Contract Enforcement and Young Firm Capital Structure: A Global Perspective*

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Abstract

We study young firms' short- and long-term leverage dynamics across developed and developing countries to infer the severity of firm-level financial constraints via the lens of a model with endogenous borrowing constraints. Using balance-sheet data for private firms, we document two novel facts: (i) short-term leverage increases, while long-term leverage decreases over the early life cycle of firms; and (ii) the rise in short-term leverage persists for more time periods in less developed economies. We build a model of endogenous borrowing constraints to reconcile these observations. The model implies that short-term funding gaps vis-a-vis a benchmark unconstrained economy are biased measures of the severity of firm-level financial constraints. Guided by the model, we estimate that the bias accounts for 17.2% (10.6%) in low (high) contract-enforcement economies. We use the model to quantify the severity of firm-level financial constraints in countries that lack balance sheet data to inform the design of government-assisted credit programs.

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1 Introduction

Young firms fuel economic growth and dynamism as they account for a large share of job creation in the economy (Haltiwanger et al., 2013; Ayyagari et al., 2014). Since the majority of this growth is financed via debt, it is of critical importance to understand the dynamics of young firms' leverage and the potential constraints these firms face in raising debt, as credit assistance programs around the globe rely crucially on firm-level estimates of financial constraints. In this paper, we provide a new theoretical and empirical framework to quantify financial constraints for young firms.

We begin by documenting leverage dynamics for young private firms across countries of varying income levels using the Orbis database. Two distinct observations emerge. First, long-term leverage declines as a firm ages. Second, short-term leverage increases during the early life cycle of a firm and then declines. We develop a theory of endogenous borrowing constraints to reconcile these two phenomena. In the model, a typical firm chooses scale (capital) at inception and labor every period, while facing an exogenous death shock. The firm has to raise debt in order to finance all payments and can renege on its obligations at any period—i.e. contract enforcement is limited.

Assuming that (i) the firm incurs a small fixed cost to underwrite every contract and (ii) the net present value of the project is negative in the first year of operation, we derive a unique optimal financial structure: long-term debt to finance initial capital and a series of short-term loans to finance per-period working capital needs. Our uniqueness result contributes to the dynamic contract literature (Albuquerque and Hopenhayn (2004); Clementi and Hopenhayn (2006); DeMarzo and Fishman (2007); Verani (2018)) by ruling out the usual multiplicity of implementations; in particular, we rule out the use of short-term debt for everything where a long asset (capital) would be financed by rolling over short-term debt. Furthermore, for any level of capital (scale), the optimal contract is in the spirit of Albuquerque and Hopenhayn

(2004) and it features front-loading of the payments of long-term debt and strictly increasing level of short-term debt over the life cycle, until a firm becomes unconstrained. Moreover, optimal maturity and scale reflect a crucial trade-off: on the one hand, higher long-term debt allows for higher scale and higher future dividend payments, on the other hand, higher debt levels delay the dividend payments further.

We use the framework to study leverage dynamics across developed and developing countries. Motivated by evidence obtained by the Cost of Doing Business Database provided by the World Bank, we assume that differences in the level of development reflect differences in the degree of contract enforcement—specifically, recovery rates on loans are persistent and systematically higher in richer countries. Under this assumption, the model predicts that a typical firm in a less developed economy has lower initial scale, lower short- and long-term leverage, and lower debt maturity. In turn, short-term leverage is increasing over the life cycle of the firm, and this increase persists for longer in less developed economies. We verify that these patterns are present in cross-country firm-level data.

Given that the model reconciles observed leverage dynamics across countries, we use it to quantify cross-country differences in the degree of financial constraints. Specifically, we measure short-term debt funding gaps, which we define as the difference between a firm's actual borrowing and its unconstrained borrowing benchmark. We show that funding gaps can be decomposed into two components. The first component is the difference between the short-term debt level that a firm raises at the end of the early life cycle (when it becomes unconstrained) and the level of short-term debt that it raises each year during the early life cycle. This object is measurable ex post using firm-level data for any given country. The second component is the distortion in the required debt level at the end of the early life cycle. This object is much more challenging to measure as it corresponds to the counterfactual debt the firm would have required if it were to be created in an economy with near perfect enforcement. In other words, this target distortion reflects the distortion in initial scale that

occurs due to the limited enforcement of contracts and measuring this requires cross-country comparison to get a definition of a benchmark economy with perfect enforcement.

To quantify both objects, we assume that the U.S. is a benchmark economy characterized by perfect contract enforcement. This choice is motivated in part by the existing literature that argues that a typical U.S. firm is unconstrained (ex. Pugsley and Hurst (2011), Bhandari et al. (2024)) and follows a strategy typically employed by the macro development literature in computing outcomes relative to the U.S. (ex. Hsieh and Klenow (2009)). When we calibrate the model's parameters to match the life-cycle of a typical firm in high- and low-enforcement countries, we find that the unobserved (or counterfactual) funding gap accounts for 17% of the entire funding gap for a typical firm in low-enforcement economy, which is nearly twice the 10% statistic that we obtain for a high-enforcement economy. This suggests that financial frictions would be even more mismeasured in developing economies due to the unobserved component.

Arguments related to the importance of measuring firms' financial constraints are at the forefront of both academic and policy work. Small and young firms are, on average, more likely to be financially constrained than mature firms (Hadlock and Pierce, 2010), which calls for government-assisted credit programs. A recent extensive World Bank report prepared by Didier and Cusolito (2024) focuses on measuring financial constraints using firm-level data. Our paper directly speaks to this work and complements it by emphasizing the importance of a model-based quantification of the severity of financial constraints in the presence of firm scale distortions at birth.

Since we study financial constraints, we cannot assume them, they need to be a result of the model. For this reason, we build on the endogenous borrowing constraints literature (Albuquerque and Hopenhayn (2004), Cooley et al. (2004)) to construct a model where financial constraints are a result of limited enforcement. This gives us a framework where we can get a notion of "more severe financial constraints" by comparing the model's be-

havior under different levels of enforcement. While an endogenous borrowing constraint can also result from agency problems (Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007), Verani (2018), etc.), we focus on enforcement frictions because a large literature in international corporate finance (discussed below) documents that cross-country differences in capital structure are explained by differences in the quality of enforcement institutions.

By focusing on the characterization of financing gaps, our approach is closer in spirit to the exercises performed by the empirical capital structure literature (see Graham and Leary (2011) for a review); in this tradition, leverage is explained by outcomes (size, capital stock, profits, etc.). In contrast, the empirical literature on financial frictions focuses on outcome distortions (ex. profits, forgone investment, employment) to infer financial constraints.[1] The seminal paper of Fazzari et al. (1988) interprets cash flow sensitivity of investment as indicative of financial constraints; the critique of Kaplan and Zingales (1997) is also based on whether the firms identified as constrained are unable to fund their desired investment. However, the empirical implementation of Kaplan and Zingales (1997) used by Lamont et al. (2001) is based on the existence of a financial constraint factor that explains stock returns, therefore, while motivated by the forgone investment approach, the implementation is closer to one of forgone profits. The index of financial constraints in Whited and Wu (2006) is constructed through a structural estimation of an Euler Equation governing firms' investment decisions. Dao and Liu (2017) study the effect of financial constraints on job creation. A recent paper by Cherchye et al. (2020) proposes a measure of financial constraints by studying forgone profits. Relative to these papers, our paper is the first to develop a framework to measure financial constraints that is applicable to private and young firms[2]

[1] A notable exception is Almeida et al. (2004) who study the sensitivity of cash flows. In their model, financially constrained firms should increase their cash holdings in response to a cash flow innovation. However, the reason that motivates the savings decisions is an investment opportunity in the future; hence, while the test does not involve investment directly, increases in cash holdings reflect a reduction in future forgone investment.

[2] The methodology in Cherchye et al. (2020) is also applicable to private firms while the literature has typically studied financial constraints on public firms.

and methodologically we are, to the best of our knowledge, the first to claim the need of cross country comparisons to measure financial constraints.

In our model, both long- and short-term debt emerge endogenously. Firms will optimally match the maturities of assets and liabilities: long-term debt will be used to fund the initial capital acquisition, and short-term debt will be used to fund working capital needs. This disciplines how to map the model to the data, since there is a direct correspondence between both. Crucially, we show that the joint dynamic behavior of short- and long-term leverage will give us information about financial constraints. In other words, long- and short-term leverage behave differently in young and constrained firms; therefore, focusing on total leverage as is typical in the firm life-cycle literature is with loss of generality when studying young firms. Specifically, according to our model, constrained firms during their early life cycle are characterized by decreasing long-term leverage and increasing short-term leverage as they age.

A relatively new literature that studies financing over the life cycle of firms documents that private firms' short and long-term leverage is linearly decreasing in age. Specifically, Dinlersoz et al. (2019) rely on a quadratic specification for private firms in the U.S. and find that short-leverage is always decreasing. Additionally, Derrien et al. (2021) document for French firms that in the first ten years of operations, total leverage decreases with age; and recently Kochen (2022) documents that leverage (defined as short plus long leverage) is decreasing in age for European countries. Relative to this literature, we focus on the joint dynamics of short- and long-term leverage in the early life cycle of firms and we document a new fact: during the first years of firms' operations, short leverage increases while long leverage decreases. Relatedly, Arellano et al. (2012) study the relationship between financial development and firm dynamics along the size distribution of firms and find that the difference between young and old firm outcomes is more pronounced in less developed economies. They report a negative relationship between leverage (measured as long leverage)

and financial development.

Since financial constraints are endogenous in our model and emerge from enforcement frictions, we have a well-defined notion in the model of "more severe financial constraints". The natural counterpart of this comparative static of the model in the data is the capital structure of firms in different countries. This is precisely what the international corporate finance literature studies; it investigates the extent to which the traditional explanatory variables for firms' capital structure, developed using U.S. data, are applicable to other countries. Two messages emerge from this literature. First, the same variables that explain capital structure choices in the U.S. also explain capital structure choices in other countries—Rajan and Zingales (1995) document this for developed economies and Booth et al. (2001) for less developed countries. Second, countries' institutions affect financing patterns, and they do so more than industry affiliation (Demirgüç-Kunt and Maksimovic, 1999; Fan et al., 2012). Antzoulatos et al. (2016) study the convergence of capital structure, evaluating firm-specific, industry-specific, and economy-wide factors, and find evidence in favor of financial development (economy-wide factor) driving the firm's leverage ratio. Exploring firm size heterogeneity and financial development, Beck et al. (2008) find that small firms benefit relatively more than large firms from the institutional factors of a country. Demirgüç-Kunt and Maksimovic (1999) find that small firms obtain less long-term debt financing in countries with less-developed financial systems. Demirgüç-Kunt and Maksimovic (1998) find that a greater proportion of firms use long-term external financing in countries with higher legal system efficiency (better enforcement). Studying relatively large firms, Qian and Strahan (2007) find that stronger creditor rights are associated with longer-term lending by banks and tangible assets correlation with debt maturity increases in countries with better creditor rights. Bae and Goyal (2009) argue that enforceability matters more than the existence of creditors protection law and find that better enforceability of contracts increases loan size and lengthens loan maturity. Haselmann et al. (2010) investigate the channel through which

legal institutions are transmitted to the economy and find that collateral laws matter more than bankruptcy laws for bank lending.

A key prediction of our model is that long-term leverage is higher in more financially developed countries, as documented in the data. This is in contrast to the standard model of endogenous financial constraints, which assumes that all firms must pay the same unproductive set-up cost and predicts that firms will repay the same face value of long-term debt in less developed (lower enforcement) countries but over a longer period of time. Furthermore, the value of the firm and the total surplus of the contract between the firm and financiers (the value of assets) is increasing in enforcement. A fixed face value of long-term debt for firms across development levels and the fact that asset values are increasing in enforcement imply that in less developed countries, leverage will be higher due to a lower value of assets of the firm. To fix this counterfactual prediction, we endogenize the unproductive set-up cost by making it a productive choice. In our preferred interpretation, young firms buy upfront the capital scale they want to operate during the early life cycle. This implies that firms in less developed economies will face a higher cost of capital due to the endogenous financial frictions and choose a smaller scale. Thus, we resolve the counterfactual prediction of the traditional model and introduce a new dimension to the measurement of financial frictions.

The financial frictions and firm dynamics literature typically distinguishes between the extensive (firm dynamism) and the intensive (factor allocation between existing firms) margin. Midrigan and Xu (2014) using detailed firm-level from Korea find that the bulk of the effects of financial frictions goes through the extensive margin. The theoretical reason is that firms eventually grow out of their borrowing constraints by accumulating internal funds. In other words, the distortions along the intensive margin tend to disappear as time goes by while the extensive margin distortion persists. In our model the initial capital distortions will not exclude the firm from entering (hence we don't have distortions along the extensive

margin, but will generate a permanent distortion along the intensive margin.[3] The scale distortion in our model is permanent by construction since we don't let firms adjust capital once they enter.[4] We interpret the "persistency" of this scale distortion with care, our model is only appropriate for the early life cycle of firms (roughly from inception up to 7/8 years old), so a persistent distortion means, in the context of our model, a distortion that lasts the duration of the early life cycle of firms.

The remainder of the paper is organized as follows; Section 2 presents the data and stylized facts, Section 3 presents the theory, Section 4 analyzes financial constraints through the lens of our model and presents an illustrative example of our methodology. Section 5 discusses potential applications of our framework and concludes. All proofs, data description and alternative empirical specifications are presented in the appendix.

