Trend Inflation and the Equilibrium Real Rate

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Abstract

During a downturn, central banks traditionally stimulate the economy by lowering nominal interest rates. In recent years, however, the zero lower bound (ZLB) has constrained this. Economists have proposed raising the inflation target to provide more room to cut during downturns. These proposals are informed by the New Keynesian model, the workhorse monetary model, which implies that raising the inflation target does not impact the equilibrium real rate. I introduce heterogeneity into this model and show that this does not hold: raising the inflation target causes the equilibrium real rate to fall. I estimate that raising the inflation target from 2% to 4% lowers the equilibrium real rate by 8 basis points in a baseline calibration and up to 66 basis points under alternative calibrations. This implies that raising the inflation target is less effective at raising the nominal interest rate and providing room to cut during downturns. I also investigate how this channel affects how much it is optimal to raise the inflation target to tackle the problem of the ZLB.

I'm currently revising this paper. Some of the results may change before the conference. I do not anticipate major changes.

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1 Introduction

Across developed countries, trend inflation has mostly remained around two percent, since the late-twentieth century. However, there are two reasons why trend inflation may rise moving forward. First, as equilibrium real interest rates have fallen (Holston et al., 2017), central banks are more constrained by the zero lower bound. Raising trend inflation increases the room central banks have available to cut in downturns (Blanchard et al., 2010; Ball, 2014; Krugman, 2014). Second, with increasing government debts and greater political pressures, doubts have risen about whether central banks will be able to maintain two percent inflation in the long-term.

It is widely assumed that a rise in trend inflation will not affect the equilibrium real rate. This is a feature of the benchmark New Keynesian model Woodford (2003); Galí (2008) that is typically used for monetary policy analysis. This is also a historic concept first introduced by Fisher (1907), and is often taken for granted in policy discussions.

In this paper, I introduce a novel channel through which a rise in trend inflation lowers the equilibrium real rate. I first show that this channel intuitively holds when there is both price rigidity, such as Calvo or menu cost pricing, and household heterogeneity, such as Aiyagari households or overlapping generations. Second, I estimate the size of this channel. In the baseline calibration, I find that a rise in trend inflation from 2 percent to 4 percent lowers the equilibrium real rate by 8 basis points. However, the size of this channel can be much larger when realistic features such as heterogeneous firms, additional firm discounting, and a low intertemporal elasticity of substitution are introduced. Finally, I explore the interaction of this channel with the zero lower bound. I show that this channel worsens the problem of the zero lower bound but, despite this, does not suggest central banks implement a higher optimal inflation target due to the reduced effectiveness of raising the inflation target.

My first contribution is to demonstrate a new channel by which raising the inflation target lowers the equilibrium real rate. Once the strict assumption of a representative agent in the standard, benchmark New Keynesian model is relaxed and richer household heterogeneity is allowed for, I find that raising the inflation target lowers the equilibrium real rate. This holds under different forms of household heterogeneity including: 1. Agents have life cycle features, which was first explored in Diamond (1965) and is commonly referred to as overlapping generations (OLG). 2. Agents have idiosyncratic income shocks with a borrowing constraint, which was first considered in Aiyagari (1994) and in the context of the New Keynesian model is commonly referred to as a Heterogeneous Agent New Keynesian model (HANK). This implies that a rise in the inflation target increases the nominal interest rate by less than is typically expected. Therefore, raising the inflation target will reduce the probability of avoiding the ZLB by less than is commonly believed. The channel has two stages. First, price rigidities imply that a rise in the inflation target lowers the markup. Second, household heterogeneity implies that a fall in the markup lowers the equilibrium real rate. Therefore, I hereafter refer to the channel as the IMR (inflation markup real rate) channel.

The first part of the IMR channel is a standard, albeit often overlooked feature of the benchmark New Keynesian model. A firm's markup is the ratio of its price to its nominal marginal cost. When firms set their prices infrequently, a higher average inflation level has two opposing impacts on average markups. First, higher inflation means that when a firm does not reset its price, its markup falls by relatively more, since nominal marginal costs rise relatively quicker with higher inflation. Second, firms observe that their markups fall more quickly and therefore set their markup higher when they reset their prices. It can be shown that

with no discounting, these two effects cancel out each other, and thus average markups are unchanged by raising average inflation. However, with discounting, the first effect dominates, since firms care more about making profits in the current period and so do not want to set their current markup very high when they reset their price. Therefore, a rise in average inflation lowers the average markup. This holds under different types of price rigidities including rigidities based on time (Calvo pricing, Taylor pricing) and rigidities based on adjustment costs (menu costs, Rotemberg pricing).

In the second part of the channel, a fall in the markup lowers the equilibrium real rate once household heterogeneity is allowed for. A fall in the markup lowers firm profits and thus reduces the value of shares and of overall savings. Taking the example of heterogeneity through overlapping generations (OLG), older people rely more on savings for their consumption. Therefore, a fall in the amount of savings, all else being equal, lowers the consumption of old people relative to young people. This means the old have higher marginal utility from consuming compared with the young. Thus, there is greater competition among young people to save for when they are old, and so the price of savings rises. As the price of savings rises, the return on savings (the equilibrium real rate) falls.

This contrasts with a representative agent New Keynesian model in which a fall in the markup has no impact on the equilibrium real rate. The key difference is that the representative agent's consumption growth path must equal the aggregate consumption growth path, because there is only one agent. The aggregate consumption growth path is pinned down by exogenous factors, such as productivity and population growth. Therefore, even if a fall in the markup lowers the amount of savings, this has no impact on the representative agent's consumption growth path. Consequently, the marginal utility the representative agent receives from consuming in the future relative to now remains the same, which implies that the price of savings is unchanged, and so the return on savings (the equilibrium real rate) remains the same.

My second contribution is to estimate the size of the IMR channel through a calibrated model. I incorporate household heterogeneity through a life-cycle framework modeling each quarter of adult life. I model other parts of the economy following a standard New Keynesian setup with monopolistic firms that face Calvo pricing. In the baseline model, I find that raising the inflation target from 2% to 4% lowers the equilibrium real rate 8 basis points. I consider various extensions. Allowing for a lower intertemporal elasticity of substitution (IES) of 0.1 rather than 0.5 boosts the effect to a fall of 18 basis points, since with a low IES, agents are keener to ensure they consume at the same level when they are old as when they are young. One concern may be that this result comes from the fact that raising inflation does not affect the degree to which firms change their prices under Calvo pricing. However, modeling price rigidity with menu costs rather than Calvo pricing yields a similar or slightly stronger effect. Introducing factors that boost the negative relationship between trend inflation and the markup strengthens the relationship. For example, allowing for firms to discount the future at the higher rate implied by their cost of financing rather than the household's discount factor, strengthens the size of the effect to 28 basis points. With both high firm discounting and low IES, this rises to 66 basis points.

The IMR channel also holds within a Heterogeneous Agent New Keynesian (HANK) model. In recent years, there has been significant work involving models in which household heterogeneity is embedded into the New Keynesian model through idiosyncratic labor shocks and a borrowing constraint (for example McKay et al. 2016; Kaplan et al. 2018). These are commonly referred to as HANK models. This form of heterogeneity also leads to the IMR channel. A fall in the markup lowers firm profits and thus the supply of assets. With fewer assets in which to save, agents are, all else being equal, more likely to hit the borrowing constraint. Thus, when agents are not constrained by the borrowing constraint, they are prepared to pay a higher price for savings to

avoid the borrowing constraint, which is equivalent to a fall in the return on savings or the equilibrium real rate. I embed this HANK style of heterogeneity into my life-cycle New Keynesian model. This strengthens the effect of the rise in the inflation target on the equlibrium real rate compared to the baseline case and also allows me to qualitatively match the hump-shaped path of agents' lifetime consumption.

My third contribution is to investigate how the IMR channel interacts with the zero lower bound. I first show that, all else being equal, the IMR channel means that raising the inflation target is less effective at mitigating the zero lower bound. I next investigate how this affects the degree to which it is optimal to raise the inflation target in response to an exogenous fall in the equilibrium real rate. On the one hand, since the zero lower bound is mitigated less, it might be expected that the inflation target should be increased by more. On the other hand, raising the inflation target is less effective at mitigating the zero lower bound. I find that these two opposing forces approximately cancel out and it is optimal to raise the inflation target by a similar amount under the IMR channel compared to when there is no IMR channel.

Related Literature

My paper relates to two strands of literature. First, my paper relates to the literature examining what the optimal inflation target is in light of the zero lower bound. This is an important policy topic. As Janet Yellen said in a press conference in June 2017, "a reconsideration of [the inflation target] needs to take account not only of benefits of a higher inflation target, but also the potential costs" and that this is "one of the most important questions facing monetary policy around the world in the future, and we very much look forward to seeing research by economists that will help inform our future decisions on this". Coibion et al. (2012) compute the optimal inflation target in the presence of the zero lower bound while Andrade et al. (2019) investigate how the optimal inflation target is affected by a fall in the equilibrium real rate. Both of these papers consider optimal inflation in a New Keynesian model only with a representative agent, and therefore do not allow for the IMR channel. In concurrent work, L'Huillier and Schoenle (2019) present an alternative channel by which raising the inflation target may be less effective at mitigating the problem of the ZLB. Using the standard representative agent New Keynesian model, they argue that higher inflation induces firms to change their prices more frequently, which makes monetary policy less effective, mitigating the benefits of the rise in the inflation target. My paper contributes to this literature by showing that a rise in the inflation target is likely to mitigate the zero lower bound by less than expected due to the IMR channel.

My paper also relates to a historical literature looking at the relationship between trend inflation and the equilibrium real rate. The idea that trend inflation does not impact the equilibrium real rate, or alternatively that trend inflation and equilibrium nominal interest rates move together proportionally, dates back to Fisher (1907). This "Fisher Effect" is also often termed "superneutrality", that is, the growth rate of prices (or more traditionally money) does not impact real variables, that is, the equilibrium real rate. Mundell (1963) and Tobin (1965) provide another channel by which a rise in trend inflation lowers the equilibrium real rate breaking superneutrality. The Mundell-Tobin effect documents that when inflation rises, it becomes costlier to hold money, so agents save more in capital, leading to a rise in the price of holding capital and thus a fall in the equilibrium real rate. This effect is likely less applicable in modern central banking, where central banks pay an interest rate on reserves that, at least in theory, should imply no portfolio shift away from money when inflation rises. Other reasons why superneutrality would not hold include liquidity traps, which are modeled for example in Eggertsson et al. (2019), or tax policy, which is summarized in Feldstein (1982). My paper provides a novel channel through which changes in trend inflation may not be superneutral. This channel comes from the combination of standard features of modern macroeconomic models, that is, price rigidity and

household heterogeneity.

In section 2, I outline a simple model that captures the mechanism behind the IMR channel. I outline the full model in section 3. I discuss the model solution and calibration in section 4. I use the full model to analyze the size of the IMR channel and the factors that determine it in section 5. I consider how the IMR channel interacts with the zero lower bound in section 6. Section 8 concludes.

2 Intuition through a Simplified Model

I break the explanation for the intuition for the IMR channel into two parts. First, I demonstrate that a rise in inflation lowers the average markup through firms' pricing decisions. Next, I show that a fall in the markup lowers the equilibrium real rate through multiple forms of household heterogeneity.

2.1 Relationship between the Inflation Level and the Markup

A firm's markup, denoted m_t, is its current price, $P_t^\star$, divided by its nominal marginal cost, $\mathcal{MC}_t$:

$$m_t = \frac{P_t^\star}{\mathcal{MC}_t}.$$

A firm's profits depend on its markups. If a firm sets its markup too high, it will not make enough sales. If a firm sets its markup too low, the firm will make a lot of sales but with too little profit on each sale. When a firm has fully flexible prices, it can set its price so that its markup yields the maximum profits each period. In the common case where firms face constant elasticity of demand, the optimal markup is just $\frac{\sigma}{\sigma-1}$, where σ is the CES parameter.

Setting markups is more complex when price adjustment is infrequent. When a firm can change its prices only infrequently, it is no longer able to set the optimal flexible price markups each period. Positive trend inflation has two effects on markups. First, if a firm cannot change its price in a period, its markup will fall. This is because the firm's nominal marginal costs ($\mathcal{MC}_t$) rise (due to the rise in the price level) while its prices ($P_t^\star$) remains constant. Second, since a firm anticipates this, when it does get to change its price, it sets a markup that is higher than the optimal flexible price markup.

The impact of trend inflation on the markup depends on the degree of discounting. In the case of no discounting, a firm will weight its profits equally in current and future periods. This leads to a special case where the markup is unaffected by changing the level of inflation, since the two effects on the markup cancel out each other. However, when a firm discounts the future, it will weight its current period markup more in its decision-making. This implies that a firm sets a lower markup when it can change its price and thus that the average markup is lower with positive inflation. As the level of inflation rises, the markup falls by more.

This effect can be informally observed in the Phillips Curve, which is a standard building block of the New Keynesian model (Woodford 2003; Galí 2008). A form of the Phillips Curve is shown in equation (1).[1] When

[1]Note that the derivation of this form of the Phillips Curve involves log-linearizing around zero inflation, which is a less reasonable approximation when non-zero inflation is considered — it is used here to provide simple intuition.

trend inflation is zero ($\hat{\pi}_t = \hat{\pi}_{t+1} = 0$), the marginal cost ($\frac{\widehat{\mathcal{MC}_t}}{P_t}$) is constant, regardless of the level of discounting (β). However, if trend inflation rises ($\hat{\pi}_t = \hat{\pi}_{t+1} > 0$), the marginal cost is unchanged only if there is no discounting ($\beta = 1$). Otherwise, the rise in trend inflation implies that the marginal cost must rise ($\frac{\widehat{\mathcal{MC}_t}}{P_t} \uparrow$). The rise in the real marginal cost implies a fall in the markup ($m_t \downarrow$). Therefore, a rise in trend inflation is associated with a fall in the markup if firms discount the future.

$$\hat{\pi}_t = \kappa \frac{\widehat{\mathcal{MC}_t}}{P_t} + \beta \mathbb{E}_t[\hat{\pi}_{t+1}]. \tag{1}$$

More formally, this result can be shown to always hold under Calvo pricing, which assumes that a firm can change its price each period with some exogenous probability. Appendix E.1 demonstrates that with Calvo pricing, if a firm does not discount, the markup is constant regardless of the level of inflation. When a firm does discount, the markup falls monotonically as inflation rises.

One concern might be that this relationship holds only under the strong assumption in Calvo pricing that a firm does not change its frequency of price adjustment as inflation rises. If a firm does set prices more frequently when there is higher inflation, a firm would set its markup for shorter periods of time, on average, which means that the markup would fall by less before being changed for higher levels of inflation. However, this paper focuses on the frequency of price adjustment under low levels of inflation, and empirical evidence suggests that under low inflation there is little change in the frequency of price adjustment. For example, Gagnon (2009) demonstrates that the frequency with which firms change their price does not appear to vary below annual rates of inflation of 10% . This makes sense, because a firm is likely to change its prices for reasons other than just inflation (such as idiosyncratic demand shocks or costs).

Moreover, even when alternative price adjustment methods that allow for the frequency of price adjustment to rise under higher inflation are considered, the negative inflation-markup relationship still holds.

In the case of menu costs with idiosyncratic productivity shocks: Firms may receive positive producitivity shocks that lower their costs and imply that their markups actually rise after firms reset their price. Under low inflation, these above-reset markups can last a long time, implying higher average markups. Under higher inflation, however, the nominal costs firms face quickly rise so these above-reset markups only last a short time. Therefore, a rise in inflation implies a fall in markups, not only because firms don't want to raise their prices a lot due to discounting as in Calvo, but also because the above-reset markups last less time. Indeed, in the menu cost case, the negative relationship between inflation and the markup holds even with no discounting by firms. In section 5, I show that the relationship between inflation and the markup can actually be stronger under low inflation for menu costs than for Calvo pricing as a result of this effect.

The features needed to generate the negative inflation-markup relationship appear to generally hold in the data. To achieve a negative relationship between inflation and the markup, firms need to set their prices infrequently and discount the future. Nakamura and Steinsson (2008) demonstrate that firms change prices infrequently. Jagannathan et al. (2016) show that firms discount the future significantly. It is also worth stressing that this relationship is present in the representative agent New Keynesian model — nothing here depends on household heterogeneity.

2.2 Relationship between the Markup and the Equilibrium Real Rate

Simple Model Section 2.1 explains the determination of the markup by the pricing side of the model. Here the markup is taken as a given and instead the focus is on how changes in the markup affect the real side of a simple model.

Firms produce using a linear production function.[2] Therefore, output Y_t equals labor L_t:

$$Y_t = L_t. \tag{2}$$

The real marginal cost of firms MC_t is therefore the real wage W_t:

$$MC_t = W_t. \tag{3}$$

The markup m_t is the price divided by the nominal marginal cost, which, by definition, equals the inverse of the real marginal cost ($\frac{1}{MC_t}$), so the relation between marginal cost and wage (equation (3)) can be rewritten as:

$$\frac{1}{m_t} = W_t. \tag{4}$$

The total real profits Ω_t of firms are their real sales, which is their output minus their costs of labor:

$$\Omega_t = Y_t - W_t L_t.$$

Y_t can be replaced with L_t using equation (2). The right-hand side can be multiplied and divided by W_t to give:

$$\Omega_t = (\frac{1}{W_t} - 1) W_t L_t.$$

The markup-wage relationship (equation (4)) can be applied to yield:

$$\Omega_t = (m_t - 1)\frac{W_t}{P_t} L_t. \tag{5}$$

Asset Supply The solution is broken down into the supply and demand for assets. Asset supply is the quantity of assets available for households to hold. Asset demand is the quantity of assets that agents want to hold.

The only assets in which agents can save are shares in firms. The total real value of firms is denoted by Z_t. Therefore, asset supply, denoted A_t^s, is given by:

$$A_t^s = Z_t. \tag{6}$$

[2]Note that in this simple model all firms produce the same amount. This would be true, for example, under Rotemberg Pricing. However, under price dispersion, where different prices would imply different demands, this would not hold. That being said, the case with price dispersion generates exactly the same results but is somewhat more complicated to derive. The results are shown in appendix E.2.

By standard asset pricing, the price of buying b shares, bZ_t, must equal the next period return on those shares discounted at r_{t+1}. The next period return of those shares is the dividends received from profits ($b\Omega_{t+1}$) plus the price the shares are sold for at $t+1$ (bZ_{t+1}). Therefore:

$$Z_t = \frac{\Omega_{t+1} + Z_{t+1}}{1 + r_{t+1}}. \tag{7}$$

In the steady state, equation (7) can be rewritten as:

$$\bar{Z} = \frac{\bar{\Omega}}{\bar{r}}. \tag{8}$$

The value of shares with profits (equation (8)) can be substituted into the asset supply equation (equation (6)) to yield:

$$\bar{A}^s = \frac{\bar{\Omega}}{\bar{r}}. \tag{9}$$

Profits can then be substituted out using equation (5):

$$\bar{A}^s = \frac{\bar{m} - 1}{\bar{r}} \bar{W} \bar{L}. \tag{10}$$

To make the problem simpler, relative assets a are defined as assets relative to labor income:

$$a = \frac{A}{WL}. \tag{11}$$

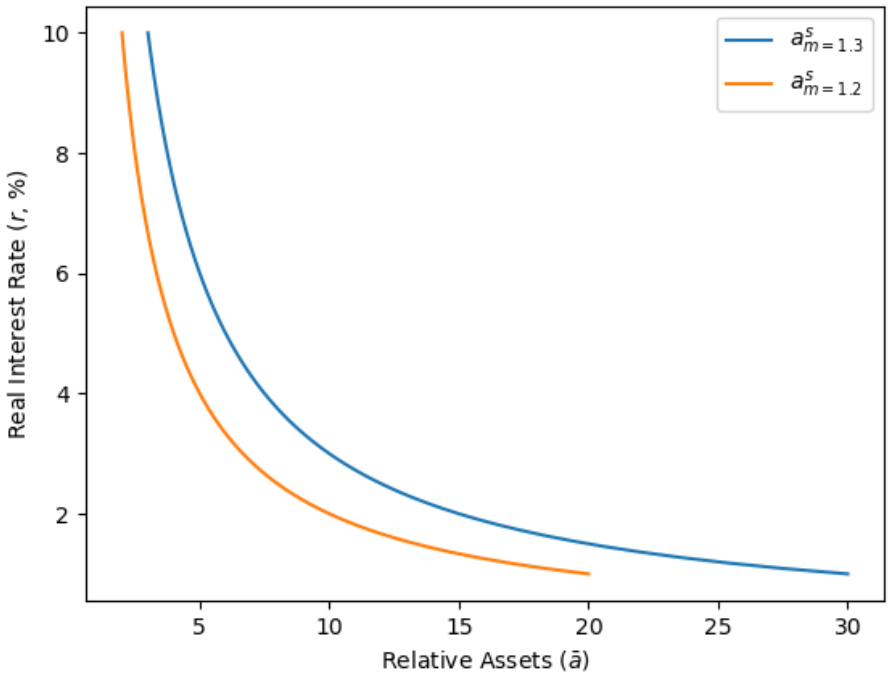
The supply of assets (equation (10)) can be rewritten in relative terms to get:

$$\bar{a}^s = \frac{\bar{m} - 1}{\bar{r}}. \tag{12}$$

Two features can be observed. First, in equation (12) a rise in $\bar{r}$ lowers $\bar{a}^s$. This makes sense, because higher discounting implies the discounted sum of profits is lower so the value of firms falls. Equation (12) is plotted in figure 1. The blue curve represents $\bar{a}^s$ with $\bar{m} = 1.3$, and the orange curve represents $\bar{a}^s$ with $\bar{m} = 1.2$. Since raising $\bar{r}$ lowers $\bar{a}^s$, the curve has a downward slope. It may seem strange that the supply curve is downward sloping, but this is because the vertical axis is the return on assets. The return on assets is like the inverse of the price of assets (since the return that agents make on assets falls as the price of those assets rises). If the curve was drawn with the price of assets on the vertical axis, it would have the usual upward-sloping supply curve.

