, and their portfolios converge back down to the classical Mertonian benchmark. This complete trajectory—from zero, to a peak, and back to the benchmark—naturally traces out the hump-shaped lifecycle investment path.

Furthermore, this mechanism yields richer economic implications. We prove the existence of a "poverty trap region" defined by an aggregate intertemporal function $\mathcal{T}$, which combines labor hedging and REP effects. Within this region, household portfolios fall significantly below their myopic demand, while outside this region they exceed myopic demand. This REP-dominated risk aversion behavior toward aggregate shocks leads households to rationally choose idiosyncratic-risk assets with lower return-risk ratios to avoid systematic risk, causing their wealth growth to stagnate relative to the overall economy and trapping them in a self-reinforcing region. Second, through the introduction of risky assets, our model resolves the theoretical challenge in continuous-time Aiyagari (1994)-Bewley (1986)-Huggett (1993) (ABH) models where the wealth distribution degenerates to a mass point, providing a new channel for generating realistic wealth inequality. Finally, using the non-insurability of labor as background risk, we demonstrate that the model preserves the precautionary saving motive, which supports household wealth accumulation and enables them to traverse different investment behavior regions over their lifecycle.

To address the "curse of dimensionality" problem arising from infinite-dimensional distributional measures, we introduce methodological innovations. First, we employ a temporary equilibrium concept whose validity is rooted in the supply side of the economy: in a recursive competitive market populated by rational firms, the certainty

equivalence principle dominates their decision-making, implying that macroeconomic dynamics are primarily determined by first moments of aggregate quantities. Therefore, focusing limited cognitive resources on predicting macroeconomic aggregates represents an efficient simplification for households. Since the REP mechanism stems from pricing non-diversifiable aggregate shocks, our conclusions remain robust to heterogeneous belief shocks or information noise. Second, we propose a novel steady state concept—the local steady state—which characterizes a special fixed point in an economy with persistent aggregate uncertainty when the instantaneous increment of shocks equals zero ($d\mathbf{W}_t = 0$). Compared to the "risky steady state" concept in the literature that relies on the entire shock history path returning to zero ($\mathbf{W}_T = 0$), this represents a more general and powerful analytical tool, as it serves as a short-term fixed point for economic dynamics and provides effective terminal conditions for solving the household's optimization problem. The completely deterministic steady state serves merely as a benchmark for our analysis. Finally, to further highlight the clarity and robustness of our model mechanisms, we derive analytical solutions for key model results under homogeneity assumptions. This enables convenient comparison of the magnitudes of three forces in the portfolio and explicit derivation of the aggregate intertemporal function $\mathcal{T}$.

The remainder of this paper is organized as follows. Section 2 develops the theoretical framework. Section 3 defines the equilibrium concept. Section 4 presents core theoretical results concerning portfolio choice and wealth dynamics. Section 5 visualizes the model's key mechanisms through numerical simulations. Section 6 concludes.

1.1 Literature Reviews

Our work is situated at the confluence of three dynamic research areas: CTHA macroeconomics, the mathematical theory of MFG with common noise, and the development of advanced computational methods to solve high-dimensional economic models.

This paper first contributes to the CTHA literature that builds on the seminal Aiyagari-Bewley-Huggett framework. The continuous-time approach pioneered in this context by Achdou et al. (2022), offers significant advantages over discrete-time formulations by avoiding integer problems (Stokey, 2008) and generating sparse problems. Subsequent research, such as Ahn et al. (2018) and Fernández-Villaverde et al. (2023), has extended these models to incorporate multiple assets and financial frictions. However, a fundamental challenge persists: these frameworks often abstract from a meaningful asset distribution or, when they include it, struggle to endogenously generate empirically documented patterns like the limited participation of low-wealth households in risky asset markets.

Methodologically, our framework departs from standard CTHA models by formally adopting the structure of a mean-field game with common noise. The mathematical foundations of MFG theory were established by Lasry and Lions (2007), with substantial literature focusing on systems without aggregate uncertainty. The introduction of common noise, which is essential for studying aggregate risk, represents a major theoretical leap. Carmona, Delarue, et al. (2018) and, most comprehensively, Cardaliaguet et al. (2019) characterized the resulting system of coupled backward-forward stochastic partial differential equations. While this literature provides a rigorous mathematical apparatus, its application to address core macroeconomic questions of portfolio allocation and wealth inequality remains a nascent field. Our paper helps bridge this gap by using the BSPDE formulation to derive new economic insights.

Finally, our research is situated against the backdrop of the rapidly advancing frontier of computational macroeconomics. For CTHA models without common noise, a range of powerful techniques now exists, including the finite difference methods used by Achdou et al. (2022), the state-space iteration of Ahn et al. (2018), and the neural network approaches of Fernández-Villaverde et al. (2023). More recent innovations focusing on solution techniques include the deep learning method of Han, Yang, et al. (2021), the perturbation techniques of Bilal (2023), the PINN methods of Gu et al. (2024), and the sequence-space Jacobian methods of Bilal and Goyal (2025). However, the presence of common noise in our model generate BSPDEs, which renders these methods insufficient. The resulting BSPDEs require novel computational strategies, like deep learning methods. Han et al. (2018) pioneeres a deep learning framework to solve high-dimensional PDEs via the discretized Euler method. Recent work by Huang (2023) has adapted a deep learning-based probabilistic approach using the finite volume method to solve CTHA models. Given these computational complexities, this paper focuses on first establishing the core theoretical properties of the economic mechanism at play, thereby providing a clear and robust foundation to guide future quantitative investigations.

2 Theoretical Framework

In our framework, we model a primary aggregate shock, dW_1, as the innovation driving systematic financial asset returns. This shock can be interpreted as the high-frequency component of fundamental economic shocks (e.g., technology or news shocks) as discussed in the broader macro-finance literature. We posit that financial markets serve as the most direct and instantaneous channel for the transmission of these shocks throughout the economy. In contrast, the labor market responds to this fundamental

shock more inertially due to wage-setting behavior and employment contracts. While a richer model could feature multiple fundamental shocks with different time horizons, our more parsimonious setup—which focuses on the financial market as the primary transmission channel—is sufficient to isolate our main mechanism of interest: the pricing of the value function's exposure to systematic risk.

2.1 Economic Environment

We consider an economy populated by a continuum of infinitely-lived households indexed by $i \in [0, 1]$ with unit mass. Each household i begins with initial wealth $a_{i,0} > 0$ and initial income $y_{i,0} > 0$, giving rise to an initial joint distribution $g(a_0, y_0)$. In the economy, aggregate shocks refer to common shocks that have an impact on the state (a_t, y_t) of every household. The central friction in our framework arises from the fundamental mismatch between different economic entities in terms of their risk characteristics.

To formalize this, we first define an *systematic-risk* asset if it can be traded and its price dynamics fully and instantaneously reflect the primary common shock. Mathematically, its diffusion term is driven solely by the aggregate shock or a linear combination of the aggregate shock. We use S_t to denote the price of systematic-risk asset, and it evolves as

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dW_s, \tag{1}$$

where μ_s and σ_s are the drift and volatility parameters of the systematic-risk asset. dW_s is a standard Wiener process representing the aggregate shock to the systematic-risk asset.

Consequently, the systematic-risk asset are perfect instruments for hedging or speculating on the primary aggregate shock. These objects are typically standardized and traded in highly liquid markets, which facilitates this perfect price transmission. Standard market indices are typical examples. We adopt a partial equilibrium approach in which these return processes are exogenously determined, allowing us to isolate the risk-exposure mechanism that is central to our analysis.

In contrast, we define an *idiosyncratic-risk* asset if it can be traded but its returns are driven by asset-specific idiosyncratic shocks. This implies that its correlation coefficient with the primary aggregate shock is zero. Let S_t^N be the price of idiosyncratic-risk asset, and it evolves as

$$\frac{dS_t^N}{S_t^N} = \mu_N dt + \sigma_N dZ_{N,t}, \tag{2}$$

where μ_N and σ_N are the drift and volatility parameters of the idiosyncratic-risk asset. $dZ_{N,t}$ is an idiosyncratic Wiener process. To avoid confusion, we will use $W_t \in \mathbf{W}_t$ to

denote aggregate shocks and $Z_t \in \mathbf{Z}_t$ to denote idiosyncratic shocks.

We define a *risk-free* asset as an asset if it does not respond to any shocks. Let B_t be the price of risk-free asset, and it follows

$$\frac{dB_t}{B_t} = r_f dt, \tag{3}$$

where r_f is the risk-free rate. Such assets are typically cash or government bonds.

Applying our classification to the labor market, household labor income represents a non-tradable asset with mixed risk characteristics. The aggregate wage rate w_t contains systematic risk:

$$\frac{dw_t}{w_t} = \mu_y dt + \sigma_y dW_y, \tag{4}$$

where μ_y and σ_y are the drift and volatility parameters of the wage process, and dW_y represents an orthogonal aggregate shock to the labor market.

Individual labor supply $n_{i,t}$ remains purely idiosyncratic and non-tradable:

$$\frac{dn_{i,t}}{n_{i,t}} = \tilde{\mu}_{n_i} dt + \tilde{\sigma}_{n_i} dZ_{n_i,t}, \tag{5}$$

where $\tilde{\mu}_{n_i}$ and $\tilde{\sigma}_{n_i}$ are household-specific parameters and $dZ_{n_i,t}$ is an idiosyncratic shock.

Applying Itô's lemma to the product $y_{i,t} = w_t n_{i,t}$ and noting that wage shocks are independent of idiosyncratic labor supply shocks, household i's labor income $y_{i,t}$ follows

$$\begin{aligned} \frac{dy_{i,t}}{y_{i,t}} &= \frac{dw_t}{w_t} + \frac{dn_{i,t}}{n_{i,t}} \\ &= (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y + \tilde{\sigma}_{n_i} dZ_{n_i,t}. \end{aligned} \tag{6}$$

The feature of this labor income endowment is that its risks are incompletely insurable. While the systematic component tied to dW_s can be partially hedged by taking positions in the systematic-risk asset, the orthogonal aggregate risk dW_y and the idiosyncratic risk $dZ_{n_i,t}$ cannot be hedged through traded assets. It is this incomplete insurability that generates a powerful hedging motive in the household's portfolio problem and makes their optimal choice a non-trivial trade-off.

We decompose shock processes as

$$\begin{aligned} dW_s &= dW_1 \\ dW_y &= \rho_{sy} dW_1 + \sqrt{1 - \rho_{sy}^2} dW_2, \end{aligned} \tag{7}$$

where dW_2 is an orthogonal shock. Let $\mathbb{E}[dW_s dW_y] = \rho_{sy} dt$ denote the instantaneous correlation between the systematic-risk asset and the aggregate wage rate, where $\rho_{sy} \in [-1, 1]$.

The Cholesky decomposition ensures that dW_s and dW_y maintain their standard Wiener properties:

$$dW_2 = \frac{1}{\sqrt{1-\rho_{sy}^2}}(dW_y - \rho_{sy} dW_s). \tag{8}$$

This generates two-dimensional aggregate risk structure $d\mathbf{W}_t \equiv (dW_1, dW_2)^\top$.

2.2 The Household's Problem

Each household chooses consumption and portfolio allocation to maximize expected utility, facing the incomplete environment described above. Households' investment opportunity set consists of the systematic-risk asset S_t and the risk-free asset B_t. Households face a continuous-time intertemporal budget constraint:

$$da_t = \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\}\, dt + \omega_t a_t \sigma_s dW_s, \tag{9}$$

subject to a borrowing constraint $a_t \geq 0$ and the systematic-risk asset portfolio allocation ω_t. At the borrowing constraint, consumption cannot exceed current labor income: $0 < c_t \leq y_t$.

Households exhibit constant relative risk aversion (CRRA) preferences over consumption, characterized by the instantaneous utility function:

$$U(c_t) := \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \tag{10}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

Each household i chooses optimal consumption $c_{i,t}$ to maximize expected lifetime utility:

$$\max_{c_{i,t}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} U(c_{i,t})\, dt \right], \tag{11}$$

where $\rho > 0$ is the discount rate.

According to our framework with common noise, the household's optimal problem can be formulated as the BSPDE $v(a_t, y_t, g_t, \mathbf{W}_t)$:

$$dv_t = \left\{ u(c_t) + \mathcal{L}[v_t] + \mathbb{E}_t \left[\frac{\partial v_t}{\partial t} \right] - \mathrm{div}(\mathcal{V}_t) \right\} dt + \mathcal{V}_t d\mathbf{W}_t, \tag{12}$$

where $\mathbb{E}_t(\cdot)$ is the conditional expectation operator, $\mathrm{div}(\cdot)$ is the divergence operator, $\mathcal{V}_t$ is the adaptive process, and $\mathcal{L}$ is the infinitesimal generator of the underlying state process:

$$\begin{aligned} \mathcal{L}[v_t] \equiv & \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) \frac{\partial v}{\partial y} \\ & + \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y}. \end{aligned} \tag{13}$$

In CTHA models with only idiosyncratic shocks $\mathbf{Z}_{i,t}$,the law of large numbers ensures that the value function v_t and the distribution g_t evolve deterministically, eliminating stochastic components at the population level (Achdou et al., 2021). The expected evolution of a household's value function v_t is fully described by the HJB operator $\mathcal{L}[v_t]$ and $\mathbb{E}_t\left[\frac{\partial v_t}{\partial t}\right]$.

This deterministic framework breaks down when the common noise $\mathbf{W}_t$ is introduced. It affects all households simultaneously and cannot be eliminated by the law of large numbers. The distribution g_t now becomes a stochastic process, forcing households to condition their optimization on the entire shock history $\mathbf{W}_{\tau\{0\leq\tau\leq t\}}$. The value function must therefore evolve as a random field $v(a_t, y_t, g_t, \mathbf{W}_t)$, requiring the more general backward stochastic partial differential equation (BSPDE) presented in equation (12). The BSPDE was proposed by Peng (1992), and its application in MFG with common noise was proposed by Cardaliaguet et al. (2019). The well-posedness of our dynamic formulation follows from the Glivenko-Cantelli theorem (Rachev & Rüschendorf, 2006), and standard results in mean field theory (Cardaliaguet et al., 2019).

The transition from HJB equation to BSPDE requires households to price in aggregate shocks. We define an *aggregate shocks impact* (ASI) as the stochastic change in the value function caused by the increment of aggregate shocks $d\mathbf{W}_t$ at time t:

$$ASI_t \equiv \mathcal{V}_t d\mathbf{W}_t = \begin{bmatrix} \mathcal{V}_{1,t} \\ \mathcal{V}_{2,t} \end{bmatrix} d\mathbf{W}_t. \tag{14}$$

It represents the direct blow or boost to the household's future overall welfare prospects due to the macroeconomic innovation. When a positive aggregate shock occurs, the household's value function instantly increases by ASI_t. The magnitude of the shock is determined by an endogenous sensitivity $\mathcal{V}_t$.

In the framework, the sensitivity $\mathcal{V}_t$ must be chosen such that the resulting equation maintains the proper stochastic structure, which is the adaptive process in the BSPDE:

$$\begin{aligned} \mathcal{V}_{1,t} =& \omega_t a_t \sigma_s \frac{\partial v}{\partial a} + \sigma_y \rho_{sy} \frac{\partial v}{\partial y} \\ \mathcal{V}_{2,t} =& \sigma_y \sqrt{1-\rho_{sy}^2} \frac{\partial v}{\partial y}. \end{aligned} \tag{15}$$

The process measures how much the entire future utility will be hit when macro common shocks $d\mathbf{W}_t$ arrive. This represents the continuous value exposure of the household to systemic risk.

The component $\mathcal{V}_{1,t}$ signifies the repercussions of the primary shock dW_1 on households' value function through the risk-exposure channel. In the context of a wage rate (i.e., idiosyncratic labor income), the impact of this component is exerted exclusively through the proportion of systematic-risk asset held by households. Component $\mathcal{V}_{2,t}$ is

indicative of the aggregate shock dW_2 that are unique to the wage market and correspond to the labor endowment.

We define an *aggregate shocks premium* (ASP) as the endogenous price that households must pay to bear the welfare impact of the stochasticity of aggregate shocks $d\mathbf{W}_t$ at time t:

$$\begin{aligned} ASP_t \equiv & -\operatorname{div}(\mathcal{V}_t) = -\frac{\partial \mathcal{V}_{1,t}}{\partial a} - \frac{\partial \mathcal{V}_{2,t}}{\partial y} \\ = & -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - \sigma_y \left(\rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sqrt{1-\rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \right). \end{aligned} \tag{16}$$

It refers to the risk cost or premium that must be deducted from the drift term of the value function due to its exposure to aggregate shocks. It measures the cost of "having an aggregate uncertain future" itself. Even if the expected value of the shock is zero ($\mathbb{E}[ASI_t] = 0$), its volatility alone can harm the welfare of a risk-averse household. The ASP quantifies this harm. Its value can explicitly depend on the exposure sensitivity $\mathcal{V}_t$. Consequently, at time t, households price aggregate shocks $\mathbf{W}_t$ as $ASI_t + ASP_t dt$.

The ASI term ensures adaptation to the common noise filtration, while theASP provides the corresponding drift-correction term. This is mathematically necessary to construct a well-posed dynamic coupled system, and together these two terms ensure that the evolution of the value function and the state space remains martingale in a random field characterized by stochastic evolution. Applying the Itô-Kunita-Wentzell formula (Kunita, 1990) generates the drift-correction term ASP_t. When the common noise affects both the value function v_t and the state space (a, y), their quadratic covariance term generates an additional drift term $\operatorname{div}(\mathcal{V}_t)$. The ASP_t is precisely designed to cancel out this mathematical artifact. This ensures that the resulting BSPDE is free of spurious terms and reflects coherent economic trade-offs, making it a mathematical precondition for a well-posed optimization problem in an economy with aggregate uncertainty.

For economic interpretation, we decompose the ASP into components based on whether they are directly controllable through the optimal portfolio choice ω_t. We define the component that households can directly control as the *risk-exposure premium* (REP):

$$REP_t \equiv -\omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right). \tag{17}$$

It is the direct source of a novel effect in portfolio choice.

All other components of the ASP, which are not directly scalable by the portfolio choice ω_t, are grouped into the *risk-exposure fixed premium* (REFP):

$$REFP_t \equiv -\sigma_y \left(\rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sqrt{1-\rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \right). \tag{18}$$

The REFP captures the background risk costs arising from labor income's exposure to both the primary and the orthogonal aggregate shock. This allows us to write the total premium as

$$ASP_t = REP_t + REFP_t. \tag{19}$$

The REP quantifies the marginal utility cost arising from households' risk exposure to the systematic-risk asset under the primary aggregate shock dW_1. Households' choice of portfolio share directly scales this cost, determining the magnitude and sign of the premium. The structure of REP reveals a fundamental asymmetry in how aggregate risk affects households across the wealth distribution. The premium consists of two opposing components: a positive compensation term $a_t \frac{\partial^2 v}{\partial a^2}$ and a negative cost term $\frac{\partial v}{\partial a}$. The compensation component is scaled by the wealth factor a_t, which means low-wealth households receive minimal compensation despite facing high risk exposure through the concavity of their value function $\frac{\partial^2 v}{\partial a^2}$. In contrast, the cost component operates without wealth scaling, making low-wealth households disproportionately sensitive to changes in marginal utility $\frac{\partial v}{\partial a}$. This asymmetric structure creates a systematic disadvantage for households in the low wealth range. While their high marginal utility of wealth makes them bear substantial costs from aggregate risk exposure, the wealth-scaled compensation mechanism provides them with inadequate relief. Consequently, the REP generates heterogeneous risk premiums that depend critically on households' position in the wealth distribution, with the pronounced effects concentrated in the low-to-medium wealth segments.