2 Empirical Analysis

2.1 Data and Variables Definition

The data source is Orbis compiled by Moody's Bureau van Dijk (BvD). BvD compiles annual balance sheets and income statements of private and public firms and harmonizes them into an internationally comparable format. The data also report industry identifiers and the firm's year of incorporation, allowing us to construct the firm's age. While Orbis has extensive coverage of countries, the degree to which the data represents the population of firms varies across countries. Bajgar et al. (2020) shows that in an extended sample of countries, Orbis captures 60% of aggregate employment and output but 40% of value-added; they further identify a subset of countries (what they called "preferred sample") that is representative of the firm population. We use in our analysis those countries and restrict the analysis to

[3] A modification requiring a minimum investment in order to start operations could solve this.

[4] The idea that firms choose at inception a factor that fixes their future scale seems to be in line with the literature on persistent effects of entry conditions. See Moreira (2019).

companies with no consolidated accounts that have non zero debt; the summary statistics are presented in table 3 in the appendix.

Our objective is to document the patterns of young firms' capital structure. Guided by our model we focus on the life cycle patterns of short- and long-term leverage:

$$\text{Short Leverage} \equiv \frac{\text{Short Term Debt}}{\text{Short Term Debt} + \text{Long Term Debt} + \text{Net Worth}}$$

$$\text{Long Leverage} \equiv \frac{\text{Long Term Debt}}{\text{Short Term Debt} + \text{Long Term Debt} + \text{Net Worth}}$$

Where short-term debt are loans from financial institutions that mature in less than a year, and long-term debt are loans from financial institutions that mature in more than a year. One can define leverage in more than one way[5] (see Rajan and Zingales (1995) and Welch (2011)), the advantage of the definition adopted here is that it does not vary with omitted variables as it would be the case if we were to define short term leverage as $\frac{\text{Short Term Debt}}{\text{Total Assets}}$ where an increase in total assets funded by trade credit would result in lower leverage.

2.2 Fact 1: Financial Development, Long and Short-Leverage

Firms in more financially developed economies have higher levels of long leverage and lower levels of short term leverage. This patterns holds true for young and old firms, moreover in more developed economies the differences between young and old firms are attenuated. Figure 1 is in line with the previous literature that abstracting from life cycle considerations documents a similar patterns. Demirgüç-Kunt and Maksimovic (1999) finds that firms in developing countries have lower long-leverage ratios. Fan et al. (2012) finds that countries with more corruption and lower legal enforcement have lower debt maturities. Intuitively, in

[5]For example, defining leverage as $\frac{\text{Short Term Debt} + \text{Long Term Debt}}{\text{Net Worth}}$ captures the relative claims on the firms value held by equity and debt. If we were studying agency problems, this would be a more appropriate definition.

environments with lower law enforcement, short-term debt that allows insiders less discretion should dominate, as opposed to long-term debt that usually is accompanied by covenants and collateral clauses that rely more heavily on a country's legal institutions. Empirically, this intuition is corroborated by Fan et al. (2012) and Demirgüç-Kunt and Maksimovic (1999) that find higher low term leverage in less developed economies (weaker legal institutions, higher corruption, etc..). Arellano et al. (2012) study the relationship between financial development and firm dynamics along the size distribution of firms and find that the difference between young and old firm outcomes is more pronounced in less developed economies. They report a negative relationship between leverage (measured as long leverage) and financial development. de Jong et al. (2008) study firm-specific and country-specific factors and document the indirect effect of institutions that results in the differences in levels of the typical factors explaining capital structure. They conclude that firms in countries with a better legal environment take more debt, and country-specific factors also reinforce the effects of firm-level determinants of leverage.[6]

2.3 Fact 2: Leverage Over The Early Life Cycle of Firms

To document the behavior of capital structure over the early life cycle of firms, we run the following model:

$$y_{i,t} = \eta_i + \sum_{a \in A} \beta_a \mathcal{D}^a_{i,t} + LEMP_{i,t} + COLLAT_{i,t} + PROFIT_{i,t} + LPROD_{i,t} + \epsilon_{i,t}$$

Where $y_{i,t}$ denotes the outcomes variable of interest (short- and long-term leverage), $\mathcal{A} = \{(0-1, 2-3, \ldots, 24-25, +25\}$ corresponds to 14 age categories, $\mathcal{D}^a_{i,t}$ is a dummy variable that takes value one if firm i belongs to the age-bin a at time t. LEMP is log of employment,

[6]For example, collateral explains more long-term debt in more developed economies

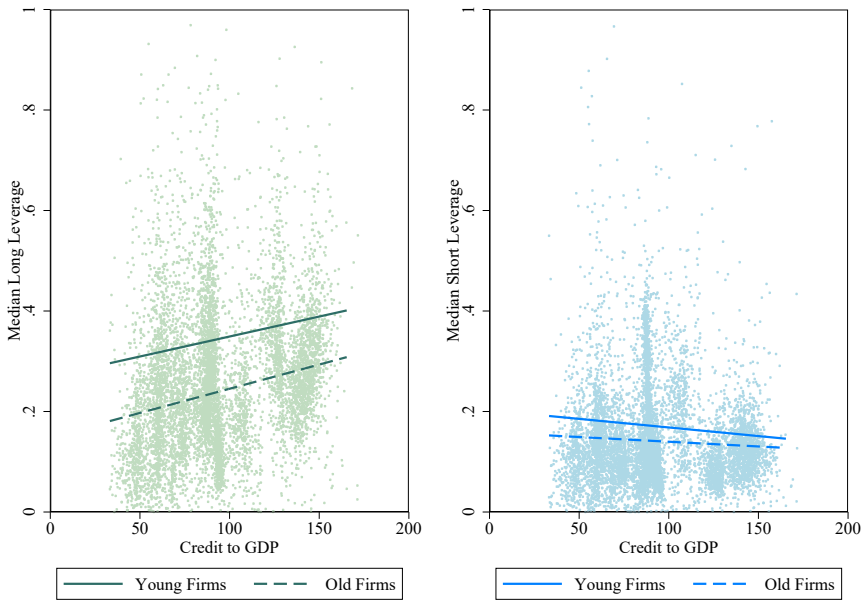


FIGURE 1: EACH OBSERVATION CORRESPONDS TO AN INDUSTRY COUNTRY AND AGE TRIPLET. YOUNG FIRMS ARE FIRMS THAT ARE 8 YEARS OLD OR LESS, AND OLD FIRMS ARE FIRMS MORE THAN 8 YEARS OLD. FITTED LINES CORRESPOND TO A LINEAR MODEL IN CREDIT TO GDP, AND INDUSTRY FIXED EFFECTS RUN OVER THE SUB-SAMPLE OF YOUNG AND OLD FIRMS.

COLLAT is tangible assets over total assets, PROFIT is caculated as EBIT over total assets and LPROD is caculated as the log of the ratio of sales to employment. The main message of figure 2 is that firms up to 7 years old present a distinctive pattern, short-leverage increases while long-leverage decreases. We will refer to this pattern as the capital structure over the early life cycle (ELC) of firms, and we will refer to the last period where age has a positive effect on age as the *duration* of the ELC.

Note that beyond the duration of the ELC, short—and long-leverage decrease with firm age. Hence, the discussion of whether one should measure short-, long—or total leverage when studying life cycle effects is not of first-order importance. In this sense, models with only short-term debt (whose typical data counterpart is total leverage) are likely to be a good representation of firms' debt dynamics beyond, but not during, the ELC.

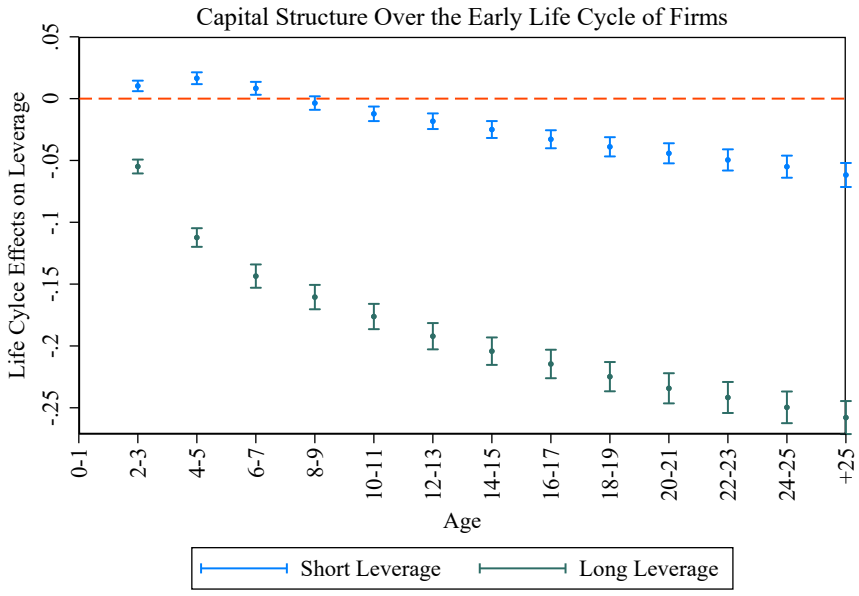


FIGURE 2: LIFE CYCLE PATTERNS OF CAPITAL STRUCTURE OF FIRMS FOR WHICH WE HAVE AT LEAST 3 OBSERVATIONS AND HAVE POSITIVE DEBT. CONFIDENCE INTERVALS ARE AT 90% AND STANDARD ERRORS ARE CLUSTERED AT COUNTRY-INDUSTRY-YEAR.

3 Model

Motivated by the stylized facts from international corporate finance, we develop a model of firm dynamics with endogenous borrowing constraints. The model generates leverage dynamics consistent with our empirical findings and provides a framework for measuring the severity of financial constraints over the early life cycle of firms.

3.1 Environment and Optimal Financial Structure

Firms operate a decreasing returns to scale technology with a quasi-fixed factor (capital K) and a flexible factor (labor n). The firm can only choose capital K at inception, after which this factor remains fixed. Each period a is divided into two sub-periods, morning and night. During the morning, firms commence production, pay for inputs, and make savings decisions. During the night, production and revenues are realized, and firms decide whether to continue operations or strategically default. Conditional on continuation, firms make dividend, savings, and debt repayment decisions. At the end of each period, firms face an exogenous exit shock with probability $1 - \rho$.

Assumption 1 (Property Rights) *There exists an enforcement technology ξ in the economy that can be applied to promises codified in contracts. Registering a transaction in a contract requires paying a fixed cost $\kappa_C > 0$. The technology ξ imposes a penalty $(1 - \xi)$ when the borrower engages in strategic default, but not to default induced by exit shocks.*

Assumption 2 (Transaction Costs) *The contracting costs $\kappa_C > 0$ are not too high:*

$$\kappa_C < S_0(K(\xi_L), z_L)\frac{1-\rho}{\rho}$$

where $S_0(K(\xi_L), z_L)$ denotes the short-term debt that the least productive firm (z_L) would request in the first year of operations in the country with worst enforcement (ξ_L).

Assumption 3 (Minimum Scale) *Parameters are such that:*

$$\frac{\beta\rho(1+r_f)+\beta(1-\rho)}{(1-\beta\rho)(1+r_f)} > \frac{1-\eta}{\eta\alpha}$$

Throughout the main text, we assume that capital has no collateral value ($\phi = 0$) to focus on the pure effects of enforcement frictions. The general model with collateral ($\phi > 0$) is presented in the Appendix, where we show that the key mechanisms and empirical predictions remain robust.

Proposition 1 (Financial Structure) *Under Assumptions 1-3, the optimal financial structure consists of:*

1. *Long-term debt to fund the initial capital acquisition*
2. *A sequence of short-term loans to fund working capital needs*

Proof. See Appendix A.1.1 ■

This result establishes the optimality of different debt instruments for different purposes, corresponding to the corporate finance principle of maturity matching. The model provides a disciplined approach to leverage measurement, with clear empirical counterparts for both types of debt.

For notational simplicity, we denote long-term debt as $L = \{L_0; \{q_a\}_{a\geq 0}\}$ and short-term loans as $S = \{\{S_a; (1+r_f)S_a\}\}_{a=0}^{\infty}$, where q_a represents the coupon payment at age a for long-term debt, and S_a represents short-term borrowing at age a.

3.2 Firm's Problem Under Limited Enforcement

When enforcement is imperfect ($\xi < 1$), firms can engage in strategic default, keeping $(1-\xi)$ of realized revenues. Financiers only sign contracts that are enforceable, meaning they don't incentivize the firm to strategically default.

Let V_a denote the value of the firm at age a, defined as the expected present value of all future profits:

$$V_a = \sum_{j=0}^{\infty} (\beta\rho)^j \left(z \left[K^\alpha \left(\frac{R_{a+j}}{w} \right)^{1-\alpha} \right]^\eta - (1+r_f) R_{a+j} - q_{a+j} \right)$$

The firm's problem under limited enforcement is to maximize the expected present value of dividends:

$$V_0 = \max_{\{K, \{q_a, S_a, n_a\}_{a=0}^{\infty}\}} \sum_{a=0}^{\infty} (\beta\rho)^a \left(z \left[K^\alpha \left(\frac{n_a}{w} \right)^{1-\alpha} \right]^\eta - (1+r_f) S_a - q_a \right)$$

subject to:

$$w_a n_a \leq S_a \tag{WCC)}$$

$$p_k K (1+r_f) \leq \sum_{a=0}^{\infty} (\beta\rho)^a q_a \tag{PCF}$$

$$0 \leq q_a \leq z \left[K^\alpha \left(\frac{R_a}{w} \right)^{1-\alpha} \right]^\eta - (1+r_f) S_a \tag{NNC}$$

$$(1-\xi) z \left[K^\alpha \left(\frac{S_a}{w} \right)^{1-\alpha} \right]^\eta \leq V_a \quad \forall a \tag{LE}$$

where (WCC) is the working capital constraint, (PCF) is the participation constraint of financiers, (NNC) is the non-negativity constraint on coupons, and (LE) is the limited enforcement constraint.

