Redistricting with Endogenous Candidates

Paola Moscariello

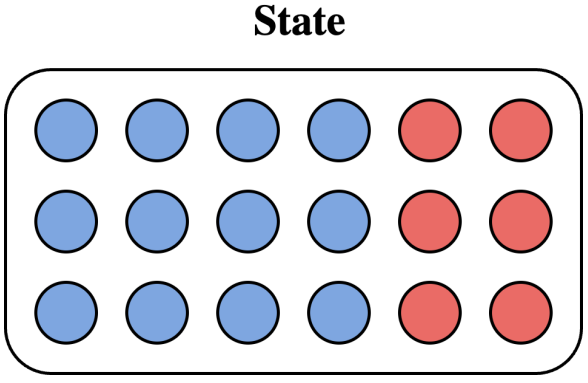
Princeton University

Introduction

- In the US, congressional district boundaries drawn by political partisans
- Partisan gerrymandering: "the practice of dividing a geographic area into electoral districts, often of highly irregular shape, to give one political party an unfair advantage by diluting the opposition's voting strength"
- This paper: gerrymandering when district composition affects candidates' policy positions, and thus voting behavior, at the district level

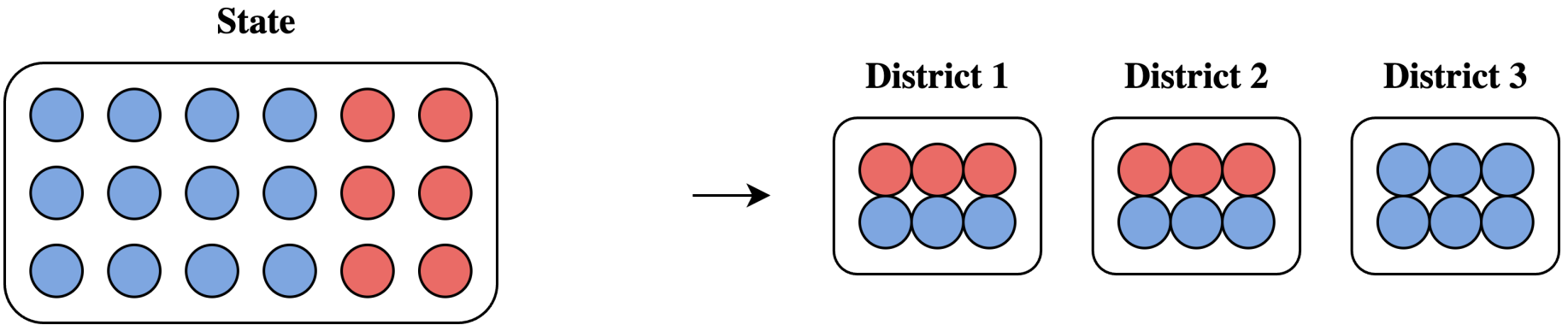
Recap: Standard Redistricting

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$\frac{1}{3}$ Republicans and $\frac{2}{3}$ Democrats

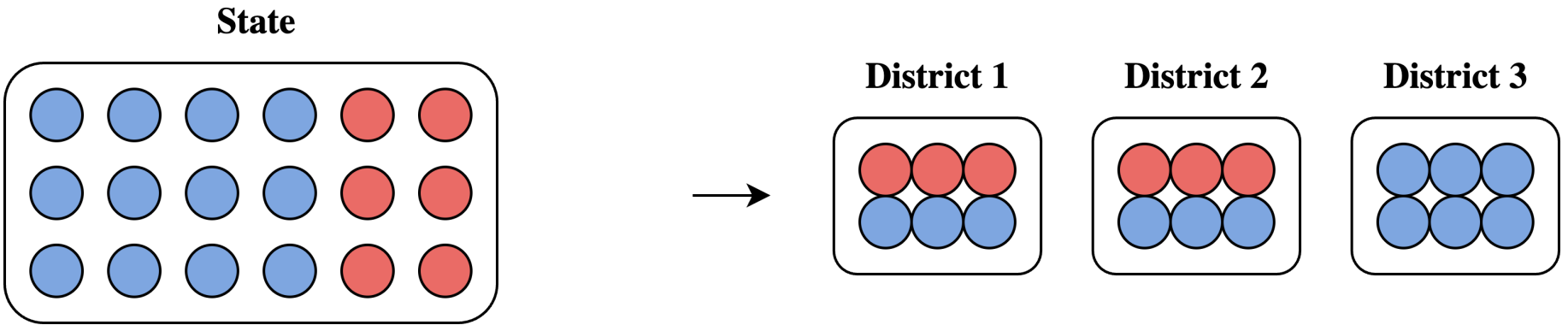
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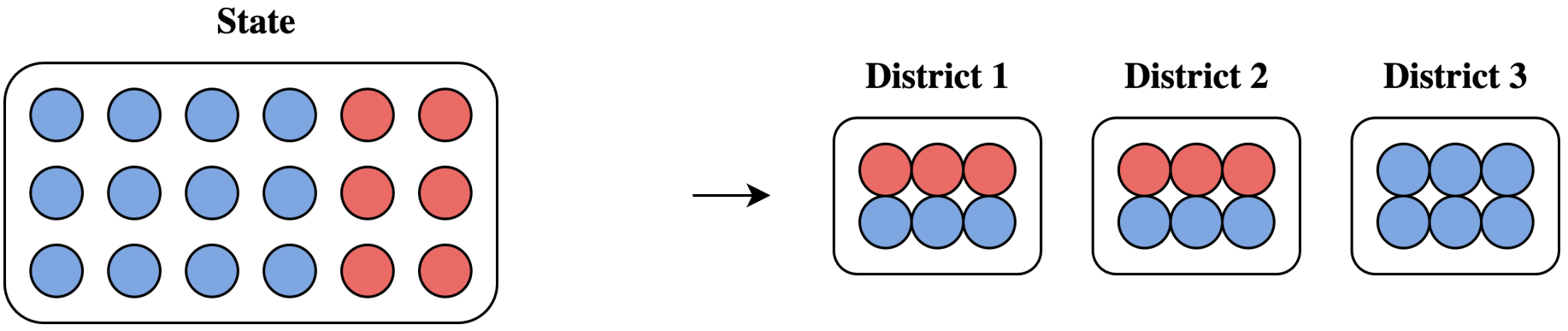


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- Considers party affiliation but not preference intensities
- Underlying assumption: voting behavior is fixed

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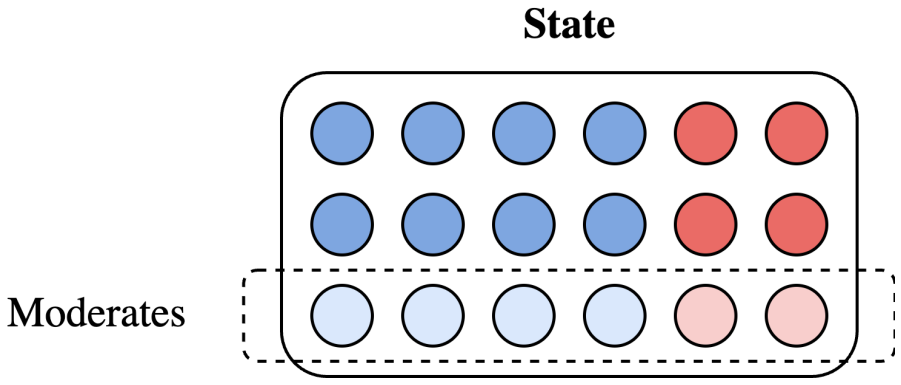


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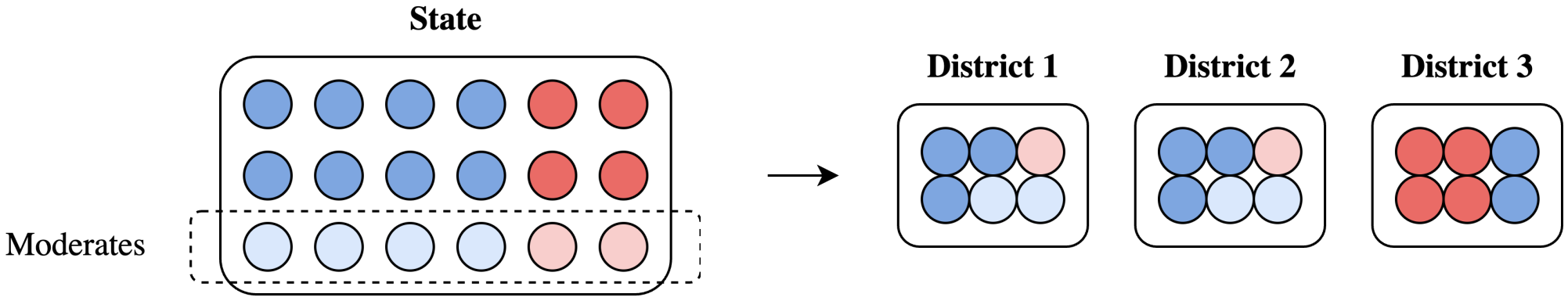
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- Considers party affiliation but not preference intensities
- Underlying assumption: voting behavior is fixed
- Behavior might be endogenous to districts through primary elections...

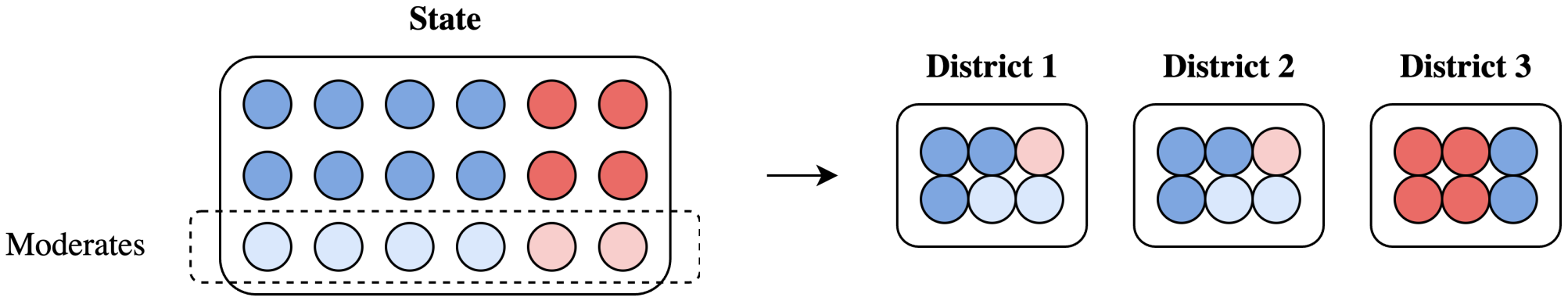
Taking Electoral Incentives into Account



