Temperature Shocks and Climate Change: A Conceptual Analysis*

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Abstract: This paper addresses the challenge of accurately modeling and estimating climate change damages. Time-series approaches rely on weather shocks, while cross-sectional analyses capture climatic differences but suffer from omitted variable bias. Climate is defined as the statistical pattern of weather that persists over time and allows for adaptation, unlike unpredictable weather realizations. To assess econometric approaches, I (i) integrate forward-looking adaptation into a quantitative integrated assessment model of climate change permitting an analytic solution and (ii) generalize the insights based on a novel dynamic envelope argument. I show how a careful time series (or panel) estimation can comprehensively identify the cost of climate change, including adaptation costs that are not directly observable in either a time series approach or a cross-section. Compared to earlier approaches, the indirect identification of adaptation costs does not rely on cross-sectional variation. The paper is also the first to explicitly derive a formula for the social cost of carbon under forward-looking adaptation. This result is not only insightful in itself but valuable for carefully interpreting prevailing envelope theorem arguments in the literature.

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1 Introduction

Over the past few decades, research on the economic impacts of climate change has grown substantially, with a surge of recent studies. Yet, significant conceptual challenges remain. Cross-sectional analyses, which compare countries in different climate zones, struggle with numerous confounding factors due to the inherent differences in geographic, economic, and social conditions. Conversely, time-series approaches have been criticized for their difficulty in effectively distinguishing between the effects of short-term temperature shocks and the long-term trend of rising global temperatures. This distinction is crucial, as societies can typically adapt more effectively to gradual climate change than to stochastic temperature fluctuations. This paper contributes to the empirical and theoretical literature on climate change damages by addressing this conceptual challenge and modeling the relationship between forward-looking adaptation, observed climate damages, and the resulting social costs.

To address the conceptual identification challenge, I integrate forward-looking adaptation into a detailed, quantitative, and analytically tractable integrated assessment model of climate change. A representative consumer responds to a (potentially) time-changing weather distribution (climate). By picking convenient but quantitatively motivated functional forms, I can solve for the forward-looking adaptation investments in closed form and derive the resulting observed damages as a function of temperature realizations and climatic moments. I show how different econometric approaches pick up different parts of the full social cost. Maybe surprisingly, a careful weather-based time-series approach outperforms cross-sectional or long-difference methods in key aspects, even in the absence of confounding factors.

I relate my closed-form analytic results to static and dynamic envelope arguments. As discussed in the literature, these arguments suggest that, as a result of optimal adaptation, we can neglect adaptation on the margin. However, it is crucial to clarify both the observable variables and the type of variation under which we can neglect explicitly accounting for adaptation. The envelope conditions state that, if our data include both adaptation damages and adaptation costs, these cancel out in the overall variation as a result of optimizing behavior. However, in most cases, data capture the adapted impact of temperature on GDP (or crop yields or...) and does not explicitly account for the re-routing of resources to adaptation measures. In such cases, observing anticipated variations in temperature does not capture the social cost, as they incorporate the benefit but neglect the cost of adaptation. Yet, an unanticipated shock can

yield variations that prevent adaptation, allowing us to apply the envelope condition because the marginal cost of adaptation equals the additional cost incurred when not adapting to the unanticipated shock.

Within a deterministic model, unanticipated shocks cannot be consistently included. Moving to a stochastic framework, the paper models climate as the statistics of weather. Embracing this idea, previous work defines weather as climate plus a mean zero shock, defining climate as the expected weather. Yet, climate is not merely a value but a distribution, and people adapt not only to mean temperature but also to weather volatility and possible downside risks. Thus, the variance of the mean-zero shock distinguishing climate from weather in such approaches is itself part of what defines climate, as are other characteristics of the distribution of weather.

In many cases, the adaptation-inclusive social costs can be identified or approximated based on deviations from expected temperature. Thus, it turns out that common time series approaches do well in inferring the comprehensive cost of climate change, whereas cross-sectional or long-difference-based approaches tend to miss the cost of adaptation. However, it is not only the deviation of temperature from the trend that matters for adaptation, investments, and observed climate damages, but also the higher-order moments of the distribution. Instead of using deviations from the trend, or controlling for expectations, I show that the best estimator of the total social costs of climate change controls for all the moments of the weather distribution (climate). We can think about the residual variation of temperature, after controlling for expectation, variance, skew and higher order moments, as an instrument identifying the *unadapted variation required to incorporate the cost of adaptation* into our damage estimates and the social cost of carbon (SCC).

Most adaptation decisions are dynamic, requiring foresight and persistent investments to shield against climatic changes and weather extremes. Unlike reactive measures, proactive investments in adaptation capital—such as air conditioning, housing insulation, or drought-resistant crops (including genetic modifications)—provide ongoing protection against increasing temperatures. For example, California's substantial investments required to prevent its electric grid from sparking wildfires illustrate the enduring nature of effective adaptation strategies in the face of evolving climate risks.

Incorporating forward-looking stock investment into adaptation is often regarded as one of the most challenging aspects of empirical climate damage estimation (Carleton et al. 2024). By contrast, immediate flow adaptation has received substantial atten-

tion in the recent literature. I show that immediate flow adaptation is actually much harder—if not impossible—to quantify without directly observing adaptation costs; adaptation by not leaving the house on a bad weather day or turning up the AC requires explicit cost observation (or a rather unusual natural experiment). As I will explain, the envelope theorem generally requires unexpected temperature fluctuations that either do not leave sufficient time to adapt or that do not make it worthwhile to adapt. Such fluctuations can be used to infer the social cost of climate change under forward-looking stock adaptation.

The underlying base model is the Analytic Climate Economy (Traeger 2023). Its general production system accommodates that of Golosov et al. (2014), DICE,To comprise DICE's production and abatement system, I have to assume exogenous business-as-usual emissions in the DICE model, which does not explicitly model energy use. and more detailed energy transition models that comprise a variety of fossil fuels and renewable energy sources and production sectors (Traeger 2022). The representative agent optimally decides on investment, labor and capital allocation across sectors, and fossil fuel use. The model also accounts for potential scarcity rents of fossil fuels (Hotelling rents). A state-of-the-art climate emulation delivers a good approximation of the emission impulse response and the distribution of the transient climate response. As a contribution to the IAM literature, the paper breaks new ground by explicitly solving a comprehensive quantitative IAM for the optimal forward-looking adaptation strategy and showing how optimal adaptation changes the observed damages and the SCC. While I focus on the full-fledged general equilibrium model to identify the social costs and optimal carbon tax, the results extend seamlessly to simpler partial observation models, addressing specific impacts such as crop yields, productivity, sea-level rise, or other damage dimensions.

Literature. The Ricardian approach, introduced by Mendelsohn et al. (1994) has sought to estimate climate impacts by comparing economic outcomes across locations with different climatic conditions while accounting for cross-sectional confounders (Schlenker et al. 2005). However, persistent concerns about omitted variable bias have led an alternative strand of research to favor time series and panel data methods that incorporate fixed effects and leverage short-term weather fluctuations as a source of identifying variation (Deschênes & Greenstone 2007). Subsequent literature points out that short-term weather shocks may not fully capture the long-term effects of climate change due to the potential for adaptation (Hsiang 2016, Auffhammer 2018, Hogan &

Schlenker 2024). In particular, it is generally hard or impossible to observe the cost of adaptation directly.

In the context of flow adaptation, Guo & Costello (2013), Hsiang (2016), Deryugina & Hsiang (2017), and Carleton et al. (2022) discuss how the static envelope theorem can, under some assumptions, eliminate the need to directly observe adaptation. These illustrations of the envelope argument, apart from Mérel et al. (2024), rely on using temperature or expected temperature as the climate state, assuming that weather is a mean-zero fluctuation around the temperature state. Formalizing climate more carefully, it is the statistics of temperature realizations. In particular, the variance of weather itself is part of the climate, as are skewness and tail risks of the underlying distribution. Mérel et al. (2024) elaborate on this issue and formally identify assumptions under which standard econometric approaches can correctly identify the full effect of climate change under flow adaptation. A crucial assumption is that payoffs are quadratic. This assumption implies, in suitable settings, that uncertainty does not affect decisions (risk equivalence principle), reviving the temperature-based envelope argument in the context of flow adaptation.

The present paper further elaborates on the interpretation of the envelope argument in different contexts. In particular, it clarifies that the envelope theorem does not imply that the costs of the marginal adaptation project have no bearing on the social cost of climate change (or carbon). It then extends the envelope theorem in three important dimensions. First, it embeds the envelope theorem into a forward-looking dynamic setting. Second, it includes stock adaptation, i.e., investments into persistent adaptation measures. Third, it develops the envelope argument in the real-world context where climate is captured by the distribution of weather rather than temperature or expected temperature.

Three projects have recently taken on related challenges of identifying comprehensive climate damage estimates in a dynamic setting with adaptation. Lemoine (2018) takes envelope-type reasoning to a long-run steady state limit. He shows how, in this approach, persistent adaptation capital can destroy identification even if we can observe adaptation costs. Both Lemoine (2018) and, in a flow adaptation setting, Shrader (2023) argue that controlling for weather forecasts can reduce a built-in bias of standard econometric approaches. Deriving a more general dynamic envelope argument as well as a closed-form solution, the present paper re-emphasizes the latter point and explains the importance of controlling not only for expected weather ("forecasts") but also for

higher-order moments of the distribution when identifying comprehensive estimates of climate impact. Rudik et al. (2024) introduce a dynamic envelope argument in a context where migration across sectors plays a role analogous to adaptation. Relying on a structural setting, they show that migration flows are a sufficient statistics for households' expected future welfare.[1] In yet unpublished work, Costello & Collie (2025) use a dynamic envelope theorem where climate corresponds to expected temperatures to estimate the impact of climate change on global fisheries.

These approaches share the present paper's goal of addressing the long-voiced concern that dynamic adaptation creates a wedge between the impacts of volatile weather and those of a persistent change in climate. They either challenge the informational content of current climate damage estimates or argue that the marginal value of adaptation is negligible and can be disregarded. The present paper takes a unique approach by connecting forward-looking adaptation decisions with an analytic derivation of their implications for damages, costs, empirical observations, and the comprehensive social cost of climate change and CO_2 emissions in an integrated assessment. The resulting closed-form solution provides a rigorous framework for illustrating, interpreting, and extending key conjectures from the literature. As importantly, this paper resolves the conceptual identification challenge in a general setting with stock adaptation, relying on a novel envelope argument that consistently formalizes climate as the distribution of temperatures.

A series of numeric integrated assessment models incorporate adaptation in various ways. For instance, Bosetti et al. (2006) develop a framework in which adaptation emerges endogenously through learning-by-doing and investments in research and development. Tol (2007) analyze adaptation to sea level rise, incorporating both protective measures and displacement responses. De Bruin et al. (2009) extend the DICE model to explicitly represent adaptation as a decision variable. Diaz (2016) and Depsky et al. (2023) quantify the economic impact of sea level rise under scenarios with and without adaptation. Dietz & Lanz (2022) build an integrated assessment model that analyzes a series of adaptation mechanisms, including demographic transitions, technological progress, shifts in land use, and sectoral reallocation. Lab (2023) develop a spatially detailed IAM that simulates climate damages damages across multiple sec-

[1] In their setting, the envelope theorem states that the marginal response of the value function with respect to expected temperature before the temperature realization is the same as the migration re-weighted expected marginal response of the value function after the temperature realization and the resulting migration decision.

tors incorporating adaptation into the damage functions. I refer to the excellent recent handbook chapter by Carleton et al. (2024) on adaptation to climate change for a more detailed discussion of several hundred papers on the topic.

2 The Deterministic World

This section develops a basic IAM with forward-looking adaptation investments and a closed-form solution for the SCC.

2.1 Production, Damages, and Adaptation

Let gross output be a function of capital $\boldsymbol{K}_t$, labor $\boldsymbol{N}_t$, energy inputs $\boldsymbol{E}_t$, and exogenously evolving technology $\boldsymbol{A}_t$. I use bold letters to indicate that these inputs are generally vectors characterizing the inputs of a multi-sector economy. I characterize climate damages as a fraction $D(T_{1,t})$ of gross output. These damages are a function of the current atmospheric surface temperature increase above 1900 levels, $T_{1,t}$. For now, this temperature can represent a global average or regional temperatures over a given period. Then, production after damages

$$Y_t = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)[1 - D(T_{1,t})].$$

I assume that the production function $F(\cdot)$ is homogeneous in capital of some degree $\kappa \in (0, 1)$.[2] As explained in Traeger (2022), this functional form covers the production and abatement structure of several common integrated assessment models such as Golosov et al. (2014) and Nordhaus (2017) as well as a more complex production structure relying on several sectors and energy inputs with distinct and time-changing degrees of energy substitutability.[3]

It is helpful to specify damages in the form

$$D(\tau_{1,t}) = 1 - \exp[-d(T_{1,t})]$$

[2] $F(\boldsymbol{A}_t, \boldsymbol{N}_t, \gamma\boldsymbol{K}_t, \boldsymbol{E}_t, i_t) = \gamma^\kappa F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, i_t); \forall \gamma \in \mathbb{R}_+$. I also assume that the production optimization problem is well-defined.

[3] Golosov et al. (2014) assume a final goods sector with a Cobb-Douglas production function that uses a CES energy composite of oil, coal, and renewables. DICE's production model by Nordhaus (2017) is a special case of the present structure when using an exogenous calibration of business-as-usual emissions and reformulating the abatement cost model into an explicit production model. Traeger (2022) calibrates a corresponding production model with three final sectors, five primary energy sources, an electricity sector, and time-changing sector-specific substitutabilities across energy sources.

From the empirical perspective, the function $d(T_{1,t})$ represents the damages estimated using log-output

$$\log Y_t = \log(F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t)) - d(T_{1,t}).$$

From the integrated assessment perspective, the function $d(T_{1,t})$ captures a generally convex damages function, whereas the transformation $D(\tau_{1,t}) = 1 - \exp[-d(T_{1,t})]$ limits damages to not exceed full production. A second order Taylor expansion $1 - \exp[-x] \approx x - \frac{1}{2}x^2$ illustrates that, to first order, we can interpret $d(T_{1,t})$ itself as the percentage loss of output, but actual damages $D(\cdot)$ are somewhat less convex than $d(\cdot)$ would suggest.

To obtain closed-form solutions, my structural analysis requires particular assumptions about the shape of the damage function $d(T_{1,t})$. Traeger (2021) shows that

$$d^{\text{na}}(T_{1,t}) = \xi_0(\underbrace{\exp[\xi_1 T_{1,t}]}_{\equiv \tau_{1,t}} - 1) = \xi_0(\tau_{1,t} - 1), \tag{1}$$

where "na" abbreviates *n*o explicit *a*daptation, nicely approximates Rennert et al.'s (2022) mean damages. This study follows the National Academies of Sciences & Medicine's (2017) advice on updating the SCC and was employed, among others, in the recent update of the US federal SCC (EPA 2023). Their global damage estimate aggregates country-level and regional damage estimates for agricultural yield, mortality, sea-level rise, and energy impact across 10 000 Monte Carlo simulations. The left panel of Figure 1 compares Rennert et al.'s (2022) simulated damages with the damage function $D(\tau_{1,t}) = 1 - \exp[-d^{na}(T_{1,t})]$ fitted over the temperature interval $[1, 4]$. I will use variations of these damages for quantitative illustrations, yet the main point of Figure 1 is to illustrate that, once I have to pick a particular functional form, the exponential formulation in (1) seems reasonable and, at least here, fits the data far better than the common quadratic form.[4]

Equation (1) introduces the abbreviation $\tau_{1,t} = \exp[\xi_1 T_{1,t}]$. This transformation is not only notationally convenient but also plays a special role in solving the forward-

[4]Ever since Nordhaus (1994), a quadratic function, or one over a quadratic function, has been a prominent structural assumption in integrated assessment models and even in meta-analyses of empirical damage estimates (Howard & Sterner 2017). Here, the common "one over" transformation as in $D(\tau_{1,t}) = \frac{1}{1+aT_{1,t}^2}$ plays a similar role as the $1 - \exp[-d(T_{1,t})]$ transformation above; it limits damages to not exceed full production (or eventually exceed full production plus capital stock). The (double-) exponential function presented here is initially less and eventually more convex than the quadratic function.

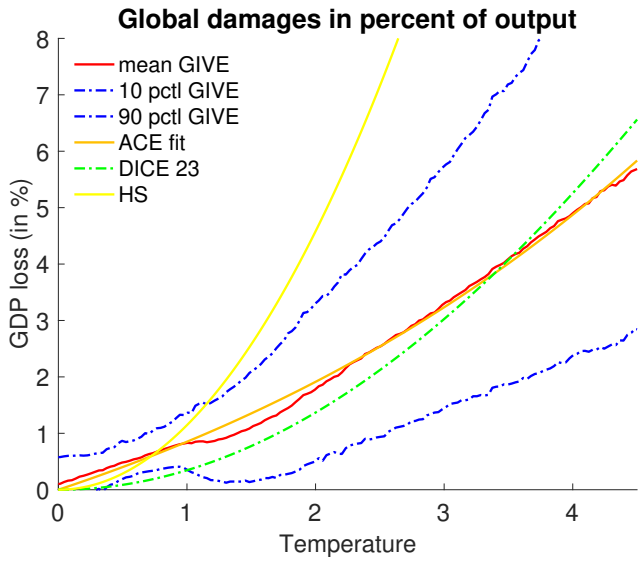


Figure 1: Mean and percentiles of Rennert et al.'s (2022) 10,000 Monte Carlo run analysis of global damages over temperature (GIVE model), and ACE fit using the damage function $D(T_{1,t}) = 1 - \exp[-d^{na}(T_{1,t})]$ with $\xi = 0.033$ The graph also shows the DICE 2023 damage function and Howard & Sterner's (2017) preferred estimate from a meta-analysis (HS).

looking IAM as $\tau_{1,t}$ rather than $T_{1,t}$ will have a constant shadow value.[5] As is common in IAMs, function (1) does not explicitly model adaptation. Most IAMs do not carefully discuss whether these reduced-form damage functions, which exclude explicit adaptation, are meant to represent damages before or after adaptation measures have been implemented. The prevailing interpretation is that these models aim to represent damages after adaptation.

This study extends the structural assumptions governing damages to include adaptation explicitly. To understand the relationship between the actual damage function, estimated damages, and the implied social costs, including the SCC, it is very helpful to obtain closed-form solutions to the forward-looking optimization problem, which involves optimal adaptation along with other variables such as capital allocation, labor allocation, and resource use. I am not aware of any previous successful attempt to solve such a problem. Given this context, I take some liberty in choosing functional forms to extend equation (1) to

[5]See Traeger (2023) and the present Appendix for details. The present analysis focuses on the free damage parameter ξ_0. The parameter ξ_1 also affects the model's non-linear temperature dynamics as explained in Traeger (2023); here, I take it as given.

$$d_t(\tau_{1,t}, i_t, A_t) = \underbrace{\xi_0(\tau_{1,t} - 1)}_{\substack{\text{non-adaptable} \\ \text{damage component}}} + \underbrace{\gamma \overbrace{\exp(\lambda \tau_{1,t})}^{\text{damage}} \overbrace{\exp(-\chi(A_t + i_t))}^{\text{adaptation}}}_{\substack{\text{partially adaptatable} \\ \text{damage component}}} . \tag{2}$$

The first part of the damages function maintains a damage component as in equation (1), excluding explicit adaptation. Likely candidates for such a component include non-market damages to vulnerable ecosystems. Of course, the component could also be zero. The novel part of the damage function interacts an even steeper damage function with an adaptation function. The parameters ξ_0 and γ scale the magnitude of the two different damage components, λ scales the steepness of the adaptable damages, and χ measures the effectiveness of adaptation measures. A_t is the adaptation capital and i_t the current adaptation investment. The adaptation capital follows the equation of motion

$$A_{t+1} = (1 - \delta_A)(A_t + i_t).$$

The parameter δ_A reflects the depreciation rate. I choose the joint appearance of $A_t + i_t$ to obtain the flow adaptation case in the limit of full depreciation, $\delta_A = 1$.[6] So far, such flow adaptation models received the most attention in the literature. In these models, the only role of adaptation expenses i_t is to lower current damages in equation (2). In general, investments in adaptation reduce damages in multiple periods. The exponential form implies that no amount of adaptation stock can increase GDP beyond output levels observed in the absence of climate change.[7]

At this point, the functional forms are qualitatively reasonable. Most importantly, these functional forms allow for an analytic solution for optimal adaptation investment, their implications for damage estimation, and the SCC. Investment expenses i_t are

[6] Thanks to depreciating both A_t and i_t in the equation of motion for abatement capital, abatement capital under $\delta_A = 1$ is always zero at the beginning of a period. If I adopted the common capital building equation $A_{t+1} = (1 - \delta_A)A_t + i_t$, then the capital persistence multiplier $(1 - \beta(1 - \delta_A))$ appearing in many of the future results would be replaced by the multiplier $\left(1 - \frac{\beta}{1+\beta\delta_A}\right)$, which is the only resulting change. However, the latter multiplier would not collapse to unity under full depreciation because today's adaptation payments reduce both current damages and the next period's damages before decaying.