The term $\mathbb{E}_t[\frac{\partial v_t}{\partial t}]$ captures how distributional changes affect individual value functions:

$$\mathbb{E}_t\left[\frac{\partial v_t}{\partial t}\right] \equiv \iint \frac{\delta v_t}{\delta g(a,y)} \frac{\partial g(a,y)}{\partial t} da\, dy, \tag{20}$$

where $\frac{\delta v_t}{\delta g(a,y)}$ denotes the Fréchet derivative, quantifying the marginal impact of distributional changes on the value function.

The stochastic Kolmogorov Forward (SKF) equation governing the evolution of the wealth-income distribution g_t takes the form:

$$dg_t = (\mathcal{L}^* g_t)dt - \mathrm{div}(\mathcal{V}_t g_t) d\mathbf{W}_t, \tag{21}$$

where $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$:

$$\begin{aligned}\mathcal{L}^*[g_t] \equiv &- \left\{ \frac{\partial}{\partial a}\Big(\left((\omega_t(\mu_s - r_f) + r_f)\, a_t + y_t - c_t\right) g_t \Big) + \frac{\partial}{\partial y}\Big((\mu_y + \tilde{\mu}_{n_i}) g_t \Big) \right\} \\ &+ \left\{ \frac{\partial^2}{\partial a^2}\left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) + \frac{\partial^2}{\partial a \partial y}\Big(\omega_t a_t \rho_{sy} \sigma_s \sigma_y g_t \Big) \right\}.\end{aligned} \tag{22}$$

In a standard CTHA model with only idiosyncratic risk, the value function v_t of household i depends only on its own individual state (a_i, y_i). The aggregate distribution g_t evolves deterministically according to the law of large numbers, and individual households take this aggregate distribution as given when making their optimization decisions. The dynamic programming problem contains solely the drift term $\mathcal{L}[v_t]$ and the term $\mathbb{E}_t\left[\frac{\partial v_t}{\partial t}\right]$, which is generated by the evolution of individual states.

However, the introduction of common noise $\mathbf{W}_t$ fundamentally changes the nature of the problem. The aggregate distribution g_t now becomes a stochastic process, and all households' decisions must be made contingent on both their individual state (a, y) and the increment of aggregate shocks $d\mathbf{W}_t$. In the mean field game equilibrium, each household optimizes against the aggregate distribution g_t, recognizing that this distribution itself evolves stochastically due to the common noise affecting all agents simultaneously.

The divergence of $\mathcal{V}_t g_t$ has components:

$$\begin{aligned} \operatorname{div}(\mathcal{V}_{1,t} g_t) =& \frac{\partial}{\partial a}\left(\omega_t a_t \sigma_s g_t\right) + \frac{\partial}{\partial y}\left(\sigma_y \rho_{sy} g_t\right) \\ \operatorname{div}(\mathcal{V}_{2,t} g_t) =& \frac{\partial}{\partial y}\left(\sigma_y \sqrt{1-\rho_{sy}^2}\, g_t\right). \end{aligned} \tag{23}$$

The density satisfies the normalization $\iint g_t(a, y)\, da\, dy = 1$. It is crucial to distinguish the divergence term $\operatorname{div}(\mathcal{V}_t g_t)$ in the SKF equation, which governs the advection of the density g_t under common shocks, from the drift premium term $\operatorname{div}(\mathcal{V}_t)$ in the BSPDE, which adjusts the value function v itself. Consequently, the value function, previously defined as $v(a_t, y_t)$ depending exclusively on the individual state, now takes the form $v(a_t, y_t, g_t, \mathbf{W}_t)$, where g_t represents the aggregate distribution of all households in the economy.

We now define macro aggregates as follows:

$$\begin{aligned} K_t &= \iint \omega_i a_{i,t} g_t(a, y)\, da\, dy \\ B_t &= \iint (1-\omega_{i,t}) a_{i,t} g_t(a, y)\, da\, dy, \end{aligned} \tag{24}$$

where the economy has initial macro variables K_0 and B_0. These conditions ensure that all financial assets are held by households.

Although our model is a partial equilibrium model in which households are price takers, the evolutionary dynamics of the distribution determine the dynamics of the household's value function in this coupled system. Thus, each household needs to estimate the distribution measure.

3 Equilibrium

In the full rational expectations paradigm, the household's value function v_t is a McKean-Vlasov equation that depends on the entire cross-sectional distribution g_t. Solving the optimization problem would require each household to forecast the evolution of this infinite-dimensional object, a computationally intractable task that exemplifies an extreme form of the curse of dimensionality (Krusell & Smith, 1998; Dend Haan, 1996; Fernández-Villaverde et al., 2024). This modeling approach, while theoretically rigorous, also strains credulity, as it contradicts substantial empirical evidence suggesting that real-world agents employ considerably simpler forecasting heuristics.

To overcome these challenges, we motivate our modeling choice for household beliefs by first considering the supply side of the economy. We can consider a recursive competitive market consisting of identical firms whose profit function makes them want to forecast the aggregate output decisions of other firms just like them to choose their output. These firms decide their output level by solving a Bellman equation, as well as capital investment. Under this market structure, the certainty equivalence principle would apply to firms' optimization problems. Consequently, producers' decisions, which ultimately determine the aggregate dynamics of the capital stock K_t, would depend only on the conditional expectations of relevant variables, not on their variance or higher-order moments. This framework is proposed by Lucas Jr and Prescott (1971), Stokey and Lucas Jr (1989), Prescott and Mehra (2005), Sargent (2009), and Ljungqvist and Sargent (2018).

Since the actual law of motion for capital is thus primarily driven by its first moment, it provides a strong economic rationale for why households, as price-takers in the economy, would find it sufficient to focus their "limited cognitive resources" (Enke, 2024) and "limited information processing capabilities" (Javed & Sutton, 2024) on this single, crucial aggregate statistic. This approach aligns with the bounded rationality framework inspired by the learning literature (Evans & Honkapohja, 2012) and the approximate aggregation method of Krusell and Smith (1998). We therefore assume that households exhibit "sparse attention" (Gabaix, 2014, 2020): they focus on K_t instead of tracking the full distribution g_t.

This simplification means that the term in the household's problem that captures the impact of distributional changes, $\mathbb{E}_t[\frac{\partial v}{\partial t}]$, is replaced by the household's expectation of the evolution of the aggregate capital stock they track. We formalize this using a *perceived law of motion* (PLM), where households form beliefs about the dynamics of K_t. For analytical tractability and to establish a clear benchmark, we assume a consensus belief where each household's expectation for the evolution of aggregate capital is given

by

$$d\hat{K}_{i,t} \equiv \hat{\mu}_{k_i}(K_t, B_t)dt + \hat{\sigma}_{k_i}(X_{i,t})d\mathbf{Z}_{k_i,t}, \tag{25}$$

where $X_{i,t} := \{a_{i,t}, y_{i,t}\}$ is the individual state and $d\mathbf{Z}_{k_i,t}$ is idiosyncratic belief shocks. Here, we follow Grossman and Shiller (1982) and assume costless information, precluding idiosyncratic beliefs shocks ($\hat{\sigma}_{k_i} = 0$). The broader literature on idiosyncratic beliefs includes studies by Shiller et al. (1984), De Long et al. (1990), Anderson et al. (2005), Jouini and Napp (2007), and Bhamra and Uppal (2014). This framework also connects to recent bounded rationality work by Moll (2024) and Moll and Ryzhik (2025), who demonstrate how households can use price systems to predict distributional moments.

It is worth emphasizing that this simplification from the infinite-dimensional distribution g_t to its first moment K_t does not compromise the framework's central conclusions. One might argue that households could form subjective beliefs about higher-order moments, such as the variance of the wealth distribution, or that arbitrary shocks to individuals might alter the aggregate dynamics. However, any such belief heterogeneity, conditional expectations, or individual shocks remain idiosyncratic from the household's optimization perspective. This subjective uncertainty does not introduce new stochastic terms into the BSPDE that respond to common noise, but instead enters as deterministic drift-adjustment terms. Here, the term arising from its variance is analogous to additional precautionary savings motives. While incorporating such effects might quantitatively change households' optimal behavior, they cannot eliminate the wealth-dependent REP mechanism that drives our core results on limited participation and wealth inequality, as this mechanism stems fundamentally from the the law of large numbers that make idiosyncratic shocks do not cause random shocks to the aggregate distribution g_t.

This bounded rationality assumption naturally leads to a temporary equilibrium concept, bridging the macroeconomic literature on limited information processing (Maćkowiak & Wiederholt, 2009, 2015) with microeconomic evidence on wealth-dependent expectation formation (Malmendier & Nagel, 2016).

Definition 1. A *temporary equilibrium* consists of given asset prices $\{S_t, B_t\}$, labor income y_t, and the distribution g_t such that:

1. Each household i chooses consumption-portfolio choice $\{c_{i,t}, \omega_{i,t}\}$ to maximize its value function $v_{i,t}$ (26) by its status $X_{i,t}$;

2. The distribution g_t evolves according to the SKF equation (27);

3. The PLM and macro aggregates in the market are given by (25) and (24), respectively;

4. Asset prices and labor income evolve as (1), (3), (6), respectively.

The temporary equilibrium is characterized by the coupled system:

$$\begin{aligned} dv_t = &\max_{c,\omega} \left\{ u(c_t) + \{[\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) \frac{\partial v}{\partial y} + \hat{\mu}_k \frac{\partial v}{\partial \hat{K}} \right\} dt \\ &+ \left\{ \frac{1}{2}(\omega_t a_t \sigma_s)^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2) \frac{\partial^2 v}{\partial y^2} + \omega_t a_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} \right\} dt \\ &- \left\{ \omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) + \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} + \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial^2 v}{\partial y^2} \right\} dt \\ &+ \begin{bmatrix} \omega_t a_t \sigma_s \frac{\partial v}{\partial a} + \sigma_y \rho_{sy} \frac{\partial v}{\partial y} \\ \sigma_y \sqrt{1 - \rho_{sy}^2} \frac{\partial v}{\partial y} \end{bmatrix} d\mathbf{W}_t \end{aligned} \tag{26}$$

$$\begin{aligned} dg_t = &- \left\{ \frac{\partial}{\partial a} \left(\left((\omega_t(\mu_s - r_f) + r_f) a_t + y_t - c_t \right) g_t \right) + \frac{\partial}{\partial y} \left((\mu_y + \tilde{\mu}_{n_i}) g_t \right) + \frac{\partial}{\partial \hat{K}} \left(\hat{\mu}_k g_t \right) \right\} dt \\ &+ \left\{ \frac{\partial^2}{\partial a^2} \left(\frac{1}{2}(\omega_t a_t \sigma_s)^2 g_t \right) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2) g_t \right) + \frac{\partial^2}{\partial a \partial y} \left(\omega_t a_t \rho_{sy} \sigma_s \sigma_y g_t \right) \right\} dt \\ &- \begin{bmatrix} \frac{\partial}{\partial a} \left(\omega_t a_t \sigma_s g_t \right) + \frac{\partial}{\partial y} \left(\sigma_y \rho_{sy} g_t \right) \\ \frac{\partial}{\partial y} \left(\sigma_y \sqrt{1 - \rho_{sy}^2} g_t \right) \end{bmatrix} d\mathbf{W}_t. \end{aligned} \tag{27}$$

It is worth noting that temporary equilibrium permits systematic deviations between subjective belief expectations and macro aggregates, relaxing the fixed-point requirement of rational expectations equilibrium. This concept, originally developed by Hicks (1946) and Lindahl (1939) and formalized by Grandmont (1977, 1989), has found renewed application in modern macroeconomic models Woodford (2013), Piazzesi and Schneider (2016), García-Schmidt and Woodford (2019), and Farhi and Werning (2019).

We establish our analysis around two complementary steady-state concepts that provide analytical benchmarks for the dynamic system.

Definition 2. A *local steady state* is a steady state of the temporary equilibrium 1 containing the following features:

1. A moment of stochastic calm, where the instantaneous increment of aggregate shocks are zero ($d\mathbf{W}_t = 0$);

2. Persistent underlying aggregate shocks ($\sigma(\mathbf{W}_t) > 0$);

3. The satisfaction of the standard transversality condition:

$$\lim_{T \to \infty} \mathbb{E}_t \left[e^{-\rho(T-t)} v_t \right] = 0. \tag{28}$$

The LSS is characterized by the following system:

$$\rho v_t = \max_{c,\omega} \left\{ u(c) + \mathcal{L}[v_t] + \mathbb{E}_t \left[\frac{\partial v}{\partial t} \right] + ASP_t \right\}. \tag{29}$$

$$\frac{\partial g}{\partial t} = \mathcal{L}^*[g_t]. \tag{30}$$

Definition 3. A *determined steady state* (DSS) is a fixed point of the equilibrium conditions when the volatility of aggregate shocks vanishes ($\sigma(\mathbf{W}_t) = 0$).

The DSS is characterized by the following system:

$$\rho v_t = \max_{c,\omega} u(c_t) + \{r_f a_t + y_t - c_t\} \frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i}) \frac{\partial v}{\partial y} + \frac{1}{2} \tilde{\sigma}^2_{n_i} \frac{\partial^2 v}{\partial y^2} \tag{31}$$

$$\frac{\partial g_t}{\partial t} = -\frac{\partial}{\partial a}\Big((r_f a_t + y_t - c_t) g_t \Big) - \frac{\partial}{\partial y}\Big((\mu_y + \tilde{\mu}_{n_i}) g_t \Big) + \frac{\partial^2}{\partial y^2} \left(\frac{1}{2} \tilde{\sigma}^2_{n_i} g_t \right). \tag{32}$$

The LSS characterizes a specific, continuously occurring instantaneous fixed points in the economy. It is a short-term state in which the economy continues to move along inertia after receiving a shock until the next moment when the new shock increment changes. This allows us to construct a tractable and economically meaningful terminal value function for the finite-time approximation of the household problem.

The LSS represents a fundamentally different equilibrium concept from the DSS. While the DSS describes a hypothetical world where aggregate risk is permanently absent, assuming the aggregate shock volatility is identically zero across all time ($\sigma(\mathbf{W}_t) = 0$). This effectively reduces the stochastic model to its deterministic counterpart, providing a baseline where households face no aggregate uncertainty and consequently hold only risk-free assets ($\omega_t = 0$), as there is no asset with a risk premium. In contrast, the LSS characterizes moments of calm within an inherently stochastic economy where aggregate uncertainty persists ($\sigma(\mathbf{W}_t) > 0$), yet no new shock instantaneously materializes ($d\mathbf{W}_t = 0$). This crucial distinction means that households in the LSS make decisions while fully aware of ongoing aggregate risk and its potential future realization, leading them to demand risk premia and hold diversified portfolios. Therefore, while the DSS provides a static, non-stochastic anchor, the LSS offers a dynamic, instantaneous snapshot of equilibrium within a continuously uncertain environment.

Our definition of the LSS significantly generalizes and refines existing "risky steady state" approaches, such as those by Futia (1982), Duffie et al. (1994), Schenk-Hoppé and Schmalfuss (2001), Coeurdacier et al. (2011), and Fernández-Villaverde et al. (2023). Prior analyses often relied on the highly restrictive assumption that the cumulative shock process realizes at zero $\mathbf{W}_T = 0$. This strong assumption confines the analysis to a single, zero-probability path where the entire history of shocks must perfectly cancel out to achieve a deterministic long-term fixed point and invariant distributions. Our

framework permits a far more powerful and generalizable approach. We define the LSS by the much weaker condition that only the instantaneous increment of the shock is zero $d\mathbf{W}_t = 0$, while the cumulative shock $\mathbf{W}_T$ can be any arbitrary constant reflecting the realized history. For the household's optimization, the instantaneous stochastic welfare impact $ASI_t = \mathcal{V}_t d\mathbf{W}_t$ is nullified by $d\mathbf{W}_t = 0$ alone. Meanwhile, the cost of persistent uncertainty ASP_t remains fully active. Consequently, the stationary equation defining the LSS value function v_L differs fundamentally from the HJB equation in the DSS, as it retains the ASP. It drives the risk-exposure premium effects central to our analysis. By anchoring our finite-horizon BSPDE with the LSS terminal condition, we ensure that household optimization paths correctly internalize the pricing of aggregate risk, maintaining theoretical coherence even over truncated time horizons. This specification allows our framework to capture both the transitional dynamics toward long-term equilibrium and the steady-state properties of an economy with permanent aggregate risk.

4 Theoretical Results

The theoretical framework constructed for the household's problem is fundamentally different from the firm's problem. Under the linear-quadratic (LQ) framework, a core conclusion is the establishment of the certainty equivalence principle (Kwakernaak & Sivan, 1972; Chow, 1981; Hansen, Sargent, et al., 2004). This principle states that when the objective function is quadratic and the state space dynamics are linear, the stochastic optimal decision rule is a deterministic problem (Ljungqvist & Sargent, 2018). This means that risk returns and aversion only affect the firm's welfare level, but have no first-order effect on the optimal policy.

Our model breaks the classical principle at two levels. First, households have non-quadratic preferences and face endogenous and multiplicative rather than additive aggregate shocks, which is sufficient to make the decision rule risk-dependent. In addition, we propose the model with common noise, which makes the household's problem itself evolve into a stochastic process. Consequently, the optimization problem is to price the endogenous stochasticity of aggregate shocks.

It is this break from certainty that makes the structure and price of risk a first-order factor affecting household decision-making and gives rise to the core mechanism of this article, the REP effect.

4.1 The REP effect and the Price of Aggregate Shocks

This section aims to analyze the optimal portfolio decisions of households when

facing aggregate risk. These decisions originate from the optimization problem under the temporary equilibrium (26), which is mathematically formulated by the BSPDE (12).