Second, note that in equation (12) a fall in the markup $\bar{m}$ lowers the relative asset supply $\bar{a}^s$ when the real interest rate $\bar{r}$ is held constant. This makes sense, because when the markup falls, the value of firms falls and thus the value of owning shares in firms falls. This can also be seen in figure 1. Observe that the fall in the markup shifts the relative asset supply curve left, from the blue curve with markup 1.3 to the orange curve with markup 1.2.

Figure 1: Relative Asset Supply under a Fall in the Markup



Asset Demand: 1. Representative Agent Next, the shape of the asset demand is considered under three different household structures: (1) representative agent, (2) heterogeneity through overlapping generations, and (3) heterogeneity through idiosyncratic labor.

In all standard representative agent problems, an Euler condition of a form similar to the following is derived:

$$u'(C_t) = \beta(1 + r_{t+1})u'(C_{t+1}). \tag{13}$$

A steady state equilibrium requires that a representative agent consumes the same amount over time. If $1+r_{t+1}$ is more (less) than β, consumption must be growing (falling). Therefore, the only way a steady state is possible is if:

$$\beta(1 + \bar{r}) = 1.$$

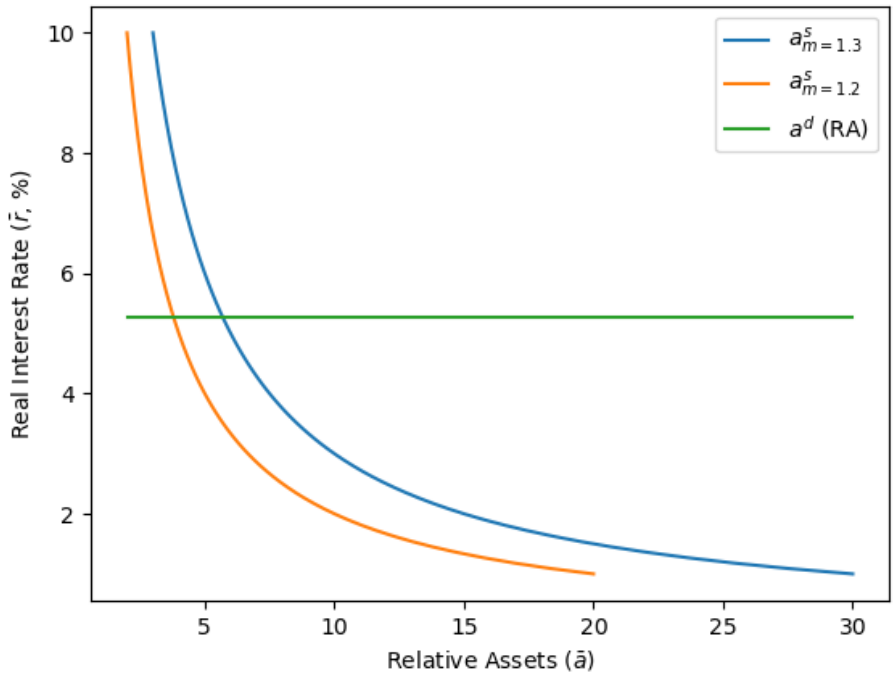
This can be rewritten as:

$$\bar{r} = \frac{1}{\beta} - 1. \tag{14}$$

Equation (14) is plotted in figure 2. As in figure 1, the impact of a fall in the markup is considered. Observe that the asset demand is just a horizontal line since $\bar{r}$ is always pinned down. Thus, a shift left in the supply of assets reduces the quantity of assets held by the household but has no impact on $\bar{r}$.

The assumption of a representative agent pins down the real interest rate regardless of changes in the supply of assets. The agent's Euler condition dictates that the real interest rate is an increasing function of savers'

Figure 2: Equilibrium under a Fall in the Markup: 1. Representative Agent



consumption in the future relative to now. The intuition is that the only way agents will consume more in the future than they do now is if they get a high real interest rate on their savings. Otherwise they would save less and consume more now. With a representative agent, however, growth in the consumption of the agent equals growth in aggregate consumption. And aggregate consumption is pinned down by exogenous parameters, such as exogenous productivity, so the growth of aggregate consumption is unchanged. Therefore, since the growth in aggregate consumption is unchanged, the growth in the representative agent's consumption is unchanged, and this means that the real interest rate must stay the same. Therefore, even if average inflation rises, which from the firm side leads to a fall in the quantity of assets in which agents can save, this has no impact on the real interest rate in the long run.

Asset Demand: 2. Overlapping Generations Now, household heterogeneity is introduced. The reason household heterogeneity makes a difference is that it implies that the agent's consumption and aggregate consumption can be different. This implies that even though the economy may remain on the same balanced growth path in the steady state, a change in the supply of assets can affect the agent's consumption in the future relative to now, which by the Euler condition feeds back to adjust the real interest rate.

A simple overlapping generations model based on Diamond (1965) is considered here to illustrate this point. In every period, a new generation is born. Each generation lives for two periods and then dies. The utility of a young agent is given by:

$$\log(C_{1,t}) + \beta \log(C_{2,t+1}). \tag{15}$$

Log utility is used for simplicity. Young agents work L units and devote their income to either consumption

$C_{1,t}$ or asset purchases A_{t+1}:

$$C_{1,t} + A_{t+1} = W_t. \tag{16}$$

Old agents merely consume $C_{2,t+1}$ from their available assets. Their available assets are the assets from when they were young on which they have earned a return of r_{t+1}:

$$C_{2,t+1} = (1 + r_{t+1})A_{t+1}. \tag{17}$$

The amount the young save can be solved for by inputting equations (16) and (17) into equation (15) and then taking first-order conditions. This yields:

$$A_{t+1} = \frac{\beta}{1+\beta} W_t L, \tag{18}$$

which shows that the young save some constant fraction of their income each period. It is simpler to rewrite the agent's demand for assets in relative terms; thus equation (18) is divided by labor income and also written in steady state terms to yield:

$$\bar{a}^d = \frac{\beta}{1+\beta}. \tag{19}$$

In this case, the demand for savings is perfectly inelastic to changes in $\bar{r}$. This is a special case (due to log utility and the inclusion of only two periods). In the full model, demand for relative assets is not perfectly inelastic. However, the generation structure in the full model is still such that the elasticity of demand is not perfectly elastic and thus the real interest rate changes in response to a shift left in the demand for assets.

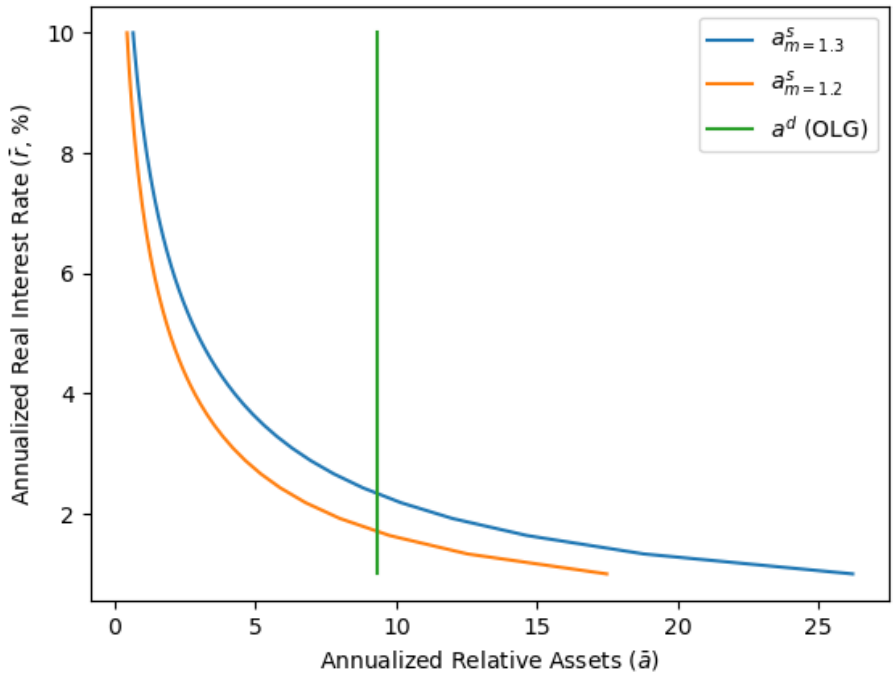
Equation (19) is plotted in figure 3, in which the impact of a fall in the markup is considered (as in figure 1). Observe that the asset demand is just a vertical line since $\bar{a}^d$ is fixed. Thus, a shift left in the supply of assets lowers $\bar{r}$ but has no impact on the quantity of assets demanded by the agent. The relative asset supply on this graph looks different from previous asset demand/supply graphs because each period represents a generation and lasts 25 to 30 years, making it necessary to rescale the curves to get back to an annual basis. The non-annualized case is shown in appendix E.3.

This is very different from the representative agent case in which a fall in the supply of assets does not impact the real interest rate but reduces equilibrium assets. The reason is that when the supply of assets falls, it reduces the consumption of the old relative to the young. Thus, young people want to save more to pass more consumption on to when they are old. Consequently, they are prepared to pay more to save, which is equivalent to accepting a lower return on their savings and thus lower real interest rates.

Asset Demand: 3. Idiosyncratic Labor In this paper, household heterogeneity is primarily introduced through overlapping generations. However, an extension with idiosyncratic labor is considered, and it is worthwhile to demonstrate that a similar intuition explains why the channel holds in this case. This form of heterogeneity was first introduced by Aiyagari (1994) and has been studied extensively in recent years in the context of the New Keynesian model (Werning 2015; McKay et al. 2016; Gornemann et al. 2016; Kaplan et al. 2018; Auclert 2019; Achdou et al. 2017) using what are frequently referred to as HANK models.

Here a simple model is considered for the purposes of exposition. There are many agents, each denoted with

Figure 3: Equilibrium under a Fall in the Markup: 2. OLG



subscript i. Agents live forever and maximize their lifetime utility:

$$\max \sum_{t=0}^{\infty} \mathbb{E}_0 \left[\beta^t u(C_{i,t})\right].$$

Agents receive a wage W from the amount they work $L_{i,t}$, which varies over time and across agents, and some real return r on assets $A_{i,t}$. Agents spend their money on consumption $C_{i,t}$ and assets for the next period. Note that there are no aggregate shocks, which is why W and r have no time subscripts. Their budget constraint is:

$$C_{i,t} + A_{i,t+1} = (1+r)A_{i,t} + WL_{i,t}.$$

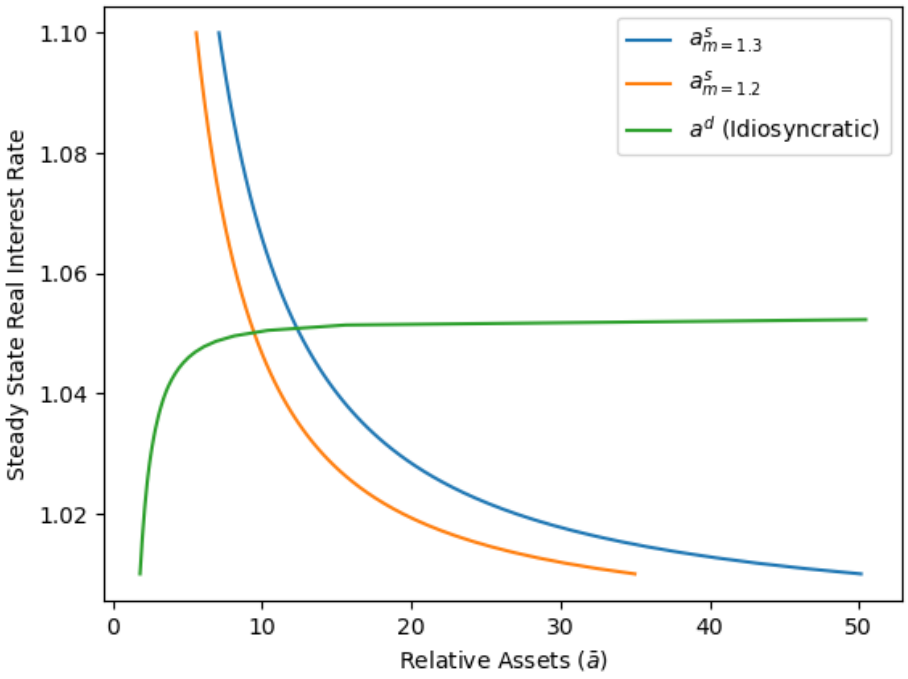
A key additional feature is that agents face some borrowing constraint, which is set to be 0, and this limits the amount they may borrow each period:

$$A_{i,t+1} \geq 0.$$

This problem can be solved by value function iteration. Again, to easily separate out the supply and the demand for assets, the problem needs to be rewritten in terms of relative assets (equation (11)):

$$\max \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u(c_{i,t})],$$

Figure 4: Equilibrium under a Fall in the Markup: 3. Idiosyncratic Labor



s.t.

$$a_{i,t+1} = (1+r)a_t + \frac{L_{i,t}}{L} - c_{i,t},$$
$$a_{i,t+1} \geq 0.$$

The asset demand $\bar{a}^d$ can then be computed for any equilibrium real rate $\bar{r}$.

$\bar{a}^d$ is plotted in figure 4 where a fall in the markup is considered (as in figure 1). A shift left in the asset supply due to a fall in the markup leads to a fall in relative assets and a fall in the equilibrium real rate.

Again, introducing this form of heterogeneity yields results that are very different from those of the representative case. This is because a fall in assets implies that there are fewer assets in which agents can save to avoid hitting their borrowing constraint in subsequent periods. Therefore, agents are keener to save. They thus pay a higher price for savings, which is equivalent to receiving a lower return on their savings and a lower real interest rate.

The degree to which a shift left in assets lowers the equilibrium real rate depends on how many agents are close to the borrowing constraints. When the level of assets is high (low), a fall in assets implies it is a little (lot) more difficult to save for the future, so it raises the price of savings a little (lot) and thus lowers the equilibrium real rate a little (lot). This can be seen in figure 4, which shows that the relative asset demand is steeper when relative assets are low.

3 Model

I now introduce the main model, which is used to assess the degree to which a change in trend inflation affects the real interest rate and to conduct welfare analysis.

3.1 Households

I start by describing the general overlapping generations framework. Each agent lives for M periods. Agents born in different periods overlap. An agent is denoted by their age in periods, so an agent born i periods ago is denoted as i. Therefore, the M cohorts in any given period are denoted as $0, \ldots, M-1$. Each period, new agents are born (cohort 0), the oldest agents from the previous period (cohort $M-1$ at time $t-1$) have died, and all other generations mature from cohort i to $i+1$.

The population of the cohort born at time t is defined as N_t. The total population is defined as $\mathbb{N}_t$, and thus $\mathbb{N}_t = \sum_{i=0}^{M-1} N_{t-i}$. It is assumed that the population grows at a constant rate of n so that $N_{t+1} = (1+n)N_t$, and thus $\mathbb{N}_{t+1} = (1+n)\mathbb{N}_t$.

An agent of cohort i at time t has a budget constraint given by equation (20). The consumption and labor of agent i at time t are denoted by $C_{i,t}$ and $L_{i,t}$, respectively. An agent of cohort i works for $L_{i,t}$. W_t is the real wage paid at time t for each unit of work. An agent can invest in bonds, capital, or shares in firms. $B_{i,t}, K_{i,t}$ are, respectively, the bonds and capital held by agents of cohort i at the start of period t (so they were chosen at $t-1$ when that agent was of cohort $i-1$). Bonds are in nominal terms, and a bond issued at $t-1$ that pays out at t ($B_{i,t}$) pays interest rate i_{t-1} at time t. Capital is in real terms, and agents get a real return of r_t from selling their capital to the firm at time t. $\tilde{\omega}_{i,t}$ is the number of shares of the composite firm that agent i owns at the start of time t. The total number of shares issued is 1, so $\tilde{\omega}_{i,t}$ also represents the proportion of the composite firm owned by an agent of cohort i at time t. The price of a share is $\tilde{Z}_t$, and a share pays out a proportional amount of the firm's total profits $\tilde{\Omega}_t$ each period. It is assumed that the agent starts with zero assets, so $K_{i,0} = B_{i,0} = \tilde{\omega_{i,0}} = 0$. For ease of notation, the gross return of capital and the gross nominal interest rate are also defined as $R_t = 1 + r_t$ and $I_t = 1 + i_t$, respectively.

$$C_{i,t} + \frac{B_{i+1,t+1}}{P_t} + K_{i+1,t+1} + \tilde{Z}_t\tilde{\omega}_{i+1,t+1} \leq W_t L_{i,t} + I_{t-1}\frac{B_{i,t}}{P_t} + R_t K_{i,t} + (\tilde{\Omega}_t + \tilde{Z}_t)\tilde{\omega}_{i,t}. \tag{20}$$

The agent's lifetime utility function when they are in cohort k is given by equation (21). CRRA utility is used (equation (22)). Both endogenous and exogenous labor are allowed for. In the exogenous labor case, the labor supply is fixed by each cohort so that $L_{i,t} = L_i \forall t$ and the disutility of labor term $v(L_{i,t})$ does not appear in the utility function. In the endogenous labor case, the disutility of labor is given by equation (23), where η is the elasticity of labor supply.

$$\mathbb{E}_t[\sum_{i=k}^{M-1} \beta^{i-k}[u(C_{i,t}) - v_i(L_{i,t})]], \tag{21}$$

$$u(C_{i,t}) = \frac{C_{i,t}^{1-\gamma}}{1-\gamma}, \tag{22}$$

$$v_i(L_{i,t}) = \frac{1}{1+\eta} x_i L_{i,t}^{1+\eta}, \tag{23}$$

Therefore, an agent of age k faces the following problem:

$$\max_{\{C_{i,t+i}, B_{i+1,t+i+1}, K_{i+1,t+i+i}, \tilde{\omega}_{i+1,t+i+1}\}_{i=k}^{M-1}} \mathbb{E}_t[\sum_{i=k}^{M-1} \beta^{i-k}[u(C_{i,t}) - v(L_{i,t})]],$$

s.t. $\forall i \in k, \ldots, M-1$:

$$C_{i,t+i} + \frac{B_{i+1,t+i+1}}{P_{t+i}} + K_{i+1,t+i+1} + \tilde{Z}_{t+i}\tilde{\omega}_{i+1,t+i+1} \leq W_{t+i}L_{i,t+i} + I_{t-1}\frac{B_{i,t+i}}{P_{t+i}} + R_{t+i}K_{i,t+i} + (\tilde{\Omega}_{t+i} + \zeta\tilde{Z}_{t+i})\tilde{\omega}_{i,t+i},$$

$$B_{M,t+M-k}, K_{M,t+M-k}, \omega_{M,t+M-k} \geq 0.$$

First-order conditions are applied. This yields arbitrage conditions on bonds (equation (24)), capital (equation (25)), and shares (equation (26)). The inclusion of ξ in the condition on bonds (equation (24)) is a non-structural method to capture the additional benefits of holding government debt, such as increased liquidity and safety, which imply it offers a lower return. This follows the example of Krishnamurthy and Vissing-Jorgensen (2012); Fisher (2015); Mehrotra (2019); Mehrotra and Sergeyev (2019). When $\xi > 0$, agents get additional utility from holding safe bonds, and the return on safe bonds is thus lower than the return on risky assets and shares. When $\xi = 0$, the Euler condition is standard and there are no additional benefits of holding government debt. Inflation is defined in the usual way ($1 + \pi_t = \Pi_{t+1} = \frac{P_{t+1}}{P_t}$). $\forall i \in 0, \ldots, M-2$:

$$u'(C_{i,t}) = \beta\mathbb{E}_t[u'(C_{i+1,t+1})\frac{I_t}{\Pi_{t+1}}(1+\xi)], \tag{24}$$

$$u'(C_{i,t}) = \beta\mathbb{E}_t[R_{t+1}u'(C_{i+1,t+1})], \tag{25}$$

$$\tilde{Z}_t u'(C_{i,t}) = \beta\mathbb{E}_t[u'(C_{i+1,t+1})(\tilde{\Omega}_{t+1} + \tilde{Z}_{t+1})]. \tag{26}$$

With endogenous labor, there is also a labor-leisure condition $\forall i \in 0, \ldots, M-1$ given by equation (27).

$$W_t u'(C_{i,t}) = v'(L_{i,t}). \tag{27}$$

To make the model tractable, aggregate variables are written in per capita terms. Define $\omega_{i,t} = \mathbb{N}_t\tilde{\omega}_{i,t}$, so that $\omega_{i,t}$ represents the proportional per capita holdings of an agent of cohort i at time t of firm shares. Then define Z_t as the price of a per capita share in firms, that is $Z_t = \frac{\tilde{Z}_t}{\mathbb{N}_t}$, and Ω_t to be the profits paid by a per capita share in firms, that is $\Omega_t = \frac{\tilde{\Omega}_t}{\mathbb{N}_t}$. Equations (20) and (26) become, respectively,

$$Z_t u'(C_{i,t}) = \beta\mathbb{E}_t[u'(C_{i+1,t+1})(1+n)(\Omega_{t+1} + Z_{t+1})], \tag{28}$$

$$C_{i,t} + \frac{B_{i+1,t+1}}{P_t} + K_{i+1,t+1} + \frac{Z_t\omega_{i+1,t+1}}{1+n} \leq W_t L_{i,t} + I_{t-1}\frac{B_{i,t}}{P_t} + R_t K_{i,t} + (\Omega_t + Z_t)\omega_{i,t}. \tag{29}$$

Next, the budget constraint is rewritten more simply. To do this, first define the amount that agents of cohort

i have available at the start of t from savings they made in $t-1$ as $T_{i,t}$ (equation (30)) and the amount that agents of cohort i save at t for $t+1$ as $S_{i+1,t+1}$ (equation (31)). This implies that equation (29) can be rewritten as equation (32).

$$T_{i,t} = I_{t-1}\frac{B_{i,t}}{P_t} + R_t K_{i,t} + (\Omega_t + Z_t)\omega_{i,t}, \tag{30}$$

$$S_{i+1,t+1} = \frac{B_{i+1,t+1}}{P_t} + K_{i+1,t+1} + \frac{Z_t \omega_{i+1,t+1}}{1+n}, \tag{31}$$

$$C_{i,t} + S_{i+1,t+1} \leq W_t L_{i,t} + T_{i,t}. \tag{32}$$

Aggregated labor (L_t) is defined by equation (33). Aggregated capital (K_t), bonds (B_t), and savings held at t (T_t) are defined similarly. Aggregated savings made at t (S_{t+1}) are defined in equation (34). Equations (30) and (31) can then be aggregated to yield equations (35) and (36). Details on these aggregation steps are given in appendix A.1.

$$L_t = \frac{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i L_{i,t}}{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i}, \tag{33}$$

$$S_{t+1} = \frac{\sum_{i=0}^{M-1}\frac{1}{(1+n)^i} S_{i+1,t+1}}{\sum_{i=0}^{M-1}\frac{1}{(1+n)^i}}, \tag{34}$$

$$T_t = I_{t-1}\frac{B_t}{P_t} + R_t K_t + \Omega_t + Z_t, \tag{35}$$

$$S_{t+1} = (1+n)\frac{B_{t+1}}{P_t} + (1+n)K_{t+1} + Z_t. \tag{36}$$

The share of savings made by an agent of cohort i at t is defined as $s_{i+1,t+1} = \frac{S_{i+1,t+1}}{S_{t+1}}$. Equation (34) can be rewritten as equation (37), which demonstrates that the shares of savings must sum to 1 across cohorts. To a first-order approximation, agents of different cohorts care only about their total savings each period but are indifferent regarding whether these savings are held in capital, shares, or bonds. Here it is specified that $T_{i,t} = s_{i,t}T_t$. This implies that a cohort's share of savings held at the start of period t equals their share of the savings made at the end of period $t-1$, suggesting that equation (32) can be rewritten as equation (38).

$$1 = \frac{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i L_{i,t}}{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i}, \tag{37}$$

$$C_{i,t} + s_{i+1,t+1}S_{t+1} = W_t L_{i,t} + s_{i,t}T_t. \tag{38}$$

3.2 Firms

The model has a standard firm setup with a competitive goods firm, aggregating monopolistic intermediate firms with Calvo pricing.