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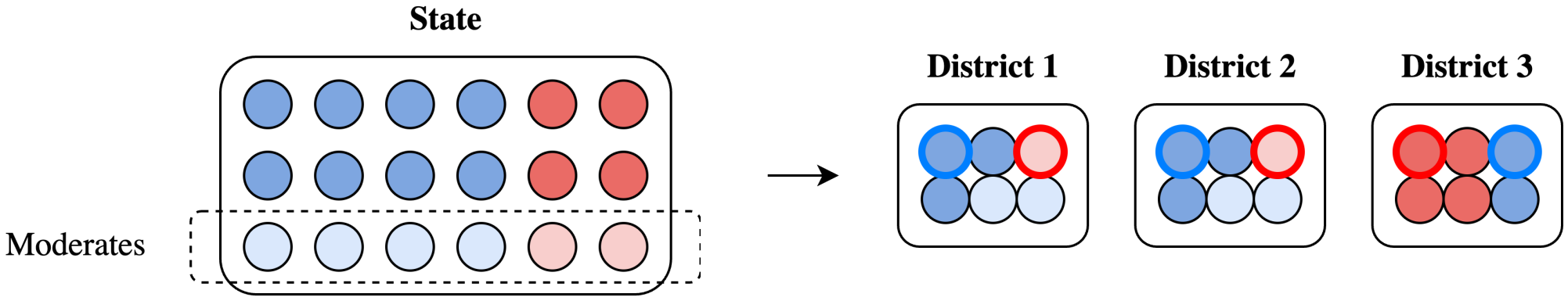


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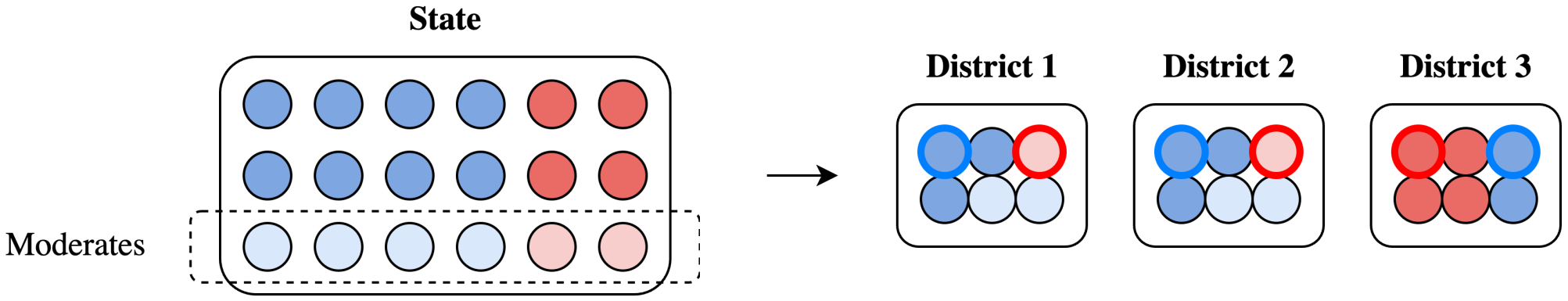
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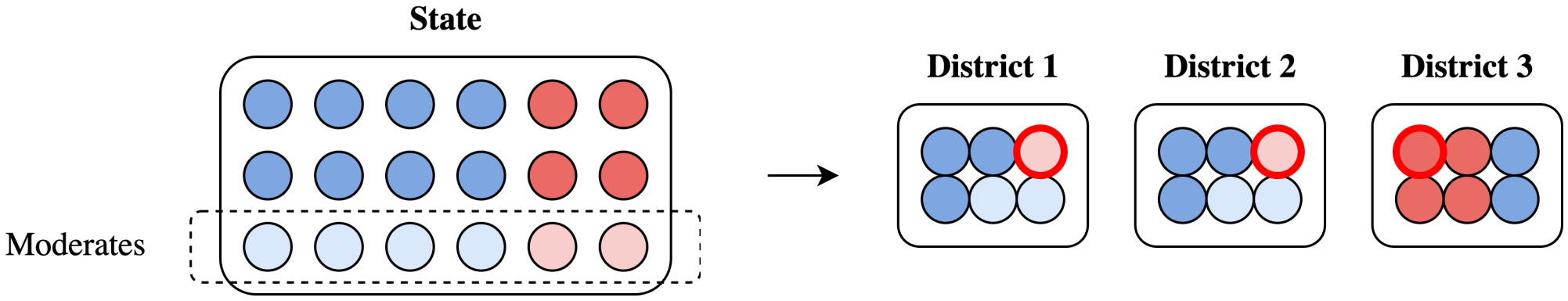
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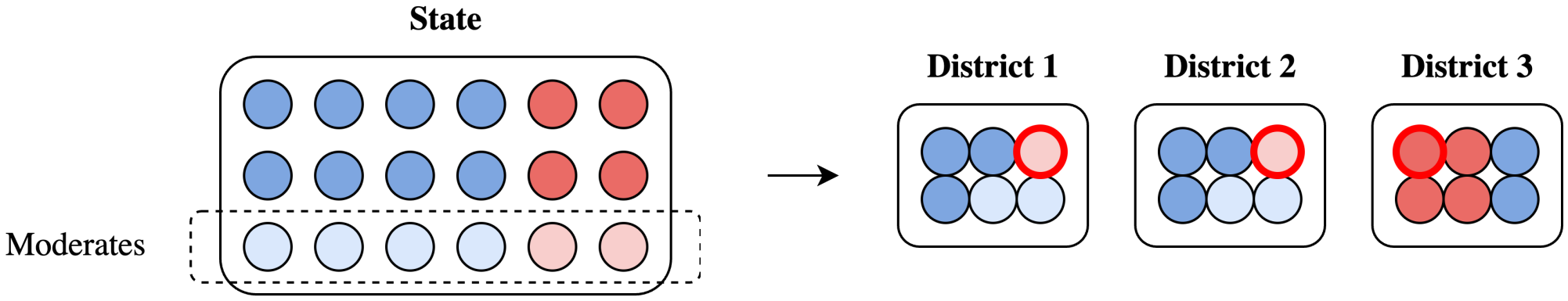
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- Rep/Dem candidate is moderate or extremist, depending on party majority
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- Gerrymandering is even more powerful: win ALL districts

The Model: Redistricting with Primary Elections

The Agents

- Single-peaked voters' preferences, ideal points distribution $\phi \in \Delta([\underline{v}, \overline{v}])$
  - Strictly increasing and continuous CDF
  - Median $v^m = 0$
- Two parties: Republican and Democratic
- Voter affiliation:

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- Designer allocates voters into districts to maximize districts won by Republicans

Districts and Redistricting Plans

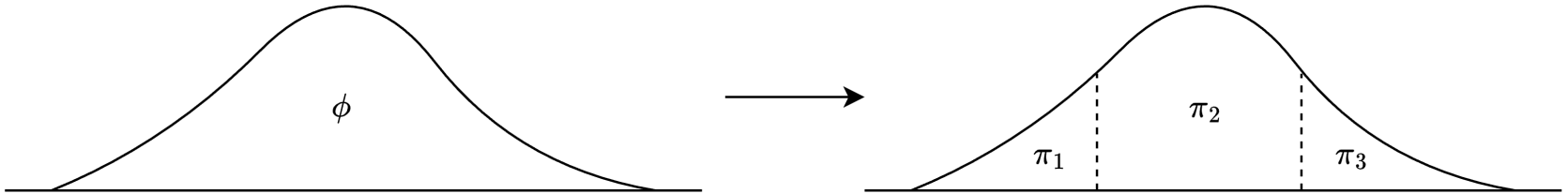
- A district is a distribution of preferences $\pi \in \Delta(\mathbb{R})$
- A redistricting plan is a distribution of districts $\mathcal{H} \in \Delta(\Delta(\mathbb{R}))$ s.t.

$$\int \pi d\mathcal{H}(\pi) = \phi \quad \rightarrow \quad \text{Feasibility}$$

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$$\pi_1\mathcal{H}(\pi_1) + \pi_2\mathcal{H}(\pi_2) + \pi_3\mathcal{H}(\pi_3) = \phi$$

Elections in District π

What determines whether a district is won?

- Two-stage elections: Party Primaries and General Elections
- In the first stage:
  - voters $v < k$ select D with position $c_{\pi,D}$
  - voters $v \geqslant k$ select R with position $c_{\pi,R}$
- In the second stage all voters select $c_\pi \in \{c_{\pi,D}, c_{\pi,R}\}$

For this talk: Assume primary candidates position at party medians

Microfoundation

Elections in District π

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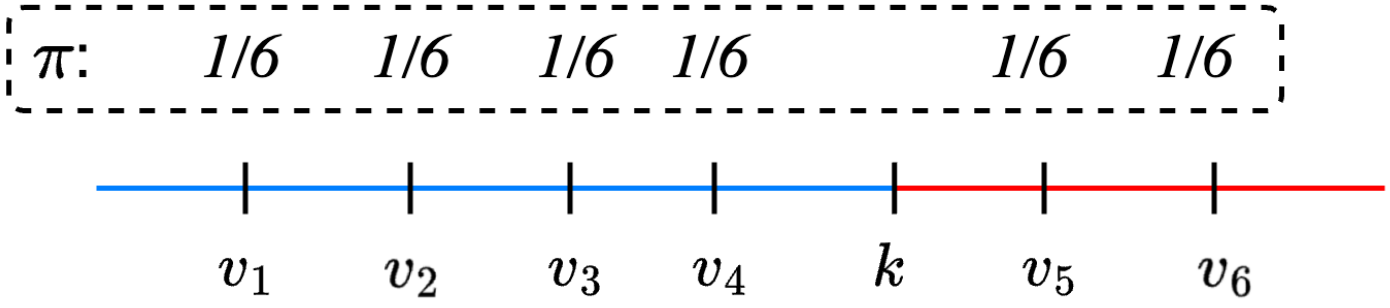
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Need candidates to respond to party base

