A Theory of International Official Lending*

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Abstract

We examine international official lending using a repeated game model of sovereign debt in the presence of two frictions: limited enforcement and moral hazard. First, the sovereign country can renege on their debt obligation. Second, lenders cannot perfectly observe whether the sovereign uses the borrowed funds for consumption or export production. The constrained optimal allocation (COA) prescribes imperfect insurance and employs dynamic incentives in designing optimal contract. We show that the COA can be implemented as a competitive equilibrium featuring official lending and private lending. Quantitative analysis demonstrates that official debt is countercyclical, while private debt is procyclical, consistent with empirical evidence. Our finding accounts for the endogenous composition of creditors and highlights the role of international official lending in sovereign debt markets.

Keywords: International official lending; sovereign debt; default risk; moral hazard

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1 Introduction

International official lending, provided by multinational institutions such as the International Monetary Fund (IMF) and the World Bank, as well as through intergovernmental agreements, serves as an important source of external financing. The magnitutde of official lending is comparable to the private debt. Moreover, official lending plays a particularly crucial role during economic downturns. It tends to surge in times of economic distress, natural disasters, or wars. Furthermore, international official lending often steps in when private capital withdraws. [1] In recent years, official lending has resurged following the global financial crisis and the COVID-19 pandemic. Another driver of this increase is the growth of China, which provides financing to developing countries mainly in the form of official loans.[2]

A key distinction between official lending and private debt is that official lenders can extend loans to countries in arrears, meaning that sovereigns in default on private debt can still access financial support from official lenders. Institutions like the IMF and World Bank have formal policies allowing lending to countries with arrears under specific conditions, ensuring continued access to liquidity during crises.[3] Another important feature of official lending is conditionality and monitoring. Unlike private creditors, official lenders often impose policy requirements, known as policy conditionality, and conduct regular assessments of borrowing countries' macroeconomic policies and fiscal health. These measures aim to mitigate moral hazard, ensuring that funds are used productively rather than diverted for short-term consumption or inefficient projects. [4]

We develop a theoretical model to examine the role of international official lending within

[1] Horn et al. (2020) document the patterns and determinants of official international lending from 1790 to 2015, showing that official lenders often provide a stabilizing force during crises when private flows retreat.

[2] China had become the world's largest official creditor, surpassing the World Bank and the IMF as of 2017. China's share of low-income country debt has increased from 18% in 2010 to 49% in 2021, Horn et al. (2021) study the size, terms, and destination of Chinese official international lending at the loan level.

[3] See https://www.imf.org/en/About/FAQ/imf-policies-on-sovereign-arrears for the IMF policies. Arellano and Barreto (2024) finds that the official debt tends to increase during default episodes.

[4] Multilateral official lenders such as IMF often require borrowers to provide periodic financial and operational reports and may conduct visits to evaluate the progress of funded programs. Some bilateral official lenders, such as China, stringent monitoring mechanisms, sometimes requiring borrowers to set up accounts under the lender's direct oversight.

a repeated game framework of sovereign debt with production. The model incorporates two key frictions in international lending: limited enforcement and moral hazard. First, sovereign debt is subject to limited enforcement, as countries can renege on their obligation. Second, international lending is subject to information frictions, as lenders cannot directly observe how funds are used. This creates an incentive for the borrower to divert resources for consumption rather than productive investment – a moral hazard problem. Due to these frictions, official lending plays an essential role in the sovereign's debt portfolio in order to achieve constrained efficiency. Our model shares some key features with Dovis (2019), which examines the optimal design of sovereign debt contracts under limited commitment and private information. Like his framework, our model highlights the role of dynamic incentives in sustaining debt contracts. However, we incorporate moral hazard in international lending and explicitly investigate how official lending contributes to the implementation of optimal contract. We find that official lending is crucial to achieving constrained efficiency.

In the model, sovereign borrowers produce domestic consumption goods and tradable export goods using labor and intermediate goods. The economy is subject to relative productivity shocks for the export sector. The sovereign borrows intermediate inputs for production. It makes the decision about the allocation of production factors before the realization of productivity. Crucially, the productivity and allocation of resources between domestic consumption and export production are private information. We assume lenders have noisy signals about the unobservable states, but their information remains imperfect. As a result, moral hazard arises because the sovereign has an incentive to misallocate resources or divert funds, knowing that lenders cannot fully verify its decisions.

We begin by solving two planning problems: the first-best allocation and the constrained optimal allocation (COA). In the first-best scenario, the planner allocates resources to achieve efficiency. Risk-neutral lenders provide complete insurance to the risk-averse sovereign borrower against productivity shocks. The realization of a productivity shock has no effect on the allocation. However, such risk sharing is unfeasible in the presence of limited en-

forcement and moral hazard. With limited enforcement, the sovereign can deviate from the planner's prescribed allocations by increasing current consumption and going to autarky if the continuation value is low. Additionally, moral hazard arises because the sovereign has an incentive to allocate more resources towards domestic consumption at the expense of producing exports. A constrained planning problem takes these frictions into account by designing an optimal contract that is subject to incentive compatibility and participation constraints. The planner provides dynamic incentives for the sovereign while deterring deviations by imposing the threat of autarky for any detectable misallocation. We solve the constrained planning problem and characterize the constrained optimal allocation using a promise utility approach, following Kocherlakota (1996).

The solution to the constrained planner's problem is history dependent, with continuation value, consumption, and lending increasing with promised utility. Unlike Dovis (2019), the stationary distribution of the model does not support a continuation value for the sovereign close to efficiency because of the interaction between the participation constraint and the incentive compatibility constraint. As a result, the level of intermediate goods under COA is always below the level in the first-best case. Furthermore, while in Dovis (2019) autarky serves as the lower bound of the sovereign's value, in our model, the planner must always provide a continuation value strictly above the value of autarky. This is because the sovereign determines allocation before the realization of the productivity shock. Due to the liminte enforcement, if the continuation value is too low, the sovereign is tempted to deviate from the efficient allocation by overconsuming today and going autarky in the future. To prevent such an outcome, the planner must provide continuation value that is higher than the value of autarky. This result plays a crucial role in our analysis of official debt, as it underscores the need for official lending to sustain constrained efficiency. In addition, our model captures the role of monitoring or conditionality in official lending. When lenders receive a negative noisy signal, the sovereign is granted a higher promised utility despite low export production, compared to cases where the signal indicates a high productivity shock. Essentially, the

lender accommodates excusable defaults but imposes harsher penalties for repudiation that occurs despite favorable signals.[5]

We study the decentralization of the constrained optimal allocations and demonstrate that the COA can be implemented as an equilibrium outcome of a sovereign debt model with an endogenous creditors portfolio. The sovereign borrower can borrows from three types of lenders using long-term noncontingent debt. First, there are two types of official lenders, both of whom receive noisy signals about the sovereign's underlying state. Multilateral official lending is non-defaultable, reflecting the seniority of such loans. Bilateral official lending is concessional with an exogenous rate of debt reduction that the bilateral official lender may offer.[6] The third type of creditors are private lenders. The sovereign can borrow private debt, which is defaultable. The sovereign chooses to default on private debt when the value of continued repayment is lower than the value of default. However, even after defaulting on private debt, the sovereign can still access official lending. Each type of debt serves a distinct purpose: multilateral official debt provides commitment since it is nondefaultable, bilateral official debt facilitates monitoring through concessionality when a negative signal is observed, and private debt introduces some state contingency as the sovereign can default when its sustainability constraint binds, typically following a bad shock. We show that these characteristics of official and private debt make the three types of debt necessary and sufficient to decentralize the COA and ensure the best possible constrained outcome.

In the quantitative analysis, we calibrate the model to a typical emerging economy and match key statistics such as default probability, output drop in default, and share of official debt. The model generates an endogenous default cost due to the reduction in imported intermediate goods. The model can account for the negative correlation between sovereign spreads. We examine the dynamics of the economy around default episodes. When a negative shock occurs, consumption, imported intermediates, and GDP decline, and the sovereign

[5] Grossman and Huyck (1988) develops a theoretical model in which lenders distinguish excusable default and unjustifiable repudiation.

[6] Schlegl et al. (2019) find that the IMF and other multilateral creditors are the most senior. Official bilateral debt is more junior to private sovereign debt and has a lower recovery rate.

spread on private debt increases. Around default, the official multilateral debt increases, while the private debt decreases, consistent with empirical findings on the countercyclicality of official debt and the procyclicality of private debt. This pattern arises because multilateral official debt has a low interest rate, providing insurance in downturns. In contrast, the private debt provides a valuable contingency that the sovereign prefers in good state. Overall, the total debt burden increases and the portfolio shifts towards the official debt in default. Furthermore, in the model, the default decision on the private debt is endogenous. The sovereign defaults when the sustainability constraint binds. The dynamics of sovereign spreads on private debt is consistent with the data.

We also examine the economy with only one or two types of creditors. Without multilateral official lending, the COA cannot be implemented. The sovereign attains less consumption smoothing and the allocation is less efficient. A special case is when no creditors receive the noisy signals about the underlying state and the sovereign's action. In this case, the welfare frontier lies strictly within that of the baseline model. While the associated COA can still be implemented with two types of debt, it is less efficient. Our baseline model, which incorporates noisy signal, captures a key feature of international official lending: the ability of official creditors to monitor the use of funds. This enhanced monitoring allows them to provide better insurance for sovereign borrowers, improving overall welfare.

Literature

Our study contributes to a substantial body of literature on sovereign debt, including works by Eaton and Gersovitz (1981), Arellano (2008), Aguiar and Gopinath (2006), and Yue (2010), which explore incomplete markets and sovereign debt dynamics. Following this line of research, several papers incorporate official lending in the quantitative analysis of sovereign debt, such as Boz (2011), Fink and Scholl (2016), Dellas and Niepelt (2016), and Kirsch and Rühmkorf (2017). These papers study the role of non-defaulting multinational

loans. Recently, Arellano and Barreto (2024) examine the role of long-term concessional official debt. Our work is different from these papers that assume incomplete markets are incomplete and an exogenously set of assets as given. We study the best equilibrium outcome of a sovereign debt game and shed light on the role of official lending in the presence of moral hazard. In addition, we do not need to impose any exogenous default cost as in these papers. Our model is a production economy and generates endogenous default cost which is in line with Mendoza and Yue (2012).

Our model is also related to the literature using the optimal contracting approach to study sovereign debt, such as Atkeson (1991), Dovis (2019), Luo and Wang (2018), Aguiar et al. (2009).[7] In particular, this paper is related to the literature on dynamic contracting with limited enforcement and informational frictions. Dovis (2019) studies the dynamic theory of sovereign debt with a lack of commitment and private information. He proposes an implementation of the constraint optimal allocation using sovereign debt of different maturities. Muller et al. (2019) study sovereign debt and structural reforms with limited enforcement and moral hazard. The COA in their model is decentralized using renegotiable GDP-linked one-period debt. Compared to these papers, we develop a dynamic theory of sovereign debt in the presence of limited enforcement and moral hazard. We show that a portfolio of multilateral official lending, bilateral official lending, and private debt can implement the COA. Lastly, Abraham et al. (2024) show the Financial Stability Fund for a union of sovereign countries is long-term contracts that are optimal in the presence of limited enforcement and moral hazard problem. Our model share similar frictions, yet have different focuses.

Furthermore, our research is related to studies on official debt and China's lending. There has been an extensive discussion regarding the role of the IMF. Jeanne et al. (2008) provides a detailed account of the history of IMF lending and presents a theoretical model to examine the role of the IMF, the rationale for conditionality, and the conditions under which IMF-

[7] In addition, several papers in the literature builds theoretical models of sovereign debt with asymmetric information. See Sandleris (2008).

induced moral hazard can arise. Our paper shares the view of the IMF as "delegated monitor" as in Tirole (2002) and treats the official debt as part of the risk-sharing arrangement as in Chami et al. (2004). In our model, official lenders can observe a noisy signal about the sovereign borrower's state. More recent empirical studies provide a comprehensive analysis of multilateral and bilateral official debt. Schlegl et al. (2019) examines the different seniority of official debt. Horn et al. (2020) find that the official lending is countercyclical. Lastly, Horn et al. (2021) provides a detailed account of the features of the loan from China's government. Kondo et al. (2022) particularly focus on the impact of Chinese loans on private debt.