Final Goods Firm There is a single competitive final goods firm that aggregates intermediate goods in different industries to produce a final good through constant elasticity of substitution (CES) production (equa-

tion (39)).

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \tag{39}$$

Cost minimization implies that final goods firm has the usual CES demand for intermediate goods given by equation (40). Combining equations (39) and (40) allows the computation of the CES price aggregator (equation (41)).

$$Y_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma}, \tag{40}$$

$$P_t = (\int_0^1 P_{i,t}^{1-\sigma})^{\frac{1}{1-\sigma}}. \tag{41}$$

Intermediate Goods Firms Cost Minimization Firms minimize their costs (equation (42)) subject to their production function (equation (43)). A_t represents productivity, $K_{i,t}$ represents firm i's capital, and $L_{i,t}$ represents firm i's labor.

$$\min_{K_{i,t}, L_{i,t}} (R_t - 1 + \delta) K_{i,t} + W_t L_{i,t}, \tag{42}$$

s.t.

$$Y_{i,t} = A_t K_{i,t}^{\alpha} L_{i,t}^{1-\alpha}. \tag{43}$$

The marginal cost of producing one additional unit of output is defined as MC_t. Cost minimization (and noting that $\frac{K_{i,t}}{L_{i,t}}$ is identical across firms) implies equations (44) and (45).

$$MC_t = \frac{r_t + \delta}{\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}}, \tag{44}$$

$$MC_t = \frac{W_t}{(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}}. \tag{45}$$

Noting that $\frac{K_{i,t}}{L_{i,t}} = \frac{K_t}{L_t}$, inputting equation (40) into equation (43) yields an equation for aggregate profits (equation (46)) that includes a term for price dispersion (equation (47)).

$$\nu_t Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}, \tag{46}$$

$$\nu_t = \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} di. \tag{47}$$

We can rewrite equations (44) and (45) using equation (46) to get equations (48) and (49).

$$\alpha MC_t \nu_t = \frac{(r_t + \delta) K_t}{Y_t}, \tag{48}$$

$$(1-\alpha) MC_t \nu_t = \frac{W_t L_t}{Y_t}. \tag{49}$$

Being able to vary the profitability of firms is useful for calibrating the model. A tax (subsidy) on firm production is introduced to enable such variation. MC_t represents the marginal cost of producing one good

before tax. However, intermediate firms are taxed on their marginal production and pay $(1+\tau)MC_t$ instead, where τ is the rate of tax. To avoid potential issues with how the receipts (costs) of the tax (subsidy) are distributed, it is assumed that the receipts (costs) are paid back to (taken from) the firm as a lump sum. Profits of an individual firm are then denoted by equation (50). Equation (50) can be aggregated to yield equation (51). Since the taxes are paid back to the firm, they play no direct part in the aggregated profit equation. There is, however, an indirect impact through the before-tax marginal cost. By the properties of CES demand, intermediate firms always set approximately the same markup over their after-tax marginal costs ($\approx \frac{\sigma}{\sigma-1}$). Thus, even if firms pay higher taxes, their after-tax marginal costs remain the same, which implies that higher taxes must lower intermediate firms' before-tax marginal costs (MC_t) and thus raise profits.

$$\Omega_{i,t} = \frac{P_{i,t}}{P_t} Y_{i,t} - (1+\tau) MC_t Y_{i,t} + \tau MC_t Y_t \nu_t, \tag{50}$$

$$\Omega_t = Y_t (1 - MC_t \nu_t). \tag{51}$$

Intermediate Goods Firm Profit Maximization Each intermediate goods firm is monopolistic. A firm can change its price only with probability λ. It faces CES demand from the final goods firm. A firm discounts a period j periods in the future by $\bar{R}^{-j}$. Their profits are thus given by equation (52) (where terms that are not functions of the firm's chosen price are ignored).

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \bar{R}^{-j} (1-\lambda)^j \left[\frac{P_t^\star Y_{i,t+j}}{P_{t+j}} - (1+\tau) MC_{t+j} Y_{i,t+j} \right] \right]. \tag{52}$$

Equation (52) is maximized subject to equation (40) to yield equation (53):

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} (1-\lambda)^j \bar{R}^{-j} Y_{t+j} \left(\frac{P_{t+j}}{P_t} \right)^\sigma \left[\frac{P_t^\star}{P_t} \frac{P_t}{P_{t+j}} - \frac{\sigma}{\sigma-1} (1+\tau) MC_{t+j} \right] \right] = 0. \tag{53}$$

Equation (53) is rewritten as equation (54), where two additional recursive variables (U_t, V_t) are defined through equations (55) and (56).

$$U_t \frac{P_t^\star}{P_t} = V_t, \tag{54}$$

$$U_t = Y_t + \mathbb{E}_t [\Pi_{t+1}^{\sigma-1} (1-\lambda) \frac{1}{\bar{R}} U_{t+1}], \tag{55}$$

$$V_t = Y_t \frac{\sigma}{\sigma-1} (1+\tau) MC_t + \mathbb{E}_t [\Pi_{t+1}^{\sigma} (1-\lambda) \frac{1}{\bar{R}} V_{t+1}]. \tag{56}$$

Equations (41) and (47) are recursively rewritten as equations (57) and (58), respectively.

$$1 = \lambda (\frac{P_t^\star}{P_t})^{1-\sigma} + (1-\lambda) \Pi_t^{\sigma-1}, \tag{57}$$

$$\nu_t = (1-\lambda) \nu_{t-1} \Pi_t^\sigma + \lambda \left(\frac{P_t^\star}{P_t} \right)^{-\sigma}. \tag{58}$$

3.3 Monetary and Fiscal Policy

In the steady state, the only component of monetary policy that matters is the inflation target $\pi^\star$. Once shocks are introduced, as they are in section 6, it becomes necessary to specify a monetary policy rule. I consider a simple case in which central banks react to inflation only with coefficient ϕ_π. Following Coibion et al. (2012) and Andrade et al. (2019), I incorporate inertia. Thus, ρ_i of the interest rate set in the current period follows what was set in the preceding period, while $1-\rho_i$ is a response to inflation in the current period. The ideal policy rule is given by equation (59). However, the central bank is sometimes constrained by the zero lower bound (equation (60)).

$$I_t^\star = {I_{t-1}^\star}^{\rho_i}\left(\bar{\Pi}\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\right)^{1-\rho_i}, \tag{59}$$

$$I_t = \max\{I_t^\star, 1\}. \tag{60}$$

The government is assumed to have no debt or savings, so $B_t = 0$ in equilibrium.

4 Model Solution and Calibration

This section describes the solution of the model.

4.1 Full Conditions

This subsection first summarizes the DSE model.

The household's problem is characterized by $2M+5$ conditions: $M-1$ Euler condition(s) (equation (25)), two arbitrage conditions (equations (24) and (28)), the sum of labor (equation (33)), the quantities of savings and assets (equations (35) and (36)), the sum of savings shares (equation (37)), and M simplified budget constraints (equation (38)). The firm's cost minimization problem is summarized by four conditions: the two cost minimization conditions (equations (48) and (49)), the definition of output (equation (46)), and the definition of profits (equation (51)). The firm's profit maximization problem is summarized by five conditions: equations (54) to (58). There are two monetary policy conditions (equations (59) and (60)). In total, there are $2M+16$ conditions. These correspond to the following variables:

$$\{C_{i,t}\}_{i=0}^{M-1}, \{s_{i,t}\}_{i=1}^{M-1}, I_t, \Pi_t, R_t, W_t, MC_t, K_t, L_t, Y_t, Z_t, \Omega_t, S_t, T_t, \nu_t, \frac{P_t^\star}{P_t}, U_t, V_t, I_t^\star$$

4.2 Steady State

The steady state is computed here in a manner similar to how it is computed in section 2. The demand and the supply of relative assets are computed as a function of the real interest rate. Relative assets are defined in equation (61). Equilibrium is the point where the demand and the supply of relative assets intersect.

$$a_t = \frac{S_{t+1}}{W_t L_t}. \tag{61}$$

The supply of relative assets comes from the firm side because assets are made up of shares and capital, and the firm side determines the value of the shares and the amount of capital that needs to be supplied. Note that although demand for capital is commonly given as coming from firms, in this context the firm is actually supplying the opportunity for the household to own capital as an asset. As the real interest rate rises, the value of firms falls and firms rent less capital from households, so the supply of assets falls. The derivation of the supply of relative assets can be broken into two parts. First, the steady state marginal cost can be computed from firms' profit maximization conditions — the derivation is given in appendix B.1. Second, the supply of assets can be computed from firms' cost minimization conditions taking as given the steady state marginal cost — the derivation is given in appendix B.2.

The demand for relative assets comes from the household side, since households wish to hold assets to save for the future. As the real interest rate rises, whether the demand for assets increases depends on whether a substitution effect or income effect dominates. The substitution effect is that agents gain more from saving and therefore want to save more when the real interest rate rises. The income effect is that when the real interest rate rises, the old become richer and thus the young want to save less. The derivation of the demand for relative assets is given in appendix B.3.

4.3 Shocks and Log-linearized Conditions

In this subsection, shocks are incorporated into the log-linearized versions of the full set of model conditions found in section 4.1.

Shocks similar to those in Coibion et al. (2012) are used. The shocks are to technology, the risk premium, and the Phillips Curve, and they are denoted as $\epsilon_{a,t}$, $\epsilon_{q,t}$, and $\epsilon_{m,t}$, respectively, with standard deviations of the shocks given as σ_a, σ_q, and σ_m, respectively. These follow AR(1) processes:

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t},$$

$$\hat{q}_t = \rho_q \hat{q}_{t-1} + \epsilon_{q,t},$$

$$\hat{m}_t = \rho_m \hat{m}_{t-1} + \epsilon_{m,t}.$$

Denote the log-linearization of X around the steady state as $\hat{X}$:

$$\hat{X}_t = \log(X_t) - \log(\bar{X}).$$

Household conditions (equations (24), (25), (28), (33) and (35) to (38)) are log-linearized to yield:

$$\mathbb{E}_t[\hat{C}_{i+1,t+1}] = \frac{1}{\gamma}(\mathbb{E}_t[\hat{R}_{t+1}] + \hat{q}_t) + \hat{C}_{i,t},$$

$$\mathbb{E}_t[\hat{R}_{t+1}] = \hat{I}_t - \mathbb{E}_t[\hat{\Pi}_{t+1}],$$

$$\hat{Z}_t + \mathbb{E}_t[\hat{R}_{t+1}] = \frac{\bar{\Omega}}{\bar{\Omega} + \bar{Z}}\mathbb{E}_t[\hat{\Omega}_{t+1}] + \frac{\bar{Z}}{\bar{\Omega} + \bar{Z}}\mathbb{E}_t[\hat{Z}_{t+1}],$$

$$\hat{L}_t = \frac{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i \hat{L}_{i,t}}{\sum_{i=0}^{M-1}(\frac{1}{1+n})^i},$$

$$\bar{T}\hat{T}_t = \bar{R}\frac{\bar{B}}{\bar{P}}\widehat{\left(\frac{B_t}{P_{t-1}}\right)} + \bar{R}\bar{K}(\hat{R}_t + \hat{K}_t) + \bar{\Omega}\hat{\Omega}_t + \bar{Z}\hat{Z}_t,$$

$$\bar{S}\hat{S}_{t+1} = \frac{\bar{B}}{\bar{P}}\widehat{\left(\frac{B_{t+1}}{P_t}\right)} + \bar{K}\hat{K}_{t+1} + \bar{Z}\hat{Z}_t,$$

$$0 = \frac{\sum_{i=0}^{M-1}\frac{1}{(1+n)^i}\bar{s}_i\hat{s}_{i,t}}{\sum_{i=0}^{M-1}\frac{1}{(1+n)^i}},$$

$$\bar{C}_i\hat{C}_{i,t} + \bar{s}_{i+1}\bar{S}[\hat{s}_{i+1,t+1} + \hat{S}_{t+1}] = \bar{W}\bar{L}_i\hat{W}_t + \bar{s}_i\bar{T}[\hat{s}_{i,t} + \hat{T}_t].$$

Firm cost minimization conditions (equations (46), (48), (49) and (51)) are log-linearized to yield:

$$\hat{\nu}_t + \hat{MC}_t = \frac{\bar{R}}{\bar{R} - 1 + \delta}\hat{R}_t + \hat{K}_t - \hat{Y}_t,$$

$$\hat{\nu}_t + \hat{MC}_t = \hat{W}_t + \hat{L}_t - \hat{Y}_t,$$

$$\hat{Y}_t + \hat{\nu}_t = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{L}_t,$$

$$\hat{\Omega}_t = \hat{Y}_t - \frac{\bar{MC}\bar{\nu}}{1 - \bar{MC}\bar{\nu}}(\hat{MC}_t + \hat{\nu}_t).$$

Firm profit maximization conditions (equations (54) to (58)) are log-linearized to yield:

$$\hat{U}_t + \widehat{\frac{P_t^\star}{P_t}} = \hat{V}_t,$$

$$\bar{U}\hat{U}_t = \bar{Y}\hat{Y}_t + \bar{\Pi}^{\sigma-1}(1-\lambda)\frac{1}{\bar{R}}\bar{U}\mathbb{E}_t[(\sigma-1)\hat{\Pi}_{t+1} + \hat{U}_{t+1}],$$

$$\bar{V}\hat{V}_t = \bar{Y}\frac{\sigma}{\sigma-1}(1+\tau)\overline{MC}(\hat{Y}_t + \widehat{MC}_t + \hat{m}_t) + \bar{\Pi}^{\sigma}(1-\lambda)\frac{1}{\bar{R}}\bar{V}\mathbb{E}_t[\sigma\hat{\Pi}_{t+1} + \hat{V}_{t+1}],$$

$$\lambda\overline{\left(\frac{P^\star}{P}\right)^{1-\sigma}}\widehat{\frac{P_t^\star}{P_t}} = (1-\lambda)\bar{\Pi}^{\sigma-1}\hat{\Pi}_t,$$

$$\bar{\nu}\hat{\nu}_t = (1-\lambda)\bar{\nu}\bar{\Pi}^{\sigma}(\hat{\nu}_{t-1} + \sigma\hat{\Pi}_t) - \sigma\lambda\overline{\left(\frac{P^\star}{P}\right)^{-\sigma}}\widehat{\frac{P_t^\star}{P_t}}.$$

The monetary rule (equation (59)) is log-linearized to yield:

$$\hat{I}_t^\star = \rho_i\hat{I}_{t-1}^\star + (1-\rho_i)\phi_\pi\hat{\Pi}_t,$$

$$\hat{I}_t = \max\{\hat{I}_t^\star, -\log(\bar{I})\}.$$

4.4 Simulation Method

This subsection explains how to simulate the model taking into account the zero lower bound (ZLB).

First, note that a standard dynamic stochastic general equilibrium (DSGE) model has been derived. Therefore, a first-order linear perturbation of the conditions given in section 4.3 can be applied. However, any first-order perturbation will ignore the ZLB in section 4.3.

To capture the impact of the ZLB, a method similar to that of Guerrieri and Iacoviello (2015) is used. The basic idea is that agents' choices at time t are solved for by finding what they would do without shocks in the future if there were no ZLB. If the central bank would set a nominal interest rate below zero in this case, then the estimation is rerun with the central bank constrained in certain periods to be at the ZLB. This process continues until a situation in which the central bank is not setting a nominal interest rate below zero is found. The exact algorithm used is as follows:

1. Set t to be the first period of the simulation and i to be 0.
2. Guess the following regime: The ZLB binds for any period t to $t + i - 1$. The ZLB does not bind for any period $t + i$ to $t + 19$. Solve backwards from $t + 19$ through t to get the policy function for each period under the guessed regime.[3]
3. Using the policy functions computed in step 2, solve out for the path of the economy from t to $t + 19$.
4. If the guess was incorrect in the specific sense that the ZLB does not bind from $t + i$ to $t + 19$,
   (a) Raise i by 1 so that the regime allows for one additional period at the ZLB.
   (b) Go back to step 3.
5. Take agents' choices at t to be the simulation values for t. Now set t to be the next period of the simulation, set $i = 0$, and go back to step 2.

The difference between my method and that of Guerrieri and Iacoviello (2015) is how I describe the central bank's course of action if the ZLB is expected to bind. In my method, if a central bank knows that it would hit the ZLB in the future without additional shocks, it will lower its nominal interest rate to zero in all preceding periods.[4] Guerrieri and Iacoviello's algorithm implies that the ZLB should bind only in periods when the nominal interest rate would have been below zero according to the monetary rule. This means the ZLB can bind in nonconsecutive periods and in the future even if it doesn't bind today.

Guerrieri and Iacoviello mention that their method is a general first–pass method, and I find there are advantages to using the method outlined here in this specific case. First, I think it is realistic to assume that if a central bank knows that the economy is likely to hit the ZLB in the future, it will lower nominal interest rates to zero immediately rather than waiting. Second, my method works significantly more quickly, which is necessary since I am working with a relatively large model due to the life-cycle framework.

4.5 Calibration

The model is calibrated to match moments from the period 1987 through 2006. This timespan was chosen because it captures a period of relative stability (the Great Moderation) before the Great Recession, and because

[3] The computations used to solve backwards from $t + 19$ to t is the same as in Guerrieri and Iacoviello (2015).

[4] My algorithm could also be considered a simpler version of the one of Andrade et al. (2019), who also adapt the algorithm of Guerrieri and Iacoviello (2015). They allow for the ZLB to start binding at some period $t_1 \geq t$ and then stop binding at some later period $t_2 > t_1$. I effectively set that $t_1 = t$ always.

it allows me to study how optimal inflation changed before and after the recent apparent fall in the real interest rate. The reason these specific years were chosen is because they correspond to Alan Greenspan's tenure as chairman of the Federal Reserve.

Panel A of table 1 presents the values of parameters that determine the steady state and were selected from data or standard values. For households, each period is set to represent a quarter. $M = 220$ is calibrated to capture each quarter of life of an adult between the ages of 24 and 78. The simulation begins at age 24 to avoid having to worry about how to capture college. Agents' last year of life is 78 because life expectancy in the United States is 78 years. The discount factor ($\beta = 0.976 = \frac{1}{1.0246}$) is set to correspond to the value the discount factor would need to take in a representative agent model to yield the average fed funds rate after inflation for the period 1987 through 2006 ($\bar{r} = 2.456$). With exogenous labor supply, $\bar{L}_i$ (hours worked by each age) is set to match the average number of hours worked of a person of that age in the American Time Use Survey for the years 2003 through 2016 (data are available only from 2003 onward, and it is useful to use many years to get smooth estimates). With endogenous labor supply, x_i in the disutility of labor function (equation (23)) is set so that when $\beta\bar{R} = 1$, $\bar{L}_i$ matches the hours worked in the exogenous case. This is explained in appendix C.1. The elasticity of intertemporal substitution ($\frac{1}{\gamma} = 0.5$) and the Frisch elasticity of labor supply ($\eta = 3$) are set to be standard values. Population growth and the relative preference for bonds over other assets ($n = 0, \xi = 0$) are not included in the baseline calibration.

For firms, the frequency of price change ($\lambda = 0.24$) is set to correspond to a monthly frequency of price change of 8.7% based on Nakamura and Steinsson (2008). The elasticity of substitution between goods ($\sigma = 8$) is set between the lower and upper bounds of 6 and 10, respectively, used in Carvalho et al. (2016). The rate of depreciation ($\delta = 0.025$) is set to a standard value.