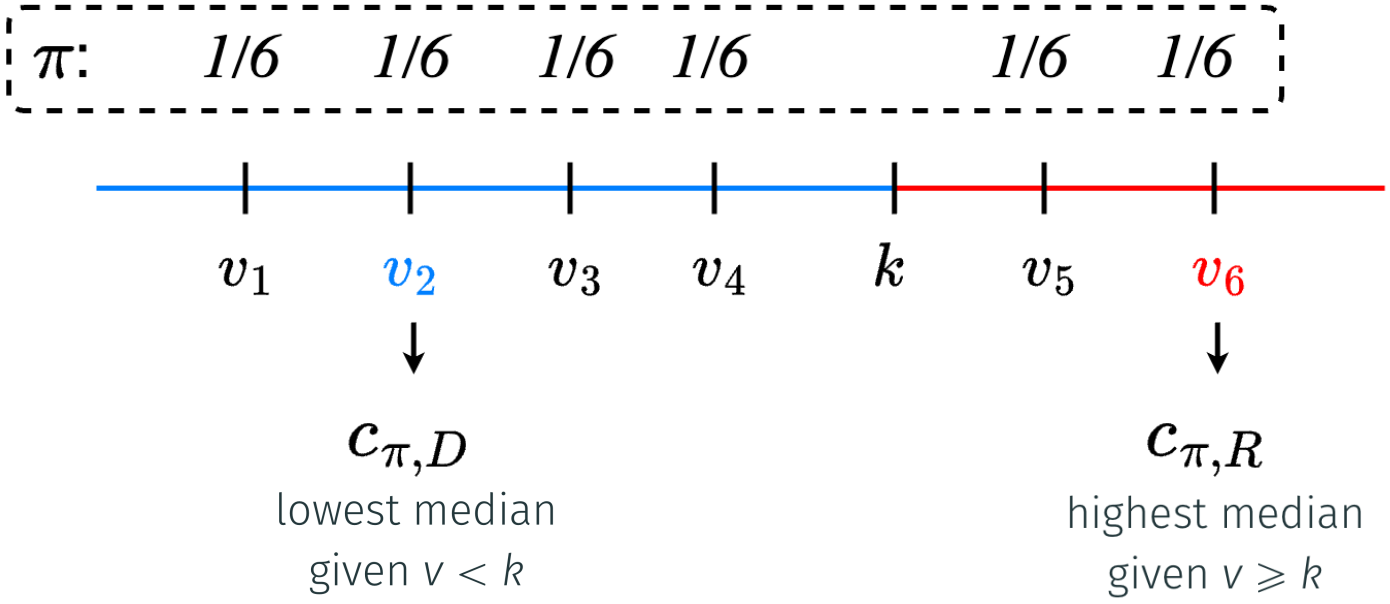
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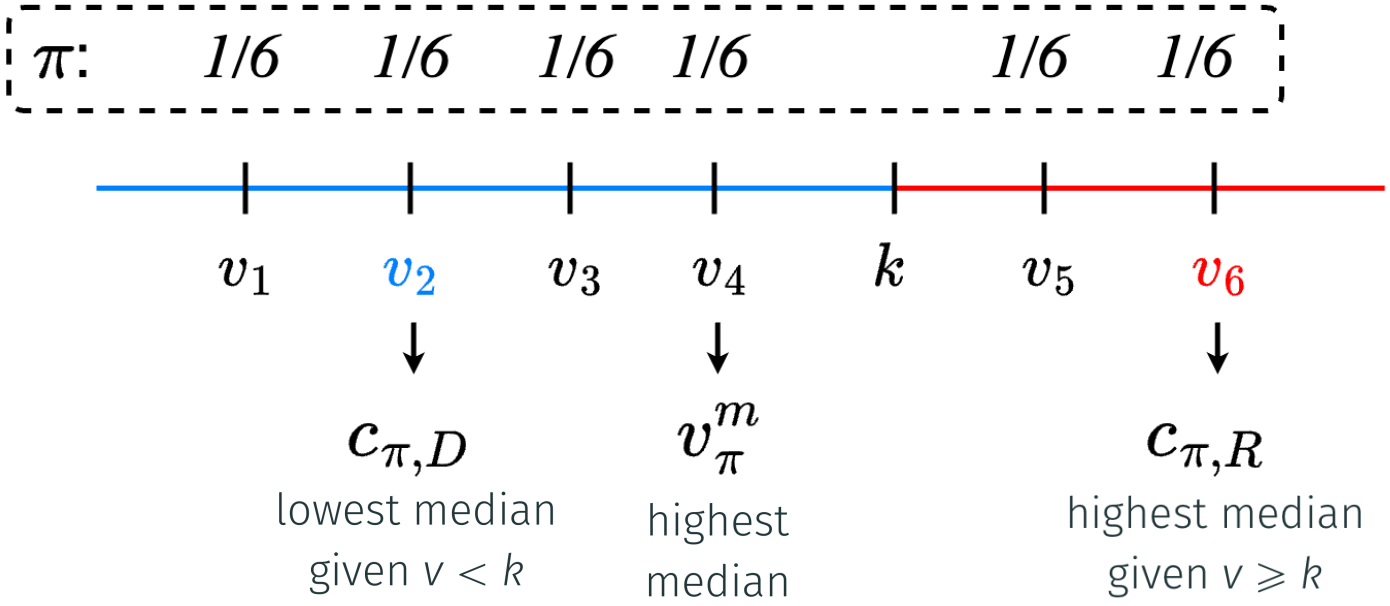
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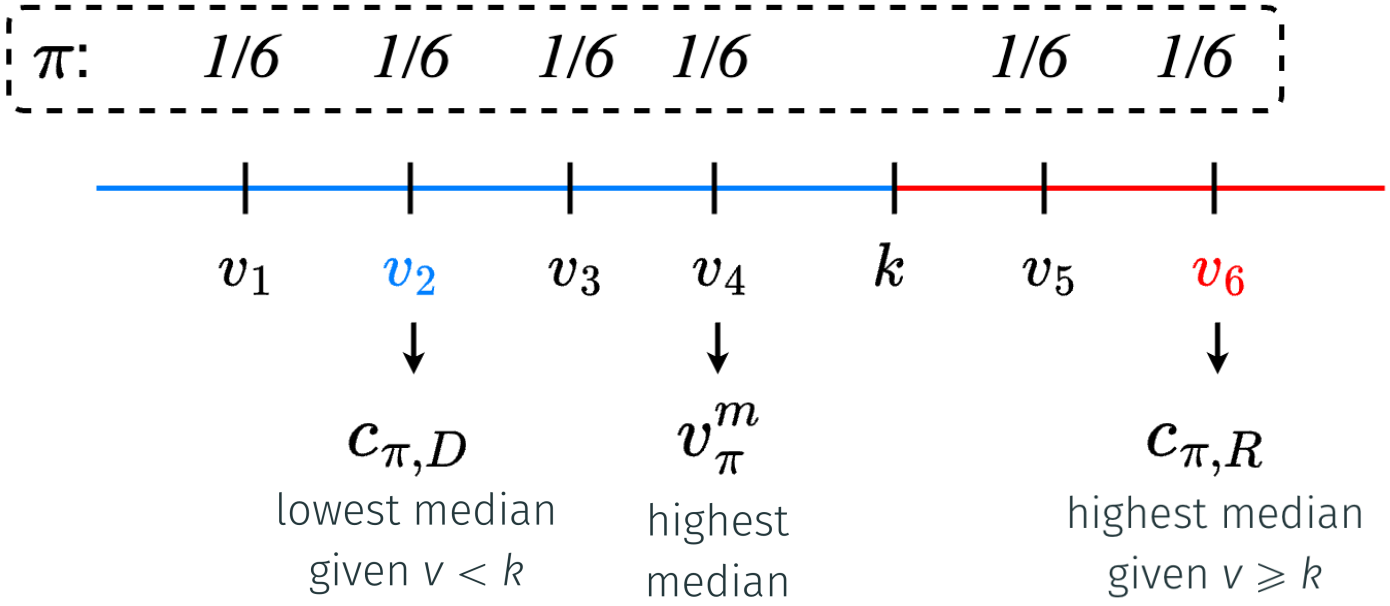
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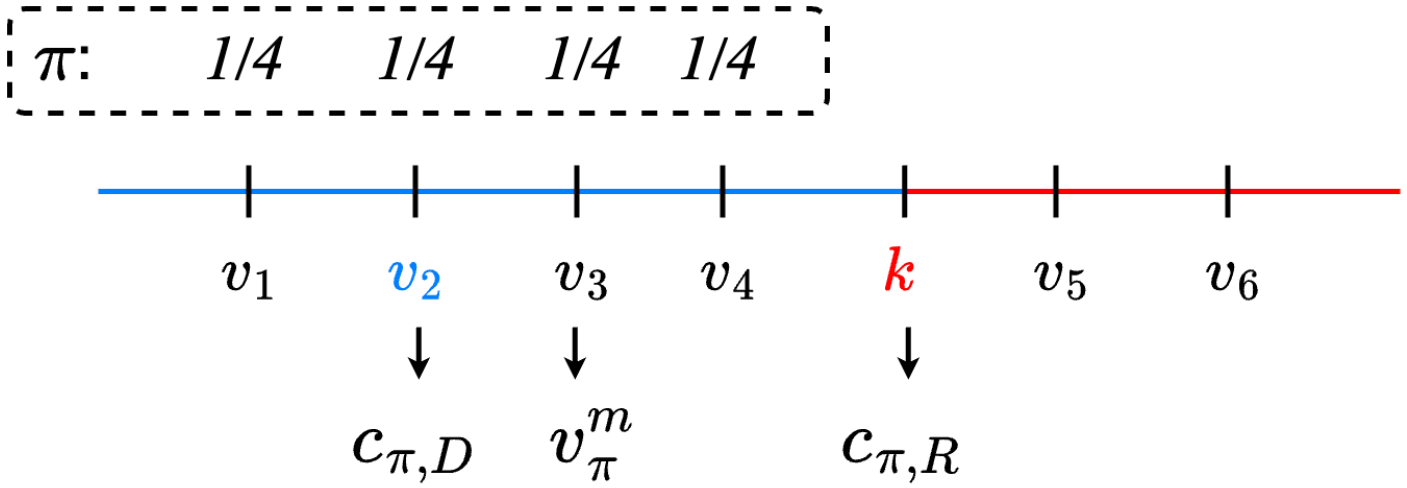
Elections in District π



Disclaimer: Tie-breaking "in favor" of Republicans. Does not matter given continuum of voters

Elections in District π

- If there are no Republicans in π...



- Same for Democrats

Allowing for Uncertainty

- Redistricting happens every 10 years
- Designer faces uncertainty about voters' preferences
- Aggregate shock: All voters experience common shock $\omega \to v - \omega$
- $\omega \sim \gamma \in \Delta(\mathbb{R})$, with CDF G increasing and continuous on $[2(\underline{v} - \overline{v}), 2(\overline{v} - \underline{v})]$
- $\pi \to \pi^{\omega}$

Timing



The Redistricter's Problem

- Designer's problem is:

$$\max_{\mathcal{H}\in\Delta(\Delta([\underline{v},\overline{v}]))} \int\int \mathbb{1}\left(v^m_{\pi^\omega} - \frac{c_{\pi^\omega,D} + c_{\pi^\omega,R}}{2} \geqslant 0\right) d\mathcal{H}(\pi)d\gamma(\omega) \tag{RP}$$

$$s.t. \int \pi d\mathcal{H}(\pi) = \phi$$

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How do we solve it?