The household's optimization problem yields the envelope condition for optimal consumption:

$$c_t^* = (u')^{-1}\left(\frac{\partial v}{\partial a}\right). \tag{33}$$

However, if the household decision set includes the systematic-risk asset, the optimal portfolio choice can change fundamentally. Taking the first-order condition (FOC) with respect to the portfolio share ω_t yields:

$$\omega_t^* = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left[\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)\frac{\partial v}{\partial a} + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_s}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right]. \tag{34}$$

The final term in the optimal portfolio choice arises directly from the results of the household's optimization over the REP. The first term is myopic demand, and the second term is hedging demand that constitutes the form with asset returns and labor supply, which was first proposed by Merton (1969). We formalize this novel mechanism as follows:

Definition 4. The *risk-exposure premium effect* is the component of a household's optimal portfolio share in the systematic-risk asset, given by $\Omega^*(a_t)$, which directly arises from the risk-exposure premium (17) within their optimization problem when choosing among a set of feasible investments including the systematic-risk asset and risk-free asset:

$$\Omega^*(a_t) \equiv \frac{1}{a_t \sigma_s}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right). \tag{35}$$

The REP effect captures how agents actively manage their portfolios to correct the increment of BSPDE generated by the primary aggregate shocks dW_1 in assets. To characterize this mechanism, we introduce the concept of effective relative risk aversion.

Definition 5. The *effective relative risk aversion* (ERRA) captures the curvature-to-slope ratio of the BSPDE with respect to wealth:

$$\eta(a_t) \equiv -a_t \frac{v_{aa}}{v_a}, \tag{36}$$

where $v_a \equiv \partial v/\partial a$ and $v_{aa} \equiv \partial^2 v/\partial a^2$.

The ERRA measures a household's preference for risk as reflected in its lifetime welfare function, moving beyond the constant risk aversion parameter γ that applies only to single-period consumption. It endogenously evolves as households price aggregate shocks $ASI_t + ASP_t dt$. Given the standard conditions ($a > 0$, $v_a > 0$, $v_{aa} < 0$), we

have $\eta(a_t) > 0$. The form of $\eta(a_t)$ is similar to the Arrow-Pratt measure of relative risk aversion ($R(c_t) \equiv -c_t u''/u'$), but the difference is that $R(c_t)$ measures instantaneous risk preference. The earliest derivation of the ERRA form comes from Merton (1969).

By substituting the ERRA definition (36) into the REP equation (17) and the REP effect equation (35), we obtain a more interpretable form:

$$REP_t = \omega_t \sigma_s \big(\eta_t - 1\big) \frac{\partial v}{\partial a} \tag{37}$$

and

$$\Omega^*(a_t) = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta_t}\right). \tag{38}$$

This representation reveals that the REP effect's sign and magnitude are critically determined by how the ERRA, $\eta(a_t)$, compares to the critical threshold of unity. The interaction between the state-dependent η_t and the $1/a_t$ scaling factor generates a rich, non-linear portfolio response that exhibits a distinctive hump-shaped profile across the wealth distribution.

The evolution of the REP effect can be characterized in three distinct phases. In the low-wealth region, as households approach the borrowing constraint, their ERRA falls towards zero ($\eta_t \to 0$). When $\eta_t < 1$, the term $(1 - 1/\eta_t)$ becomes large and negative, resulting in a strongly negative REP effect, $\Omega^*(a_t) < 0$. This creates a powerful disincentive to invest in the systematic-risk asset, as the risk-exposure premium overwhelms other portfolio motives.

At the critical threshold where $\eta(a_t) = 1$, the term $(1 - 1/\eta_t)$ equals zero, causing the REP effect to vanish completely, $\Omega^*(a_t) = 0$. This represents a pivotal wealth level where household risk preferences reach a critical balance, and portfolio allocation is determined solely by classical myopic and hedging demands. In the medium-to-high wealth region where $\eta_t > 1$, the term $(1 - 1/\eta_t)$ becomes positive, generating a positive REP effect, $\Omega^*(a_t) > 0$. Initially, as wealth increases beyond the critical point, both the marginal utility sensitivity and the risk aversion curvature remain substantial, creating rich interactions that drive the REP effect to its peak positive values. This provides strong additional incentives to hold the systematic-risk asset beyond standard portfolio motives.

However, as wealth continues to grow ($a_t \to \infty$), the $1/a_t$ scaling factor in equation (38) becomes the dominant force. Despite the persistent positive sign of $(1 - 1/\eta_t)$, the diminishing wealth factor causes the REP effect to gradually decline and ultimately converge to zero from above. This complete evolutionary pattern—from strongly negative values at low wealth, through zero at the critical point, rising to positive peaks in the medium wealth range, and finally declining toward zero at high wealth—creates the characteristic hump-shaped profile of the REP effect across the wealth distribution. This

non-monotonic relationship fundamentally alters standard portfolio theory predictions and provides a novel mechanism for understanding the "hump saving behavior" (Graaff, 1950; Ando & Modigliani, 1963) and "hump-shaped profile" of portfolio allocation in life-cycle models (Heaton & Lucas, 2000; Poterba & Samwick, 2001; Faig & Shum, 2002; Ameriks & Zeldes, 2011).

This cross-sectional hump-shaped profile has a compelling life-cycle interpretation, allowing our framework to intervene in a long-standing debate in household finance. The relationship between age and portfolio risk-taking is a subject of much disagreement, with a notable tension between empirical observations and the predictions of many canonical theoretical models. In life-cycle models, Bodie et al. (1992) early studies on portfolio decision making between asset returns and labor income. To reconcile theory with this evidence, a strand of literature calibrates income processes using micro-data (Jagannathan, Kocherlakota, et al., 1996; Davis & Willen, 2000; Campbell et al., 2001; Haliassos & Michaelides, 2003; Cocco et al., 2005), but typically requires a strong negative correlation between labor income and stock returns ($\rho < 0$). However, the robustness of this explanation is challenged by some work. Estimates of this critical correlation parameter are notoriously unstable (Guiso et al., 1996; Bertaut & Haliassos, 1997; Heaton & Lucas, 2000; Poterba, 2001; Ameriks & Zeldes, 2011) and vary dramatically across countries, industries, and time horizons, with many studies finding values close to zero or even positive (Campbell et al., 2001; Angerer & Lam, 2009; Betermier et al., 2012; Huggett & Kaplan, 2016). Recent studies on this topic include Benzoni et al. (2007), Hubener et al. (2016), Addoum (2017), Fagereng et al. (2017), Michaelides and Zhang (2017), Chang et al. (2018), Gomes (2020), Catherine (2022), and Catherine et al. (2024).

Our REP mechanism offers a novel and complementary explanation rooted not in human capital, but in financial wealth dynamics and the endogenous pricing of aggregate risk. To the extent that a household follows a typical life path of wealth accumulation—a path driven by motives such as precautionary savings, which our model endogenously generates (see Proposition 5 for a formal proof)—they will naturally traverse the wealth distribution. A young, low-wealth household starts in the region where $\eta_t < 1$, and the powerful negative REP effect suppresses their participation in the systematic-risk asset. As they enter their mid-career, higher-wealth years, their wealth crosses the threshold where $\eta_t > 1$, causing the REP effect to turn positive and boost their risky asset holdings. Finally, approaching retirement with peak wealth, the $1/a_t$ scaling factor ensures the REP effect diminishes, and their allocation converges toward the standard Merton's benchmark. Therefore, the REP effect translates a household's journey across the wealth distribution into the life-cycle portfolio pattern, providing a new perspective

on this disagreement based on households' strategic response to aggregate shocks.

4.2 ERRA and Portfolio Allocation at Both Ends

To understand the wealth-dependent nature of portfolio choice, we can link the ERRA and observable economic behavior. By deriving its relationship with the marginal propensity to consume out of wealth, we can rigorously analyze portfolio allocation at the extreme ends of the wealth distribution.

We exploit the envelope condition (33) and derive:

$$v_{aa} = u''(c_t^*)\frac{\partial c_t^*}{\partial a}, \tag{39}$$

and utilize the standard CRRA specification $\gamma = -c_t^* u''/u'$. Substituting equations (39) and γ into (36) yields:

$$\eta(a_t) = -a_t \cdot \frac{v_{aa}}{u'(c_t^*)} = -a_t \cdot \frac{\partial c_t^*}{\partial a} \cdot \frac{u''(c_t^*)}{u'(c_t^*)} = \frac{a_t}{c_t^*} \cdot \mathrm{MPC}_a \cdot \gamma, \tag{40}$$

where $\mathrm{MPC}_a \equiv \partial c_t^*/\partial a$ is the marginal propensity to consume out of wealth. This reveals that the ERRA transforms a household's inherent, constant risk preference γ into observed, time-varying risk-taking behavior at different wealth levels through an endogenous, state-varying the elasticity of consumption with respect to wealth $\epsilon_{c,a} \equiv (a_t/c_t^*) \cdot \mathrm{MPC}_a$. Its form is originally derived from Merton (1969, 1973).

4.2.1 ERRA and Portfolio Choice Near the Borrowing Constraint

As households approach their borrowing constraint $\underline{a}$, their optimal consumption is determined primarily by labor income rather than financial wealth, implying $a_t/c_t^* \to 0$ as $a \to \underline{a}$.

Proposition 1 (**ERRA at the Borrowing Constraint**)**.** For any household with finite constant relative risk aversion ($\gamma < \infty$) and bounded marginal propensity to consume out of wealth ($\mathrm{MPC}_a < \infty$), the effective relative risk aversion vanishes at the borrowing constraint:

$$\lim_{a \to 0^+} \eta(a_t) = 0. \tag{41}$$

Proof. See Appendix A.1. □

Corollary 1 (**Portfolio Choice at the Borrowing Constraint**)**.** The risk-exposure premium effect exhibits singular behavior at the borrowing constraint:

$$\lim_{a \to 0^+} \Omega^*(a_t) = -\infty, \tag{42}$$

leading to optimal portfolio choice:

$$\lim_{a\to 0^+} \omega_t^* = \begin{cases} 0 & \text{under short-selling restrictions} \\ -\infty & \text{without short-selling restrictions.} \end{cases} \tag{43}$$

Proof. See Appendix A.2. □

As household wealth approaches zero, the proposition reveals that its ERRA $\eta(a_t)$ also approaches zero near the borrowing constraint. According to its expression (36), when financial wealth is negligible, a household becomes extremely insensitive to risk aversion compared to the marginal return of wealth. Such households exhibit risk-seeking behavior. In the continuous-time portfolio choice framework, pure myopic demand is usually expressed as $\frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\eta(a_t)}$. As $\eta(a_t)$ approaches zero, the myopic demand becomes very large. This is consistent with the idea that the poor have a strong theoretical motivation to hold high-risk-premium assets in the absence of other complex interactions.

This corollary reveals strange household behavior in allocating the systematic-risk asset under borrowing constraints. When a household faces a risky asset with excess returns, its myopic demand becomes very large, theoretically prompting it to take on extremely high risk exposure. However, once the systematic-risk asset is introduced, households must pay the REP for the risk of aggregate shocks they bear. Near the borrowing constraint, this corollary shows that the REP effect $\Omega^*(a_t)$ tends to negative infinity and far exceeds the influence of myopic demand and hedging demand for labor income. This overwhelming negative force causes households to avoid the asset with aggregate risk. The result is no participation when there are short-selling restrictions and a large amount of short-selling when there are none.

This mechanism can be understood through the lens of a household's internal conflict between short-term temptations and long-term prudence. This dramatic tension arises from two powerful, opposing forces dictated by the household's proximity to the borrowing constraint. On one hand, the household faces the myopic demand of infinite magnitude. As Proposition 1 shows, η_t approaches zero for a low-wealth household. Consequently, their myopic demand for the risky asset, proportional to $1/\eta_t$, becomes extraordinarily large. This reflects the powerful short-term impulse to escape poverty through a high-risk, high-return gamble—a seemingly rational desire to go "all-in" when immediate circumstances are dire. On the other hand, the household, as a rational and forward-looking agent, must also consider its intertemporal hedging demand, which is precisely what the REP effect $\Omega^*(a_t)$ captures. For a household with negligible financial assets, the most catastrophic long-run risk is not simple market volatility, but a fundamental shift in the economic environment that leaves them permanently behind. A positive aggregate shock, while benefiting everyone, raises the entire society's wealth level,

making their own meager position feel even more precarious. The systematic asset is, by definition, positively correlated with this specific long-run risk. The infinitely negative REP effect therefore represents an urgent hedging demand to short this unfavorable future state.

In this battle of infinities, long-term prudence triumphs over short-term temptation. The overwhelming negative hedging demand completely dominates the myopic desire for high returns. The household strategically avoids the systematic asset not because they are ignorant of its excess returns, but because they correctly price its role as a carrier of a devastating long-run risk to their welfare. This provides a new, reasonable explanation based on endogenous risk pricing for the "limited participation" documented in the literature (Vissing-Jørgensen, 2002; Campbell, 2006). The REP mechanism provides a more fundamental explanation of the behavior of the poor in systematic-risk asset markets.

4.2.2 ERRA and Portfolio Allocation at High Wealth Levels

We now analyze the limiting case as wealth grows arbitrarily large, which provides a crucial benchmark for understanding the behavior of the most affluent households. In this regime, the influence of labor income diminishes, and financial wealth dynamics become paramount. Consistent with a long line of research in continuous-time finance and macroeconomics, consumption and wealth become proportional at the limit.

Theorem 1 (**Asymptotic Consumption-Wealth Ratio**)**.** Let $c^*(a_t, y_t, \hat{K}_t)$ be the optimal consumption policy derived from the solution to the BSPDE (26). The consumption-to-wealth ratio converges in the mean-square sense to a deterministic constant $\bar{k}_L$ as wealth a_t goes to infinity:

$$\lim_{a\to\infty} \mathbb{E}_t \left[\left| \frac{c^*(a_t, y_t, \hat{K}_t)}{a_t} - \bar{k}_L \right|^2 \right] = 0, \tag{44}$$

where $\bar{k}_L$ is the optimal consumption-wealth ratio in the LSS without labor income or aggregate state dependence, given by:

$$\bar{k}_L = \frac{1}{\gamma} \left(\rho + (\gamma - 1) \left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2} \right) \right). \tag{45}$$

Proof. See Appendix B.1 and B.2. □

Proposition 2 (**ERRA at High Wealth Levels**)**.** For households with positive and finite CRRA coefficient ($0 < \gamma < \infty$) and satisfying Theorem 1 assumption, the effective relative risk aversion converges to the CRRA parameter:

$$\lim_{a\to\infty} \eta(a_t) = \gamma. \tag{46}$$

Proof. See Appendix B.3. □

Corollary 2 (**Portfolio Choice at High Wealth Levels**)**.** As wealth increases without bound, the risk-exposure premium effect vanishes, and portfolio allocation converges to the classical baseline ratio:

$$\lim_{a\to\infty} \Omega^*(a_t) = 0, \qquad \lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2}\frac{1}{\gamma}. \tag{47}$$

Proof. See Appendix B.4. □

The theorem 1 echoes the classic findings of Merton (1969, 1973) and aligns with modern analyses like Benhabib et al. (2015) and Achdou et al. (2022), who show that policy functions become asymptotically linear for large wealth levels.

This asymptotic behavior can be interpreted through the same "myopic vs. long-term" lens, but for the wealthy, the narrative shifts from one of internal conflict to one of strategic alignment. Unlike the poor, whose short-term desires are pitted against their long-term fears, the affluent household's myopic and intertemporal motives work in concert to encourage participation in the systematic-risk asset. The wealthy household's myopic demand is prudent and stable. As Proposition 2 shows, their ERRA $\eta(a_t)$ converges to the constant CRRA parameter γ. This results in a moderate, positive portfolio allocation, $(\mu_s - r_f)/\gamma\sigma_s^2$, which aligns with the standard Mertonian benchmark for capturing the equity premium. There is no desperate, infinite gamble, only a calculated short-term investment decision.

Crucially, their intertemporal hedging demand, represented by the REP effect, also becomes positive. A wealthy household's primary long-run risk is not a decline in relative status, but a stagnation of the overall economy that would devalue their vast financial capital. A positive aggregate shock signals a healthier, more productive future investment opportunity set. To hedge against the risk of a low-growth future, they strategically seek more exposure to the states of the world where their capital will be most productive. This positive REP effect is a rational, forward-looking decision to "double down" on systemic prosperity. For a significant range of high wealth, their total desired allocation is the sum of a positive myopic demand and a positive hedging demand, leading them to hold even more of the systematic asset than predicted by standard models.

However, this alignment does not last indefinitely. As wealth approaches infinity ($a \to \infty$), the intertemporal hedging component of their decision, while remaining positive, is ultimately rendered negligible by the $1/a_t$ scaling factor in its definition. The economic intuition is that for an agent with near-infinite capital, the marginal benefit of hedging fluctuations in the future investment opportunity set becomes vanishingly

small compared to the sheer scale of their current wealth. Long-term strategic concerns are dwarfed by the magnitude of the existing portfolio.

This leads to two key implications. First, the most significant portfolio heterogeneity driven by the REP mechanism is concentrated among low-to-moderate wealth households, where the tension between myopic and hedging demands is most acute. Second, for the wealthiest households at the upper tail of the distribution, their behavior converges back to the classic, myopic representative-agent benchmark. The dynamics of wealth inequality are therefore driven by fundamentally different forces across the distribution: by the novel REP hedging mechanism for the majority, and by standard portfolio theory for the ultra-rich.

4.3 The Poverty Trap, Capital Accumulation, and Precautionary Savings

The analysis in the preceding section has established the clear 'boundary conditions' for portfolio behavior across the wealth distribution. At the lower tail, households are endogenously driven out of risky asset markets, while at the upper tail, their allocations converge to the classical Merton benchmark.

This stark divergence in behavior at the extremes naturally raises a crucial question: What are the consequences for wealth dynamics and inequality for the households situated between these two poles? Having established why poor households do not participate, we now turn to analyze how this choice, driven by the REP mechanism, impacts their long-term capital accumulation and potentially locks them in a poverty trap. We next explore these distributional dynamics, demonstrating how the model's core mechanism sorts agents and gives rise to self-reinforcing patterns of inequality.

4.3.1 Risk-exposure Preference and the Poverty Trap

To isolate how risk-exposure tolerance shapes portfolio choice and creates potential poverty traps, we construct a clear comparative environment. Consider two distinct, hypothetical investment universes available to households, both of which include the risk-free asset B_t. The asset set F contains only the systematic-risk asset S_t. This represents a market portfolio, such as a broad stock market index fund. The asset set N contains only idiosyncratic-risk assets S_t^N. This could represent an asset class whose risks are primarily idiosyncratic or driven by orthogonal factors.

We first analyze a household's choice when constrained to invest in only one of these universes. We then examine the benefits of diversification by allowing the household to form a portfolio combining both asset types. This setup allows us to precisely identify

the premium that households demand for bearing direct aggregate risk and how this demand can lead to seemingly suboptimal choices.