Proposition 2 (Optimal Financial Contract) *For any scale K, a solution to the problem features:*

1. *Long-term debt payments structure:*

$$q_a = \begin{cases} \pi_a(K) & \text{if } a < T(K) \\ \gamma\pi_a(K) & \text{if } a = T(K) \\ 0 & \text{if } a > T(K) \end{cases}$$

where $\pi_a(K) = z\left[K^\alpha \left(\frac{S_a(K)}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)S_a(K)$ *and* $\gamma \in (0,1]$ *is the fraction of profits paid as debt service in the last period of debt repayment. We will refer to* $T(K)$ *as the maturity of long term debt (i.e. the last period where* $q_a > 0$*).*

2. *Increasing short-term debt for* $a \in [0, \hat{a}]$ *where* $\hat{a} \leq T(K)$*:*

$$S_a(K) = \begin{cases} \left[\frac{(\beta\rho)^{T(K)+1-a}}{1-\beta\rho}\frac{(1-\gamma)(1-\beta\rho)+\beta\rho}{(1-\xi)(1-\beta\rho)}\right]^{\frac{1}{\eta(1-\alpha)}} S^u(K,z) & \text{if } a \leq \hat{a} \\ S^u(K,z) & \text{if } a > \hat{a} \end{cases}$$

where $S^u(K,z)$ *represents the unconstrained level of short-term debt:*

$$S^u(K,z) \equiv \left[\frac{\eta(1-\alpha)}{(1+r_f)} z \left(\frac{1}{w}\right)^{\eta(1-\alpha)}\right]^{\frac{1}{1-\eta(1-\alpha)}} K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}$$

3. *Dividends at each age are determined by* $d_a = \pi_a - q_a$ *and* $n_a = S_a/w$

Proof. See Appendix A.1.1 ■

The optimal contract has several key features that help explain our empirical findings on leverage dynamics. First, long-term debt payments are front-loaded, with the firm paying all profits toward debt until maturity. Second, dividends are back-loaded, with no dividends paid until debt is fully repaid. Third, and most importantly for our analysis, short-term debt increases until reaching the unconstrained level at age $\hat{a}$, while long-term debt decreases as

the firm pays it down. This creates a pattern of increasing short-term leverage and decreasing long-term leverage during what we define as the Early Life Cycle (ELC).

It is worth noting that while the solution is unique up to age $\hat{a}$, multiple implementations are possible afterward. We focus on the implementation that prioritizes early repayment of long-term debt, which maximizes balance sheet capacity. This approach would be optimal if firms faced even a small probability of new investment opportunities in the future. The characterization of the ELC duration ($\hat{a}$) is particularly valuable as it allows us to empirically identify when firms become unconstrained in their short-term borrowing, which we will later use to measure funding gaps across countries with different enforcement levels.

3.3 Early Life Cycle Duration

Proposition 2 implies that firms are financially constrained at age a (i.e., $S_a < S^u$) if:

$$(\beta\rho)^{T+1-a} < (1-\beta\rho)\frac{(1-\gamma)(1-\beta\rho)+\beta\rho}{(1-\xi)(1-\beta\rho)}$$

A key insight of our model is the characterization of the Early Life Cycle (ELC) duration, denoted $\hat{a}$. This represents the age at which firms are no longer financially constrained in their short-term borrowing:

$$\hat{a} = \left\lceil T+1-\frac{\ln\left((1-\beta\rho)\frac{(1-\gamma)(1-\beta\rho)+\beta\rho}{(1-\xi)(1-\beta\rho)}\right)}{\ln(\beta\rho)} \right\rceil$$

where $\lceil\cdot\rceil$ denotes the ceiling function.

The derivative of ELC duration with respect to enforcement is:

$$\frac{\partial\hat{a}}{\partial\xi} = \frac{\partial T}{\partial\xi} + \frac{1}{(1-\xi)\ln(\beta\rho)}$$

Since $\ln(\beta\rho) < 0$, the direct effect $\frac{1}{(1-\xi)\ln(\beta\rho)}$ is negative, while the indirect effect through

maturity $\frac{\partial T}{\partial \xi}$ is positive.

Proposition 3 (Discrete Changes in Optimal Maturity) *The optimal maturity $T(\xi)$ is a non-decreasing function of enforcement quality ξ. For any continuous change in ξ, the resulting change in optimal maturity ΔT can only take values in $\{0, 1\}$:*

$$\frac{\partial T(\xi)}{\partial \xi} \in \{0, 1\}$$

Proof. See Appendix A.1.2 ■

Proposition 3 has important implications for the relationship between $\hat{a}$ and ξ. Since $\frac{\partial T(\xi)}{\partial \xi} \in \{0, 1\}$, we can bound the effect of enforcement on ELC duration:

$$\frac{\partial \hat{a}}{\partial \xi} \leq 1 + \frac{1}{(1-\xi)\ln(\beta\rho)}$$

For typical parameter values (e.g., $\beta\rho \approx 0.5$ and $\xi \geq 0.1$), $\frac{\partial \hat{a}}{\partial \xi} \leq 0$, indicating that enforcement weakly reduces ELC duration. This implies that firms in countries with stronger enforcement institutions exit their early life cycle faster, becoming financially unconstrained in their short-term borrowing earlier in their life cycle.

3.4 Optimal Scale Under Limited Enforcement

Having characterized the ELC duration, we now address the optimal scale determination. Due to the non-differentiable nature of the problem, we first solve an auxiliary problem where we optimize over discrete pairs of scale K and maturity T, assuming full debt repayment at maturity ($\gamma = 0$). We then refine this solution by optimizing the partial repayment factor γ.

Definition 1 (Feasible Discrete Scale and Maturity) *Let $\mathcal{F}$ be the set of pairs of T and maximum feasible scale K_T attainable with long term debt of maturity T. And define the value of the firm with capital K_T and long term debt maturity T by $V(T)$.*

Proposition 4 (Optimal Scale and Maturity) *Let* $\tilde{T} = argmax_T V(T)$.

1. *Optimal scale* $K^* \in [K_{\tilde{T}}, K_{\tilde{T}+1})$ *such that*

$$K^* = K_{T^*} \cdot \left(\frac{1}{\beta\rho}\right)^{\frac{1-\eta(1-\alpha)}{\eta\alpha}}$$

2. *Optimal last period coupon:*

$$\gamma^* = 1 - \frac{(\beta\rho)^{T^*+1}}{\pi_{T^*}} \left(1 - \frac{K^*}{K_{T^*+1}}\right) [1-\eta(1-\alpha)] K^{*\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$$

3. *Optimal maturity* $T^* = \tilde{T} + \mathbf{1}_{\{\gamma>0\}}$

Proof. See Appendix A.1.3 ■

The fundamental trade-off in maturity choice is that increasing maturity from T to $T+1$ sacrifices dividends at $T+1$ but increases dividends from $T+2$ onward through higher scale $K_{T+1} > K_T$.

Proposition 5 (Enforcement, Scale, and Maturity) *The maximum feasible scale* K_T *for any given maturity* T *is strictly increasing in the enforcement parameter* ξ. *Additionally, the optimal maturity* $T^*(\xi)$ *is weakly increasing in* ξ. *Consequently, economies with lower enforcement exhibit both lower optimal firm scale* K^* *and shorter debt maturities.*

Proof. See Appendix A.1.4 ■

Interestingly, the relationship between scale and ELC duration operates primarily through the debt maturity channel. This happens because the direct effects of scale on enforcement constraints largely offset each other. Specifically, both the scale effect (increased short-term borrowing needs) and the value effect (enhanced continuation value) are proportional to $K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}$, causing them to cancel out in our framework.

This creates a clean theoretical prediction: the effect of scale on ELC duration works primarily through its impact on the optimal debt maturity structure. Consequently, countries with stronger enforcement institutions will have firms with larger scale and longer debt maturities, but shorter ELC durations. This reflects how enforcement quality affects both the firm's initial investment decision and its subsequent financial constraints.

3.5 Capital Structure and Leverage Dynamics

To analyze how capital structure evolves, we define leverage ratios to match our empirical measures. Let ℓ_a^S and ℓ_a^L denote short-term and long-term leverage at age a respectively:

$$\ell_a^S = \frac{S_a}{A_a} \qquad \text{and} \qquad \ell_a^L = \frac{L_a}{A_a}$$

where S_a represents short-term debt, L_a represents long-term debt, V_a net-worth and $A_a = V_a + S_a + L_a$ represents total assets. Based on our characterization of the optimal financial contract, we derive the following leverage dynamics:

Proposition 6 (Leverage Dynamics) *In an environment with enforcement quality ξ, the leverage ratios of a firm evolve during the early life cycle (ages $a \leq \hat{a}(\xi)$) as follows:*

$$\frac{\partial \ell_a^L}{\partial a} < 0 \quad \textit{and} \quad \frac{\partial \ell_a^S}{\partial a} > 0$$

Proof. See Appendix A.1.5 ■

The intuition behind these dynamics is that young firms start with high long-term leverage to fund their initial capital investment. As they age, they gradually pay down this long-term debt while simultaneously relaxing their short-term borrowing constraints, which increases short-term leverage. These opposing movements in leverage ratios continue until age $\hat{a}(\xi)$, when firms become unconstrained in their short-term borrowing.

3.6 Capital Structure and Development

Our model generates predictions about how enforcement quality affects leverage ratios and capital structure patterns across countries:

Proposition 7 (Enforcement and Leverage) *The relationship between enforcement quality ξ and leverage ratios is characterized by:*

1. For initial long-term leverage ($a = 0$):

$$\frac{\partial \ell_0^L}{\partial \xi} > 0 \quad for \quad \xi < \xi^* \tag{1}$$

where ξ^* is a threshold value of enforcement above which financial constraints cease to bind even for newborn firms.

2. For any age a during the early life cycle:

$$\frac{\partial \ell_a^L}{\partial \xi} > 0 \quad and \quad \frac{\partial \ell_a^S}{\partial \xi} < 0 \quad for \quad \xi < \xi^* \tag{2}$$

The cross-country implications of our model offer clear predictions: firms in economies with stronger enforcement institutions exhibit higher long-term leverage and lower short-term leverage compared to their counterparts in weaker enforcement environments. Our model reveals that the optimal scale-maturity choice varies systematically with enforcement quality. As enforcement improves, firms can support both larger initial scale and longer debt maturities simultaneously. This occurs because better enforcement allows firms to credibly commit to repaying larger debt amounts over extended periods, increasing their debt capacity at inception and enabling greater upfront investment.

This scale-maturity relationship contrasts sharply with predictions from traditional models with exogenous setup costs. In those frameworks, firms in less developed economies would paradoxically require longer debt maturities to finance identical fixed costs, producing coun-

terfactual predictions about debt maturity across development levels. Our model with endogenous scale choice correctly predicts the empirical pattern that firms in more developed economies utilize both higher initial scale and longer debt maturities.

A particularly interesting feature of our model is the higher short-term leverage observed in less developed countries despite their lower absolute levels of short-term debt. This occurs because enforcement quality affects the denominator of the leverage ratio (total assets) relatively more than the numerator (debt levels). Specifically, when enforcement weakens, long-term debt scales approximately linearly with capital, while asset values grow more slowly due to the concavity of the production function. This differential scaling generates higher short-term leverage ratios in less developed economies, even as absolute debt levels remain lower.

Enforcement quality also shapes the dynamic pattern of leverage over firms' life cycles. In economies with higher enforcement quality, firms exit their early life cycle faster (smaller $\hat{a}(\xi)$), meaning the period characterized by increasing short-term leverage and decreasing long-term leverage is shorter. This explains our empirical finding that the distinctive leverage pattern of young firms persists longer in less developed economies.

These capital structure patterns provide a novel window into the severity of financial constraints across countries. The joint dynamics of short-term and long-term leverage—particularly the duration of their opposite movements—reveal valuable information about firms' financial constraints that cannot be captured by examining total leverage alone.

3.7 The Early Life Cycle of Firms and Financial Development

In order to test the effects of enforcement in the ELC of firms we divide our sample in two according to recovery rates from the World Banks Doing Business indicators. We will define as high (low) enforcement economies that have recovery rates above (below) the median. Figure 3 show the results, we can observe that the duration of ELC is higher in economies with lower enforcement. This is in line with our theory and will motivate how we construct our measure of financial constraints.