Solution

Solution

Mapping to Optimal Transport Problem

Focus on Subset of Feasible Plans

Proposition 1

Any district π in an optimal plan is such that $\mathbf{supp}(\pi) = \{v', v''\}$*, with:*

$$v' \geqslant 0 \geqslant v''$$

$$\pi(\{v'\}) = \pi(\{v''\})$$

→ Matching voters above the median with voters below the median

Proof

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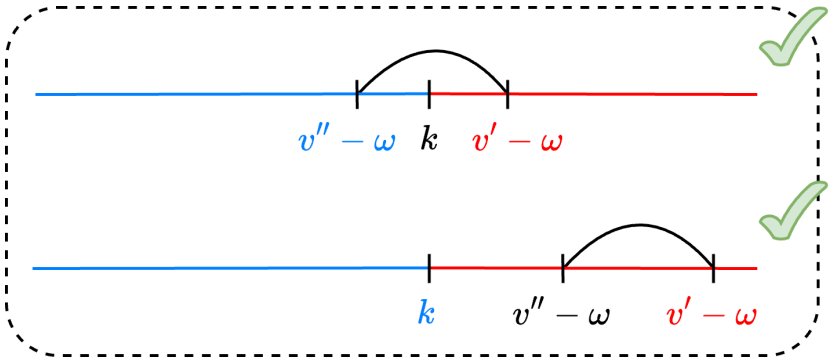
- Define $\phi' = \phi(\cdot | v \geqslant 0)$ and $\phi'' = \phi(\cdot | v \leqslant 0)$
- $\mathcal{M}(\phi', \phi'')$ set of joint distributions over v', v'' with marginals ϕ', ϕ''
- Designer's problem becomes...

Reduction to Optimal Transport Problem

$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int\int d\tau(v', v'') d\gamma(\omega)$$

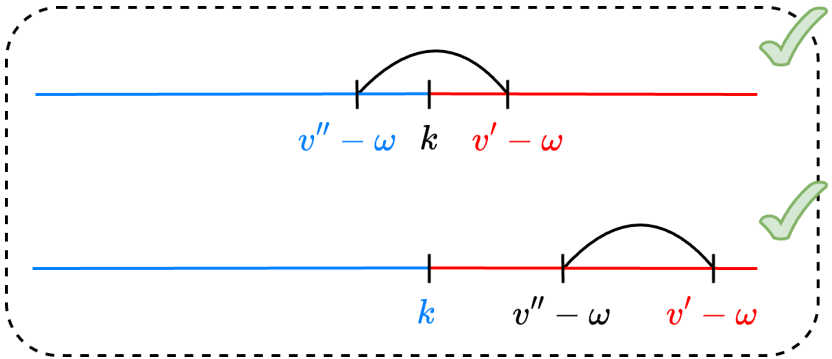
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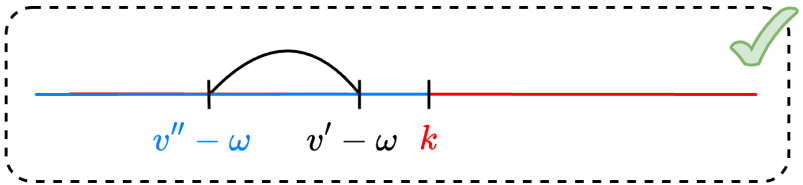
$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int \int \underbrace{\mathbb{1}(v' - \omega \geqslant k)}_{\text{50\% Rep voters}} + \qquad d\tau(v', v'')d\gamma(\omega)$$



Reduction to Optimal Transport Problem

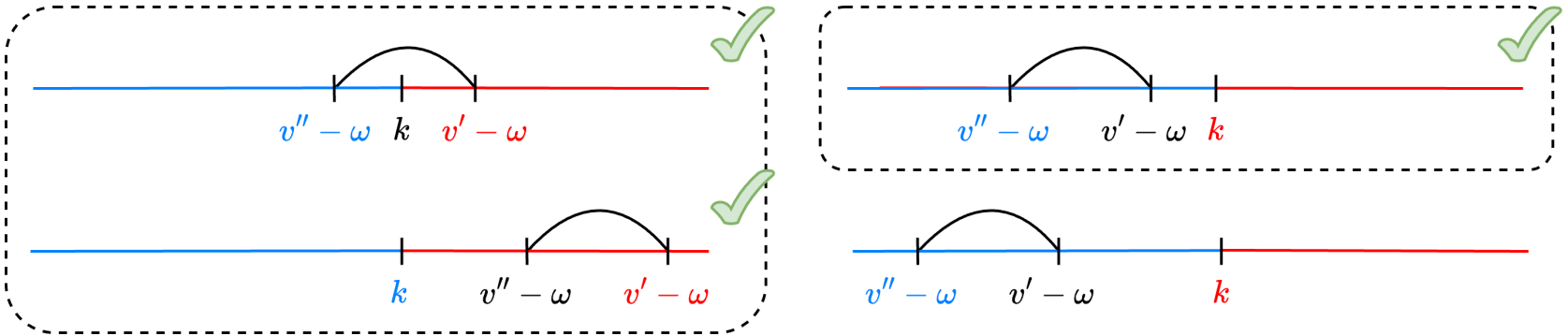
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- After some algebra...

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- This is an optimal transport problem

Theorem 1: Recap

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Theorem 1

RP is equivalent to OTP

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Solution

Solution to Optimal Transport Problem

A Closer Look at OTP

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Trade off:

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Option 2 All districts with moderate $v' - v''$

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$\rightarrow$ Depends on concavity of G

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Marginal cost of decreasing $v' - v''$ in a district where it's high

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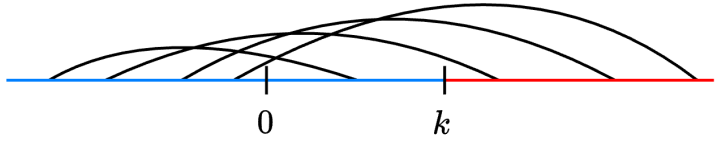
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G strictly concave $\rightarrow$ **Option 2** $\rightarrow$ positive assortative matching

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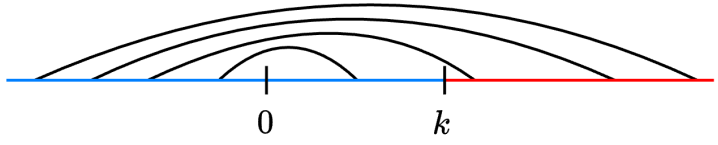
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G strictly convex $\rightarrow$ **Option 1** $\rightarrow$



negative assortative matching

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Theorem 2: S-shaped and Symmetric Shock

G S-shaped $\rightarrow$

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$$\text{concave if } \overbrace{2v' - v'' - k > 0}^{\text{District won when } \omega = 0} \rightarrow$$

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Theorem 2: S-shaped and Symmetric Shock

$$\text{concave if } \overbrace{2v' - v'' - k > 0}^{\text{District won when } \omega=0} \rightarrow$$

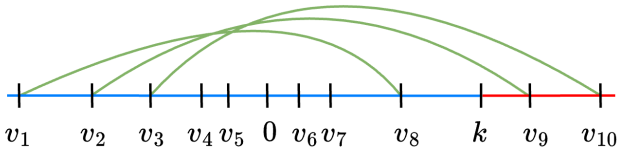
G S-shaped $\rightarrow$

$$\text{convex if } \underbrace{2v' - v'' - k < 0}_{\text{District lost when } \omega=0} \rightarrow$$

Theorem 2: S-shaped and Symmetric Shock

District won when $\omega=0$

concave if $\overbrace{2v' - v'' - k > 0}$ $\rightarrow$



G S-shaped $\rightarrow$

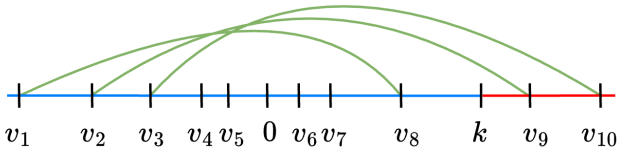
convex if $\underbrace{2v' - v'' - k < 0}$ $\rightarrow$

District lost when $\omega=0$

Theorem 2: S-shaped and Symmetric Shock

District won when $\omega=0$

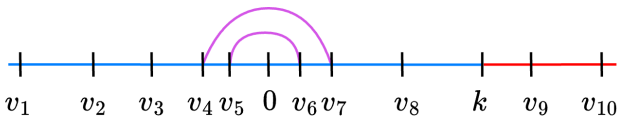
concave if $\overbrace{2v' - v'' - k > 0}$ →



G S-shaped →

convex if $\underbrace{2v' - v'' - k < 0}$ →

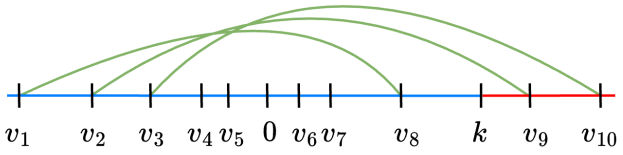
District lost when $\omega=0$



Theorem 2: S-shaped and Symmetric Shock

District won when $\omega=0$

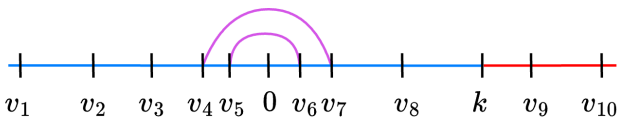
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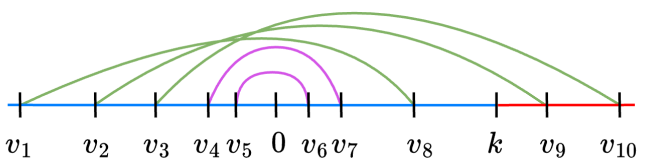
G S-shaped →

convex if $\underbrace{2v' - v'' - k < 0}$ →

District lost when $\omega=0$



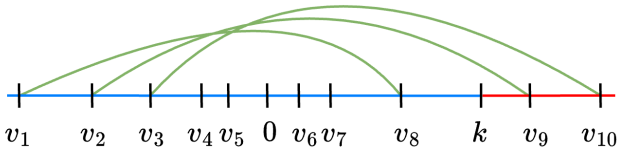
- Solution is a combination of positive and negative assortative matching



Theorem 2: S-shaped and Symmetric Shock

District won when $\omega=0$

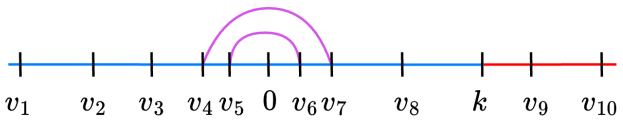
concave if $\overbrace{2v' - v'' - k > 0}$ →



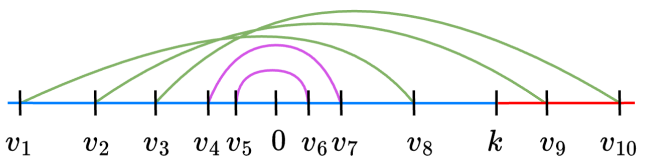
G S-shaped →

convex if $\underbrace{2v' - v'' - k < 0}$ →

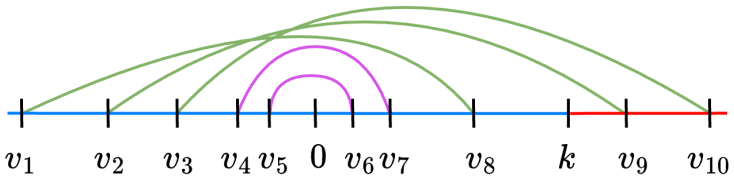
District lost when $\omega=0$



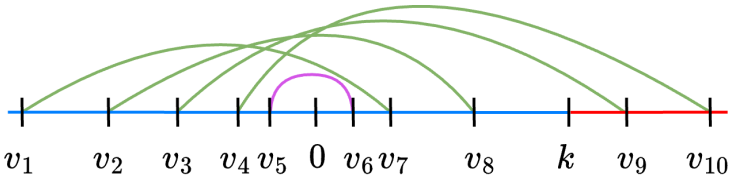
- Solution is a combination of positive and negative assortative matching
- ...But which one?