[7] Although certain investments could enhance quality of life even at preindustrial temperature levels, this distinction helps differentiate between general investments and adaptation investments specifically aimed at combating climate change.

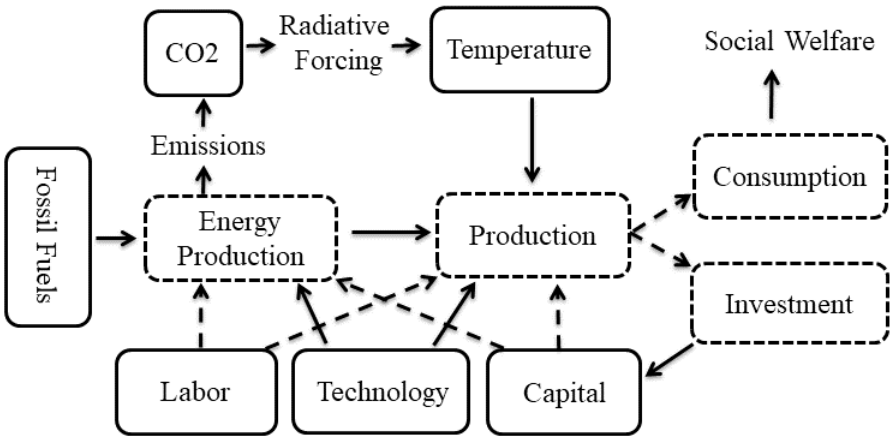


Figure 2: The structure of ACE and most IAMs. Solid boxes are state variables, dashed boxes are flows, and dashed arrows mark choice variables.

deducted from output generating net output

$$Y_t^{net} = Y_t \exp(-\eta\ i_t), \tag{3}$$

which consumers optimally split between consumption and investment.[8] The subsequent analysis assumes an interior solution to the adaptation investment problem with

$$i_t > 0\ ,$$

which exists for reasonable but not for all parameter specifications.

Appendix A details the remaining components of the IAM illustrated in Figure 2, which is directly adapted from Traeger (2023). Specifically, the model integrates an updated carbon cycle by Eftekhari et al. (2024) and employs a nonlinear ocean-atmosphere temperature model. These components together produce a carbon impulse response consistent with state-of-the-art IAMs (Traeger 2021). The model accounts for potential scarcity of some of the fossil-fuel inputs generating a positive Hotelling rent.

[8]The coefficient η converts between a share of output and consumption equivalents (or USD). The share-based approach in equation (3) captures that, as production grows, so does the cost of adaptation, which offsets damages that are similarly proportional to output. While alternative formulations are plausible, including a time-dependent conversion factor η that offsets (or enhances) some of the growth adjustment over time, the present form facilitates a relatively simple solution to the forward-looking problem.

2.2 Results in a Deterministic World

A representative agent maximizes log-utility over an infinite time horizon with time preference discount factor β. The following proposition states the optimal adaptation investment and the resulting shape of the optimally adapted damage function.

Proposition 1 *The optimal investment into adaptation capital is*

$$i_t^* = \frac{1}{\chi}\Big[\log(\gamma) - \log\Big(\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\Big) + \lambda\tau_{1,t}\Big] - A_t. \tag{4}$$

The resulting optimally adapted damages are

$$d_t(\tau_{1,t}, i_t^*, A_t) = \xi_0(\tau_{1,t}-1) + \frac{\eta}{\chi}\left(1-\beta(1-\delta_A)\right). \tag{5}$$

The optimal adaptation investment stated in equation (19) increases exponentially in temperature together with $\tau_{1,t} = \exp(\xi_1 T_{1,t})$. This increase is proportional to the ratio $\frac{\lambda}{\chi}$, which reflects the relative steepness of damage escalation (λ) compared to the effectiveness of adaptation investment (χ). Additionally, investment increases logarithmically with the magnitude of the adaptable damage component, represented by γ. Conversely, investment decreases logarithmically with the ratio $\frac{\eta}{\chi}$, the cost of adaptation relative to its effectiveness. These costs are further adjusted by a factor that incorporates the persistence of abatement capital, reducing effective costs in proportion to the discounted adaptation capital persistence $\beta(1-\delta_A)$. Finally, a higher stock of adaptation capital A_t already in place reduces the need for further investment.

These investment dependencies reflect the complex interactions of the forward-looking adaptation decision. While these drivers of investment are interesting by themselves, the main purpose of this model is to demonstrate how they affect experienced and observed damages. Equation (5) states the observed damages as a function of temperature under the assumption of an optimal abatement trajectory. Surprisingly, the adaptive component of the damage function becomes independent of temperature.

Let me assume an idealized setting with many identical regions (or worlds). These regions coincide in all parameters and produce the same level of gross output, but differ in terms of the prevailing temperature level $\tau_{1,t}$ and the resulting output after damages Y_t. Even if we guess the correct functional forms and regress log-GDP on $\tau_{1,t}$, an exponential transformation of temperature, equation (5) shows that all we can hope to pick up in our regression is the relation

$$\log Y_t = ... - \xi_0\tau_{1,t} = ... - \xi_0 \exp[\xi_1 T_{1,t}], \tag{6}$$

where I suppress observables that are independent of temperature. Thus, even in the first-best case of perfectly controlling for confounding factors, the cross-section would only pick up the non-adaptable part of damages.

At the same time, equation (5) suggests that also the adaptable damage component reduces GDP. This reduction is proportional to the effective costs of adaptation, i.e., the persistence and adaptation effectiveness adjusted adaptation costs $\frac{\eta}{\chi}(1-\beta(1-\delta_A))$. The higher these costs, the lower the adaptation level and the larger the damages. In the end, we are interested in the social cost of a temperature increase and, more specifically, the social cost of the persistent temperature increase triggered by an additional ton of CO_2 emissions. The following proposition clarifies which parameters we have to estimate in order to calculate the SCC.

Proposition 2 *The SCC under optimal adaptation is*

$$SCC = \underbrace{\frac{\beta Y_t^{net}}{M_{pre}}}_{\text{cost base}} \Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1-\beta(1-\delta_A)\Big)}\Big) \underbrace{\big[(\mathbf{1}-\beta\mathbf{\Phi})^{-1}\big]_{1,1}}_{\text{carbon dynamics}} \underbrace{\sigma^{forc}\big[(1-\beta\boldsymbol{\sigma})^{-1}\big]_{1,1}}_{\text{temperature dynamics}} \tag{7}$$

where the underlined contributions are new to the model with adaptable damages.

Proposition 2 extends the base formula in Traeger (2023) by introducing the underlined term. The first term, labeled 'cost base,' relates (discounted) net output Y_t^{net} to the preindustrial carbon concentration M_{pre}. This term sets the scale and defines the units by linking output to CO_2 levels.

The second-to-last term, labeled 'carbon dynamics,' corresponds to the top-left element of the inverted matrix. As explained in Traeger (2023), it represents a discounted sum over the additional atmospheric CO_2 in all future periods resulting from the release of one extra ton of CO_2 today. The magnitude of this multiplier depends on the discount rate, but it is always substantially greater than unity, thereby increasing the SCC—an outcome driven by the mass conservation of carbon. Similarly, the final term, labeled 'temperature dynamics,' reflects the future heat exchange induced by an additional unit of radiative forcing.[9] This final temperature dynamics term is always smaller than unity because the oceans act as a thermal buffer, delaying warming and, consequently, the associated damages.

[9]This interpretation derives from expressing the inverted matrices using the Neumann series. The result is a double sum because every increase in atmospheric carbon in the future induces additional radiative forcing, and each additional unit of radiative forcing causes further warming in all subsequent periods, interacting with the oceans, which continue to provide substantial cooling.

Turning to the components of interest, the parameter ξ_0 represents the magnitude of non-adaptable damage. Among the terms inside the bracket, it is the only one that remains in a standard model without adaptation. Here, ξ_0 is the semi-elasticity of output with respect to an exponential increase in temperature. Multiplying ξ_0 by the cost base, we obtain a social cost of carbon (SCC) that would apply in the absence of carbon cycle dynamics, meaning CO_2 would not persist in the atmosphere, and oceanic processes would not delay warming.

The novel underlined term further increases the SCC by accounting for the adaptable damage component, which rises with the cost of building adaptation capital, η. Similar to the temperature dependence of optimal investment, this contribution to the SCC also increases with the ratio of the steepness of damages to the effectiveness of adaptation capital, $\frac{\lambda}{\chi}$. The subsequent term in brackets adjusts these costs for capital persistence, leading to an interesting and uncommon finding: this SCC contribution decreases with patience (i.e., a higher discount factor, β). The reason is that a more patient decision-maker invests more in adaptation capital, thereby mitigating the damage caused by an additional ton of CO_2 released into the atmosphere.

Proposition 2 characterizes the damage terms necessary to capture the social costs. Based on these findings, I can now revisit the distinction between what is observable and what is required to identify the social costs. Equation (6) demonstrates that observing $\log Y_t$—that is, gross output after climate damages—as a function of temperature is insufficient to identify any of the key terms needed to compute the social cost under adaptation, even in a perfectly controlled cross-section. Since adaptation is costly, merely observing post-adaptation damages provides no information about the additional adaptation costs incurred due to an extra ton of CO_2 emissions. I will elaborate on this point further in the following section.

Given the somewhat discouraging observational limitations of a cross-section, I now turn to what can be observed in a time series. In a purely deterministic world, an econometrician would still only capture the relationship in equation (6), failing to account for the adapted portion of damages. However, the situation changes when recalling the common critique put forth against time series models: they tend to capture weather fluctuations rather than persistent changes of the climate. While many forms of adaptation are possible in response to persistent climate shifts, similar investments are often impractical or undesirable for short-term weather fluctuations. The deterministic model of this section is not fully equipped to distinguish between climate change and

temperature shocks. Nevertheless, equation (2) offers useful insights and conjectures before transitioning to a more comprehensive model in Section 3.

Assume that adaptation capital responds to an average temperature trend $\bar{\tau}_{1,t}$. Now consider a short-term temperature shock $\tau_{1,t}$ for which optimal adaptation capital has not been built—either due to time constraints or because investing in capital for a one-time shock was not worthwhile. In this case, the final multiplier in equation (2) simplifies to a constant, $\omega \equiv \exp(-\chi(A_t + i_t)) < 1$, leading to the following variation in log-GDP:

$$\log Y_t = ... - \xi_0 \tau_{1,t} - \gamma \exp(\lambda \tau_{1,t})\ \omega \tag{8}$$

where I have again omitted contributions independent of temperature change. Interestingly, this "surprise variation" allows the econometrician to uncover the parameter λ characterizing the steepness of damages, which is unobservable not only in the cross-section but also in the more comprehensive level of damages characterized by equation (5). The inability to adapt in response to the shock paradoxically provides a crucial step toward identifying the SCC under adaptation.

At first glance, equation (8) suggests that many of the underlined parameters in equation (7) remain unidentified. However, so far, I have evaluated the relationship between GDP and temperature at an arbitrary investment level, $\omega \equiv \exp(-\chi(A_t + i_t))$. When evaluated at the optimal investment level equation (19) delivers that $\omega^* = \frac{1}{\gamma}\frac{\eta}{\chi}\big(1 - \beta(1-\delta_A)\big)\exp(-\lambda\bar{\tau}_{1,t})$ and, thus

$$\log Y_t = ... - \xi_0 \tau_{1,t} - \frac{\eta}{\chi}\big(1 - \beta(1-\delta_A)\big)\exp(\lambda(\tau_{1,t} - \bar{\tau}_{1,t})). \tag{9}$$

For a marginal surprise deviation from the trend $\Delta\tau \equiv \tau_{1,t} - \bar{\tau}_{1,t}$, we observe the regression coefficient $-\left[\xi_0 + \frac{\eta}{\chi}\big(1 - \beta(1-\delta_A)\big)\right]$ because $\exp(\lambda(\tau_{1,t} - \bar{\tau}_{1,t})) \approx \Delta\tau$. Thus, a surprise variation due to a weather shock ultimately promises to identify all the damage terms required to calculate the social costs. Section 2.3 shows that this result is not coincidental but follows from the envelope theorem.

2.3 Interpreting Output Variation and the Envelope Theorem

Guo & Costello (2013) and a series of subsequent papers point out that optimal adaptation cancels out in the observation of output damages, reasoning that relies on the envelope theorem. It is crucial to carefully interpret the envelope theorem in this context to avoid overlooking the significant value and costs associated with adaptation.

This section connects the general principles of the envelope theorem to the present findings. I first address the common static reasoning typically invoked in this context, and then extend the analysis to incorporate a dynamic version of the envelope theorem, highlighting its implications for estimating climate damages when explicit adaptation costs remain unobserved.

Static envelope theorem. To illustrate the workings of the static envelope theorem in this context, consider that net output is a function of temperature T_t and investment into adaptation i_t:

$$Y_t^{net}(T_t, i_t^*, ...),$$

where optimal investment i_t^* is itself a function of current temperature. Thus, the total change in output resulting from a small change in temperature is

$$\frac{dY_t^{net}}{dT_t} = \frac{\partial Y_t^{net}}{\partial T_t} + \underbrace{\frac{\partial Y_t^{net}}{\partial i_t}}_{=0} \frac{\partial i_t^*}{\partial T_t} = \frac{\partial Y_t^{net}}{\partial T_t}. \tag{10}$$

The second term cancels because the decision maker optimizes investment

$$\max_{i_t} Y_t^{net}(T_t, i_t, ...) \quad \Rightarrow \quad \frac{\partial Y_t^{net}}{\partial i_t} = 0, \tag{11}$$

which is an application of the envelope theorem: we can neglect the temperature dependence of the optimized investment control when calculating the total derivative.

Equation (11) is sometimes interpreted as a finding that the marginal adaptation project has no value, and equation (10) has been interpreted to imply that observed damages alone are sufficient to measure the social cost of a temperature increase, without the need to explicitly account for unobserved adaptation efforts. While the use of the envelope theorem and the above interpretations provide valuable insights, they have to be matched carefully with their precise interpretation in the climate change context.

The value of adaptation. First, it is helpful to more carefully interpret equation (11) as stating that the economic net value of expanding adaptation by one additional unit is zero, which differs from the statement that the marginal value of adaptation is zero. To illustrate this distinction, consider the following analogy: suppose my marginal valuation for a chocolate bar is \$1, and so is the price at the store. If I purchase the chocolate bar, the net benefit of this transaction would be zero, as the \$1 I pay equals the \$1 value I assign to the bar. However, the value of that chocolate

bar remains \$1, even though the net benefit of acquiring one more bar under these circumstances is zero. Similarly, it is helpful to expand the content of equation (11) to the full "transactional details". Using a static version of the production model in the present paper, equation (11) becomes

$$\max_{i_t} Y_t^{net}(A_t, \tau_t, i_t, ...) \quad \Rightarrow \quad \underbrace{\chi\gamma \exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t))}_{\text{marginal benefits}} = \underbrace{\eta}_{\text{marginal costs}} \quad , \quad (12)$$

where costs and benefits are in fractions of output and $\tau_{1,t} = \exp(\xi_1 T_{1,t})$. In general, the (marginal) value of an adaptation project would be equivalent to either side of equation (12), a strictly positive value.

The interpretation of output: Y_t^{net} vs Y_t. My notation carefully distinguishes between Y_t^{net} and Y_t to emphasize that the envelope theorem applies only to Y_t^{net}, i.e., output net of adaptation expenses. However, we usually do not observe adaptation expenses explicitly and, thus, observe Y_t rather than Y_t^{net}. Equation (6) shows that regressing $\log Y_t$ on temperature will only identify non-adaptable climate damages and fall short of the social cost characterized in Proposition 2. The intuition why the envelope theorem only applies to Y_t^{net} is simple: the representative consumer maximizes consumption rather than gross output (and so does the decentralized economy absent market failures). For the same reason, the change in $\log Y_t$ falls short of measuring the social cost: re-routing resources that were used for consumption (or productive investment) to abatement efforts is not properly reflected in total output.

However, if we were to observe output net of adaptation investments, then we find

$$\log Y_t^{net} = ... - \xi_0\tau_{1,t} - \eta\frac{\lambda}{\chi}\tau_{1,t} = ... - \Big(\xi_0 + \eta\frac{\lambda}{\chi}\Big)\tau_{1,t} \quad (13)$$

in the present model, where I used equation (19) specifying optimal investment into adaptation as a function of temperature (neglecting other dependencies). In a static framework, equation (23) correctly identifies the marginal damages entering the SCC in equation (7). In a setting with adaptation capital, measuring the variation of Y_t^{net} in temperature misses the impact of capital persistence on the SCC, potentially overestimating the social cost.

Using the envelope theorem for the identification of damages. The envelope theorem is usually called upon because (or when) we do not observe adaptation expenditure explicitly. Then, the crucial step in applying the envelope theorem is to observe that $\frac{\partial Y_t}{\partial T_t} = \frac{\partial Y_t^{net}}{\partial T_t}$: here we are merely looking at the direct impact of temperature. As a

result, we can use equation (10) in combination with the observable total output Y_t to (try to) measure $\frac{\partial Y_t}{\partial T_t}$ and get at the welfare relevant but generally unobservable total variation of Y_t^{net}

$$\frac{dY_t^{net}}{dT_t} = \frac{\partial Y_t^{net}}{\partial T_t} = \frac{\partial Y_t}{\partial T_t}.$$

However, observing the partial derivative $\frac{\partial Y_t}{\partial T_t}$ is another key challenge. Within a deterministic model, a rational, forward-looking agent continuously adapts to climate conditions. Consequently, the unadapted variation, which is encoded in the partial derivative $\frac{\partial Y_t}{\partial T_t} = -\frac{\partial d_t(\tau_{1,t}, i_t, A_t)}{\partial \tau_t}\frac{\partial \tau_t}{\partial T_t}$ remains unobservable. Leaving the present framework, however, this result gives hope that unanticipated shocks can help us identify the relevant damage components. Of course, a natural candidate for such unanticipated changes in temperature are random weather shocks, a conjecture I will carefully analyze in Section 3.