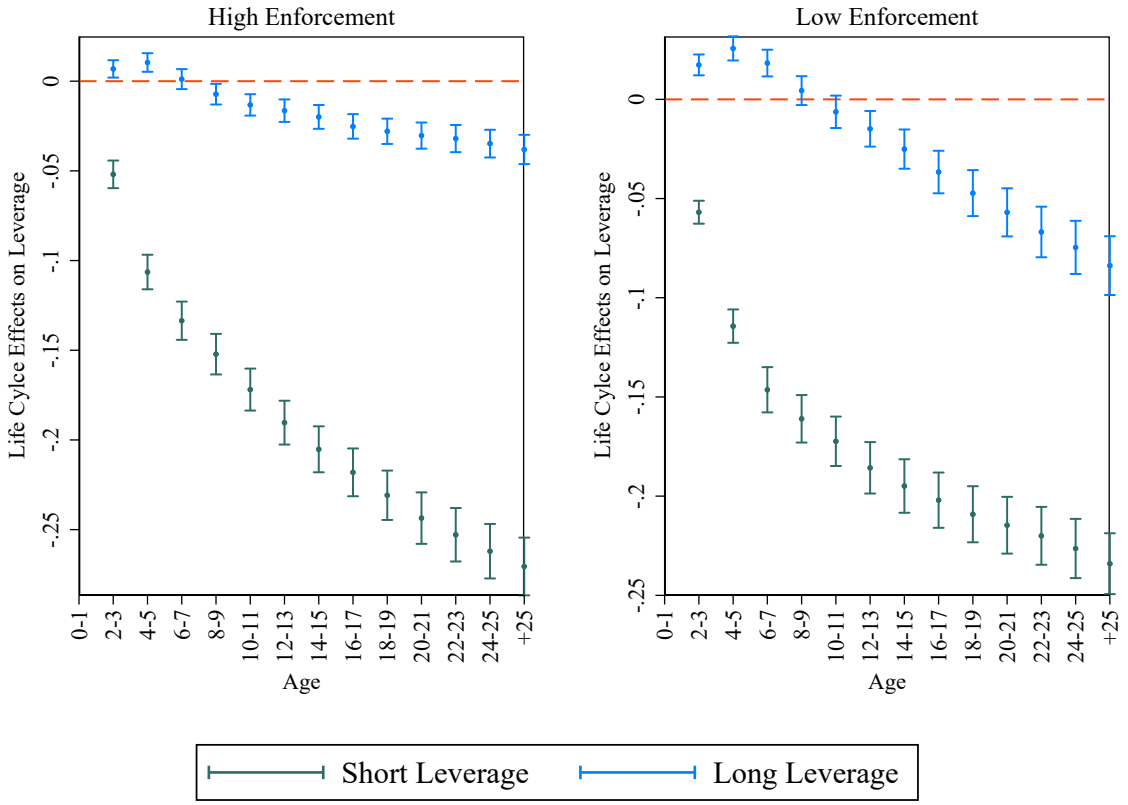


Figure 3: Life Cycle Patterns of Capital Structure of firms for which we have at least 3 observations and have positive debt. Confidence intervals are at 90% and standard errors are clustered at country-industry-year. Enforcement is defined as recovery rates (cents on the dollar) from World Banks Doing Business indicators. High enforcement means recovery rates above the median (62.3%)

4 Measurement of Financial Frictions: A Decomposition

Financial constraints are distortions in outcomes and firm decisions due to funding gaps. In our model, we can define long-term and short-term funding gaps and examine their effects on investment, employment, and profits. Let $S_a(K, z, \xi, p_k, \alpha)$ denote the level of short-term debt obtained by a firm of age a, scale K, productivity z, in a country with enforcement ξ, capital price p_k, and in an industry with capital intensity α. Let $S(z, 1, p_k, \alpha)$ denote the short-term debt a firm would obtain under perfect enforcement. The short-term and long-term funding gaps (Γ^S, Γ^L) are defined as:

$$\Gamma^S = S(K(1, \cdot), z, 1, p_k, \alpha) - S_a(K(\xi, \cdot), z, \xi, p_k, \alpha)$$
$$\Gamma^L = p_k[K(1, z, p_k, \alpha) - K(\xi, z, p_k, \alpha)]$$

Our model allows us to connect these funding gaps directly to distortions in firm outcomes such as employment, investment, and profits. (The detailed expressions for these outcome distortions can be found in the Appendix.)

4.1 Funding Gaps

The exact mapping from funding gaps to outcome gaps depends on model details. For this reason, we focus on funding gap measurements that are less model-dependent. Consider the case of unproductive and exogenous setup costs (I_0), and let $(\tilde{\Gamma}^S(I_0), \tilde{\Gamma}^L(I_0))$ denote the

funding gaps for that class of models:

$$\tilde{\Gamma}^{S,a}(I_0) = S(I_0, z, 1, p_k, \alpha) - S_a(I_0, z, \xi, p_k, \alpha)$$
$$\tilde{\Gamma}^{L}(I_0) = p_k \left[I_0 - \frac{L_0}{p_k(1+r_f)} \right]$$

By construction, the long-term funding gap can take two values $\tilde{\Gamma}^L \in \{0, p_k I_0\}$; it equals zero for active firms and $p_k I_0$ for projects not undertaken. This funding gap corresponds to what the literature calls distortions along the extensive margin. The short-term funding gap takes values in $\tilde{\Gamma}^S \in [0, S(I_0, z, 1, p_k, \alpha)]$ and converges to zero as firms mature, corresponding to distortions along the intensive margin.

In our model, instead of fixed entry costs, we have a flexible initial scale choice K, so the long-term funding gap can take values in $\Gamma^L \in [0, K(z, 1, p_k, \alpha)]$. This funding gap is monotonically decreasing in enforcement quality ξ for a given industry:

$$\frac{\partial \Gamma^L}{\partial \xi} = -p_k \frac{\partial K(z, \xi, p_k, \alpha)}{\partial \xi} < 0$$

Measuring the long-term funding gap requires data from firms at inception, which is difficult to obtain. For employment creation, we focus on the short-term funding gap, which we decompose as:

$$\begin{aligned}
\Gamma^{S,a} &= S(K(1,\cdot), z, 1, p_k, \alpha) - S_a(K(\xi,\cdot), z, \xi, p_k, \alpha) \\
&= \underbrace{\left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \right] w z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{1}{1-\eta(1-\alpha)}}}_{\Omega^S(K(\xi,\cdot)) \text{ (Short Term Debt Target Distortion)}} \\
&+ \underbrace{S_a(K(\xi,\cdot), z, 1, p_k, \alpha) - S_a(K(\xi,\cdot), z, \xi, p_k, \alpha)}_{\tilde{\Gamma}^{S,a}(K(\xi,\cdot)) \text{ (Observable Short Funding Gap)}}
\end{aligned}$$

This decomposition separates the short-term funding gap into two components:

- $\tilde{\Gamma}^{S,a}(K(\xi,\cdot))$: The observable short funding gap—measurable from data given sufficient observations.

- $\Omega^S(K(\xi,\cdot))$: The short-term debt target distortion—a counterfactual object capturing the gap in desired short-term debt due to distortions in scale choice from being born in a country with enforcement frictions.

Proposition 8 *In a given industry* (α)*, the total funding gap* $\Gamma^{S,a}$ *and the short-term debt target gap* $\Omega^S(K(\xi,\cdot))$ *are monotonic in financial development* ξ*, but the observable short-term funding gap* $\tilde{\Gamma}^{S,a}(K(\xi,\cdot),\cdot)$ *is not.*

Figure 4 illustrates the behavior of funding gaps across different levels of development and capital intensities. The non-monotonicity of the observable funding gap has important implications for policy. For high capital intensity industries ($\alpha = 0.35$), using the funding gap observed in more developed economies ($\xi = 0.7$) to estimate the gap in less developed economies ($\xi = 0.1$) would overestimate the subsidized loans required. Conversely, for low capital intensity industries ($\alpha = 0.25$), the same approach would underestimate the funding gap in less developed economies.

Since the unobservable target distortion $\Omega^S(K(\xi,\cdot))$ requires cross-country comparison, we need to establish a benchmark economy with perfect enforcement ($\xi = 1$). Using the U.S. economy as this benchmark, we can infer the target distortion from:

$$\Omega^S(K(\xi,\cdot)) = w[n^u(1,K(1),\alpha,z) - n^u(1,K(\xi),\alpha,z)]$$

This expression shows that while data on firms at inception is scarce, employment at the end of a firm's early life cycle provides a viable alternative for measuring financial constraints.

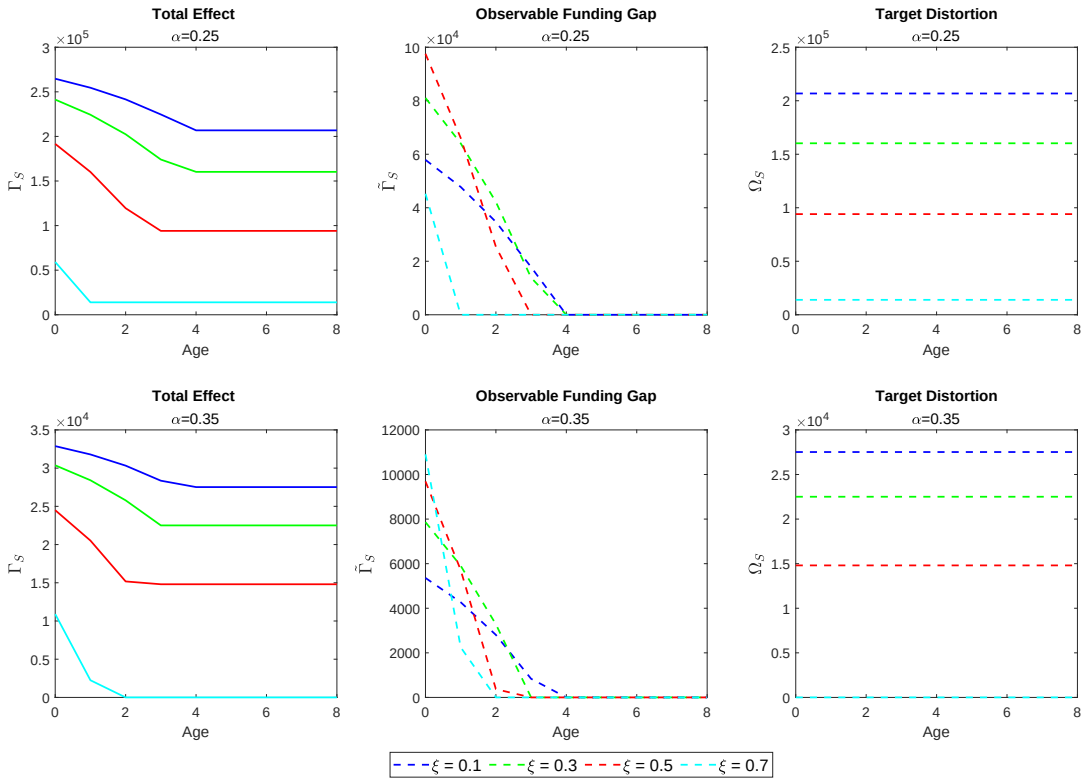


FIGURE 4: FUNDING GAPS ACROSS LEVELS OF DEVELOPMENT (ξ) FOR LOW AND HIGH CAPITAL INTENSITY ($\alpha \in \{0.25, 0.35\}$) INDUSTRIES.

4.2 Measuring Total Distortions

To implement our approach empirically, we:

1. Estimate the duration of the early life cycle $\hat{a}$ (the first period where age no longer has a positive effect on leverage).

2. Define the conditionally unconstrained average short-term debt based on this estimate.

3. Compute the observable funding gap as the NPV of deviations between observed and target short-term debt.

4. Calculate the target distortion using employment gaps between firms at age $\hat{a}$ in a given country-industry and comparable U.S. firms.

5. Scale this employment gap by wages and adjust for the duration of the early life cycle.

6. Sum these components to estimate the total funding gap.

Table 1 illustrates this methodology for high and low enforcement economies. The results reveal that in lower enforcement economies: (i) the early life cycle lasts longer; (ii) target scale is lower despite examining older firms; (iii) the unobservable scale distortion is relatively more important—accounting for 17.2% of the total funding gap compared to 10.6% in high enforcement economies. This suggests that inferring financial constraints using only observable funding gaps produces less biased measures in developed economies.

Table 1: Severity of Financial Constraints

	High Enforcement	Low Enforcement
$\hat{a}$	5	7
$S_{\hat{a}}$	403947.5	525969.1
$n_{\hat{a}}$	16.01	14.59
n^{US}	16.9	16.9
w	34087	25883
S_1	279467.9	238098.5
S_2	256712.4	277746
S_3	310945.5	327072.1
S_4	343335.1	383022.1
S_5	403947.5	419104.8
S_6	-	481067.3
S_7	-	525969.1
$\tilde{\Gamma}^S$	917,979	1,239,722
Ω^S	108,801	258,381
Γ^S	1,026,780	1,498,104
$\frac{\tilde{\Gamma}^S}{\Gamma^S}$	89.4%	82.8%
$\frac{\Omega^S}{\Gamma^S}$	10.6%	17.2%

Notes: w denotes annual wages in dollars (Source: International Labor Organization). n^{US} is the average employment over 1998-2017 of firms in US of age 6-10 with fewer than 500 employees. The calculations for High and Low enforcement economies also consider firms with 500 employees or less (Source: US Census, Business Dynamics Statistics). $n_{\hat{a}}$ denotes the average employment of firms at age $\hat{a}$. The discount factor assumes $\beta = 0.98$ and survival probability $\rho = 0.85$.

The greater importance of the unobservable target distortion at lower development levels highlights distortions in firms' initial scale choices. While policymakers can address the long-term funding gap through upfront transfers to firms, they can also affect it indirectly through credible subsidized short-term credit programs, which would influence both long-term debt access and scale choice.

5 Conclusion

How severe are financial constraints for young firms? This paper proposes a methodology to estimate the full effect of financial constraints emerging from enforcement problems. The paper has four messages. First, studying long and short leverage, instead of total leverage, is important to quantify the severity of financial constraints over the early life cycle of firms. Moreover, the duration of the negative co-movement of short and long leverage defines the duration of the early life cycle. Second, a model of endogenous borrowing constraints that aims to match the international corporate finance stylized facts requires firms to make a productive initial investment choice. Third, a consequence of the initial productive choice of firms is that short-term borrowing constraints don't fully capture the severity of financial frictions in the economy. Fourth, to measure the full effect of financial frictions over the early life cycle of firms, one needs to implement cross-country comparisons and the estimate the duration of the early life cycle of firms.