Theorem 2: S-shaped and Symmetric Shock

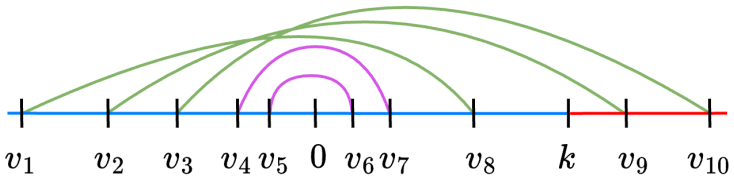


Option 1 Fewer positive matches, but with higher distance each



Option 2 More positive matches, but with lower distance each

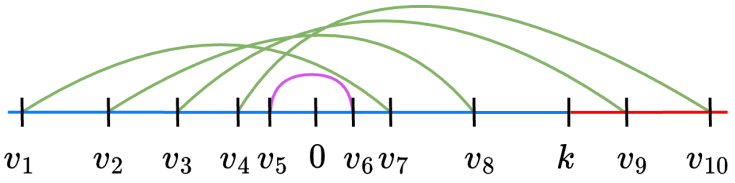
Theorem 2: S-shaped and Symmetric Shock



Fewer won when $\omega=0$

Option 1 Fewer positive matches, but with higher distance each

safer when bad ω

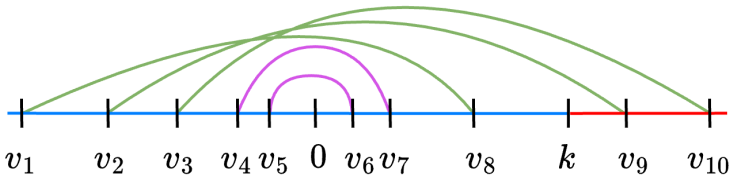


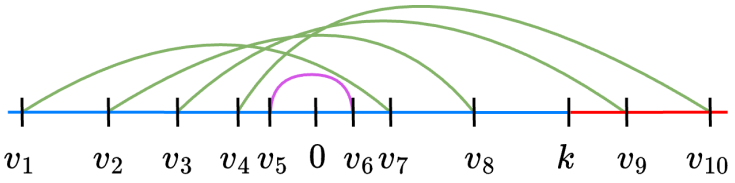
More won when $\omega=0$

Option 2 More positive matches, but with lower distance each

riskier when bad ω

Theorem 2: S-shaped and Symmetric Shock



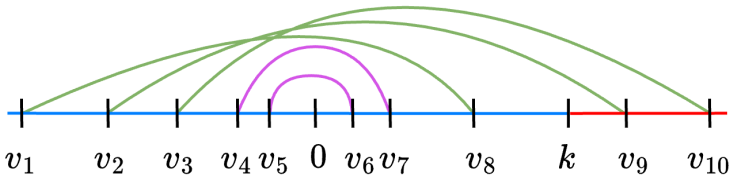


Option 1 Fewer positive matches (fewer won when $\omega=0$), but with higher distance (safer when bad ω) each

Option 2 More positive matches (more won when $\omega=0$), but with lower distance (riskier when bad ω) each

G symmetric $\rightarrow$ **Option 2**

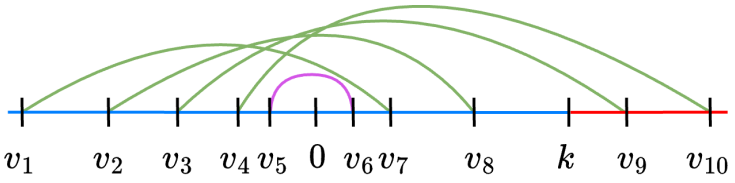
Theorem 2: S-shaped and Symmetric Shock



Fewer won when $\omega=0$

Option 1 Fewer positive matches, but with higher distance each

safer when bad ω



More won when $\omega=0$

Option 2 More positive matches, but with lower distance each

riskier when bad ω

G symmetric → **Option 2**

Intuition: OTP linear in number of districts won + symmetry → As if shock doesn't matter

Theorem 2: S-shaped and Symmetric Shock

Theorem 2

There is a unique solution τ to OTP. Moreover, τ is such that:

- $\tau = \alpha\tau^+ + (1-\alpha)\tau^-$ *for appropriate:*
  - $\alpha \in [0,1]$
  - τ^+ *positive assortative*
  - τ^- *negative assortative*
- *There is* $v', v'' \in \mathsf{supp}(\tau^+)$ *such that* $(1-\alpha)(2v' - v'' - k) = 0$

Theorem 2: S-shaped and Symmetric Shock

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FOC: Either $\tau = \tau^+$ or $2v' - v'' - k \geqslant 0$ binding for at least one v', v''

Theorem 2: S-shaped and Symmetric Shock

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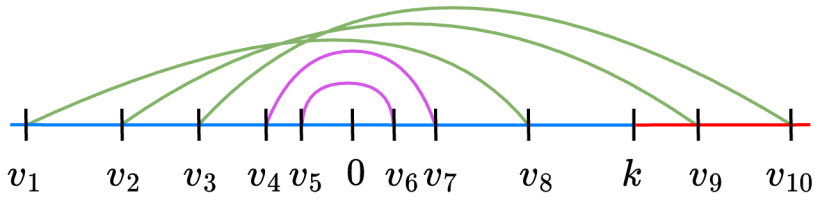
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FOC: Either $\tau = \tau^+$ or $2v' - v'' - k \geqslant 0$ binding for at least one v', v''

Intuitively: Solution must "maximize" positive assortative matches

Comparative Statics and Implications for the U.S. House

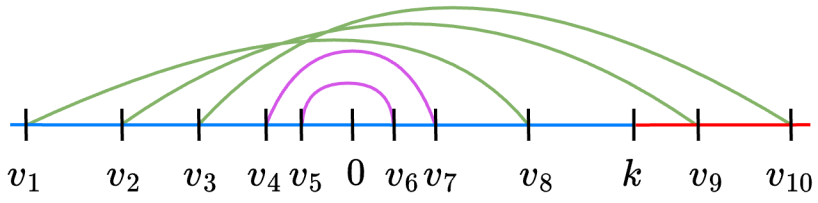
Some Comparative Statics



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- $\tau = \alpha\tau^+ + (1-\alpha)\tau^-$
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Some Comparative Statics

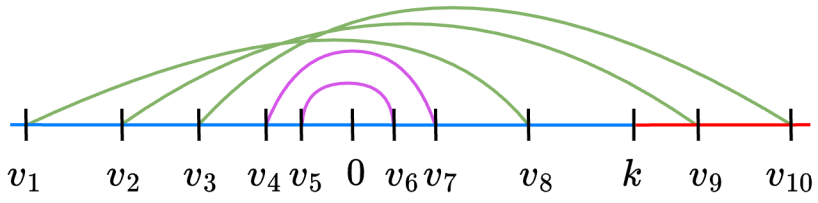


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- $\tau = \alpha\tau^+ + (1-\alpha)\tau^-$
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- More Republican voters in $\phi \rightarrow$ More positive matches $\rightarrow$ Better for designer

Some Comparative Statics

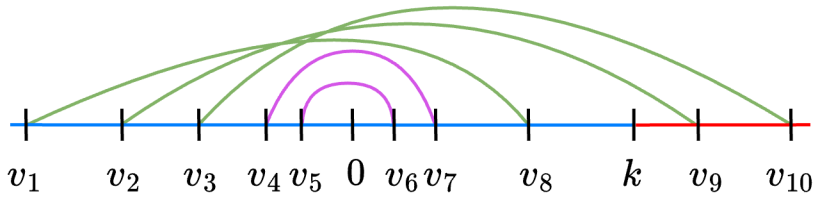


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- Median preserving spread of $\phi \rightarrow$ More positive matches $\rightarrow$ Better for designer

Some Comparative Statics



There is a unique solution τ to OTP:

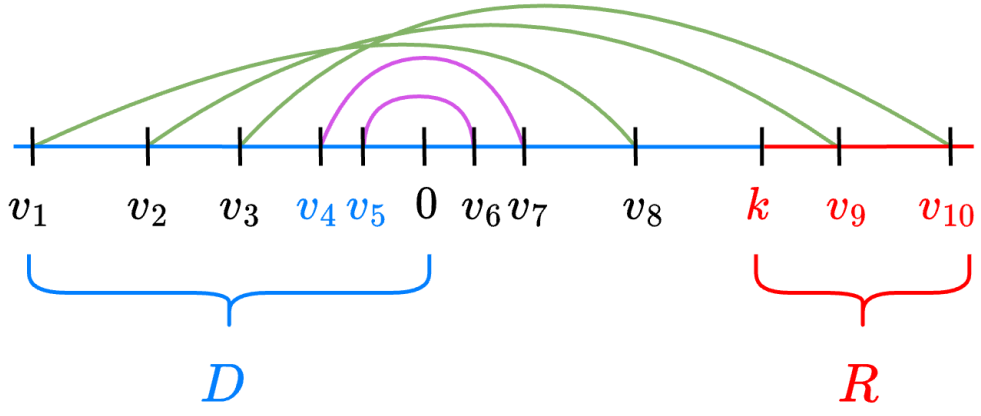
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- More Republican voters in $\phi \rightarrow$ More positive matches $\rightarrow$ Better for designer
- Median preserving spread of $\phi \rightarrow$ More positive matches $\rightarrow$ Better for designer