Panel B of table 1 presents the steady state parameters that are calibrated. The coefficient of capital in the Cobb-Douglas production function ($\alpha = 0.34$) is calibrated so that the labor share equals 0.62, which was the average US labor share in the period 1987 through 2006. Note the labor share and capital share do not sum to 1 due to the presence of the profit share, which averaged 6.06% during this period. The tax on marginal production is calibrated to fix the real interest rate under 2% inflation to equal 2.456% in the steady state. τ is varied across each model (including extensions) to obtain the same real interest rate under 2% inflation. Taxing (subsidizing) marginal production implies that firms' after-tax marginal cost is higher (lower), but firms always set the same markup on after-tax marginal costs, so a tax (subsidy) on marginal production implies lower (higher) before-tax marginal costs and thus higher (lower) profits. Higher (lower) profits imply a higher (lower) supply of assets and thus a higher (lower) real interest rate. In the baseline case, $\tau = -0.108$, implying marginal production is actually subsidized.

Panel C of table 1 presents the parameters affecting only the simulation (but not the steady state) that are chosen. The response of interest rates to deviations from the inflation target ($\phi_\pi = 2.5$) is set to be the same as it is in Coibion et al. (2012). The inertia in the monetary policy rule ($\rho_i = 0.66$) is set to be the same as it is in Andrade et al. (2019).

Panel D of table 1 presents the parameter values for parameters that affect only the simulation and are estimated through Bayesian methods. The size and type of the shocks in the simulation affect how much it makes sense to raise the inflation target in response to the ZLB. Moreover, on the household side, this model is quite different from standard New Keynesian models. Thus, for each shock, the size and persistence of the shock are estimated through Bayesian methods. They are estimated over the 1987–2006 period using US

Table 1: Calibration Summary

Panel A: Steady State Parameters: Chosen

Description	Parameter	Value	Source
Number of cohorts	M	220	Life expectancy USA
Discount factor	β	0.976	rep. agent $\bar{r} = 2.456$
Relative labor supply	L_i/x_i	-	ATUS
Elasticity of income substitution	$1/\gamma$	0.5	
Frisch elasticity labor supply	η	3	
Frequency price change	λ	0.24	Nakamura and Steinsson (2008)
Elasticity subs. between goods	σ	8	Carvalho et al. (2016)
Depreciation	δ	0.025	

Panel B: Steady State Parameters: Calibrated

Description	Parameter	Value	Target
Cobb-Douglas capital coeff.	α	0.34	Labor share
Tax on marginal production	τ	-0.108	$\bar{r} = 2.46$

Panel C: Simulation Parameters: Chosen

Description	Parameter	Value	Source
Policy rule response inflation	ϕ_π	2.5	Coibion et al. (2012)
Policy rule persistence	ρ_i	0.66	Andrade et al. (2019)

Panel D: Simulation Parameters: Bayesian Estimation

Description	Parameter	Distribution	Prior mean	Prior s.d.	Post. mean	Post. s.d.
Prod. shock s.d.	σ_a	Inv. Gamma	0.009	0.01	0.0067	0.0023
Risk prem. shock s.d.	σ_q	Inv. Gamma	0.0024	0.005	0.0034	0.0023
Phillips shock s.d.	σ_m	Inv. Gamma	0.0014	0.005	0.0279	0.0026
Prod. shock pers.	ρ_a	Beta	0.9	0.02	0.8791	0.0268
Risk prem. shock pers.	ρ_q	Beta	0.947	0.02	0.9136	0.0289
Phillips shock pers.	ρ_m	Beta	0.9	0.02	0.9258	0.0129

HP–filtered output and inflation. The estimation uses the standard Bayesian DSGE technique in which the policy functions under given parameters are used to construct log likelihoods through the Kalman filter. The estimation is conducted through a $10,000$-period simulation from which the first $1,000$ periods are discarded. The priors for the shock standard deviations are, where possible, set to be the values of the shocks in Coibion et al. (2012). The results of the Bayesian estimation show that the risk premium and Phillips Curve shocks are of greater magnitude than the priors while the productivity shock is weaker. The persistence of each shock is not very different from its priors.

5 Measuring the IMR Channel

This section measures the size of the IMR channel in the long run and the factors that determine it. Table 2 reports the steady state real interest rate under different trend inflation levels. Table 3 reports how raising trend inflation from 2% to 4% affects a variety of different economic variables. The reason for the focus on raising inflation from 2% to 4% is that this is what many prominent economists have proposed (Blanchard et al. 2010; Ball 2014; Krugman 2014).

Table 2: Real Interest Rate (r %) under Different Trend Inflation Levels

Trend Inflation	0%	1%	2%	3%	4%	5%
A. Baseline	2.501	2.482	2.456	2.423	2.379	2.319
B. Endog. Labor	2.490	2.476	2.456	2.431	2.397	2.351
C. Low IES	2.569	2.519	2.456	2.376	2.275	2.147
D. Menu Costs	2.601	2.513	2.456	2.398	2.339	2.277
E. Menu Costs (Internal)	2.553	2.499	2.456	2.413	2.371	2.327
F. High Firm Discount	2.610	2.544	2.456	2.336	2.166	1.911
G. Multiple Sectors	2.561	2.524	2.456	2.332	2.145	-
H. Low IES + High Firm Discount	2.847	2.677	2.456	2.169	1.786	1.257
I. Low IES + Multiple Sectors	2.721	2.623	2.456	2.185	1.857	-
J. Idio. Shocks 1 Type	2.509	2.486	2.456	2.417	2.364	2.292
K. Idio. Shocks 2 Types	2.525	2.495	2.456	2.405	2.336	2.242

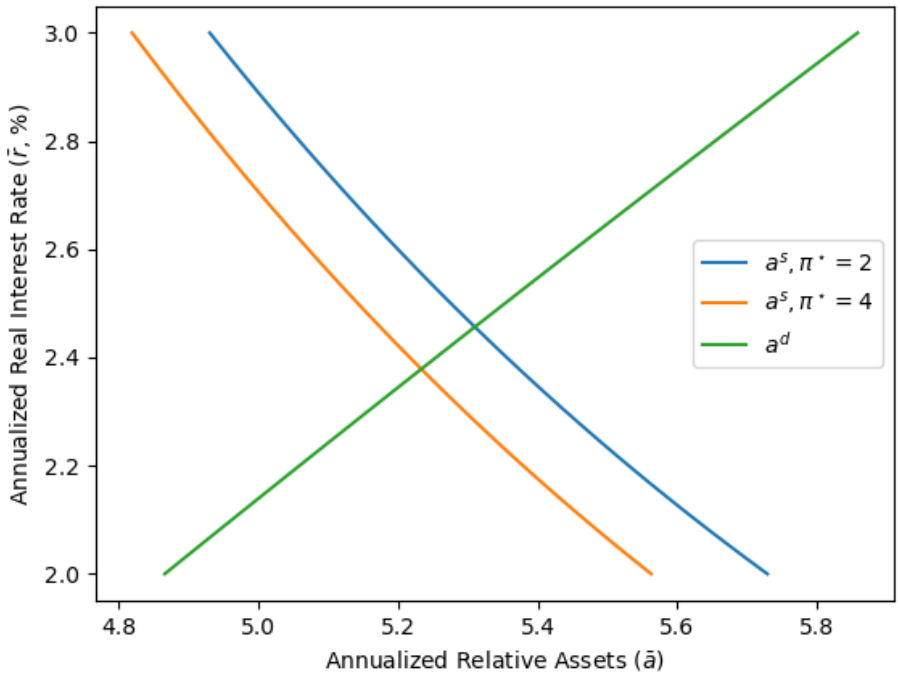
"-" represents that a steady state did not exist, which is a possibility in the multisector Calvo model. The table shows how a rise in trend inflation in percentage terms affects the steady state real interest rate in percentage terms.

Table 3: Impact of Raising Inflation from 2% to 4%

Change in Variables (p.p.)	$\Delta\frac{\bar{\Omega}}{\bar{Y}}$	$\Delta\overline{MC}$	$\Delta\bar{\nu}$	$\Delta\log(\frac{\bar{C}_{old}}{\bar{C}_{young}})$	$\Delta\log\bar{K}$	$\Delta\log(\bar{L})$	$\Delta\log(\bar{Y})$	$\Delta\log(\bar{W})$	$\Delta\bar{r}$
A. Baseline	-0.211	-0.524	0.621	-1.045	-1.045	0.000	-1.876	-1.661	-0.076
B. Endog. Labor	-0.213	-0.523	0.621	-0.797	-0.798	0.465	-1.485	-1.733	-0.058
C. Low IES	-0.197	-0.529	0.621	-0.487	0.186	0.000	-1.457	-1.257	-0.177
D. Menu Costs	-0.313	0.100	-0.004	-1.578	1.790	0.000	0.623	0.857	-0.115
E. Menu Costs (Internal)	-0.231	0.100	-0.004	-1.150	1.405	0.000	0.491	0.727	-0.084
F. High Firm Discount	-0.738	-0.308	0.621	-3.901	2.354	0.000	-0.716	0.034	-0.284
G. Multiple Sectors	-0.787	-1.522	1.901	-4.186	-2.015	0.000	-5.305	-4.506	-0.304
H. Low IES + High Firm Discount	-0.693	-0.326	0.621	-1.805	7.029	0.000	0.874	1.577	-0.656
I. Low IES + Multiple Sectors	-0.624	-1.571	1.885	-1.613	1.378	0.000	-4.112	-3.478	-0.586
J. Idio. Shocks 1 Type	-0.210	-0.525	0.621	-1.124	-0.874	0.000	-1.818	-1.605	-0.090
K. Idio. Shocks 2 Types	-0.209	-0.529	0.621	-1.336	-0.535	0.000	-1.704	-1.494	-0.118

The table analyzes how a rise in trend inflation from 2% to 4% affects different steady state variables. The changes in variables are in percentage points. The variables are the profit share ($\frac{\bar{\Omega}}{\bar{Y}}$), the marginal cost ($\overline{MC}$), the price dispersion ($\bar{\nu}$), a measure of the consumption of the old relative to the young ($\log(\frac{\bar{C}_{old}}{\bar{C}_{young}})$), capital ($\bar{K}$), labor ($\bar{L}$), output ($\bar{Y}$), real wage ($\bar{W}$), real interest rate ($\bar{r}$). $\frac{\bar{C}_{old}}{\bar{C}_{young}}$ represents the fraction of mean consumption of the old half of agents (those agents aged 46.5–78) over the young half of agents (those agents aged 24–46.25).

Figure 5: Baseline Calibration: Asset Supply and Demand

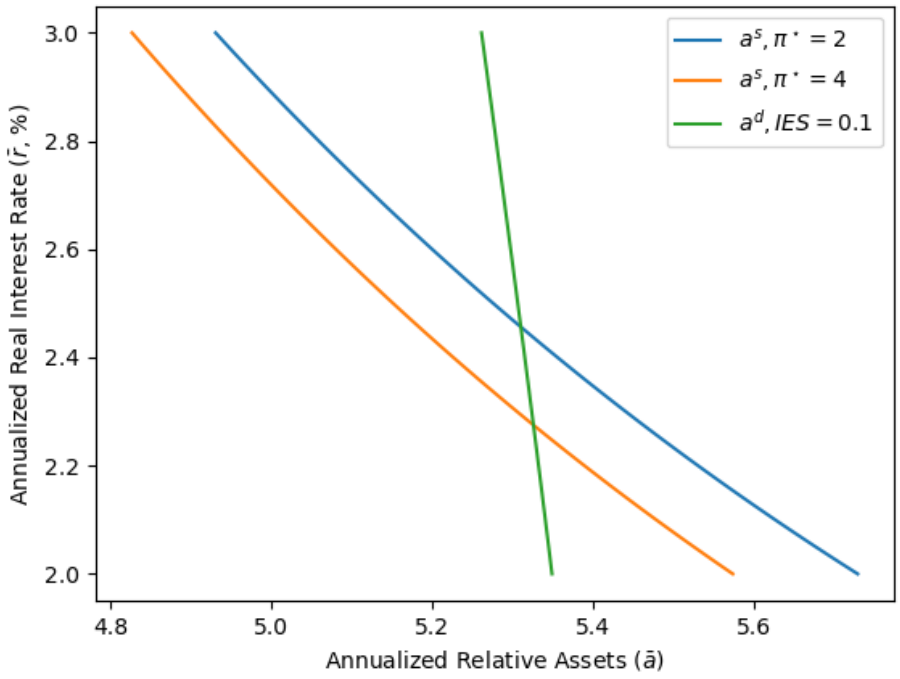


Model A reports results for the baseline model. In table 2, it can be observed that where inflation rises from 2% to 4% , the real interest rate in the steady state falls by 8 basis points. Figure 5 plots this rise in inflation. It can be seen that the dynamics are exactly the same as in the simple model, that is, figure 3. A rise in inflation lowers the profitability of firms, which reduces the value of shares and thus lowers the supply of assets for a given real interest rate. As a result, the supply of assets shifts left for any given real interest rate. Since there are fewer assets in which the young can save, they accept a lower real return in order to save, implying a fall in the real interest rate. Note that the reason the assets supply is downward sloping is because the vertical axis is the real interest rate, which increases when the price of assets falls. Table 2 also indicates that as inflation increases, the real interest rate falls relatively more. This is because as inflation rises, profits fall relatively more.

Table 3 outlines in detail what happens to the economy when trend inflation rises from 2% to 4% . A rise in inflation lowers the profit share 0.21 p.p. The rise in inflation has two impacts: It lowers the marginal cost faced by firms 0.52p.p. (which actually raises profits), but this is dominated by the rise in price dispersion of 0.62p.p., which implies firms with lower profitability are selling relatively more. Consumption of the old half of agents falls by 1.05% relative to the young half's consumption. There are two potential effects on capital. First, the marginal cost at which firms operate encourages lower capital use, but second, the real interest rate also falls, which means capital becomes cheaper for firms. In the baseline case, the first effect dominates, and capital falls 1.05% . There are two effects on output: first, the fall in capital lowers output; second, the rise in price dispersion means effective aggregate output falls. Thus, output falls 1.88p.p. Wages fall, since the fall in capital implies labor is less effective.

Model B introduces endogenous labor. In this case, the real interest rate falls by 6 basis points, which is a smaller decrease than in the baseline case. The reason for this is that when inflation rises and profitability falls,

Figure 6: IES = 0.1: Asset Supply and Demand



meaning there are fewer assets in which to save, the old choose to work relatively more. As a result, labor increases 0.47% . The increase in labor also means that capital is more effective, which is why capital falls less than in the baseline case.

Recent estimates from Best et al. (2018) suggest that the intertemporal elasticity of substitution (IES) is 0.1. Model C lowers the IES from 0.5 to 0.1. Table 2 shows that this strengthens the effect of raising inflation on the real interest rate to 18 basis points. The intuition is that when the supply of savings falls, agents are much more reluctant to lower their consumption when they are old relative to when they are young. Thus, they accept a much lower return on savings so they can transfer consumption to when they are older. This is why in table 3 the fall in consumption is smaller for the old than it is for the young. Capital also rises, since the young save in the form of capital so they can transfer consumption to when they are old. Figure 6 plots the effect of a rise in inflation from 2% to 4% on asset supply and demand under an IES of 0.1 rather than 0.5. The plot demonstrates that the fall in the IES causes the demand for assets to tilt backward. The intuition is that with low IES, the income effect dominates; that is, under a low real interest rate, agents consume less when they are old, so want to transfer more assets to when they are old, even though they get a lower return on these assets.

Models D and E introduce menu costs. The reason for considering menu costs is that one might be skeptical about firms not changing the frequency with which they set prices when inflation is high, as in Calvo. Model D specifies this in the usual way where if firms pay a menu cost, it reduces their profits, which magnifies the feedback from inflation to profits. In model E, I consider an alternative where menu costs are internalized so firms take the menu costs into account when they set prices but the menu costs do not affect profits. Appendix F.1 explains how menu costs are introduced into the model.

Table 2 demonstrates that with menu costs, the effects are actually slightly larger at low levels of inflation than they are with Calvo pricing, even when the menu costs are internalized. When inflation increases from 2% to 4% the real interest rate falls by 12 basis points when menu costs are external and 8 basis points when they are internal. The effects are also significantly stronger under menu costs when inflation changes from 0% to 2% . These results occur because after adjusting their price, some firms will receive idiosyncratic shocks that lower their costs, implying that their markup increases above the markup to which they reset when they pay the menu cost. Since firms face fixed costs of price adjustment, they may choose to wait rather than adjust, implying a high average markup under low inflation. Thus, when inflation rises under low inflation, it means these above-reset markups dissipate more quickly, lowering profits by more than they do under Calvo. The results in table 2 show this effect is particularly strong under low levels of inflation.

Table 3 shows some differences in how other economic variables respond to a rise in inflation from 2% to 4% under menu costs. The marginal cost rises while price dispersion remains unchanged, implying a similar overall change in the profit share. Since the marginal cost rises at the same time as the real interest rate falls, capital actually rises in this case. This in turn implies that output and wages rise.

Model F considers the case where firms discount by the real interest rate that corresponds to their cost of financing rather than the discount factor of the household. This implies that firms care less about the future and thus are more reluctant to raise their price. Consequently, under higher inflation levels, markups fall by more, and the effect on the real interest rate is greater. Appendix F.2 explains how this alternative form of discounting is introduced into the model. From table 2, it can be seen that a rise in inflation from 2% to 4% implies that the real interest rate falls 28 basis points in this case.

As Nakamura and Steinsson (2010) demonstrate, multiple sectors can have a big impact on the degree of monetary non-neutrality. Model G introduces multiple sectors with different frequencies of price change. These are introduced into the main model in appendix F.5, respectively. This feature implies that a rise in inflation lowers the markup by more, which lowers the real interest rate by more. A rise in inflation from 2% to 4% lowers the real interest rate 30 basis points with multiple sectors.

Introducing multiple features can boost the degree to which raising inflation lowers the real interest rate. Model H reports the case with a low IES and where firms discount by the cost of their financing rather than the discount factor of the household. Model I reports results with a low IES and multiple sectors. A rise in inflation from 2% to 4% lowers the real interest rate 66 basis points in model H and 59 basis points in model I.

Models J and K introduce idiosyncratic shocks into the baseline model. There are two reasons for studying this case. First, there has been great focus on this form of heterogeneity in recent years. Second, introducing idiosyncratic shocks allows for a better match of the consumption path of cohorts across their lifetime. Appendix F.6 explains how idiosyncratic shocks are introduced into the main model. The probability that an agent is unemployed at time t is 5% if the agent was employed at $t-1$. The probability that agent is unemployed at time t is 50% if the agent was unemployed at time $t-1$. Unemployment allows the agent to work for only 10% of the hours of employment. Model J considers the case where all agents have identical preferences with a discount factor of 0.97. Model K considers the case with the same unemployment risk as model H, but where there are two types of equally-weighted agents: a patient type with a discount factor of 0.98 and an impatient type with a discount factor of 0.94. Table 2 demonstrates that these changes somewhat strengthen the relationship between inflation and the real interest rate. A rise in the inflation target from 2% to 4% lowers the equilibrium real rate by 9 and 12 points, respectively, in Models J and K.

Figure 7: Baseline Calibration: Consumption Path

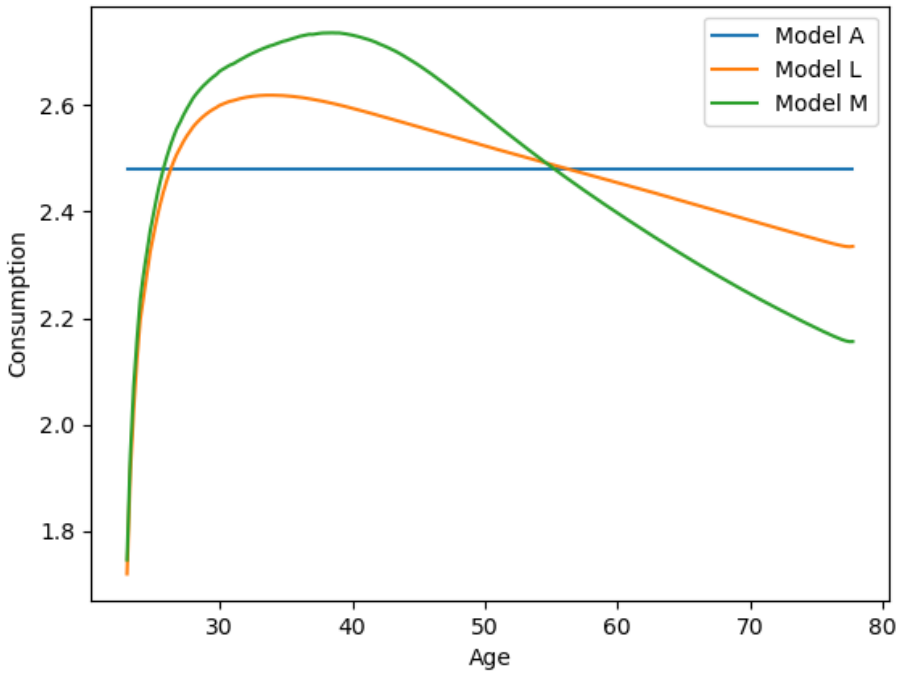


Figure 7 shows the consumption paths of the cohorts across their lives in the baseline model, model J, and model K. The consumption path in the baseline model is effectively a straight line. This comes from the Euler condition, which dictates that the consumption path is determined by $(\beta\bar{R})^{\frac{1}{\gamma}}$ in the steady state. Thus, if $\bar{R} > \frac{1}{\beta}$, the consumption path is always increasing; if $\bar{R} < \frac{1}{\beta}$, the consumption path is always decreasing; and if $\bar{R} = \frac{1}{\beta}$, the consumption path is flat. Introducing idiosyncratic shocks implies that at the start of agents' lives they save in order to insure against possible unemployment, which explains the increase in their consumption as they age when they are young. In model J, agents are assumed to have a lower discount factor, and so the consumption path is decreasing once idiosyncratic shocks are not important, which explains the decrease in their consumption as they age when they are old. Thus, through idiosyncratic shocks and a lower discount factor, it is possible to generate a hump-shaped consumption path. Introducing a second type of impatient agents allows for a greater fall in consumption as agents age, since the impatient agents have saved little for when they are old.