Dynamic envelope theorem and social cost. Finally, let me move to a dynamic version of the envelope theorem, which is most easily spelled out in a dynamic programming formulation. Abstracting from the full complexity model, I focus on the essential states of temperature and adaptation capital, dropping even standard capital (see footnote also **??**). Moreover, I move from mere output to utility and a welfare measure for the social cost of a temperature increase. The corresponding Bellman equation is

$$V(T_t, A_t) = \max_{i_t} U(C_t, T_t, i_t, A_t) + \beta V(T_{t+1}, A_{t+1}),$$

where the utility function's arguments abbreviate the dependencies of the essential arguments of the underlying production system. Then, for the optimized problem, we find that the social cost of a marginal temperature increase in welfare units is

$$\frac{\partial V(T_t, A_t)}{\partial T_t} = \frac{\partial U(C_t, T_t, i_t, A_t)}{\partial T_t} + \underbrace{\frac{\partial U(C_t, T_t, i_t, A_t)}{\partial i_t}\frac{\partial i_t^*}{\partial T_t} + \beta\frac{\partial V(T_{t+1}, A_{t+1})}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial i_t}\frac{\partial i_t}{\partial T_t}}_{=0 \text{ by the envelope theorem/FOC for adaptation investments}}$$

$$+\beta\frac{\partial V(T_{t+1}, A_{t+1})}{\partial T_{t+1}}\frac{\partial T_{t+1}}{\partial T_t} \quad = \quad \sum_{l=t}^{\infty}\beta^{l-t}\frac{\partial U(C_l, T_l, i_l, A_l)}{\partial T_l}\frac{\partial T_l}{\partial T_t}\,, \tag{14}$$

where the last equality recursively substitutes out the value function (assuming well-behaved convergence). The term $\frac{\partial T_l}{\partial T_t}$ abbreviates temperature dynamics and the persistence of a current temperature shock. Returning to the (full-complexity) setting of

this paper, I find[10]

$$\begin{aligned}\frac{\partial U(C_t, T_t, i_t, A_t)}{\partial T_t} &= -\frac{\partial d_t(\tau_{1,t}, i_t, A_t)}{\partial \tau_t}\frac{\partial \tau_t}{\partial T_t} \\ &= \Big[\underbrace{\xi_0}_{\substack{\text{non-adaptable} \\ \text{damage component}}} + \underbrace{\lambda\gamma \overbrace{\exp(\lambda\tau_{1,t})}^{\text{damage}} \overbrace{\exp(-\chi(A_t + i_t))}^{\text{adaptation}}}_{\substack{\text{partially adaptatable} \\ \text{damage component}}} \Big] \xi_1 \underbrace{\exp[\xi_1 T_{1,t}]}_{=\tau_{1,t}}, \\ &= \Big[\underbrace{\xi_0}_{\substack{\text{non-adaptable} \\ \text{damage component}}} + \underbrace{\eta\frac{\lambda}{\chi}\Big(1 - \beta(1 - \delta_A)\Big)}_{\substack{\text{partially adaptatable} \\ \text{damage component}}} \Big] \xi_1 \underbrace{\exp[\xi_1 T_{1,t}]}_{=\tau_{1,t}},\end{aligned} \tag{15}$$

where the final row evaluates marginal utility at the optimal investment level i_t^*. As suggested by the dynamic envelope theorem, the expression in squared brackets correctly picks up the social cost of a temperature increase that also appears as the central damage component in the SCC in Proposition 2. Calculating the social cost of *carbon*, the proposition contains additional terms characterizing temperature dynamics ("$\frac{\partial T_l}{\partial T_t}$") and carbon dynamics, and it translates the social cost back from utils into consumption units/USD.[11]

It is crucial to observe that equation (15) takes the partial derivative at some given adaptation level *before* inserting the optimal adaptation level. Thus, we can recover crucial elements of the social cost by evaluating either the net output variation Y_t^{net}, which includes directly observing adaptation expenses, or, thanks to the envelope theorem, by observing an unadapted variation at the previously optimal adaptation level. It turns out that, among these two options, only the unadapted variation also picks up the dynamic multiplier capturing the persistence of abatement capital.

As discussed in connection to the static envelope theorem, the issue with this result is that a rational forward-looking agent will generally adapt, and the econometrician will not observe such an unadapted variation, at least in a deterministic world. Here, the econometrician only observes the (here non-existent) variation of damages *after*

[10] The first equality uses equation (B.2) from Appendix B.2.

[11] Keeping this initial exposition as simple as possible, I abstracted from damages affecting capital accumulation. The full ACE model contains such damages, but the optimizing consumer splits total damages equally between consumption and investment damages. As a result, the full model differs from the above formula merely by a joint multiplicative factor. Moreover, the terms canceling in equation (14) based on the envelope theorem/first order condition usually contain another term specifying the value function derivative with respect to capital and measuring forgone investment as a result of adaptation expenditure.

adaptation whereas equation (15) relies on the variation (partial derivative) *before* adaptation. As we will see in the next section, stochastic weather fluctuations can resolve this issue.

3 Uncertain Climate Evolution and Temperature Shocks

Up to this point, temperatures have evolved deterministically, allowing agents to adapt optimally to any realized temperature level. As discussed in the previous section, random weather shocks are promising candidates for identifying adaptation-inclusive costs of climate change. In addition, the uncertainty over the evolution of climate itself affects optimal adaptation investments. The present section introduces an uncertain climate evolution and weather shocks. I first present the stochastic framework underlying temperature dynamics, followed by the modified damage function, the objective function, and, finally, the results.

3.1 Temperature Dynamics

Temperature evolution is uncertain for many reasons. I focus on the uncertain temperature response of atmospheric CO_2 concentrations, also known as climate sensitivity. The underlying stochastic process is highly autoregressive because bad news about climate feedbacks today imply that climate feedbacks tomorrow are most likely higher than expected as well. This base model of stochastic climate feedbacks and its implications for the evalution of global average surface temperature is adopted from Traeger (2021), where it is calibrated to IPCC's (2013) and IPCC's (2021) transient climate sensitivity. I summarize this model below. The only extension in the present context is the addition of serially uncorrelated temperature shocks that represent weather and, in the next section, idiosyncratic regional shocks.

The adopted model uses a modified version of Gourieroux & Jasiak's (2006)'s autoregressive gamma process, denoted by y_t, and an expectation adjusted shift of this process, z_t. Incorporating this stochastic process into the equation of motion (31) for

temperatures, see Appendix A, yields

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \Big(\underbrace{\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}}_{\text{deterministic forcing}} + \underbrace{z_{t+1}}_{\substack{\text{persistent}\\ \text{stochastic}\\ \text{deviation}}} + \underbrace{\Delta_t}_{\substack{\text{new}\\ \text{volatility}\\ \text{term}}} \Big)\boldsymbol{e}_1.$$

The unit vector $\boldsymbol{e}_1$ ensures that feedbacks drive atmospheric temperature. Traeger (2023) summarizes how to transform a non-linear temperature-ocean interaction as well as the standard non-linear radiative forcing question into an equation that is linear in $\boldsymbol{\tau}_t$, and Appendix A gives a brief summary. The shifted autoregressive process z_t is summarized by its conditional expectation and variance:

$$\mathbb{E}_t\, z_{t+1} \quad = \underbrace{\gamma^z z_t}_{\text{persistence}} + \underbrace{\epsilon(c,\delta_\tau)\left(\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)}_{\text{expectation correction}} \tag{16}$$

$$\mathbb{V}\text{ar}_t\, z_{t+1} = \mathbb{V}\text{ar}_t\, \tau_{t+1} = \mathbb{V}\text{ar}_t\, y_{t+1} = c\Big[\ \underbrace{2\gamma^z y_t}_{\substack{\text{variance}\\ \text{persistence}}} + \underbrace{\delta_\tau\left(\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)}_{\text{endogenous forcing uncertainty}}\ \Big]. \tag{17}$$

Equation (16) represents the expectation's deviation from the deterministic model. The first term shows the persistence of learning shocks, reflecting updates to our understanding of nature's response to CO_2. Stronger-than-expected feedbacks in the current period imply stronger feedbacks in the future, quantified by the autoregression coefficient γ^z. The final term aligns the expected temperature evolution with its deterministic counterpart. The atmospheric temperature in degrees Celsius, $T_{1,t+1} = \frac{1}{\xi_1}\log(\tau_{1,t+1})$, is concave in $\tau_{1,t+1}$. By Jensen's inequality, a mean zero shock to $\tau_{1,t+1}$ would reduce the expected temperature $T_{1,t+1}$. The "expectation correction" adjusts the stochastic process upward to account for this non-linearity, ensuring the long-term deterministic and expected temperatures match.

Equation (17) describes the uncertainty in atmospheric temperature evolution, which increases with climate system perturbations ($\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau$) and is autoregressive. Greater forcings from fossil fuel burning and other greenhouse gases raise uncertainty about future temperatures. The parameter $\eta_\tau \in (0,1)$ ensures low uncertainty in the unperturbed preindustrial system ($M_{1,t} = M_{pre}$ and $G_t = 0$). The parameter δ_τ scales endogenous uncertainty from climate perturbations relative to the autoregressive component. The process y_t is the underlying autoregressive gamma process, with variance persistence characterized by γ^z. If uncertainty is high in one period, it likely

remains high in subsequent periods. The parameter c scales the overall variance of the autoregressive gamma process. The expectation correction term in (16) increases with uncertainty, requiring a larger correction due to temperature expectation non-linearity. Thus, the correction term grows with the perturbation term $\frac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau$ and the uncertainty-scaling parameters c and δ_τ, described by the function $\epsilon(c, \delta_\tau)$.

New to the present paper is only the novel volatility term Δ_t which I assume to be iid. This stochastic process represents weather shock as compared to the persistent stochastic process governing global average temperatures. The novel term introduces the necessary distinction to explore how the adaptation and damage response differs between an uncertain persistent temperature evolution and expected but stochastic weather shocks.

3.2 Objective Function and Timing of Decisions

The purpose of this paper is to explore the implications of forward-looking behavior and climate adaptation for damage estimation and the social cost of carbon (SCC). Under uncertainty, such forward-looking behavior is effectively modeled using Bellman's dynamic programming equation. This approach simplifies the (infinite horizon) intertemporal optimization problem by breaking it down into a straightforward two-period problem. Instead of optimizing across infinitely many possible futures, the challenge becomes finding the value function $V(\cdot)$, which captures future discounted welfare based on the current states of the economy and climate.

In the present model, these states are a vector $\boldsymbol{\tau}_t$ characterizing exponential temperature in the atmosphere and different ocean layers, a vector $\boldsymbol{M}_t$ characterizing the carbon concentration in the atmosphere and different carbon reservoirs, the vectors $\boldsymbol{R}_t$ characterizing the resources left in the ground, adaptation capital A_t, total (non-adaptation) capital in the economy K_t, and additional informational state variables $\boldsymbol{I}_t$ that characterize stochastic processes (e.g., last period's shock in an autoregressive model). I also introduce time itself as an additional state variable to permit additional exogenous time trends. The Bellman equation is

$$\begin{aligned} &V(K_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, \boldsymbol{I}_t, t) \\ &= \max_{\boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t, i_t} \mathbb{E}_t \Big(\log C_t + \beta V(K_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, A_{t+1}, \boldsymbol{I}_{t+1}, t+1) \Big) \Big] \Big). \end{aligned} \tag{18}$$

The decision maker optimizes the labor distribution across sectors $\boldsymbol{N}_t$, capital use

$\boldsymbol{K}_t$ across sectors, the energy input vector $\boldsymbol{E}_t$, and the investments into adaptation capital i_t.[12] The log-utility $\log C_t$ represents the current consumption benefit and $\beta V(\cdot)$ represents future utility.

The expectation operator $\mathbb{E}_t$ in the Bellman (18) represents the expectations conditional on period-t information. It is common to see the expected value operator only in front of the value function $V(\cdot)$. In the present setting, it is crucial that the expected value is applied to the full right side of the Bellmen equation, and after the maximization operator. The agents take their investment decisions, including for adaptation capital, before the realization of temperatures. The damages in equation (2) depend on temperature τ_{t+1}, which is stochastic. As a result, the agents can only adapt to expected temperatures.

The more common timing of the Bellman equation (18) places the expected value operator only in front of the value function $V(\cdot)$. In this case, the agents can always perfectly control their current consumption and shocks merely affect future capital accumulation. Such a setting would be unrealistic in the present context of temperature shocks. Heat waves cause hospitalizations and mortality because individuals are struck by a heatwave, not (or at least less so) because it leads to a lower capital accumulation. The present timing permits that shocks directly affect an individual's consumption level and current welfare experience, and it forces individuals to choose adaptation levels under uncertainty over the actual temperature realization. They will no longer be able to fully adapt to any given temperature realization.

3.3 Results in an Uncertain World

The following proposition states the optimal adaptation investment and the resulting shape of the optimally adapted damage function in the uncertain world laid out above.

Proposition 3 *The optimal investment into adaptation capital is*

$$i_t^* = \frac{1}{\chi}\Big[\log(\gamma) - \log\Big(\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\Big) + \mathrm{CGF}_{\tau_{t+1}}(\lambda)\Big] - A_t. \tag{19}$$

where $\mathrm{CGF}_{\tau_{t+1}}(\lambda)$ *denotes the cumulant generating function of the conditional distri-*

[12] The total capital employed $K_t = \sum_i K_{i,t}$ determines investment and, thus, consumption. The Appendix splits the decision into a consumption (and investment) rate and a distribution of capital across sectors.

bution of τ_{t+1}. The resulting optimally adapted expected *damages are*

$$\mathbb{E}_t\, d_t(\tau_{1,t+1}, i_t^*, A_t) = \xi_0(\mathbb{E}_t\, \tau_{1,t+1} - 1) + \frac{\eta}{\chi}\big(1 - \beta(1 - \delta_A)\big)\ . \tag{20}$$

Optimal investment follows a similar structure as under certainty. The key difference is that investment does not respond directly to temperature but rather to all moments of the climate distribution, encapsulated in the cumulant generating function

$$\mathrm{CGF}_{\tau_{t+1}}(\lambda) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!}\lambda^n = \lambda\, \mathbb{E}_t\, \tau_{1,t+1} + \frac{1}{2}\lambda^2\, \mathbb{V}\mathrm{ar}_t(\tau_{1,t+1}) + \frac{1}{6}\lambda^3\, \mathbb{S}\mathrm{kew}(\tau_{1,t+1}) + ... \tag{21}$$

where κ_n denotes the n-th cumulant of τ_{t+1}, conditional on time t information.[13] The first cumulant corresponds to the expected value, the second to the variance, and the third captures skewness, measuring asymmetry in the distribution. Higher-order cumulants differ slightly in normalization from central moments but convey qualitatively similar information.

Ignoring higher-order moments, the investment under uncertainty would respond to expected temperatures in the same way that an investment under certainty responds to realized temperature. However, in the presence of uncertainty, also variance, skewness, and higher-order moments ("fat tails") of the temperature distribution can significantly influence optimal protective adaptation investment.

Equation (20) shows that optimally adapted expected damages under uncertainty take the same form as damages under certainty, with realized temperature simply replaced by its expected value. This result arises from the specific functional forms used here and does not hold in general. Without adaptation, damages would be significantly higher under uncertainty. However, the decision-maker increases adaptation investment to offset the higher expected damages, ensuring that adapted damages remain the same in expectation.

Equation (20) also reveals that observing expected damages is hardly more helpful than observing damages in the deterministic world. Fortunately, we observe realized rather than expected temperatures.

Proposition 4 *Observed damages at a given temperature realization $\hat{\tau}_{1,t+1}$ are*

$$d_t(\tau_{1,t+1}, i_t^*, A_t) = \xi_0(\tau_{1,t+1} - 1) + \frac{\eta}{\chi}\big(1 - \beta(1 - \delta_A)\big) \exp\left(\lambda\tau_{1,t+1} - \mathrm{CGF}_{\tau_{t+1}}(\lambda)\right). \tag{22}$$

[13] The cumulant generating function is the logarithm of the moment generating function, i.e., $\mathrm{CGF}_{\tau_{t+1}}(\lambda) \equiv \log\left[\mathbb{E}_t e^{\lambda\tau_{t+1}}\right]$. This representation allows for a convenient expansion in terms of moments.

Under uncertainty, observed damages promise to reveal the factor $\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)$ that we can conjecture to, once again, play an important role for the social costs of a temperature increase. Using equation (21), I can rewrite equation (22) in the form

$$d_t(\tau_{1,t+1}, i_t^*, A_t) = \xi_0\tau_{1,t+1} + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\exp\left(\lambda\big(\tau_{1,t+1}-\mathbb{E}_t\,\tau_{1,t+1}\big) - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!}\lambda^n\right).$$

Let me first ignore higher-order moments and define a small deviation from expectations (or "the trend") as $\Delta\tau \equiv \mathbb{E}_t\,\tau_{1,t+1}-\tau_{1,t+1}$. Then, a *marginal deviation* delivers precisely the identification conjectured in equation (9), where it was based on an unanticipated shock in the deterministic model, $\log Y_t = ... - \Big[\xi_0 + \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\Big]\Delta\tau$. Thus, ignoring higher-order moments, the fact that weather is stochastic does indeed allow the econometrician to identify crucial components of the adapted damage function.

In general, the time series econometrician observes

$$\log Y_t = \xi_0\,\mathbb{E}_t\,\tau_{1,t+1} - \xi_0\Delta\tau - \frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)\exp\left(\lambda\Delta\tau - \sum_{n=2}^{\infty}\frac{\kappa_n}{n!}\lambda^n\right),$$

and has to control for the higher-order moments of the temperature distribution. Fortunately, these moments are readily observable in the temperature data. The following proposition confirms that the now observable effective cost factor $\frac{\eta}{\chi}\big(1-\beta(1-\delta_A)\big)$ remains the most crucial contribution to social costs deriving from the adaptable damage component.

Proposition 5 *The SCC under optimal adaptation and climate uncertainty is*

$$SCC = \underbrace{\frac{\beta Y_t^{net}}{M_{pre}}\left(\xi_0 + \frac{\eta\lambda}{\chi}\big(1-\beta(1-\delta_A)\big)\right)\Big(\sigma^{forc}}_{\text{as under certainty}} + \underbrace{\Big(1+\frac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau)}_{\text{non-linearity uncertainty correction}}\Big)$$
$$\Big(1+\big[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\big]_{1,1}\Big)\big[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\big]_{1,1} + SCC^{endo}$$

with the additional contribution

$$SCC^{endo} = \tfrac{\beta Y_t^{net}}{M_{pre}}h\delta_\tau\tfrac{\eta\lambda}{\chi}\big(1-\beta(1-\delta_A)\big)\Big(\tfrac{\gamma^z}{1-\beta\gamma^z}\tfrac{\lambda hc}{1-\lambda hc} - 1 - \tfrac{\log(1-\lambda hc)}{\lambda hc}\Big)\big[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\big]_{1,1}$$

capturing the endogeneity of climate change uncertainty, i.e., that emissions not only cause warming but also increase climate change uncertainty.

The first part of the expression coincides with the SCC in Proposition 2. The second part, labeled "non-linearity uncertainty correction," arises as a result of the stochastic fluctuations implied by uncertainty. Climate damages are a convex function of temperature. As a result, a symmetric shock to temperature causes a higher welfare loss than the expected temperature. In the expected damage equation (20), this convexity is embedded in the transformation $\tau_{1,t+1} = \exp(\xi_1 T_{1,t+1})$ and results from the unadapted damage componentand is a function of the parameters characterizing the climate distribution as well as the discount rate: a more patient decision-maker will pay more attention to the persistent (high γ^z) impacts of today's temperature shock. This SCC contribution is not *directly* observed by the econometrician but will generally be picked up in the structural model using the estimated damages to simulate forward a calculation of the SCC.

Finally, Proposition 5 identifies a third contribution to the SCC. Emitting carbon does not merely increase the level of temperature, but it changes the temperature distribution, also increasing the uncertainty over future climate. As explained in Traeger (2021), the endogeneity of uncertainty is a crucial component of the SCC that is generally overlooked in studies simulating expected damages under a given distribution of the future climate. This contribution interacts with adaptation, but eliciting the endogeneity of climate uncertainty is an orthogonal exercise relying on scientific rather than economic data.

3.4 Identifying Climate Damage Measures That Include Adaptation Costs

Excellent reviews of the empirical work identifying climate impacts include Hsiang (2016), Auffhammer (2018), and Hogan & Schlenker (2024). One of the most recent attempts to include unobserved adaptation costs follows an approach suggested by Butler & Huybers (2013) and further developed by Carleton et al. (2022). It uses (slightly refined) versions of a panel regression of the form

$$\log Y_{nt} = \alpha_n + \underbrace{\left(\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 \bar{T}_n + \text{“further controls”}\right)}_{\equiv \boldsymbol{\beta}^\dagger} \boldsymbol{f}(T_{nt}) + \boldsymbol{\zeta} \boldsymbol{X}_{nt} + \delta_t + \epsilon_{nt} \qquad (23)$$

where I adapted the outcome variable to be log-output rather than, e.g., mortality in Carleton et al. (2022), and omitted more refined group-specific controls such as, e.g., country-year fixed effects instead of year dummies δ_t when observations are at

the county level. The location fixed effects α_n control for unobserved time-invariant location-specific characteristics and $\boldsymbol{X_{nt}}$ denotes explicitly observed controls. While I continue to use log-output as the outcome variable, the analogous reasoning applies to crop-yields, mortality, energy use, and other fields of climate impacts and adaptation measures.

In the empirical context, we should think of T_{nt} as sufficient statistics of the weather realization over the relevant time period, e.g., the course of a year. E.g., Carleton et al.'s (2022) function $f(T_{nt})$ captures different powers of daily average temperatures summed over the course of the year. Alternatively, one could include heat days, cold days, or use temperature bins. Important for the present framework is that these realizations reflect weather realizations as opposed to climate. Climate will be the distribution of the realizations T_{nt}, and our ultimate goal is to evaluate how impacts change with a changing climate.