Corollary 1

If ϕ is "spread out" enough, the designer wins ALL districts

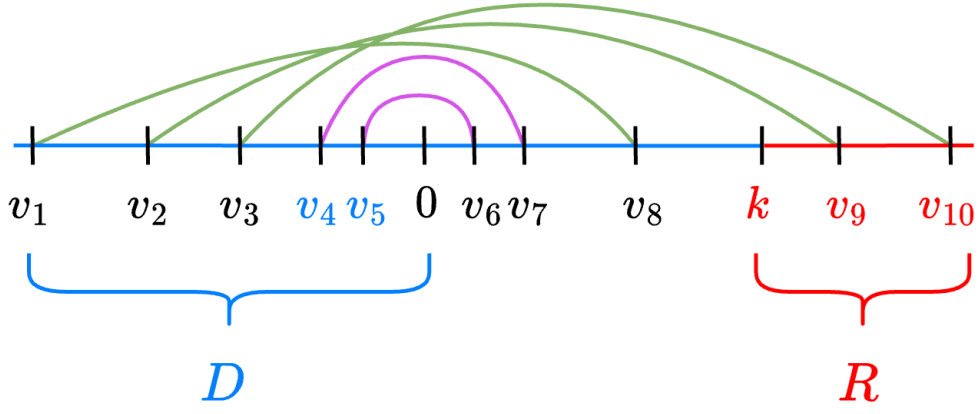
Polarization in the U.S. House



Proposition 2

Distribution of representatives has a gap in $[-\omega, k]$

Polarization in the U.S. House



Proposition 2

Distribution of representatives has a gap in $[-\omega, k]$

Dilution of moderate voters' power

Extensions

Extensions

Primary Elections

- Convex combination of party medians and median → same characterization
- Different party quantile q → "three-wise matching" (in the paper)

Objective Function

- Designer has policy preferences → pack supporters
- Majoritarian objective → non-linear OT → More negative assortment

Other

- Idiosyncratic uncertainty → means instead of medians
- Exogenous/Endogenous turnout

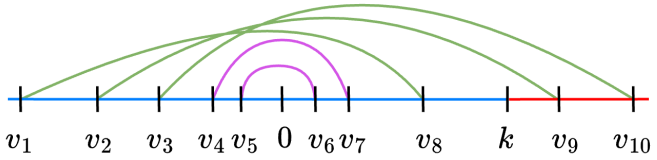
Conclusions

Recap

Optimal redistricting with endogenous candidates' positions

- Designer's objective depends on districts' medians AND conditional medians

→ optimal transport characterization:



- Standard gerrymandering can backfire
- Designer exploits extreme opponents to turn moderate opponents into supporters

Some Implications:

More spread out electorate
→ stronger gerrymandering

Polarized U.S. House

For Future Research

- Empirical follow-up: "Supply Responses to Redistricting" (with Herman and Longuet-Marx)
  - Preliminary findings: candidates respond to district composition

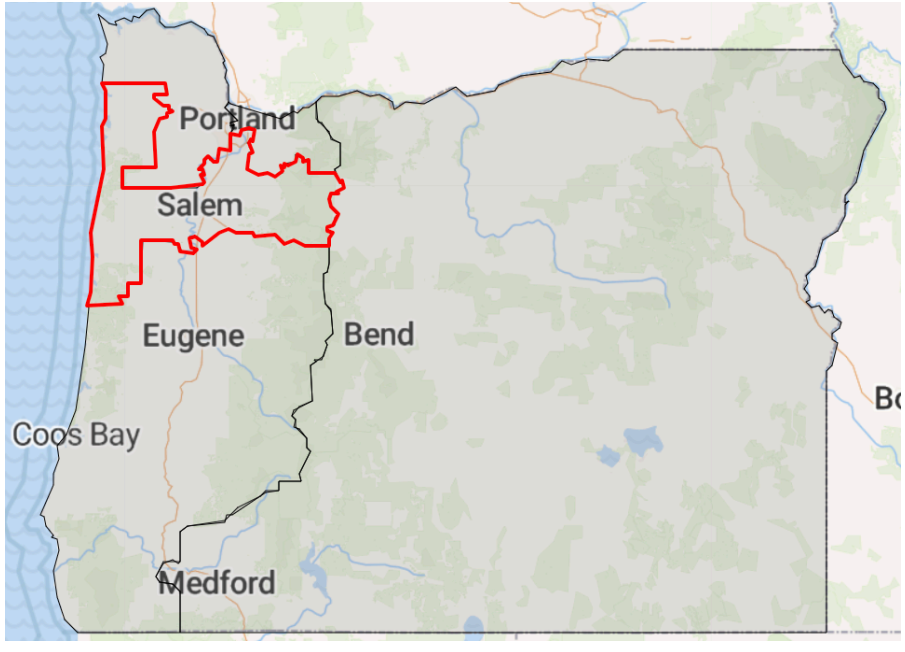
Application to other contexts

- Agenda setting in groups
- Information design with dependence on more than one statistic
  - Multiple signals
  - Behavioral preferences

Extra

Case Study: Oregon 5th Congressional District

Until January 2, 2023



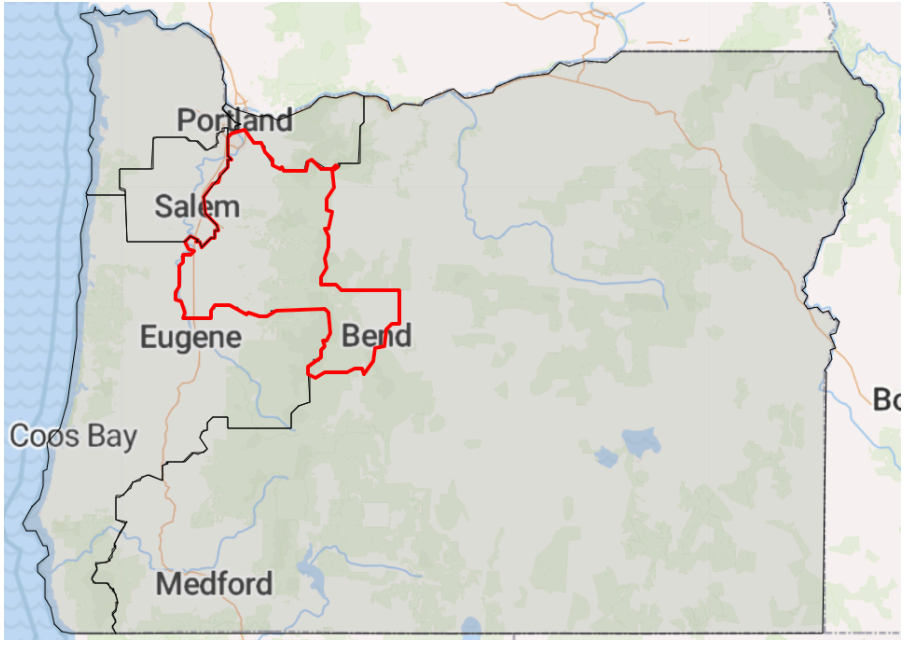
- Democratic gerrymandering
- 53.6% vs 43.9%
- 7th term incumbent *Kurt Shrader*
- Moderate Democrat, endorsed by Biden

Case Study: Oregon 5th Congressional District

Until January 2, 2023



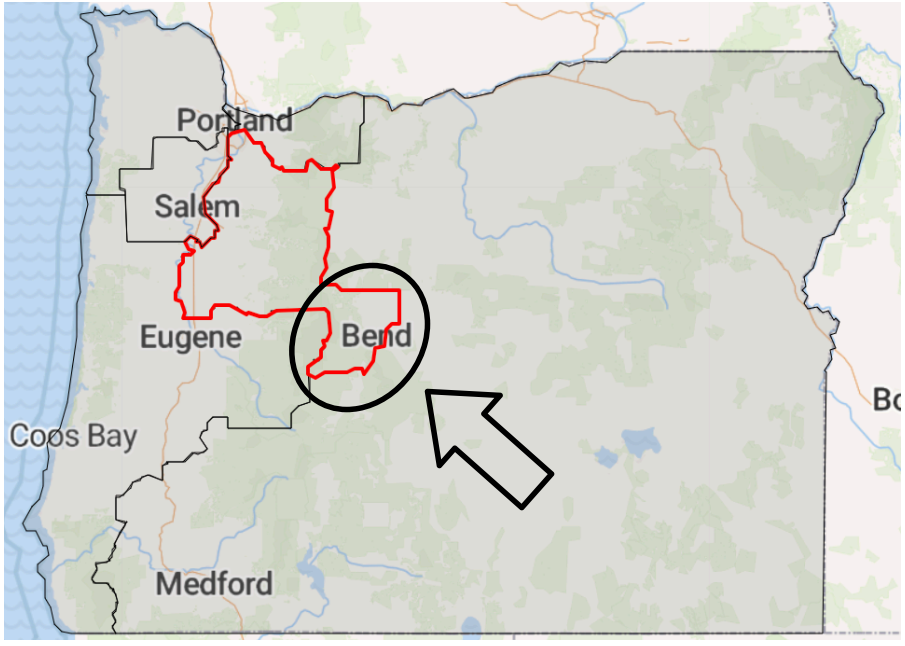
From January 3, 2023



Case Study: Oregon 5th Congressional District

From January 3, 2023

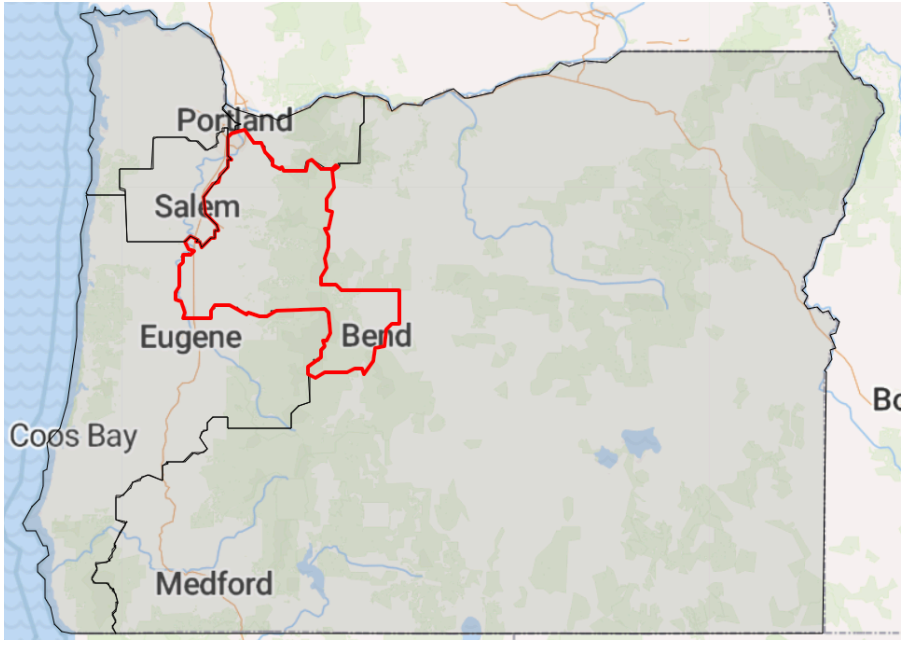
- Democratic gerrymandering
- 53.2% vs 44.4%
- Progressive Democrat *Jamie McLeod-Skinner* has 40 points advantage in Deschute county over incumbent