6 Optimal Inflation Target under the IMR Channel

I am currently in the process of rewriting this section. The main point of it is:

- I investigate the interaction between the IMR channel and the zero lower bound (ZLB)
- I show that the IMR channel means that under a given increase in trend inflation, the probability of hitting the ZLB remains higher

- However, the degree to which we should raise the optimal inflation target is largely unaffected by the IMR channel. This is because while the benefit of raising nominal interest rates is higher under the IMR channel, the impact of raising trend inflation on nominal interest rates is diminished, that is, less than one-to-one. These two effects approximately cancel out meaning the optimal inflation target remains similar once the IMR channel is accounted for.

Many economists argue that the recent fall in the real interest rate makes it optimal for central banks to raise their inflation targets to help avoid the zero lower bound (ZLB). This section considers how the IMR channel affects how much it is optimal to raise the inflation target.

The reason many economists have argued the inflation target should be increased is because the equilibrium real rate has fallen for reasons exogenous to the IMR channel, such as, demographics or the global savings glut. To aid clarity, the exogenous real interest rate in an economy is defined as the steady state real interest rate on bonds that the economy would experience under a 0% inflation target. This exogenous real rate is independent of the IMR channel, since it always considers the real interest rate at the same level of inflation. The reason the real interest rate on bonds is considered is because it determines whether the ZLB binds or not. $\pi^\star(r)$ is then defined to represent the optimal inflation target when the exogenous real rate is r. The way the exogenous real rate is varied across a given model is by changing the degree to which agents hold safe bonds through the ξ parameter (by equation (24)). For example, if a low exogenous real rate is desired, it is assumed that agents obtain additional utility from safe bonds (and thus ξ is positive), which implies that the real interest rate on bonds is lower than it is on capital or shares.

The next difficulty involves computing the counterfactual of what the optimal inflation target would be without the IMR channel. $\pi^\star_f(r)$ is defined to represent the optimal inflation target when the real interest rate on bonds is fixed as r under any level of inflation. The IMR channel implies that a rise in inflation lowers the real interest rate on bonds. To fix the real interest rate as r under any level of inflation, a counteracting force is needed. Thus, ξ is adjusted (by equation (24)) to hold constant the real interest on bonds. For example, if inflation rises, this lowers the real interest rate on capital and shares, so ξ needs to fall, which lowers the relative value of safe bonds and therefore raises the real interest rate on bonds to hold it at r.

To compute $\pi^\star(r)$ and $\pi^\star_f(r)$, I employ a grid search method where the grid is refined using estimates of $\hat{\pi}^\star(r)$. The method is explained in appendix D.1. Two models are considered. The first is the basic model with endogenous labor, which corresponds to model B in table 2. The second is an extended model that allows for a stronger IMR channel through a lower intertemporal elasticity of demand ($\gamma = 0.1$) and additional firm discounting (which is explained in appendix F.2). This corresponds to model J in table 2, except with endogenous rather than exogenous labor supply.

The results under different exogenous real rates are displayed in table 4. The second and third columns display the results for the basic model, and the fourth and fifth columns display the results for the extended model. The second and fourth columns represent the case with the IMR channel, that is, $\pi^\star(r)$; while the third and fifth columns represent the case without the IMR channel, that is, $\pi^\star_f(r)$. In both the basic and the extended models, there is a clear monotonic increase in the optimal inflation rate as the exogenous real interest falls. This corresponds to the analysis by Andrade et al. (2019). I focus on the case where the exogenous real rate falls from 2% to -1% . In the basic model, when the exogenous real rate falls by 3p.p. from 2% to -1% , the optimal inflation target rises by 1.996p.p. from 0.738% to 2.734% . Thus, the optimal inflation target rises by 67% of the fall in the exogenous real rate. In the extended model, when the exogenous real rate falls from 2% to -1% , the

Table 4: Optimal Inflation at Different Real Interest Rates

	Basic Model		Extended Model	
r	With IMR ($\pi^\star(r)$)	Without IMR ($\pi_f^\star(r)$)	With IMR ($\pi^\star(r)$)	Without IMR ($\pi_f^\star(r)$)
-3.0	4.280	4.249	3.884	3.849
-2.5	3.892	3.869	3.664	3.633
-2.0	3.503	3.486	3.402	3.372
-1.5	3.116	3.104	3.106	3.078
-1.0	2.734	2.726	2.787	2.759
-0.5	2.361	2.356	2.453	2.425
0.0	2.000	1.997	2.114	2.087
0.5	1.654	1.652	1.778	1.753
1.0	1.327	1.325	1.455	1.433
1.5	1.020	1.019	1.155	1.137
2.0	0.738	0.738	0.887	0.875
2.5	0.484	0.484	0.659	0.655
3.0	0.261	0.263	0.481	0.488

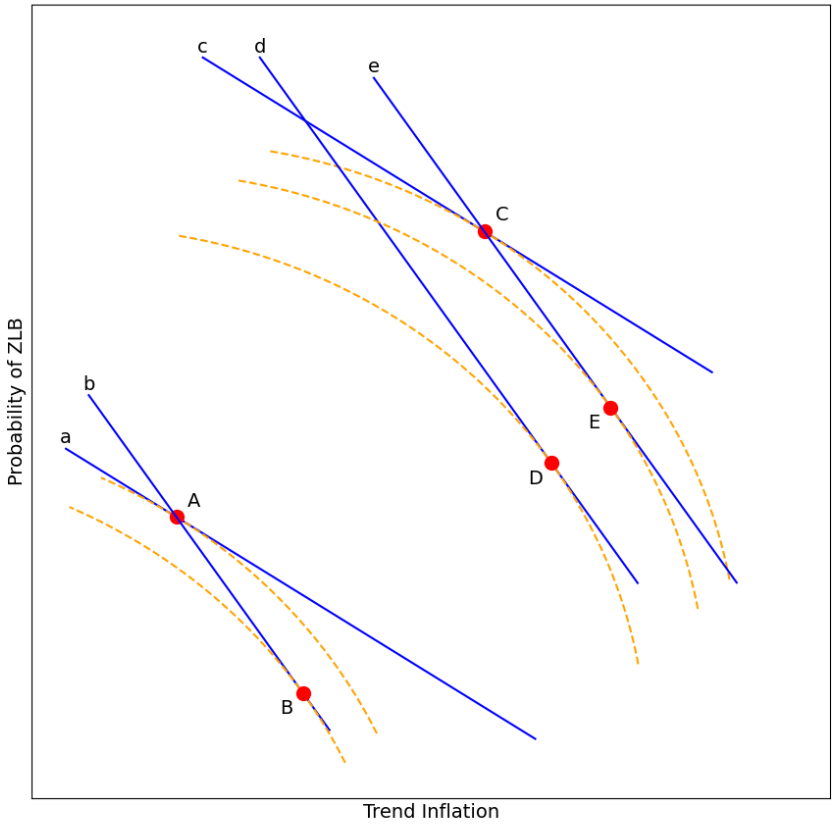
optimal inflation target rises by 1.900p.p. from 0.887% to 2.787% ; that is, 63% of the fall in the exogenous real rate.

A fall in the exogenous real interest rate causes optimal inflation to rise more under the IMR channel but only slightly more. With a high exogenous real rate, table 4 demonstrates that the optimal inflation target is very similar with and without the IMR channel, paritcularly in the basic model. However, as the exogenous real rate falls, the optimal inflation target rises more under the IMR channel, but only slightly more. Focusing on the case where the real interest rate on bonds falls from 2% to -1% : in the basic model, the optimal inflation target rises only 0.8 basis points more with the IMR channel, even though the real interest rate falls an additional 1.8 basis points.[5] This is an increase of only 44% of the additional fall in the real interest rate. In the extended model, the optimal inflation target rises only an additional 0.8 basis points, even though the real interest rate falls an additional 10.6 basis points, which is an increase of 8% of the additional fall in the real interest rate. This is much less than the percentage increase in optimal inflation following a general increase in the exogenous real rate, which are approximately 67% and 63% for the basic and extended models, respectively. Thus, there is a question as to why the optimal inflation target does not rise more due to the IMR channel.

To explore this further, figure 8 graphically decomposes the different effects of a fall in the exogenous real rate with and without the IMR channel. A mathematical decomposition is also provided in appendix D.2. The social planner can pick the inflation target $\pi^\star$, which is displayed on the x axis. This determines the frequency with which the economy hits the ZLB, which is displayed on the y axis. In a given model, there is a constraint (similar to a budget constraint) that dictates the feasible values of inflation and ZLB probabilities. This is downward sloping, since higher inflation implies a lower ZLB probability. Lines (a) and (c) represent the feasible values of inflation and the ZLB probability under a high and low exogenous real rate, respectively. It can be observed that line (a) has a strictly lower probability of hitting the ZLB than does line (c) for a given inflation level since the real interest on bonds is higher and thus the nominal interest rate is higher for any given inflation level. The dashed lines represent indifference curves of the social planner. The shape of the indifference curve reflects the fact that the social planner does not like hitting the ZLB but also does not like high inflation. The optimal inflation level is then the point where the feasible values of inflation and probability of ZLB are tangential to an indifference curve. Thus, when the exogenous real rate falls, taking into account

[5]The computation here is $(\pi^\star(-0.01) - \pi^\star(0.02)) - (\pi_f^\star(-0.01) - \pi_f^\star(0.02))$.

Figure 8: Graphical Decomposition of Impact of IMR Channel on Response of Optimal Inflation to a Fall in the Exogenous Real Rate



the IMR channel, the optimum for the economy moves from point A to point C.

Now the case without the IMR channel is considered. Line (b) represents the feasible values of inflation and ZLB probabilities without the IMR channel when the real interest rate on bonds is fixed (by adjusting the utility from safe bonds) at the same value it takes in point A. Thus, line (b) passes through point A because if inflation is at $\pi^{\star}_{A}$ and the real interest rate on safe bonds is the same as at A (so the utility from safe bonds is the same), the economic situation is identical to A. It can also be seen that line (b) has a steeper gradient than line (a). This is because under line (b), a rise in inflation does not lower the real interest rate, implying that nominal interest rates rise by more and thus that the probability of hitting the ZLB falls by more than it does under line (a). Line (e) represents the feasible values of inflation and ZLB probabilities when the real interest rate on bonds is fixed at the same value it takes at point C. Thus, similarly to line (b), line (e) passes through point C and has a steeper gradient than line (c). Note that the real rate at line (e) therefore captures both the fall in the exogenous real rate, but also the additional impact of the IMR channel; that is, $\pi^{\star}_{C} > \pi^{\star}_{A}$ so the IMR channel implies the real rate falls by more than just the fall in the exogenous real rate. Line (d) captures the feasible values of inflation and ZLB probabilities when the real interest rate is fixed at the level it was at A minus only

the impact of the fall in the exogenous real rate on bonds due to the increased desire for safe bonds. Therefore, The real interest rate in line (d) is higher than in line (e) since line (e) captures the IMR channel, while line (d) does not. Thus, the probability of hitting the ZLB is lower for (d) than for (e) for any given inflation target level. Without the IMR channel, the impact of the fall in the exogenous real rate moves the optimum of the economy from point B to point D.

The direct effect of the IMR channel can be isolated. A fall in the exogenous real rate moves the optimum of the economy $\pi_A^\star$ to $\pi_C^\star$ and $\pi_B^\star$ to $\pi_D^\star$, respectively, with and without the IMR channel. The difference between these changes is decomposed into three effects in equation (62). The first effect $(\pi_E^\star - \pi_D^\star)$ isolates the direct impact of how the additional fall in the equilibrium real rate due to the IMR channel raises the optimal inflation target. It does this because line (e) captures the fall in the real interest rate due to both the fall in the exogenous real rate and the IMR channel, while line (d) only captures the fall in the real interest rate due to the fall in the exogenous real rate. Therefore, $\pi_E^\star > \pi_D^\star$ because the IMR channel lowers the real rate and thus raises the probability of hitting the ZLB, implying inflation should be higher.

$$\underbrace{(\pi_C^\star - \pi_A^\star)}_{\text{Change in Optimal Inflation with the IMR Channel}} = \underbrace{(\pi_D^\star - \pi_B^\star)}_{\text{Change in Optimal Inflation without the IMR Channel}} + \underbrace{(\pi_E^\star - \pi_D^\star)}_{\text{Pure effect of fall in } r} + \underbrace{(\pi_B^\star - \pi_A^\star)}_{\text{Substitution effect 1}} - \underbrace{(\pi_E^\star - \pi_C^\star)}_{\text{Substitution effect 2}} . \quad (62)$$

There are two additional indirect effects. Taking as given the exogenous real rate, the optimal inflation target would be higher without the IMR channel, because raising the inflation target does not lower the equilibrium rate and is thus more effective in preventing the ZLB. This comparison is effectively a substitution effect because the "price" of lowering the probability of hitting the ZLB is lower without the IMR channel. The first indirect effect $(\pi_B^\star - \pi_A^\star)$ captures this substitution effect before the fall in the exogenous real rate. Line (a) allows for the IMR channel so raising inflation above $\pi_A^\star$ lowers the real interest rate; line (b) fixes the real interest rate at the level it takes at A. In line (b) it is more beneficial to raise the inflation target since nominal rates rise one-to-one, as opposed to less than one-to-one with the IMR channel, so $\pi_B^\star > \pi_A^\star$. The second indirect effect $(\pi_E^\star - \pi_C^\star)$ captures this substitution effect after the fall in the exogenous real rate. Line (c) allows for the IMR channel while line (e) fixed the real interest rate at the level it takes at C. Therefore, in line (e), it is more beneficial to raise the inflation target, so $\pi_E^\star > \pi_C^\star$. Consequently, the impact of the IMR channel can be broken down into the direct effect of the additional fall in the real interest rate and two substitution effects representing how the IMR channel affects the relative benefits of raising the inflation target.

The direct effect of the IMR channel raises the inflation target by a similar degree to a similarly sized exogenous fall in the equilibrium real rate. The direct effect of the IMR channel is found by comparing the optimal inflation target with and without the additional fall in the equilibrium real rate due to the IMR channel $(\pi_E^\star - \pi_D^\star)$. Table 5 considers the decomposition in equation (62) in the context of my simulation. It demonstrates that the direct impact of the IMR channel is to raise the optimal inflation target by 1.2 and 6.8 basis points, respectively, in the basic and extended models. The IMR channel causes the equilibrium real rate to fall by an additional 1.7 and 10.4 basis points, respectively, in the basic and extended models. Therefore, the direct effect of the IMR channel is to raise the inflation target by 71 and 66 percent of the fall in the equilibrium real rate in the basic and extended models, respectively. This approximately corresponds to the degree to which it has been shown a fall in the exogenous real rate raises the inflation target — 67% and 63% in the basic and extended models when the exogenous real rate falls from 2% to -1%, respectively.

Table 5: Decomposition of IMR Channel Results

Impact of Fall in r_B from 2% to -1%	Basic	Extended
$\Delta\pi^\star$ with IMR ($\pi^\star_C - \pi^\star_A$)	1.996	1.900
= $\Delta\pi^\star$ without IMR ($\pi^\star_D - \pi^\star_B$)	1.989	1.889
+ Effect of Fall in r only ($\pi^\star_E - \pi^\star_D$)	0.012	0.068
+ Substitution Effect 1 ($\pi^\star_B - \pi^\star_A$)	0.002	0.004
- Substitution Effect 2 ($\pi^\star_E - \pi^\star_C$)	-0.008	-0.061

The indirect effects of the IMR channel through the two substitution effects cumulatively reduce the impact of the IMR channel on the optimal inflation target. The first substitution effect captures the fact that, without the IMR channel, the inflation target before a fall in the exogenous real rate would have been higher without the IMR channel, since it is more effective to raise the inflation target without the IMR channel. This raises the impact of the IMR channel all else being equal. However, this substitution effect is very small. Table 5 demonstrates it only raises the inflation target by 0.2 and 0.4 basis points in the basic and extended models, respectively ($\pi^\star_B - \pi^\star_A$). The second substitution effect captures the fact that, without the IMR channel, the inflation target after a fall in the exogenous real rate would be higher without the IMR channel since it is more effective to raise the inflation target without the IMR channel. This lowers the impact of the IMR channel all else being equal. This substitution effect is larger. Table 5 demonstrates it lowers the inflation target by 0.8 and 6.1 basis points in the basic and extended models, respectively ($\pi^\star_E - \pi^\star_C$). The intuition for why the second substitution effect is so much stronger is that, after a fall in the exogenous real rate, the inflation target is higher, which implies the IMR channel is stronger; that is, a rise in the inflation target lowers the equilibrium real rate by more as can be observed in table 2. Therefore, raising the inflatoin target is therefore less effective at preventing the ZLB. Consequently, removing the IMR channel would raise the inflation target by more after the fall in the exogenous real rate than before. Therefore, the dominant indirect effect is that raising the inflation target is less effective at preventing the ZLB following a fall in the exogenous real rate, reducing the degree to which it is optimal to raise the inflation target.

Overall, the IMR channel has two main effects on how much a central bank should raise the inflation target in response to a fall in the exogenous real rate. First, a fall in the exogenous real rate prompts a higher inflation target and an additional fall in the real rate, and thus the central bank has an increased incentive to raise the nominal interest rate. Second, raising the inflation target is less effective at raising the nominal interest rate, implying that even though there is an increased incentive to raise the nominal interest rate, it may not make sense to do so. My simulations imply the first effect dominates the second effect, but only slightly, and that the IMR channel implies that it is optimal to raise the inflation target by only a little more in response to a fall in the exogenous real rate.

7 Other Evidence for the IMR Channel

[I intend to add more details here.]

It is difficult to assess how long-run changes in trend inflation affect the equilibrium real rate, but some empirical evidence supports the IMR channel. Across developed countries in recent years, both inflation and the real interest rate have fallen. This would contradict the IMR channel, by which inflation and equilibrium real rates should move in opposite directions, if the fall in inflation is the only channel affecting the equilibrium real

rate. However, various studies propose many other reasons for the lower real interest rates across developed countries.. They include demographic changes (Carvalho et al. 2016; Gagnon et al. 2016), global savings glut (Caballero and Farhi 2017), secular stagnation (Eggertsson et al. 2019), low productivity growth (Yi and Zhang 2017), and high inequality (Lancastre 2018). Indeed, it is puzzling that real interest rates have not fallen by more. Gagnon (2009) argues that demographic factors alone can explain the fall in the equilibrium real rate, while Eggertsson et al. (2019) argue that real rates should be much lower. An ideal assessment of the IMR channel's empirical validity would involve random variation in inflation targets, not just short-run inflation, but it does not seem possible to directly identify such random variation. However, some authors have attempted to identify the long-run relationship between inflation and the equilibrium real rate through structural methods. In a structural VAR model using US data, King and Watson (1997) show that an increase in long-run inflation leads to a decrease in the long-run real interest rate under a variety of different restrictions. Rapach (2003) extends the analysis to 14 countries with a richer structural model and finds similar qualitative results.

8 Conclusion

In recent years, many economists have proposed raising the inflation target. They argue that this will raise the average nominal interest rate and thus reduce the probability of hitting the zero lower bound (ZLB). It is a general assumption and a feature of the benchmark New Keynesian model that raising the inflation target has no impact on the equilibrium real rate.

I show that once heterogeneity is introduced into a standard New Keynesian model through either generational features or idiosyncratic shocks, raising the inflation target will lower the equilibrium real rate. In my baseline calibration of a New Keynesian model with life-cycle features, a rise in the inflation target from 2% to 4% lowers the equilibrium real rate between 6 and 66 basis points depending on the calibration. Even though the IMR channel implies that raising the inflation target will reduce the problem of the ZLB by less, it also implies that raising the inflation target is less marginally beneficial. In a simulation, I find these two effects roughly cancel out each other and that it is optimal to raise the inflation target by only slightly more in response to an exogenous fall in the equilibrium real rate.

The results of this paper provide valuable insights for policymakers. It implies that raising the inflation target may be less effective than expected in tackling the problems posed by the fall in the equilibrium real rate in recent years. However, despite the lower effectiveness of the policy, it may still make sense to raise the inflation target to a similar degree. My paper adds to the growing literature demonstrating the importance of considering monetary policy outcomes in models that allow for realistic heterogeneity across households.

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Appendices

A Model Details

This section provides additional details on the derivations of section 3.