Proposition 3 (**Preference for idiosyncratic-risk Assets**)**.** For any household with identical wealth and income characteristics, if the following conditions are met:

$$\mathcal{T} \equiv a_t \rho_{sy} \sigma_y \frac{\partial^2 v}{\partial a \partial y} + \big(\eta(a_t) - 1\big) \frac{\partial v}{\partial a} < 0. \tag{48}$$

Then:

(i) The optimal portfolio weights satisfy the strict ordering:

$$\omega_F^* < \omega_N^*. \tag{49}$$

(ii) For equivalent welfare levels, households demand a strictly higher Sharpe ratio from systematic-risk assets:

$$\mathcal{S}_F > \mathcal{S}_N \tag{50}$$

with the required Sharpe ratio premium given by:

$$\mathcal{S}_F - \mathcal{S}_N = -\frac{\mathcal{T}}{a_t v_a} > 0. \tag{51}$$

Proof. See Appendix C.1. □

The difference $\mathcal{S}_F - \mathcal{S}_N$ represents the additional Sharpe ratio premium that households demand to hold systematic-risk assets. This premium compensates for the welfare cost of aggregate shocks to poverty-trap households. The magnitude of this premium depends positively on the severity of the combined effect $|\mathcal{T}|$ and the household's marginal utility of wealth v_a, and inversely on the household's wealth level a_t. It implies that the inequality becomes more pronounced as household wealth approaches the borrowing constraint.

Now we consider a constrained asset set P which is a convex set consisting of the asset set F and the asset set N with the weight $\beta \in (0, 1)$ on asset F:

$$P = \beta F + (1 - \beta) N. \tag{52}$$

Corollary 3 (**Preference for Constrained Assets**)**.** For any household with identical wealth and income characteristics, if the following conditions are met:

$$\mathcal{T} \equiv a_t \rho_{sy} \sigma_y \frac{\partial^2 v}{\partial a \partial y} + \big(\eta(a_t) - 1\big) \frac{\partial v}{\partial a} < 0. \tag{53}$$

Then:

(i) For equivalent risk-return characteristics, the optimal portfolio weights satisfy:

$$\omega_P^* > \omega_F^*. \tag{54}$$

(ii) The required Sharpe ratio for portfolio P lies strictly between the individual asset requirements:

$$\mathcal{S}_N < \mathcal{S}_P < \mathcal{S}_F \tag{55}$$

with the specific relationship:

$$\mathcal{S}_P - \mathcal{S}_F = \frac{\mathcal{T}}{a_t v_a} \left(1 - \frac{\beta \sigma_s}{\sqrt{\beta^2 \sigma_s^2 + (1-\beta)^2 \sigma_N^2}} \right) < 0. \tag{56}$$

These results provide a theoretical foundation for behaviors that might otherwise appear puzzling, offering a new perspective on the poverty trap mechanism rooted in the logic of intertemporal hedging. The condition $\mathcal{T} < 0$ defines a "poverty trap zone" in the wealth distribution. Within this zone, a household's total intertemporal hedging demand—driven by the combined desire to hedge both labor income risk and the REP effect—is strongly negative. For low-wealth households, this negativity is overwhelmingly dictated by the REP component ($\eta_t < 1$), reflecting a profound desire to hedge against the adverse long-term welfare consequences of a positive aggregate shock.

This powerful negative hedging demand manifests as a rational hedging premium. As shown in equation (51), households trapped by the condition $\mathcal{T} < 0$ demand a strictly higher Sharpe ratio from the systematic asset. They are not being irrational; they are explicitly pricing the long-run risk that this asset carries. Faced with a choice, they will not select the asset with the best risk-return profile (the market index 'F') if it forces them to bear a risk they desperately wish to hedge. Instead, they rationally choose assets like 'N' or 'P', even with inferior Sharpe ratios. In effect, they are willingly paying an insurance premium in the form of forgone expected returns. This "premium" buys them insurance against the catastrophic long-run risk of being left permanently behind in a wealthier society.

The rational choice, however, has a tragic long-term consequence: it creates a self-reinforcing poverty trap. By systematically "paying" this insurance premium each period, households in this zone achieve slower wealth accumulation than their peers who can afford to hold the systematic asset. This slower growth ensures they remain trapped in the very low-wealth region where the negative hedging demand is most severe, locking them into a cycle of risk aversion and stagnant wealth. This finding also bridges a gap with the literature on lottery-like stocks. Our framework demonstrates that the poor's apparent preference for assets with high idiosyncratic risk may not stem

from cognitive biases, but from a rational aversion to the systemic risk that our REP mechanism precisely quantifies.

This result presents a novel resolution to an apparent dilemma within the household finance literature. The standard theoretical framework, as articulated in works such as Viceira (2001) and Campbell and Viceira (2002), suggests that employed households should take larger portfolio allocations to risky assets. The logic is twofold: first, their high marginal utility of consumption implies a strong desire for wealth growth; second, because their financial wealth is small relative to their human capital, portfolio risk contributes little to their overall risk profile. This would imply a high optimal allocation to risky assets. Building on this foundation, Campbell (2018) further proposes that, given any risk preference in models with labor income, households should increase their demand for risky assets that can hedge their nonfinancial income risk and should reduce demand for assets that are positively correlated with those risks.

Our framework not only affirms this myopic demand but reveals why it does not translate into holding systematic assets. The missing piece in the traditional view is the powerful, countervailing intertemporal hedging demand, which our term $\mathcal{T}$ fully captures. While a poor household's myopic impulse screams for high-return assets, we show that their rational, forward-looking self identifies an even more powerful need to hedge the long-run risk associated with aggregate shocks. The negative REP effect is so dominant at low wealth that the total hedging demand ($\mathcal{T} < 0$) overwhelms the myopic drive for growth.

Consequently, their desire for high returns is not extinguished; it is redirected. Low-wealth households are not inherently risk-averse. They rationally shun the specific, welfare-damaging risk of systematic assets while channeling their risk-taking appetite towards assets where the risk is primarily idiosyncratic and does not trigger the severe long-term hedging penalty. This explains how a household can simultaneously exhibit a high desire for wealth growth while holding little to no assets in the market index—a sophisticated behavior our model attributes to the endogenous pricing of and hedging against aggregate shocks.

We also provide theoretical perspectives on the following areas. Poor households exhibit risk-seeking characteristics similar to the loss domain of prospect theory, which has been studied in Kai-Ineman, Tversky, et al. (1979), Kahneman and Tversky (2013), Thaler and Johnson (1990), McDermott et al. (2008), Fiegenbaum and Thomas (1988), N. C. Barberis (2013), and Tversky and Kahneman (1992). This behavior may cause poor households' portfolios to be "squeezed into" an asset full of specific risks, similar to a lottery. Assets as lotteries study was conducted by Kumar (2009), N. Barberis and Huang (2008), Boyer and Vorkink (2014), Bali et al. (2011), and Eraker and Ready

(2015). Research on poverty and risk behavior has also proved that the poor tend to invest in scattered and informal specific risk sets (Banerjee & Duflo, 2007; Hsieh & Klenow, 2009).

4.3.2 Capital Accumulation

While the analysis above reveals that the REP mechanism creates a 'poverty trap' by suppressing risky asset holdings for low-wealth households, this does not automatically imply that aggregate risk reduces the total capital stock. The impact on households outside the trap, who may increase their holdings, must also be considered. To understand the net effect on the macroeconomy, we must shift our perspective from the cross-sectional heterogeneity of portfolio choices to the aggregate level. We now ask a different question: How does the introduction of aggregate financial risk itself, compared to a world entirely devoid of DSS, shape the economy's overall capacity for capital accumulation?

Proposition 4 (**Capital Accumulation under Aggregate Financial Risk**)**.** After a major financial risk dW_1 is introduced into an economy, the capital stock in temporary equilibrium K_t^T is greater than the DSS capital stock K_t^D without the risk at any moment:

$$K_t^T \geq K_t^D. \tag{57}$$

Proof. See Appendix C.3. □

This proposition explains why economies with more volatile financial markets (if risk premia are positive) tend to exhibit higher capital stocks than deterministic economies. The theoretical framework addresses the degenerate problem of wealth distribution in the continuous-time ABH framework. A major challenge in the CTHA model without common shocks is that the optimal portfolio allocation leads to $\omega_D^* = 0$ and total capital stock is zero ($K_t^D = 0$). This leads to the fact that under the borrowing constraint, the wealth distribution degenerates to a point mass and the economy is boring, as described in Achdou et al. (2022). This makes it possible to generate a realistic wealth distribution without relying on arbitrary preference heterogeneity.

4.3.3 Precautionary Savings

Our theoretical framework is naturally extended to examine how uninsurable idiosyncratic labor income risk affects household savings behavior in the presence of aggregate shocks. This extension is particularly important for understanding the excess capital accumulation of the macroeconomy in response to idiosyncratic shocks. Huggett and

Ospina (2001) emphasized that this accumulation is the total amount of precautionary savings in the economy. The incomplete market model has a larger steady-state capital stock than the complete market model because agents cannot insure their labor (Ljungqvist & Sargent, 2018).

Proposition 5 (**Precautionary Savings against Uninsurable Labor Risk**)**.** The introduction of uninsurable idiosyncratic labor income risk ($\tilde{\sigma}^0_{n,i} > 0$) induces households to hold strictly higher levels of assets s^*_L compared to an otherwise identical economy s^*_{NL} where such idiosyncratic risk is absent ($\tilde{\sigma}_{n,i} = 0$):

$$s^*_L > s^*_{NL}. \tag{58}$$

Proof. See Appendix C.4. □

Higher savings lead to greater asset accumulation over time. We have established that uninsurable idiosyncratic labor income risk creates a precautionary savings motive that leads to higher asset accumulation compared to an economy without such risk. This result emerges naturally from the convexity of the additive value function and represents a fundamental insurance mechanism in dynamic economies with incomplete markets. The magnitude of this effect depends on the household's position in the wealth distribution and the degree of prudence in their preferences, with lower-wealth households exhibiting stronger precautionary responses due to their higher marginal utility of wealth and the convexity of the value function.

This result holds regardless of the correlation between systematic-risk assets and labor income, as long as the idiosyncratic component cannot be perfectly hedged through the tradeable assets. Even in the presence of aggregate shocks and partial hedging opportunities through portfolio choice, idiosyncratic risk creates additional welfare costs that households can only mitigate through self-insurance via asset accumulation.

This proposition reflects the fundamental insurance principle in dynamic consumption theory. When households face uninsurable background risk, they respond by accumulating additional assets to self-insure against adverse income realizations. This precautionary behavior makes the expected marginal utility of consumption higher under uncertain income than under its certainty equivalent. Early work on precautionary savings was done by Miller (1974) and Sibley (1975). This research was further developed by Zeldes (1989), Kimball (1991), Kimball (1990), and Wang (2003). Carroll and Kimball (1996) summarized the class of hyperbolic absolute risk aversion (HARA) utility functions based on the discrete Bellman framework. This proposition is also related to risk vulnerability that adding an background risk to wealth makes risk-averse households behave in a more risk-averse way with respect to any other independent risk (Gollier & Pratt, 1996).

4.4 Homogeneity in the LSS

To more directly understand the core mechanism of the model and obtain an analytical solution, we introduce a assumption: in the LSS, the value function $v(a, y)$ is homogeneous with respect to the household's financial wealth a and labor income y:

$$v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x), \tag{59}$$

where $h(x)$ is an unknown function that is only related to the asset-income ratio $x_t \equiv a_t/y_t$. This assumption allows us to simplify the household's decision-making problem into a single state variable problem that is only related to the asset-income ratio x. Some studies using similar log-linear methods include Viceira (2001) and Campbell and Viceira (2002). This setting allows us to handles cross-derivatives of the value function and derive explicit expressions for the ERRA, REP, and their impact on the optimal portfolio.

Lemma 1 (**Distorted Effective Relative Risk Aversion**)**.** Under the assumption that the value function is homogeneous, the effective relative risk aversion $\eta(a_t)$ is equal to a *distorted risk aversion function* $\tilde{\eta}(x_t)$ determined only by the asset-income ratio x:

$$\eta(a_t) = \tilde{\eta}(x_t) \equiv -x_t \frac{h''(x)}{h'(x)}. \tag{60}$$

Proof. See Appendix D.1. □

The lemma 1 shows that whether from a general form or under a more structural homogeneous assumption, the core variable that determines a household's effective risk attitude is jointly determined by the elasticity of consumption with respect to wealth $\epsilon_{c,a}$ and the basic risk aversion coefficient γ. We now use it to derive the explicit forms of the REP effect and the optimal portfolio allocation under the homogeneity assumption.

Lemma 2 (**Homogeneous REP Effect and Optimal Portfolio Allocation**)**.** Under the assumption that the value function is homogeneous,
(i) The risk-exposure premium effect $\Omega^*(a_t)$ simplifies to a function $\tilde{\Omega}(x_t, y_t)$:

$$\tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t \sigma_s} \left(1 - \frac{1}{\tilde{\eta}(x_t)}\right). \tag{61}$$

(ii) The optimal portfolio allocation ω_t^* simplifies to a function $\tilde{\omega}^*(x_t, y_t)$:

$$\tilde{\omega}^*(x_t, y_t) = \frac{1}{\tilde{\eta}(x_t)} \left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s y_t}(\tilde{\eta}(x_t) - \gamma) + \frac{1}{x_t y_t \sigma_s}(\tilde{\eta}(x_t) - 1)\right]. \tag{62}$$

Proof. See Appendix D.2. □

This lemma provides an explicit analytical solution for the household's portfolio choice under the homogeneity assumption. It clearly demonstrates how the optimal allocation $\tilde{\omega}^*$ is governed by the asset-income ratio x_t through the endogenous risk aversion $\tilde{\eta}(x_t)$ and scaled by the level of labor income y_t. The results in Lemma 1 and Lemma 2 provide the foundation for deriving analytical solutions for the mechanisms. We now formalize these results.

Proposition 6 (Homogeneous Form of the Poverty Trap Condition)**.** Under the assumption that the value function is homogeneous, the poverty trap condition function $\mathcal{T}$ defined in equation (48) simplifies to a function $\tilde{\mathcal{T}}(x_t)$ which is explicitly dependent on the household's asset-income ratio:

$$\tilde{\mathcal{T}}(x_t) = \left[x_t \rho_{sy} \sigma_y \big(\tilde{\eta}(x_t) - \gamma\big) + \big(\tilde{\eta}(x_t) - 1\big)\right] < 0. \tag{63}$$

Proof. See Appendix D.3. □

Corollary 4 (**Analytical Sharpe Ratio Premium**)**.** Under the homogeneity assumption, the Sharpe ratio difference between a diversified portfolio P and a systematic-risk asset F, as defined in Corollary 3 and $\beta \in [0, 1)$, has the following analytical solution:

$$\begin{aligned} &\mathcal{S}_{\mathcal{P}} - \mathcal{S}_{\mathcal{F}} \\ =&\frac{1}{x_t y_t}\left[x_t \rho_{sy} \sigma_y \big(\tilde{\eta}(x_t) - \gamma\big) + \big(\tilde{\eta}(x_t) - 1\big)\right] \left(1 - \frac{\beta \sigma_s}{\sqrt{\beta^2 \sigma_s^2 + (1-\beta)^2 \sigma_N^2}}\right) < 0. \end{aligned} \tag{64}$$

Proof. See Appendix D.4. □

These results provide a clear, analytical form for the condition that drives households into the poverty trap. The sign of $\tilde{\mathcal{T}}(x_t)$ depends explicitly on the household's effective risk aversion $\tilde{\eta}(x_t)$ relative to its baseline risk aversion γ and the neutrality level of unity. The explicit analytical solution for the premium households require to alter their portfolio composition. It precisely quantifies how a household at state (x_t, y_t) with effective risk aversion $\tilde{\eta}(x_t)$ trades off the higher Sharpe ratio of a systematic-risk asset against the welfare cost of its associated aggregate risk, which is captured by the $\tilde{\mathcal{T}}(x_t)$ function.

5 Stationary Analysis of the REP Mechanism

To translate our theoretical framework into quantitative predictions, we parameterize the model and visualize the core economic mechanisms. Our parameterization is

disciplined by the theoretical properties derived in Section 4. Specifically, we adopt a parsimonious functional form for the household consumption rule, $c(a)$, that is consistent with the theoretical properties derived in the preceding sections. We adopt the form $c(a) = (c_0^p + (\bar{k}_L \cdot a)^p)^{1/p}$, which ensures that consumption is sustained by a base level c_0 at low wealth and becomes asymptotically proportional to wealth ($c(a) \approx \bar{k}_L \cdot a$) for affluent households, consistent with Theorem 1. This formulation generates a smooth, monotonic elasticity of consumption with respect to wealth $\epsilon_{c,a}(a)$, which in turn ensures that the ERRA, $\eta(a) = \gamma \cdot \epsilon_{c,a}(a)$, adheres to the boundary conditions established in Propositions 1 and 2. The subsequent analysis illustrates how this endogenous risk aversion shapes portfolio decisions across the wealth distribution.

For our baseline simulations, we set the model parameters as follows: the decay rate p is 0.8; the coefficient of CRRA, γ, is 3; the time discount rate, ρ, is 0.05. We assume an annualized expected return for risky assets, μ_s, of 8%, and a risk-free rate, r_f, of 3%. The mean income, y_0, is normalized to 1.0. Asset volatilities are set at $\sigma_s = 0.30$ for risky assets and $\sigma_y = 0$ for income. For hand-to-mouth households at the lower end of the asset spectrum, consumption c_0 equals their income.

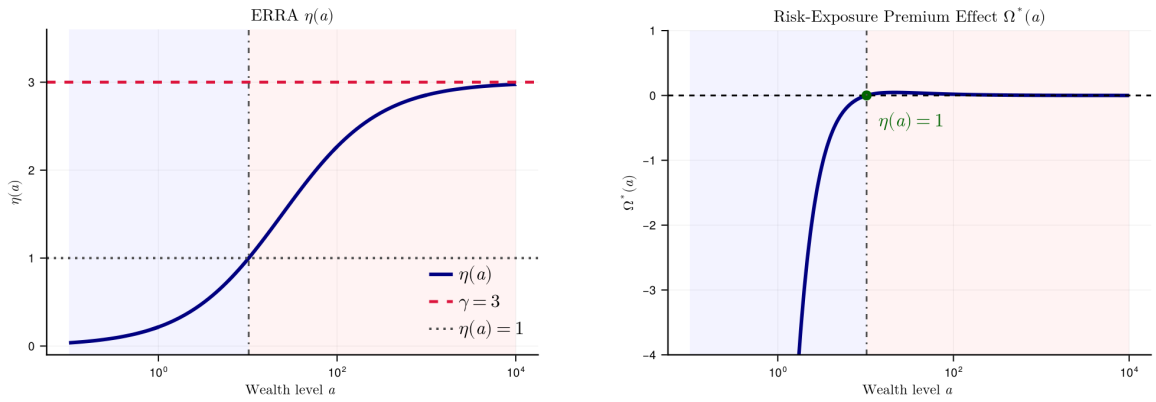
5.1 REP and its Implications

The cornerstone of our mechanism is the state-dependent nature of households' REP. Figure 1a plots the ERRA, $\eta(a)$, as a function of household wealth, a. The figure illustrates the theoretical predictions of our model.