Neglecting the details in the parenthesis for the moment, the regression coefficient $\boldsymbol{\beta^\dagger}$ measures the impact of weather on the outcome variable. Applying the regression to equation (22), the regression coefficient identifies the variation

$$\frac{\partial d_t(\tau_{1,t+1}, i_t^*, A_t)}{\partial T_t} = \left[\xi_0 + \frac{\eta}{\chi}\lambda\big(1-\beta(1-\delta_A)\big)\exp\left(\lambda\tau_{1,t}^n\right)\exp(-\operatorname{CGF}_{\tau_t}^n(\lambda))\right]\frac{\partial\tau_t^n}{\partial T_t^n} \quad (24)$$

where I added location indices on the right side to point out that, at the very least, temperatures and climate are location-specific.[14] How precisely the regression would pick up the present damage specification somewhat depends on the permitted non-linearity of the regression. Most crucially, the common specification in equation (23) would struggle with the fact that it employs the same non-linear specification for the interaction with both $\boldsymbol{\gamma_0}$ and $\boldsymbol{\gamma_0}\bar{T}_n$. In the present model, $\boldsymbol{\gamma_0}$ would pick up the non-adaptable component ξ_0, which has a different steepness of damages as compared to the adaptable component. But this shortcoming is easily fixed by allowing for different non-linearities, say functions $\boldsymbol{f}_0(T_{nt})$ and $\boldsymbol{f}_t(T_{nt})$.

The coefficient $\boldsymbol{\gamma_1}$ plays a crucial role when unpacking the parentheses that make up the regression coefficient $\boldsymbol{\beta^\dagger}$. The hope is that the interaction of weather $\boldsymbol{f}(T_{nt})$ and climate $\bar{T}_n$ identify the effectiveness and, thus, cost of adaptation. Intuitively, as we adapt to expected weather, we become less vulnerable. E.g., if we consistently

[14] I also changed the timing of the temperature label to the present period, which is better aligned with standard notation in empirical settings where we discuss the ex-post realization of the temperature in period t.

experience hotter days, then we adopt AC and are less vulnerable to hotter days. Based on this reasoning, we can follow Carleton et al. (2022) and try to identify adaptation costs. For this purpose, they take the derivative of $\boldsymbol{\beta}^{\dagger}$ w.r.t. the mean temperature $\bar{T}_n$. Realizing that the cumulant generating function $\text{CGF}^n_{\tau_t}(\lambda)$ contains the summand $\lambda\bar{T}_n$. Thus, up to some non-linearity, the (indirect) observation of adaptation costs using the interaction between weather and mean temperatures indeed works. That said, a potential oversight is that the original coefficient $\boldsymbol{\beta}^{\dagger}$ already contains a proxy of adaptation costs. In particular, I note that we can only obtain a proxy of adaptation costs in $\boldsymbol{\beta}^{\dagger}$ if we observe shocks that are not adapted for. But then, this proxy of adaptation costs is already part of the resulting impact estimate even without explicitly backing them out and adding them explicitly once again.

As compared to the surprise shock under certainty discussed in equation (15) in Section 2.3, a challenge in the stochastic setting remains that it is non-trivial to determine the precise temperature level at which to evaluate equation (24). As already pointed out by Mérel et al. (2024) in the static setting, simply using expected temperatures is generally not sufficient.

3.5 General Discussion and Dynamic Envelope Theorem

In general, the value function depends on climate, i.e., the (changing) distribution determining weather. I denote the probability distribution over temperature by $P(T_t; \boldsymbol{m}_t)$ and characterize the distribution by the vector of moments $\boldsymbol{m}_t = [m_{1,t}, m_{2,t}, \ldots, m_{k,t}]$ specifying mean, variance, skewness,... of the temperature distribution. Here, I assume that all distributional moments are finite and that, for practical purposes, we can approximate the distribution sufficiently well with a finite number of moments. As already in Section 2.3, I simplify the presentation and drop states that are not of immediate importance to the reasoning. The resulting Bellman equation is

$$V(\boldsymbol{m}_t, A_t) = \max_{i_t} \mathbb{E}\left[U(T_t, i_t, A_t) + \beta V(\boldsymbol{m}_{t+1}, A_{t+1})\right].$$

Optimizing abatement expenses i_t results in the first-order condition

$$\mathbb{E}\left[\frac{\partial U(T_t, i_t, A_t)}{\partial i_t} + \beta \frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial i_t}\right] = 0. \tag{25}$$

I assume that the interior solution $i_t^* = i_t^*(\boldsymbol{m}_t, A_t, K_t)$ exists and maximizes the objective. Investment can only depend on the moments of the distribution, not on the

realized temperature, which is unknown to the decision maker at the time of investing.

A change in climate corresponds to a change in the moments of the weather distribution. Differentiating the value function with respect to the vector of moments $\boldsymbol{m}_t$ results in a vector of changes[15]

$$\frac{\partial V(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t} = \frac{\partial}{\partial \boldsymbol{m}_t}\mathbb{E}\left[U(T_t, i_t^*(\boldsymbol{m}_t, A_t, K_t), A_t)\right] + \beta\frac{\partial}{\partial \boldsymbol{m}_t}\mathbb{E}\left[V(\boldsymbol{m}_{t+1}, A_{t+1})\right].$$

Spelling out the different contributions in detail and canceling those that add up to zero based on the first order condition for investment, equation (25), i.e., explicitly writing down and canceling the terms that can be omitted based on the envelope theorem, I find the following forward-looking welfare response to a change in the climate

$$\begin{aligned}
\frac{\partial V(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t} = {} & \cancel{\mathbb{E}\left[\frac{\partial U(T_t, i_t^*(\boldsymbol{m}_t, A_t), A_t)}{\partial i_t}\right]\frac{\partial i_t^*(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t}} \\
& + \int U(T_t, i_t^*(\boldsymbol{m}_t, A_t), A_t)\frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t. \\
& + \beta\mathbb{E}\left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1})}{\partial \boldsymbol{m}_{t+1}}\frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t}\right] \\
& + \cancel{\beta\mathbb{E}\left[\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1})}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial i_t}\right]\frac{\partial i_t^*(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t}} \\
& + \int V(\boldsymbol{m}_{t+1}(T_t), A_{t+1})\frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t. \qquad (26)
\end{aligned}$$

The first non-vanishing term quantifies how changes in the probability density function governing temperatures alter the representative agent's expected utility in the current period.

The second non-vanishing term captures the effect on future welfare when a shift in the current climate influences the future climate. Typically, we are not concerned with arbitrary climate change but rather with anthropogenic shifts driven by greenhouse gas emissions. These emissions determine both the specific change $d\boldsymbol{m}_t$ and its evolution over time, given by $\frac{\partial \boldsymbol{m}_{t+1}}{\partial \boldsymbol{m}_t}$.

A simple yet insightful model of anthropogenic climate change assumes that a ton of carbon emitted today raises the expected temperature in all future periods by the same amount. This permanent temperature response to emissions is known as the transient

[15] In the end, we are generally interested in the particular combination of changes of moments generated by our greenhouse gas emissions, mapping this vector of change back into a single value.

climate response to cumulative carbon emissions (TCRE). Under this assumption, a ton of carbon emissions affects only the first moment of the temperature distribution, i.e., the mean temperature $\bar{T}_t$. Moreover, since this carbon-induced climate shift is permanent, the only relevant moment satisfies $\frac{\partial m_{t+1}}{\partial m_t} = \frac{\partial \bar{T}_{t+1}}{\partial \bar{T}_t} = 1$.

Finally, the third and final non-vanishing term arises in situations where current temperature realizations affect the assumed future distribution of climate change. This potential dependence of the moments $\boldsymbol{m}_{t+1}(T_t)$ on the temperature realization arises in cases where endogenous learning occurs through temperature observations. In such scenarios, a high-temperature realization may also update our beliefs about the future climate. To simplify the presentation, from here onward, I will assume that next period's climate only depends on shifts in current climate but not on current temperature realizations, excluding such endogenous learning specifications.

Let me now make the abstract equation (26) operational, which requires addressing two conceptual challenges. First, we do not observe utility levels. Second, we generally do not know the value function $V(\cdot)$ either. Moving from utility to an observable, we first have to pick a form for the utility function. Staying with the model laid out in the previous sections, I have

$$U(T_t, i_t^*(\boldsymbol{m}_t, A_t), A_t) = f_t - \eta i_t^* - d_t(\tau_{1,t}, i_t^*, A_t),$$

where I collect production characteristics and capital- (or income-) dependent terms in the additively separable variable f_t.[16] The arguably more challenging concern is that equation (26) relies on the absolute values of utility and damages. Empirically, we cannot expect to identify such levels. Fortunately, I can employ integration by parts to transform the relevant expression as follows (see Appendix for details)

$$\begin{aligned}
&\int U(T_t, i_t^*(\boldsymbol{m}_t, A_t), A_t) \frac{\partial P(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t}\, dT_t \\
&\quad = -\int \frac{\partial U(T_t, i_t^*(\boldsymbol{m}_t, A_t), A_t)}{\partial T_t} \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} P(T_t; \boldsymbol{m}_t)\, dT_t \\
&\quad = \int \frac{\partial d_t(\tau_{1,t}, i_t, A_t)}{\partial \tau_t} \frac{\partial \tau_t}{\partial T_t} \frac{\partial CDF(T_t; \boldsymbol{m}_t)}{\partial \boldsymbol{m}_t} P(T_t; \boldsymbol{m}_t)\, dT_t.
\end{aligned}$$

The final row employs the observable variation of output with weather shocks and a

[16] See equation (B.2) in Appendix B.2 for the precise expression of utility in this model: $u(x_t) = \log C_t = \log x_t + \kappa \log K_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - d_t(\tau_{1,t}, i_t, A_t)$.The additive separability of income (or capital) crucially relies on the convenient assumption of log-utility.

measure of climate change that we obtain from climate change projection, where the cumulative distribution function $CDF(T_t; \boldsymbol{m}_t)$ replaces the earlier probability density function $P(T_t; \boldsymbol{m}_t)$ to characterize the change in climate.

Second, I have to eliminate the unknown value function $V(\cdot)$ from equation (26). By excluding the case where future climate depends not only on current climate, but on current weather realization, I already eliminate the final term.[17] I can recursively eliminate the value function derivatives $\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1})}{\partial \boldsymbol{m}_{t+1}}$ following an approach pioneered in Jensen & Traeger (2014). Here, I use equation (26) to recursively replace $\frac{\partial V(\boldsymbol{m}_{t+1}, A_{t+1})}{\partial \boldsymbol{m}_{t+1}}$. Then, the following proposition holds.

Proposition 6 *For simplicity, I assume that the beliefs about future climate change are not affected by the current temperature realizations causing the damage. Then, the comprehensive forward-looking welfare costs of climate change can be calculated as*

$$\frac{\partial V(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t} = -\,\mathbb{E}_t\left[\sum_{l=t}^{\infty} \beta^{l-t} \frac{\partial U(T_l, i_l^*, A_l)}{\partial T_l} \frac{\partial CDF(T_l; \boldsymbol{m}_l)}{\partial \boldsymbol{m}_l} \frac{\partial \boldsymbol{m}_l}{\partial \boldsymbol{m}_t}\right], \tag{27}$$

where $\boldsymbol{m}_t$ is a vector of moments describing the climate and $\frac{\partial \boldsymbol{m}_l}{\partial \boldsymbol{m}_t}$ abbreviates the causal impact of a current change on future climate.

In the case of the ACE model discussed above, the marginal utility loss translates into $\frac{\partial d_t(\tau_{1,t}, i_t, A_t)}{\partial \tau_l} \frac{\partial \tau_l}{\partial T_l}$. It might be most helpful to translate Proposition 6 into a recipe to calculate the social cost of carbon. Once again, let me illustrate the procedure using the simple TCRE model, assuming that a ton of CO_2 emitted today (merely) shifts the expected temperature up by a constant amount in all future periods. Today's best guess for this number is a permanent temperature increase of $1.65{*}10^{-12}$ degree Celsius per ton of carbon emitted (IPCC 2021). Then, a ton of CO_2 emitted only affects the first moment on the climate distribution, and we only need the first entry of the vector of derivatives $\frac{\partial V(\boldsymbol{m}_t, A_t)}{\partial \boldsymbol{m}_t}$. Let's denote this first moment, the mean global surface temperature, by $\bar{T}_t$.[18] The TCRE model suggests that the increase is permanent and,

[17] The stochastic ACE model used for the examples discussed above includes an endogenous response of future beliefs on present observations and, thus, gives rise to additional terms in the SCC.

[18] Here, I assume that T_t refers to actual temperature. If T_t or τ_t is interpreted as a sufficient statistics for an annual weather realization (see footnote **??**) then the present experiment becomes a bit more complex.

thus, $\frac{\partial \bar{T}_l}{\partial T_t} = 1$ (covering the only relevant entry of $\frac{\partial \boldsymbol{m}_l}{\partial \boldsymbol{m}_t}$). Then[19]

$$SCC = -\frac{1.65 * 10^{-12}}{MU_t} \mathbb{E}_t \left[\sum_{l=t}^{\infty} \beta^{l-t} \frac{\partial U(T_l, i_l^*, A_l)}{\partial T_l} \frac{\partial CDF(T_l; \boldsymbol{m}_l)}{\partial \bar{T}_l} \right] \tag{28}$$

$$= \frac{1.65 * 10^{-12}}{MU_t} \mathbb{E}_t \left[\sum_{l=t}^{\infty} \beta^{l-t} \frac{\partial d_t(\tau_{1,t}, i_t, A_t)}{\partial \tau_l} \frac{\partial \tau_l}{\partial T_l} \frac{\partial CDF(T_l; \boldsymbol{m}_l)}{\partial \bar{T}_l} \right].$$

where a division by marginal utility MU_t translates the SCC into consumption equivalents. More interesting models, such as ACE or the empirical simulations used in Carleton et al. (2022) give rise to more sophisticated SCC calculations using more elaborate implementations of the dynamics $\frac{\partial \boldsymbol{m}_l}{\partial \boldsymbol{m}_t}$. A challenge in evaluating expressions such as in equations (27) and (28) is that the possible trajectories under the expectations generally depend on the forward-looking decisions taken along the way as climate evolves. If climate evolves deterministically and only the weather remains stochastic, such simulations are generally straightforward. If the climate response to emissions itself is uncertain, then future decisions have to be conditioned on future realized climate change and the evaluation usually requires a recursive stochastic model implementation.

The main takeaway of this section is that, also under uncertainty and with stock adaptation, we can identify the crucial damage-related components of the social cost of climate change merely from observing weather shocks. By observing unadapted variation, we obtain an impact measure that is inclusive of adaptation costs (or rather of an equivalent magnitude). Importantly, this identification strategy does not rely on cross-sectional information and is, thus, less susceptible to omitted variable bias than previously suggested approaches.

4 Conclusion

This paper advances the conceptual foundations of estimating climate change damages by integrating adaptation explicitly into a full-scale, analytic, and forward-looking integrated assessment model. Standard econometric approaches, including cross-sectional and time-series methods, capture different aspects of climate damages, but neither

[19] I keep the general argument of the cumulative distribution $CDF(T_l; \boldsymbol{m}_l)$ because all moments can change exogenously over time, yet, we are only interested in the marginal shift of the mean causal to the ton of carbon emitted.

fully accounts for the role of adaptation. Even idealized cross-sectional estimates that were able to resolve the omitted variable bias fall short of estimating the true costs of climate change because they cannot measure the full costs of adaptation. By contrast, unexpected temperature variations can serve as an effective tool for identifying adaptation-inclusive climate costs using a refined time-series approach.

While the envelope theorem is sometimes interpreted as implying that adaptation has no marginal value and could be neglected in the estimation, the present paper shows that the cost of adaptation plays a crucial role and that, somewhat paradoxically, its value can be identified precisely through unexpected and unadapted variations because of the envelope theorem. By explicitly modeling forward-looking and persistent adaptation responses in a quantitative framework, this analysis provides a clear method for interpreting adaptation's contribution to the social cost of climate change.

I show that adaptation depends not only on expected temperature trends but on the full climate distribution, including higher-order moments such as variance, skewness, and tail risks. This has important implications for empirical damage estimation: rather than relying solely on deviations from the trend, more reliable damage estimates should incorporate the full statistical structure of climate variation. Moving seamlessly from estimation to the SCC in an uncertain world, I integrate climate damage estimates with necessary corrections for the non-linear welfare effects of temperature uncertainty and the endogeneity of climate risk. This approach ensures that uncertainty is not merely a residual consideration but an integral component of the assessment. The framework strengthens the connection between empirical damage estimation and quantitative policy assessment by offering a systematic and consistent way to move between reduced-form econometric estimation and structural assessments of the social cost of carbon.

Comment for the NBER submission: I am aware that the presentation governing the connection between the theoretical results and hands-on estimation approaches can (and will) be improved for the meetings (if this paper will be included).

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Appendix

A Summary of Remaining IAM Components

Emissions and Resources. The first d entries of the energy vector $\boldsymbol{E}_t$ are fossil-fuel-based, measured by their carbon content, and summarized in the vector $\boldsymbol{E}_t^d$. Total emissions from these fossil fuels are $\sum_{i=1}^{I^d} E_{i,t}$, and exogenous emissions from land conversion, forestry, and agriculture are denoted by E_t^{exo}. Fossil fuel use reduces the resource stocks in the ground $\boldsymbol{R}_t \in \mathbb{R}_+^{I^d}$:

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d,$$

with initial stock levels $\boldsymbol{R}_0$ given. If a resource is scarce along the optimal path, I assume that its use is stretched over an infinite time horizon.

Capital Accumulation. The base model invests production left after climate damages and consumption

$$I_t = Y_t^{net}[1 - D_t(T_{1,t})] - C_t.$$

The full depreciation assumption by Golosov et al. (2014) would correspond to $K_{t+1} = I_t$. Such a full depreciation assumption has two main shortcomings: (i) it misses the remaining capital from previous periods, and (ii) it fails to account for how current investments increase future capital availability. The equation

$$K_{t+1} = I_t \left[\frac{1 + g_{k,t}}{\delta_k + g_{k,t}} \right], \tag{29}$$

provides a "quick fix" using a correction factor that is greater than one when the depreciation factor δ_k is below one. It enhances both the availability of capital in the next period and the payoff from current investments in terms of future capital. Traeger (2023) shows that equation (29) coincides with the standard capital accumulation equation

$$K_{t+1} = Y_t[1 - D_t(T_{1,t})] - C_t + (1 - \delta_k)K_t.$$

if the exogenous capital growth approximation $g_{k,t} = \frac{K_{t+1}}{K_t} - 1$ is accurate, or if $\delta_k = 1$. This approach adjusts the capital dynamics and capital-output ratio to align with macroeconomic observations, making decision-makers aware of additional future returns from capital investments.

Carbon Cycle and Greenhouse Effect. Carbon cycles through atmospheric and non-atmospheric reservoirs, with atmospheric carbon $M_{1,t}$ and reservoirs $M_{2,t}, ..., M_{m,t}$ governed by:

$$\boldsymbol{M}_{t+1} = \boldsymbol{\Phi}\boldsymbol{M}_t + \boldsymbol{e}_1(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}).$$

Increased atmospheric carbon causes radiative forcing, modeled by

$$Q_t = \eta \frac{\log \frac{M_{1,t}+G_t}{M_{pre}}}{\log 2}.$$

where η defines the greenhouse effect's strength and G_t captures the radiative forcing from other, exogenous greenhouse gases in CO_2 equivalents.

Temperature Dynamics. Apart from atmospheric temperature $T_{1,t}$, the model uses a finite number of ocean layers $T_{i,t}, i \in \{2, ..., l\}, l \in \mathbb{N}$ to capture the atmosphere-ocean heat exchange that substantially cools our surface at the moment. Abbreviating the atmospheric equilibrium temperature resulting from the radiative forcing level Q_t by $T_{0,t} = \frac{s}{\eta}Q_t$, I let temperatures between different layers adjust using a generalized mean (see Traeger (2023) for details)

$$T_{i,t+1} = \mathfrak{M}_i^{\boldsymbol{\sigma}}(T_{i,t}, T_{i-1,t}, T_{i+1,t}) \text{ for } i \in \{1, ..., l\}. \tag{30}$$

The weight matrix $\boldsymbol{\sigma}$ defines heat flow. Using the transformation $\tau_{i,t} = \exp(\xi_1 T_{i,t})$, $i \in 1, ..., l$, already introduced in the main text for atmospheric temperatures, and stacking them into the vector $\boldsymbol{\tau}_t$, the transformed diffusion model becomes linear

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t} + G_t}{M_{pre}}\boldsymbol{e}_1 \tag{31}$$

enabling an analytic solution.