Lastly, our paper draws on the literature on endogenous incomplete markets due to limited enforcement, such as Alverez and Jermann (2000), Kehoe and Perri (2004), Albuquerque and Hopenhayn (2004). Our analysis shows that the interaction of limited enforcement and moral hazard determines the dynamic incentive provision. The COA can be decentralized using official lending. We also conduct welfare analysis to provide policy insights related to international official lending.

The remainder of the paper is organized as follows. Section 2 describes the model environment. Section 3 characterizes the first-best and constrained optimal allocation. Section 4 provides the main result of decentralization. Section 5 characterizes the set of debt including official lending that implementes the COA. Section 6 contains a quantitative analysis of the model economy. Section 7 concludes.

2 Model Environment

We now present the model environment for a small open economy. Time is discrete and indexed by $t = 0, 1, ...$ There are three types of agents in the economy: a representative risk-averse domestic household, a benevolent sovereign government, and a large number of risk-neutral foreign lenders. In addition, there are three types of goods: a domestic consumption good (non-tradable), an export good, and an intermediate good. As in Dovis

(2019), the borrower uses intermediate goods as input for the production of consumption and export goods. However, the borrower can privately allocate imported intermediate inputs to the non-tradable sector, which creates a moral hazard problem as in Atkeson (1991) and Muller et al. (2019). Foreign lenders do not directly observe the borrower's allocation or the relative productivity shock to the export sector. Instead, they receive a noisy signal about the productivity shock.

2.1 Preferences and Technology

All agents are infinitely-lived. The risk-averse domestic households value a stochastic sequence of consumption of the domestic good, $\{c_t\}_{t=0}^{\infty}$, according to

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t\right),$$

where the period utility function $U(c)$ has constant relative risk aversion, $U(c) = \dfrac{c^{1-\gamma}}{1-\gamma}$, with $\gamma > 1$.

The benevolent sovereign government makes decisions on behalf of the domestic representative household, and we will call domestic agents "the borrower" when it causes no confusion.

Risk-neutral foreign lenders value their consumption of the export good, $\{x_t\}_{t=0}^{\infty}$, according to

$$\mathbf{E}_0 \sum_{t=0}^{\infty} \beta^t x_t.$$

. We assume the borrower and lenders have the same discount factor β which can be relaxed as long as the borrower has a lower discount factor than the lender.

Foreign lenders have a large endowment of intermediate goods, which can be transformed into foreign consumption goods in a one-to-one ratio. The relative price between the export and the intermediate good is fixed at one in the international market. Each domestic house-

hold is endowed with one unit of labor. A domestic production technology transforms the intermediate good and labor into the domestic consumption good, c_t, and export good, x_t, as follows:

$$c_t \leq F\left(m_{c,t}, l_{c,t}\right), \quad \text{and} \quad x_t \leq A_t F\left(m_{x,t}, l_{x,t}\right), \tag{2.1}$$

$$m_{c,t} + m_{x,t} \leq m_t, \quad l_{c,t} + l_{x,t} \leq 1, \tag{2.2}$$

where m_t is the total intermediate goods used in domestic production, $m_{c,t}$ and $m_{x,t}$ are the units of the intermediate good allocated to the production of the consumption and export good, and $l_{c,t}$ and $l_{x,t}$ are the units of domestic labor allocated to consumption and export production, respectively.

The production function F has constant return to scale; it is increasing and twice continuously differentiable, with $F_m(m,l) > 0$ and $F_{mm}(m,l) < 0$; and it satisfies the Inada condition $\lim_{m\to 0} F_m(m,l) = +\infty, \forall l > 0$. Finally, $F(0,1) > 0$, so strictly positive output can be produced in autarky (without imported intermediate goods).[8] For notational simplicity, we denote $f(m) \equiv F(m,1)$.

The relative productivity of the export sector, A_t, is independently and identically distributed over time. As in Dovis (2019), we let A_t take only two values, $A_t \in \{A_L, A_H\}$ with $0 < A_L < A_H$.[9] Denote $p = Pr\left(A_t = A_H\right)$. We normalize the expected relative productivity $\mathbf{E}(A_t)$ to one.

Due to the constant-return-to-scale property of the production technology, (2.1)-(2.2) can be represented by the following aggregate resource constraint

$$x_t = A_t\left(f\left(m_t\right) - c_t\right), \tag{RC}$$

where c_t and x_t are nonnegative.[10] Hence, we can assume that the borrower directly chooses

[8] We further assume $F(0,1) \neq 1$ for technical convenience.

[9] As in Dovis (2019), the characterization of the efficient allocation does not depend on the two-shock assumption, and it can be extended. This assumption will play a role in the implementation, which will be discussed in Section 5

[10] The optimal allocation of the intermediate good and labor requires that $m_{c,t}/l_{c,t} = m_{x,t}/l_{x,t} =$

domestic consumption c_t, and x_t is produced according to (RC). The value of c_t is a sufficient statistic for the borrower's allocation decision.

2.2 Information and the noisy signal

The total intermediate goods m_t supplied by foreign lenders and the export x_t produced by borrower are public information. However, the productivity shock A_t is unobservable for lenders, which indicates that lenders cannot infer the borrower's domestic consumption c_t from export x_t. The borrower privately decides the allocation of inputs between the consumption and export sectors before the realization of A_t. That is, c_t is privately decided by the borrower and cannot be conditional on A_t. This creates a classic moral hazard problem: The borrower is always willing to serve domestic needs (c_t) at the expense of the export sector (x_t).[11]

Furthermore, we assume that there exists a public noisy signal $s_t \in \{L, H\}$ in each period after the realization of A_t. This signal is informative about the productivity shock A_t, with $Pr(s_t = L|A_t = A_L) = Pr(s_t = H|A_t = A_H) = a \in (0.5, 1)$, where a refers to the accuracy of the signal. Hence, according to (RC), foreign lenders can form a posterior belief about c_t using their information about m_t, x_t and s_t. The introduction of a noisy signal to provide additional information is standard in the literature, as in Holmström (1979). We apply this setting to study the monitoring role of official lenders, which will be made clear in Section 5.

It is worth mentioning two extreme cases. The first case is the uninformative case where $a = 0.5$. In this circumstance, the noisy signal does not reveal additional information on the realization of A_t, and is therefore irrelevant. The second case is the complete information case, where $a = 1$. In this circumstance, foreign lenders know A_t for sure and can therefore infer the value of c_t correctly. Hence, there will be no moral hazard problem.

m_t, leading to aggregate resource constraint above because $x_t = A_t F(m_{x,t}, l_{x,t}) = A_t l_{x,t} f(m_t) = A_t (f(m_t) - l_{c,t} f(m_t)) = A_t (f(m_t) - c_t)$

[11] Dovis (2019) is a model with private information only. There is no moral hazard because the borrower chooses the allocation after observing the productivity shock.

2.3 Timing

The timing of events within period t is as follows:

1. Foreign lenders supply intermediate goods $m_t \geq 0$.
2. The borrower chooses c_t.
3. The productivity shock A_t is realized (and x_t is realized as $A_t\left(f\left(m_t\right)-c_t\right)$).
4. The noisy signal s_t is realized.

Define $h^{-1}=\emptyset$ and $h^t=h^{t-1}\cup\left\{A_t,s_t\right\}$ for $t\geq 0$. An allocation is a stochastic process

$$\sigma\equiv\left\{m\left(h^{t-1}\right),c\left(h^{t-1}\right)\right\}_{t=0}^{+\infty},$$

which describes the dynamics of imports and domestic consumption. According to (RC), exports under the allocation are given by

$$x\left(A_t,h^{t-1}\right)=A_t\left[f\left(m\left(h^{t-1}\right)\right)-c\left(h^{t-1}\right)\right].$$

3 Efficient Allocation

In this section, we define and characterize a constrained optimal allocation as the solution to a social planner's problem, which maximizes the lenders' expected profits subject to incentive constraints. First of all, we define the continuation utility for the borrower and lenders associated with an allocation σ after h^t as

$$v\left(\sigma,h^t\right)=\sum_{j=1}^{\infty}\sum_{h^t\subseteq h^{t+j}}\beta^{j-1}Pr\left(h^{t+j}|h^t\right)U\left(c\left(h^{t+j-1}\right)\right)$$

and

$$L\left(\sigma, h^t\right) = \sum_{j=1}^{\infty} \sum_{h^t \subseteq h^{t+j}} \beta^{j-1} Pr\left(h^{t+j} | h^t\right) \left[x\left(A_{t+j}, h^{t+j-1}\right) - m\left(h^{t+j-1}\right)\right],$$

respectively. σ will be omitted whenever it causes no confusion.

3.1 Incentive Constraints

First, consider the restriction imposed by the fact that c_t is privately chosen by the borrower. If the borrower chooses to consume $c_t' = (1 - A_L/A_H) f(m_t) + A_L/A_H c_t \geq c_t$, then $x_t = A_L(f(m_t) - c_t)$ when $A_t = A_H$. Hence, when the borrower produces exports $x_t = A_L(f(m_t) - c_t)$, foreign lenders cannot distinguish whether there has been a low productivity shock or the borrower has deviated by choosing higher consumption. To incentivize the borrower to produce export goods, the promised continuation utility should be lower when x_t is low, effectively serving as a punishment for the borrower. In addition, foreign lenders can rely on the noisy signal s_t to monitor the borrower's behavior. Lastly, we assume that detectable deviations are punished by the worst equilibrium payoff v_a, as it is standard in the literature. Therefore, the associated incentive compatibility constraint for all t and h^t is:

$$\begin{aligned}
&U\left(c\left(h^{t-1}\right)\right) + \beta p\left[av\left(h^{t-1}, A_H, H\right) + (1-a)v\left(h^{t-1}, A_H, L\right)\right] \\
&+ \beta(1-p)\left[(1-a)v\left(h^{t-1}, A_L, H\right) + av\left(h^{t-1}, A_L, L\right)\right] \\
&\geq U\left(\left(1 - \frac{A_L}{A_H}\right) f\left(m\left(h^{t-1}\right)\right) + \frac{A_L}{A_H} c\left(h^{t-1}\right)\right) \\
&+ \beta p\left[av\left(h^{t-1}, A_L, H\right) + (1-a)v\left(h^{t-1}, A_L, L\right)\right] + \beta(1-p)v_a,
\end{aligned} \tag{IC}$$

where v_a is the value of permanent autarky given by

$$v_a = \frac{U(f(0))}{1-\beta}.$$

Second, the restriction imposed by the limited enforcement of lenders is captured by the sustainability constraint. The borrower can renege on the relationship, choose not to produce export goods, and stay in autarky starting from the next period. The associated sustainability constraint for all t and h^t is:

$$\begin{aligned} & U\left(c\left(h^{t-1}\right)\right)+\beta p\left[av\left(h^{t-1},A_H,H\right)+(1-a)v\left(h^{t-1},A_H,L\right)\right] \\ & +\beta(1-p)\left[(1-a)v\left(h^{t-1},A_L,H\right)+av\left(h^{t-1},A_L,L\right)\right] \\ & \geq U\left(f\left(m\left(h^{t-1}\right)\right)\right)+\beta v_a. \end{aligned} \tag{SUST}$$

Lastly, lenders can always choose not to lend. Hence, the continuation utility for lenders is no smaller than 0 since lenders can choose to leave the international financial market at any time. The associated participation constraint for all t and h^t is:

$$L\left(h^t\right)\geq 0. \tag{PC}$$

Definition 1 An allocation σ is implementable if and only if it satisfies (IC), (SUST), and (PC).

3.2 Recursive formulation

Define $\Sigma=\{\sigma|\sigma \text{ is an implementable allocation}\}$.[12] Abusing notation, we denote $L(\sigma)$ and $v(\sigma)$ as the continuation utility of lenders and the borrower evaluated at h^{-1}, respectively. Let $\overline{v}=sup\{v(\sigma)|\sigma\in\Sigma\}$.[13] As in Muller et al. (2019) and Dovis (2019), we focus on the constrained optimal allocation (COA) which is implementable and maximizes the continuation utility of lenders given the continuation utility of the borrower. A COA is therefore a solution to the following program, where $v\in[v_a,\overline{v}]$ is the period 0 utility promised to the

[12] Σ is nonempty as the autarkic allocation with $m\left(h^{t-1}\right)=0$, $c\left(h^{t-1}\right)=f(0)$ and $x\left(A_t,h^{t-1}\right)=0$ is always implementable.