Case Study: Oregon 5th Congressional District

From January 3, 2023

- Shrader loses the Democratic primary to McLeod-Skinner
- McLeod-Skinner loses general elections to moderate Republican *Lori Chavez-DeRemer*

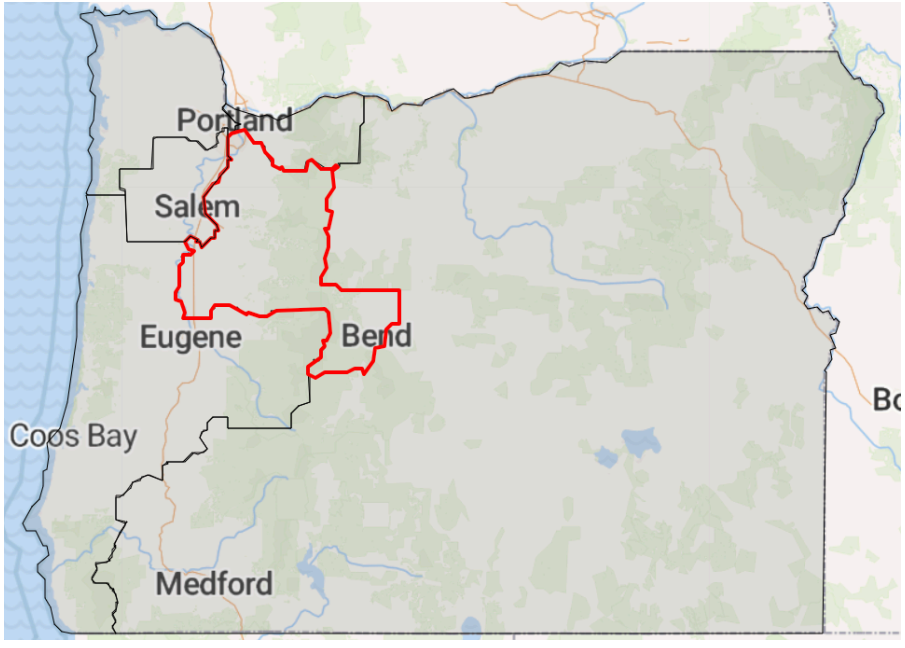


Case Study: Oregon 5th Congressional District

From January 3, 2023

Democrats ate their own [...] and now a standout Republican candidate will face-off against a far-too-liberal activist in Jamie McLeod-Skinner.

- Dan Conston for The Washington Post

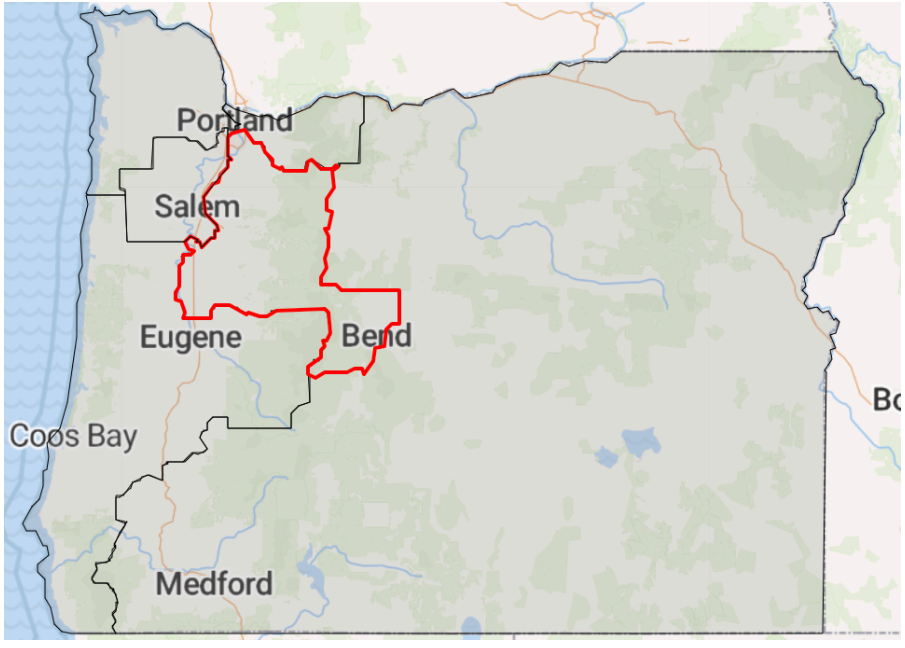


Case Study: Oregon 5th Congressional District

From January 3, 2023

Democrats ate their own [...] and now a standout Republican candidate will face-off against a far-too-liberal activist in Jamie McLeod-Skinner.

- Dan Conston for The Washington Post



Is gerrymandering is less powerful than we thought?

Elections in District π: Primaries

- Primary positions:

$$(c_{\pi,D}, c_{\pi,R}) = \begin{cases} (v^m_{\pi,D}, k) & \text{if } \operatorname{supp}\pi \subseteq (-\infty, 0] \\ (k, v^m_{\pi,R}) & \text{if } \operatorname{supp}\pi \subseteq [0, \infty) \\ (v^m_{\pi,D}, v^m_{\pi,R}) & \text{otherwise} \end{cases}$$

- $v^m_{\pi,D}$ the lowest median of π conditional on $v < 0$
- $v^m_{\pi,R}$ the highest median of π conditional on $v \geqslant 0$

Elections in District π: General Elections

- General elections: Voters vote for the candidate closer to them
- v_π^m highest median of π. Then:

$$c_P = \begin{cases} c_{\pi,D} & \text{if } v_\pi^m < \frac{c_{\pi,R}+c_{\pi,D}}{2} \\ c_{\pi,R} & \text{if } v_\pi^m \geqslant \frac{c_{\pi,R}+c_{\pi,D}}{2} \end{cases}$$

Elections in District π: Microfoundations

- Single-peaked voters' utilities $u(\cdot, v_i)$, $v_i \in [\underline{v}, \overline{v}]$
  - v^m_π median
  - $v^m_{\pi,R}$ median given $v_i \geqslant k$
  - $v^m_{\pi,D}$ median given $v_i < k$

- Two candidates per party, purely office motivated, commit to policy before elections

- Voters and candidates believe v^m_π to have distribution $H \in \Delta([\underline{v}, \overline{v}])$
  - In addition, candidates know $v^m_{\pi,R}, v^m_{\pi,D}$

Elections in District π: Microfoundations

Primary Elections

- Given position y of opposing nominee, voter t votes for candidate maximizing

$$U(x; y, t) = u(x, t)p(x, y) + u(y, t)(1 - p(x, y))$$

where $p(x, y)$ probability of x winning against y

General Election

- Voters vote for candidate closer to them

Elections in District π: Microfoundations

Linear setting

Suppose $u(x, v_i) = -|x - v_i|$ and H is uniform on $[\underline{v}, \overline{v}]$

Proposition 3

There exists a unique Nash equilibrium where the Republican and Democratic candidates set positions $c_{\pi,R} = v^m_{\pi,R}$ and $c_{\pi,D} = v^m_{\pi,D}$, respectively.

Elections in District π: Microfoundations

Convex setting

$$u(x, v_i) = e^{-\alpha|x-v_i|},\ \alpha > 0,\ H \sim \mathcal{N}(v^m_\pi, \sigma)$$

Proposition 4 (Owen and Grofman (2006))

If $\sigma \geqslant \frac{\max\left\{1-e^{1-\alpha(v^m_\pi - v^m_{\pi,D})}, 1-e^{\alpha(v^m_{\pi,R}-v^m_\pi)}\right\}}{\alpha\sqrt{2\pi}}$, *there exists a unique Nash equilibrium where the Republican and Democratic candidates set positions* $c_{\pi,R} = v^m_{\pi,R}$ *and* $c_{\pi,D} = v^m_{\pi,D}$, *respectively.*

Return

Proof Sketch: Step 1

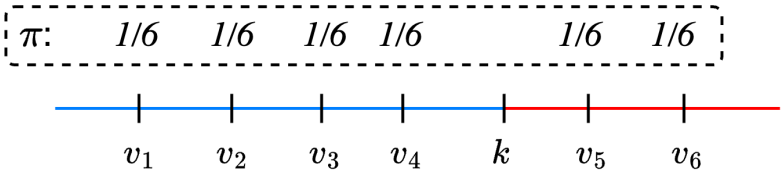
Intuition:

Step 1 For any district π, $\exists\ v', v''$:

$$\mathsf{supp}(\pi) = \{v', v''\}$$
$$\pi(\{v'\}) = \pi(\{v''\})$$

Proof Sketch: Step 1

Intuition:



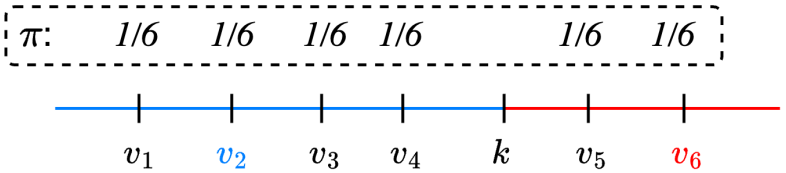
suppose $|\mathsf{supp}(\pi)| > 2$

Step 1 For any district π, $\exists\ v', v''$:

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Proof Sketch: Step 1

Intuition:



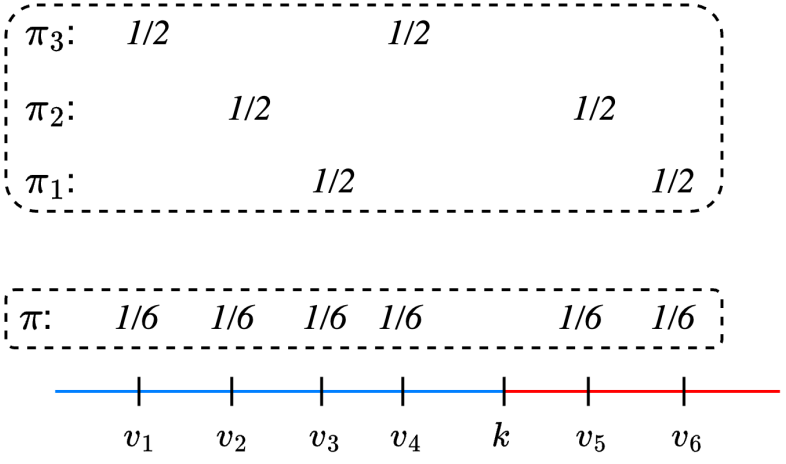
$c_{\pi,R} - v_\pi^m > v_\pi^m - c_{\pi,D} \rightarrow \pi$ is lost

Step 1 For any district π, $\exists\, v', v''$:

$$\text{supp}(\pi) = \{v', v''\}$$
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Proof Sketch: Step 1