B Derivations

B.1 Climate system

I transform the temperature's equation of motion (30) for layer layer $i \in \{1, ..., l\}$ to

$$T_{i,t+1} = \frac{1}{\xi_1} \log \Big((1 - \sigma_{i,i-1} - \sigma_{i,i+1}) \exp[\xi_1 T_{i,t}] + \sigma_{i,i-1} \exp[\xi_1 T_{i-1,t}] + \sigma_{i,i+1} \exp[\xi_1 T_{i+1,t}] \Big) .$$

Using the definitions $\sigma_{ii} = 1 - \sigma_{i,i-1} - \sigma_{i,i+1}$ and $\tau_{i,t} = \exp(\xi_1 T_{i,t})$ I find

$$\begin{aligned} \exp(\xi_1 T_{i,t+1}) &= \sigma_{i,i-1} \exp[\xi_1 T_{i,t}] + \sigma_{i,i-1} \exp[\xi_1 T_{i-1,t}] + \sigma_{i,i+1} \exp[\xi_1 T_{i+1,t}] \\ \Rightarrow \tau_{i,t+1} \quad &= \sigma_{i,i} \tau_{i,t} + \sigma_{i,i-1} \tau_{i-1,t} + \sigma_{i,i+1} \tau_{i+1,t}, i \in \{2, ..., l\}, \end{aligned}$$

still using $\sigma_{l,l+1} = 0$ for notational convenience (see footnote **??**). Noting that

$$\exp[\xi_1 T_{0,t}] = \exp\left[\xi_1 \frac{s}{\eta} Q_t\right] = \exp\left[\xi_1 \frac{s}{\log 2} \log \frac{M_{1,t} + G_t}{M_{pre}}\right] = \frac{M_{1,t} + G_t}{M_{pre}},$$

the equation for atmospheric temperature ($i = 1$) becomes

$$\tau_{1,t+1} = \sigma_{1,1} \tau_{1,t} + \sigma_{1,0} \frac{M_{1,t} + G_t}{M_{pre}} + \sigma_{1,2} \tau_{2,t} .$$

Note that the linearity in $M_{1,t}$ requires $\xi_1 = \frac{\log 2}{s}$ as stated in the proposition. Then, using the definition $\sigma^{forc} = \sigma_{1,0}$, the individual equations of motion for generalized temperature can be collected into the vector equation

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma} \boldsymbol{\tau}_t + \sigma^{forc} \frac{M_{1,t} + G_t}{M_{pre}} \boldsymbol{e}_1 . \tag{B.1}$$

B.2 Setting up the optimization problem

Let the production inputs be $\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t \in \mathbb{R}_+^I$ and let the first $I^d \leq I$ energy sources $E_1, ..., E_{I^d}$ be fossil fuels generating emissions with stock levels $\boldsymbol{R}_t \in \mathbb{R}_+^{I_d}$ left in the ground (all measured in terms of their CO_2 content). Let $\boldsymbol{M}_t \in \mathbb{R}_+^m$ characterize carbon in the different reservoirs and $\boldsymbol{T}_t \in \mathbb{R}_+^l$ comprise atmospheric global average temperature $T_{1,t}$ and temperature in the different ocean layers. The components of

the normalized vector $\boldsymbol{\mathcal{K}}_t \equiv \frac{\boldsymbol{K}_t}{K_t}$ satisfy $\sum_{i=1}^{I} \mathcal{K}_{i,t} = 1$ and I define log-capital as $k_t \equiv \log K_t = \log\left(\sum_{i=1}^{I} K_{i,t}\right)$. I work with transformed exponential temperature states $\tau_{i,t} \equiv \exp(\xi_i T_{i,t}),\ i \in \{1, ..., L\}$ that I collect in the vector $\boldsymbol{\tau}_t \in \mathbb{R}^L$. Anticipating that optimal consumption and investment will (in most cases) be proportional to net production I define the consumption rate $x_t = \frac{C_t}{Y_t^{net}}$.

The assumption stated verbally in the main text that production is homogenous of some degree κ in capital translates into the formal statement that the exits $\kappa \in (0, 1)$ such that

$$Y_t^{gross} = F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{K}_t, \boldsymbol{E}_t) = K_t^{\kappa} F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t).$$

Expressing utility in terms of the consumption rate results in

$$\begin{aligned} u(x_t) \equiv \log C_t &= \log x_t + \log Y_t^{gross} - \eta i_t + \log[1 - D_t(T_{1,t})] \\ &= \log x_t + \kappa \log K_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - d_t(\tau_{1,t}, i_t, A_t). \end{aligned} \tag{B.2}$$

The Bellman is

$$\begin{aligned} V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, t) = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E_t}, i_t} &\log x_t + \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t \\ &- d_t(\tau_{1,t}, i_t, A_t) + \beta V(k_{t+1}, \boldsymbol{\tau}_{t+1}, \boldsymbol{M}_{t+1}, \boldsymbol{R}_{t+1}, A_{t+1}, t+1) \end{aligned} \tag{B.3}$$

subject to the following linear equations of motion

$$\begin{aligned} k_{t+1} &= \kappa k_t + \log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) - \eta i_t - d_t(\tau_{1,t}, i_t, A_t) + \log(1 - x_t) \\ &\quad + \log[1 + g_{k,t}] - \log[\delta_k + g_{k,t}] \end{aligned} \tag{B.4}$$

$$\boldsymbol{M_{t+1}} = \boldsymbol{\Phi} \boldsymbol{M}_t + \left(\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\right) \boldsymbol{e}_1 \tag{B.6}$$

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma} \boldsymbol{\tau}_t + \sigma^{forc} \frac{M_{1,t} + G_t}{M_{pre}} \boldsymbol{e}_1 \tag{B.7}$$

$$\boldsymbol{R}_{t+1} = \boldsymbol{R}_t - \boldsymbol{E}_t^d \tag{B.8}$$

$$A_{t+1} = (1 - \delta_A) A_t + i_t$$

and constraints

$$\sum_{i=0}^{I} N_{i,t} = 1,\ N_{i,t} \geq 0,\ \sum_{i=1}^{I_K} \mathcal{K}_{i,t} = 1,\ \mathcal{K}_{i,t} \geq 0,\ \boldsymbol{R}_t \geq 0 \tag{B.9}$$

and the initial states given. Following DACE, I assumes that the optimal labor and capital allocation across sectors has an interior solution and that potentially scarce resources are stretched over the infinite time horizon along the optimal path, avoiding boundary value complications.

B.3 Solving the dynamic programming problem

The following derivation extends DACE's solution approach by abatement decisions and abatement capital. I show that the following (affine) value function solve the Bellman equation (B.3)

$$V(k_t, \boldsymbol{\tau}_t, \boldsymbol{M}_t, \boldsymbol{R}_t, A_t, t) = \varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_t. \tag{B.10}$$

Inserting the value function's trial solution (equation B.10) and the next period states (equations B.4-B.8) into the (deterministic) Bellmann equation (B.3) delivers

$$\begin{aligned}
\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_t = \max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}_t}, \boldsymbol{E_t}, i_t} & \log x_t + \beta\varphi_k \log(1-x_t) \\
& + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \underline{-(1+\beta\varphi_k)(\eta i_t + d_t(\tau_{1,t}, i_t, A_t))} \\
& + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + + \lambda_t^{N}\big(1 - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
& + \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^d\right) + \underline{\beta\varphi_A\left((1-\delta_A)A_t + i_t\right)} + \beta\varphi_{t+1} \\
& + \beta\boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta\boldsymbol{\varphi}_\tau^\top \left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right),
\end{aligned} \tag{B.11}$$

where I underlined the novel terms gernerated or altered by adaptation. Maximizing the right hand side of the Bellman equation over the first four controls/control vectors delivers as in DACE

$$x^* = \frac{1}{1+\beta\varphi_k} \tag{B.12}$$

$$\mathcal{K}_{i,t} = \frac{\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\sum_{i=1}^{I_K} \sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} \tag{B.13}$$

$$N_{i,t} = \frac{\sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\sum_{i=1}^{I_N} \sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)} \tag{B.14}$$

$$E_{i,t} = \frac{(1+\beta\varphi_k)\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\beta(\varphi_{R,i,t+1} - \varphi_{M,1})} \tag{B.15}$$

where the shadow cost (Hotelling rent) of non-scarce or non-fossil energy inputs is $\varphi_{R,i,t+1} = 0$ and I defined the elasticities $\sigma_{Y,\mathcal{K}_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}{\partial \mathcal{K}_{i,t}} \frac{\mathcal{K}_{i,t}}{F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t)}$,

$\sigma_{Y,N_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial N_{i,t}} \frac{N_{i,t}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}$, and $\sigma_{Y,E_i}(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \equiv \frac{\partial F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}{\partial E_{i,t}} \frac{E_{i,t}}{F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t)}$.

The new first order conditions governing abatement investment i_t are

$$\frac{\partial}{\partial i_t}\left[-(1+\beta\varphi_k)(\eta i_t + d_t(\tau_{1,t}, i_t, A_t)) + \beta\varphi_A\left((1-\delta_A)A_t + i_t\right)\right] = 0$$

with $d_t(\tau_{1,t}, i_t, A_t) = \xi_0(\tau_{1,t} - 1) + \gamma \exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t))$ implying

$$\begin{aligned}
&-(1+\beta\varphi_k)\eta - (1+\beta\varphi_k)\big(-\gamma\chi \exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t))\big) + \beta\varphi_A = 0 \\
\Rightarrow\ &-(1+\beta\varphi_k)\big(-\gamma\chi \exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t))\big) = (1+\beta\varphi_k)\eta - \beta\varphi_A \\
\Rightarrow\ &\gamma\chi \exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t)) = \eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)} \\
\Rightarrow\ &\exp(\lambda\tau_{1,t}) \exp(-\chi(A_t + i_t)) = \frac{1}{\gamma\chi}\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) \\
\Rightarrow\ &-\chi(A_t + i_t) = -\log(\gamma\chi) + \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) - \lambda\tau_{1,t} \\
\Rightarrow\ &i_t^* = \frac{1}{\chi}\Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \lambda\tau_{1,t}\Big] - A_t
\end{aligned} \tag{B.16}$$

For any given set of shadow values, we can solve the system of equations (B.13-B.15) for the optimal controls $\boldsymbol{N}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, $\boldsymbol{\mathcal{K}}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, and $\boldsymbol{E}_t^*(\boldsymbol{A}_t, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$. After noting that these solutions are independent of the state variables I will suppress the arguments. I can now plug these solutions, x^*, and equation (B.24) for the optimal abatement investment back into the Bellman equation (B.3). For this purpose, I first simplify some of the resulting expressions. Given optimal

abatement levels i_t^*, the damage function becomes

$$
\begin{aligned}
d_t(\tau_{1,t}, i_t^*, A_t) &= \xi_0(\tau_{1,t}-1) + \gamma \exp(\lambda\tau_{1,t}) \exp\Big(\\
&\qquad -\chi\Big(A_t + \frac{1}{\chi}\Big[\log(\gamma\chi) - \log\Big(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big] - A_t\Big)\Big) \\
&= \xi_0(\tau_{1,t}-1) + \gamma \exp(\lambda\tau_{1,t}) \exp\Big(-\chi\Big(\frac{1}{\chi}\Big[\log(\gamma\chi) - \log\Big(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big]\Big)\Big) \\
&= \xi_0(\tau_{1,t}-1) + \gamma \exp(\lambda\tau_{1,t}) \exp\Big(-\Big[\log(\gamma\chi) - \log\Big(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big) + \lambda\tau_{1,t}\Big]\Big) \\
&= \xi_0(\tau_{1,t}-1) + \gamma \exp(\lambda\tau_{1,t}) \frac{1}{\gamma\chi}\Big(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big) \exp(-\lambda\tau_{1,t}) \\
&= \xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\Big(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big). \qquad \text{(B.17)}
\end{aligned}
$$

Equipped with equations (B.24) and (B.25) the novel terms in the Bellman equation

become

$$
\begin{aligned}
&- (1+\beta\varphi_k)(\eta i_t^* + d_t(\tau_{1,t}, i_t^*, A_t)) + \beta\varphi_A \left((1-\delta_A)A_t + i_t^*\right) \\
&= \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) i_t^* - (1+\beta\varphi_k) d_t(\tau_{1,t}, i_t^*, A_t) + \beta\varphi_A(1-\delta_A)A_t \\
&= \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) \left(\frac{1}{\chi}\Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \lambda\tau_{1,t}\Big] - A_t \right) \\
&\quad -(1+\beta\varphi_k)\left(\xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right)\right) + \beta\varphi_A(1-\delta_A)A_t \\
&= \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) \frac{1}{\chi}\Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right)\Big] \\
&\quad + \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) \frac{1}{\chi}\lambda\tau_{1,t} - \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) A_t \\
&\quad -(1+\beta\varphi_k)\xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\left((1+\beta\varphi_k)\eta - \beta\varphi_A\right) + \beta\varphi_A(1-\delta_A)A_t \\
&= \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) \frac{1}{\chi}\Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right)\Big] \\
&\quad +(1+\beta\varphi_k)\xi_0 + \frac{1}{\chi}\left((1+\beta\varphi_k)\eta - \beta\varphi_A\right) \\
&\quad +\Big(\left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right) \frac{\lambda}{\chi} - (1+\beta\varphi_k)\xi_0\Big)\tau_{1,t} \\
&\quad +\Big(\beta\varphi_A(1-\delta_A) - \left(\beta\varphi_A - (1+\beta\varphi_k)\eta\right)\Big) A_t
\end{aligned}
\tag{B.18}
$$

With this preparation, I plug the solutions for the control variables back into the

Bellman equation (B.3)

$$
\begin{aligned}
&\big(\varphi_k - (1+\beta\varphi_k)\kappa\big)k_t + \big(\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top\big)\boldsymbol{R}_t + \big(\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top\big)\boldsymbol{M}_t \\
&+ \left(\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \left((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\right)\boldsymbol{e}_1^\top\right)\boldsymbol{\tau}_t \\
&+ \underline{\Big(\varphi_A - \big(\beta\varphi_A(1-\delta_A) - (\beta\varphi_A - (1+\beta\varphi_k)\eta)\big)\Big)A_t} \\
&+\varphi_t \;=\; \beta\varphi_{t+1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (B.19)\\
&\qquad + \;\log x_t^*(\varphi_k) + \beta\varphi_k\log(1-x_t^*(\varphi_k)) + (1+\beta\varphi_k)\xi_0 \\
&\qquad + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*) \\
&\qquad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) - \beta\boldsymbol{\varphi}_{R,t+1}^\top\boldsymbol{E}_t^{d*} \\
&\qquad + \beta\varphi_{M,1}\left(\textstyle\sum_{i=1}^{I^d} E_{i,t}^* + E_t^{exo}\right) + \beta\varphi_{\tau,1}\tfrac{\sigma^{forc}}{M^{pre}}G_t \\
&\qquad + \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\tfrac{1}{\chi}\Big[\log(\gamma\chi) - \log\Big(\eta - \tfrac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big)\Big] + \tfrac{1}{\chi}\left((1+\beta\varphi_k)\eta - \beta\varphi_A\right)}
\end{aligned}
$$

where I underline the novel terms with respect to DACE.[20] This equation had to hold for all levels of the state variables, which requires that the coefficients in front of the state variables vanish. Thus, the shadow values have to satisfy as already in DACE

$$\varphi_k - (1+\beta\varphi_k)\kappa = 0 \qquad \Rightarrow \varphi_k = \frac{\kappa}{1-\beta\kappa} \tag{B.20}$$

$$\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top = 0 \Rightarrow \boldsymbol{\varphi}_M^\top = \frac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top(\boldsymbol{1} - \beta\boldsymbol{\Phi})^{-1} \tag{B.21}$$

$$\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top = 0 \qquad \Rightarrow \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ .$$

[20]Equation (B.18) also includes the terms $(1+\beta\varphi_k)\xi_0\tau_{1,t}$ and $(1+\beta\varphi_k)\xi_0$ in addition to the ones underlined above. These are terms that derive from the original (non-adaptable) part of the damage function which is also part of DACE.

and for the new and altered equations

$$\begin{aligned}
&\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top = 0\\
&\quad\Rightarrow \boldsymbol{\varphi}_\tau^\top = -\Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&\varphi_A - \beta\varphi_A(1-\delta_A) + (\beta\varphi_A - (1+\beta\varphi_k)\eta) = 0\\
&\quad\Rightarrow \varphi_A + \beta\delta_A\varphi_A - (1+\beta\varphi_k)\eta = 0\\
&\quad\Rightarrow \delta_A(1+\beta\varphi_A) = (1+\beta\varphi_k)\eta\\
&\quad\Rightarrow \varphi_A = \frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta = \frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}\ ,
\end{aligned} \tag{B.22}$$

where I inserted the shadow value of log-capital in the final step. Inserting the solution for the shadow values of capital and abatement capital into the shadow value of an (exponential) temperature increase yields

$$\begin{aligned}
\boldsymbol{\varphi}_\tau^\top &= -\Big((1+\beta\varphi_k)\xi_0 - \underline{\Big(\beta\frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta - (1+\beta\varphi_k)\eta\Big)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&= -(1+\beta\varphi_k)\Big(\xi_0 - \underline{\Big(\frac{\beta\eta}{1+\beta\delta_A} - \eta\Big)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&= -(1+\beta\frac{\kappa}{1-\beta\kappa}\,)\Big(\xi_0 + \underline{\eta\left(1 - \frac{\beta}{1+\beta\delta_A}\right)\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&= -\frac{1}{1-\beta\kappa}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1 - \frac{\beta}{1+\beta\delta_A}\Big)}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\ .
\end{aligned} \tag{B.23}$$

As in DACE, the Bellman equation (B.42) turns into a condition for the affine constant of the value function of the form $\varphi_t - \beta\varphi_{t+1} = ...$ which can always be met by appropriately choosing the sequence of shadadow values $\varphi_0, \varphi_1, \varphi_2, ...$.[21] By equations (B.12) and (B.43) the optimal consumption rate is $x^* = 1 - \beta\kappa$.

The SCC is the shadow cost of atmospheric carbon in money-measured consumption units. Inserting the shadow value of atmospheric temperature (first entry of $\boldsymbol{\tau}_t$-vector

[21] See DACEfor details. In case of a positive Hotelling rent, $\boldsymbol{\varphi}_{R,0}^\top$ is pinned down by eventually exhausting the scarce resource (otherwise it is always zero). The Bellman equation (B.42) does not directly pin down the initial value of the affine constant φ_0. The additional condition $\lim_{t\to\infty}\beta^t V(\cdot) = 0 \Rightarrow \lim_{t\to\infty}\beta^t\varphi_t = 0$ pins down this initial value φ_0, ensuring that the value function is normalized just as the infinite sum of optimized utility (Stokey & Lucas 1989, chapter 4.1). The discussion in this paper does not depend on these constants.

in equation (B.46)) into equation (B.44) delivers

$$\boldsymbol{\varphi}_M^\top = -\frac{\beta\sigma^{forc}}{M_{pre}}\frac{1}{1-\beta\kappa}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1-\frac{\beta}{1+\beta\delta_A}\Big)}\Big)\boldsymbol{e}_1^\top(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}.$$

To translating the shadow value from utils into consumption equivalents I use $du_t = \frac{1}{C_t}dC_t = \frac{1}{x^* Y_t^{net}}dC_t \Rightarrow dC_t = (1-\beta\kappa)Y_t^{net}du_t$ and find

$$\begin{aligned} SCC &= -(1-\beta\kappa)Y_t^{net}\varphi_{M,1} \\ &= Y_t^{net}\frac{\beta\sigma^{forc}}{M_{pre}}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1-\frac{\beta}{1+\beta\delta_A}\Big)}\Big)\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}. \end{aligned}$$

Finally, I use equations (B.43) and (B.45) to eliminate the shadow values in equations (B.24) and (B.25). Observing that

$$\frac{\beta\varphi_A}{(1+\beta\varphi_k)} = \frac{\beta\frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}}{(1+\beta\frac{\kappa}{1-\beta\kappa})} = \frac{\beta\frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}}{(\frac{1}{1-\beta\kappa})} = \frac{\beta\eta}{(1+\beta\delta_A)}$$

I find

$$i_t^* = \frac{1}{\chi}\Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\eta}{(1+\beta\delta_A)}\right) + \lambda\tau_{1,t}\Big] - A_t \tag{B.24}$$

$$d_t(\tau_{1,t}, i_t^*, A_t) = \xi_0(\tau_{1,t}-1) + \frac{1}{\chi}\left(\eta - \frac{\beta\eta}{(1+\beta\delta_A)}\right). \tag{B.25}$$

B.4 Capital Depreciation

This section derives the capital equation of motion (29), quantifies the correction factor, and discusses the model's implication that the consumption *rate* (but not level) is unaffected by climate states. The usual capital accumulation equation, enriched by climate damages, is

$$K_{t+1} = Y_t[1-D_t(T_{1,t})] - C_t + (1-\delta_k)K_t\ .$$

Defining the consumption rate $x_t = \frac{C_t}{Y_t[1-D_t(T_{1,t})]}$ and recognizing that $Y_t[1-D_t(T_{1,t})] - C_t = K_{t+1} - (1-\delta_k)K_t$ implies[22]

$$K_{t+1} = Y_t[1-D_t(T_{1,t})](1-x_t)\left[1 + \frac{1-\delta_k}{\frac{K_{t+1}}{K_t} - (1-\delta_k)}\right]\ .$$

[22]The step uses $K_{t+1} = Y_t[1-D_t(T_{1,t})] - Y_t[1-D_t(T_{1,t})]x_t + (Y_t[1-D_t(T_{1,t})] - C_t) \times \frac{(1-\delta_k)K_t}{Y_t[1-D_t(T_{1,t})]-C_t}$.