[13] We assume that $\overline{v}>v_a$ to avoid trivial discussions.

borrower:

$$L(v) = \max_{\sigma \in \Sigma} L(\sigma), \quad s.t. \quad v(\sigma) = v. \tag{L}$$

Following Abreu et al. (1990) and Kocherlakota (1996), the problem in (L) admits a recursive formulation with the borrower's promised utility, v, as a state variable. Let $v^{i,j}$ $(i, j \in \{L, H\})$ represent the promised continuation utility to the borrower, where i refers to the realization of the productivity shock A_t, and j refers to the realization of the noisy signal s_t. A COA solves the following recursive problem for $v \in [v_a, \overline{v}]$:

$$\begin{aligned} L(v) = \max_{m,c,(v^{i,j})_{i,j=L,H}} & f(m) - c - m + \beta p \left[aL\left(v^{H,H}\right) + (1-a)L\left(v^{H,L}\right) \right] \\ & + \beta(1-p)\left[(1-a)L\left(v^{L,H}\right) + aL\left(v^{L,L}\right) \right], \end{aligned} \tag{3.1}$$

subject to

$$\begin{aligned} v = U(c) + \beta p \left[a v^{H,H} + (1-a) v^{H,L} \right] \\ + \beta(1-p)\left[(1-a) v^{L,H} + a v^{L,L} \right], \end{aligned} \tag{3.2}$$

$$v \geq U\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c \right) + \beta p \left[a v^{L,H} + (1-a) v^{L,L} \right] + \beta(1-p) v_a, \tag{3.3}$$

$$v \geq U(f(m)) + \beta v_a, \tag{3.4}$$

$$L(v) \geq 0 \quad \forall v \in [v_a, \overline{v}], \tag{3.5}$$

$$v^{i,j} \in [v_a, \overline{v}]. \tag{3.6}$$

Equation (3.2) is the constraint that the recursive allocation delivers a value of v to the borrower (the promise-keeping constraint). Inequalities (3.3), (3.4) and (3.5) are recursive versions of the incentive compatibility constraint (IC), the sustainability constraint (SUST) and the participation constraint (PC), respectively. The promised continuation utility must be generated by an implementable allocation, which gives (3.6). The function $L(v)$ is the largest continuation utility that lenders can achieve given that the continuation utility of the borrower is v. Hence, $L(v)$ trace the frontier of utility possibilities.

The solution to the above recursive problem is $\left\{m(v), c(v), (v^{i,j}(v))_{i,j=L,H}\right\}$.

3.3 First-best allocation

We start by characterizing the first-best allocation given by the program (3.1)-(3.2). The result is summarized in the following proposition.[14]

Proposition 1 (First-Best Allocation) Under (3.2), the solution to (3.1) satisfies:

(i) (*Production efficiency*) $m_{FB}(v) = m^*$, where m^* is given by $f'(m^*) = 1$.

(ii) (*Perfect insurance*) $v^{i,j}_{FB}(v) = v$ $(i, j = L, H)$.

The first-best allocation yields production efficiency and perfect insurance: the supply of intermediate goods m reaches the efficient level m^*; the allocation is stationary and the borrower enjoys constant consumption irrespective of the history of productivity shocks. The economy achieves efficiency, as no distortion is needed for incentive provision.

Introducing the sustainability constraint (3.4) will not change the main implications of Proposition 1. As in Aguiar and Amador (2011), the supply of intermediate goods reaches efficiency at least in the long run, and the consumption path does not depend on the realization of productivity shocks. This highlights the importance of moral hazard and the incentive compatibility constraint (3.3) in generating distortions in allocations.

3.4 Constrained optimal allocation (COA)

Next, we characterize the COA. The planning problem in (3.1)-(3.2) is subject to the IC (3.3), the SUST (3.4), the PC (3.5) and the feasibility constraint (3.6). To establish our results, we first make the following assumptions:

Assumption 1 $\dfrac{f'(m) - 1}{U'(f(m))f'(m)}$ decreases in m.

[14] The proof (and all the following proofs) can be found in the online Appendix.

Assumption 2 p is sufficiently large and a is sufficiently below 1.

Assumption 1 is to ensure that the problem is convex so that randomization is not optimal[15]. Assumption 2 is to ensure that the incentive compatibility constraint is always binding, so that our results can differ from the first-best allocation. When p is large or a is small, it is less likely that a deviation is detected, and the borrower is more willing to deviate. We will show that Assumption 2 is satisfied for realistic parameters in Section 6.

The COA satisfies the following properties:

Proposition 2 (Constrained Optimal Allocation) Given Assumption 1-2, under (3.2)-(3.6), the solution to (3.1) satisfies:

(i) (*Production inefficiency*) $m(v) < m^*$.

(ii) (*Imperfect insurance*) $v^{H,H}(v) = v^{H,L}(v) > v^{L,L}(v) > v^{L,H}(v) > v_a$.

Different from Dovis (2019), Proposition 2 suggests that COA is characterized by persistent production inefficiency and imperfect insurance. First, the supply of intermediate goods m never reaches the efficient level m^* from the first-best case. Compared to Dovis (2019), the borrower is more willing to deviate if m is large. In addition, dynamic incentives are required to overcome the moral hazard problem, which leads to endogenous fluctuations in imports and domestic consumption. Hence, the COA fails to achieve perfect risk sharing. Furthermore, Part (ii) highlights the dynamic nature of the COA: high exports lead to higher continuation utility, encouraging more imports and consumption, while low exports reduce future utility and lead to harsher consequences. If low exports are "excusable" and likely due to external shocks (indicated by a signal $s_t = L$), lenders are more lenient, allowing better risk-sharing. However, if the signal $s_t = H$, stricter penalties are imposed to deter undesir-

[15] Dovis (2019) uses the same assumption.

able behavior. Furthermore, $v^{i,j}(v) > v_a$ regardless of v. As the borrower can compensate the lenders by lowering c, there is no need for autarky, which is socially costly. As a result, COA avoids complete autarky, even temporarily, justifying "rescue lending" during crises when borrowers face exclusion from markets. This feature of the COA is a key reason why the implementation involves the official lending as shown in the following sections.

To clarify the role of the noisy signal, two extreme scenarios are considered. With complete information ($a = 1$), all deviations are detected and penalized, eliminating the moral hazard problem and allowing efficient outcomes. The first-best allocation is achieved. On the other hand, with uninformative signals ($a = 0.5$), decisions cannot depend on the signal, $v^{L,H} = v^{L,L}$. The uninformative case is Pareto-inferior to the COA. This aligns with the principle of informativeness, which states that the incorporation of informative signals into the optimal contract improves the welfare (Holmström, 1979).

The above results are summarized in the following figures (Figure 3.1). The figure in the upper left traces the frontier of utility possibilities $L(v)$. Because $L(v)$ decreases with v, there is no sequential inefficiency. The figure in the upper right shows the policy function for intermediate imports, $m(v)$. The figure on the bottom left shows the policy function for domestic consumption, $c(v)$. Both imports and domestic consumption are increasing functions of the promised utility v. The figure on the bottom right depicts the law of motion for the borrower's value, $v^{i,j}(v)$. As Proposition 2 indicates, the future utility for the borrower increases with today's value. Higher exports are associated with higher future utility. When exports are low, the future utility also depends on the signal. Finally, as shown in all the figures, the lower bound for the borrower's utility is strictly above v_a. The lower bound is determined by the fixed point of $v^{L,H}(v)$.

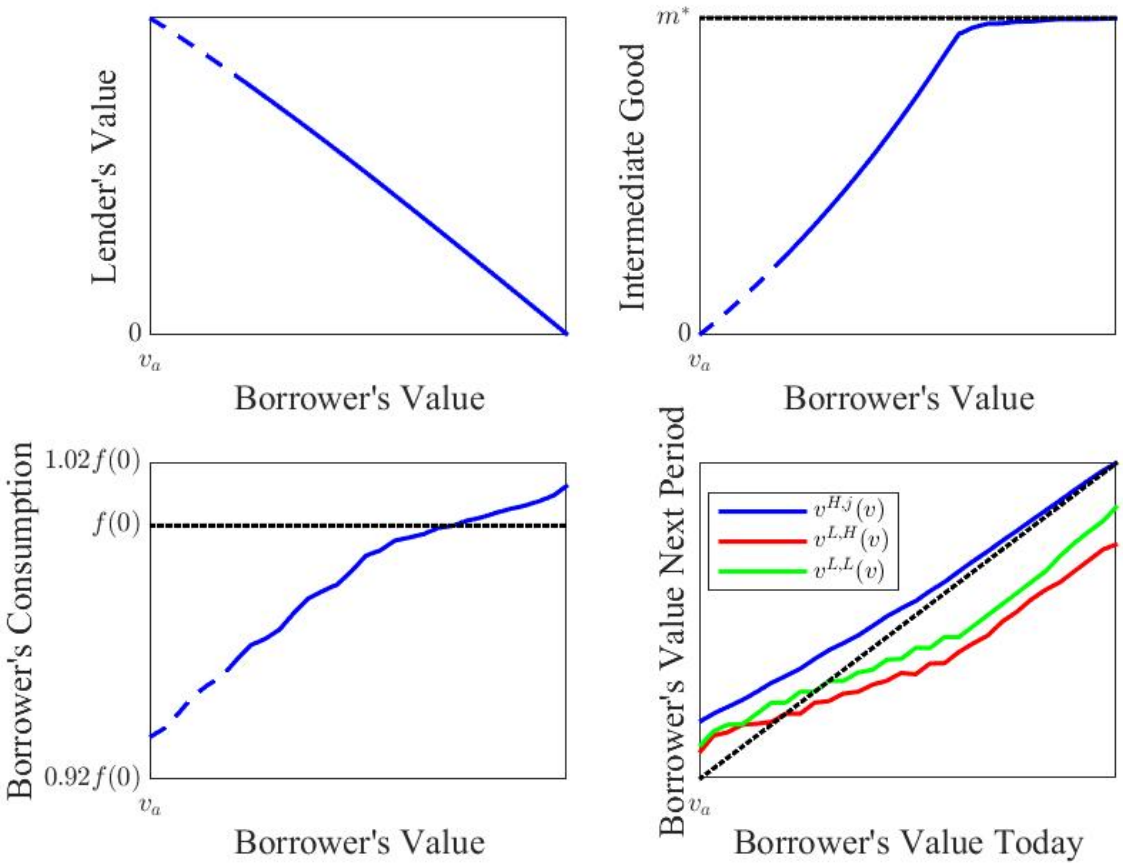


Figure 3.1: The dynamics of the COA.

4 Decentralization

In this section, we describe how the COA can be decentralized in a sustainable market equilibrium. Consider a game between competitive foreign lenders (creditors and exporters of the intermediate good), competitive domestic firms, and a benevolent domestic government. In equilibrium, the benevolent domestic government borrows internationally.

We consider the set of securities that the sovereign borrower can issue, distinguishing between official debt and private debt, each with its typical characteristics. We take these characteristics, along with a set of rules governing the treatment of the sovereign in default, as given features of the environment. Given these rules, we derive the implications for the evolution of debt and bond prices that support the efficient allocation.[16]

4.1 Sovereign Debt Contracts

First, the government borrows from two types of official lenders (multilateral official lenders and bilateral official lenders) and private lenders. The key distinction between official debt and private debt is that private debt is subject to the government's default risk, whereas official debt is non-defaultable (though official lenders can choose to provide debt reductions).[17] Denote different types of debt as d_t^i, their prices as q_t^i, and the repayment decisions as X_t^i ($i \in \{OM, OB, M\}$, where M refers to private lenders or market lenders). The government actively manages its debt portfolio to maximize domestic welfare, taking the price of debt as given. The government can also default on its private debt obligations, and the interest spread reflects the probability of default.

Private lenders The government can issue long-term private debt in period t that promises an infinite and decaying stream of coupons at a constant rate $\delta^M \in [0, 1]$. In particular, the coupon payment on the private debt issued at t is $\left(1 - \delta^M\right)^{k-1}$ units of the

[16]The fact that the constrained efficient allocation can be implemented under these rules suggests that they do not impose restrictive limitations on the outcomes.