A.1 Aggregating Household Conditions

Define per capita capital K_t as the per capita capital (equation (64)). This can be computed by summing the population-weighted capital held by each cohort ($\sum_{i=0}^{M-1} N_{t-i}T_{i,t}$) divided by the total population ($\mathbb{N}_t$). This equation can be written more simply in terms of n. These steps are shown in equation (64).

$$K_t = \frac{\sum_{i=0}^{M-1} N_{t-i}K_{i,t}}{\mathbb{N}_t} = \frac{\sum_{i=0}^{M-1} N_{t-i}K_{i,t}}{\sum_{i=0}^{M-1} N_{t-i}} = \frac{\sum_{i=0}^{M-1} N_t \frac{1}{(1+n)^i}K_{i,t}}{\sum_{i=0}^{M-1} N_t \frac{1}{(1+n)^i}} = \frac{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}K_{i,t}}{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}}. \tag{63}$$

Other per-capita variables are defined similarly: per capita total labor (L_t), per capita savings held at t (T_t, equation (64)), per capita savings made at t (S_{t+1}, equation (34)), aggregate bonds (B_t, equation (65)), aggregate shares (ω_t, equation (66)).

$$T_t = \frac{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}T_{i,t}}{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}}. \tag{64}$$

$$B_t = \frac{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}B_{i,t}}{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}}, \tag{65}$$

$$\omega_t = \frac{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}\omega_{i,t}}{\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}}. \tag{66}$$

Next, recall that the total holdings of shares ($\tilde{\omega}_{i,t}$) in a firm must sum to 1; that is, $\sum_{i=0}^{M-1} N_{t-i}\tilde{\omega}_{i,t} = 1$. Applying the definition of $\omega_{i,t} = \tilde{\omega}_{i,t}\mathbb{N}_t$ implies that the aggregate per capita holdings of shares in a firm ω_t must also always equal 1. This is shown formulaically in equation (67).

$$\omega_t = \frac{\sum_{i=0}^{M-1} N_{t-i}\omega_{i,t}}{\mathbb{N}_t} = \frac{\sum_{i=0}^{M-1} N_{t-i}\tilde{\omega}_{i,t}\mathbb{N}_t}{\mathbb{N}_t} = \sum_{i=0}^{M-1} N_{t-i}\tilde{\omega}_{i,t} = 1. \tag{67}$$

Inputting equation (30) into equation (64) and then applying equations (63) and (65) to (67) yields equation (35). Equation (35) states that the total assets held at t equal the total return on bonds, capital, and shares. Similarly, inputting equation (31) into equation (34) and then applying equations (63) and (65) to (67) yields equation (36).[6]

[6]To derive equation (36), the following steps can be made for bonds, capital, and shares:

$$\frac{\sum_{i=0}^{M-1} N_{t-i}K_{i+1,t+1}}{\mathbb{N}_t} = \frac{\sum_{i=1}^{M} N_{t-i}K_{i,t+1}}{\mathbb{N}_t} = \frac{\sum_{i=0}^{M-1} N_{t-i}K_{i,t+1}}{\mathbb{N}_t} = (1+n)\frac{\sum_{i=0}^{M-1} N_{t-i}K_{i,t+1}}{\mathbb{N}_{t+1}} = (1+n)K_{t+1}.$$

B Steady State Details

This section provides fuller derivations of the steady state. The conditions are summarized in section 4.2.

B.1 Markup and Inflation Target

First, note that the steady state inflation level equals the target inflation level: $\bar{\Pi} = \Pi^\star$. Next, equation (57) can be rewritten to express the optimal relative price as a function of steady state inflation:

$$\frac{\overline{P^\star}}{P} = \left(\frac{1-(1-\lambda)\bar{\Pi}^{\sigma-1}}{\lambda}\right)^{\frac{1}{1-\sigma}}. \tag{68}$$

Next, $\bar{U}, \bar{V}$ can be expressed as functions of $\bar{Y}, \overline{MC}, \bar{\Pi}, \bar{R}$:

$$\bar{U}\frac{\overline{P^\star}}{P} = \bar{V},$$

$$\bar{U} = \bar{Y}\left(1 - \frac{1-\lambda}{\bar{R}}\bar{\Pi}^{\sigma-1}\right), \tag{69}$$

$$\bar{V} = \bar{Y}\frac{\sigma}{\sigma-1}(1+\tau)\overline{MC}\left(1 - \frac{1-\lambda}{\bar{R}}\bar{\Pi}^{\sigma}\right). \tag{70}$$

Combining equations (68) to (70) and appendix B.1 allows the marginal cost to be rewritten as a function of $\bar{\Pi}, \bar{R}$:

$$\overline{MC} = \left(1 - \frac{1-\lambda}{\bar{R}}\bar{\Pi}^{\sigma-1}\right)\left(\frac{1-(1-\lambda)\bar{\Pi}^{\sigma-1}}{\lambda}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{\sigma-1}(1+\tau)\left(1 - \frac{1-\lambda}{\bar{R}}\bar{\Pi}^{\sigma}\right)\right)^{-1}.$$

The steady state price dispersion is given by equation (71):

$$\nu_t = \frac{1}{1-(1-\lambda)\bar{\Pi}^{\sigma}}\lambda\left(\frac{\overline{P^\star}}{P}\right)^{-\sigma}. \tag{71}$$

B.2 Relative Asset Supply

Equation (72) follows from equation (36) and the definition of a^s:

$$\bar{a}^s = \frac{(1+n)\bar{K} + \bar{Z}}{\bar{W}\bar{L}}. \tag{72}$$

Inputting equation (26) into equation (72) yields equation (73):

$$\bar{a}^s = \frac{(1+n)\bar{K} + \frac{\bar{\Omega}}{\frac{\bar{R}}{1+n}-1}}{\bar{W}\bar{L}}. \tag{73}$$

The first equality is just an adjustment of the summation index. The second equality uses $K_{0,t+1} = K_{M,t+1} = 0$ to adjust the summation beginning and starting points. The third equality adjusts $\mathbb{N}_t$. The fourth equality is just a definition.

Inputting equation (51) into equation (73) yields equation (74), where m is defined such that $m = \frac{1}{\bar{MC}\bar{\nu}}$ (it is akin to the markup):

$$\bar{a}^s = \frac{(1+n)\bar{K} + \frac{\bar{Y}(1-\frac{1}{m})}{\frac{\bar{R}}{1+n}-1}}{\bar{W}\bar{L}}. \tag{74}$$

Equations (48) and (49) can be combined to yield equation (75). Inputting equations (49) and (75) into equation (74) yields equation (76):

$$\frac{\bar{K}}{\bar{W}\bar{L}} = \frac{\alpha}{1-\alpha}\frac{1}{\bar{R}-1+\delta}, \tag{75}$$

$$\bar{a}^s = (1+n)\frac{\alpha}{1-\alpha}\frac{1}{\bar{R}-1+\delta} + \frac{m-1}{\frac{\bar{R}}{1+n}-1}\frac{1}{1-\alpha}. \tag{76}$$

We observe that $\bar{a}^s$ is an increasing function of m. This makes sense, because when the markup rises, profits are higher and thus the value of firms is higher and the quantity of assets supplied rises.

B.3 Relative Asset Demand

Relative Consumption The total savings that agents hold at the start of a period must equal the amount they saved in the previous period plus the return. This yields equation (77). This is shown formally by applying arbitrage conditions on bonds and shares (equations (24) and (28)) to equations (35) and (36):

$$\bar{T} = \bar{R}\bar{S}. \tag{77}$$

Applying equation (77) to the individual period budget constraint (equation (38)) and then iterating yields the intertemporal steady state budget constraint equation (78):

$$\sum_{i=0}^{M-1}\frac{\bar{C}_i}{\bar{R}^i} = \bar{W}\sum_{i=0}^{M-1}\frac{\bar{L}_i}{\bar{R}^i}. \tag{78}$$

The Euler condition (equation (25)) can be rewritten as equation (79):

$$\bar{C}_{i+1} = (\beta\bar{R})^{\frac{1}{\gamma}}\bar{C}_i. \tag{79}$$

Iterating over equation (79) yields equation (80):

$$\bar{C}_i = (\beta\bar{R})^{\frac{i}{\gamma}}\bar{C}_0. \tag{80}$$

Inputting this back into equation (78) and simplifying yields equation (81):

$$\bar{C}_0 = \left(\sum_{i=0}^{M-1}\beta^{\frac{i}{\gamma}}\bar{R}^{\frac{i(1-\gamma)}{\gamma}}\right)^{-1}\bar{W}\sum_{i=0}^{M-1}\frac{\bar{L}_i}{\bar{R}^i}. \tag{81}$$

Therefore, applying equation (79) to equation (81) yields an expression for consumption in any period in terms

of $\bar{R}$:

$$\bar{C}_j = (\beta\bar{R})^{\frac{j}{\gamma}} \left(\sum_{i=0}^{M-1} \beta^{\frac{i}{\gamma}} \bar{R}^{\frac{i(1-\gamma)}{\gamma}} \right)^{-1} \bar{W} \sum_{i=0}^{M-1} \frac{\bar{L}_i}{\bar{R}^i}. \tag{82}$$

Relative consumption for each cohort i is defined as consumption by that cohort divided by labor income:

$$\bar{c}_i = \frac{\bar{C}_i}{\bar{W}\bar{L}}.$$

Thus, equation (82) can be rewritten as equation (83):

$$\bar{c}_j = (\beta\bar{R})^{\frac{j}{\gamma}} \left(\sum_{i=0}^{M-1} \beta^{\frac{i}{\gamma}} \bar{R}^{\frac{i(1-\gamma)}{\gamma}} \right)^{-1} \sum_{i=0}^{M-1} \frac{\frac{\bar{L}_i}{\bar{L}}}{\bar{R}^i}. \tag{83}$$

Endogenous Labor Relative Labor Supply In the case with exogenous labor, relative consumption for each cohort is rewritten purely in terms of $\bar{R}$ (equation (83)). However, in the case with endogenous labor, $\frac{\bar{L}_i}{\bar{L}}$ are endogenous, so the labor part of equation (83), that is $\sum_{i=0}^{M-1} \frac{\bar{L}_i}{\bar{L}} \frac{1}{\bar{R}^i}$, needs to be rewritten in terms of $\bar{R}$ only.

Substituting the labor-leisure condition (equation (27)) into the Euler condition (equation (25)) yields the intertemporal labor supply condition:

$$\frac{v'(L_{i,t})}{W_t} = \beta R_{t+1} \frac{v'(L_{i,t+1}}{W_{t+1}}. \tag{84}$$

Applying the steady state and the disutility of working function to equation (84) yields:

$$x_i \bar{L}_i^{\eta} = \beta\bar{R} x_{i+1} \bar{L}_{i+1}^{\eta}. \tag{85}$$

Rewriting equation (85):

$$\bar{L}_{i+1} = \left(\frac{1}{\beta\bar{R}} \right)^{\frac{1}{\eta}} \left(\frac{x_i}{x_{i+1}} \right)^{\frac{1}{\eta}} \bar{L}_i. \tag{86}$$

Iterating over equation (86):

$$\bar{L}_i = \left(\frac{1}{\beta\bar{R}} \right)^{\frac{i}{\eta}} \left(\frac{x_0}{x_i} \right)^{\frac{i}{\eta}} \bar{L}_0. \tag{87}$$

Next, note that the (population-weighted) total labor supply is given by:

$$\bar{L} = \left(\sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \right)^{-1} \sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \bar{L}_i. \tag{88}$$

Inputting equation (87) into equation (88) yields:

$$\bar{L} = \left(\sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \right)^{-1} \sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \left(\frac{1}{\beta\bar{R}} \right)^{\frac{i}{\eta}} \left(\frac{x_0}{x_i} \right)^{\frac{i}{\eta}} \bar{L}_0. \tag{89}$$

Substituting out $\bar{L}_0$ from equation (87) using equation (89) yields the relative labor supplied by each cohort given by equation (90).

$$\frac{\bar{L}_j}{\bar{L}} = \left(\sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \left(\frac{1}{\beta \bar{R}} \right)^{\frac{i}{\eta}} \left(\frac{x_0}{x_i} \right)^{\frac{i}{\eta}} \right)^{-1} \left(\frac{1}{\beta \bar{R}} \right)^{\frac{j}{\eta}} \left(\frac{x_0}{x_i} \right)^{\frac{j}{\eta}} \sum_{i=0}^{M-1} \frac{1}{(1+n)^i}. \tag{90}$$

The economic intuition is that when $\bar{R}$ is higher, old agents supply relatively less labor because they are already getting a high return on their savings and therefore don't need to work as much.

Relative Asset Demand Cohort relative assets are defined in equation (91):

$$\bar{a}_i = \frac{\bar{S}_i}{\bar{W}\bar{L}}. \tag{91}$$

Equation (92) can be rewritten in terms of each cohort's relative assets and relative consumption:

$$\bar{c}_i + \bar{a}_{i+1} = \frac{\bar{L}_i}{\bar{L}} + \bar{R}\bar{a}_i. \tag{92}$$

Note that $\bar{a}_0 = \bar{a}_M = 0$ (since agents start with zero assets and have no need for assets when they are dead) and we have already computed $\bar{c}_i$ in terms of $\bar{R}$. Therefore, this yields $M-1$ equations from equation (92) and $M-1$ unknowns $\bar{a}_1, \ldots, \bar{a}_{M-1}$. Thus, if we know $\bar{R}$, $\bar{a}_i$ can be solved for by iterating over equation (92) starting from the beginning or end.

Finally, note that by equation (34), the demand for relative assets is given by equation (93):

$$\bar{a}^d = \left(\sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \right)^{-1} \sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \bar{a}_i. \tag{93}$$

C Calibration Details

This section provides more details on the calibration explained in section 4.5.

C.1 Endogenous Labor Supply

Substituting the labor-leisure condition (equation (27)) into the Euler condition (equation (25)) yields the intertemporal labor supply condition:

$$\frac{v'(L_{i,t})}{W_t} = \beta R_{t+1} \frac{v'(L_{i,t+1})}{W_{t+1}}. \tag{94}$$

In the steady state this becomes:

$$x_i \bar{L}_i^{\eta} = \beta \bar{R} x_{i+1} \bar{L}_{i+1}^{\eta}. \tag{95}$$

Equation (95) can be rewritten as:

$$x_{i+1} = \frac{1}{\beta \bar{R}} x_i \left(\frac{\bar{L}_i}{\bar{L}_{i+1}} \right)^{\eta}. \tag{96}$$

Iterating over this yields:

$$x_i = \frac{1}{(\beta \bar{R})^i} x_0 \left(\frac{\bar{L}_0}{\bar{L}_i} \right)^{\eta}.$$

x_0 is a free variable, since the x terms are relative, so x_0 is set to be 1. To keep things simple, I set $x_i \forall i > 0$ so that the labor supply is the same as the exogenous case parameterization when $\beta \bar{R} = 1$, so:

$$x_i = \left(\frac{\bar{L}_0}{\bar{L}_i} \right)^{\eta},$$

where the $\bar{L}$ ratios are the same as in the exogenous labor case.

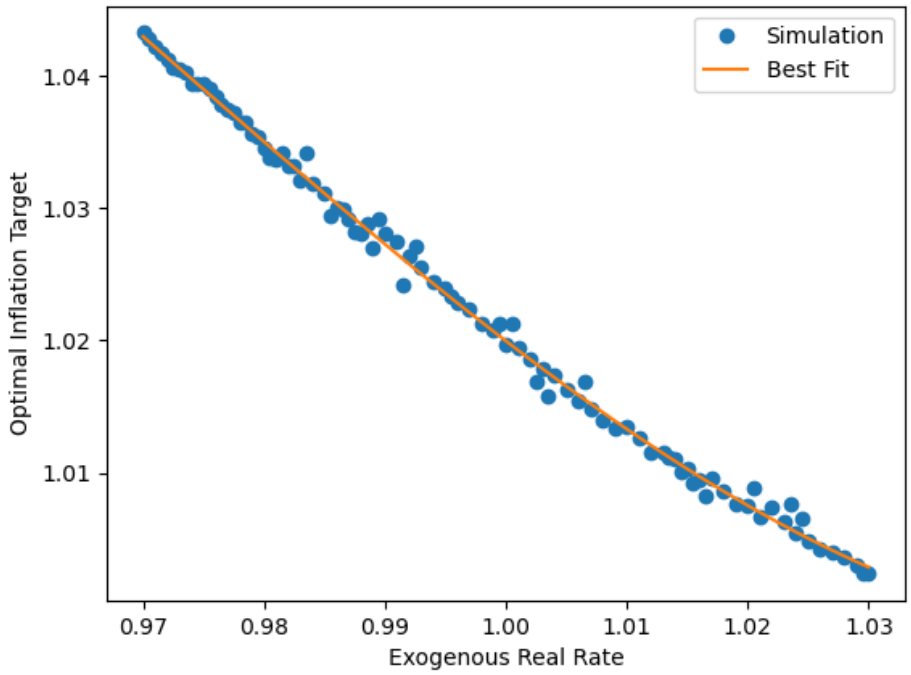
D Simulation Details

D.1 Simulation Methods Details

I now explain how I compute $\pi^\star(r)$ (the case allowing for the IMR channel). Each simulation is for $10,000$ periods using the same seeded shocks as in the method outlined in section 4.4. The case with endogenous labor is considered (Model B in sec:inflation-real-results). Welfare is then computed by applying the utility of consumption (equation (22)) and the disutility of working (equation (23)).

1. For each $r \in \{-0.03, -0.029, -0.028, \ldots, 0.028, 0.029, 0.03\}$, compute the welfare for each inflation level that would yield a positive nominal interest rate for $\pi^\star \in$: $\{0, 0.005, 0.01, \ldots, 0.07, 0.075, 0.08\}$.
2. Set $\pi^\star(r)$ for each estimated r as the inflation level that yielded that highest welfare for that r. Estimate $\hat{\pi}^\star(r)$ by doing a linear regression of $\pi^\star$ on r.
3. For each $r \in \{-0.03, -0.029, -0.028, \ldots, 0.028, 0.029, 0.03\}$, compute the welfare for each inflation level that would yield a positive nominal interest rate in $\{-0.02+\hat{\pi}^\star(r), -0.019+\hat{\pi}^\star(r), \ldots, 0.019+\hat{\pi}^\star(r), 0.02+\hat{\pi}^\star(r)\}$.
4. Using the same method as in step 2, recompute $\hat{\pi}^\star(r)$.
5. For each $r \in \{-0.03, -0.029, -0.028, \ldots, 0.028, 0.029, 0.03\}$, compute the welfare for each inflation level that would yield a positive nominal interest rate in $\{-0.005+\hat{\pi}^\star(r), -0.0048+\hat{\pi}^\star(r), \ldots, 0.0048+\hat{\pi}^\star(r), 0.005+\hat{\pi}^\star(r)\}$.
6. Using the same method as in step 2, except with a cubic rather than a linear regression, recompute $\hat{\pi}^\star(r)$.
7. For each $r \in \{-0.03, -0.0295, -0.029, \ldots, 0.029, 0.0295, 0.03\}$, compute the welfare for each inflation level that would yield a positive nominal interest rate in $\{-0.001 + \hat{\pi}^\star(r), -0.0009 + \hat{\pi}^\star(r), \ldots, 0.0009 + \hat{\pi}^\star(r), 0.001 + \hat{\pi}^\star(r)\}$.

Figure 9: Graph of Optimal Inflation by Exogenous Real Interest Rate with Line of Best Fit



8. Using the same method as in step 2, except with a cubic rather than a linear regression, recompute $\hat{\pi}^\star(r)$. The estimated $\pi^\star(r)$ is then $\hat{\pi}^\star(r)$.

I follow mostly similar steps when computing $\pi_f^\star(r)$ (the case excluding the IMR channel). However, since the estimates are close to $\pi^\star(r)$ and the simulation takes a long time to run, I use the estimates from step 4 of the method for $\pi^\star(r)$ as a starting point and then continue from step 5.

Figure 9 shows a plot of the points $(r, \pi^\star(r))$ where $\pi^\star(r)$ is the π that yielded the highest welfare for a given r together with $\hat{\pi}^\star(r)$ from step 8.

D.2 Mathematical Decomposition of Impact of IMR Channel on Response of Optimal Inflation to a Fall in the Exogenous Real Rate

Here a fall in the exogenous real rate from r_h to r_l is investigated. It can immediately be observed that $\pi_A^\star = \pi^\star(r_h)$ and $\pi_C^\star = \pi^\star(r_l)$.

To aid the mathematical decomposition, additional definitions are required. $r^b(\pi)$ is defined to represent the steady state real interest rate on bonds when trend inflation is at the level π. In general, it can be observed that $r^b(\pi^\star(r)) < r$. The reason for this is that r represents the real interest rate that the economy would take when trend inflation is zero. However, taking into account the zero lower bound, the optimal inflation target will be positive and thus the real interest rate on bonds will actually be lower than in the zero trend inflation case. Note that it also follows that $\pi_f^\star(r^b(\pi^\star(r)))$ represents the optimal inflation target when the real interest rate

on bonds is fixed as $r^b(\pi^\star(r))$ where $r^b(\pi^\star(r))$ represents the real interest rate when inflation is at its optimal level and the real interest rate on bonds is not fixed. Therefore, $\pi_b^\star = \pi_f^\star(r^b(\pi^\star(r_h)))$, $\pi_e^\star = \pi_f^\star(r^b(\pi^\star(r_l)))$, and $\pi_d^\star = \pi_f^\star(r^b(\pi^\star(r_h)) - (r_h - r_l))$.

One additional minor note for clarity: table 4 and table 5 yield slightly different results for how a fall in the exogenous real rate affects optimal inflation. Consider, for example, the case in which the exogenous real rate falls from 2% to -1% . Columns 3 and 5 of table 4 consider how the optimal inflation responds to specific levels of the exogenous real rate at 2% and -1% , that is, $\pi_f^\star(2), \pi_f^\star(-1)$; whereas table 5 considers the optimal inflation level when the real interest rate is fixed as the same as what it would be in the IMR case under 2% and a corresponding fall in the exogenous real rate, that is, $\pi_f^\star(r^b(\pi^\star(r_h)))$ and $\pi_f^\star(r^b(\pi^\star(2)) - 3)$. Given this difference, a fall in the exogenous real rate is found to be 0.8 basis points in table 4 and 0.7 basis points in table 5 for the basic model.

E Intuition in a Simplified Model Details

This section provides additional details relating to section 2.

E.1 Calvo Pricing: Inflation and the Markup

Here, the standard Calvo pricing setup is assumed, and the relationship between inflation and the markup is derived. This is an example of the inflation-markup relationship discussed in section 2.1.