Defining the capital growth rate $g_{k,t} = \frac{K_{t+1}}{K_t} - 1$, I obtain the equation of motion for capital (29) stated in the main text.

I evaluate the correction factor based on the Penn World Tables (**?**). For 2019, currently the latest year of the Penn Word Tables 10.0, the global depreciation parameter is $\delta_k^{annual} = 0.0439$, so $\delta_k \approx 0.44$. Averaging capital growth over the past 10 years (using $2017USD$ values) delivers $g_k^{annual} = 0.02949$ or $g_k \approx 0.29$. The resulting correction factor is $\left[\frac{1+g_{k,t}}{\delta_k+g_{k,t}}\right] \approx 1.8$. (for a 5 year time step the correction factor would be 3.1, and for an annual time step 14). For the US, $\delta_k^{annual} = 0.046$ and $g_k^{annual} = 0.013$, resulting in a decadal correction factor of 1.9 (or a correction factor of 3.6 for a five year time step).

B.5 Stochastic shocks and regional versus global impacts

Under uncertainty, all equations of motion are stochastic. Formally, the state variables become functions on a probability space, whose filtration $\mathcal{F}$ is generated by the stochastic climate and damage evolution. My notation suppresses this dependence. All processes are adapted to the filtration $\mathcal{F}$: we learn only what we have already observed. Formally, conditioning on time t information means conditioning expectations on the filtration at time t: $\mathbb{E}_t(\cdot) \equiv \mathbb{E}(\cdot\,|\,\mathcal{F}_t)$).

I model global average surface temperature using an expectation-corrected autoregressive gamma process. Introduced by Gourieroux & Jasiak (2006), it is a Poisson mixture of gamma distributions

$$\frac{y_{t+1}}{c}|(Z, X_t) \sim gamma(\nu_t + Z), \text{ where } Z|y_t \sim Poisson\left(\frac{\gamma y_t}{c}\right) \tag{B.26}$$

for $c, \gamma, \nu_t > 0$ in all periods. The autoregressive gamma process y_t draws from a standard gamma distribution with scale c and modulates the shape parameter drawing from a Poisson distribution. The one-step-ahead expectation of the process are

$$\begin{aligned}\mathbb{E}_t(y_{i,t+1}|y_t) &= \nu_t c + \gamma y_t \\ \mathbb{V}\text{ar}_t(y_{i,t+1}|y_t) &= \nu_t c^2 + 2c\gamma y_t\end{aligned}$$

and its moment generating function is

$$\text{MGF}_{y_{t+1}}(u) \equiv \mathbb{E}\left(\exp(uy_{t+1})|y_t\right) = \exp\left(-\nu_t \log(1-uc)\right) \exp\left(\tfrac{u}{1-uc}\gamma y_t\right) .$$

The subsequent implementation is, apart from the additional iid shocks, based on Traeger's (2021) expectation-corrected shifted gamma process. For this purpose, I pick

$$\nu_t = \frac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) ,$$

which results in the expectation and variance

$$\begin{aligned}\mathbb{E}\, y_{t+1} \quad &= \gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) \\ \mathbb{V}\mathrm{ar}\, y_{t+1} &= c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big]\end{aligned} \tag{B.27}$$

and moment generating function

$$\mathrm{MGF}_{y_{t+1}}(u) = \mathbb{E}\left(\exp(uy_{t+1})|y_t\right) = \exp\left(-\tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-uc) + \tfrac{u}{1-uc}\gamma^z y_t\right) \tag{B.28}$$

The following deterministic process neutralizes the temperature expectations to those of the deterministic model

$$y^o_{t+1} = \gamma^z y^o_t + \left(\delta_\tau - \epsilon(c,\delta_\tau)\right)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right), \tag{B.29}$$

where $\epsilon(c,\delta_\tau)$ is a given function that Traeger (2021) calibrates together with the parameters c and δ_τ based on the IPCC's (2013) RCP scenarios and IPCC's (2021) transient climate response.

The expectation adjusted process $z_t \equiv y_t - y^o_t$ has the expectation and variance

$$\mathbb{E}_t\, z_{t+1} \quad = \gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) \tag{B.30}$$

$$\mathbb{V}\mathrm{ar}_t\, z_{t+1} = \mathbb{V}\mathrm{ar}_t\, y_{t+1} = c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big]. \tag{B.31}$$

Traeger (2021), uses the global average temperature dynamics

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \left(\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + hz_{t+1}\right)\boldsymbol{e}_1. \tag{B.32}$$

The present paper introduces the additional idiosyncratic regional shocks Δ_{t+1}, resulting in the equation of motion for regional temperatures

$$\boldsymbol{\tau}_{t+1} = \boldsymbol{\sigma}\boldsymbol{\tau}_t + \left(\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + hz_{t+1} + \Delta_{t+1}\right)\boldsymbol{e}_1 \tag{B.33}$$

with expectations

$$\begin{aligned}
\mathbb{E}_t\,\boldsymbol{\tau}_{t+1} \quad &= \boldsymbol{\sigma}\boldsymbol{\tau}_t + \left(\sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + h\gamma^z z_t + h\epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right) + \mathbb{E}_t\,\Delta_{t+1}\right)\boldsymbol{e}_1 \\
\mathbb{V}\mathrm{ar}_t\,\tau_{1,t+1} &= h^2\,\mathbb{V}\mathrm{ar}_t\,z_{t+1} + \mathbb{V}\mathrm{ar}_t\,\Delta_{t+1} = h^2\,\mathbb{V}\mathrm{ar}_t\,y_{t+1} + \mathbb{V}\mathrm{ar}_t\,\Delta_{t+1} \\
&= h^2 c\Big[2\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\Big] + \mathbb{V}\mathrm{ar}_t\,\Delta_{t+1}.
\end{aligned}$$

stated in equations (B.32) and (B.31).

The Bellman equation is

$$\begin{aligned}
V(K_t,\boldsymbol{\tau}_t,\boldsymbol{M}_t,\boldsymbol{R}_t,A_t,\boldsymbol{I}_t,t) = \max_{x_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t,i_t,i_t^\Delta} \mathbb{E}_t\Big(&\log x_t + \kappa k_t - \eta i_t - d_t(\tau_{1,t},i_t,A_t) \\
&+ \log F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t) + \beta V(K_{t+1},\boldsymbol{\tau}_{t+1},\boldsymbol{M}_{t+1},\boldsymbol{R}_{t+1},A_{t+1},\boldsymbol{I}_{t+1},t+1)\Big)
\end{aligned}$$

which is subject to the following equations. The equations of motion (B.5) for carbon and (B.8) for the resource stock are the same as before. The equation of motion for temperature changes to equation (B.33), which, in addition, relies on the equations of motion (B.26) and (B.29) for the stochastic temperature updates. The constraints (B.9) remain. In addition, there is the new equation of motion (**??**) for the dedicated adaptation capital fighting weather shocks. Investments into this new capital stock also reduce $Y_t^{net} = Y_t\exp(-\eta i_t - \eta_t^\Delta i_t^\Delta)$ and, thus, change the equation of motion (B.4) for capital to

$$\begin{aligned}
k_{t+1} = \kappa k_t + \log F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t) - \eta i_t - \eta_t^\Delta i_t^\Delta - d_t(\tau_{1,t},i_t,i_t^\Delta,A_t,A_t^\Delta,\Delta_t) + \log(1-x_t) \\
+ \log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]
\end{aligned}$$

and utility as a function of the consumption rate $x_t = \frac{C_t}{Y_t^{net}}$ to

$$u(x_t) = \log x_t + \kappa\log K_t + \log F(\boldsymbol{A}_t,\boldsymbol{N}_t,\boldsymbol{\mathcal{K}}_t,\boldsymbol{E}_t) - \eta i_t - \eta_t^\Delta i_t^\Delta - d_t(\tau_{1,t},i_t,i_t^\Delta,A_t,A_t^\Delta,\Delta_t),$$

which now also contain the more complex damage function.

Once again, I use an affine trial solution for the value function, adding the new informational state variables y_t and y_t^o

$$\begin{aligned}
V(k_t,\boldsymbol{\tau}_t,\boldsymbol{M}_t,\boldsymbol{R}_t,A_t,A_t^\Delta,y_t,y_t^o,t) = \\
\varphi_k k_t + \boldsymbol{\varphi}_M^\top\boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top\boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top\boldsymbol{R}_t + \varphi_A A_t + \varphi_A^\Delta A_t^\Delta + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t.
\end{aligned}$$

The changed equation of motion and the novel terms change the Bellman equation (B.11) to the form

$$
\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_A^\Delta A_t^\Delta + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}_t}, \boldsymbol{E_t}, i_t} \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + \lambda_t^N\big(1 - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^d\right) + \underline{\beta\varphi_A\left((1-\delta_A)A_t + i_t\right)} + \underline{\beta\varphi_A^\Delta\left((1-\delta_A^\Delta)A_t^\Delta + i_t^\Delta\right)} + \beta\varphi_{t+1} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta\boldsymbol{\varphi}_\tau^\top \left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right) \\
&\quad - \mathbb{E}_t \underline{(1+\beta\varphi_k)(\eta i_t + \eta_t^\Delta i_t^\Delta + d_t(\tau_{1,t+1}, i_t, A_t))} \\
&\quad + (-\beta h\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \qquad \text{from } \tau_{1,t+1} \text{ and } y_{t+1}^o \\
&\quad + \mathbb{E}_t(\beta h\varphi_{\tau,1} + \beta\varphi_y^\tau) y_{t+1} \qquad \text{from } \tau_{1,t+1} \text{ and } y_{t+1} \\
&\quad + \mathbb{E}_t\, \beta\varphi_{\tau,1}\Delta_{t+1} \qquad \text{from } \tau_{1,t+1}
\end{aligned}
\tag{B.34}
$$

where the last three lines collect the novel terms of the stochastic setting. In addition, the damage function is now more complex and in the argument of an expected value operator yielding

$$
\begin{aligned}
\mathbb{E}_t(1+\beta\varphi_k) d_t(\tau_{1,t+1}, i_t, A_t) = {} & \mathbb{E}_t(1+\beta\varphi_k)\xi_0(\tau_{1,t+1} - 1) \\
& + \mathbb{E}_t(1+\beta\varphi_k)\gamma \exp(\lambda\tau_{1,t+1}) \exp(-\chi(A_t + i_t)) \\
& + \mathbb{E}_t(1+\beta\varphi_k)\gamma^\Delta \exp(\lambda^\Delta \Delta_{t+1}) \exp(-\chi(A_t^\Delta + i_t^\Delta)) \; .
\end{aligned}
$$

I evaluate these three damage terms one after the other. First, by equation (B.33) for the next period temperature, the non-adaptable damage term is

$$
\begin{aligned}
\mathbb{E}_t(1+\beta\varphi_k)\xi_0(\tau_{1,t+1} - 1) = {} & (1+\beta\varphi_k)\xi_0\, \mathbb{E}_t\left(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + h z_{t+1} + \Delta_{t+1}\right) \\
& -(1+\beta\varphi_k)\xi_0 \\
= {} & (1+\beta\varphi_k)\underbrace{\xi_0\Big(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Big)}_{\equiv\tilde{\xi}_0(\boldsymbol{\tau}_t, M_{1,t})} - (1+\beta\varphi_k)\xi_0 \\
& +(1+\beta\varphi_k)\xi_0\; \mathbb{E}_t(h z_{t+1} + \Delta_{t+1}) \\
= {} & (1+\beta\varphi_k)\tilde{\xi}_0(\boldsymbol{\tau}_t, M_{1,t}) - (1+\beta\varphi_k)\xi_0 + (1+\beta\varphi_k)\xi_0\; \mathbb{E}_t\, \Delta_{t+1} \\
& +(1+\beta\varphi_k)\xi_0 h\left(\gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right),
\end{aligned}
$$

where I used equation (B.30) in the final step, and $\boldsymbol{\sigma}_{1\cdot}$ denotes the first row vector of the heat transfer matrix $\boldsymbol{\sigma}$. The adaptable damage term caused by τ_{t+1} is

$$
\begin{aligned}
&\mathbb{E}_t(1+\beta\varphi_k)\gamma \exp(\lambda\tau_{1,t+1}) \exp(-\chi(A_t+i_t)) \\
&= \mathbb{E}_t(1+\beta\varphi_k)\gamma \exp\left(\lambda\left(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}} + hz_{t+1} + \Delta_{t+1}\right)\right) \exp(-\chi(A_t+i_t)) \\
&= (1+\beta\varphi_k) \underbrace{\gamma \exp\left(\lambda\left(\boldsymbol{\sigma}_{1\cdot}\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\right)\right)}_{\equiv\tilde{\gamma}(\boldsymbol{\tau}_t, M_{1,t})} \mathbb{E}_t \exp\left(\lambda\big(hz_{t+1}+\Delta_{t+1}\big)\right) \exp(-\chi(A_t+i_t)) \\
&= (1+\beta\varphi_k)\tilde{\gamma}(\boldsymbol{\tau}_t, M_{1,t}) \exp(-\chi(A_t+i_t)) \mathbb{E}_t \exp\left(\lambda\big(hy_{t+1} - hy^o_{t+1} + \Delta_{t+1}\big)\right) \\
&= (1+\beta\varphi_k)\tilde{\gamma}(\boldsymbol{\tau}_t, M_{1,t}) \exp(-\chi(A_t+i_t)) \exp\big(-\lambda h y^o_{t+1}\big) \mathbb{E}_t \exp\left(\lambda\big(hy_{t+1}+\Delta_{t+1}\big)\right) \\
&= (1+\beta\varphi_k) \underbrace{\tilde{\gamma}(\boldsymbol{\tau}_t, M_{1,t}) \exp\left(-\lambda h\left(\gamma^z y^o_t + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\right)}_{\equiv\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y^o_t)} \exp(-\chi(A_t+i_t)) \\
&\qquad \mathbb{E}_t \exp\left(\lambda\big(hy_{t+1}+\Delta_{t+1}\big)\right) \\
&= (1+\beta\varphi_k)\,\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y^o_t) \exp(-\chi(A_t+i_t)) \mathbb{E}_t \exp\left(\lambda\big(hy_{t+1}+\Delta_{t+1}\big)\right) \\
&= (1+\beta\varphi_k)\,\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y^o_t) \exp(-\chi(A_t+i_t)) \operatorname{MGF}_{y_{t+1}}(\lambda h) \operatorname{MGF}_{\Delta_{t+1}}(\lambda),
\end{aligned}
$$

where $\operatorname{MGF}_{y_{t+1}}(\lambda h)$ is the moment generating function in equation (B.28) evaluated at λh and $\operatorname{MGF}_{\Delta_{t+1}}(\lambda)$ is the moment-generating function of the (so far arbitrary) shock Δ_t evaluated for the parameter λ. Finally, the adaptable damage term caused by Δ_{t+1} is

$$
\begin{aligned}
&\mathbb{E}_t(1+\beta\varphi_k)\gamma^\Delta \exp(\lambda^\Delta\Delta_{t+1}) \exp(-\chi^\Delta(A^\Delta_t + i^\Delta_t)) \\
&= (1+\beta\varphi_k)\gamma^\Delta \exp(-\chi^\Delta(A^\Delta_t + i^\Delta_t)) \mathbb{E}_t \exp(\lambda^\Delta\Delta_{t+1}) \\
&= (1+\beta\varphi_k)\gamma^\Delta \exp(-\chi^\Delta(A^\Delta_t + i^\Delta_t)) \operatorname{MGF}_{\Delta_{t+1}}(\lambda^\Delta).
\end{aligned}
$$

Plugging these results into the Bellman equation (B.34) delivers

$$
\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_A^\Delta A_t^\Delta + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\max_{x_t, \boldsymbol{N_t}, \boldsymbol{\mathcal{K}_t}, \boldsymbol{E_t}, i_t, i_t^\Delta} \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t, \boldsymbol{\mathcal{K}}_t, \boldsymbol{E}_t) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}\big) + \lambda_t^N\big(1 - \textstyle\sum_{i=1}^{I_N} N_{i,t}\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^d\right) + \underline{\beta\varphi_A\left((1-\delta_A)A_t + i_t\right)} + \underline{\beta\varphi_A^\Delta\left((1-\delta_A^\Delta)A_t^\Delta + i_t^\Delta\right)} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top \left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t} + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta\boldsymbol{\varphi}_\tau^\top \left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right) \\
&\quad \underline{-(1+\beta\varphi_k)(\eta i_t + \eta_t^\Delta i_t^\Delta)} + \beta\varphi_{t+1} \\
&\quad \underline{-(1+\beta\varphi_k)\tilde{\xi}_0(\boldsymbol{\tau}_t, M_{1,t}) + (1+\beta\varphi_k)\xi_0 - (1+\beta\varphi_k)\xi_0\ \mathbb{E}_t\,\Delta_{t+1}} \\
&\qquad \underline{-(1+\beta\varphi_k)\xi_0 h\left(\gamma^z z_t + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)} \\
&\quad \underline{-(1+\beta\varphi_k)\,\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\ \exp(-\chi(A_t+i_t))\ \mathrm{MGF}_{y_{t+1}}(\lambda h)\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda)} \\
&\quad \underline{-(1+\beta\varphi_k)\gamma^\Delta \exp(-\chi^\Delta(A_t^\Delta + i_t^\Delta))\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda^\Delta)} \\
&\quad + (-\beta h\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \\
&\quad + \mathbb{E}_t(\beta h\varphi_{\tau,1} + \beta\varphi_y^\tau) y_{t+1} \\
&\quad + \mathbb{E}_t\,\beta\varphi_{\tau,1}\Delta_{t+1}.
\end{aligned}
\tag{B.35}
$$

The first order condition for the abatement investment i_t now has the form

$$
\begin{aligned}
&\frac{\partial}{\partial i_t}\Big[\ \beta\varphi_A i_t - (1+\beta\varphi_k)\eta i_t \\
&\qquad -(1+\beta\varphi_k)\,\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\ \exp(-\chi(A_t+i_t))\ \mathrm{MGF}_{y_{t+1}}(\lambda h)\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda)\Big] = 0.
\end{aligned}
$$

This equation is equivalent to the previous first order condition equation (B.24), which contains the term $-(1+\beta\varphi_k)\gamma\exp(\lambda\tau_{1,t})\exp(-\chi(A_t+i_t))$, when replacing

$$
\gamma\exp(\lambda\tau_{1,t}) \quad \to \quad \hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda h)\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda)
$$

or, equivalently,

$$
\log(\gamma) + \lambda\tau_{1,t} \quad \to \quad \log(\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) + \log(\mathrm{MGF}_{y_{t+1}}(\lambda h)) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))
$$

yielding

$$
\begin{aligned}
i_t^* = \frac{1}{\chi}\Big[& \log(\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) \\
& + \log(\mathrm{MGF}_{y_{t+1}}(\lambda h)) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big] - A_t \qquad \text{(B.36)} \\
= \frac{1}{\chi}\Bigg[& \log(\gamma) + \left(\lambda\left(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\right)\right) \\
& + \Bigg(- \lambda h\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\Bigg) \\
& + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda)) \\
& + \Bigg(- \tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-\lambda h c) + \tfrac{\lambda h}{1-\lambda h c}\gamma^z y_t\Bigg)\Bigg] - A_t
\end{aligned}
$$

where the final transformation uses equation (B.28) to evaluate $\log(\mathrm{MGF}_{y_{t+1}}(\lambda h))$ and reinserts the definition for

$$
\begin{aligned}
\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o) = \gamma \exp & \left(\lambda\left(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\right)\right) \\
& \exp\Bigg(- \lambda h\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\Bigg)
\end{aligned}
$$

Analogously, the investment i_t^Δ into the shock-protecitve capital A_t^Δ delivers the first order condition

$$
\frac{\partial}{\partial i_t^\Delta}\Big[\ \beta\varphi_A^\Delta i_t^\Delta - (1+\beta\varphi_k)\eta_t^\Delta i_t^\Delta - (1+\beta\varphi_k)\gamma^\Delta \exp(-\chi^\Delta(A_t^\Delta + i_t^\Delta))\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda)\Big] = 0.
$$

This equation has the same structure as the first order condition equation (B.24) replacing

$$
\gamma\exp(\lambda\tau_{1,t}) \quad \rightarrow \quad \gamma^\Delta\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda)
$$

or, equivalently,

$$
\log(\gamma) + \lambda\tau_{1,t} \quad \rightarrow \quad \log(\gamma^\Delta) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))
$$

yielding

$$
i_t^{\Delta *} = \frac{1}{\chi^\Delta}\Big[\ \log(\gamma^\Delta) + \log(\chi^\Delta) - \log\left(\eta_t^\Delta - \frac{\beta\varphi_A^\Delta}{(1+\beta\varphi_k)}\right) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big] - A_t^\Delta.
$$

The changes in the equations will generally lead to updates in the shadow values, see below, but it does not change the mechanics of solving the system of equations for the optimal controls $\boldsymbol{N}_t^*(\boldsymbol{A}_t\,, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, $\boldsymbol{\mathcal{K}}_t^*(\boldsymbol{A}_t\,, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$, and $\boldsymbol{E}_t^*(\boldsymbol{A}_t\,, \varphi_k, \boldsymbol{\varphi}_M, \boldsymbol{\varphi}_{R,t+1})$. Most importantly, these solutions remain independent of the state variables, and hence, I will suppress their arguments.