[17]It is important to note that the classification of official lenders into multilateral and bilateral categories is not essential. What matters is that some financing remains available even when the sovereign defaults on private debt and that some creditors can observe the noisy signal.

export good in period $t+k$, with $k \geq 1$, following Hatchondo and Martinez (2009). δ^M is the parameter that determines the duration of the long-term private debt. The borrower can default on the private debt. By defaulting, the government does not pay the coupon due in that period.[18] The price of the private debt reflects the probability of future default. Private lenders are assumed to be competitive and risk-neutral. Therefore, the price of the private debt, q_t^M, is given by

$$q_t^M = \beta \mathbf{E}_t \left(X_{t+1}^M + \left(1 - \delta^M\right) q_{t+1}^M \right), \tag{PR-M}$$

where $X_{t+1}^M = 0$ if the borrower defaults at period $t+1$, and $X_{t+1}^M = 1$ if the borrower does not default. It is important to note that it is the borrower who makes the default decision X_t^M, consistent with the "willingness to pay" problem in the study of sovereign debt on private debt.

Official lenders There are two types of official lenders: official multilateral lenders and official bilateral lenders. Similar to private debt, official debts are long-term debts with duration $\delta^{OM} \in [0,1]$ for official multilateral debt and duration $\delta^{OB} \in [0,1]$ for official bilateral debt. The liquidity provided by official multilateral lenders is risk-free: the borrower simply cannot default on official multilateral debt. Hence, $X_t^{OM} = 1$, and the pricing rule of the official multilateral debt is given by

$$q_t^{OM} = \beta \mathbf{E}_t \left(X_{t+1}^{OM} + \left(1 - \delta^{OM}\right) q_{t+1}^{OM} \right) = \frac{\beta}{1 - (1 - \delta^{OM})\,\beta}. \tag{PR-OM}$$

This price is constant and hence denoted as q^{OM}. On the other hand, official bilateral lenders offer a debt reduction of $1 - r_{OB}$ ($r_{OB} \in [0,1)$) to the borrower if the low export is "excusable". That is, a debt reduction is granted if the noisy signal $s_t = L$ and the export x_t are lower than expected. Hence, $X_t^{OB} = r_{OB}$ if low export is "excusable", and $X_t^{OB} = 1$

[18] This assumption is without loss of generality. The present value of the market debt is lower in default which implies a debt reduction.

otherwise. The pricing rule of the official bilateral debt is given by

$$q_t^{OB} = \beta \mathbf{E}_t \left(X_{t+1}^{OB} \left[1 + \left(1 - \delta^{OB} \right) q_{t+1}^{OB} \right] \right) = \frac{\beta \left[1 - (1-p) a (1 - r_{OB}) \right]}{1 - (1 - \delta^{OB}) \beta \left[1 - (1-p) a (1 - r_{OB}) \right]}. \tag{PR-OB}$$

This price is also time invariant. We denote it by q^{OB} whenever it causes no confusion.

4.2 Government and domestic market

In addition to issuing debt, the government collects lump-sum tax on labor to achieve the desired allocation of inputs between the consumption and export sectors.[19] In addition, the government also purchases intermediate inputs from the international market and sells to competitive domestic firms.

Domestic competitive firms purchase intermediate inputs from the government and hire labor to produce consumption and export goods. The profit-maximizing problems for firms in the consumption and export sectors are

$$\max_{m_{c,t}, l_{c,t}} p_{c,t} F \left(m_{c,t}, l_{c,t} \right) - p_{i,t} m_{c,t} - w_t l_{c,t}, \tag{4.1}$$

and

$$\max_{m_{x,t}, l_{x,t}} A_t F \left(m_{x,t}, l_{x,t} \right) - p_{i,t} m_{x,t} - w_t l_{x,t}, \tag{4.2}$$

where $p_{c,t}$, $p_{i,t}$, and w_t are the relative prices of consumption good, intermediate good, and the competitive wage for labor, respectively. Export good is assumed to be the numeraire.

As there is no saving decision in our setting, the market clear condition for consumption is

$$w_t - p_{c,t} \tau_t = p_{c,t} F \left(m_{c,t}, l_{c,t} \right), \tag{4.3}$$

where τ_t is the lump-sum tax on labor.

[19] We assume that the tax cannot be observed by foreign lenders. Thus there still exists an information asymmetry between lenders and the borrower.

Denote the budget revenue of the government as B_t, and we have

$$B_t = p_{i,t} m_t + \tau_t. \tag{4.4}$$

As domestic household does not consume export goods, the market clear condition for export goods is

$$B_t = A_t F\left(m_{x,t}, l_{x,t}\right). \tag{4.5}$$

Given the taxes, the consumption of the representative household is $c_t = \left(w_t - p_{c,t}\tau_t\right)/p_{c,t}$, and the export is $x_t = A_t F\left(m_{x,t}, l_{x,t}\right)$. Finally, markets for intermediate inputs and labor clear.

$$m_{c,t} + m_{x,t} = m_t, \quad l_{c,t} + l_{x,t} = 1. \tag{4.6}$$

Based on (4.1)-(4.6), we can solve for the relative prices and the output of each sector. The results are summarized in the following lemma.

Lemma 1 (Domestic Market)

(i) Prices: $p_{c,t} = 1$, $p_{i,t} = f'\left(m_t\right)$, and $w_t = f\left(m_t\right) - m_t f'\left(m_t\right)$.

(ii) Domestic consumption: $c_t = f\left(m_t\right) - m_t f'\left(m_t\right) - \tau_t$.

(iv) Exports and budget revenue: $x_t = B_t = A_t\left(f\left(m_t\right) - c_t\right)$.

Lemma 1 states how the government's policy affects economic activities. By setting a lump-sum tax τ_t on labor, the government can direct the allocation of intermediate inputs and labor. More specifically, given m_t, c_t and x_t are uniquely pinned down by τ_t.

The sequence of events within each period t is as follows:

1. The government imports intermediate goods, m_t, and sells them to domestic firms at the price $p_{i,t}$.

2. The government sets the lump-sum tax rate τ_t.

3. The productivity shock A_t is realized and the production of c_t and x_t occurs.

4. The noisy signal s_t is realized.

5. The government chooses the default decision X_t^M, borrowing decisions d_{t+1}^i, and next period's imports m_{t+1}.

Given the government's import and tax policy, the allocation of intermediate goods and labor between the consumption and export sectors is determined by the competitive market.

The government is benevolent and maximizes the utility of the representative household. We assume that the imports of intermediate goods are subject to the working capital constraint. To import m_{t+1} in period $t+1$, the government is required to make the payment βm_{t+1} at period t.[20] According to Lemma 1, the government's revenue equals $A_t\left(f\left(m_t\right)-c_t\right)$, and the budget constraint of the government is given by

$$\begin{aligned} & A_t\left(f\left(m_t\right)-c_t\right)+q_t^M\left[d_{t+1}^M-\left(1-\delta^M\right)d_t^M\right] \\ +&q_t^{OB}\left[d_{t+1}^{OB}-\left(1-\delta^{OB}\right)X_t^{OB}d_t^{OB}\right]+q_t^M\left[d_{t+1}^{OM}-\left(1-\delta^{OM}\right)d_t^{OM}\right] \\ =&\sum_{i\in\{M,OB,OM\}} X_t^i d_t^i+\beta m_{t+1}. \end{aligned} \tag{BC}$$

4.3 The sustainable equilibrium

The government's decision on default, borrowing and import is summarized as the policy $\pi_t \equiv \left(X_t^M,\left(d_{t+1}^i\right)_{i\in\{M,OB,OM\}},m_{t+1}\right)$. The government also decides the tax, τ_t.

Let $h_p^{-1}=\left(\left(d_0^i\right)_{i\in\{M,OB,OM\}},m_0\right)$ be the initial outstanding debt and imports. The public history up to period $t\geq 0$ is $h_p^t=\left(h_p^{t-1},x_t,s_t,\pi_t\right)$. We further define the history when the government decides its policy π_t as $h_s^t=\left(h_p^{t-1},x_t,s_t\right)$. The strategy for the government and the debt prices are all functions of the relevant histories.[21]

[20] Consistent with Mendoza and Yue (2012), the interest rate on the working capital credit is risk free rate.

[21] Specifically, the tax decision, τ_t, is a function of public history h_p^{t-1}; the default, borrowing and import decision, π_t, is a function of the history h_s^t, because π_t is made after the realization of production x_t and the noisy signal s_t.

The problem for the government is to maximize the utility of the representative household, taken as given the price schedule of sovereign debts, $(q_t^i)_{i\in\{M,OB,OM\}}$, after any history h_p^t. The strategy for the government solves the government's problem

$$W\left(h_p^{t-1}\right) = \max_{\tau_t,\pi_t(x_t,s_t)} U\left(c_t\right) + \beta \mathbf{E}\left[W\left(h_p^t\right)|h_p^{t-1},\pi_t\left(x_t,s_t\right)\right] \tag{G}$$

subject to $d_{t+1}^i < \overline{D}$ and the budget constraint for the government (BC). Borrowings are bounded by a large positive constant $\overline{D}$ to rule out a Ponzi scheme. Domestic consumption c_t is a function of m_t and τ_t based on Lemma 1.

Now we can define the sustainable equilibrium. Given initial outstanding debt and imports, a sustainable equilibrium is a strategy for the government, $\left(\tau_t\left(h_p^{t-1}\right),\pi_t\left(h_s^t\right)\right)_{t=0}^{\infty}$ and prices for the government debt, $(q_t^i)_{i\in\{M,OB,OM\}}$, such that (i) given debt prices $(q_t^i)_{i\in\{M,OB,OM\}}$, the government's strategy is the optimal policy associated with (G) for all histories h_p^{t-1} and h_s^t; (ii) given the government's strategy, the debt prices $(q_t^i)_{i\in\{M,OB,OM\}}$ satisfy (PR-M), (PR-OB), and (PR-OM).

The associated equilibrium result is denoted by $\mathbf{y} = (\sigma, \mathbf{g}, \mathbf{q})$, where σ is the equilibrium allocation of m, c, and x, $\mathbf{g}$ is the government policy τ and π, and $\mathbf{q}$ is the bond prices. Given the government policy $\mathbf{g}$, the equilibrium allocation σ is a stochastic process in terms of h^t, as defined in Section 2.4.

We characterize the sustainable equilibrium outcome $\mathbf{y}$ in the following lemma.

Lemma 2 (Sustainable Equilibrium) Given the initial history h_p^{-1}, $\mathbf{y}$ is a sustainable equilibrium outcome if and only if:

(i) The associated equilibrium allocation σ is an implementable allocation.

(ii) The associated government policy g satisfies the budget constraint (BC).

(iii) The debt prices q satisfy (PR-M), (PR-OB), and (PR-OM).

There may exist multiple equilibrium. We focus only on the efficient equilibrium. Lemma 2 allows for the characterization of the efficient equilibrium based on the results in Section 3: Lemma 2 shows that the associated allocation of a sustainable equilibrium is an implementable allocation with the budget constraint of the government. If the budget constraint does not impose an additional constraint on the problem, the COA can be achieved under efficient sustainable equilibrium. We will show next that the structure of the international financial market and the set of debt instruments available is crucial to achieve the constrained efficiency because the government's budget relies heavily on external financing opportunities.

5 Implementation with Official and Private Debt

In this section, we will show how the combination of private lenders and official lenders helps to implement the COA under a sustainable equilibrium. We construct a sustainable equilibrium that replicates the COA, using the continuation utility of the government, v, as the state variable. As the COA is implementable, Lemma 2 suggests that we only need to verify whether or not the government budget constraint is satisfied.

5.1 Dynamics of domestic economy

We first construct the allocation that determines the dynamics of domestic consumption and the allocation of imported intermediate goods. First, we need to determine the continuation utility. Given the continuation utility today, v, the continuation utility of the next period, v', is given by

$$v' = \begin{cases} v^{H,H}(v), & A = A_H, s = H; \\ v^{H,L}(v), & A = A_H, s = L; \\ v^{L,H}(v), & A = A_L, s = H; \\ v^{L,L}(v), & A = A_L, s = L. \end{cases} \tag{5.1}$$

Now let the imports of the government be given by $m(v)$. Set the tax of the government as

$$\tau(v) = f\left(m(v)\right) - m(v)f'\left(m(v)\right) - c(v). \tag{5.2}$$

Under this import and tax policy, it is easy to verify that today's domestic consumption is given by $c(v)$. Hence, this allocation is the COA.