We conclude by discussing future avenues of research. We focused on comparing the average firm in the relatively high- and low-enforcement economies in our sample. The model is flexible enough to implement our methodology across industries and countries. We leave for future work the estimation of the severity of financial constraints across industries and countries. In preliminary simulations, our model indicates that the ranking of the severity of financial constraints could not be stable across development levels. We take this as a

theoretical possibility and leave the empirical plausibility of this for future work.

While our model can be used to calculate the funding gap that a credit program could target, note that it is only an ex-ante calculation to approximate the severity of financial constraints. Since we abstract from any asymmetric information problems, we caution against using the model to evaluate welfare gains from credit programs. However, we highlighted that credible announcements of subsidized short-term credit could reduce the long-term debt funding gap and increase the initial scale choice of firms. Exploring this implementation of a credit program might be interesting for international organizations that offer subsidized credit and require institutional reforms. Intuitively, our model suggests that one can get all the effects of a long-term debt program[7] with a credible subsidized short-term program sustained across time. However, the implementation using short-term debt gives a threat of discontinuing the program to international organizations. This threat could discipline the local government to implement institutional reforms.

The work of Lian and Ma (2021) and Cloyne et al. (2019) suggest that firms in their early life cycle are different: they rely more on asset-backed lending[8]. Building on the intuition that young firms are fundamentally different, we studied how the capital structure of a young firm buying capital with collateralized long-term debt reveals information about the severity of financial constraints. But what happens beyond the early life cycle? Here we considered only one investment opportunity; however, in a model where new projects arrive at later stages (for example, the opportunity of opening a new establishment), we could study whether the initial scale distortion disappears or persists. The literature on firm size and development (Hsieh and Klenow, 2014; Poschke, 2018; Bento and Restuccia, 2021) would suggest the existence of permanent effects of initial distortions. However, if collateral seems

[7]Conditional on financiers having deep pockets, or, in other words, conditional on the funding gap being purely driven by enforceability.

[8]While Lian and Ma (2021) documents this directly, Cloyne et al. (2019) find that this difference has important economic consequences, they show that no dividend paying and young firms investment account for more than 20% of the response of investment to monetary policy while accounting for less than 10% of aggregate investment;

to be mainly relevant during the early life cycle, from which channel would the initial scale distortion be operating if it is going to be persistent beyond the early life cycle? We leave the extension of this model in the previously highlighted directions for future work.

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A Appendix: Theory

A.1 Proofs of Propositions

A.1.1 Proof of Proposition 1

The optimality of this financial structure follows from comparing the present value of costs across different financing arrangements.

For intra-period funding needs, a single-period loan costs $\kappa_C + (1 + r_f)R_a$ each period. Discounting these recurring costs yields a present value of:

$$\sum_{t=0}^{\infty}(\beta\rho)^t[\kappa_C + (1 + r_f)R_a] = \frac{\kappa_C}{1 - \beta\rho} + \frac{(1 + r_f)R_a}{1 - \beta\rho}$$

In contrast, a multi-period loan covering the same recurring needs costs κ_C once, but charges a premium for firm exit risk:

$$\kappa_C + (1 + r_f)\sum_{t=0}^{\infty}(\beta\rho)^t\frac{R_a}{\rho} = \kappa_C + \frac{(1 + r_f)R_a}{\rho(1 - \beta\rho)}$$

The difference between these costs is:

$$\frac{\kappa_C}{1 - \beta\rho} - \kappa_C + \frac{(1 + r_f)R_a}{1 - \beta\rho} - \frac{(1 + r_f)R_a}{\rho(1 - \beta\rho)} = \frac{\beta\rho\kappa_C}{1 - \beta\rho} - \frac{(1 - \rho)(1 + r_f)R_a}{\rho(1 - \beta\rho)}$$

By Assumption 2, $\kappa_C < S_0(K(\xi_L), z_L)^{\frac{1-\rho}{\rho}}$, which ensures that for typical working capital needs, the transaction cost savings don't outweigh the exit risk premium, making single-period loans optimal.

For capital acquisition, which requires multiple periods to repay (by Assumption 3), a multi-period loan incurs a single transaction cost κ_C, while a sequence of T single-period loans would incur $\sum_{t=0}^{T-1}(\beta\rho)^t\kappa_C$ in transaction costs. Since $T > 1$, the multi-period loan dominates.

Proof of Proposition 2

The firm's problem is to choose $\{q_a, R_a, K\}_{a=0}^{\infty}$ to maximize:

$$\max_{\{q_a,R_a,K\}_{a=0}^{\infty}} \sum_{a=0}^{\infty}(\beta\rho)^a \left(z\left[K^\alpha \left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_a - q_a \right)$$

subject to:

$$(1+r_f)p_k K \leq \sum_{a=0}^{\infty}(\beta\rho)^a q_a \tag{PCF}$$

$$0 \leq q_a \leq z\left[K^\alpha \left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_a \tag{NNC}$$

$$(1-\xi)z\left[K^\alpha \left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta \leq V_a \tag{LE}$$

where $V_a = \sum_{j=0}^{\infty}(\beta\rho)^j \left(z\left[K^\alpha \left(\frac{R_{a+j}}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_{a+j} - q_{a+j} \right)$ is the continuation value of the firm at age a.

Let λ denote the Lagrange multiplier for the participation constraint of financiers (PCF), μ_a^H and μ_a^L for the upper and lower bounds of the non-negativity constraint (NNC), and ψ_a for the limited enforcement constraint (LE).

The first-order conditions with respect to q_a and R_a are:

$$\partial q_a \ : \ -(\beta\rho)^a + \lambda(\beta\rho)^a + \mu_a^L - \mu_a^H - \sum_{j=0}^{a}(\beta\rho)^j \psi_{a-j} = 0 \tag{FOC-q}$$

$$\begin{aligned}\partial R_a \ : \ & \left[(\beta\rho)^a + \mu_a^H + \sum_{j=0}^{a}(\beta\rho)^j \psi_{a-j}\right]\left[\frac{\eta(1-\alpha)}{R_a} z \left[K^\alpha \left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right] \\ & = \psi_a(1-\xi)\frac{\eta(1-\alpha)}{R_a} z \left[K^\alpha \left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta \end{aligned} \tag{FOC-R}$$

From (FOC-R), we define the unconstrained level of short-term debt:

$$R^u(K,z) \equiv \left[\frac{\eta(1-\alpha)}{(1+r_f)} z \left[K^\alpha \left(\frac{1}{w}\right)^{1-\alpha}\right]^\eta\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

Note that $\psi_a = 0 \iff R_a = R^u(K,z)$, which follows from (FOC-R).

From the first-order condition for q_a (FOC-q), we can isolate λ:

$$\lambda = 1 + \frac{\mu_a^H - \mu_a^L}{(\beta\rho)^a} + \sum_{j=0}^{a} \psi_{a-j} \left(\frac{1}{\beta\rho}\right)^{a-j} \quad \forall a$$

Since λ is a constant multiplier, comparing its expression across periods a and $a+1$ yields:

$$\beta\rho(\mu_a^H - \mu_a^L) - \psi_{a+1} = \mu_{a+1}^H - \mu_{a+1}^L$$

Applying this recursively, we can establish:

$$(\beta\rho)^a \mu_0^H - \sum_{j=1}^{a}(\beta\rho)^{a-j}\psi_j = (\mu_a^H - \mu_a^L)$$

This expression characterizes the dynamics of the contract, we characterize it in following

claims:

Claim 1 *In a solution with $K > 0$, we must have $\mu_0^H > 0$.*

Proof. We proceed by contradiction. Suppose that $\mu_0^H = 0$. From the first-order condition with respect to q_0:

$$-1 + \lambda + \mu_0^L - \mu_0^H - \psi_0 = 0$$

Substituting $\mu_0^H = 0$:

$$\lambda = 1 - \mu_0^L + \psi_0$$

Since $\mu_0^L \geq 0$ and $\psi_0 \geq 0$ (as Lagrange multipliers), we have two cases to consider:

Case 1: If $\mu_0^L > 0$, then $q_0 = 0$ by complementary slackness. The participation constraint of financiers requires:

$$(1 + r_f)p_k K \leq \sum_{a=0}^{\infty} (\beta\rho)^a q_a = \sum_{a=1}^{\infty} (\beta\rho)^a q_a$$

From our recursive relationship for the multipliers:

$$(\beta\rho)^a \mu_0^H - \sum_{j=1}^{a} (\beta\rho)^{a-j} \psi_j = (\mu_a^H - \mu_a^L)$$

With $\mu_0^H = 0$, this implies $(\mu_a^H - \mu_a^L) \leq 0$ for all $a \geq 1$ (since all $\psi_j \geq 0$). This means either $q_a = 0$ for all $a \geq 1$ (if $\mu_a^L > 0$), or q_a are below their upper bounds. Either way, the participation constraint cannot be satisfied for any $K > 0$.

Case 2: If $\mu_0^L = 0$, then from our recursive relationship with $\mu_0^H = 0$:

$$-\sum_{j=1}^{a}(\beta\rho)^{a-j}\psi_j = (\mu_a^H - \mu_a^L)$$

Since $\psi_j \geq 0$ for all j, we have $(\mu_a^H - \mu_a^L) \leq 0$ for all $a \geq 1$. For any $K > 0$, the firm must generate positive coupon payments to satisfy the participation constraint:

$$(1 + r_f)p_k K \leq \sum_{a=0}^{\infty}(\beta\rho)^a q_a$$

But with $(\mu_a^H - \mu_a^L) \leq 0$ for all a, the upper bound on coupons never binds, and implies that $q_a = 0 \ \forall a$ is a feasible solution, but this would contradict the PCF for any $K > 0$. Therefore, we must have $\mu_0^H > 0$. ■

Claim 2 *In a solution with $K > 0$, there exists a finite age $\hat{a}$ such that:*

$$(\beta\rho)^{\hat{a}}\mu_0^H \leq \sum_{j=1}^{\hat{a}}(\beta\rho)^{\hat{a}-j}\psi_j$$

Proof. We proceed by contradiction. Suppose that for all ages a:

$$(\beta\rho)^a\mu_0^H > \sum_{j=1}^{a}(\beta\rho)^{a-j}\psi_j \qquad \forall a$$

From our derived recursive relationship, this implies $\mu_a^H - \mu_a^L > 0$ for all a. Since $\mu_a^L \geq 0$, we have $\mu_a^H > 0$ for all a, meaning the upper bound on coupon payments always binds. Consequently, $q_a = \pi_a$ for all a and $V_a = \beta\rho V_{a+1}$ for all a. This implies that the firm pays zero dividends, hence $V_a = 0$. The enforcement constraint then implies $R_a = 0 \ \forall a$ and consequently $\pi_a = q_a = 0$, which contradicts the participation constraint of financiers for any $K > 0$. ■

Claim 2 implies that we can have two cases at $a = \hat{a}$: $\mu_{\hat{a}}^H = 0$ and $\mu_{\hat{a}}^L = 0$, or, $\mu_{\hat{a}}^H = 0$ and $\mu_{\hat{a}}^L > 0$. And it also implies $\mu_a^H - \mu_a^L \leq 0$ for all $a \geq \hat{a}$

Claim 3 *Claims 1 and 2 imply* $\mu_a^H - \mu_a^L \leq 0$ *for all* $a \geq \hat{a}$.

Proof. Recall the recursive relationship for multipliers derived from the first-order conditions:

$$(\beta\rho)^a \mu_0^H - \sum_{j=1}^{a} (\beta\rho)^{a-j} \psi_j = (\mu_a^H - \mu_a^L) \tag{3}$$

From Claim 2, we know there exists a finite age $\hat{a}$ such that:

$$(\beta\rho)^{\hat{a}} \mu_0^H \leq \sum_{j=1}^{\hat{a}} (\beta\rho)^{\hat{a}-j} \psi_j \tag{4}$$

Substituting this inequality into the recursive relationship at age $\hat{a}$:

$$(\beta\rho)^{\hat{a}} \mu_0^H - \sum_{j=1}^{\hat{a}} (\beta\rho)^{\hat{a}-j} \psi_j = (\mu_{\hat{a}}^H - \mu_{\hat{a}}^L) \tag{5}$$

Given the constraint in Claim 2, we have:

$$(\mu_{\hat{a}}^H - \mu_{\hat{a}}^L) \leq 0 \tag{6}$$

This establishes the result for $a = \hat{a}$.