Before plugging the solution back into the Bellman equation (B.39), I first simplify the terms corresponding to adaptable damages after inserting the optimal investment levels (the last two underlined terms in equation B.39). First, for the adaptable component of temperature-driven damage, I find using the first line of equation (B.36)

$$
\begin{aligned}
&\underline{-(1+\beta\varphi_k)\,\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\exp(-\chi(A_t + i_t^*))\,\mathrm{MGF}_{y_{t+1}}(\lambda h)\,\mathrm{MGF}_{\Delta_{t+1}}(\lambda)} \\
&= -(1+\beta\varphi_k)\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda h)\,\mathrm{MGF}_{\Delta_{t+1}}(\lambda)\exp\Big(-\chi\Big(A_t + \frac{1}{\chi}\Big[\log(\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) \\
&\quad + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \log(\mathrm{MGF}_{y_{t+1}}(\lambda h)) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big] - A_t\Big)\Big) \\
&= -(1+\beta\varphi_k)\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)\,\mathrm{MGF}_{y_{t+1}}(\lambda h)\,\mathrm{MGF}_{\Delta_{t+1}}(\lambda)\exp\Big(-\Big(\log(\hat{\gamma}(\boldsymbol{\tau}_t, M_{1,t}, y_t^o)) \\
&\quad + \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \log(\mathrm{MGF}_{y_{t+1}}(\lambda h)) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big)\Big) \\
&= -(1+\beta\varphi_k)\exp\Big(-\log(\chi) + \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right)\Big)\Big) \\
&= -(1+\beta\varphi_k)\frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right), \qquad \text{(B.37)}
\end{aligned}
$$

which coincides with the corresponding term in the deterministic model's equation (B.25). Analogously, the adaptable component of the damages responding directly to

the shock becomes

$$
\begin{aligned}
&\underline{-(1+\beta\varphi_k)\gamma^{\Delta}\exp(-\chi^{\Delta}(A_t^{\Delta}+i_t^{\Delta}))\ \mathrm{MGF}_{\Delta_{t+1}}(\lambda^{\Delta})} \\
&= -(1+\beta\varphi_k)\,\mathrm{MGF}_{\Delta_{t+1}}(\lambda^{\Delta})\gamma^{\Delta}\exp\Big(-\chi^{\Delta}\Big(A_t^{\Delta}+\frac{1}{\chi^{\Delta}}\Big[\log(\gamma^{\Delta})+\log(\chi^{\Delta}) \\
&\quad -\log\left(\eta_t^{\Delta}-\frac{\beta\varphi_A^{\Delta}}{(1+\beta\varphi_k)}\right)+\log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big]-A_t^{\Delta}\Big)\Big) \\
&= -(1+\beta\varphi_k)\gamma^{\Delta}\exp\Big(-\Big(\log(\gamma^{\Delta})+\log(\chi^{\Delta})-\log\left(\eta_t^{\Delta}-\frac{\beta\varphi_A^{\Delta}}{(1+\beta\varphi_k)}\right)\Big)\Big) \\
&= -(1+\beta\varphi_k)\exp\Big(-\log(\chi^{\Delta})+\log\left(\eta_t^{\Delta}-\frac{\beta\varphi_A^{\Delta}}{(1+\beta\varphi_k)}\right)\Big) \\
&= -(1+\beta\varphi_k)\frac{1}{\chi^{\Delta}}\left(\eta_t^{\Delta}-\frac{\beta\varphi_A^{\Delta}}{(1+\beta\varphi_k)}\right) \qquad \text{(B.38)}
\end{aligned}
$$

Before returning to solve the Bellman equation, I note that equations (B.37) and (B.38) give us the expected damages stated in the proposition, using $\frac{\beta\varphi_A}{(1+\beta\varphi_k)}=\frac{\beta\eta}{1+\beta\delta_A}$:

$$
\begin{aligned}
\mathbb{E}_t\, d_t(\tau_{1,t+1}, i_t, A_t) &= \mathbb{E}_t\,\xi_0(\tau_{1,t+1}-1)+\mathbb{E}_t\,\gamma\exp(\lambda\tau_{1,t+1})\exp(-\chi(A_t+i_t)) \\
&\quad +\mathbb{E}_t\,\gamma^{\Delta}\exp(\lambda^{\Delta}\Delta_{t+1})\exp(-\chi(A_t^{\Delta}+i_t^{\Delta})) \\
&= \xi_0(\mathbb{E}_t\,\tau_{1,t+1}-1)+\frac{1}{\chi}\left(\eta-\frac{\beta\eta}{1+\beta\delta_A}\right)+\frac{1}{\chi^{\Delta}}\left(\eta_t^{\Delta}-\frac{\beta\eta_t^{\Delta}}{1+\beta\delta_A}\right)\ .
\end{aligned}
$$

Returning to the Bellman equation, I rearrange the first three underlined terms in equation (B.39) as follows

$$
\begin{aligned}
&+\underline{\beta\varphi_A\left((1-\delta_A)A_t+i_t^*\right)}+\underline{\beta\varphi_A^{\Delta}\left((1-\delta_A^{\Delta})A_t^{\Delta}+i_t^{\Delta^*}\right)}\underline{-(1+\beta\varphi_k)(\eta i_t^*+\eta_t^{\Delta}i_t^{\Delta^*})} \\
&= +\underline{\beta\varphi_A(1-\delta_A)A_t}+\underline{\beta\varphi_A^{\Delta}(1-\delta_A^{\Delta})A_t^{\Delta}}+\underline{(\beta\varphi_A-(1+\beta\varphi_k)\eta)i_t^*}+\underline{(\beta\varphi_A^{\Delta}-(1+\beta\varphi_k)\eta_t^{\Delta})i_t^{\Delta^*}},
\end{aligned}
$$

I plug the solutions for the optimized control variables back into the Bellman equation

(B.39) finding

$$
\begin{aligned}
&\varphi_k k_t + \boldsymbol{\varphi}_M^\top \boldsymbol{M}_t + \boldsymbol{\varphi}_\tau^\top \boldsymbol{\tau}_t + \boldsymbol{\varphi}_{R,t}^\top \boldsymbol{R}_t + \varphi_A A_t + \varphi_A^\Delta A_t^\Delta + \varphi_{y^o}^\tau y_t^o + \varphi_y^\tau y_t + \varphi_t = \\
&\quad \log x_t + \beta\varphi_k \log(1-x_t) + (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t^*, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*) \\
&\quad + \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) + \lambda_t^{\mathcal{K}}\big(1 - \textstyle\sum_{i=1}^{I_K} \mathcal{K}_{i,t}^*\big) + \lambda_t^N\big(1 - \textstyle\sum_{i=1}^{I_N} N_{i,t}^*\big) \\
&\quad + \beta\boldsymbol{\varphi}_{R,t+1}^\top \left(\boldsymbol{R}_t - \boldsymbol{E}_t^{\boldsymbol{d}*}\right) + \underline{\beta\varphi_A(1-\delta_A)A_t} + \underline{\beta\varphi_A^\Delta(1-\delta_A^\Delta)A_t^\Delta} \\
&\quad + (\beta\varphi_A - (1+\beta\varphi_k)\eta)\Bigg(\frac{1}{\chi}\Bigg[\log(\gamma) + \lambda\Bigg(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Bigg) \\
&\qquad \underline{+ \Bigg(-\lambda h\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)\Bigg)} \\
&\qquad \underline{+ \log(\chi) - \log\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))} \\
&\qquad \underline{+ \Bigg(-\tfrac{\delta_\tau}{c}\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\log(1-\lambda hc) + \tfrac{\lambda h}{1-\lambda hc}\gamma^z y_t\Bigg)\Bigg] - A_t\Bigg) \\
&\quad + \underline{(\beta\varphi_A^\Delta - (1+\beta\varphi_k)\eta_t^\Delta)\Bigg(\frac{1}{\chi^\Delta}\Big[\log(\gamma^\Delta) + \log(\chi^\Delta) - \log\left(\eta_t^\Delta - \frac{\beta\varphi_A^\Delta}{(1+\beta\varphi_k)}\right)} \\
&\qquad \underline{+ \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda))\Big] - A_t^\Delta\Bigg)} + \beta\varphi_{t+1} \qquad \text{(B.39)} \\
&\quad + \beta\boldsymbol{\varphi}_M^\top\left(\boldsymbol{\Phi}\boldsymbol{M}_t + \big(\textstyle\sum_{i=1}^{I^d} E_{i,t}^* + E_t^{exo}\big)\boldsymbol{e}_1\right) + \beta\boldsymbol{\varphi}_\tau^\top\left(\boldsymbol{\sigma}\boldsymbol{\tau}_t + \sigma^{forc}\tfrac{M_{1,t}+G_t}{M_{pre}}\boldsymbol{e}_1\right) \\
&\quad \underline{-(1+\beta\varphi_k)\xi_0\Big(\boldsymbol{\sigma}_1.\boldsymbol{\tau}_t + \sigma^{forc}\frac{M_{1,t}+G_t}{M_{pre}}\Big) + (1+\beta\varphi_k)\xi_0 - (1+\beta\varphi_k)\xi_0\ \mathbb{E}_t\,\Delta_{t+1}} \\
&\qquad \underline{-(1+\beta\varphi_k)\xi_0 h\left(\gamma^z(y_t - y_t^o) + \epsilon(c,\delta_\tau)\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right)} \\
&\quad \underline{-(1+\beta\varphi_k)\frac{1}{\chi}\left(\eta - \frac{\beta\varphi_A}{(1+\beta\varphi_k)}\right)} \\
&\quad \underline{-(1+\beta\varphi_k)\frac{1}{\chi^\Delta}\left(\eta_t^\Delta - \frac{\beta\varphi_A^\Delta}{(1+\beta\varphi_k)}\right)} \\
&\quad + (-\beta h\varphi_{\tau,1} + \beta\varphi_{y^o}^\tau)\left(\gamma^z y_t^o + (\delta_\tau - \epsilon(c,\delta_\tau))\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \\
&\quad + (\beta h\varphi_{\tau,1} + \beta\varphi_y^\tau)\left(\gamma^z y_t + \delta_\tau\left(\tfrac{M_{1,t}+G_t}{M_{pre}} - \eta_\tau\right)\right) \\
&\quad + \beta\varphi_{\tau,1}\,\mathbb{E}_t\,\Delta_{t+1}.
\end{aligned}
$$

where the second-to-last line used equation (B.27) to evaluate $\mathbb{E}_t\, y_{t+1}$. Most importantly, the Bellman equation above is linear in the state variables. As a result, I can match the coefficients of the states on both sides of the equation and the affine trial

solution for the value function indeed solves the dynamic programming problem. The expressions that are independent of the states can be matched as before using an appropriate sequence of the value functions time varying constants $\varphi_0, \varphi_1, ...$ as before. I obtain the states' shadow values by carrying out the coefficient matching one step at a time. I first not that the terms containing the resource stock $\boldsymbol{R}_t$, the overall capital k_t and the abatement capital A_t are identical to those appearing in the deterministic Bellman equation (B.42) and still return

$$\varphi_k = \frac{\kappa}{1-\beta\kappa} \; ; \qquad \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0} \; ; \qquad \varphi_A = \frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta = \frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)} \; .$$

Moreover, the terms governing the new adaptation capital A_t^Δ are structurally the same as for A_t and give

$$\varphi_A^\Delta = \frac{1+\beta\varphi_k}{1+\beta\delta_A^\Delta}\eta_t^\Delta = \frac{\eta_t^\Delta}{(1-\beta\kappa)(1+\beta\delta_A^\Delta)} \; .$$

The remaining equations matching the state coefficients and determining their shadow

values become

$$
\begin{aligned}
\boldsymbol{\varphi}_\tau^\top &= \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} - \Big((1+\beta\varphi_k)\xi_0 - (\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot} \\
\Rightarrow \boldsymbol{\varphi}_\tau^\top &= -\Big((1+\beta\varphi_k)\xi_0 - (\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\,[(1-\beta\boldsymbol{\sigma})^{-1}] \\
&= -\Big((1+\beta\varphi_k)\big(\xi_0+\eta\tfrac{\lambda}{\chi}\big) - \beta\varphi_A\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\,[(1-\beta\boldsymbol{\sigma})^{-1}] \\
&= -\Big(\tfrac{1}{1-\beta\kappa}\big(\xi_0+\eta\tfrac{\lambda}{\chi}\big) - \tfrac{\beta\eta}{(1-\beta\kappa)(1+\beta\delta_A)}\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\,[(1-\beta\boldsymbol{\sigma})^{-1}] \\
&= -\tfrac{1}{1-\beta\kappa}\Big(\xi_0+\eta\tfrac{\lambda}{\chi} - \tfrac{\beta\eta}{1+\beta\delta_A}\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\,[(1-\beta\boldsymbol{\sigma})^{-1}] \\
&= -\tfrac{1}{1-\beta\kappa}\Big(\xi_0+\eta\tfrac{\lambda}{\chi}\big(1-\tfrac{\beta}{1+\beta\delta_A}\big)\Big)\boldsymbol{\sigma}_{1\cdot}\,[(1-\beta\boldsymbol{\sigma})^{-1}] \\
\boldsymbol{\varphi}_M^\top &= \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} + \beta\Big(\tfrac{\sigma^{forc}}{M_{pre}}\varphi_{\tau,1} + \underbrace{\tfrac{1}{M_{pre}}(\varphi_{y^o}^\tau - h\varphi_{\tau,1})(\delta_\tau - \epsilon(c,\delta_\tau)) + \tfrac{1}{M_{pre}}(\varphi_y^\tau + h\varphi_{\tau,1})\delta_\tau}_{=\,\frac{1}{M_{pre}}\left[(h\varphi_{\tau,1}-\varphi_{y^o}^\tau)\epsilon(c,\delta_\tau)+\varphi_y^\tau\delta_\tau\right]} \\
&\quad + (\beta\varphi_A - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\big[\lambda\sigma^{forc} - \lambda h(\delta_\tau - \epsilon(c,\delta_\tau))\big]\tfrac{1}{M_{pre}} \\
&\quad - (\beta\varphi_A - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{\delta_\tau}{c}\log(1-\lambda hc)\tfrac{1}{M_{pre}} \\
&\quad - (1+\beta\varphi_k)\xi_0\sigma^{forc}\tfrac{1}{M_{pre}} - (1+\beta\varphi_k)\xi_0 h\epsilon(c,\delta_\tau)\tfrac{1}{M_{pre}} \\
&= \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} + \beta\Big(\tfrac{\sigma^{forc}}{M_{pre}}\varphi_{\tau,1} + \tfrac{1}{M_{pre}}\big[(h\varphi_{\tau,1}-\varphi_{y^o}^\tau)\epsilon(c,\delta_\tau)+\varphi_y^\tau\delta_\tau\big] \\
&\quad + (\beta\varphi_A - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{1}{M_{pre}}\Big(\lambda\sigma^{forc} - \lambda h(\delta_\tau - \epsilon(c,\delta_\tau)) - \tfrac{\delta_\tau}{c}\log(1-\lambda hc)\Big) \\
&\quad - (1+\beta\varphi_k)\xi_0\tfrac{1}{M_{pre}}\Big(\sigma^{forc} + h\epsilon(c,\delta_\tau)\Big)\Big)\boldsymbol{e}_1^\top \\
\varphi_{y^o}^\tau &= -(\beta\varphi_A - (1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\lambda h\gamma^z + (1+\beta\varphi_k)\xi_0 h\gamma^z + (\beta\varphi_{y^o}^\tau - \beta h\varphi_{\tau,1})\gamma^z \\
\varphi_y^\tau &= (\beta\varphi_A - (1+\beta\varphi_k)\eta)\frac{1}{\chi}\frac{\lambda h}{1-\lambda hc}\gamma^z - (1+\beta\varphi_k)\xi_0 h\gamma^z + (\beta h\varphi_{\tau,1} + \beta\varphi_y^\tau)\gamma^z
\end{aligned}
$$

Ultimately, I am interested in the shadow value of atmospheric carbon. This shadow value depends on all the other shadow values above and I am going to solve the expressions successively.