There is a continuum of intermediate firms indexed by i that produce differentiated intermediate goods. There is a competitive final goods firm with constant elasticity of substitution (CES) production with CES parameter σ. The final goods firm's demand for good i is represented by:

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} Y_t, \tag{97}$$

where

$$\int_0^1 P_{i,t} Y_{i,t} di = P_t Y_t, \tag{98}$$

$$P_t = \left(\int_0^1 P_{i,t}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}, \tag{99}$$

$$Y_t = \left(\int_0^1 Y_{i,t}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}.$$

Intermediate firms have some probability λ of updating their price each period. Firms have the following maximization problem:

$$\max_{P_t^\star, \{Y_{t+k}^\star\}_{k=0}^\infty} \mathbb{E}_t[\sum_{k=0}^{\infty} (1-\lambda)^k \beta^k [\frac{P_t^\star Y_{t+k}^\star}{P_{t+k}} - MC_{t+k} P_{t+k} Y_{t+k}^\star]], \tag{100}$$

s.t. $\forall k$

$$Y^{\star}_{t+k} = Y_{t+k}(\frac{P^{\star}_t}{P_{t+k}})^{-\sigma}. \tag{101}$$

Inputting equation (101) into equation (100) and taking FOCs yields:

$$\mathbb{E}_t[\sum_{k=0}^{\infty}(1-\lambda)^k\beta^k((1-\sigma)P_t^{\star-\sigma}Y_{t+k}P_{t+k}^{\sigma-1} + \sigma MC_{t+k}Y_{t+k}P_t^{\star-\sigma-1}{P_{t+k}}^{\sigma}]].$$

In the steady state, this can be rewritten as:

$$\sum_{k=0}^{\infty}(1-\lambda)^k\beta^k\bar{\Pi}^{k\sigma}\left[\overline{\left(\frac{P^{\star}}{P}\right)}\frac{1}{\bar{\Pi}^k} - \frac{\sigma}{\sigma-1}\overline{MC}\right].$$

This can be further simplified to get:

$$\overline{MC} = \frac{\sigma-1}{\sigma}\frac{1-(1-\lambda)\beta\bar{\Pi}^{\sigma}}{1-(1-\lambda)\beta\bar{\Pi}^{\sigma-1}}\overline{\left(\frac{P^{\star}}{P}\right)}. \tag{102}$$

A price dispersion term is defined:

$$\nu_t = \int_0^1\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma}di. \tag{103}$$

This can be rewritten as:

$$\nu_t = \lambda\left(\frac{P^{\star}_t}{P_t}\right)^{-\sigma} + (1-\lambda)\nu_{t-1}\Pi_t^{\sigma}.$$

In the steady state:

$$\bar{\nu} = \frac{\lambda}{1-(1-\lambda)\bar{\Pi}^{\sigma}}\overline{\left(\frac{P^{\star}}{P}\right)}^{-\sigma}.$$

Multiplying both sides of equation (102) by $\bar{\nu}$ yields:

$$\overline{MC}\bar{\nu} = \frac{\sigma-1}{\sigma}\frac{1-(1-\lambda)\beta\Pi^{\star\sigma}}{1-(1-\lambda)\beta\Pi^{\star\sigma-1}}\overline{\left(\frac{P^{\star}}{P}\right)}^{1-\sigma}\frac{\lambda}{1-(1-\lambda)\bar{\Pi}^{\sigma}}. \tag{104}$$

Equation (99) can be rewritten as:

$$1 = \lambda\left(\frac{P^{\star}_t}{P_t}\right)^{1-\sigma} + (1-\lambda)\Pi_t^{\sigma-1},$$

where $\Pi_t = \frac{P_t}{P_{t-1}}$ and also $\pi_t = \Pi_t - 1$. In the steady state:

$$\left(\frac{P^{\star}}{P}\right)^{1-\sigma} = \frac{1}{\lambda}[1-(1-\lambda)\bar{\Pi}^{\sigma-1}]. \tag{105}$$

Inputting equation (105) into equation (104) and simplifying yields:

$$\overline{MC}\bar{\nu} = \frac{\sigma - 1}{\sigma}\left[\frac{1-(1-\lambda)\bar{\Pi}^{\sigma-1}}{1-(1-\lambda)\beta\bar{\Pi}^{\sigma-1}}\right]\left[\frac{1-(1-\lambda)\beta\bar{\Pi}^{\sigma}}{1-(1-\lambda)\bar{\Pi}^{\sigma}}\right]. \tag{106}$$

$\bar{m}$ is defined to be $\bar{m} = \left(\overline{MC}\bar{\nu}\right)^{-1}$, which is a measure of the effective markup (this is further discussed in appendix E.2). Therefore, equation (106) can be rewritten as:

$$\bar{m} = \frac{\sigma}{\sigma - 1}\left[\frac{\bar{\Pi}^{1-\sigma}-(1-\lambda)\beta}{\bar{\Pi}^{1-\sigma}-(1-\lambda)}\right]\left[\frac{\bar{\Pi}^{-\sigma}-(1-\lambda)}{\bar{\Pi}^{-\sigma}-(1-\lambda)\beta}\right]. \tag{107}$$

Informal: Raising Inflation Lowers the Markup Raising $\bar{\Pi}$ increases the first square bracket but lowers the second. When $\beta < 1$, the second square bracket dominates since $\bar{\Pi}^{-\sigma}$ changes by more than $\bar{\Pi}^{1-\sigma}$, so raising average inflation lowers the markup.

Note that here the markup is effectively the average markup weighted by sales. This is why the results are qualitatively different from those of Ascari and Sbordone (2014), who consider the non-weighted average markup and show that it can rise when inflation rises.

Formal: Raising Inflation Lowers the Markup The result that raising inflation lowers the markup when $\beta < 1$ is now shown formally. First derivatives are applied:

$$\frac{d\bar{m}}{d\bar{\Pi}} = \left[\frac{(1-\sigma)\bar{\Pi}^{-\sigma}}{\bar{\Pi}^{1-\sigma}-(1-\lambda)\beta} - \frac{(1-\sigma)\bar{\Pi}^{-\sigma}}{\bar{\Pi}^{1-\sigma}-(1-\lambda)} - \frac{\sigma\bar{\Pi}^{-\sigma-1}}{\bar{\Pi}^{-\sigma}-(1-\lambda)} + \frac{\sigma\bar{\Pi}^{-\sigma-1}}{\bar{\Pi}^{-\sigma}-(1-\lambda)\beta}\right]$$
$$\frac{\sigma}{\sigma-1}\left[\frac{\bar{\Pi}^{1-\sigma}-(1-\lambda)\beta}{\bar{\Pi}^{1-\sigma}-(1-\lambda)}\right]\left[\frac{\bar{\Pi}^{-\sigma}-(1-\lambda)}{\bar{\Pi}^{-\sigma}-(1-\lambda)\beta}\right].$$

Concentrating on the first square bracket:

$$\frac{d\bar{m}}{d\bar{\Pi}} \propto -\frac{(1-\sigma)\bar{\Pi}^{-\sigma}(1-\lambda)(1-\beta)}{(\bar{\Pi}^{1-\sigma}-(1-\lambda)\beta)(\bar{\Pi}^{1-\sigma}-(1-\lambda))} - \frac{\sigma\bar{\Pi}^{-\sigma-1}(1-\lambda)(1-\beta)}{(\bar{\Pi}^{-\sigma}-(1-\lambda))(\bar{\Pi}^{-\sigma}-(1-\lambda)\beta)}.$$

Rearranging and removing $(1-\lambda)(1-\beta)$ (which would not be possible if $\beta = 1$):

$$\frac{d\bar{m}}{d\bar{\Pi}} \propto \frac{\bar{\Pi}^{-\sigma}}{(\bar{\Pi}^{1-\sigma}-(1-\lambda)\beta)(\bar{\Pi}^{1-\sigma}-(1-\lambda))} - \frac{\sigma}{\sigma-1}\frac{\bar{\Pi}^{-\sigma-1}}{(\bar{\Pi}^{-\sigma}-(1-\lambda))(\bar{\Pi}^{-\sigma}-(1-\lambda)\beta)}.$$

Substituting out $\frac{\sigma}{\sigma-1}$ using equation (107) and rearranging to get:

$$\frac{d\bar{m}}{d\bar{\Pi}} \propto \bar{\Pi}^{-\sigma}(\bar{\Pi}^{-\sigma}-(1-\lambda))^2 - \bar{m}\bar{\Pi}^{-\sigma-1}(\bar{\Pi}^{1-\sigma}-(1-\lambda))^2.$$

Simplifying:

$$\frac{d\bar{m}}{d\bar{\Pi}} \propto \bar{\Pi}^{-\sigma}(\bar{\Pi}^{-\sigma}-(1-\lambda))^2 - \bar{m}\bar{\Pi}^{-\sigma-1}(\bar{\Pi}^{1-\sigma}-(1-\lambda))^2. \tag{108}$$

$\bar{m} \geq 1$, otherwise firms exit the market and $\frac{d\bar{m}}{d\bar{\Pi}}$ is decreasing in $\bar{m}$. Therefore, $\frac{d\bar{m}}{d\bar{\Pi}}$ takes its highest possible value when $\bar{m} = 1$. It is shown that even in this case $\frac{d\bar{m}}{d\bar{\Pi}} < 0$, and thus the markup always decreases in inflation. Under $\bar{m} = 1$ $\frac{d\bar{m}}{d\bar{\Pi}}$ (equation (108)) simplifies to yield:

$$\frac{d\bar{m}}{d\bar{\Pi}}_{\bar{m}=1} \propto \bar{\Pi}^{-3\sigma} + (1-\lambda)^2\bar{\Pi}^{-\sigma} - [\bar{\Pi}^{1-3\sigma} + \bar{\Pi}^{-\sigma-1}(1-\lambda)^2],$$

$$\frac{d\bar{m}}{d\bar{\Pi}}_{\bar{m}=1} \propto (\bar{\Pi}-1)((1-\lambda)^2 - \bar{\Pi}^{-2\sigma}).$$

Then note that by equation (107) $\bar{\Pi}^{-\sigma} \geq 1-\lambda$ since this is required for $\bar{m} > 1$. Thus, $\frac{d\bar{m}}{d\bar{\Pi}} < 0$, so it has been demonstrated that under Calvo pricing, when average inflation rises, the average markup always falls when $\beta < 1$.

E.2 Asset Demand with Price Dispersion

In the simple model under the assumption that firms set identical prices, profits were derived to be (equation (5)):

$$\Omega_t = (m_t - 1)\frac{W_t}{P_t}L_t.$$

This section shows this also holds when firms do not set identical prices.

Intermediate firms produce output $Y_{i,t}$ by a linear production function over labor $L_{i,t}$:

$$Y_{i,t} = L_{i,t}. \tag{109}$$

Inputting equation (97) into equation (109) and aggregating yields:

$$Y_t\nu_t = L_t, \tag{110}$$

where ν_t is defined by equation (103) and:

$$L_t = \int_0^1 L_{i,t}.$$

The real marginal cost MC_t of intermediate firms is just their wage bill (W_t):

$$MC_t = W_t. \tag{111}$$

Each intermediate firm has the following real profits $\Omega_{i,t}$:

$$\Omega_{i,t} = \frac{P_{i,t}Y_{i,t}}{P_t} - MC_tY_{i,t}. \tag{112}$$

Inputting equation (97) into equation (112) and aggregating (and applying equation (98)) yields:

$$\Omega_t = (1 - MC_t\nu_t)Y_t, \tag{113}$$

where

$$\Omega_t = \int_0^1 \Omega_{i,t} di$$

Next, m_t is defined to be $\frac{1}{MC_t \nu_t}$. We input this into equation (113):

$$\Omega_t = (1 - \frac{1}{m_t}) Y_t. \tag{114}$$

m_t represents the inverse of real costs of firms, which is a measure of the average markup across firms. When $m_t = 1$ (the competitive case), firms make no profits. However, as the markup rises, $m_t \uparrow$, firms make higher profits.

Next, the RHS of equation (114) is multiplied by $\frac{W_t}{MC_t}$ (which equals 1 by equation (111)), and then equation (110) is applied to get:

$$\Omega_t = (1 - \frac{1}{m_t} \frac{W_t L_t}{MC_t \nu_t},$$
$$\Omega_t = (m_t - 1) W_t L_t.$$

Thus, profits are the same as in the case without price dispersion.

E.3 OLG in a Non-annualized Model

Figure 10 shows the case where the OLG model is non-annualized, unlike figure 3 where it is annualized. The real interest rate is very high since it represents the return from one generation to the next.

F Model Extensions

This section describes how extensions to the main model (section 3) are introduced.

F.1 Menu Costs

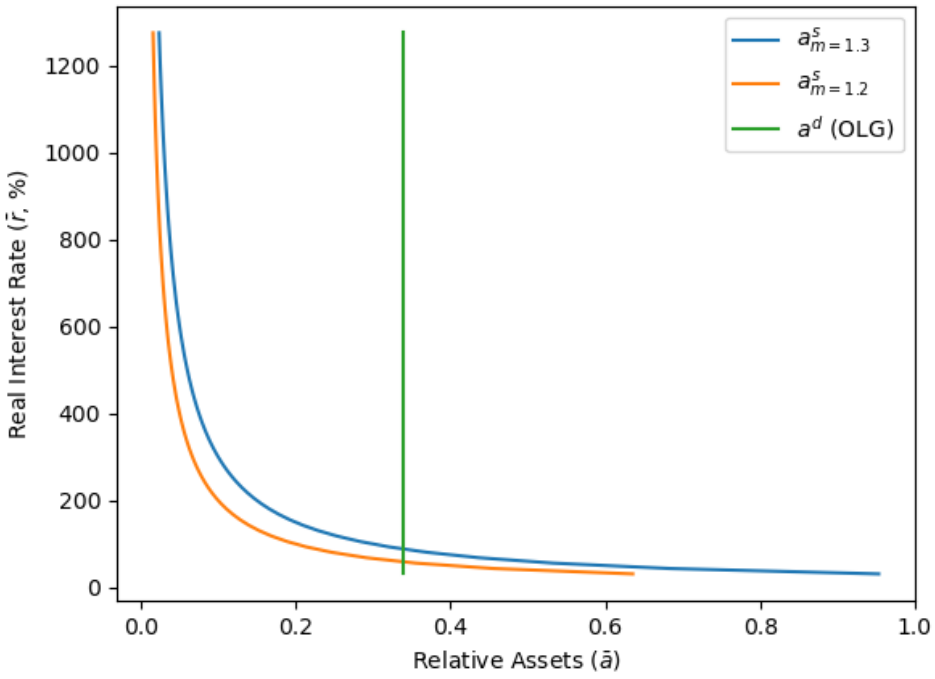
It is interesting to consider how changing the price rigidity affects the relationship between inflation and the real interest rate. In the main model, Calvo pricing was considered. Here, menu costs are considered instead.

Intermediate Firm Problem The same basic structure as under Calvo pricing is assumed. There is a competitive final goods firm with CES production that aggregates intermediate goods into a final good. There are monopolistic intermediate firms.

Idiosyncratic shocks are introduced to generate continuous distributions of price changes. To introduce shocks, it is assumed that intermediate firms have idiosyncratic productivity $A_{i,t}$. Therefore, their output $Y_{i,t}$ is given by equation (115):

$$Y_{i,t} = A_{i,t} K_{i,t}^{\alpha} L_{i,t}^{1-\alpha}, \tag{115}$$

Figure 10: Equilibrium under a Fall in the Markup: 2. OLG (Not Annualized)



where

$$\log(A_{i,t}) = \rho_{ai} \log(A_{i,t-1}) + (1 - \rho_{ai}) \log(\bar{A}) + \epsilon_{i,t},$$

and $\epsilon_{i,t} \sim N(0, \sigma_{ai})$.

Under $A_{i,t} = 1$, firms have the same cost minimization problem as in section 3.2 (when $A_t = 1$). There are no aggregate shocks, so real aggregate variables are constant and do not vary with time. The marginal cost of a firm with $A_{i,t} = 1$ is defined as MC, which is constant. Thus, the idiosyncratic marginal cost can be rewritten as equation (116).

$$MC_{i,t} = \frac{MC}{A_{i,t}}. \tag{116}$$

An intermediate goods firm pays a fixed cost f that is a function of aggregate output Y if it changes its price. The firm's profit function is then given by equation (117):

$$\Omega_{i,t} = \frac{P_{i,t}}{P} Y_{i,t} - MC_{i,t} Y_{i,t} - fY \mathbb{1}_{pricechange,i}. \tag{117}$$

Inputting the demand from the final goods firm (equation (40)) and the idiosyncratic marginal cost (equation (116)) into equation (117) yields equation (118):

$$= Y \left[\left(\frac{P_{i,t}}{P} \right)^{1-\sigma} - \frac{MC}{A_{i,t}} \left(\frac{P_{i,t}}{P_t} \right)^{-\sigma} \right]. \tag{118}$$

The problem can then be aggregated. The relative price of a firm $p_{i,t} = \frac{P_{i,t}}{P_t}$ is defined to get equation (119):

$$\Omega_{i,t} = Y \left[p_{i,t}^{1-\sigma} - \frac{MC}{A_{i,t}} p_{i,t}^{-\sigma} - f \mathbb{1}_{pricechange,i} \right]. \tag{119}$$

Relative profits, that is a firm's profits relative to output, are defined as $\omega_{i,t} = \frac{\Omega_{i,t}}{Y}$. A firm's profits are then a function of idiosyncratic productivity and a firm's price, which are defined as a function ω in equation (120):

$$\omega_{i,t} = \frac{\Omega_{i,t}}{Y} = \omega(p_{i,t}, A_{i,t}) = p_{i,t}^{1-\sigma} - \frac{MC}{A_{i,t}} p_{i,t}^{-\sigma}. \tag{120}$$

Each period, firms have two choices:

1. Pay a menu cost μY, in which case a firm will get to reset its price at time t, and firms receive profits of $Y\omega(p_{i,t}, A_{i,t}) - \mu)$ in the current period.
2. Do not pay a menu cost, in which case a firm will not reset its price and will use the same price as in the previous period. Thus, $p_{i,t} = \frac{P_{i,t-1}}{P_t} = \frac{p_{i,t-1}}{\Pi}$, where Π is gross inflation $\frac{P_{t+1}}{P_t}$ that is constant in the steady state and so does not require a time subscript. In this case, firms receive profits of $Y\omega\left(\frac{p_{i,t-1}}{\Pi}, A_{i,t}\right)$ in the current period.

Therefore, the problem is rewritten as follows (where profits have been divided by Y to simplify the problem and where 0 represents the current period and -1 represents the previous period):

$$V(p_{i,-1}, A_{i,0}) = \max \left\{ \max_{p_{i,0}} \omega(p_{i,0}, A_{i,0}) + \beta \mathbb{E}\left[V\left(\frac{p_{i,0}}{\Pi^\star}, A_{i,1} \right) \right] - \mu, \omega\left(\frac{p_{i,-1}}{\Pi}, A_{i,0} \right) + \beta \mathbb{E}\left[V\left(\frac{p_{i,-1}}{\Pi^\star}, A_{i,1} \right) \right] \right\}.$$

This value function iteration problem can then be solved to give firm price distributions given a marginal cost MC and menu cost f.

Aggregate Conditions Equation (41) can be rewritten as equation (121):

$$1 = \int_0^1 p_{i,t}^{1-\sigma}. \tag{121}$$

Equation (121) holds only when MC is at the correct level. If MC is too high, firms will set their prices too high. If MC is too low, firms will set their prices too low.

The aggregate profit share can be computed by aggregating equation (119). To do this, aggregate profits are defined as $\Omega = \int_0^1 \Omega_{i,t} di$. Then integrating over i and applying equation (121) yields equation (122), where ν is defined in equation (123):

$$\frac{\Omega}{Y} = 1 - MC\nu - fYprob_{pricechange}, \tag{122}$$

$$\nu = \int_0^1 \frac{p_i^{-\sigma}}{A_i}. \tag{123}$$

In equation (122), it can be observed that if a rise in inflation raises the probability of a price change, the profit share falls. In practice, menu costs may represent internal costs of changing prices, so it may be preferable to exclude this impact on the profit share, in which case the profit share is given simply by equation (124):

$$\frac{\Omega}{Y} = 1 - MC\nu - fYprob_{pricechange}. \tag{124}$$

Calibration As in the main model, $\sigma = 8$. The standard deviation of productivity shocks (σ_{ai}) and the persistence of productivity shocks (ρ_{ai}) are set to be respectively 0.0428 and 0.66 as in Nakamura and Steinsson (2008). The model is calibrated to set the monthly frequency of price changes to be 8.7% (which also is based on Nakamura and Steinsson 2008). The model calibration proceeds with two iterative steps. The first step is that the MC are computed that satisfy equation (121) under a low and a high value of f. The second step is that the f is computed that yields 8.7% frequency of price change under the low and high MC that have been computed. The steps are alternated until convergence is achieved.

F.2 Firm Additional Discounting

Firms may discount at a greater rate than households do, since they face funding constraints that mean their cost of borrowing may be higher than $\bar{r}$. To incorporate this feature into the model, a wedge is added to the degree to which firms borrow. Thus, rather than discount by $\frac{1}{R}$, firms instead discount by $\frac{\beta_f}{R}$.

The choice of β_f is based on the weighted average cost of capital (WACC), which is the average a company is expected to pay to finance its assets.[7] Jagannathan et al. (2016) estimate that it was 8% in 2003, when the expected 10-year rate on real bonds (r^e) was 2.8p.p. Graham and Harvey (2011) estimate it was 10.0% in 2011Q1, when r^e was 2.2% . Graham and Harvey (2012) estimate it was 9.3% in 2012Q2, when r^e was 1.3% . From these three surveys, the average wedge between WACC and the expected real rate is 7p.p. Therefore, a (quarterly) firm discount of $\beta_f = \left(\frac{1}{1.07}\right)^{\frac{1}{4}}$ is applied.

F.3 Firm Death

In the baseline model, the value of shares in firms is very high, which implies a high real interest rate, unless we apply a significant subsidy on marginal production. One way to get a more reasonable level for firm value is to assume that with some probability firms die each period.

To model firm death, it is assumed that each period after paying out profits, only a fraction ζ of firms survive. The firms that do not survive are restructured and sold by entrepreneurs who are not modeled. Therefore, agents receive a payout of $\zeta\tilde{Z}_t$ only on the shares they bought in the previous period and sell in the current period. Then equation (26) becomes equation (125), which, using the same steps as in the main model, simplifies to equation (126):

$$\tilde{Z}_t u'(C_{i,t}) = \beta\mathbb{E}_t[u'(C_{i+1,t+1})(\tilde{\Omega}_{t+1} + \zeta\tilde{Z}_{t+1})], \tag{125}$$

$$Z_t u'(C_{i,t}) = \beta\mathbb{E}_t[u'(C_{i+1,t+1})(1+n)(\Omega_{t+1} + \zeta Z_{t+1})]. \tag{126}$$

[7] This is used since it is the cost to the firm of not obtaining funds earlier by setting a lower markup.