To show the inequality holds for all $a > \hat{a}$, we use induction and the recursive relationship between consecutive periods:

$$\beta\rho(\mu_a^H - \mu_a^L) - \psi_{a+1} = \mu_{a+1}^H - \mu_{a+1}^L \tag{7}$$

For the base case $a = \hat{a}$, we have:

$$\beta\rho(\mu_{\hat{a}}^H - \mu_{\hat{a}}^L) - \psi_{\hat{a}+1} = \mu_{\hat{a}+1}^H - \mu_{\hat{a}+1}^L \tag{8}$$

Since $\mu_{\hat{a}}^H - \mu_{\hat{a}}^L \leq 0$ (as established above) and $\psi_{\hat{a}+1} \geq 0$ (being a Lagrange multiplier), we have:

$$\beta\rho(\mu_{\hat{a}}^H - \mu_{\hat{a}}^L) - \psi_{\hat{a}+1} \leq 0 - \psi_{\hat{a}+1} \leq 0 \tag{9}$$

Therefore:

$$\mu_{\hat{a}+1}^H - \mu_{\hat{a}+1}^L \leq 0 \tag{10}$$

Assuming the inequality holds for some $a \geq \hat{a}$, i.e., $\mu_a^H - \mu_a^L \leq 0$, we can apply the same logic to show it holds for $a + 1$:

$$\beta\rho(\mu_a^H - \mu_a^L) - \psi_{a+1} = \mu_{a+1}^H - \mu_{a+1}^L \tag{11}$$

Since $\mu_a^H - \mu_a^L \leq 0$ by assumption and $\psi_{a+1} \geq 0$, we have:

$$\mu_{a+1}^H - \mu_{a+1}^L \leq 0 \tag{12}$$

By induction, we have shown that $\mu_a^H - \mu_a^L \leq 0$ for all $a \geq \hat{a}$. ■

Claim 4 *For $a \geq \hat{a}$, we have $\psi_a = 0$.*

Proof. From Claim 3, we know that $\mu_a^H - \mu_a^L \leq 0$ for all $a \geq \hat{a}$. Since $\mu_a^L \geq 0$ (as a Lagrange multiplier), this implies $\mu_a^H = 0$ for all $a \geq \hat{a}$.

For contradiction, suppose $\psi_a > 0$ for some $a \geq \hat{a}$. Then:

1. From the first-order condition for R_a, when $\psi_a > 0$, we have $R_a < R^u(K, z)$.

2. Since $\mu_a^H = 0$ for $a \geq \hat{a}$, the firm is no longer constrained to pay all profits as coupons, meaning $q_a \leq \pi_a$ and the firm could pay positive dividends.

3. Consider the firm's profit function $\pi(R) = z\left[K^\alpha \left(\frac{R}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R$, with derivative:

$$\frac{d\pi}{dR} = \eta(1-\alpha)z\left[K^\alpha \left(\frac{R}{w}\right)^{1-\alpha}\right]^\eta \frac{1}{R} - (1+r_f)$$

   For $R_a < R^u(K,z)$, this derivative is positive, so the firm would strictly benefit from redirecting some dividend funds to increase R_a toward $R^u(K,z)$

4. This reallocation would strictly increase profits, maintaining the same total outflow, contradicting the optimality of $q_a \leq \pi_a$ and $\mu_a^H = 0$ for $a \geq \hat{a}$

Therefore, we must have $\psi_a = 0$ for all $a \geq \hat{a}$. ■

Claim 5 *For any $K > 0$, a coupon structure that repays debt as soon as possible features:*

$$q_a = \begin{cases} \pi_a(K) & \text{if } a < T(K) \\ \in [0, \pi_a(K)] & \text{if } a = T(K) \\ 0 & \text{if } a > T(K) \end{cases}$$

is a solution to the optimal contracting problem, where $T(K) \geq \hat{a}$ and $\pi_a(K) = z\left[K^\alpha \left(\frac{R_a(K)}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_a(K)$. However, any other structure of q_a after $\hat{a}$ that satisfies the participation constraint of financiers (PCF) would also be optimal.

Proof. From Claims 1, 3, and 4, we can characterize the Lagrange multipliers as follows:

$$\mu_a^H > 0 \text{ for } a < \hat{a}$$

$$\mu_a^H = 0 \text{ for } a \geq \hat{a}$$

$$\psi_a \geq 0 \text{ for } a < \hat{a}$$

$$\psi_a = 0 \text{ for } a \geq \hat{a}$$

For $a < \hat{a}$, since $\mu_a^H > 0$, the upper bound on coupon payments binds, meaning $q_a = \pi_a(K)$. For $a \geq \hat{a}$, the pattern of q_a is not uniquely pinned down by the first-order conditions.

From the first-order condition with respect to q_a for $a \geq \hat{a}$:

$$-(\beta\rho)^a + \lambda(\beta\rho)^a + \mu_a^L - \mu_a^H - \sum_{j=0}^{a}(\beta\rho)^j\psi_{a-j} = 0$$

Substituting $\mu_a^H = 0$ and noting that $\psi_j = 0$ for $j \geq \hat{a}$:

$$-(\beta\rho)^a + \lambda(\beta\rho)^a + \mu_a^L - \sum_{j=0}^{\min(a,\hat{a}-1)}(\beta\rho)^j\psi_{a-j} = 0$$

Since λ is a constant multiplier, there are multiple possible coupon structures that could satisfy these conditions. However, we can show that a debt structure where the firm repays its debt as soon as possible is consistent with equilibrium.

Specifically, assume the firm pays the maximum possible coupon $q_a = \pi_a(K)$ for all $a < T(K)$, where $T(K)$ is the earliest period where the participation constraint of financiers is satisfied:

$$
\begin{aligned}
(1+r_f)p_kK &= \sum_{a=0}^{T(K)} (\beta\rho)^a q_a \\
&= \sum_{a=0}^{T(K)-1} (\beta\rho)^a \pi_a(K) + (\beta\rho)^{T(K)} q_{T(K)}
\end{aligned}
$$

Under this debt structure, $q_a = 0$ for all $a > T(K)$ and $q_{T(K)} \in [0, \pi_{T(K)}(K)]$ is determined by the remaining amount needed to satisfy the participation constraint. This corresponds to the firm paying off its debt as quickly as possible.

For $a \geq \hat{a}$, the firm is indifferent between different coupon payment schedules as long as the present value of total payments remains the same, because:

1. For $a < \hat{a}$, we already established that $\mu_a^H > 0$ implies $q_a = \pi_a(K)$. 2. For $a \geq \hat{a}$, the firm is indifferent between paying coupons and dividends when $\lambda = 1$. 3. Even if $\lambda \neq 1$, the firm can adjust the timing of coupon payments to satisfy the PCF without affecting optimality, as long as the present value of the payments equals $(1+r_f)p_kK$.

The front-loaded structure (repaying debt as soon as possible) is a natural and simple solution, but the firm would be equally satisfied with any other coupon structure that satisfies the PCF after $\hat{a}$. This is because once the enforcement constraint no longer binds (after $\hat{a}$), the time pattern of payments between dividends and coupons becomes irrelevant to firm value maximization, as long as the total present value of payments to financiers satisfies the PCF.

Therefore, while we focus on the debt repayment structure that pays off debt as soon as

possible:

$$q_a = \begin{cases} \pi_a(K) & \text{if } a < T(K) \\ \in [0, \pi_a(K)] & \text{if } a = T(K) \\ 0 & \text{if } a > T(K) \end{cases}$$

where $T(K) \geq \hat{a}$. ■

Claim 6 *The optimal dividend structure features:*

$$d_a = \begin{cases} 0 & \textit{if } a < T(K) \\ \in [0, \pi_u(K)] & \textit{if } a = T(K) \\ \pi_u(K) & \textit{if } a > T(K) \end{cases}$$

Proof. The dividend at age a is given by:

$$d_a = \pi_a(K) - q_a$$

From Claim 5, we have:

$$q_a = \begin{cases} \pi_a(K) & \text{if } a < T(K) \\ \in [0, \pi_a(K)] & \text{if } a = T(K) \\ 0 & \text{if } a > T(K) \end{cases}$$

For $a < T(K)$, $q_a = \pi_a(K)$, so $d_a = 0$. For $a = T(K)$, $q_a \in [0, \pi_a(K)]$, so $d_a \in [0, \pi_a(K)]$.

For $a > T(K)$, $q_a = 0$, so $d_a = \pi_a(K)$. Additionally, from Claim 4, we know that $\psi_a = 0$ for $a \geq \hat{a}$, which implies $R_a = R^u(K, z)$ for $a \geq \hat{a}$. Since $T(K) \geq \hat{a}$, we have $R_a = R^u(K, z)$ for $a > T(K)$, and thus $\pi_a(K) = \pi_u(K)$ for $a > T(K)$.

Therefore, the optimal dividend structure is:

$$d_a = \begin{cases} 0 & \text{if } a < T(K) \\ \in [0, \pi_u(K)] & \text{if } a = T(K) \\ \pi_u(K) & \text{if } a > T(K) \end{cases}$$

■

Claim 7 *The optimal short-term debt structure features:*

$$R_a(K) = \begin{cases} \left[\frac{(\beta\rho)^{T(K)+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \omega_0 R_u(K,z) & \text{if } a \leq \hat{a} \\ R_u(K,z) & \text{if } a > \hat{a} \end{cases}$$

where $\omega_0 = [1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$.

Proof. From Claim 4, we know that $\psi_a = 0$ for $a \geq \hat{a}$, which implies $R_a = R^u(K,z)$ for $a \geq \hat{a}$. This establishes the second case in our expression.

For $a < \hat{a}$, the enforcement constraint binds ($\psi_a > 0$), and we have:

$$(1-\xi)z\left[K^\alpha\left(\frac{R_a}{w}\right)^{1-\alpha}\right]^\eta = V_a$$

From Claim 5, we know that $q_a = \pi_a(K)$ for $a < T(K)$, which implies $d_a = 0$ for

$a < T(K)$. Therefore, for $a < \min(\hat{a}, T(K))$:

$$\begin{aligned} V_a &= \sum_{j=0}^{\infty} (\beta\rho)^j d_{a+j} \\ &= \sum_{j=T(K)-a}^{\infty} (\beta\rho)^j d_{a+j} \\ &= \sum_{j=T(K)-a}^{\infty} (\beta\rho)^j \pi_u(K) \\ &= \frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \pi_u(K) \end{aligned}$$

Substituting this into the binding enforcement constraint:

$$\begin{aligned} (1-\xi) z \left[K^{\alpha} \left(\frac{R_a}{w} \right)^{1-\alpha} \right]^{\eta} &= \frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \pi_u(K) \\ z \left[K^{\alpha} \left(\frac{R_a}{w} \right)^{1-\alpha} \right]^{\eta} &= \frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \frac{\pi_u(K)}{1-\xi} \end{aligned}$$

Recalling that $\pi_u(K) = [1-\eta(1-\alpha)] \cdot \frac{1+r_f}{\eta(1-\alpha)} \cdot R_u(K,z)$ and defining $\omega_0 = [1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$, we can solve for R_a:

$$\begin{aligned} \left(\frac{R_a}{R_u(K,z)} \right)^{\eta(1-\alpha)} &= \frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \frac{1}{1-\xi} \\ \frac{R_a}{R_u(K,z)} &= \left[\frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \frac{1}{1-\xi} \right]^{\frac{1}{\eta(1-\alpha)}} \\ R_a &= \left[\frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho} \frac{1}{1-\xi} \right]^{\frac{1}{\eta(1-\alpha)}} R_u(K,z) \end{aligned}$$

Note that since $\beta\rho < 1$, R_a is increasing in a for $a < \hat{a}$. Specifically, $R_a < R_{a+1} < \ldots < R_{\hat{a}} = R_u(K,z)$. This matches the empirical observation of increasing short-term debt during the early life cycle.

Since the factor $\left[\frac{(\beta\rho)^{T(K)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}$ approaches 1 as a approaches $\hat{a}$, we can identify $\hat{a}$ as the smallest integer such that:

$$\left[\frac{(\beta\rho)^{T(K)-\hat{a}}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \geq 1$$

This gives us $\hat{a} = \left\lceil T(K) + 1 - \frac{\ln\left((1-\beta\rho)\frac{1-\xi}{[1-\eta(1-\alpha)]}\right)}{\ln(\beta\rho)} \right\rceil$, where $\lceil\cdot\rceil$ denotes the ceiling function.

Therefore, the optimal short-term debt structure is:

$$R_a(K) = \begin{cases} \left[\frac{(\beta\rho)^{T(K)+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \omega_0 R_u(K,z) & \text{if } a \leq \hat{a} \\ R_u(K,z) & \text{if } a > \hat{a} \end{cases}$$

■

Claim 8 *The optimal employment structure follows the dynamics of short-term debt:*

$$n_a = \begin{cases} \frac{R_a(K)}{w} & \textit{if } a \leq \hat{a} \\ \frac{R_u(K)}{w} & \textit{if } a > \hat{a} \end{cases}$$

Proof. This follows directly from the working capital constraint (WCC): $w_a n_a \leq R_a$. Since the firm wants to maximize output, it will use all available short-term financing for labor, implying $w_a n_a = R_a$, or equivalently, $n_a = \frac{R_a}{w}$.

Therefore, the employment pattern directly mirrors the dynamics of short-term debt:

$$n_a = \begin{cases} \frac{R_a(K)}{w} & \text{if } a \leq \hat{a} \\ \frac{R_u(K)}{w} & \text{if } a > \hat{a} \end{cases}$$

■

In summary, Claims 5, 6, 7, and 8 establish the four key features of the optimal financial

contract stated in Proposition 2:

1. Front-loading of long-term debt payments (Claim 5) 2. Back-loading of dividends (Claim 6) 3. Increasing short-term debt during the early life cycle (Claim 7) 4. Employment that follows the dynamics of short-term debt (Claim 8)

This completes the proof of Proposition 2.

A.1.2 Proof of Proposition 3

Let $V_0(K, T, \xi)$ denote the firm's value at age 0 as a function of scale K, maturity T, and enforcement quality ξ. For each maturity T, define $K_T(\xi)$ as the maximum feasible scale under enforcement quality ξ.

The optimal maturity $T^*(\xi)$ satisfies:

$$\frac{V_0(K_{T^*+1}(\xi), T^*+1, \xi)}{V_0(K_{T^*}(\xi), T^*, \xi)} < 1 \text{ and } \frac{V_0(K_{T^*}(\xi), T^*, \xi)}{V_0(K_{T^*-1}(\xi), T^*-1, \xi)} \geq 1 \tag{13}$$

First, we show that $T^*(\xi)$ is non-decreasing in ξ. Since higher enforcement quality relaxes the financing constraints, we have $K_T(\xi') \geq K_T(\xi)$ for any $\xi' > \xi$ and any maturity T. Additionally, V_0 is increasing in both K and ξ.