5.2 Dynamics of debt and default

We now characterize the dynamics of debt and default. We have shown in Section 4.4 that the government's default decision X^M is chosen after the shock A and the signal s are realized. Therefore, X^M is a function of the continuation utility of the next period, v'. We denote it as $X^M\left(v'\right)$. We assume that the government chooses to default, $X^M(v') = 0$, if and only if $v' \leq v^*$, where v^* is defined as

$$v^* = \sup\left\{\hat{v} \mid v = U(f(m(v))) + \beta v_a, \forall v \in [v_a, \hat{v}]\right\}.$$

That is, the government chooses to default if the (SUST) constraint is binding in the next period:

$$v' = U(f(m(v'))) + \beta v_a.$$

This default setting can be rationalized by the following protocol: if the government defaults today, there is a positive probability that the government is excluded from the international financial market tomorrow. Hence, the government will default only when v' is sufficiently small and the (SUST) is binding. Therefore, default is more likely to occur after the realization of a negative productivity shock since $v' = v^{L,j}(v)$ is lower.

Given the default decision of the government, the price for private debt is derived from (PR-M):

$$q^M(v) = \beta \mathbf{E}\left(X^M(v') + \left(1 - \delta^M\right) q^M\left(v'\right)\right). \tag{5.3}$$

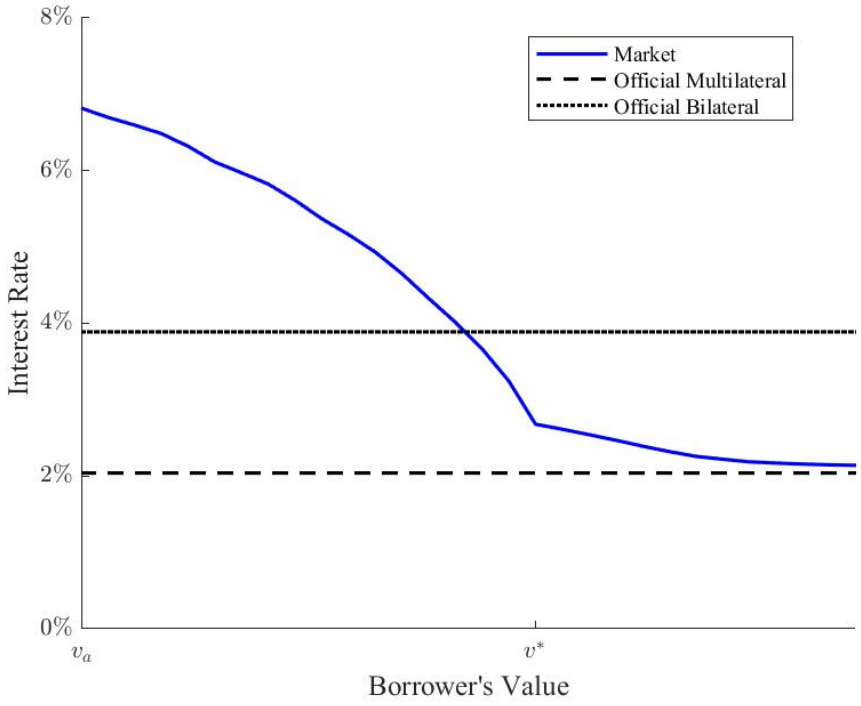


Figure 5.2: Interest Rate.

Hence, the price of the debt reflects the probability of default. We have the following lemma regarding the debt price.

Lemma 3 If v' increases in v for all $v \in [v_a, \overline{v}]$, then for any continuation utility $v \in [v_a, \overline{v}]$, price of private debt $q^M(v) < \beta / \left[1 - \left(1 - \delta^M\right)\beta\right]$ and is increasing in v.

This lemma suggests that the spread of private debt increases after a negative shock due to the higher probabilities of future default, and the interest rate is larger than the risk-free rate due to positive default probability. Figure 5.2 shows the interest rates of different debts. Figure 5.2 illustrates the interest rate on the private debt, in comparison with the interest rates of official multilateral and official bilateral debt.

To ensure that the government's policy satisfies the budget constraint (BC), we have the following lemma.

Lemma 4 (Budget Constraint) The budget constraint of the government (BC) is satisfied if and only if the government's debt holding $d^i(v)$ ($i \in \{M, OB, OM\}$) is such that:

$$\begin{aligned}&\left[X_t^M + \left(1-\delta^M\right) q_t^M\right] d_t^M + X_t^{OB}\left[1+\left(1-\delta^{OB}\right) q^{OB}\right] d_t^{OB} + \left[X_t^{OM} + \left(1-\delta^{OM}\right) q^{OM}\right] d_t^{OM} \\ =&\mathbf{E}_t\left[\sum_{j=0} \beta^j \left(A_{t+j}\left(f(m_{t+j}) - c_{t+j}\right) - m_{t+j}\right)\right] + m_t, \quad \forall t.\end{aligned}$$

The RHS of the above equation is the total expected profit of lenders plus m_t. From Section 3, we know that the total expected profit can be written as $L(v)$, which is a function of the promised utility of the borrower.

Applying Lemma 4, we can determine the debt holdings, $d^i(v)$ ($i \in \{M, OB, OM\}$), of the government.

$$\begin{aligned}&\left[X^M\left(v^{H,H}(v)\right) + \left(1-\delta^M\right) q^M\left(v^{H,H}(v)\right)\right] d^M(v) + D^{OB}(v) + D^{OM}(v) \\ =&A_H\left[f\left(m(v)\right) - c(v)\right] + \beta L\left(v^{H,H}(v)\right); \\ &\left[X^M\left(v^{L,H}(v)\right) + \left(1-\delta^M\right) q^M\left(v^{L,H}(v)\right)\right] d^M(v) + D^{OB}(v) + D^{OM}(v) \\ =&A_L\left[f\left(m(v)\right) - c(v)\right] + \beta L\left(v^{L,H}(v)\right); \\ &\left[X^M\left(v^{L,L}(v)\right) + \left(1-\delta^M\right) q^M\left(v^{L,L}(v)\right)\right] d^M(v) + r_{OB} D^{OB}(v) + D^{OM}(v) \\ =&A_L\left[f\left(m(v)\right) - c(v)\right] + \beta L\left(v^{L,L}(v)\right),\end{aligned} \tag{5.4}$$

where $D^{OB}(v) = \left[1+\left(1-\delta^{OB}\right) q^{OB}\right] d^{OB}(v)$ and $D^{OM}(v) = \left[1+\left(1-\delta^{OM}\right) q^{OM}\right] d^{OM}(v)$. In sum, we have the following lemma regarding debt holdings of the government.

Lemma 5 (Debt Holdings) There exists a debt portfolio $d^i(v)$ ($i \in \{M, OB, OM\}$) that satisfies (5.4).

We have now constructed a sustainable equilibrium, summarized in Proposition 3.

Proposition 3 The strategy constructed by (5.1)-(5.4) constitutes a sustainable equilibrium. The associated equilibrium allocation is the COA.

Proposition 3 shows that the combination of private lenders and official lenders in the international financial market helps implement the COA. We now discuss in detail how the implementation works.

Under a negative productivity shock, risk-neutral lenders should bear some losses, facilitating risk-sharing. The decline in the price of the private debt after the negative shock imposes a capital loss on the lenders and a capital gain on the borrower. The borrower's gain is based on the fact that the borrower is more likely to default after a negative productivity shock. This defaultability introduces state contingency, allowing partial risk-sharing between borrowers and lenders. However, due to the moral hazard problem, restricting the borrower's consumption is necessary. Borrowing from official multilateral lenders provides a credible commitment mechanism, as these debts are non-defaultable, making the borrower's budget more restrictive.

Additionally, official bilateral debt plays a crucial monitoring role, with repayments conditional on the received signal. In sum, all three types of debt are essential for implementing the COA: private debt enables risk-sharing, official multilateral debt enforces commitment, and official bilateral debt ensures monitoring. Their combined presence allows the borrower greater flexibility in managing its budget while maintaining financial discipline, achieving constrained optimal outcome.

We summarize the above discussion in the following corollary.

Corollary 1. Without any of the three types of debts, the COA is not an equilibrium allocation.

6 Quantitative Anlysis

We next take the model to the data to assess its empirical relevance. Table 6.1 shows the calibration of parameters. The risk-aversion coefficient $\gamma = 2$ and the discount factor $\beta = 0.98$ as in the standard RBC literature. The production function $F(m, l)$ takes a standard CES form $F(m, l) = \left[\alpha m^{\rho} + (1-\alpha) l^{\rho}\right] / \alpha$, implying a statically efficient level of intermediate goods of $m^* = 1$. Mendoza and Yue (2012) find that during default, intermediate good imports drop by 20% and GDP falls about 5% below trend. To match these empirical patterns, ρ is set to 0.974 to match the contraction of imports and α is set to 0.26 to match the contraction of domestic production in default. p is set to 0.945 to match the default probability of private debt at 3%, and A_L is set to 0.7 to match the volatility of real exchange rate. The recovery rate for bilateral official lending, r_{OB}, is set to 0.4 to match the empirical literature on official lending (Schlegl et al., 2019). Schlegl et al. (2019) documents that the share of official bilateral debt is around 17% on average. We set the precision of the noisy signal, a, to 0.55 to match the share of official bilateral lending. Finally, the duration of private debt, δ^M, is set to 0.045 as in Hatchondo and Martinez (2009). The duration of official debts can take any values without affecting any results in our model.

Calibrated parameters	Value	Source
Risk aversion γ	2	
Discount factor β	0.98	
Production function para ρ	0.974	Drop of intermediate inputs
Production function para α	0.26	Output drop
Productivity p	0.945	Default probability
Productivity A_L	0.7	Real depreciation rate
Recovery rate r_{OB}	0.4	Schlegl et al. (2019)
Signal precision a	0.55	Share of official bilateral debt
Duration δ^M	0.045	Hatchondo and Martinez (2009)

Table 6.1: Calibrated parameters.

We compute first and second moments based on the above parameters. The results are summarized in Table 6.2. The output standard deviation is 6.1% as typical for emering

economies. It should be noted that the volatility of consumption is 0.5% and there is positive correlation between net export and output. This indicates a high degree of risk sharing, though imperfect, between lenders and the borrower under the COA. As shown in the subsequent dynamics of sovereign debt, the access to official lending is a main reason for the sovereign country to smooth consumption even in default. The interest rate on private debt is negatively correlated with the output, consistent with the data. Furthermore, the average duration of a default period is 3.8 years, which arises in the model endogenously. Lastly, we examine the composition of the debt portfolio. In particular, the share of multilateral official debt and private debt are non-target statistics and are in line with the data.

Model Statistics	
Output Std. dev.	6.1%
Consumption Std. dev.	0.5%
Interest Rate Mean	2.2%
Corr(NX,Y)	97.8%
Corr(R,Y)	-15.3%
Default Statistics	
Default Probability	3.0%
Default Duration	3.8 years
Debt Statistics	
Total Debt to GDP Ratio	29.1%
Private Debt to GDP Ratio	11.4%
Official Multilateral Debt to GDP Ratio	12.4%
Official Bilateral Debt to GDP Ratio	5.3%

Table 6.2: Model statistics.

We compute the equilibrium and examine the impulse responses of imports, domestic output, domestic consumption, debt prices, and debt holdings around default. The impulse responses, summarized in Figure 6.3 and Figure 6.4, illustrate the average of simulated dynamics around defaults on private debt. Typically, the country defaults after a series of negative productivity shocks, and imports, output, and consumption all decline during default.

Debt prices drop during default due to an increased probability of future default. As mar-

ket financing becomes prohibitively expensive, borrowing from official multilateral lenders that offer cheaper and concessional financing becomes more attractive. This substitution toward official debt aligns with findings in the literature (Boz, 2011; Kirsch and Rühmkorf, 2017; Cordella and Powell, 2021). Consequently, our model replicates the stylized fact that private debt is procyclical while official debt is countercyclical, as shown in Figure 6.4. This result is consistent with empirical studies on official lending (Horn et al., 2020; Arellano and Barreto, 2024), which identify countercyclicality as a key feature of official lending.