First, consistent with Proposition 1, as wealth approaches the borrowing constraint ($a \to 0$), ERRA $\eta(a)$ converges to zero. This occurs because consumption is almost entirely determined by non-financial income as same as wealth is small, making the marginal utility of financial wealth nearly constant concerning changes in a. Second, in line with Proposition 2, as wealth becomes large, $\eta(a)$ asymptotically converges to the underlying preference parameter, γ, which is set to 3 in this calibration.

It is worth noting that the transition of $\eta(a)$ across the critical value of unity dictates the sign and impact of the REP effect, $\Omega^*(a) \propto \frac{\eta(a)-1}{\eta(a)}$. As shown in Figure 1a, for households with wealth below a certain threshold, $\eta(a) < 1$, inducing a negative premium that powerfully suppresses demand for risky assets. Conversely, for wealthier households with $\eta(a) > 1$, the premium becomes positive, reflecting an endogenously hedging behavior against aggregate shocks.

Figure 1b isolates the REP effect, $\Omega^*(a)$. The singular nature of this term at low wealth levels is apparent. As $a \to 0$, $\eta(a) \to 0$, causing $\Omega^*(a)$ to diverge to $-\infty$. This provides a powerful endogenous force for non-participation in the risky assets market. For wealthier households, as $\eta(a)$ approaches γ, the term $\frac{\eta(a)-1}{\eta(a)}$ stabilizes. However, the



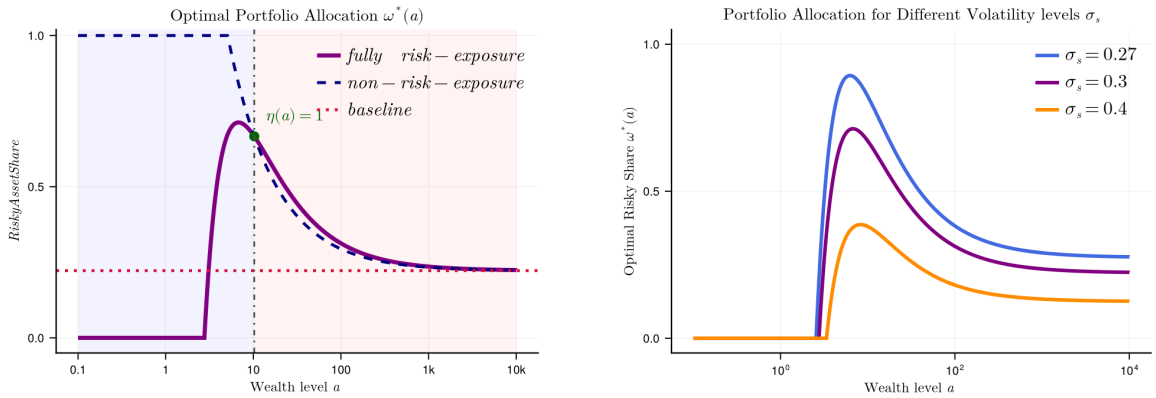
(a) (b)

Figure 1: Effective Relative Risk Aversion and the REP Effect

presence of the $1/a$ scaling factor in the definition of $\Omega^*(a)$ ensures that the effect itself vanishes, approaching zero from above.

5.2 Optimal Portfolio Allocation and Endogenous Limited Participation

The wealth-dependent ERRA $\eta(a)$, channeled through the REP mechanism, produces a rich pattern of portfolio holdings across the wealth distribution. Figure 2a depicts the optimal risky asset share, ω_t^*, and contrasts it with the benchmark: the classical Merton (1969, 1973) portfolio choice problem based on no risk in labor income.



(a) (b)

Figure 2: Optimal Portfolio Allocation

The results provide a clear visualization of our central theoretical claims. At low levels of wealth, the REP effect $\Omega^*(a)$ is strongly negative, as shown in Figure 2a. This

force overwhelms the myopic demand, driving the optimal allocation ω_t^* to the zero lower bound. This provides a novel, endogenous rationale for the empirically documented phenomenon of limited stock market participation among low-wealth households, a direct consequence of Corollary 1. Our model generates this result without imposing exogenous constraints or assuming extreme correlations between labor income and asset returns.

As wealth increases, $\eta(a)$ surpasses one, and the REP effect becomes positive, leading to a distinctive "hump" shape in the optimal portfolio. For the wealthiest households, two effects ensure convergence to the baseline ratio in Corollary 2. First, the REP effect $\Omega^*(a)$ vanishes due to the $1/a$ scaling factor. Second, $\eta(a)$ converges to γ. Consequently, as predicted by Corollary 2, the optimal portfolio share ω_t^* robustly converges to the baseline ratio, demonstrating that standard portfolio theory serves as a good approximation for the affluent but fundamentally mischaracterizes the behavior of the less wealthy facing common shocks in risky assets.

Figure 2b further explores the properties of the optimal portfolio choice by presenting a comparative statics exercise to the volatility of risky assets, σ_s. An increase in volatility unambiguously lowers the demand for risky assets across all wealth levels, a standard result. Note that the figure illustrates a more subtle point: the wealth threshold at which households begin to participate in the risky assets market increases with volatility. This is because higher volatility amplifies the cautionary effect of the REP mechanism at low wealth levels, making households even more reluctant to hold risky assets.

5.3 Implications for Wealth Dynamics

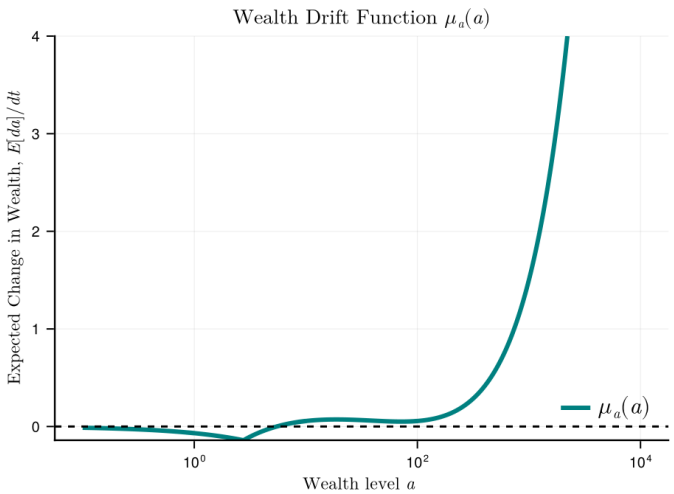


Figure 3: Wealth Accumulation Dynamics

Finally, we examine how these wealth-dependent portfolio choices translate into wealth accumulation dynamics. Figure 3 plots the expected instantaneous change in wealth, or the wealth drift function, $\mu_a(a) = \mathbb{E}[da]/dt$. The drift function is negative for a significant range of low to moderate wealth levels, indicating an endogenous force pushing these households toward the borrowing constraint. This "poverty trap" dynamic is a direct result of the limited participation induced by the REP mechanism; by holding only the risk-free asset, these households have lower expected returns.

However, at very high wealth levels, the portfolio share converges to the baseline ratio, and the wealth drift becomes strongly positive and convex. This convexity implies that the wealthiest households accumulate wealth at an accelerating rate, creating a powerful mechanism for the emergence of a fat-tailed wealth distribution. The interplay between the poverty trap at the lower assets and the explosive accumulation at the upper assets, both driven by the endogenous portfolio responses to common shocks, provides a novel channel for generating and sustaining wealth inequality.

It is imperative to acknowledge that the analysis presented in this subsection is predicated on the assumption that all households possess a common benchmark income y_0. The reality is that a household's income y usually corresponds to their wealth a. This will certainly make the poverty trap worse.

6 Conclusion

This paper confronts a foundational challenge in modern macroeconomics: the difficulty of constructing models that simultaneously generate realistic wealth inequality and endogenous portfolio heterogeneity in the presence of aggregate risk. We demonstrate that moving from the standard HJB framework to a mean field game with common shocks provides a resolution. This approach, characterized by a coupled system of a BSPDE and an SKF equation, reveals a previously unrecognized mechanism in portfolio choice: the risk-exposure premium.

Our central theoretical result is that the risk-exposure premium, which arises purely from the interaction between individual optimization and aggregate dynamics, fundamentally reshapes household risk-taking in a wealth-dependent manner. For households near borrowing constraints, the mechanism generates an extreme aversion to systemic risk, providing a novel, endogenous explanation for limited stock market participation. For the wealthiest households, this effect vanishes, and their behavior converges to the predictions of classical portfolio theory. This differential impact offers a new channel through which aggregate shocks drive wealth distribution dynamics, allowing the model to sustain significant inequality even under standard preference assumptions.

The implications of this framework are broad. Economic theory bridges the gap between heterogeneous agent macroeconomics and household finance, establishing the feedback from aggregate risk as a first-order determinant of cross-sectional portfolio choice. This paper has focused on isolating this theoretical mechanism in a partial equilibrium setting. The natural next step for this research agenda is to embed the REP channel within a general equilibrium model. Such a model, once solved numerically, would allow for a disciplined quantitative assessment against macroeconomic and financial data, making it possible to measure the importance of this new channel and to analyze the distributional consequences of various economic policies.

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A. Proofs of Main Results at the Borrowing Constraint

A.1 Proof of Proposition 1

Proof. To prove the proposition, we analyze the limit of each component of the product in (40) as wealth a_t approaches the borrowing constraint from above, which we take to be $a_t \to 0^+$.

The assumptions stated in the proposition are:

1. The coefficient of constant relative risk aversion, γ, is a finite positive constant.
2. The marginal propensity to consume out of wealth, $\frac{\partial c}{\partial a} \equiv \frac{\partial c_t^*}{\partial a_t}$, is bounded, i.e., there exists some $M < \infty$ such that $\frac{\partial c}{\partial a} \leq M$ as $a_t \to 0^+$.

Furthermore, we consider a household with a positive stream of non-financial income, $y_t > 0$. As the household's financial wealth a_t approaches zero, consumption does not fall to zero. Instead, it is sustained by labor income. Thus, optimal consumption c_t^* converges to a strictly positive value, $c^*(0, y_t) > 0$. Consequently, the limit of the wealth-to-consumption ratio is:

$$\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} = \frac{\lim_{a_t \to 0^+} a_t}{\lim_{a_t \to 0^+} c_t^*} = \frac{0}{c^*(0, y_t)} = 0. \tag{65}$$

According to the equation (40), we can now evaluate the limit of $\eta(a_t)$:

$$\begin{aligned} \lim_{a_t \to 0^+} \eta(a_t) &= \lim_{a_t \to 0^+} \left(\frac{a_t}{c_t^*} \frac{\partial c}{\partial a} \gamma \right) \\ &= \left(\lim_{a_t \to 0^+} \frac{a_t}{c_t^*} \right) \cdot \left(\lim_{a_t \to 0^+} \frac{\partial c}{\partial a} \right) \cdot \gamma \\ &= (0) \cdot (\text{bounded value}) \cdot (\text{finite constant}) \\ &= 0. \end{aligned} \tag{66}$$

This establishes that the effective relative risk aversion vanishes as the household's wealth approaches the borrowing constraint. □

A.2 Proof of Corollary 1

Proof. The proof is presented in two parts, corresponding to the two limits stated in the corollary.

The REP term is given by equation (38):

$$\Omega_t^*(a_t) = \frac{1}{a_t \sigma_s} \frac{\eta(a_t) - 1}{\eta(a_t)} = \frac{1}{a_t \sigma_s} \left(1 - \frac{1}{\eta(a_t)} \right). \tag{67}$$

We analyze its limit as $a_t \to 0^+$. From Proposition 1, we have $\lim_{a_t \to 0^+} \eta(a_t) = 0$. Since for any $a_t > 0$, wealth is positive, consumption is positive, $\frac{\partial c}{\partial a} > 0$ (for a non-Giffen good), and $\gamma > 0$, it follows that $\eta(a_t) > 0$. Consequently, the limit is approached from the right, i.e., $\eta(a_t) \to 0^+$.

Let's examine the terms in (67):

1. $\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s} = +\infty$, assuming asset volatility σ_s is a positive constant.
2. $\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta(a_t)}\right)$. Since $\eta(a_t) \to 0^+$, the term $1/\eta(a_t) \to +\infty$. Thus, the expression tends to $-\infty$.

The limit of the product is therefore:

$$\lim_{a_t \to 0^+} \Omega^*(a_t) = \left(\lim_{a_t \to 0^+} \frac{1}{a_t \sigma_s}\right) \cdot \left(\lim_{a_t \to 0^+} \left(1 - \frac{1}{\eta(a_t)}\right)\right) = (+\infty) \cdot (-\infty) = -\infty. \tag{68}$$

This confirms that the REP term diverges to negative infinity at the borrowing constraint.

The optimal portfolio allocation ω_t^* from equation (34) can be expressed as the sum of the REP term $\Omega^*(a_t)$ (35) and the classical portfolio choice

$$\omega_t^C \equiv \omega_t^* - \Omega^*(a_t). \tag{69}$$

The classical component can be expressed in terms of ERRA as:

$$\begin{aligned} \omega_t^C =& -\frac{1}{a_t}\left(-\frac{a_t}{\eta(a_t)}\frac{1}{v_a}\right)\left[\left(\frac{\mu_s - r_f}{\sigma_s^2}\right)v_a + \left(\frac{\rho_{sy}\sigma_y}{\sigma_s}\right)v_{ay}\right] \\ =& \frac{1}{\eta(a_t)}\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right]. \end{aligned} \tag{70}$$

The full portfolio is then:

$$\omega_t^* = \frac{1}{\eta(a_t)}\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right] + \frac{1}{a_t\sigma_s}\frac{\eta(a_t) - 1}{\eta(a_t)}. \tag{71}$$

To evaluate the limit, we combine terms over a common denominator, $a_t\sigma_s\eta(a_t)$:

$$\omega_t^* = \frac{a_t\sigma_s\left[\frac{\mu_s - r_f}{\sigma_s^2} + \frac{\rho_{sy}\sigma_y}{\sigma_s}\frac{v_{ay}}{v_a}\right] + \eta(a_t) - 1}{a_t\sigma_s\eta(a_t)}. \tag{72}$$

Let us analyze the limit of the numerator and the denominator as $a_t \to 0^+$.

As $a_t \to 0^+$, the first term $a_t\sigma_s[\dots] \to 0$ because the bracketed expression is assumed to be bounded. From Proposition 1, $\eta(a_t) \to 0$. Thus, the numerator converges to:

$$\lim_{a_t \to 0^+} (a_t\sigma_s[\dots] + \eta(a_t) - 1) = 0 + 0 - 1 = -1. \tag{73}$$

As $a_t \to 0^+$, both $a_t \to 0$ and $\eta(a_t) \to 0$. The denominator $a_t\sigma_s\eta(a_t) \to 0$. Since $a_t > 0$, $\sigma_s > 0$, and $\eta(a_t) > 0$, the denominator approaches 0 from the positive side (0^+).

Combining these results, the limit of the *unconstrained* optimal portfolio is:

$$\lim_{a_t \to 0^+} \omega^*_{t,\text{unc}} = \frac{-1}{0^+} = -\infty. \tag{74}$$

This shows that the underlying optimal behavior is to take an unboundedly large short position in risky assets.

However, the corollary states the portfolio choice *vanishes*. This result holds under the empirically relevant institutional constraint that households cannot take infinite short positions, often modeled simply as a no-short-sale constraint, $\omega^*_t \geq 0$.

Let $\omega^*_{t,\text{unc}}$ be the unconstrained solution derived above. The constrained optimal portfolio choice is given by:

$$\omega^*_t = \max(0, \omega^*_{t,\text{unc}}). \tag{75}$$

Since we have shown that $\lim_{a_t \to 0^+} \omega^*_{t,\text{unc}} = -\infty$, there exists a $\delta > 0$ such that for all $a_t \in (0, \delta)$, $\omega^*_{t,\text{unc}}(a_t) < 0$. In this region, the optimal constrained choice is the corner solution:

$$\omega^*_t(a_t) = 0 \quad \text{for } a_t \in (0, \delta). \tag{76}$$

Consequently, the limit of the constrained portfolio choice is:

$$\lim_{a_t \to 0^+} \omega^*_t = 0. \tag{77}$$

□

B. Proofs of Main Results at High Wealth Levels

This appendix provides rigorous proof for the main result in the text concerning the asymptotic behavior of the household's optimal consumption-wealth ratio. We prove that, as a household's wealth approaches infinity, its consumption-wealth ratio converges in the L^2 sense to a deterministic constant determined by the economy's fundamental parameters. The proof relies on a scale transformation of the governing BSPDE, modern SPDE regularity theory, and martingale convergence theorems.

B.1 Preliminaries and Statement of the Main Theorem

The agent's value function $v(a, y, \hat{K}, t)$ is the unique solution to the following BSPDE, given in the main text as the equation (26):

$$
\begin{aligned}
-dv_t = \sup_{c_t,\omega_t} \Big\{ & u(c_t) + \left([\omega_t(\mu_s - r_f) + r_f]a_t + y_t - c_t\right) v_a + \frac{1}{2}(\omega_t a_t \sigma_s)^2 v_{aa} \\
& + \mu_y v_y + \frac{1}{2}\sigma_y^2 v_{yy} + \omega_t a_t \rho_{sy}\sigma_s\sigma_y v_{ay} + \hat{\mu}_k v_{\hat{K}} - \omega_t \sigma_s \left(a_t v_{aa} + v_a\right) \\
& - \sigma_y \rho_{sy} v_{ay} - \sigma_y \sqrt{1-\rho_{sy}^2}\, v_{yy} \Big\} dt - ASI_t,
\end{aligned}
$$

where for brevity we denote partial derivatives by subscripts (e.g., $v_a \equiv \frac{\partial v}{\partial a}$) and the martingale component by ASI_t. The optimal policy $c^*(a, y, \hat{K}, t)$ is derived from the first-order condition $u'(c^*) = v_a$.

Our proof requires standard assumptions on the boundedness and smoothness of the model's coefficients.

Assumption 1 (Regularity of Coefficients)**.** The utility function is of the CRRA form, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 1$. The market and income process coefficients $(\mu_s, r_f, \sigma_s, \mu_y, \sigma_y, \rho_{sy}, \hat{\mu}_k)$ are assumed to be uniformly bounded and sufficiently smooth (e.g., globally Lipschitz continuous) in the state variables.

Under this assumption, standard results from the theory of MFG with common noise guarantee the existence and uniqueness of a solution $v \in C^{2,2,2,1}$ in the state space $(a, y, \hat{K}, t)$, which possesses sufficient regularity for the following analysis. Note that if the state variables were to include distribution measures, this framework would extend to a McKean-Vlasov equation, demanding more stringent regularity assumptions on the coefficients to guarantee the existence and uniqueness of solutions, as discussed in works like Cardaliaguet et al. (2019) and Achdou et al. (2021).

B.2 Proof of Theorem 1

The proof proceeds in four steps. First, we define a scale-invariant value function and derive the BSPDE it satisfies. Second, we perform an asymptotic analysis on this new BSPDE, showing that its stochastic, income-related, and macro-state-dependent terms vanish in the limit. Third, we characterize the resulting limiting deterministic HJB equation. Finally, we use the convergence of the value functions to establish the L^2 convergence of the policy functions.