Suppose, by contradiction, that for $\xi' > \xi$, we have $T^*(\xi') < T^*(\xi)$. This implies:

$$\frac{V_0(K_{T^*(\xi')+1}(\xi'), T^*(\xi')+1, \xi')}{V_0(K_{T^*(\xi')}(\xi'), T^*(\xi'), \xi')} < 1 \tag{14}$$

However, since $T^*(\xi') < T^*(\xi)$, we have $T^*(\xi')+1 \leq T^*(\xi)$. The optimality condition for $T^*(\xi)$ implies:

$$\frac{V_0(K_{T^*(\xi')+1}(\xi), T^*(\xi')+1, \xi)}{V_0(K_{T^*(\xi')}(\xi), T^*(\xi'), \xi)} \geq 1 \tag{15}$$

Given that $K_T(\xi') \geq K_T(\xi)$ and V_0 is increasing in ξ, we have:

$$\frac{V_0(K_{T^*(\xi')+1}(\xi'), T^*(\xi')+1, \xi')}{V_0(K_{T^*(\xi')}(\xi'), T^*(\xi'), \xi')} \geq \frac{V_0(K_{T^*(\xi')+1}(\xi), T^*(\xi')+1, \xi)}{V_0(K_{T^*(\xi')}(\xi), T^*(\xi'), \xi)} \geq 1 \tag{16}$$

This contradicts our initial assumption. Therefore, $T^*(\xi)$ is non-decreasing in ξ.

Next, we show that $\frac{\partial T^*(\xi)}{\partial \xi} \in \{0, 1\}$. Consider an infinitesimal increase from ξ to $\xi + d\xi$. Since T^* is non-decreasing and takes integer values, the change ΔT must be a non-negative integer. We need to show that $\Delta T \leq 1$.

Suppose, by contradiction, that $\Delta T \geq 2$, so $T^*(\xi + d\xi) \geq T^*(\xi) + 2$. This implies:

$$\frac{V_0(K_{T^*(\xi)+1}(\xi + d\xi), T^*(\xi)+1, \xi + d\xi)}{V_0(K_{T^*(\xi)}(\xi + d\xi), T^*(\xi), \xi + d\xi)} \geq 1 \tag{17}$$

since otherwise $T^*(\xi) + 1$ would be optimal under $\xi + d\xi$.

The change in enforcement from ξ to $\xi + d\xi$ affects the feasible scale K_T through relaxed constraints. However, for an infinitesimal change $d\xi$, the effect on feasible scale is also infinitesimal. Given the smooth nature of the value function V_0 with respect to K and ξ, and the optimality condition at ξ:

$$\frac{V_0(K_{T^*(\xi)+1}(\xi), T^*(\xi)+1, \xi)}{V_0(K_{T^*(\xi)}(\xi), T^*(\xi), \xi)} < 1 \tag{18}$$

the ratio cannot jump from strictly less than 1 to greater than or equal to 1 for an infinitesimal change in ξ unless $T^*(\xi)$ was exactly at an indifference point.

Therefore, $\Delta T \leq 1$ for any infinitesimal change, which implies $\frac{\partial T^*(\xi)}{\partial \xi} \in \{0, 1\}$.

A.1.3 Proof of Proposition 4

We first define the set of feasible discrete scales and maturities:

$$\mathcal{F} = \{(T, K_T) \mid K_T \text{ is the maximum feasible scale attainable with maturity } T\} \tag{19}$$

For each pair $(T, K_T) \in \mathcal{F}$, we compute the firm value:

$$V(T) = \frac{(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]K_T^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \tag{20}$$

Let $\tilde{T} = \arg\max_T V(T)$ be the maturity that maximizes firm value among discrete scales.

Part 1: Optimal scale

Given $\tilde{T}$, we need to determine whether to choose maturity $\tilde{T}$ or $\tilde{T}+1$. This decision depends on the ratio:

$$\frac{V(\tilde{T}+1)}{V(\tilde{T})} = \frac{\frac{(\beta\rho)^{\tilde{T}+2}}{1-\beta\rho}[1-\eta(1-\alpha)]K_{\tilde{T}+1}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{\frac{(\beta\rho)^{\tilde{T}+1}}{1-\beta\rho}[1-\eta(1-\alpha)]K_{\tilde{T}}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}} \tag{21}$$

$$= \beta\rho \left(\frac{K_{\tilde{T}+1}}{K_{\tilde{T}}}\right)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \tag{22}$$

Since $\tilde{T}$ is the optimal discrete maturity, we know:

$$\beta\rho \left(\frac{K_{\tilde{T}+1}}{K_{\tilde{T}}}\right)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} < 1 \tag{23}$$

This implies that:

$$\left(\frac{K_{\tilde{T}+1}}{K_{\tilde{T}}}\right)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} < \frac{1}{\beta\rho} \tag{24}$$

Taking the power of both sides by $\frac{1-\eta(1-\alpha)}{\eta\alpha}$:

$$\frac{K_{\tilde{T}+1}}{K_{\tilde{T}}} < \left(\frac{1}{\beta\rho}\right)^{\frac{1-\eta(1-\alpha)}{\eta\alpha}} \tag{25}$$

Therefore, the optimal scale K^* satisfies:

$$K^* = K_{\tilde{T}} \cdot \left(\frac{1}{\beta\rho}\right)^{\frac{1-\eta(1-\alpha)}{\eta\alpha}} \tag{26}$$

as long as $K^* < K_{\tilde{T}+1}$, which is guaranteed by the inequality derived above.

Part 2: Optimal last period coupon

The optimal last period coupon factor γ^* must satisfy the participation constraint of financiers with equality. This requires:

$$(1+r_f)p_k K^* = \sum_{t=0}^{\tilde{T}-1}(\beta\rho)^t\pi_t + (\beta\rho)^{\tilde{T}}\gamma^*\pi_{\tilde{T}} \tag{27}$$

$$\tag{28}$$

Solving for γ^*:

$$\gamma^* = \frac{(1+r_f)p_k K^* - \sum_{t=0}^{\tilde{T}-1}(\beta\rho)^t\pi_t}{(\beta\rho)^{\tilde{T}}\pi_{\tilde{T}}} \tag{29}$$

This can be further simplified to:

$$\gamma^* = 1 - \frac{(\beta\rho)^{\tilde{T}+1}}{\pi_{\tilde{T}}} \left(1 - \frac{K^*}{K_{\tilde{T}+1}}\right) [1 - \eta(1-\alpha)] K^{* \frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \quad (30)$$

Part 3: Optimal maturity

If $\gamma^* = 0$, then the optimal maturity is $T^* = \tilde{T}$. If $\gamma^* > 0$, then $T^* = \tilde{T} + 1$ because the firm pays a fraction of profits at time $\tilde{T} + 1$. Therefore:

$$T^* = \tilde{T} + \mathbf{1}_{\{\gamma^* > 0\}} \quad (31)$$

A.1.4 Proof of Proposition 5

Part 1: Maximum feasible scale is increasing in enforcement

For any fixed maturity T, the maximum feasible scale K_T is determined by the binding enforcement constraint at age $a = 0$:

$$(1 - \xi) z \left[K_T^{\alpha} \left(\frac{S_0}{w}\right)^{1-\alpha}\right]^{\eta} = V_0(K_T, T) \quad (32)$$

Given our characterization of short-term debt:

$$S_0(K_T) = \left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho} \frac{(1-\gamma)(1-\beta\rho) + \beta\rho}{(1-\xi)(1-\beta\rho)}\right]^{\frac{1}{\eta(1-\alpha)}} S_u(K_T, z) \quad (33)$$

Taking the total derivative of the enforcement constraint with respect to ξ:

$$-z\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^\eta+(1-\xi)z\eta\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^{\eta-1}K_T^\alpha\left(\frac{1}{w}\right)^{1-\alpha}(1-\alpha)S_0^{-\alpha}\frac{\partial S_0}{\partial\xi}+ \tag{34}$$

$$(1-\xi)z\eta\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^{\eta-1}\left(\frac{S_0}{w}\right)^{1-\alpha}\alpha K_T^{\alpha-1}\frac{\partial K_T}{\partial\xi}=\frac{\partial V_0(K_T,T)}{\partial K_T}\frac{\partial K_T}{\partial\xi} \tag{35}$$

Solving for $\frac{\partial K_T}{\partial\xi}$:

$$\frac{\partial K_T}{\partial\xi}=\frac{z\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^\eta-(1-\xi)z\eta\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^{\eta-1}K_T^\alpha\left(\frac{1}{w}\right)^{1-\alpha}(1-\alpha)S_0^{-\alpha}\frac{\partial S_0}{\partial\xi}}{\frac{\partial V_0(K_T,T)}{\partial K_T}-(1-\xi)z\eta\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^{\eta-1}\left(\frac{S_0}{w}\right)^{1-\alpha}\alpha K_T^{\alpha-1}} \tag{36}$$

The numerator is positive because: 1. The first term is positive: $z\left[K_T^\alpha\left(\frac{S_0}{w}\right)^{1-\alpha}\right]^\eta>0$ 2. The second term is negative: $\frac{\partial S_0}{\partial\xi}>0$ because improved enforcement relaxes the borrowing constraint

The denominator is positive because: 1. $\frac{\partial V_0(K_T,T)}{\partial K_T}>0$ (firm value increases with scale) 2. The second term is positive

Therefore, $\frac{\partial K_T}{\partial\xi}>0$, which proves that the maximum feasible scale K_T is strictly increasing in the enforcement parameter ξ.

Part 2: Optimal maturity is weakly increasing in enforcement

We've already proven in our earlier proposition on Discrete Changes in Optimal Maturity that $T^*(\xi)$ is non-decreasing in ξ, and specifically:

$$\frac{\partial T^*(\xi)}{\partial\xi}\in\{0,1\} \tag{37}$$

This follows from the fact that for each maturity T, the maximum feasible scale $K_T(\xi)$ is increasing in ξ, making higher maturities relatively more attractive as enforcement improves.

Part 3: Scale effect and value effect offsetting

To see why the scale effect and value effect largely offset each other, we examine the ELC duration formula:

$$\hat{a} = \left\lceil T + 1 - \frac{\ln\left((1-\beta\rho)\frac{(1-\gamma)(1-\beta\rho)+\beta\rho}{(1-\xi)(1-\beta\rho)}\right)}{\ln(\beta\rho)} \right\rceil \tag{38}$$

Note that K does not appear explicitly in this formula. The scale effect increases short-term borrowing needs proportionally to $K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}$, which tightens the enforcement constraint. However, the value effect increases the continuation value of the firm by the same factor, relaxing the enforcement constraint.

This proportionality can be seen by examining the enforcement constraint:

$$(1-\xi)z\left[K^{\alpha}\left(\frac{S_a}{w}\right)^{1-\alpha}\right]^{\eta} \leq V_a \tag{39}$$

Both the left side (output) and right side (continuation value) scale with $K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}$, causing the direct effects of scale on the ELC duration to approximately cancel out.

Therefore, the primary effect of scale on ELC duration operates through the maturity channel T^*, creating the clean relationship that countries with stronger enforcement have larger firms, longer debt maturities, but shorter ELC durations.

A.1.5 Proof of Proposition 6

We will prove by contradiction that short-term leverage increases and long-term leverage decreases during the early life cycle.

First, recall the definitions of short-term and long-term leverage:

$$\ell_a^S = \frac{S_a}{A_a} = \frac{S_a}{V_a + S_a + L_a}$$
$$\ell_a^L = \frac{L_a}{A_a} = \frac{L_a}{V_a + S_a + L_a}$$

Contradiction Argument for Short-Term Leverage

Suppose, contrary to our claim, that $\ell^S_{a+1} - \ell^S_a \leq 0$ for some $a < \hat{a}$ during the early life cycle. This would imply:

$$\frac{S_{a+1}}{V_{a+1} + S_{a+1} + L_{a+1}} \leq \frac{S_a}{V_a + S_a + L_a}$$

Cross-multiplying, we get:

$$S_{a+1}(V_a + S_a + L_a) \leq S_a(V_{a+1} + S_{a+1} + L_{a+1})$$

Expanding and rearranging:

$$S_{a+1}V_a + S_{a+1}S_a + S_{a+1}L_a \leq S_aV_{a+1} + S_aS_{a+1} + S_aL_{a+1}$$

$$S_{a+1}V_a + S_{a+1}L_a \leq S_aV_{a+1} + S_aL_{a+1}$$

$$S_{a+1}V_a - S_aV_{a+1} \leq S_aL_{a+1} - S_{a+1}L_a$$

$$S_{a+1}V_a - S_aV_{a+1} \leq S_aL_{a+1} - S_{a+1}L_a$$

From Proposition 2, we know that during the ELC:

$$S_{a+1} > S_a \quad \text{(short-term debt increases)}$$

$$V_{a+1} > V_a \quad \text{(firm value increases)}$$

$$L_{a+1} < L_a \quad \text{(long-term debt decreases)}$$

These relationships imply:

$$S_{a+1}V_a - S_aV_{a+1} = S_aV_a\left(\frac{S_{a+1}}{S_a} - \frac{V_{a+1}}{V_a}\right)$$

Since $S_{a+1} > S_a$ and $V_{a+1} > V_a$, the sign of $(S_{a+1}V_a - S_aV_{a+1})$ depends on the relative growth rates of S and V.