Firm survival ($\zeta = 0.985$) is calibrated so that the profit share equals 6.06% , which was its average value in the 1987–2006 period.

F.4 Intermediate Goods

An approach similar to that of Basu (1995) is followed to introduce intermediate goods into the model. This impacts only the firm cost minimization part of the model. A "roundabout" production structure is assumed where final goods are used as intermediate inputs in the production of other goods. Thus, $M_{i,j,t}$ represents the amount of the final good used in production by firm i in sector j at time t, while other elements are defined similarly to how they are defined in the main model:

$$Y_{i,t} = M_{i,t}^{\chi}(A_t {K_{i,t}}^{\alpha} {L_{i,t}}^{1-\alpha})^{1-\chi}. \tag{127}$$

Intermediate firms minimize their costs. They face the following problem:

$$\min_{K_{i,t},L_{i,t}} (R_t - 1 + \delta)K_{i,j,t} + W_t L_{i,j,t} + M_{i,j,t},$$

s.t. equation (127).

We define $M_t = \int_0^1 M_{i,t}$. Taking FOCs and noting that the ratios $\frac{K_{i,t}}{L_{i,t}}, \frac{K_{i,t}}{M_{i,t}}$ are the same across firms yields equations (128) to (130).

$$MC_t = \frac{r_t}{(1-\chi)\alpha A_t^{1-\chi} K_t^{(1-\chi)\alpha-1} L_t^{(1-\chi)(1-\alpha)} M_t^{\chi}}, \tag{128}$$

$$MC_t = \frac{W_t}{(1-\chi)(1-\alpha) A_t^{1-\chi} K_t^{(1-\chi)\alpha} L_t^{(1-\chi)(1-\alpha)-1} M_t^{\chi}}, \tag{129}$$

$$MC_t = \frac{1}{\chi A_t^{1-\chi} K_t^{(1-\chi)\alpha} L_t^{(1-\chi)(1-\alpha)} M_t^{\chi-1}}. \tag{130}$$

Inputting equation (40) into equation (127) and noting that the ratios $\frac{K_{i,t}}{L_{i,t}}, \frac{K_{i,t}}{M_{i,t}}$ are constant across firms allows equation (127) to be rewritten as equation (131), where ν_t is defined in the same way as it is in equation (47):

$$\nu_t Y_t = M_t^{\chi}(A_t K_t^{\alpha} L_t^{1-\alpha})^{1-\chi}. \tag{131}$$

Next, equation (131) can be inputted into equations (128) to (130) to replace M_t with Y_t, which yields equations (132) and (133):

$$(1-\chi)\alpha\nu_t MC_t = \frac{r_t K_t}{Y_t}, \tag{132}$$

$$(1-\chi)(1-\alpha)\nu_t MC_t = \frac{W_t L_t}{Y_t}, \tag{133}$$

$$\chi\nu_t MC_t = \frac{M_t}{Y_t}. \tag{134}$$

Output can also be rewritten without intermediate goods by first rewriting equation (130) as equation (135)

and then inputting equation (135) into equation (131) to get equation (136):

$$M_t^{1-\chi} = MC_t \chi A_t^{1-\chi} K_t^{(1-\chi)\alpha} L_t^{(1-\chi)(1-\alpha)}, \tag{135}$$

$$Y_t \nu_t = A_t K_t^{\alpha} L_t^{1-\alpha} (MC_t \chi)^{\frac{\chi}{1-\chi}}. \tag{136}$$

It is also useful to note that inputting equations (128) to (130) into equation (136) yields equation (137):

$$MC_t = \frac{1}{A_t^{1-\chi}} \left(\frac{R_t - 1 + \delta}{\alpha(1-\chi)} \right)^{(1-\chi)\alpha} \left(\frac{W_t}{(1-\chi)(1-\alpha)} \right)^{(1-\chi)(1-\alpha)} \left(\frac{1}{\chi} \right)^{\chi}. \tag{137}$$

Profits simplify in a manner similar to the way they do in the main model to yield equation (51).

Calibration Without price rigidity or price dispersion, equation (134) becomes equation (138). Nakamura and Steinsson (2010) point out that the average revenue share of intermediate inputs in the US private sector using CPI expenditure weights was 52% . Thus, $\frac{M_t}{Y_t}$ can be estimated to be 0.52, while $\sigma = 8$. Thus, χ is estimated to be 0.594.

$$\chi = \frac{\sigma}{\sigma - 1} \frac{M_t}{Y_t}. \tag{138}$$

F.5 Multiple Sectors

Multiple sectors are introduced in this section. The key difference between sectors is in their price rigidity. Sectors are also allowed to differ in size. This follows models by Carvalho et al. (2016) and Nakamura and Steinsson (2010).

Final Goods Firm There is a single competitive final goods firm that aggregates goods in different industries to produce a final good. There are J industries in total, denoted by $1, \ldots, J$. The final goods firm has CES production, and each industry has a weight a_j (equation (139)). The CES parameter is allowed to be different for the aggregation across sectors (σ_2) relative to within sectors (σ):

$$Y_t = \left(\sum_{j=1}^{J} a_j^{\frac{1}{\sigma_2}} Y_{j,t}^{\frac{\sigma_2 - 1}{\sigma_2}} dj \right)^{\frac{\sigma_2}{\sigma_2 - 1}}. \tag{139}$$

Therefore, the final goods firm has the usual CES demand (taking into account industry weights) for each industry good given by equation (140). The price aggregator also takes the usual form given by equation (141). Note that weights a_j need to be added for each industry:

$$Y_{j,t} = a_j \left(\frac{P_{j,t}}{P_t} \right)^{-\sigma_2} Y_t, \tag{140}$$

$$P_t = \left(\sum_{j=1}^{J} a_j P_{j,t}^{1-\sigma_2} dj \right)^{\frac{1}{1-\sigma_2}}. \tag{141}$$

Industry Aggregator A perfectly competitive firm aggregates all the intermediate goods in that industry to produce the good for sector j. The sector firm has the following production function:

$$Y_{j,t} = \left(\int_0^1 Y_{i,j,t}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}.$$

Therefore, the industry aggregator has the usual CES demand for each intermediate good given by equation (142). The price aggregator also takes the usual form given by equation (143):

$$Y_{i,j,t} = Y_{j,t} \left(\frac{P_{i,j,t}}{P_{j,t}} \right)^{-\sigma}, \tag{142}$$

$$P_{j,t} = \left(\int_0^1 P_{i,j,t}^{1-\sigma} di \right)^{\frac{1}{1-\sigma}}. \tag{143}$$

Intermediate Firm Profit Maximization Firms in each industry j have an exogenous λ_j probability of updating their price each period (Calvo pricing). When they do get to change their price, firms maximize equation (144) subject to the demand for their good from the industry aggregator firm (equation (142)). Therefore, firms maximize equation (144) subject to the demand from industry aggregator firms (equation (142)):

$$\max_{P_{j,t}^\star, Y_{i,j,t}} \sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k [\frac{P_{j,t}^\star Y_{i,j,t+k}}{P_{t+k}} - MC_{t+k} Y_{i,j,t+k}]. \tag{144}$$

Taking FOCs:

$$\mathbb{E}_t[\sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k [(1-\sigma) {P_{j,t}^\star}^{-\sigma} P_{j,t+k}^\sigma \frac{1}{P_{t+k}} Y_{j,t+k} + \sigma MC_{j,t+k} {P_{j,t}^\star}^{-\sigma-1} P_{j,t+k}^\sigma Y_{j,t+k}]]. \tag{145}$$

Rearranging:

$$\mathbb{E}_t[\sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k P_{j,t+k}^\sigma Y_{j,t+k} [\frac{P_{j,t}^\star}{P_{t+k}} - \frac{\sigma}{\sigma-1} MC_{j,t+k}]]. \tag{146}$$

Inputting equation (140) and dividing by $P_{j,t}^\sigma$:

$$\mathbb{E}_t[\sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k \left(\frac{P_{j,t+k}}{P_{j,t}} \right)^\sigma \left(\frac{P_{j,t+k}}{P_{t+k}} \right)^{-\sigma_2} Y_{t+k} [\frac{P_{j,t}^\star}{P_{j,t}} \frac{P_{j,t}}{P_t} \frac{P_t}{P_{t+k}} - \frac{\sigma}{\sigma-1} MC_{j,t+k}]]. \tag{147}$$

Next, define:

$$U_{j,t} = \mathbb{E}_t[\sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k \left(\frac{P_{j,t+k}}{P_{j,t}} \right)^\sigma \left(\frac{P_{j,t+k}}{P_{t+k}} \right)^{-\sigma_2} Y_{t+k} \frac{P_t}{P_{t+k}}],$$

$$V_{j,t} = \mathbb{E}_t[\sum_{k=0}^{\infty} \left(\frac{1}{\bar{R}} \right)^k (1-\lambda_j)^k \left(\frac{P_{j,t+k}}{P_{j,t}} \right)^\sigma \left(\frac{P_{j,t+k}}{P_{t+k}} \right)^{-\sigma_2} Y_{t+k} \frac{\sigma}{\sigma-1} MC_{j,t+k}].$$

Equation (147) can then be rewritten as the first-order condition (equation (148)) plus two auxiliary equations

(equations (149) and (150)):

$$U_{j,t}\frac{P^{\star}_{j,t}}{P_{j,t}}\frac{P_{j,t}}{P_t} - V_{j,t} = 0, \tag{148}$$

$$U_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\sigma_2} Y_t + \mathbb{E}_t[\frac{1}{\bar{R}}(1-\lambda_j)\Pi^{\sigma}_{j,t+1}\Pi^{-1}_{t+1}U_{j,t+1}], \tag{149}$$

$$V_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\sigma_2} Y_t\frac{\sigma}{\sigma-1}MC_t + \mathbb{E}_t[\frac{1}{\bar{R}}(1-\lambda_j)\Pi^{\sigma}_{j,t+1}V_{j,t+1}]. \tag{150}$$

Aggregate price dispersion is defined by equation (151). The price dispersion in industry j is defined by equation (152):

$$\nu_t = \sum_{j=1}^{J} a_j \left(\frac{P_{j,t}}{P_t}\right)^{-\sigma_2} \nu_{j,t}dj, \tag{151}$$

$$\nu_{j,t} = \int_0^1 \left(\frac{P_{i,j,t}}{P_{j,t}}\right)^{-\sigma} di. \tag{152}$$

Under Calvo pricing, Equations (143) and (152) can be rewritten as equations (153) and (154), respectively. There is a relationship between inflation in an industry and the relative price in that industry that holds by definition and is shown in equation (155). And equation (141) can be rewritten as equation (156):

$$1 = \lambda_j \left(\frac{P^{\star}_{j,t}}{P_{j,t}}\right)^{1-\sigma} + (1-\lambda_j)\Pi^{\sigma-1}_{j,t}, \tag{153}$$

$$\nu_{j,t} = \lambda_j \left(\frac{P^{\star}_{j,t}}{P_{j,t}}\right)^{-\sigma} + (1-\lambda_j)\nu_{j,t-1}\Pi^{\sigma}_{j,t}, \tag{154}$$

$$\Pi_{j,t} = \frac{P_{j,t}}{P_t}\frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{j,t-1}}, \tag{155}$$

$$1 = \sum_{j=1}^{J} a_j \left(\frac{P_{j,t}}{P_t}\right)^{1-\sigma_2}. \tag{156}$$

Steady State Introducing multiple sectors impacts the steady state marginal cost only in the steady state, that is, appendix B.1. It has no impact on the demand of relative assets or the supply of relative assets given the markup. To derive the revised steady state, first note that:

$$\bar{\Pi} = \bar{\Pi}_j = \Pi^{\star}.$$

Equation (153) can be rewritten to get a steady state equation for $\overline{\frac{P^{\star}_j}{P_j}}$, which is shown in equation (157). Equation (154) can be rewritten to get a steady state equation for $\bar{\nu}_j$, which is shown in equation (158):

$$\overline{\left(\frac{P^{\star}_j}{P_j}\right)} = \left(\frac{1-\frac{1-\lambda_j}{\Pi^{\star 1-\sigma}}}{\lambda_j}\right)^{\frac{1}{1-\sigma}}, \tag{157}$$

$$\bar{\nu}_j = \frac{1}{1-(1-\lambda_j)\Pi^{\star\sigma}}\lambda_j\overline{\left(\frac{P^{\star}_j}{P_j}\right)}^{-\sigma}. \tag{158}$$

In the steady state equations (148) to (150) become, respectively:

$$\overline{\frac{P_j}{P}} = \frac{\bar{V}_j}{\bar{U}_j}\left(\overline{\frac{P_j^\star}{P_j}}\right)^{-1}, \tag{159}$$

$$\bar{U}_j = \frac{1}{1-\frac{1}{\bar{R}}(1-\lambda_j)\bar{\Pi}^{\sigma-1}}\left(\overline{\frac{P_j}{P}}\right)^{-\sigma_2}\bar{Y}, \tag{160}$$

$$\bar{V}_j = \frac{1}{1-\frac{1}{\bar{R}}(1-\lambda_j)\bar{\Pi}^{\sigma}}\left(\overline{\frac{P_j}{P}}\right)^{-\sigma_2}\bar{Y}\frac{\sigma}{\sigma-1}\overline{MC}. \tag{161}$$

Equations (160) and (161) can be inserted into equation (159) to derive equation (162):

$$\overline{\left(\frac{P_j}{P}\right)} = \frac{\sigma}{\sigma-1}\frac{1-(1-\lambda_j)\frac{1}{\bar{R}}\Pi^{\star\sigma-1}}{1-(1-\lambda_j)\frac{1}{\bar{R}}\Pi^{\star\sigma}}\overline{\left(\frac{P_j^\star}{P_j}\right)}^{-1}\overline{MC}. \tag{162}$$

Equation (141) can be rewritten as equation (163). $\overline{\left(\frac{P_j}{P}\right)}$ can then be input from equation (162) into equation (163) to get equation (164):

$$\int_0^1 a_j\overline{\left(\frac{P_j}{P}\right)}^{1-\sigma_2}dj = 1, \tag{163}$$

$$(\overline{MC})^{1-\sigma_2}\int_0^1 a_j\left[\frac{\sigma}{\sigma-1}\frac{1-(1-\lambda_j)\frac{1}{\bar{R}}\Pi^{\star\sigma-1}}{1-(1-\lambda_j)\frac{1}{\bar{R}}\Pi^{\star\sigma}}\overline{\left(\frac{P_j^\star}{P_j}\right)}^{-1}\right]^{1-\sigma_2}dj = 1. \tag{164}$$

$\overline{MC}$ can be backed out from equation (164). $\overline{\left(\frac{P_j}{P}\right)}$ can then be found from equation (162). $\bar{\nu}$ can be backed out by its definition (equation (151)).

Calibration The industry weights and frequencies of price adjustment are set to match the analysis with nine sectors in Nakamura and Steinsson (2010). The elasticity of substitution between industries (σ_2) is set to be the same as σ.

F.6 Idiosyncratic Shocks

This section explains how idiosyncratic shocks are introduced into the model.

Idiosyncratic shocks are considered only in the steady state case. As section 5 shows, in the steady state it is possible to break down the model into the demand for assets from households and the supply of relative assets from firms. Idiosyncratic shocks affect only the household side, and thus it is necessary to consider only the derivation of the demand for assets.

Asset Demand with Idiosyncratic Shocks Notation is similar to that in section 3.1. The cohort is represented by i, and time is represented by t. Cohorts have the same life cycle as in section 3.1. Since there are idiosyncratic shocks within cohorts, it is necessary to consider how individuals within a cohort will respond. For each

cohort, it is therefore assumed that there is a continuum of individuals denoted by h between 0 and 1 for each cohort i.

Agents can invest in the same assets with the same returns as in section 3.1, and to keep things simple, it is assumed there is no additional utility on holding bonds ($\xi = 0$). From section 3.1, it can also be observed that agents are indifferent to a first-order approximation (and also in the steady state) between holding different types of assets and (assuming $\xi = 0$) all assets yield the same return to a first-order approximation. Consequently, it does not matter in the steady state which type of asset agents hold, since agents are indifferent and the assets yield the same returns. Consequently, it is necessary to consider only the total savings held by a given individual h of cohort i at time t, which is represented by $S_{h,i,t}$. As in the main model, it is assumed that agents start with zero assets, so $S_{h,i,t} = 0$. Their consumption and labor supply are represented by $C_{h,i,t}$ and $L_{h,i,t}$, respectively. Agents work for a real wage W and receive a real return R on their assets. Consequently, an agent's budget constraint is given by equation (165):

$$C_{h,i,t} + S_{h,i+1,t+1} \leq WL_{h,i,t} + RS_{h,i,t}. \tag{165}$$

Agents have an exogenous supply of labor. The amount that agents work depends on whether they are employed or unemployed. If they are employed, they work for eL_i, and if they are unemployed, they work for uL_i. L_i differs by cohort. Whether agents are employed or unemployed is determined by a Markov process. e, u, and the Markov process are such that the average labor supplied across a cohort is L_i. Consequently, the total labor supplied by households is also represented by equation (33).

Agents have Epstein-Zin utility. The advantage of using this utility framework is that it allows the effects of risk aversion and income elasticity of substitution to be separated. An agent's utility is thus defined recursively. Their relative risk aversion is denoted by γ, and their intertemporal elasticity of substitution is denoted by ρ:

$$V_{i,t} = \left((1-\beta)C_{h,i,t}^{1-\rho} + \beta(\mathbb{E}_t[V_{i,t+1}^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}})\right)^{\frac{1}{1-\rho}}. \tag{166}$$

Therefore, an agent of age k faces the following problem:

$$\max_{\{C_{h,i,t+i}, S_{h,i+1,t+i+1}\}_{i=k}^{M-1}} \mathbb{E}_t[V_{i,t}], \tag{167}$$

s.t. $\forall i \in k, \ldots, M-1$

$$V_{i,t} = \left((1-\beta)C_{h,i,t}^{1-\rho} + \beta(\mathbb{E}_t[V_{i,t+1}^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}})\right)^{\frac{1}{1-\rho}},$$

$$C_{h,i,t} + S_{h,i+1,t+1} \leq WL_{h,i,t} + RS_{h,i,t}, \tag{168}$$

$$S_{M,t+M} \geq 0.$$

As in the main model, the problem is rewritten in relative terms, that is, divided by labor income. Relative assets, consumption and savings are defined in equations (61), (169) and (170), respectively:

$$c_{h,i,t} = \frac{C_{h,i,t}}{WL}, \tag{169}$$

$$s_{h,i,t} = \frac{S_{h,i,t}}{WL}. \tag{170}$$

The problem can then be rewritten as[8]:

$$\max_{\{C_{h,i,t+i},S_{h,i+1,t+i+1}\}_{i=,}^{M-1}} \mathbb{E}_t[V_{i,t}], \tag{171}$$

s.t. $\forall i \in k, \ldots, M-1$

$$\begin{aligned} V_{i,t} &= \left((1-\beta)c_{h,i,t}^{1-\rho} + \beta(\mathbb{E}_t[V_{i,t+1}^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}})\right)^{\frac{1}{1-\rho}}, \\ c_{h,i,t} + s_{h,i+1,t+1} &\leq \frac{L_{h,i,t}}{L} + Rs_{h,i,t}, \\ s_{M,t+M} &\geq 0. \end{aligned} \tag{172}$$

The problem can then be solved by iterating over value functions. The value of savings and labor income in the final period of the agent's life is given by the utility of consuming all the remaining assets of the agent:

$$V_{i,M-1} = ((1-\beta)(\frac{L_{M-1}}{L} + Rs_{M-1})^{1-\rho})^{\frac{1}{1-\rho}}.$$

Then, working backwards the value of an agent of cohort i's savings and labor income can be computed using the value of an agent of cohort $i+1$'s savings and labor income.

Observe that the distribution of $s_{h,i}$ can be computed for every cohort i given the value of R. Therefore, the relative savings is effectively a function of R. And therefore the demand for relative assets is simply the population-weighted sum of relative savings by each cohort (equation (173)):

$$a^d(R) = \left(\sum_{i=0}^{M-1} \frac{1}{(1+n)^i}\right)^{-1} \sum_{i=0}^{M-1} \frac{1}{(1+n)^i} \int_0^1 s_{h,i+1}(R)dh. \tag{173}$$

Multiple Types The case where there are two different types of households, patient and impatient, is also considered. The asset supply for the patient type (denoted by $a_p^d(R)$) and for the impatient type (denoted by $a_i^d(R)$) can each be computed separately using the same steps as when there is only one type of household facing idiosyncratic shocks. The aggregate demand for assets is then simply $a^d(R) = 0.5a_p^d(R) + 0.5a_i^d(R)$.

Parameterization L_i is set the same as it is in the exogenous labor case in section 4.5. The probability that an agent is employed in the current period if they were employed in the previous period is set as 0.95. The probability that an agent is unemployed in the current period if they were unemployed in the previous period is set as 0.5. The probability that the youngest cohort begins as unemployed is set as 9.09% , which is the steady state of the Markov system. Thus, an individual in any cohort has an unconditional 9.09% chance of being unemployed. u is set as $0.1e$, which implies that $u = 0.109$ and $e = 1.09$.

Two cases are considered. First, a case with a single type of agent with $\beta = 0.97$ is considered. The reason β is lower than in the case without idiosyncratic shocks is because this implies that agents consume less as they get older, which fits with the latter part of agents' hump-shaped consumption path. Second, a case with two types of agents is considered. Half the agents are patient, with $\beta = 0.98$, and half the agents are impatient, with $\beta = 0.94$.

[8]This has been simplified by cancelling a constant in the utility function.