Proof. **Step 1: The Scale Transformation** We introduce a scaling parameter $\xi > 0$ and define a new value function v_ξ which normalizes the original value function v by the agent's scale. The transformation applies only to the individual-specific state variables (a, y), not the aggregate state $\hat{K}$.

Definition 6. For any $\xi > 0$, the scaled value function $v_\xi : (0,\infty) \times \mathbb{R} \times \mathbb{R} \times [0,T] \to \mathbb{R}$ is defined as:

$$v_\xi(a, y, \hat{K}, t) := \frac{1}{\xi^{1-\gamma}} v(\xi a, \xi y, \hat{K}, t). \tag{78}$$

The key to the proof is to understand the dynamics of v_ξ.

Lemma 3. The scaled value function $v_\xi(a, y, \hat{K}, t)$ satisfies a BSPDE where the coefficients of the labor income process and the magnitude of the martingale term are scaled down by powers of ξ.

Proof of Lemma 3. We derive the dynamics of v_ξ by applying the chain rule to its definition. The partial derivatives of v_ξ relate to those of v as follows:

$$\begin{aligned}
\frac{\partial v_\xi}{\partial a} &= \xi^\gamma v_a(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial a^2} &= \xi^{\gamma+1} v_{aa}(\xi a, \dots), \\
\frac{\partial v_\xi}{\partial y} &= \xi^\gamma v_y(\xi a, \dots), & \frac{\partial^2 v_\xi}{\partial y^2} &= \xi^{\gamma+1} v_{yy}(\xi a, \dots), \\
\frac{\partial^2 v_\xi}{\partial a \partial y} &= \xi^{\gamma+1} v_{ay}(\xi a, \dots), & \frac{\partial v_\xi}{\partial \hat{K}} &= \xi^{\gamma-1} v_{\hat{K}}(\xi a, \dots).
\end{aligned} \tag{79}$$

Let $c_\xi = c/\xi$ and $\omega_\xi = \omega$ be the scaled controls. Substituting these into the original BSPDE (78), evaluating at $(\xi a, \xi y, \hat{K}, t)$, and multiplying the entire equation by $\xi^{\gamma-1}$ yields the BSPDE for $v_\xi(a, y, \hat{K}, t)$. The drift term, for instance, transforms as:

$$\xi^{\gamma-1} u(c) = u(c/\xi) = u(c_\xi). \tag{80}$$

And the wealth dynamics term becomes:

$$\begin{aligned}
&\xi^{\gamma-1} \left([\omega(\mu_s - r_f) + r_f](\xi a) + (\xi y) - (\xi c_\xi) \right) v_a(\xi a, \dots) \\
&\qquad = \left([\omega(\mu_s - r_f) + r_f] a + y - c_\xi \right) v_{\xi,a}(a, \dots).
\end{aligned} \tag{81}$$

After transforming all terms, the labor income drift and volatility terms are scaled: $\mu_y \mapsto \mu_y/\xi$ and $\sigma_y \mapsto \sigma_y/\xi$. The martingale term $dM_{\xi,t}$ for v_ξ has coefficients that are also scaled down, which is the subject of the next step. □

Step 2: Asymptotic Analysis of the Scaled BSPDE

We now analyze the BSPDE for v_ξ in the limit as $\xi \to \infty$. We show that all terms that differentiate the agent's problem from the classic Merton problem vanish.

Lemma 4 (Vanishing Martingale Component)**.** Let $M_{\xi,t}$ be the martingale component of the solution to the scaled BSPDE. Then,

$$\lim_{\xi \to \infty} \mathbb{E}\left[\sup_{s \in [0,t]} |M_{\xi,s}|^2 \right] = 0. \tag{82}$$

Proof of Lemma 4. The martingale term $dM_{\xi,t}$ is a linear combination of the original stochastic drivers, with coefficients depending on the scaled derivatives. A typical coefficient is the one on $dW_{2,t}$, which includes the term $\frac{\sigma_y}{\xi}\rho_{sy}v_{\xi,y}$. The homogeneity of the value function implies that for some constant K, $|v_y(a,y,\hat{K},t)| \leq Ka^{-\gamma}$. Thus,

$$\left|\frac{\sigma_y}{\xi}\rho_{sy}v_{\xi,y}(a,\dots)\right| = \left|\frac{\sigma_y}{\xi}\rho_{sy}\xi^{\gamma}v_y(\xi a,\dots)\right| \leq \frac{|\sigma_y\rho_{sy}|}{\xi}\xi^{\gamma}K(\xi a)^{-\gamma} = \frac{K|\sigma_y\rho_{sy}|}{a^{\gamma}}\frac{1}{\xi}. \tag{83}$$

The squared coefficient decays at a rate of ξ^{-2}. Applying this to all components of the martingale term, the quadratic variation $\langle M_\xi\rangle_t$ satisfies:

$$\mathbb{E}[\langle M_\xi\rangle_t] = \mathbb{E}\left[\int_0^t \|\mathbf{Z}_{\xi,s}\|_F^2 ds\right] \leq \frac{Ct}{\xi^2} \tag{84}$$

for some constant C independent of ξ, where $\|\cdot\|_F$ is the Frobenius norm. By the Burkholder-Davis-Gundy (BDG) inequality, there exists a universal constant C_{BDG} such that:

$$\mathbb{E}\left[\sup_{s\in[0,t]}|M_{\xi,s}|^2\right] \leq C_{\text{BDG}}\mathbb{E}[\langle M_\xi\rangle_t] \leq \frac{C_{\text{BDG}}Ct}{\xi^2}. \tag{85}$$

Taking the limit as $\xi\to\infty$, the right-hand side converges to zero, which proves the lemma. □

The deterministic terms related to labor income $(\mu_y/\xi, \sigma_y/\xi)$ and the macro-state $(\hat{\mu}_k v_{\hat{K}}/\xi^{1-\gamma})$ also vanish. The former is immediate from the scaling. For the macro-state term, we posit that in the limit of infinite wealth, the agent's value is insensitive to marginal changes in the aggregate state, implying $\lim_{\xi\to\infty} v_{\xi,\hat{K}} = 0$.

Step 3: Convergence to the Deterministic Benchmark Problem

As $\xi\to\infty$, the BSPDE for v_ξ converges in distribution to a deterministic partial differential equation. This limiting equation is the familiar HJB equation for an infinitely-lived agent with no labor income, corresponding to the LSS optimal problem:

$$\begin{aligned}&\rho v_S(a)\\ =&\sup_{c,\omega}\left\{u(c) + [(\omega(\mu_s - r_f) + r_f)a - c - \omega\sigma_s]v_S'(a) + [\frac{1}{2}(\omega a\sigma_s)^2 - \omega a\sigma_s]v_S''(a)\right\}.\end{aligned} \tag{86}$$

The solution to this HJB equation is $v_S(a) = B\frac{a^{1-\gamma}}{1-\gamma}$, where B is a constant. The associated optimal consumption-wealth ratio is the constant $\bar{k}_L$ given in the statement of Theorem 1.

To derive $\bar{k}_L$, we substitute the proposed solution $v_S(a) = B\frac{a^{1-\gamma}}{1-\gamma}$ and the CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ into the HJB equation (86). The first and second derivatives of $v_S(a)$ are:

$$\begin{aligned}v_S'(a) &= Ba^{-\gamma}\\ v_S''(a) &= -\gamma Ba^{-\gamma-1}.\end{aligned} \tag{87}$$

First, we find the optimal consumption c^*. The first-order condition with respect to c is $u'(c) - v_S'(a) = 0$, which gives:

$$c^{-\gamma} = Ba^{-\gamma} \implies c^* = B^{-1/\gamma}a. \tag{88}$$

Let $\bar{k}_L = B^{-1/\gamma}$, so $c^* = \bar{k}_L a$.

Next, we find the optimal portfolio share ω^*. The first-order condition with respect to ω is:

$$[(\mu_s - r_f)a - \sigma_s]v_S'(a) + (\omega\sigma_s^2 a^2 - \sigma_s a)v_S''(a) = 0. \tag{89}$$

Substitute the derivatives $v_S'(a)$ and $v_S''(a)$:

$$[(\mu_s - r_f)a - \sigma_s](Ba^{-\gamma}) + (\omega\sigma_s^2 a^2 - \sigma_s a)(-\gamma Ba^{-\gamma-1}) = 0. \tag{90}$$

Dividing by $Ba^{1-\gamma}$ (since $B \neq 0$ and $a \neq 0$):

$$\omega^*(a) = \frac{\mu_s - r_f}{\gamma\sigma_s^2} + \frac{\gamma - 1}{\gamma a \sigma_s}. \tag{91}$$

Now, substitute $c^* = \bar{k}_L a$, $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2} + \frac{\gamma-1}{\gamma a\sigma_s}$, $v_S(a)$, $v_S'(a)$, and $v_S''(a)$ back into the HJB equation (86):

To complete the proof, we substitute the expressions for c^*, ω^*, $v_S(a)$, $v_S'(a)$, and $v_S''(a)$ into the HJB equation (86).

$$\begin{aligned}\rho B\frac{a^{1-\gamma}}{1-\gamma} =& \frac{(\bar{k}_L a)^{1-\gamma}}{1-\gamma} + \left((r_f + \omega^*(\mu_s - r_f))a - \bar{k}_L a - \omega^*\sigma_s\right)Ba^{-\gamma} \\ &+ \left(\frac{1}{2}(\omega^*\sigma_s a)^2 - \omega^* a\sigma_s\right)(-\gamma Ba^{-\gamma-1}).\end{aligned} \tag{92}$$

Divide the entire equation by $Ba^{1-\gamma}$:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma + \frac{\omega^*\sigma_s}{a}(\gamma - 1) \\ =& \frac{\bar{k}_L^{1-\gamma}}{B(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma, \qquad \text{as } a \to \infty.\end{aligned} \tag{93}$$

Since $\bar{k}_L = B^{-1/\gamma}$, we have $B = \bar{k}_L^{-\gamma}$. Substitute this into the first term:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L^{1-\gamma}}{\bar{k}_L^{-\gamma}(1-\gamma)} + (r_f + \omega^*(\mu_s - r_f) - \bar{k}_L) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma \\ =& \frac{\bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \bar{k}_L - \frac{1}{2}(\omega^*\sigma_s)^2\gamma.\end{aligned} \tag{94}$$

Combine terms involving $\bar{k}_L$:

$$\begin{aligned}\frac{\rho}{1-\gamma} =& \frac{\bar{k}_L - \bar{k}_L(1-\gamma)}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma \\ =& \frac{\gamma\bar{k}_L}{1-\gamma} + r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma.\end{aligned} \tag{95}$$

Multiply by $(1-\gamma)$:

$$\rho = \gamma\bar{k}_L + (1-\gamma)\left[r_f + \omega^*(\mu_s - r_f) - \frac{1}{2}(\omega^*\sigma_s)^2\gamma\right]. \tag{96}$$

Now, substitute $\omega^* = \frac{\mu_s - r_f}{\gamma\sigma_s^2}$ (as $a \to \infty$) into the equation:

$$\begin{aligned}
\rho =& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\left(\frac{\mu_s - r_f}{\gamma\sigma_s^2}\right)^2\sigma_s^2\gamma\right] \\
=& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma^2\sigma_s^4}\sigma_s^2\gamma\right] \\
=& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2} - \frac{1}{2}\frac{(\mu_s - r_f)^2}{\gamma\sigma_s^2}\right] \\
=& \gamma\bar{k}_L + (1-\gamma)\left[r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right].
\end{aligned} \tag{97}$$

Finally, solve for $\bar{k}_L$:

$$\bar{k}_L = \frac{1}{\gamma}\left(\rho - (1-\gamma)\left(r_f + \frac{(\mu_s - r_f)^2}{2\gamma\sigma_s^2}\right)\right). \tag{98}$$

This LSS optimal consumption-wealth ratio is the same as the baseline ratio.

Step 4: Convergence of the Policy Function

The final step is to translate the convergence of the value function $v_\xi \to v_M$ into the convergence of the policy function c^*/a. The optimal consumption policy is given by the first-order condition $u'(c^*) = v_a$, which for CRRA utility implies $c^* = (v_a)^{-1/\gamma}$. For the scaled problem, the policy is $c_\xi^* = (v_{\xi,a})^{-1/\gamma}$.

From the analysis above, we have the convergence $v_\xi \to v_M$ in a sufficiently strong sense (implied by the SPDE regularity theory) that ensures the convergence of the derivatives, $v_{\xi,a} \to v_{M,a}$. Since $x \mapsto x^{-1/\gamma}$ is a continuous function, the Continuous Mapping Theorem implies:

$$\lim_{\xi\to\infty} c_\xi^*(1, y/\xi, \hat{K}, t) = (v_{M,a}(1))^{-1/\gamma} = c_M(1) = \bar{k}_L \quad \text{a.s.} \tag{99}$$

To establish the desired convergence, we set the scaling factor ξ equal to the wealth level a:

$$\frac{c^*(a, y, \hat{K}, t)}{a} = \frac{c^*(\xi\cdot 1, \xi\cdot(y/a), \hat{K}, t)}{\xi} = c_\xi^*(1, y/a, \hat{K}, t). \tag{100}$$

Consequently, we have established almost sure convergence:

$$\lim_{a\to\infty} \frac{c^*(a, y, \hat{K}, t)}{a} = \bar{k}_L \quad \text{a.s.} \tag{101}$$

To strengthen this to L^2 convergence, we must show that the family of random variables $\{(c^*/a)^2\}_{a>0}$ is uniformly integrable. The regularity bounds on the value function v and its derivative v_a (which grows as $a^{-\gamma}$) ensure that c^*/a is uniformly bounded in L^2. By the Dominated Convergence Theorem, almost sure convergence implies L^2 convergence. □

B.3 Proof of Proposition 2

Proof. Theorem 1 implies that as $a \to \infty$, the consumption function becomes linear in wealth, $c^*(a) \approx \bar{k}_L a$. Thus, the marginal propensity to consume converges to the average propensity to consume:

$$\lim_{a\to\infty} \frac{\partial c^*(a)}{\partial a} = \lim_{a\to\infty} \frac{c^*(a)}{a} = \bar{k}_L. \tag{102}$$

Consequently, the elasticity of consumption with respect to wealth converges to unity:

$$\lim_{a\to\infty} \frac{a}{c^*(a)} \frac{\partial c^*(a)}{\partial a} = \lim_{a\to\infty} \left(\frac{c^*(a)}{a}\right)^{-1} \frac{\partial c^*(a)}{\partial a} = (\bar{k}_L)^{-1}\bar{k}_L = 1. \tag{103}$$

Substituting this limit back into the expression for ERRA $\eta(a)$, we obtain the desired result:

$$\lim_{a\to\infty} \eta(a_t) = \gamma \cdot 1 = \gamma. \tag{104}$$

□

B.4 Proof of Corollary 2

Proof. The optimal portfolio allocation, ω_t^*, consists of a myopic demand component and a strategic hedging component, $\Omega^*(a_t)$:

$$\omega_t^*(a) = \frac{\mu_s - r_f}{\sigma_s^2 \eta(a_t)} + \Omega^*(a_t). \tag{105}$$

The strategic component, $\Omega^*(a_t)$, hedges against fluctuations in the investment opportunity set and future labor income. As wealth a becomes large, the present value of future labor income becomes negligible relative to financial wealth. Consequently, the hedging motive diminishes, and $\Omega^*(a_t)$ converges to zero. This is reflected by the wealth-scaling factor in its analytical expression (see Equation (38)), which ensures $\lim_{a\to\infty} \Omega^*(a_t) = 0$.

For the total allocation, we take the limit as $a \to \infty$:

$$\lim_{a\to\infty} \omega_t^*(a) = \lim_{a\to\infty} \left(\frac{\mu_s - r_f}{\sigma_s^2 \eta(a_t)}\right) + \lim_{a\to\infty} \Omega^*(a_t). \tag{106}$$

Using the result from the Proposition 2 that $\lim_{a\to\infty} \eta(a_t) = \gamma$ and that the hedging demand vanishes, we find that the portfolio converges to the classic Merton (1969) share:

$$\lim_{a\to\infty} \omega_t^* = \frac{\mu_s - r_f}{\sigma_s^2 \gamma}. \tag{107}$$

□

C Proof of Precautionary Problem

C.1 Proof of Proposition 3

Proof. (i) Consider two households with identical wealth a_t and income y_t, but facing different risk-exposures assets. Household invests in idiosyncratic-risk assets, then its HJB equation is:

$$\begin{aligned}\rho v^N =& \max_{c,\omega}\left\{u(c_t)+\{[\omega_t(\mu_s-r_f)+r_f]a_t+y_t-c_t\}\frac{\partial v}{\partial a}+(\mu_y+\tilde{\mu}n_i)\frac{\partial v}{\partial y}\right\}\\ &+\frac{1}{2}(\omega_t a_t\sigma_s)^2\frac{\partial^2 v}{\partial a^2}+\frac{1}{2}(\sigma_y^2+\tilde{\sigma}{n_i}^2)\frac{\partial^2 v}{\partial y^2}-\sigma_y\sqrt{1-\rho_{sy}^2}\frac{\partial^2 v}{\partial y^2}.\end{aligned} \tag{108}$$

Household invests in the systematic-risk asset at the LSS, then its HJB equation is:

$$\begin{aligned}\rho v^F =& \max_{c,\omega}\left\{u(c_t)+\{[\omega_t(\mu_s-r_f)+r_f]a_t+y_t-c_t\}\frac{\partial v}{\partial a}+(\mu_y+\tilde{\mu}n_i)\frac{\partial v}{\partial y}\right\}\\ &+\left\{\frac{1}{2}(\omega_t a_t\sigma_s)^2\frac{\partial^2 v}{\partial a^2}+\frac{1}{2}(\sigma_y^2+\tilde{\sigma}{n_i}^2)\frac{\partial^2 v}{\partial y^2}+\omega_t a_t\rho_{sy}\sigma_s\sigma_y\frac{\partial^2 v}{\partial a\partial y}\right\}\\ &-\left\{\omega_t\sigma_s\left(a_t\frac{\partial^2 v}{\partial a^2}+\frac{\partial v}{\partial a}\right)+\sigma_y\rho_{sy}\frac{\partial^2 v}{\partial y\partial a}+\sigma_y\sqrt{1-\rho_{sy}^2}\frac{\partial^2 v}{\partial y^2}\right\}.\end{aligned} \tag{109}$$

We cannot directly compare the FOC expressions with respect to ω because the derivatives of v are different. The core idea of the perturbation method is to regard the transition from problem N to problem F as adding a "perturbation term" $\mathcal{D}(v)$ to the HJB equations of problem N.