Moreover, because repayment obligations on non-defaultable official debt require additional resources, the availability of intermediate goods imports is constrained, effectively functioning as an endogenous fiscal rule. At the same time, bilateral official lending remains essentially acyclical in the model.[22] Finally, the total debt-to-GDP ratio rises during default.

We also examine the impacts of the noisy signal on the real variables. Under a negative productivity shock, the responses of the economy depend on the signals, as shown in Figure 6.5 and Figure 6.6. The realization of an unfavorable signal ($s = H$) suggests that the borrower could deviate and negatively affect the economy more. The increase in the debt-to-GDP ratio and the share of official multilateral lending is more prominent under $s = H$.

We also examine the impact of the noisy signal on real variables. Following a negative productivity shock, the economy's response depends on the signal received, as shown in Figures 6.5 and 6.6. When an unfavorable signal ($s = H$) is realized, it indicates a higher likelihood of borrower deviation, exacerbating the economic downturn. Under $s = H$, the bilateral official lending reduces slightly more. Yet the increase in the debt-to-GDP ratio and the share of official multilateral lending is more pronounced, reflecting the heightened need for external support in adverse conditions.

Overall, our model generates an endogenous default cost of 5% of outputs and accounts for the cyclicality of private lending and official lending, which is in line with the empirical

[22] Horn et al. (2020) provides evidence that bilateral official lending increases during economic crises and natural disasters. Using exclusively bilateral loan data, Avellán and Lotti (2024) finds that bilateral official loans are acyclical relative to the borrower's business cycle.

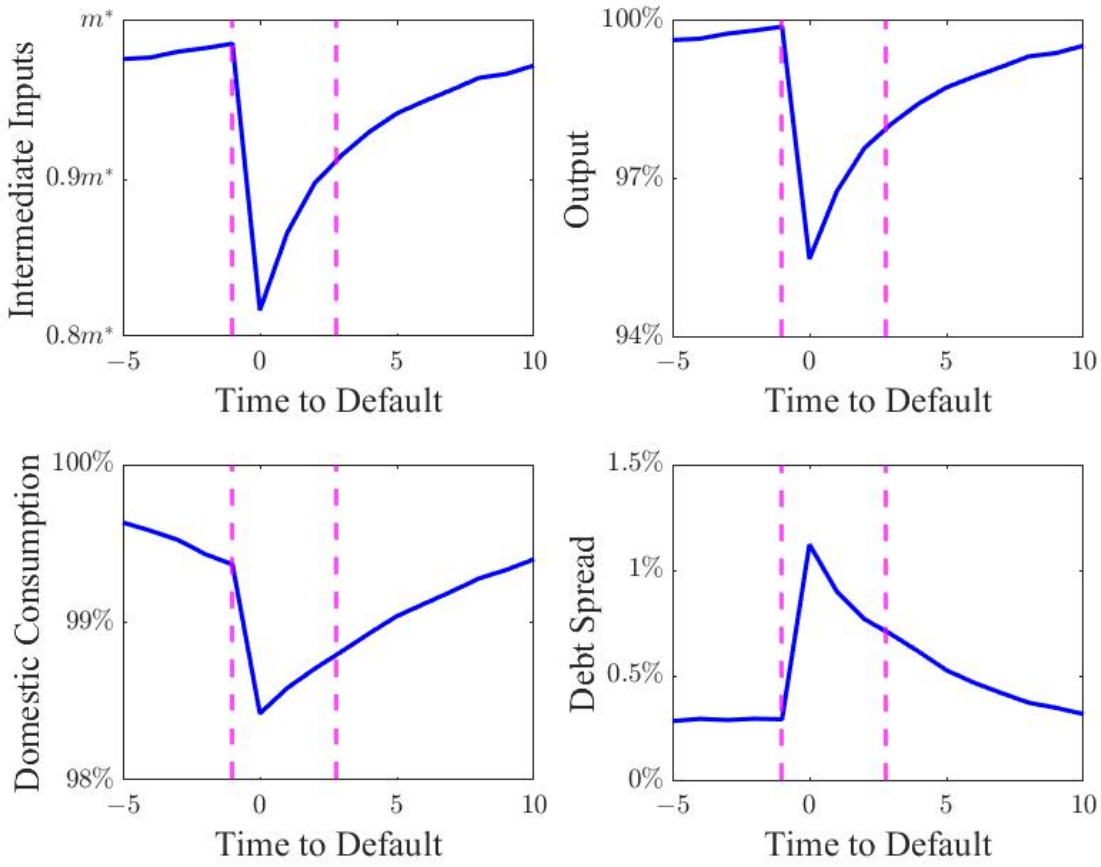


Figure 6.3: Impulse responses of imports, domestic output $f(m)$, domestic consumption, and the debt price around default. The pink lines represent the start and end of the default.

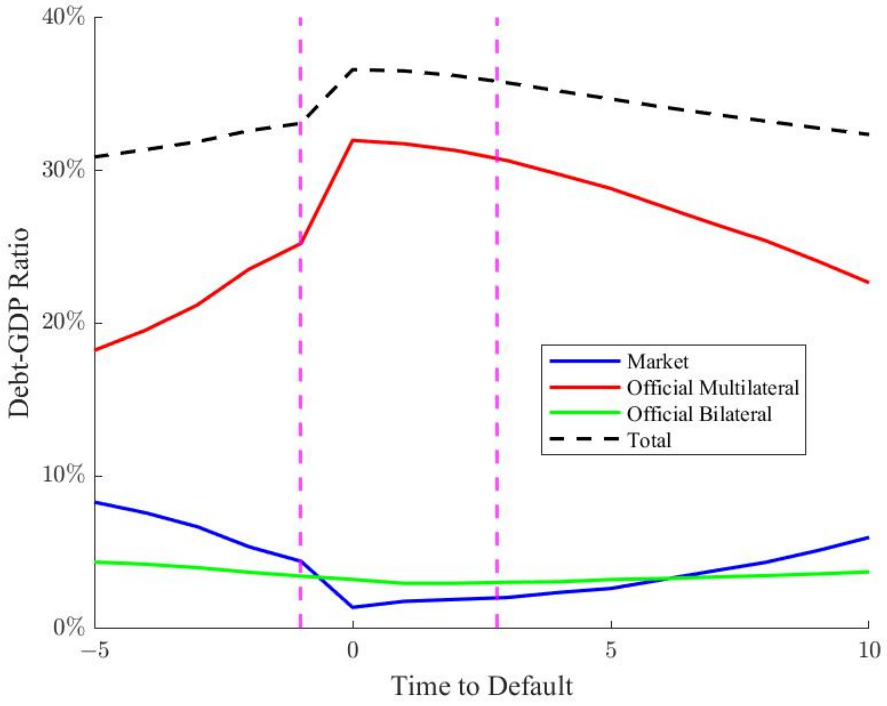


Figure 6.4: Impulse response of debt holdings around default. The pink lines represent the start and end of the default.

literature. We argue that the haircut of official bilateral lending depends on the realization of the noisy signal, which serves the role of monitoring the sovereign borrower. On the other hand, non-defaultable official multilateral lending imposes fiscal rule on the sovereign borrower to mitigate the moral hazard problem. The reason why the borrower switches to official lending is that private lending becomes more costly during default. The combination of private lending and official lending helps implement the COA.

7 Conclusion

The importance of official lending for developing and emerging economies, along with China's growing role, highlights the need to understand the dynamics of international official lending. This paper studies a repeated game of international lending under moral hazard. It illustrates the role of official lenders as liquidity providers and as monitors. We show that the existence

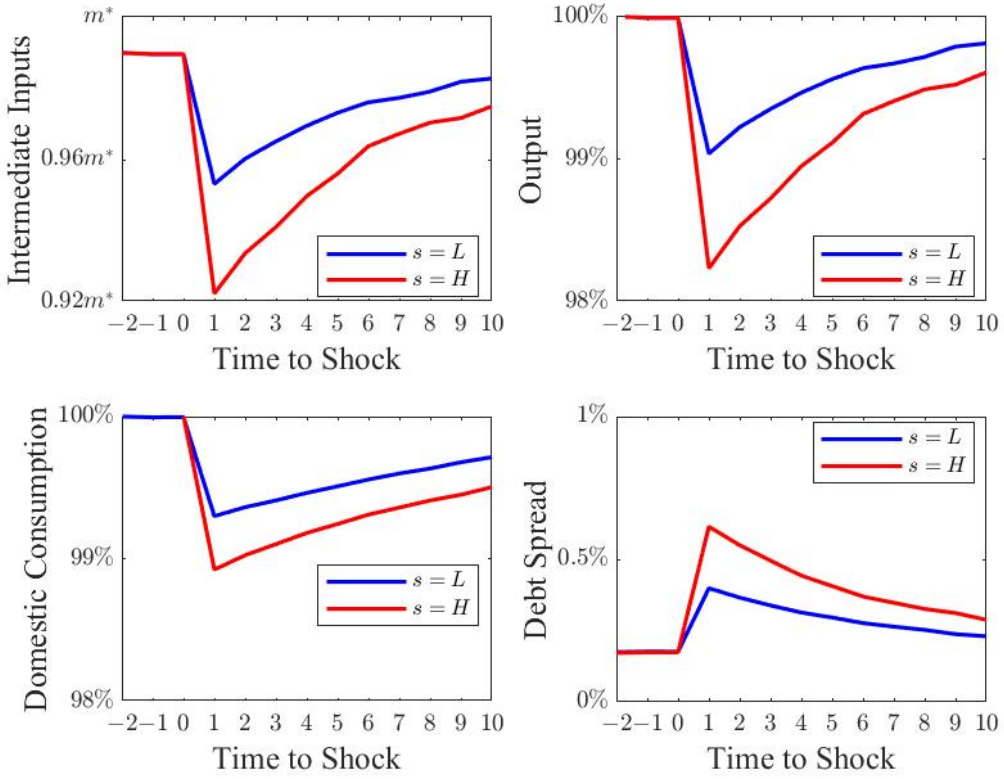
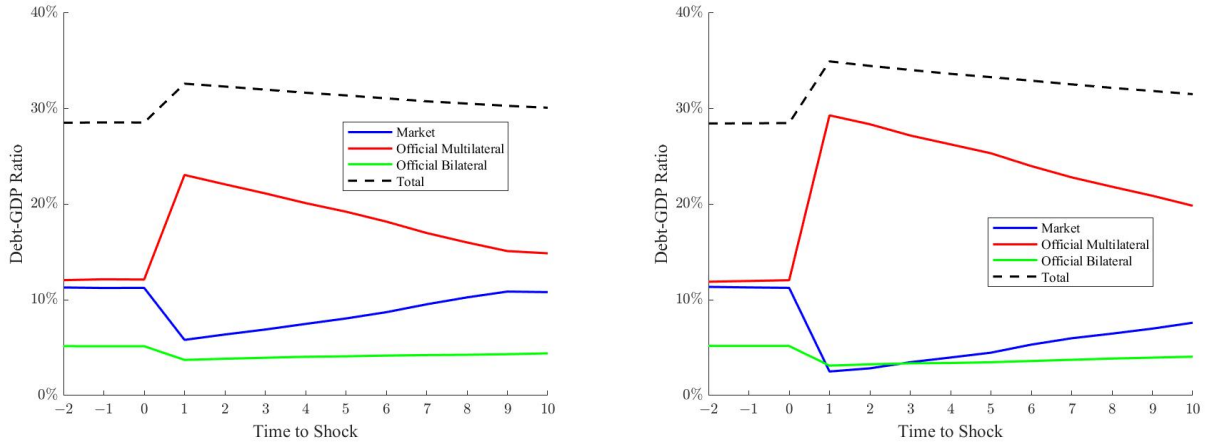


Figure 6.5: Impulse responses of imports, domestic output $f(m)$, domestic consumption, and the debt price around a negative productivity shock.