$$
\varphi_k = \frac{\kappa}{1-\beta\kappa}\ ; \qquad \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ ; \qquad \varphi_A = \frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta = \frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}\ .
$$

$$
\begin{aligned}
\varphi_{y^o}^{\tau} &= \tfrac{1}{1-\beta\gamma^z}\Big(-(\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{\lambda}{\chi}h\gamma^z+(1+\beta\varphi_k)\xi_0 h\gamma^z-\beta h\gamma^z\varphi_{\tau,1}\Big)\\
\varphi_{y}^{\tau} &= \tfrac{1}{1-\beta\gamma^z}\Big((\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{\lambda h}{1-\lambda hc}\gamma^z-(1+\beta\varphi_k)\xi_0 h\gamma^z+\beta h\gamma^z\varphi_{\tau,1}\Big)\\
\boldsymbol{\varphi}_{\tau}^{\top} &= -\Big((1+\beta\varphi_k)\xi_0-(\beta\varphi_A-(1+\beta\varphi_k)\eta)\,\tfrac{\lambda}{\chi}\Big)\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]\\
&= -\tfrac{1}{1-\beta\kappa}\Big(\xi_0+\eta\tfrac{\lambda}{\chi}\big(1-\tfrac{\beta}{1+\beta\delta_A}\big)\Big)\boldsymbol{\sigma}_{1\cdot}\left[(1-\beta\boldsymbol{\sigma})^{-1}\right]\\
\boldsymbol{\varphi}_{M}^{\top} &= \beta\boldsymbol{\varphi}_{M}^{\top}\boldsymbol{\Phi}+\tfrac{\beta}{M_{pre}}\Bigg[\sigma^{forc}\varphi_{\tau,1}+h\varphi_{\tau,1}\\
&\quad -\tfrac{\epsilon(c,\delta_\tau)}{1-\beta\gamma^z}\Big(-(\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{\lambda}{\chi}h\gamma^z+(1+\beta\varphi_k)\xi_0 h\gamma^z-\beta h\gamma^z\varphi_{\tau,1}\Big)\\
&\quad +\tfrac{\delta_\tau}{1-\beta\gamma^z}\Big((\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\tfrac{\lambda h}{1-\lambda hc}\gamma^z-(1+\beta\varphi_k)\xi_0 h\gamma^z+\beta h\gamma^z\varphi_{\tau,1}\Big)\\
&\quad +(\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\Big(\lambda\sigma^{forc}-\lambda h(\delta_\tau-\epsilon(c,\delta_\tau))-\tfrac{\delta_\tau}{c}\log(1-\lambda hc)\Big)\\
&\quad -(1+\beta\varphi_k)\xi_0\Big(\sigma^{forc}+h\epsilon(c,\delta_\tau)\Big)\Bigg]\boldsymbol{e}_1^{\top}\\
&= \beta\boldsymbol{\varphi}_{M}^{\top}\boldsymbol{\Phi}+\tfrac{\beta}{M_{pre}}\Bigg[\sigma^{forc}\varphi_{\tau,1}+h\varphi_{\tau,1}\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))\Big)\\
&\quad -\tfrac{\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))(1+\beta\varphi_k)\xi_0 h\\
&\quad +\tfrac{\gamma^z}{1-\beta\gamma^z}\big(\beta\varphi_A-(1+\beta\varphi_k)\eta\big)\tfrac{\lambda}{\chi}h\epsilon(c,\delta_\tau)\\
&\quad +\tfrac{\gamma^z}{1-\beta\gamma^z}\big(\beta\varphi_A-(1+\beta\varphi_k)\eta\big)\tfrac{1}{\chi}\tfrac{\lambda h}{1-\lambda hc}\delta_\tau\\
&\quad +(\beta\varphi_A-(1+\beta\varphi_k)\eta)\tfrac{1}{\chi}\Big(\lambda\sigma^{forc}-\lambda h(\delta_\tau-\epsilon(c,\delta_\tau))-\tfrac{\delta_\tau}{c}\log(1-\lambda hc)\Big)\\
&\quad -(1+\beta\varphi_k)\xi_0\Big(\sigma^{forc}+h\epsilon(c,\delta_\tau)\Big)\Bigg]\boldsymbol{e}_1^{\top}\\
&= \beta\boldsymbol{\varphi}_{M}^{\top}\boldsymbol{\Phi}+\tfrac{\beta}{M_{pre}}\Bigg[\varphi_{\tau,1}\Big(\sigma^{forc}+h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))\Big)\Big)\\
&\quad -\tfrac{\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))(1+\beta\varphi_k)\xi_0 h\\
&\quad -(1+\beta\varphi_k)\xi_0\Big(\sigma^{forc}+h\epsilon(c,\delta_\tau)\Big)\Bigg]\boldsymbol{e}_1^{\top}
\end{aligned}
$$

Using $\frac{\beta\varphi_A}{(1+\beta\varphi_k)} = \frac{\beta\eta}{1+\beta\delta_A}$ yields

$$\boldsymbol{\varphi}_M^\top = \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} + \tfrac{1}{1-\beta\kappa}\tfrac{\beta}{M_{pre}}\Bigg[$$

$$-\Big(1-\tfrac{\beta}{1+\beta\delta_A}\Big)\eta\tfrac{\lambda}{\chi}\Bigg(\sigma^{forc} + \big(1+\tfrac{\gamma^z}{1-\beta\gamma^z}\big)h\epsilon(c,\delta_\tau) + \Big(\tfrac{\gamma^z}{1-\beta\gamma^z}\tfrac{1}{1-\lambda hc} - 1 - \tfrac{1}{\lambda hc}\log(1-\lambda hc)\Big)h\delta_\tau$$

$$+\Big(\sigma^{forc} + h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau + +\epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\Bigg)$$

$$-\xi_0\Bigg(\sigma^{forc} + \Big(1+\tfrac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau) + \tfrac{\gamma^z}{1-\beta\gamma^z}h\delta_\tau$$

$$+\Big(\sigma^{forc} + h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau + \epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1}\Bigg)\Bigg]\boldsymbol{e}_1^\top$$

and

$$
\begin{aligned}
SCC &= -(1-\beta\kappa)Y_t^{net}\varphi_{M,1} \\
&= Y_t^{net}\frac{\beta}{M_{pre}}\Bigg(\Big(1-\frac{\beta}{1+\beta\delta_A}\Big)\eta\frac{\lambda}{\chi}\Bigg[\Big(\sigma^{forc}+h\Big(1+\frac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau++\epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \\
&\qquad +\sigma^{forc}+\Big(1+\frac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau)+\Big(\frac{\gamma^z}{1-\beta\gamma^z}\frac{1}{1-\lambda hc}-1-\frac{1}{\lambda hc}\log(1-\lambda hc)\Big)h\delta_\tau\Bigg] \\
&\qquad +\xi_0\Bigg[\Big(\sigma^{forc}+h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \\
&\qquad +\sigma^{forc}+\Big(1+\frac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau)+\frac{\gamma^z}{1-\beta\gamma^z}h\delta_\tau\Bigg]\Bigg)\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1} \\
&= Y_t^{net}\frac{\beta}{M_{pre}}\Bigg(\\
&\qquad \Big(\xi_0+\Big(1-\tfrac{\beta}{1+\beta\delta_A}\Big)\eta\tfrac{\lambda}{\chi}\Big)\Bigg[\Big(\sigma^{forc}+h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \\
&\qquad\qquad +\sigma^{forc}+\Big(1+\frac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau)+\frac{\gamma^z}{1-\beta\gamma^z}h\delta_\tau\Bigg] \\
&\quad +\Big(1-\tfrac{\beta}{1+\beta\delta_A}\Big)\eta\tfrac{\lambda}{\chi}\Bigg[+\Big(\tfrac{\gamma^z}{1-\beta\gamma^z}\Big(\tfrac{1}{1-\lambda hc}-1\Big)-1-\tfrac{1}{\lambda hc}\log(1-\lambda hc)\Big)h\delta_\tau\Bigg]\Bigg)\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1} \\
&= Y_t^{net}\frac{\beta}{M_{pre}}\Bigg(\\
&\qquad \Big(\xi_0+\Big(1-\tfrac{\beta}{1+\beta\delta_A}\Big)\eta\tfrac{\lambda}{\chi}\Big)\Bigg[\Big(\sigma^{forc}+h\Big(1+\tfrac{\beta\gamma^z}{1-\beta\gamma^z}(\delta_\tau+\epsilon(c,\delta_\tau))\Big)\Big)\left[\boldsymbol{\sigma}(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \\
&\qquad\qquad +\sigma^{forc}+\Big(1+\frac{\gamma^z}{1-\beta\gamma^z}\Big)h\epsilon(c,\delta_\tau)+\frac{\gamma^z}{1-\beta\gamma^z}h\delta_\tau\Bigg] \\
&\quad +\Big(1-\tfrac{\beta}{1+\beta\delta_A}\Big)\eta\tfrac{\lambda}{\chi}\Bigg[+\Big(\tfrac{\gamma^z}{1-\beta\gamma^z}\tfrac{\lambda hc}{1-\lambda hc}-1-\tfrac{1}{\lambda hc}\log(1-\lambda hc)\Big)h\delta_\tau\Bigg]\Bigg)\left[(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1}\right]_{1,1}
\end{aligned}
$$

I obtain the observed damages (not expected) as

$$
\begin{aligned}
d_t(\tau_{1,t+1}, i_t, A_t) &= \xi_0(\tau_{1,t+1} - 1) + \gamma \exp(\lambda \tau_{1,t+1}) \exp(-\chi(A_t + i_t^*)) \\
&\quad + \gamma^{\Delta} \exp(\lambda^{\Delta} \Delta_{t+1}) \exp(-\chi^{\Delta}(A_t^{\Delta} + i_t^{\Delta *})) \\
&= \xi_0(\tau_{1,t+1} - 1) + \gamma \exp(\lambda \tau_{1,t+1}) \exp\Bigg(- \chi \Bigg(\\
&\qquad \frac{1}{\chi} \Bigg[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\eta}{1 + \beta\delta_A} \right) + \mathrm{CGF}_{\tau_{t+1}}(\lambda) \Bigg] \Bigg) \Bigg) \\
&\quad + \gamma^{\Delta} \exp(\lambda^{\Delta} \Delta_{t+1}) \exp\Bigg(- \chi^{\Delta} \Bigg(\\
&\qquad \frac{1}{\chi^{\Delta}} \Big[\log(\gamma^{\Delta}\chi^{\Delta}) - \log\left(\eta_t^{\Delta} - \frac{\beta\eta_t^{\Delta}}{1 + \beta\delta_A} \right) + \mathrm{CGF}_{\Delta_{t+1}}(\lambda) \Big] \Bigg) \Bigg) \\
&= \xi_0(\tau_{1,t+1} - 1) + \gamma \exp(\lambda \tau_{1,t+1}) \exp\Bigg(\\
&\qquad - \Bigg[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\eta}{1 + \beta\delta_A} \right) + \mathrm{CGF}_{\tau_{t+1}}(\lambda) \Bigg] \Bigg) \\
&\quad + \gamma^{\Delta} \exp(\lambda^{\Delta} \Delta_{t+1}) \exp\Bigg(\\
&\qquad - \Big[\log(\gamma^{\Delta}\chi^{\Delta}) - \log\left(\eta_t^{\Delta} - \frac{\beta\eta_t^{\Delta}}{1 + \beta\delta_A} \right) + \mathrm{CGF}_{\Delta_{t+1}}(\lambda) \Big] \Bigg) \\
&= \xi_0(\tau_{1,t+1} - 1) + \frac{1}{\chi} \exp\left(\lambda \tau_{1,t+1} - \mathrm{CGF}_{\tau_{t+1}}(\lambda) \right) \left(\eta - \frac{\beta\eta}{1 + \beta\delta_A} \right) \\
&\quad + \frac{1}{\chi^{\Delta}} \exp\left(\lambda^{\Delta} \Delta_{t+1} - \mathrm{CGF}_{\Delta_{t+1}}(\lambda) \right) \left(\eta_t^{\Delta} - \frac{\beta\eta_t^{\Delta}}{1 + \beta\delta_A} \right)
\end{aligned}
$$

I obtain the observed damages (not expected) as

$$
\begin{aligned}
d_t(\tau_{1,t+1}, i_t, A_t) = \xi_0(\tau_{1,t+1} - 1) + \gamma \exp(\lambda \tau_{1,t+1}) \exp\Bigg(\\
&- \Big[\log(\gamma\chi) - \log\left(\eta - \frac{\beta\eta}{1+\beta\delta_A}\right) + \lambda\, \mathbb{E}_t\, \tau_{1,t+1} \\
&\quad + \log\left(\mathrm{MGF}_{y_{t+1}}(\lambda h)\right) - \lambda h\, \mathbb{E}\, y_{t+1} + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda)) - \lambda\, \mathbb{E}_t\, \Delta_{t+1} \Big] \Bigg) \\
+\gamma^{\Delta} \exp(\lambda^{\Delta} \Delta_{t+1}) \exp\Bigg(\\
&- \Big[\log(\gamma^{\Delta}\chi^{\Delta}) - \log\left(\eta_t^{\Delta} - \frac{\beta\eta_t^{\Delta}}{1+\beta\delta_A}\right) + \log(\mathrm{MGF}_{\Delta_{t+1}}(\lambda)) \Big] \Bigg) \\
= \xi_0(\tau_{1,t+1} - 1) + \frac{\eta}{\chi}\left(1 - \frac{\beta}{1+\beta\delta_A}\right) \exp\Bigg(\lambda\left(\tau_{1,t+1} - \mathbb{E}_t\, \tau_{1,t+1}\right) \\
&- \Big[\mathrm{CGF}_{y_{t+1}}(\lambda h) - \lambda h\, \mathbb{E}\, y_{t+1} + \mathrm{CGF}_{\Delta_{t+1}}(\lambda) - \lambda\, \mathbb{E}_t\, \Delta_{t+1} \Big] \Bigg) \\
+\frac{\eta_t^{\Delta}}{\chi^{\Delta}}\left(1 - \frac{\beta}{1+\beta\delta_A}\right) \exp\left(\lambda^{\Delta}\Delta_{t+1} - \mathrm{CGF}_{\Delta_{t+1}}(\lambda) \right)
\end{aligned}
$$

I note that under certainty – see DACE or work out the special case of the equations above – it is

$$
\boldsymbol{\varphi_M}^{\top\, det} - \beta \boldsymbol{\varphi}_M^{\top} \boldsymbol{\Phi} - \beta \varphi_{\tau,1} \frac{\sigma^{forc}}{M_{pre}} \boldsymbol{e}_1^{\top} = 0 \Rightarrow \boldsymbol{\varphi}_M^{\top\, det} = \frac{\beta \varphi_{\tau,1} \sigma^{forc}}{M_{pre}} \boldsymbol{e}_1^{\top} (\mathbf{1} - \beta \boldsymbol{\Phi})^{-1} \tag{B.40}
$$

$$
\boldsymbol{\varphi}_{\tau}^{\top det} = -\xi_0(1+\beta\varphi_k) \boldsymbol{e}_1^{\top} (1 - \beta\boldsymbol{\sigma})^{-1} \quad \Rightarrow \varphi_{\tau,1}^{det} = -\xi_0(1+\beta\varphi_k)[1 - \beta\boldsymbol{\sigma})^{-1}]_{1,1}
$$

which together with equation (B.43) above delivers

$$
\varphi_{M,1}^{det} = -\xi_0 \left(1 + \beta \frac{\kappa}{1-\beta\kappa}\right) \left[(1-\beta\boldsymbol{\sigma})^{-1}\right]_{1,1} \frac{\beta \sigma^{forc}}{M_{pre}} \left[(\mathbf{1} - \beta\boldsymbol{\Phi})^{-1}\right]_{1,1}.
$$

The subsequent proof relates this shadow value under certainty to the general shadow value of atmospheric carbon under uncertainty.

and for the new and altered equations

$$
\begin{aligned}
&\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top = 0\\
&\quad\Rightarrow \boldsymbol{\varphi}_\tau = -\Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&\varphi_A - \beta\varphi_A(1-\delta_A) + (\beta\varphi_A - (1+\beta\varphi_k)\eta) = 0\\
&\quad\Rightarrow \varphi_A + \beta\delta_A\varphi_A - (1+\beta\varphi_k)\eta = 0\\
&\quad\Rightarrow \delta_A(1+\beta\varphi_A) = (1+\beta\varphi_k)\eta\\
&\quad\Rightarrow \varphi_A = \frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta = \frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}\ ,
\end{aligned}
\tag{B.41}
$$

$$
\begin{aligned}
&\big(\varphi_k - (1+\beta\varphi_k)\kappa\big)k_t + \big(\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top\big)\boldsymbol{R}_t + \big(\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top\big)\boldsymbol{M}_t\\
&+ \Big(\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top\Big)\boldsymbol{\tau}_t\\
&+ \underline{\Big(\varphi_A - \big(\beta\varphi_A(1-\delta_A) - (\beta\varphi_A - (1+\beta\varphi_k)\eta)\big)\Big)A_t}\\
&+\varphi_t \;=\; \beta\varphi_{t+1}\\
&\qquad+\ \log x_t^*(\varphi_k) + \beta\varphi_k\log(1-x_t^*(\varphi_k)) + (1+\beta\varphi_k)\xi_0\\
&\qquad+ (1+\beta\varphi_k)\kappa k_t + (1+\beta\varphi_k)\log F(\boldsymbol{A}_t, \boldsymbol{N}_t^*, \boldsymbol{\mathcal{K}}_t^*, \boldsymbol{E}_t^*)\\
&\qquad+ \beta\varphi_k(\log[1+g_{k,t}] - \log[\delta_k + g_{k,t}]) - \beta\boldsymbol{\varphi}_{R,t+1}^\top\boldsymbol{E}_t^{d*}\\
&\qquad+ \beta\varphi_{M,1}\Big(\textstyle\sum_{i=1}^{I^d} E_{i,t}^* + E_t^{exo}\Big) + \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M^{pre}}G_t\\
&\qquad+ \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\tfrac{1}{\chi}\Big[\log(\gamma\chi) - \log\Big(\eta - \tfrac{\beta\varphi_A}{(1+\beta\varphi_k)}\Big)\Big] + \tfrac{1}{\chi}\left((1+\beta\varphi_k)\eta - \beta\varphi_A\right)}
\end{aligned}
\tag{B.42}
$$

where I underline the novel terms with respect to DACE.[23] This equation had to hold for all levels of the state variables, which requires that the coefficients in front of the

[23]Equation (B.18) also includes the terms $(1+\beta\varphi_k)\xi_0\tau_{1,t}$ and $(1+\beta\varphi_k)\xi_0$ in addition to the ones underlined above. These are terms that derive from the original (non-adaptable) part of the damage function which is also part of DACE.

state variables vanish. Thus, the shadow values have to satisfy as already in DACE

$$\varphi_k - (1+\beta\varphi_k)\kappa = 0 \qquad \Rightarrow \varphi_k = \frac{\kappa}{1-\beta\kappa} \tag{B.43}$$

$$\boldsymbol{\varphi}_M^\top - \beta\boldsymbol{\varphi}_M^\top\boldsymbol{\Phi} - \beta\varphi_{\tau,1}\frac{\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top = 0 \Rightarrow \boldsymbol{\varphi}_M^\top = \frac{\beta\varphi_{\tau,1}\sigma^{forc}}{M_{pre}}\boldsymbol{e}_1^\top(\mathbf{1}-\beta\boldsymbol{\Phi})^{-1} \tag{B.44}$$

$$\boldsymbol{\varphi}_{R,t}^\top - \beta\boldsymbol{\varphi}_{R,t+1}^\top = 0 \qquad \Rightarrow \boldsymbol{\varphi}_{R,t} = \beta^{-t}\boldsymbol{\varphi}_{R,0}\ .$$

and for the new and altered equations

$$\begin{aligned}
&\boldsymbol{\varphi}_\tau^\top - \beta\boldsymbol{\varphi}_\tau^\top\boldsymbol{\sigma} + \Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top = 0\\
&\quad\Rightarrow \boldsymbol{\varphi}_\tau = -\Big((1+\beta\varphi_k)\xi_0 - \underline{(\beta\varphi_A - (1+\beta\varphi_k)\eta)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
&\varphi_A - \beta\varphi_A(1-\delta_A) + (\beta\varphi_A - (1+\beta\varphi_k)\eta) = 0\\
&\quad\Rightarrow \varphi_A + \beta\delta_A\varphi_A - (1+\beta\varphi_k)\eta = 0\\
&\quad\Rightarrow \delta_A(1+\beta\varphi_A) = (1+\beta\varphi_k)\eta\\
&\quad\Rightarrow \varphi_A = \frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta = \frac{\eta}{(1-\beta\kappa)(1+\beta\delta_A)}\ ,
\end{aligned} \tag{B.45}$$

where I inserted the shadow value of log-capital in the final step. Inserting the solution for the shadow values of capital and abatement capital into the shadow value of an (exponential) temperature increase yields

$$\begin{aligned}
\boldsymbol{\varphi}_\tau &= -\Big((1+\beta\varphi_k)\xi_0 - \underline{\Big(\beta\frac{1+\beta\varphi_k}{1+\beta\delta_A}\eta - (1+\beta\varphi_k)\eta\Big)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
\boldsymbol{\varphi}_\tau &= -(1+\beta\varphi_k)\Big(\xi_0 - \underline{\Big(\frac{\beta\eta}{1+\beta\delta_A} - \eta\Big)\,\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
\boldsymbol{\varphi}_\tau &= -(1+\beta\frac{\kappa}{1-\beta\kappa}\,)\Big(\xi_0 + \underline{\eta\left(1 - \frac{\beta}{1+\beta\delta_A}\right)\frac{\lambda}{\chi}}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\\
\boldsymbol{\varphi}_\tau &= -\frac{1}{1-\beta\kappa}\Big(\xi_0 + \underline{\eta\frac{\lambda}{\chi}\Big(1 - \frac{\beta}{1+\beta\delta_A}\Big)}\Big)\boldsymbol{e}_1^\top(1-\beta\boldsymbol{\sigma})^{-1}\ .
\end{aligned} \tag{B.46}$$

$$d_t(\tau_{1,t}, i_t, i_t^{\Delta}, A_t, A_t^{\Delta}, \Delta_t) = \xi_0(\tau_{1,t+1} - 1) + \gamma \exp(\lambda \tau_{1,t+1}) \exp(-\chi(A_t + i_t)) \\ + \gamma^{\Delta} \exp(\lambda^{\Delta} \Delta_{t+1}) \exp(-\chi(A_t^{\Delta} + i_t^{\Delta})) \ .$$