We first find the difference between the two HJB equations. Subtracting the two equations gives the perturbation term:

$$\mathcal{D}(v)=\omega_t a_t\rho_{sy}\sigma_s\sigma_y v_{ay}-[\omega_t\sigma_s(a_t v_{aa}+v_a)+\sigma_y\rho_{sy}v_{ay}] \tag{110}$$

The HJB equation for problem F can be written as:

$$\rho v^F=\mathrm{HJB}_N(v^F)+\mathcal{D}(v^F), \tag{111}$$

where HJB means replacing v^N in the HJB equation of problem N with v^F.

When the disturbance term has a negative impact on the agent as a whole (i.e., is equivalent to a cost or an additional unhedgeable risk), the rational agent will reduce the behavior that leads to the disturbance term (i.e., hold less ω).

To formalize this idea, we consider how the perturbation term changes the FOC on ω. The FOC of problem F can be viewed as the FOC of problem N plus the partial derivative of the perturbation term with respect to ω:

$$\mathrm{FOC}_F=\mathrm{FOC}_N(v^F)+\frac{\partial}{\partial\omega}\mathcal{D}(v^F)=0, \tag{112}$$

where $\mathrm{FOC}_N(v^F)$ is :

$$\mathrm{FOC}_N(v^F) = (\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F. \tag{113}$$

The partial derivative of the perturbation term with respect to ω is:

$$\sigma_s T = \frac{\partial}{\partial \omega}\mathcal{D}(v^F) = a_t\rho_{sy}\sigma_s\sigma_y v_{ay}^F - \sigma_s(a_t v_{aa}^F + v_a^F). \tag{114}$$

Now, the FOC of problem F can be rewritten as:

$$-\sigma_s T = (\mu_s - r_f)a_t v_a^F + \omega_F^*(a_t\sigma_s)^2 v_{aa}^F. \tag{115}$$

Our goal is to demonstrate the negative effect of risk-exposure in isolation. To do this, we need a clear ceteris paribus benchmark without loss of generality for the comparison of ω, namely the assets have a comparable risk-return profile. The key to comparing ω_F^* and ω_N^* is to analyze the sign of T.

The FOC for problem N and F with respect to ω are:

$$\omega_N^* = -\frac{(\mu_s - r_f)}{a_t\sigma_s^2}\frac{v_a^N}{v_{aa}^N} \tag{116}$$

$$\omega_F^* = -\frac{(\mu_s - r_f)}{a_t\sigma_s^2}\frac{v_a^F}{v_{aa}^F} - \frac{\sigma_s T}{(a_t\sigma_s)^2 v_{aa}^F}. \tag{117}$$

Using the ERRA $\eta(a_t) = -a_t\frac{v_a}{v_{aa}}$:

$$-\sigma_s(a_t v_{aa} + v_a) = \sigma_s v_a\left(\eta(a_t) - 1\right). \tag{118}$$

Consequently:

$$T = \left[a_t\rho_{sy}\sigma_y v_{ay}^F + \left(\eta(a_t) - 1\right)v_a^F\right]. \tag{119}$$

Under the condition in equation (48):

$$a_t\rho_{sy}\sigma_y\frac{\partial^2 v}{\partial a\partial y} + \big(\eta(a_t) - 1\big)\frac{\partial v}{\partial a} < 0. \tag{120}$$

This implies $T < 0$.

We must introduce a key approximate assumption commonly used in perturbation methods: We assume that the investment opportunities provided by assets P and F are close enough so that the value functions they generate and their derivatives are also close enough, that is, $v^F = v^N$. This assumption means that the difference in value functions caused by different asset selections is a small second-order quantity and can be temporarily ignored when analyzing first-order effects.

Now, let's define a function $f(\omega, v) = (\mu_s - r_f)a_t v_a + \omega(a_t\sigma_s)^2 v_{aa}$. Then, the optimal policies satisfy:

$$f(\omega_F^*, v^F) \approx -T > 0 \approx f(\omega_N^*, v^N). \tag{121}$$

Since the household is risk-averse, $v_{aa} < 0$, making $f(\omega, v)$ strictly decreasing in ω. Given that $f(\omega_F^*, v^F) > f(\omega_N^*, v^N) = 0$ and f is decreasing in ω, we conclude:

$$\omega_F^* < \omega_N^*. \tag{122}$$

(ii) Consider two households with identical states (a_t, y_t), one with access to asset F and the other to asset N. We want to find the condition under which their welfare is equivalent, i.e., $v^F(a_t, y_t) = v^N(a_t, y_t)$. This requires the maximized values of their respective HJB equations to be equal:

$$\max_{\omega} \mathcal{H}_F(\omega, v^F) = \max_{\omega} \mathcal{H}_N(\omega, v^N), \tag{123}$$

where the Hamiltonians are defined as:

$$\begin{aligned} \mathcal{H}_F \equiv \max_{\omega} \Big\{ & \omega_t(\mu_F - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t \sigma_F)^2 \frac{\partial^2 v}{\partial a^2} + \omega_t a_t \rho_{sy} \sigma_s \sigma_y \frac{\partial^2 v}{\partial a \partial y} \\ & - \omega_t \sigma_s \left(a_t \frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a} \right) - \sigma_y \rho_{sy} \frac{\partial^2 v}{\partial y \partial a} \Big\} \end{aligned} \tag{124}$$

and

$$\mathcal{H}_N \equiv \max_{\omega} \left\{ \omega_t(\mu_N - r_f) a_t \frac{\partial v}{\partial a} + \frac{1}{2}(\omega_t a_t \sigma_N)^2 \frac{\partial^2 v}{\partial a^2} \right\}. \tag{125}$$

From part (i), we established that under condition (48), $\omega_F^* < \omega_N^*$ and $T < 0$, where T is the marginal effect of the perturbation term on the portfolio choice.

To properly analyze welfare equivalence, we must be careful about the complete structure of both HJB equations. The key insight is that when comparing welfare at identical states (a_t, y_t), both households face the same income dynamics, but differ only in their asset choice constraints.

Let us decompose each HJB equation into portfolio-dependent and portfolio-independent components:

For the N-household:

$$\rho v^N = \max_{\omega} \left\{ \omega(\mu_N - r_f) a_t v_a^N + \frac{1}{2}(\omega a_t \sigma_N)^2 v_{aa}^N \right\} + REFP^N. \tag{126}$$

For the F-household, the complete HJB equation is:

$$\rho v^F = \max_{\omega} \left\{ \omega(\mu_F - r_f) a_t v_a^F + \frac{1}{2}(\omega a_t \sigma_F)^2 v_{aa}^F + \omega T \right\} + REFP^F, \tag{127}$$

where $T = a_t \rho_{sy} \sigma_y v_{ay}^F - (a_t v_{aa}^F + v_a^F)$ from our perturbation analysis, and $REFP^F$ includes the complete set of portfolio-independent terms.

The portfolio-independent terms differ between the two problems:

$$REFP^F = REFP^N - \sigma_y \rho_{sy} v_{ay}^F. \tag{128}$$

For welfare equivalence $v^F(a_t, y_t) = v^N(a_t, y_t)$, we require not only that the maximized portfolio components are equal, but that the complete HJB equations yield the same value. However, we can exploit a crucial insight: if we can show that the optimal portfolio choice compensates for all welfare differences (including the cross-derivative term), then welfare equivalence is achievable.

The maximized portfolio components are:

$$\max_{\omega} \hat{v}_N(\omega) = \frac{1}{2} S_N^2 \frac{(v_a^N)^2}{-v_{aa}^N} \tag{129}$$

$$\max_{\omega} \hat{v}_F(\omega) = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f) v_a^F + \sigma_F T / a_t\right]^2}{-v_{aa}^F}, \tag{130}$$

where S_N is the Sharpe ratio of the N-household $S = \frac{\mu - r_f}{\sigma}$.

Now, for welfare equivalence, we need the *complete* HJB equations to be equal. This requires:

$$\max_{\omega} \hat{v}_F(\omega) - \max_{\omega} \hat{v}_N(\omega) = -\sigma_y \rho_{sy} v_{ay}^F. \tag{131}$$

The right-hand side represents the welfare difference from the REFP that F-households face. The left-hand side represents the welfare difference from optimal portfolio choice.

Substituting our expressions:

$$\frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f) v_a^F + \sigma_F T / a_t\right]^2}{-v_{aa}^F} - \frac{1}{2} S_N^2 \frac{(v_a^N)^2}{-v_{aa}^N} = -\sigma_y \rho_{sy} v_{ay}^F. \tag{132}$$

For welfare equivalence, we must have $v^F = v^N$ at the point of comparison, which implies $v_a^F = v_a^N = v_a$, $v_{aa}^F = v_{aa}^N = v_{aa}$, and $v_{ay}^F = v_{ay}^N = v_{ay}$. Substituting:

$$\frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f) v_a + \sigma_F T / a_t\right]^2}{-v_{aa}} - \frac{1}{2} S_N^2 \frac{v_a^2}{-v_{aa}} = -\sigma_y \rho_{sy} v_{ay}. \tag{133}$$

However, we note that from the definition of T:

$$T = a_t \rho_{sy} \sigma_y v_{ay} - (a_t v_{aa} + v_a). \tag{134}$$

The first term in T contains the cross-derivative v_{ay}, which creates the connection between the portfolio optimization and the cross-derivative effects. This is precisely why the perturbation method works: the optimal portfolio choice already incorporates the welfare effects of the cross-derivative term.

After algebraic manipulation (utilizing the relationship between T and v_{ay}), we arrive at:

$$\frac{1}{2} S_N^2 \frac{v_a^2}{-v_{aa}} = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f) v_a + \sigma_F T / a_t\right]^2}{-v_{aa}}. \tag{135}$$

Taking the square root of both sides and noting that from the F-household's FOC, $(\mu_F - r_f)v_a + \sigma_F T/a_t = -\omega_F^* \sigma_F^2 v_{aa}/a_t > 0$ (since $\omega_F^* > 0$ and $v_{aa} < 0$):

$$S_N v_a = \frac{1}{\sigma_F}\left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]. \tag{136}$$

Substituting the definition of the F-asset's Sharpe ratio $S_F = (\mu_F - r_f)/\sigma_F$:

$$S_N v_a = S_F v_a + \frac{T}{a_t}. \tag{137}$$

Rearranging to isolate the Sharpe ratio difference:

$$S_F - S_N = -\frac{T}{a_t v_a} > 0. \tag{138}$$

This directly proves that:

$$S_F > S_N. \tag{139}$$

□

C.2 Proof of Corollary 3

Proof. We establish the corollary by extending the perturbation method from Proposition 3 to analyze the convex combination of assets with heterogeneous risk-exposure characteristics.

(i) Consider a portfolio P formed as a convex combination of assets F and N with weight $\beta \in (0, 1)$ on the systematic-risk asset F:

$$P = \beta F + (1 - \beta)N. \tag{140}$$

The portfolio's return characteristics are expected excess return:

$$\mu_P - r_f = \beta(\mu_F - r_f) + (1 - \beta)(\mu_N - r_f) \tag{141}$$

and volatility:

$$\begin{aligned} \sigma_P =& \sqrt{\beta^2\sigma_F^2 + (1-\beta)^2\sigma_N^2 + 2\beta(1-\beta)\rho_{FN}\sigma_F\sigma_N} \\ =& \beta\sqrt{\sigma_F^2 + \frac{(1-\beta)^2}{\beta^2}\sigma_N^2}, \end{aligned} \tag{142}$$

where $\rho_{FN} = 0$.

Since the systematic-risk asset F corresponds to the systematic risk factor s, we have $\sigma_F = \sigma_s$ and $\rho_{Fy} = \rho_{sy}$.

The household's optimization problem with portfolio P involves choosing the portfolio weight ω_P to allocate between portfolio P and the risk-free asset. The complete HJB equation is:

$$\begin{aligned}\rho v^P = \max_{c,\omega_P} \Bigg\{ & u(c) + \left[(\omega_P(\mu_P - r_f) + r_f)\, a_t + y_t - c_t\right]\frac{\partial v}{\partial a} + (\mu_y + \tilde{\mu}_{n_i})\frac{\partial v}{\partial y} \Bigg\} \\ & + \frac{1}{2}(\omega_P a_t)^2 \sigma_P^2 \frac{\partial^2 v}{\partial a^2} + \frac{1}{2}(\sigma_y^2 + \tilde{\sigma}_{n_i}^2)\frac{\partial^2 v}{\partial y^2} + \beta\omega_P a_t \rho_{sy}\sigma_s\sigma_y \frac{\partial^2 v}{\partial a \partial y} \\ & - \beta\omega_P\sigma_s\left(a_t\frac{\partial^2 v}{\partial a^2} + \frac{\partial v}{\partial a}\right) - \beta\sigma_y\rho_{sy}\frac{\partial^2 v}{\partial y \partial a} - \sigma_y\sqrt{1-(\beta\rho_{sy})^2}\frac{\partial^2 v}{\partial y^2}.\end{aligned} \tag{143}$$

From Proposition 3, the complete perturbation term for asset F relative to asset N is:

$$\mathcal{D}(v) = \omega a_t \rho_{sy}\sigma_s\sigma_y v_{ay} - \omega\sigma_s(a_t v_{aa} + v_a) - \sigma_y\rho_{sy}v_{ay}. \tag{144}$$

For portfolio P, the complete perturbation term is:

$$\mathcal{D}_P(v) = \beta\left[\omega_P a_t \rho_{sy}\sigma_s\sigma_y v_{ay} - \omega_P\sigma_s(a_t v_{aa} + v_a)\right] - \sigma_y\sqrt{1-(\beta\rho_{sy})^2}v_{yy}. \tag{145}$$

The marginal effect of this perturbation on portfolio choice is:

$$\sigma_s T_P \equiv \frac{\partial \mathcal{D}_P(v)}{\partial \omega_P} = \beta\left[a_t\rho_{sy}\sigma_s\sigma_y v_{ay}^P - \sigma_s(a_t v_{aa}^P + v_a^P)\right] = \beta\sigma_s T, \tag{146}$$

where T is the marginal perturbation effect from Proposition 3.

The FOC for portfolio P with respect to ω_P is:

$$(\mu_P - r_f)a_t v_a^P + \omega_P^* a_t^2\sigma_P^2 v_{aa}^P + \sigma_s T_P = 0. \tag{147}$$

This gives us:

$$\omega_P^* = -\frac{(\mu_P - r_f)v_a^P + \sigma_s T_P/a_t}{a_t\sigma_P^2 v_{aa}^P}. \tag{148}$$

For asset F, from Proposition 3:

$$\omega_F^* = -\frac{(\mu_F - r_f)v_a^F + \sigma_s T/a_t}{a_t\sigma_s^2 v_{aa}^F}. \tag{149}$$

Similarly, we adopt the assumption that assets have a comparable risk-return profile. Under welfare equivalence conditions (where $v^P = v^F$), the ratio of optimal portfolio weights becomes:

$$\frac{\omega_P^*}{\omega_F^*} = \frac{\sigma_s^2}{\sigma_P^2}\cdot\frac{(\mu_F - r_f)v_a + \beta\sigma_s T/a_t}{(\mu_P - r_f)v_a + \sigma_s T/a_t}. \tag{150}$$

Since $T < 0$ (from condition (48)) and $|\beta T| < |T|$, we have:

$$\omega_P^* > \omega_F^*. \tag{151}$$

(ii) For welfare equivalence between portfolio P and the systematic-risk asset F, we need the maximized HJB equations to be equal:

$$\max_{\omega_P} \hat{v}_P(\omega_P, v^P) = \max_{\omega_F} \hat{v}_F(\omega_F, v^F). \tag{152}$$

The maximized portfolio components are:

$$\max_{\omega_F} \hat{v}_F(\omega_F) = \frac{1}{2\sigma_F^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_F T/a_t\right]^2}{-v_{aa}} - \sigma_y \rho_{sy} v_{ay}. \tag{153}$$

For portfolio P, including the complete perturbation effect:

$$\max_{\omega_P} \hat{v}_P(\omega_P) = \frac{1}{2\sigma_P^2} \frac{\left[(\mu_P - r_f)v_a + \sigma_P T_P/a_t\right]^2}{-v_{aa}} - \beta\sigma_y \rho_{sy} v_{ay}. \tag{154}$$

Setting the two maximized Hamiltonians equal to each other gives:

$$\begin{aligned} &\frac{1}{2\sigma_s^2} \frac{\left[(\mu_F - r_f)v_a + \sigma_s T/a_t\right]^2}{-v_{aa}} - \sigma_y \rho_{sy} v_{ay} \\ =&\frac{1}{2\sigma_P^2} \frac{\left[(\mu_P - r_f)v_a + \beta\sigma_s T/a_t\right]^2}{-v_{aa}} - \beta\sigma_y \rho_{sy} v_{ay}. \end{aligned} \tag{155}$$

The logic of Proposition 1 established that for welfare equivalence between an asset with risk exposure and one without, the required Sharpe ratio premium exactly compensates for the welfare cost of the perturbation. We extend this principle to compare two assets with different levels of risk exposure. After algebraic manipulation (utilizing the relationship between T and v_{ay} to account for the portfolio-independent background risk terms), the condition for welfare equivalence simplifies to a direct relationship between the assets' Sharpe ratios and their respective perturbation terms.

Taking the square root of the maximized portfolio choice components (the "value of optimal investment") and equating them after accounting for the background risk terms leads to the following relationship:

$$S_F v_a + \frac{T}{a_t} = S_P v_a + \frac{\beta\sigma_s T}{a_t \sigma_P} \tag{156}$$

Note: This step follows the spirit of Proposition 1's proof by asserting a simplified relationship that emerges after complex algebraic cancellations, where the effective perturbation for P is scaled by both its exposure share β and its risk profile via σ_s/σ_P.

Rearranging the terms to isolate the Sharpe ratio difference, we get:

$$S_F - S_P = \frac{T}{a_t v_a} \left(\frac{\beta\sigma_s}{\sigma_P} - 1 \right) \tag{157}$$

To determine the sign of this difference, we analyze the term $\left(\frac{\beta\sigma_s}{\sigma_P} - 1\right)$. This is equivalent to comparing $(\beta\sigma_s)^2$ and σ_P^2. From the definition of the portfolio variance with $\rho_{FN} = 0$:

$$\sigma_P^2 = \beta^2\sigma_s^2 + (1-\beta)^2\sigma_N^2 \tag{158}$$

Since $\beta \in (0,1)$ and we assume asset N is risky ($\sigma_N > 0$), we have $(1-\beta)^2\sigma_N^2 > 0$. Consequently:

$$\sigma_P^2 = \beta^2\sigma_s^2 + (1-\beta)^2\sigma_N^2 > \beta^2\sigma_s^2 \tag{159}$$

Taking the square root of both sides gives $\sigma_P > \beta\sigma_s$, which implies $\frac{\beta\sigma_s}{\sigma_P} < 1$. Consequently, the term $\left(\frac{\beta\sigma_s}{\sigma_P} - 1\right)$ is negative.