(a) (b)

Figure 6.6: Impulse responses of debt to GDP ratio under different realizations of the noisy signal around a negative productivity shock. Panel (a) is the case where $s = L$, whereas Panel (b) is the case where $s = H$.

of a moral hazard problem on the part of the sovereign borrower constrains international lending, preventing it from achieving efficiency. The constrained optimal allocation can be implemented through a combination of private lenders, official bilateral lenders, and official multilateral lenders. Both types of official lenders act as monitors in the presence of informational friction. Official bilateral lenders mitigate the moral hazard problem by tailoring punishment based on the signal received and accept a haircut only when the default is excusable. Official multilateral lenders provide cheap credit and serve as liquidity providers for the sovereign borrower. Moreover, since official multilateral lending is non-defaultable, borrowing from official multilateral lenders restricts the borrower's future consumption, thus serving as an endogenous fiscal rule during bad times. The model accounts for countercyclical lending from the official multilateral creditors and replicates the aggregate dynamics around default including endogenous reduction in output. This theory of international official lending offers new insight into the role of official lenders and the dynamics of sovereign debt.

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A Theory of International Official Lending
Online Appendix

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1 Proposition 1

The problem is:

$$L(v) = \max_{m,c,(v^{i,j})_{i,j=L,H}} f(m) - c - m + \beta p \left[aL\left(v^{H,H}\right) + (1-a)L\left(v^{H,L}\right)\right] + \beta(1-p)\left[(1-a)L\left(v^{L,H}\right) + aL\left(v^{L,L}\right)\right], \tag{1.1}$$

subject to

$$v = U(c) + \beta p \left[av^{H,H} + (1-a)v^{H,L}\right] + \beta(1-p)\left[(1-a)v^{L,H} + av^{L,L}\right]. \tag{1.2}$$

We first prove that L is strictly concave. Instead of choosing $(m, c, v^{i,j})$, consider choosing $(m, U(c), v^{i,j})$. With this formulation, the constraint set is linear in the choice variables. Moreover, f and $-U^{-1}$ are concave functions. Hence, the objective function is strictly concave, and the constraint set is convex. By standard arguments (Stokey and Lucas Jr, 1989), L is strictly concave.

We then prove that $v^{i,j}$ is constant for all $i, j \in \{L, H\}$. This is obvious since the constraint is linear in $v^{i,j}$ and L is strictly concave. We denote this constant as v'.

Now, the problem becomes:

$$L(v) = \max_{m,c,v'} f(m) - m - c + \beta L\left(v'\right), \quad s.t.$$

$$v = U(c) + \beta v'.$$

We now prove that L is differentiable. For $v_0 \in (v_a, \overline{v})$, define $(\hat{m}(v), \hat{c}(v), \hat{v}'(v))$ on $v \in (v_0 - \epsilon, v_0 + \epsilon)$ as follows:

$$\hat{m}(v) = m\left(v_0\right),$$

$$\hat{c}(v) = U^{-1}\left(v - \beta v'\left(v_0\right)\right),$$

$$\hat{v}'(v) = v'\left(v_0\right).$$

The corresponding $\hat{L}$ is given as follows:

$$\hat{L}(v) = f\left(\hat{m}(v)\right) - \hat{m}(v) - \hat{c}(v) + \beta L\left(\hat{v}'(v)\right).$$

As $\hat{L}(v) \leq L(v)$, $\hat{L}\left(v_0\right) = L\left(v_0\right)$, and $\hat{L}(v)$ is differentiable and concave, we know that L is differentiable at v_0 by Benveniste and Scheinkman theorem. Hence, L is differentiable.

Now we can derive the first order conditions for the above problem. Suppose the Lagrange multiplier for the constraint (1.2) is λ. Hence, the first order conditions are given by:

$$m: \quad f'(m) - 1 = 0;$$

$$c: \quad U'(c) = -1/\lambda;$$

$$v': \quad L'\left(v'\right) - \lambda = 0.$$

By the envelop theorem, we have $L'(v) = \lambda$. Therefore, $L'(v) = \lambda = L'\left(v'\right)$. Since L is strictly concave, we know that $v' = v$.

Therefore, the solution to the above problem satisfies:

1. $m_{FB}(v) = m^*$.
2. $v = U(c) + \beta v' = U(c) + \beta v \implies U\left(c_{FB}(v)\right) = (1-\beta)v$.
3. $v_{FB}^{i,j}(v) = v' = v$.

Q.E.D.

2 Proposition 2

The problem is:

$$L(v) = \max_{m,c,(v^{i,j})_{i,j=L,H}} f(m) - c - m + \beta p \left[aL\left(v^{H,H}\right) + (1-a)L\left(v^{H,L}\right)\right] + \beta(1-p)\left[(1-a)L\left(v^{L,H}\right) + aL\left(v^{L,L}\right)\right], \tag{2.1}$$

subject to

$$v = U(c) + \beta p \left[av^{H,H} + (1-a)v^{H,L}\right] + \beta(1-p)\left[(1-a)v^{L,H} + av^{L,L}\right], \tag{2.2}$$

$$v \geq U\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right) + \beta p \left[av^{L,H} + (1-a)v^{L,L}\right] + \beta(1-p)v_a, \tag{2.3}$$

$$v \geq U(f(m)) + \beta v_a, \tag{2.4}$$

$$L(v) \geq 0 \quad \forall v \in [v_a, \overline{v}], \tag{2.5}$$

$$v^{i,j} \in [v_a, \overline{v}]. \tag{2.6}$$

We first show that L is strictly concave and differentiable. The proof is similar to that of Proposition 1. Instead of choosing $(m, c, v^{i,j})$, consider choosing $(U(f(m)), U(c), v^{i,j})$. As $-U^{-1}$ and $U^{-1} - (U \circ f)^{-1}$ are concave functions according to Assumption 1, the objective function is strictly concave in the choice variables. On the other hand, all constraints except for (2.3) are linear in the choice variables. Regarding (2.3), we know that

$$\begin{aligned}
&U\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right) \\
&= \frac{\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right)^{1-\gamma}}{1-\gamma} \\
&= \frac{\left[\left(1 - \frac{A_L}{A_H}\right) ((1-\gamma)U(f(m)))^{1/(1-\gamma)} + \frac{A_L}{A_H}((1-\gamma)U(c))^{1/(1-\gamma)}\right]^{1-\gamma}}{1-\gamma} \\
&= \left[\left(1 - \frac{A_L}{A_H}\right) U(f(m))^{1/(1-\gamma)} + \frac{A_L}{A_H} U(c)^{1/(1-\gamma)}\right]^{1-\gamma}.
\end{aligned}$$

This function is convex in $(U(f(m)), U(c))$ as $\gamma > 1$. Together, we know that the constraint set is convex. Hence, by standard arguments (Stokey and Lucas Jr, 1989), L is strictly concave. Differentiability of L can be established by applying the Benveniste and Scheinkman theorem, as in Proposition 1.

We then show that $L(\overline{v}) = 0$ and $L(v) \geq 0$ $(\forall v \in [v_a, \overline{v}])$. Suppose by contradiction that $L(\overline{v}) > 0$. Under this circumstance, consider the perturbation with $c = c(\overline{v}) + \epsilon$ and other variables unchanged. Since

$$U'(c) - \frac{A_L}{A_H} U'\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right) > 0,$$

the IC constraint is still satisfied. Hence, we have constructed an equilibrium with the borrower's utility larger than $\overline{v}$, a contradiction. Hence, $L(\overline{v}) = 0$. As $L(v_a) \geq 0$ and $L(v)$ is strictly concave, we can conclude that $L(v) \geq 0$ for $v \in (v_a, \overline{v})$. In this sense, we can ignore the participation constraint (2.5) as long as $v \in [v_a, \overline{v}]$.

Now we can derive the first order conditions. Denote the Lagrangian multipliers for (2.2), (2.3), and (2.4) as λ_{pk}, λ_{ic}, and λ_{sust}, respectively. The first order conditions are given by

$$m: \quad f'(m) - 1 - \lambda_{ic}\left(1 - \frac{A_L}{A_H}\right) U'\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right) f'(m) - \lambda_{sust} U'(f(m)) f'(m) = 0;$$

$$c: \quad -1 - \lambda_{pk} U'(c) - \lambda_{ic} \frac{A_L}{A_H} U'\left(\left(1 - \frac{A_L}{A_H}\right) f(m) + \frac{A_L}{A_H} c\right) = 0;$$

$$v^{H,H}: \quad L'\left(v^{H,H}\right) - \lambda_{pk} = 0;$$

$$v^{H,L}: \quad L'\left(v^{H,L}\right) - \lambda_{pk} = 0;$$

$$v^{L,H}: \quad L'\left(v^{L,H}\right) - \lambda_{pk} - \frac{a}{1-a}\frac{p}{1-p}\lambda_{ic} = 0;$$

$$v^{L,L}: \quad L'\left(v^{L,L}\right) - \lambda_{pk} - \frac{1-a}{a}\frac{p}{1-p}\lambda_{ic} = 0.$$

Moreover, the envelope theorem implies

$$L'(v) = \lambda_{pk} + \lambda_{ic} + \lambda_{sust}.$$

In Assumption 2, we assume that p is sufficiently large. To be more specific, we assume

$$p > \max\left\{ \frac{\frac{U\left(f\left(m^*\right)-m^*\right)}{1-\beta} - U\left(f\left(m^*\right) - \frac{A_L}{A_H}m^*\right) - \beta v_a}{\beta\left[\frac{U\left(f\left(m^*\right)-m^*\right)}{1-\beta} - v_a\right]}, \frac{A_L}{A_H}, \frac{1}{2} \right\}.$$

The RHS is the lower bound for p. It is obvious that the RHS is smaller than 1 so that there exists such a p.

We first show that $m(v) < m^*$ for any $v \in [v_a, \overline{v}]$. If $\lambda_{ic} > 0$ or $\lambda_{sust} > 0$, $f'(m)$ should be larger than 1 and m is smaller than m^* due to the concavity of f. Hence, we only need to rule out $\lambda_{ic} = \lambda_{sust} = 0$. Suppose by contradiction that $\lambda_{ic} = \lambda_{sust} = 0$, then the problem is reduced to the first best case. According to Proposition 1, the problem is stationary (since $v^{i,j}(v) = v$) with $m(v) = m^*$. Hence, the profit of lenders $L(v)$ is given by

$$L(v) = \frac{f\left(m^*\right) - m^* - c(v)}{1-\beta}.$$

The participation constraint $L(v) \geq 0$ implies that

$$c(v) \leq f\left(m^*\right) - m^*.$$

By assumption, $\lambda_{ic} = 0$, which implies that the IC constraint is not binding:

$$\frac{U(c(v))}{1-\beta} \geq U\left(\left(1 - \frac{A_L}{A_H}\right) f\left(m^*\right) + \frac{A_L}{A_H}c(v)\right) + \beta p \frac{U(c(v))}{1-\beta} + \beta(1-p)v_a.$$

We know that

$$\frac{1-\beta p}{1-\beta}U'(c) - \frac{A_L}{A_H}U'\left(\left(1-\frac{A_L}{A_H}\right)f\left(m^*\right)+\frac{A_L}{A_H}c\right) > 0.$$

Hence, in order that the IC constraint is not binding under v, it must be that the IC constraint is not binding when $c(v) = f\left(m^*\right) - m^*$. This implies that

$$\frac{U\left(f\left(m^*\right)-m^*\right)}{1-\beta} \geq U\left(f\left(m^*\right)-\frac{A_L}{A_H}m^*\right)+\beta p\frac{U\left(f\left(m^*\right)-m^*\right)}{1-\beta}+\beta(1-p)v_a,$$

a contradiction to the assumption that p is sufficiently large. In sum, $m(v) < m^*$.

Moreover, we can prove that $\lambda_{ic} > 0$ for all v, which suggests that the IC constraint is always binding. Suppose by contradiction that $\lambda_{ic} = 0$. Therefore, according to the first order conditions, $v^{H,H} = v^{H,L} = v^{L,H} = v^{L,L}$, and we denote it as v'. We have

$$v = U(c) + \beta v'.$$

We have already proved that it is not possible that $\lambda_{ic} = \lambda_{sust} = 0$. Hence, $\lambda_{sust} > 0$, and we have

$$v = U(f(m)) + \beta v_a.$$

Hence,

$$\begin{aligned}
&v - \beta p v' - \beta(1-p)v_a \\
=&p(v-\beta v') + (1-p)(v-\beta v_a) \\
=&pU(c) + (1-p)U(f(m)) \\
\leq&U((1-p)f(m)+pc)
\end{aligned}$$

Since $p > A_L/A_H$, we have

$$\begin{aligned} v &\leq U((1-p)f(m)+pc)+\beta pv'+\beta(1-p)v_a \\ &\leq U\left(\left(1-\frac{A_L}{A_H}\right)f(m)+\frac{A_L}{A_H}c\right)+\beta pv'+\beta(1-p)v_a, \end{aligned}$$

a contradiction to the assumption that the IC constraint is not binding. In sum, $\lambda_{ic} > 0$.