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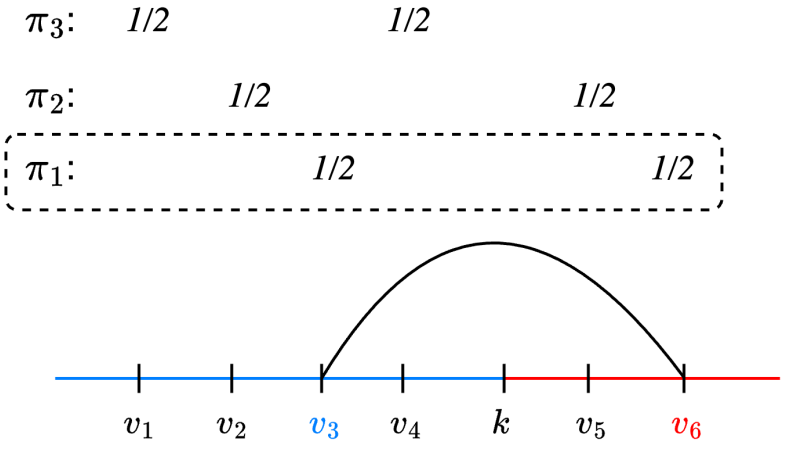
can substitute π with π_1, π_2, π_3

Step 1 For any district π, $\exists\ v', v''$:

$$\text{supp}(\pi) = \{v', v''\}$$
$$\pi(\{v'\}) = \pi(\{v''\})$$

Proof Sketch: Step 1

Intuition:



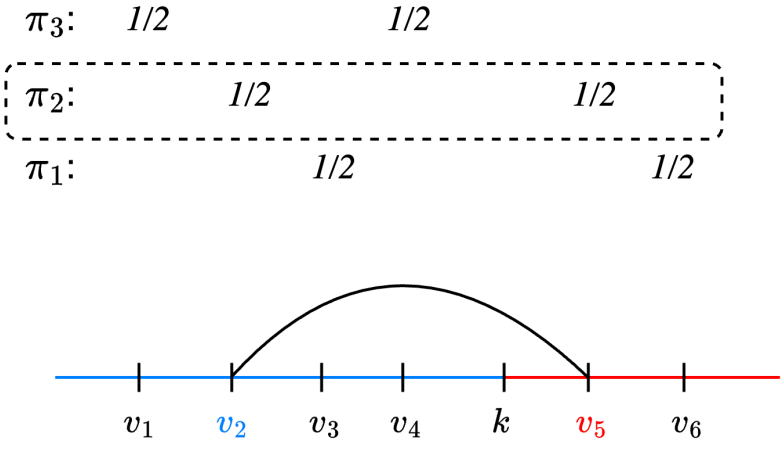
$C_{\pi_1,R} = V^m_{\pi_1} \to \pi_1$ is won

Step 1 For any district π, $\exists\, v', v''$:

$$\mathsf{supp}(\pi) = \{v', v''\}$$
$$\pi(\{v'\}) = \pi(\{v''\})$$

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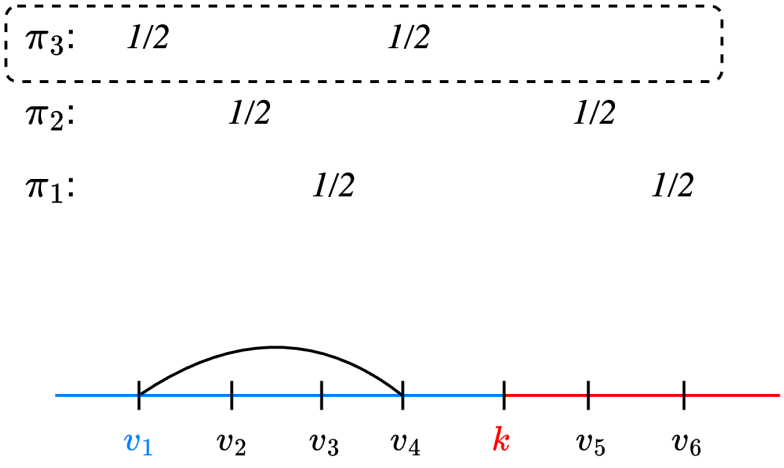
$C_{\pi_2,R} = V^m_{\pi_2} \to \pi_2$ is won

Step 1 For any district π, $\exists\ v', v''$:

$$\text{supp}(\pi) = \{v', v''\}$$
$$\pi(\{v'\}) = \pi(\{v''\})$$

Proof Sketch: Step 1

Intuition:



$c_{\pi_3,R} - v^m_{\pi_3} < v^m_{\pi_3} - c_{\pi_3,D} \rightarrow \pi_3$ is won

Step 1 For any district π, $\exists\ v', v''$:

$$\text{supp}(\pi) = \{v', v''\}$$
$$\pi(\{v'\}) = \pi(\{v''\})$$

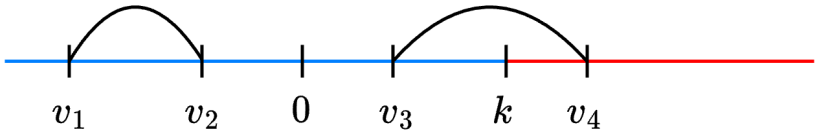
Proof Sketch: Step 2

Intuition:

Step 2 $v' \geqslant 0 \geqslant v''$

Proof Sketch: Step 2

Intuition:

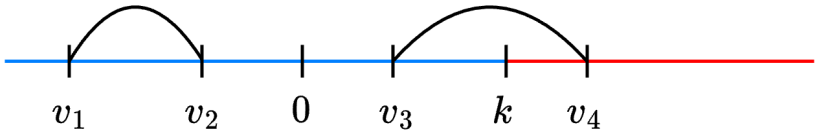


Only district with v_3, v_4 is won

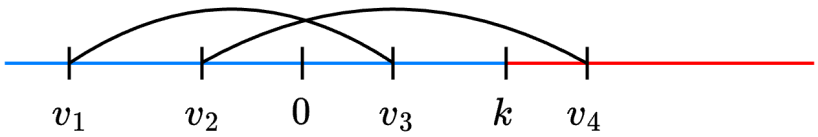
Step 2 $v' \geqslant 0 \geqslant v''$

Proof Sketch: Step 2

Intuition:



Only district with v_3, v_4 is won



Both districts are won

Step 2 $v' \geqslant 0 \geqslant v''$

Return

Reduction to Optimal Transport Problem

- Define $\phi' = \phi(\cdot | v \geqslant v^m)$ and $\phi'' = \phi(\cdot | v \leqslant v^m)$
- $\mathcal{M}(\phi', \phi'')$ set of joint distributions over v', v'' with marginals ϕ', ϕ''
- Designer's problem becomes:

$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int \int \mathbb{1}(v' - \omega \geqslant k) + \mathbb{1}(v' - \omega < k) \mathbb{1}\left(v' - \omega - \frac{v'' - \omega}{2} \geqslant k\right) d\gamma(\omega) d\tau(v', v'')$$

Reduction to Optimal Transport Problem

- Define $\phi' = \phi(\cdot | v \geqslant v^m)$ and $\phi'' = \phi(\cdot | v \leqslant v^m)$
- $\mathcal{M}(\phi', \phi'')$ set of joint distributions over v', v'' with marginals ϕ', ϕ''
- Designer's problem becomes:

$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int G(v' - k) + \int \mathbb{1}(v' - \omega < k) \mathbb{1}\left(v' - \omega - \frac{v'' - \omega}{2} \geqslant k\right) d\gamma(\omega) d\tau(v', v'')$$

Reduction to Optimal Transport Problem

- Define $\phi' = \phi(\cdot | v \geqslant v^m)$ and $\phi'' = \phi(\cdot | v \leqslant v^m)$
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- Designer's problem becomes:

$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int G(v' - k) + G\left(2v' - v'' - k\right) - G(v' - k) d\tau(v', v'')$$

Reduction to Optimal Transport Problem

- Define $\phi' = \phi(\cdot | v \geqslant v^m)$ and $\phi'' = \phi(\cdot | v \leqslant v^m)$
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$$\max_{\tau \in \mathcal{M}(\phi', \phi'')} \int G\left(2v' - v'' - k\right) d\tau(v', v'') \tag{OTP}$$

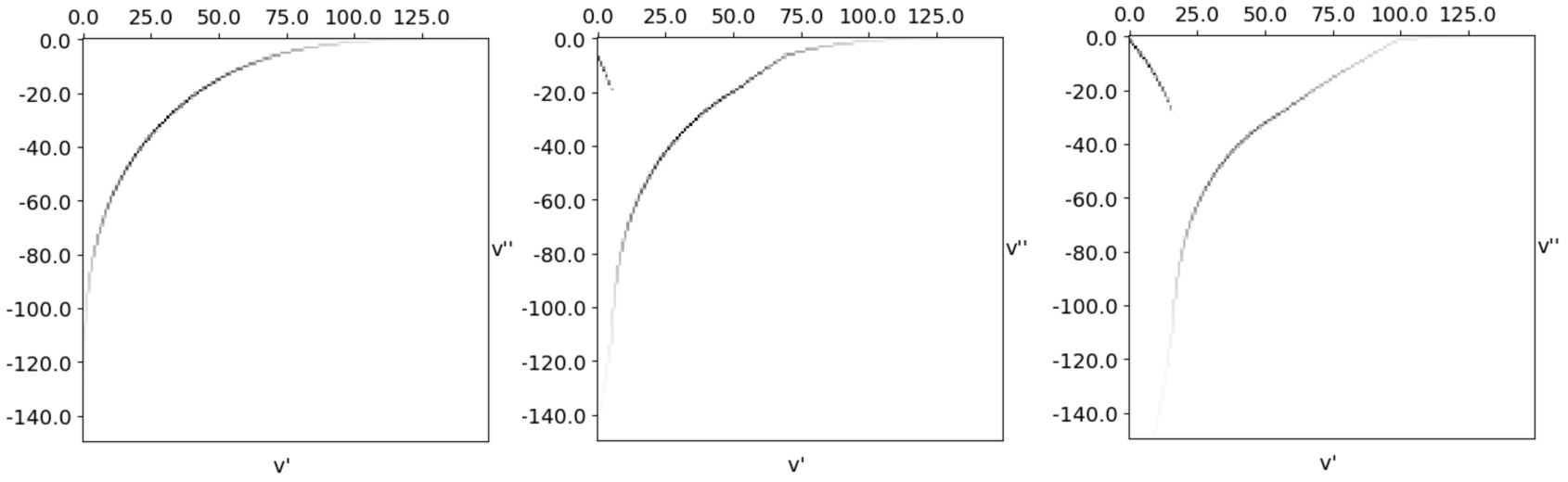
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Return

Example with Normal Shock



(a) k=60 **(b)** k=75 **(c)** k=100

Solution for simulated F normal, G normal

References

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