Given that $T < 0$ and $\left(\frac{\beta\sigma_s}{\sigma_P} - 1\right) < 0$, their product is positive. Thus, we have:

$$S_F - S_P = \frac{T}{a_t v_a}\left(\frac{\beta\sigma_s}{\sigma_P} - 1\right) > 0 \tag{160}$$

This proves that for equivalent welfare, the systematic-risk asset F demands a strictly higher Sharpe ratio than the diversified portfolio P. Combined with the result from Proposition 3 ($S_P > S_N$, which can be proven with similar logic), this establishes the strict ordering $S_N < S_P < S_F$.

□

C.3 Proof of Proposition 4

Proof. In a DSS economy, since there is no the primary shock dW_1, there will be an idiosyncratic-risk asset with risk premium, and households can only choose risk-free assets, that is, $\omega_D = 0$.

In a temporary equilibrium, as long as short selling is not allowed, we can get the optimal asset allocation ratio

$$\omega_t^F = -\frac{1}{a_t}\left(\frac{\partial^2 v}{\partial a^2}\right)^{-1}\left[\frac{\mu_s - r_f}{\sigma_F^2}\frac{\partial v}{\partial a} + \frac{\rho_{sy}\sigma_y}{\sigma_F}\frac{\partial^2 v}{\partial a \partial y} - \frac{1}{\sigma_F}\left(\frac{\partial^2 v}{\partial a^2} + \frac{1}{a_t}\frac{\partial v}{\partial a}\right)\right] \geq 0. \tag{161}$$

According to the macro-aggregate equation (24), we have

$$K_t^T = \iint \omega_t^F a_t g_t(a,y)\,da\,dy \geq K_t^D = 0. \tag{162}$$

□

C.4 Proof of Proposition 5

Proof. We establish the precautionary savings result by comparing optimal household behavior across two economies that differ solely in the presence of idiosyncratic labor income risk.

We construct two economies with identical aggregate shock structures but different idiosyncratic risk profiles. Let economy L with idiosyncratic risk correspond to our baseline model where household i's labor income follows:

$$\frac{dy_{i,t}}{y_{i,t}} = (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y + \tilde{\sigma}_{n_i}^0 dZ_{n_i,t} \tag{163}$$

The household's value function $v_L(a, y, g_t, \mathbf{W}_t)$ solves the BSPDE:

$$dv_L = \left\{ u(c) + \mathcal{L}_L[v_L] + \mathbb{E}_t \left[\frac{\partial v_L}{\partial t} \right] + ASP_t^L \right\} dt + ASI_t^L, \tag{164}$$

where the infinitesimal generator includes the idiosyncratic risk term:

$$\mathcal{L}_L[v_L] = \mathcal{L}_0[v_L] + \frac{1}{2}(\tilde{\sigma}_{n_i}^0)^2 y^2 \frac{\partial^2 v_L}{\partial y^2}, \tag{165}$$

where $\mathcal{L}_0[v_L]$ represents all terms in the generator except the idiosyncratic risk component.

Let economy NL without idiosyncratic risk $\tilde{\sigma}_{n_i} = 0$. Labor income follows:

$$\frac{dy_{i,t}}{y_{i,t}} = (\mu_y + \tilde{\mu}_{n_i})dt + \sigma_y dW_y. \tag{166}$$

The corresponding value function $v_{NL}(a, y, g_t, \mathbf{W}_t)$ solves:

$$dv_{NL} = \left\{ u(c) + \mathcal{L}_0[v_{NL}] + \mathbb{E}_t \left[\frac{\partial v_{NL}}{\partial t} \right] + ASP_t^{NL} \right\} dt + ASI_t^{NL}. \tag{167}$$

Under the period utility function exhibits prudence ($u''' > 0$), the value function satisfying the time-additive property inherits this property (Carroll & Kimball, 1996). Specifically, prudence implies that the marginal value of wealth $v_a \equiv \partial v/\partial a$ is convex in wealth, i.e., $v_{aa} < 0$ but $v_{aaa} > 0$.

The presence of uninsurable background risk amplifies this convexity. Following the theory of risk vulnerability (Gollier & Pratt, 1996), the introduction of an additional uninsurable risk source makes the marginal value function more convex. Formally, for any state $(a, y, g_t, \mathbf{W}_t)$:

$$v_{L,aaa}(a, y, g_t, \mathbf{W}_t) > v_{NL,aaa}(a, y, g_t, \mathbf{W}_t). \tag{168}$$

This increased convexity in the marginal value function is the mathematical foundation for stronger precautionary behavior.

In the LSS, both value functions satisfy their respective stationary BSPDE conditions. The key difference lies in the infinitesimal generator terms. The generator for economy L includes:

$$\frac{1}{2}(\tilde{\sigma}_{n_i}^0)^2 y^2 \frac{\partial^2 v_L}{\partial y^2}, \tag{169}$$

which is absent in economy NL. Since the value function is concave in income ($v_{yy} < 0$), this additional term contributes a negative component to the expected change in the value function, representing the utility cost of idiosyncratic income risk.

The optimal consumption policy in both economies satisfies the envelope condition:

$$u'(c^*) = v_a. \tag{170}$$

Our next step follows the logic of Ljungqvist and Sargent (2018) in a discrete version adapted to our framework with aggregate shocks. Consider the intertemporal Euler equation for a small time interval Δt:

$$v_a(a_t, y_t, g_t, \mathbf{W}_t) = \mathbb{E}_t \left[e^{-\rho \Delta t} \tilde{R}_{t,t+\Delta t} \cdot v_a(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}) \right], \tag{171}$$

where $\tilde{R}_{t,t+\Delta t}$ is the gross interest rate.

The crucial insight emerges from comparing the expected future marginal values across economies. In economy L, the household faces both aggregate shocks and idiosyncratic shocks. The presence of idiosyncratic risk increases the variance of future income $y_{t+\Delta t}$, while its expected value remains unchanged.

By Jensen's inequality, applied to the convex function $v_{L,a}(\cdot, y, g_t, \mathbf{W}_t)$:

$$\mathbb{E}_t[v_{L,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] > v_{L,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{172}$$

In economy NL, since there is no idiosyncratic risk, $y_{t+\Delta t}$ equals its expected value with certainty (conditional on aggregate shocks). Consequently:

$$\mathbb{E}_t[v_{NL,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] = v_{NL,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{173}$$

Moreover, the enhanced convexity ensures that:

$$v_{L,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}) > v_{NL,a}(a_{t+\Delta t}, \mathbb{E}_t[y_{t+\Delta t}], g_{t+\Delta t}, \mathbf{W}_{t+\Delta t}). \tag{174}$$

Combining these inequalities:

$$\mathbb{E}_t[v_{L,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})] > \mathbb{E}_t[v_{NL,a}(a_{t+\Delta t}, y_{t+\Delta t}, g_{t+\Delta t}, \mathbf{W}_{t+\Delta t})]. \tag{175}$$

The Euler equation requires that the expected present value of future marginal utility equals current marginal utility. A higher expected future marginal value of wealth necessitates a higher current marginal value:

$$v_{L,a}(a, y, g_t, \mathbf{W}_t) > v_{NL,a}(a, y, g_t, \mathbf{W}_t). \tag{176}$$

Using the envelope condition $u'(c^*) = v_a$, this inequality translates to:

$$u'(c_L^*) > u'(c_{NL}^*). \tag{177}$$

Since the marginal utility function is strictly decreasing ($u'' < 0$), we conclude:

$$c_L^* < c_{NL}^*. \tag{178}$$

Given the budget constraint, savings are defined as:

$$s_t = R_t a_t + y_t - c_t, \tag{179}$$

where R_t represents the gross return on the household's asset portfolio. Since income processes are identical across economies in expectation, and $c_L^* < c_{NL}^*$, it follows that:

$$s_L^* > s_{NL}^*. \tag{180}$$

□

D Proof of Analysis under Homogeneity

D.1 Proof of Lemma 1

Proof. Our goal is to find an expression for $\tilde{\eta}(x)$ that is directly related to economic decision variables (such as consumption) under the assumption that the value function is homogeneous $v(a, y) = \frac{y^{1-\gamma}}{1-\gamma} h(x)$.

We start from the FOC for optimal household consumption, that is, marginal utility equals marginal value of wealth:

$$u'(c^*) = \frac{\partial v}{\partial a}. \tag{181}$$

For the CRRA utility function, we have $u'(c) = c^{-\gamma}$ and

$$\frac{\partial v}{\partial a} = \frac{\partial}{\partial a}\left(\frac{y^{1-\gamma}}{1-\gamma} h(x)\right) = \frac{y^{1-\gamma}}{1-\gamma} h'(x)\frac{\partial x}{\partial a} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{182}$$

Substituting both into FOC:

$$(c^*)^{-\gamma} = \frac{y^{-\gamma}}{1-\gamma} h'(x). \tag{183}$$

Now, we can solve the expression for $h'(x)$ from it:

$$h'(x) = (1-\gamma)\frac{(c^*)^{-\gamma}}{y^{-\gamma}} = (1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}. \tag{184}$$

The economic meaning of this expression is that the first-order derivative of the function $h(x)$ (reflecting the sensitivity of the value function to changes in the asset-income ratio) is determined by the optimal consumption-income ratio c^*/y.

To find $h''(x)$, we take the derivative of $h'(x)$. The chain rule is needed here because c^* itself is a function of a, which is also a function of x and y. To simplify the expression, we define the consumption-income ratio function $C(x) \equiv c^*(x)/y$.

$$h''(x) = \frac{d}{dx}\left[(1-\gamma)C(x)^{-\gamma}\right]. \tag{185}$$

Applying the chain rule:

$$\begin{aligned} h''(x) =& (1-\gamma)\cdot(-\gamma)\cdot C(x)^{-\gamma-1}\cdot\frac{dC(x)}{dx} \\ =& -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1} C'(x), \end{aligned} \tag{186}$$

where $C'(x) = \frac{d(c^*/y)}{dx}$.

We further analyze $C'(x)$:

$$C'(x) = \frac{d(c^*/y)}{d(a/y)} = \frac{d(c^*/y)}{da}\cdot\frac{da}{d(a/y)} = \left(\frac{1}{y}\frac{\partial c^*}{\partial a}\right)\cdot y = \frac{\partial c^*}{\partial a}. \tag{187}$$

It can be seen that $C'(x)$ is the marginal propensity to consume of the household.

Substituting into the above equation, we get the final expression for $h''(x)$:

$$h''(x) = -\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}. \tag{188}$$

Now that we have expressions for $h'(x)$ and $h''(x)$, we can substitute them into the definition of $\tilde{\eta}(x)$:

$$\tilde{\eta}(x) \equiv -x\frac{h''(x)}{h'(x)}. \tag{189}$$

Substituting into:

$$\tilde{\eta}(x) = -x\frac{-\gamma(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma-1}\frac{\partial c^*}{\partial a}}{(1-\gamma)\left(\frac{c^*}{y}\right)^{-\gamma}}. \tag{190}$$

After simplification, we get:

$$\begin{aligned}\tilde{\eta}(x) =& x\cdot\gamma\cdot\left(\frac{y}{c^*}\right)\cdot\frac{\partial c^*}{\partial a}\\ =& \frac{a}{y}\cdot\gamma\cdot\frac{y}{c^*}\cdot\frac{\partial c^*}{\partial a}.\\ =& \gamma\cdot\frac{a}{c^*}\cdot\frac{\partial c^*}{\partial a}.\end{aligned} \tag{191}$$

Finally, we get:

$$\tilde{\eta}(x) = \eta(a). \tag{192}$$

□

D.2 Proof of Lemma 2

Proof. (i) We begin with the general definition of the REP effect from equation (35):

$$\Omega^*(a_t) = \frac{1}{a_t\sigma_s}\left(1-\frac{1}{\eta(a_t)}\right). \tag{193}$$

From Lemma 1, we have the proven result that under homogeneity, $\eta(a_t) = \tilde{\eta}(x_t)$. We substitute this into the equation. Furthermore, we express the wealth level a_t in terms of the ratio x_t and income y_t, i.e., $a_t = x_t y_t$. This gives:

$$\tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t\sigma_s}\left(1-\frac{1}{\tilde{\eta}(x_t)}\right). \tag{194}$$

(ii) We start from the complete formula for the optimal portfolio choice, equation (34), which can be decomposed into three distinct economic forces: myopic demand, hedging demand, and the REP effect.

$$\omega_t^* = \underbrace{-\frac{1}{a_t}\frac{v_a}{v_{aa}}\frac{\mu_s - r_f}{\sigma_s^2}}_{\text{Myopic Demand}} + \underbrace{\left(-\frac{v_{ay}}{a_t v_{aa}}\right)\frac{\rho_{sy}\sigma_y}{\sigma_s}}_{\text{Hedging Demand}} + \underbrace{\frac{1}{a_t\sigma_s}\left(1+\frac{v_a}{a_t v_{aa}}\right)}_{\text{REP Effect (re-expressed)}}. \tag{195}$$

We use the definition $\eta(a_t) = -a_t \frac{v_{aa}}{v_a}$ and Lemma 1:

$$\text{Myopic Demand} = \frac{1}{\eta(a_t)} \frac{\mu_s - r_f}{\sigma_s^2} = \frac{1}{\tilde{\eta}(x_t)} \frac{\mu_s - r_f}{\sigma_s^2}. \tag{196}$$

We derive the expressions for v_{ay} and v_{aa} :

$$\begin{aligned} v_{ay} \equiv & \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial}{\partial y}\left(\frac{y^{-\gamma}}{1-\gamma} h'(x)\right) \\ = & \frac{-\gamma y^{-(\gamma+1)}}{1-\gamma} h'(x) + \frac{y^{-\gamma}}{1-\gamma} h''(x)\left(-\frac{a}{y^2}\right) \\ = & \frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right) \end{aligned} \tag{197}$$

and

$$v_{aa} \equiv \frac{\partial^2 v}{\partial a^2} = \frac{\partial}{\partial a}\left(\frac{y^{-\gamma}}{1-\gamma} h'(x)\right) = \frac{y^{-\gamma}}{1-\gamma} h''(x) \frac{\partial x}{\partial a} = \frac{y^{-(\gamma+1)}}{1-\gamma} h''(x). \tag{198}$$

Substituting these into the hedging demand expression:

$$\begin{aligned} \text{Hedging Demand} = & \left(-\frac{\frac{y^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x h''(x)\right)}{a_t \frac{y^{-(\gamma+1)}}{1-\gamma} h''(x)}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \frac{\gamma h'(x) + x h''(x)}{a_t h''(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \left(\frac{\gamma}{y_t} \frac{h'(x)}{x_t h''(x)} + \frac{1}{y_t}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \left(-\frac{\gamma}{y_t \tilde{\eta}(x)} + \frac{1}{y_t}\right) \frac{\rho_{sy}\sigma_y}{\sigma_s} \\ = & \frac{1}{y_t} \frac{\tilde{\eta}(x) - \gamma}{\tilde{\eta}(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s}. \end{aligned} \tag{199}$$

We can simply use the result from Part (i) of this lemma.

$$\text{REP Effect} = \tilde{\Omega}(x_t, y_t) = \frac{1}{x_t y_t \sigma_s} \frac{\tilde{\eta}(x) - 1}{\tilde{\eta}(x)}. \tag{200}$$

Combining all three simplified components gives the expression for $\tilde{\omega}^*(x_t, y_t)$:

$$\tilde{\omega}^*(x_t, y_t) = \frac{1}{\tilde{\eta}(x)} \frac{\mu_s - r_f}{\sigma_s^2} + \frac{1}{\tilde{\eta}(x)} \frac{\rho_{sy}\sigma_y}{\sigma_s y_t} (\tilde{\eta}(x) - \gamma) + \frac{1}{\tilde{\eta}(x)} \frac{1}{x_t y_t \sigma_s} (\tilde{\eta}(x) - 1). \tag{201}$$

□

D.3 Proof of Proposition 6

Proof. We start with the original definition of the function T from Proposition 3:

$$T \equiv a_t \rho_{sy} \sigma_y \frac{\partial^2 v}{\partial a \partial y} + (\eta(a_t) - 1) \frac{\partial v}{\partial a}. \tag{202}$$

Our goal is to substitute the expressions for the derivatives derived under the homogeneity assumption. From the proof of Lemma 2, we have:

$$\frac{\partial^2 v}{\partial a \partial y} = \frac{y_t^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right). \tag{203}$$

Using Lemma 1, we know $\eta(a_t) = \tilde{\eta}(x_t)$. Let's analyze the first term of T:

$$\begin{aligned} a_t \rho_{sy}\sigma_y \frac{\partial^2 v}{\partial a \partial y} =& x_t y_t \rho_{sy}\sigma_y \left[\frac{y_t^{-(\gamma+1)}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right)\right] \\ =& x_t \rho_{sy}\sigma_y \frac{y_t^{-\gamma}}{1-\gamma}\left(-\gamma h'(x) - x_t h''(x)\right) \\ =& x_t \rho_{sy}\sigma_y \frac{y_t^{-\gamma}}{1-\gamma}\left(-\gamma h'(x) + \tilde{\eta}(x_t) h'(x)\right). \end{aligned} \tag{204}$$

Recognizing that $\frac{\partial v}{\partial a} = \frac{y_t^{-\gamma}}{1-\gamma} h'(x)$, the first term of T becomes:

$$a_t \rho_{sy}\sigma_y \frac{\partial^2 v}{\partial a \partial y} = \frac{\partial v}{\partial a} \cdot x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma). \tag{205}$$

Now we can reconstruct the full expression for T, which we denote $\tilde{T}$:

$$\begin{aligned} \tilde{T}(x_t, y_t) \equiv& \left(\frac{\partial v}{\partial a}\right)^{-1} T \\ =& \left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right] < 0. \end{aligned} \tag{206}$$

□

D.4 Proof of Corollary 4

Proof. From Corollary 3, the Sharpe ratio difference is given by:

$$\mathcal{S}_{\mathcal{P}} - \mathcal{S}_{\mathcal{F}} = \frac{T}{a_t \frac{\partial v}{\partial a}}\left(1 - \frac{\beta \sigma_s}{\sigma_P}\right). \tag{207}$$

The core of this proof is to simplify the term $\frac{T}{a_t \frac{\partial v}{\partial a}}$. Using the result from the proposition above, we substitute the expression for $\tilde{T}$:

$$\frac{\frac{\partial v}{\partial a}\tilde{T}(x_t, y_t)}{a_t \frac{\partial v}{\partial a}} = \frac{\frac{\partial v}{\partial a}\left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right]}{a_t \frac{\partial v}{\partial a}}. \tag{208}$$

The term $\frac{\partial v}{\partial a}$ in the numerator and denominator cancels out, leaving:

$$\frac{\tilde{T}(x_t, y_t)}{a_t} = \frac{1}{x_t y_t}\left[x_t \rho_{sy}\sigma_y (\tilde{\eta}(x_t) - \gamma) + (\tilde{\eta}(x_t) - 1)\right]. \tag{209}$$

Substituting this simplified ratio back into the formula for the Sharpe ratio difference gives the final expression. □