We then show that $v^{H,H} = v^{H,L} > v^{L,L} > v^{L,H} > v_a$. We focus on the first order conditions. It is obvious that $v^{H,H} = v^{H,L}$. Since $\lambda_{ic} > 0$ and $a > 0.5$, $v^{L,L} > v^{L,H}$ and $v^{H,L} > v^{L,L}$. Now we only need to prove $v^{L,H} > v_a$. As the system is continuous in a and we assume that a is sufficiently below 1 in Assumption 2, we can focus on the case where $a \to 0.5$. When $a \to 0.5$, we know that $v^{L,L} \to v^{L,H}$ from the first order conditions. Suppose by contradiction that $v^{L,H} = v_a$. If $c < f(m)$, then

$$v \geq U(f(m))+\beta v_a > U\left(\left(1-\frac{A_L}{A_H}\right)f(m)+\frac{A_L}{A_H}c\right)+\beta v_a.$$

This suggests that the IC constraint (2.3) is not binding, a contradiction. Now we consider $c = f(m)$. Since $\lambda_{ic} > 0$, $v^{H,H} = v^{H,L} > v^{L,L} > v^{L,H} \geq v_a$. Hence, (2.3) and (2.4) are not binding, indicating $\lambda_{ic} = \lambda_{sust} = 0$, a contradiction. In sum, $v^{H,H} = v^{H,L} > v^{L,L} > v^{L,H} > v_a$.

Now we can prove that $m(v_a) = 0$, $c(v) \in (0, f(m(v)))$, and $c(v)$ is not a constant. Consider $m(v_a)$. According to (2.4),

$$v_a \geq U(f(m(v_a)))+\beta v_a.$$

Hence, $m(v_a) = 0$. Consider $c(v)$. It is obvious that $c(v) > 0$. If $c(v) = f(m(v))$, the IC constraint is not binding given that $v^{H,H} = v^{H,L} > v^{L,L} > v^{L,H} > v_a$, a contradiction. Hence, $c(v) \in (0, f(m(v)))$. We now prove that $c(v)$ is not a constant. Suppose by contradiction that $c(v) = c^*$ for all v. Then,

$$v = \frac{U(c^*)}{1-\beta}.$$

This is a contradiction since v can take multiple values given that $\overline{v} > v_a$.

Lastly, we show that $v^* > v_a$. Consider the relaxed optimization problem without the SUST constraint (2.4). Under this circumstance, $\lambda_{sust} = 0$. According to the first order conditions, we know that

$$\frac{f'(m)-1}{U'(f(m))f'(m)} = \lambda_{ic}\left(1-\frac{A_L}{A_H}\right)\left(\frac{f(m)}{(1-A_L/A_H)\,f(m)+A_L/A_H c}\right)^{\gamma}.$$

According to Assumption 1, the LHS of the above equation decreases in m. Moreover, $\lim_{m\to 0} f'(m) = +\infty$. Hence, as long as $f(0) \neq 1$, we have

$$\lim_{m\to 0}\frac{f'(m)-1}{U'(f(m))f'(m)} = +\infty.$$

However, the RHS of the above equation is bounded above as λ_{ic} is bounded above. Hence, there exists a $m^{**} > 0$ such that the solution to the relaxed optimization problem $m \geq m^{**}$. Now we let $\hat{v} = U\left(f\left(m^{**}\right)\right) + \beta v_a > v_a$. For $v \in [v_a, \hat{v}]$, the SUST constraint must bind. Otherwise, $m(v) \geq m^{**}$ and the SUST constraint is violated. Therefore, $v^* > v_a$. For $v \in [v_a, v^*]$, as the SUST constraint is binding, we have

$$m(v) = (U \circ f)^{-1}\left(v - \beta v_a\right),$$

which is strictly increasing in v. Q.E.D.

3 Lemma 1

First, $p_{c,t} = 1$. Otherwise, if $p_{c,t} > 1$, there will be no export production; if $p_{c,t} < 1$, there will be no domestic production.

Hence, the profit maximizing problems of the firms imply:

$$F_m\left(m_{c,t}, l_{c,t}\right) - p_{i,t} = 0,$$

$$F_l\left(m_{c,t}, l_{c,t}\right) - w_t = 0,$$

$$F_m\left(m_{x,t}, l_{x,t}\right) - p_{i,t} = 0,$$

$$F_l\left(m_{x,t}, l_{x,t}\right) - w_t = 0.$$

Therefore, $m_{c,t}/l_{c,t} = m_{x,t}/l_{x,t} = m_t$, which leads to

$$p_{i,t} = f'\left(m_t\right), \quad w_t = f\left(m_t\right) - m_t f'\left(m_t\right).$$

As the household consume all his revenue, we have

$$c_t = \frac{w_t - p_{c,t}\tau_t}{p_{c,t}} = f\left(m_t\right) - m_t f'\left(m_t\right) - \tau_t.$$

As $m_{c,t}/l_{c,t} = m_{x,t}/l_{x,t} = m_t$, we have

$$x_t = A_t F\left(m_{x,t}, l_{x,t}\right) = A_t l_{x,t} f\left(m_t\right) = A_t\left[f\left(m_t\right) - l_{c,t} f\left(m_t\right)\right] = A_t\left(f\left(m_t\right) - c_t\right).$$

Since the household does not consume export goods, the government's budget revenue $B_t = x_t$. Q.E.D.

4 Lemma 2

The proof here is the same as Dovis (2019). It is obvious that (ii) and (iii) must hold for any sustainable equilibrium. Hence, we focus our attention on (i).

It is evident that (i) is a necessary condition for the equilibrium to be sustainable. Constraints (IC) and (SUST) ensure that the borrower's government has no strict incentive to engage in a detectable and undetectable deviation, respectively. This is in line with the borrower's government's optimization.

We now prove that (i) is also sufficient. Consider an outcome that satisfies the conditions

in the statement. Such outcome can be implemented as sustainable equilibrium by relying on trigger strategies that revert to the worst equilibrium after a detectable deviation from the government. We argue that the government has no incentive to deviate from the prescribed path of plays. Note the detectable deviations are not profitable, because the outcome satisfies (SUST). Moreover, undetectable deviations are not profitable, because the outcome satisfies (IC). Then, the government does not have a strict incentive to deviate from the proposed path of plays, concluding the proof.

5 Lemma 3

$X^M(v, A, s)$ can be written as $X^M\left(v'(v)\right)$ since $v'(v)$ is uniquely determined by v, A, and s. Given the government's default protocol, $X^M\left(v'\right)$ increases in $v'(v)$ and therefore in v since $v'(v)$ increases in v as assumed in Assumption 4. Let Q be the space of bounded functions $q^M : [v_a, \overline{v}] \to \left[0, \beta/\left(1 - \left(1 - \delta^M\right)\beta\right)\right]$, and let $T : Q \to Q$ be defined as:

$$(Tq^M)(v) = \beta \mathbf{E}\left(X^M(v'(v)) + \left(1 - \delta^M\right) q^M\left(v'(v)\right)\right).$$

T satisfies the Blackwell's sufficient condition for a contraction mapping. Hence, by the contraction mapping theorem, there exists a unique fixed point of T, q^M. Moreover, q^M is increasing since T maps increasing functions into increasing functions.

Now we need to show that $q^M(v)$ is smaller than the risk-free rate. For $v > v^*$, as the SUST constraint is not binding, we know that

$$L'(v) = \lambda_{pk} + \lambda_{ic} < \lambda_{pk} + \frac{a}{1-a}\frac{p}{1-p}\lambda_{ic} = L'\left(v^{L,H}\right).$$

Hence, $v^{L,H}(v) < v$ for $v > v^*$. In this sense, starting from v, the probability that the continuation utility of the borrower is no larger than v^* is positive. Therefore, the probability that the borrower defaults is positive, indicating $q^M(v)$ is smaller than the risk-free rate.

6 Lemma 4

According to the budget constraint of the government, we know that

$$\begin{aligned}
&\left(X_t^M + (1-\delta^M)q_t^M\right) d_t^M + X_t^{OB}\left(1 + (1-\delta^{OB})q_t^{OB}\right) d_t^{OB} + \left(X_t^{OM} + (1-\delta^{OM})q_t^{OM}\right) d_t^{OM} \\
=&A_t\left(f\left(m_t\right) - c_t\right) - \beta m_{t+1} + q_t^M d_{t+1}^M + q_t^{OB} d_{t+1}^{OB} + q_t^{OM} d_{t+1}^{OM}.
\end{aligned}$$

We know that

$$\begin{aligned}
&q_t^M d_{t+1}^M + q_t^{OB} d_{t+1}^{OB} + q_t^{OM} d_{t+1}^{OM} \\
=&\beta \mathbf{E}_t\left(X_{t+1}^M + (1-\delta^M)q_{t+1}^M\right) d_{t+1}^M + \beta \mathbf{E}_t\left(X_{t+1}^{OB}\left[1 + \left(1-\delta^{OB}\right) q_{t+1}^{OB}\right]\right) d_{t+1}^{OB} + \beta \mathbf{E}_t\left(X_{t+1}^{OM} + \left(1-\delta^{OM}\right) q_{t+1}^{OM}\right) d \\
=&\beta \mathbf{E}_t\left(A_{t+1}\left(f\left(m_{t+1}\right) - c_{t+1}\right) - \beta m_{t+2} + q_{t+1}^M d_{t+2}^M + q_{t+1}^{OB} d_{t+2}^{OB} + q_{t+1}^{OM} d_{t+2}^{OM}\right).
\end{aligned}$$

Hence, by iteration, we know that

$$\begin{aligned}
&\left[X_t^M + \left(1-\delta^M\right) q_t^M\right] d_t^M + X_t^{OB}\left[1 + \left(1-\delta^{OB}\right) q^{OB}\right] d_t^{OB} + \left[X_t^{OM} + \left(1-\delta^{OM}\right) q^{OM}\right] d_t^{OM} \\
=&\mathbf{E}_t\left[\sum_{j=0} \beta^j \left(A_{t+j}\left(f(m_{t+j}) - c_{t+j}\right) - m_{t+j}\right)\right] + m_t.
\end{aligned}$$

Q.E.D.

7 Lemma 5

$$d^M(v) = \frac{A_H\left[f\left(m(v)\right) - c(v)\right] + \beta L\left(v_{COA}^{H,H}(v)\right) - A_L\left[f\left(m(v)\right) - c(v)\right] - \beta L\left(v_{COA}^{L,H}(v)\right)}{X^M(v, A_H, s) + (1-\delta^M) q^M\left(v_{COA}^{H,H}(v)\right) - X^M(v, A_L, H) - (1-\delta^M) q^M\left(v_{COA}^{L,H}(v)\right)}.$$

The numerator is positive due to Assumption 3. The denominator is positive due to the fact that $X^M(v)$ and $q^M(v)$ are increasing in v. Hence, $d^M(v) > 0$.

$$\begin{aligned}
&\left(1 - r_{OB}\right) D_{OB}(v) \\
=&\beta\left[L\left(v_{COA}^{L,H}(v)\right) - L\left(v_{COA}^{L,L}(v)\right)\right] \\
+&\left[X^M(v, A_L, L) + \left(1-\delta^M\right) q^M\left(v_{COA}^{L,L}(v)\right) - X^M(v, A_L, H) - \left(1-\delta^M\right) q^M\left(v_{COA}^{L,H}(v)\right)\right] d^M(v).
\end{aligned}$$

The RHS is positive since $L(v)$ is decreasing in v and $X^M(v)$ and $q^M(v)$ are increasing. As $D^{OB}(v) = \left[1 + \left(1 - \delta^{OB}\right) q^{OB}\right] d^{OB}(v)$, we know that $d^{OB}(v) > 0$. Q.E.D.

8 Proposition 3

First, by definition, the allocation associated with the strategy is implementable. Second, according to Lemma 3 and Lemma 5, the strategy satisfies the government budget constraint. Lastly, the debt prices follow the pricing rules. Hence, the constructed strategy generates a sustainable equilibrium, and the associated allocation is the COA. Q.E.D.

References

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