For the right-hand side:

$$S_aL_{a+1} - S_{a+1}L_a = S_aL_a\left(\frac{L_{a+1}}{L_a} - \frac{S_{a+1}}{S_a}\right)$$

Since $L_{a+1} < L_a$ and $S_{a+1} > S_a$, we have $\frac{L_{a+1}}{L_a} < 1 < \frac{S_{a+1}}{S_a}$, making $(S_aL_{a+1} - S_{a+1}L_a) < 0$.

Now, from our original assumption $\ell^S_{a+1} - \ell^S_a \leq 0$, we derived:

$$S_{a+1}V_a - S_aV_{a+1} \leq S_aL_{a+1} - S_{a+1}L_a$$

But we've shown that the right-hand side is negative, which means the left-hand side must also be negative:

$$\begin{aligned} S_{a+1}V_a - S_aV_{a+1} &< 0 \\ \frac{S_{a+1}}{S_a} &< \frac{V_{a+1}}{V_a} \end{aligned}$$

This implies that the growth rate of firm value exceeds the growth rate of short-term debt during the ELC.

From Proposition 2, we have:

$$\begin{aligned} S_a &= \left[\frac{(\beta\rho)^{T(K)+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} S_u(K,z) \\ V_a &= (\beta\rho)^{T+1-a}\frac{\pi_u(K,z)}{1-\beta\rho} \end{aligned}$$

Computing the growth rates:

$$\frac{S_{a+1}}{S_a} = \left[\frac{(\beta\rho)^{T(K)-a}}{(\beta\rho)^{T(K)+1-a}}\right]^{\frac{1}{\eta(1-\alpha)}} = \left[\frac{1}{\beta\rho}\right]^{\frac{1}{\eta(1-\alpha)}}$$
$$\frac{V_{a+1}}{V_a} = \frac{(\beta\rho)^{T-a}}{(\beta\rho)^{T+1-a}} = \frac{1}{\beta\rho}$$

Since $\eta(1-\alpha) < 1$ and $\beta\rho < 1$, we have:

$$\left[\frac{1}{\beta\rho}\right]^{\frac{1}{\eta(1-\alpha)}} > \frac{1}{\beta\rho}$$

This means $\frac{S_{a+1}}{S_a} > \frac{V_{a+1}}{V_a}$, which contradicts our derived inequality $\frac{S_{a+1}}{S_a} < \frac{V_{a+1}}{V_a}$.

Therefore, our initial assumption that $\ell^S_{a+1} - \ell^S_a \leq 0$ must be false, and we conclude that $\ell^S_{a+1} - \ell^S_a > 0$ during the early life cycle.

Contradiction Argument for Long-Term Leverage

Similarly, we can prove by contradiction that long-term leverage decreases during the ELC.

Suppose, contrary to our claim, that $\ell^L_{a+1} - \ell^L_a \geq 0$ for some $a < \hat{a}$ during the early life cycle. This would imply:

$$\frac{L_{a+1}}{V_{a+1} + S_{a+1} + L_{a+1}} \geq \frac{L_a}{V_a + S_a + L_a}$$

Following similar steps as above, we derive:

$$L_{a+1}V_a - L_aV_{a+1} \geq L_aS_{a+1} - L_{a+1}S_a$$

Since $L_{a+1} < L_a$, $V_{a+1} > V_a$, and $S_{a+1} > S_a$, we have:

$$L_{a+1}V_a - L_aV_{a+1} = L_aV_a\left(\frac{L_{a+1}}{L_a} - \frac{V_{a+1}}{V_a}\right) < 0$$
$$L_aS_{a+1} - L_{a+1}S_a = L_aS_a\left(\frac{S_{a+1}}{S_a} - \frac{L_{a+1}}{L_a}\right) > 0$$

This leads to the inequality:

$$\underbrace{L_{a+1}V_a - L_aV_{a+1}}_{\text{(negative)}} \geq \underbrace{L_aS_{a+1} - L_{a+1}S_a}_{\text{(positive)}}$$

This is a contradiction. Therefore, our initial assumption that $\ell^L_{a+1} - \ell^L_a \geq 0$ must be false, and we conclude that $\ell^L_{a+1} - \ell^L_a < 0$ during the early life cycle.

This completes our proof by contradiction that during the early life cycle, short-term leverage increases and long-term leverage decreases with firm age.

A.1.6 Proof of Proposition 7 (Enforcement and Leverage)

To prove the relationship between enforcement quality ξ and leverage ratios, we analyze the derivatives of long-term and short-term leverage with respect to ξ.

A.1.7 Part 1: Initial Long-Term Leverage

For a newborn firm ($a = 0$), long-term leverage is:

$$\ell^L_0 = \frac{L_0}{A_0} = \frac{L_0}{V_0 + S_0 + L_0}$$

From our characterization of the optimal contract, long-term debt equals the present value of all coupon payments. We can break this down into payments during and after the

ELC period:

$$\begin{aligned} L_0 &= \sum_{a=0}^{T(K)} (\beta\rho)^a q_a \\ &= \sum_{a=0}^{\hat{a}-1} (\beta\rho)^a \pi_a(K,\xi) + \sum_{a=\hat{a}}^{T(K)-1} (\beta\rho)^a \pi_u(K) + (\beta\rho)^{T(K)} \gamma \pi_u(K) \end{aligned}$$

where $\hat{a}$ represents the duration of the ELC, $T(K)$ is the debt maturity, and $\gamma \in (0,1]$ is the fraction of profits paid as debt service in the last period.

Using the homogeneity of our production function, we express unconstrained profits as:

$$\begin{aligned} \pi_u(K) &= [1-\eta(1-\alpha)] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\ &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \cdot \Omega \end{aligned}$$

where $\Omega = [1-\eta(1-\alpha)] z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$ represents the scale-independent component of unconstrained profits.

For constrained profits during the ELC:

$$\begin{aligned} \pi_a(K,\xi) &= zK^{\eta\alpha} \left(\frac{S_a}{w}\right)^{\eta(1-\alpha)} - (1+r_f)S_a \\ &= \left[\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\ &\quad - \eta(1-\alpha)\left[\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\ &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \cdot \Theta_a(\xi,T) \end{aligned}$$

where

$$\Theta_a(\xi,T) = z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \left[\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right) - \eta(1-\alpha)\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}}\right]$$

We can express long-term debt, firm value, and short-term debt as:

$$
\begin{aligned}
L_0 &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \cdot \Lambda(\xi, T, \hat{a}) \\
V_0 &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \cdot \frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega \\
S_0 &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} \cdot \Phi(\xi, T)
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda(\xi, T, \hat{a}) &= \sum_{a=0}^{\hat{a}-1} (\beta\rho)^a \Theta_a(\xi, T) + \Omega \left[\sum_{a=\hat{a}}^{T-1} (\beta\rho)^a + (\beta\rho)^T \gamma \right] \\
\Phi(\xi, T) &= \left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho} \frac{1}{1-\xi} \right]^{\frac{1}{\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w} \right)^{\frac{1}{1-\eta(1-\alpha)}}
\end{aligned}
$$

Therefore, long-term leverage can be expressed as:

$$
\ell_0^L = \frac{\Lambda(\xi, T, \hat{a})}{\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega + \Phi(\xi, T) + \Lambda(\xi, T, \hat{a})}
$$

We see that K cancels out in the leverage ratio, which depends only on enforcement ξ, maturity T, and ELC duration $\hat{a}$.

Now we analyze $\frac{\partial \ell_0^L}{\partial \xi}$ considering all possible cases:

Case 1: No change in T and no change in $\hat{a}$.

In regions where infinitesimal changes in ξ affect neither T nor $\hat{a}$, we calculate the derivative of ℓ_0^L with respect to ξ:

$$\begin{aligned}
\frac{\partial \ell_0^L}{\partial \xi} &= \frac{\partial}{\partial \xi}\left(\frac{\Lambda(\xi,T,\hat{a})}{\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega+\Phi(\xi,T)+\Lambda(\xi,T,\hat{a})}\right) \\
&= \frac{\left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega+\Phi(\xi,T)+\Lambda(\xi,T,\hat{a})\right]\frac{\partial\Lambda(\xi,T,\hat{a})}{\partial\xi}-\Lambda(\xi,T,\hat{a})\left[\frac{\partial\Phi(\xi,T)}{\partial\xi}+\frac{\partial\Lambda(\xi,T,\hat{a})}{\partial\xi}\right]}{\left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega+\Phi(\xi,T)+\Lambda(\xi,T,\hat{a})\right]^2} \\
&= \frac{\left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega+\Phi(\xi,T)\right]\frac{\partial\Lambda(\xi,T,\hat{a})}{\partial\xi}-\Lambda(\xi,T,\hat{a})\frac{\partial\Phi(\xi,T)}{\partial\xi}}{\left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega+\Phi(\xi,T)+\Lambda(\xi,T,\hat{a})\right]^2}
\end{aligned}$$

To determine the sign of this derivative, we need to compute $\frac{\partial\Lambda(\xi,T,\hat{a})}{\partial\xi}$ and $\frac{\partial\Phi(\xi,T)}{\partial\xi}$. First, we calculate $\frac{\partial\Theta_a(\xi,T)}{\partial\xi}$:

$$\begin{aligned}
\frac{\partial\Theta_a(\xi,T)}{\partial\xi} &= z^{\frac{1}{1-\eta(1-\alpha)}}\left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\
&\times \frac{\partial}{\partial\xi}\left[\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)-\eta(1-\alpha)\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}}\right] \\
&= z^{\frac{1}{1-\eta(1-\alpha)}}\left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\
&\times \left[\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{(1-\xi)^2}-\eta(1-\alpha)\frac{1}{\eta(1-\alpha)}\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}-1}\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{(1-\xi)^2}\right] \\
&= z^{\frac{1}{1-\eta(1-\alpha)}}\left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{(1-\xi)^2} \\
&\times \left[1-\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}-1}\right]
\end{aligned}$$

Since $\eta(1-\alpha)<1$, we have $\frac{1}{\eta(1-\alpha)}>1$, which means $\frac{1}{\eta(1-\alpha)}-1>0$. For $\xi<\xi^*$ where constraints bind, we have $\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)<1$, implying $\left(\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}-1}<1$. Therefore, $\frac{\partial\Theta_a(\xi,T)}{\partial\xi}>0$.

Now, we compute $\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi}$:

$$\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi} = \sum_{a=0}^{\hat{a}-1} (\beta\rho)^a \frac{\partial \Theta_a(\xi,T)}{\partial \xi} > 0$$

For $\frac{\partial \Phi(\xi,T)}{\partial \xi}$:

$$\begin{aligned}
\frac{\partial \Phi(\xi,T)}{\partial \xi} &= \frac{\partial}{\partial \xi}\left[\left(\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{1}{1-\eta(1-\alpha)}}\right] \\
&= \frac{1}{\eta(1-\alpha)}\left(\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}-1} \frac{(\beta\rho)^{T+1}}{1-\beta\rho}\frac{1}{(1-\xi)^2} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{1}{1-\eta(1-\alpha)}} \\
&= \frac{1}{\eta(1-\alpha)(1-\xi)^2}\left(\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\right)\left(\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\frac{1}{1-\xi}\right)^{\frac{1}{\eta(1-\alpha)}-1} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{1}{1-\eta(1-\alpha)}} > 0
\end{aligned}$$

To determine the sign of $\frac{\partial \ell_0^L}{\partial \xi}$, we need to compare the magnitudes of $\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi}$ and $\frac{\partial \Phi(\xi,T)}{\partial \xi}$. When $\xi < \xi^*$ (enforcement frictions significantly bind), $\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi}$ dominates $\frac{\partial \Phi(\xi,T)}{\partial \xi}$ for the following reasons:

1. $\Lambda(\xi,T,\hat{a})$ captures the long-term debt capacity, which depends on the present value of all profits including unconstrained profits after the ELC. This component grows more rapidly with enforcement than short-term debt capacity.

2. The expression for $\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi}$ includes a sum over multiple periods, whereas $\frac{\partial \Phi(\xi,T)}{\partial \xi}$ captures only the instantaneous effect on short-term debt.

3. As enforcement improves, the increase in profit terms $\Theta_a(\xi,T)$ outpaces the increase in required short-term debt because better enforcement allows firms to credibly promise more long-term repayment.

Therefore, $\left[\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega + \Phi(\xi,T)\right]\frac{\partial \Lambda(\xi,T,\hat{a})}{\partial \xi} > \Lambda(\xi,T,\hat{a})\frac{\partial \Phi(\xi,T)}{\partial \xi}$, which implies $\frac{\partial \ell_0^L}{\partial \xi} > 0$ for $\xi < \xi^*$.

Case 2: Change in $\hat{a}$ but no change in T.

When ξ increases, $\hat{a}$ can decrease even if T remains constant. From the characterization

of $\hat{a}$:

$$\hat{a} = \left\lceil T + 1 - \frac{\ln\left((1-\beta\rho)\frac{(1-\gamma)(1-\beta\rho)+\beta\rho}{(1-\xi)(1-\beta\rho)}\right)}{\ln(\beta\rho)} \right\rceil$$

As ξ increases, the term $\frac{1}{1-\xi}$ increases, which decreases $\hat{a}$. This means firms reach unconstrained short-term borrowing sooner. When $\hat{a}$ decreases by one period, the function $\Lambda(\xi, T, \hat{a})$ changes as follows:

$$\begin{aligned}
\Lambda(\xi, T, \hat{a}-1) &= \sum_{a=0}^{\hat{a}-2} (\beta\rho)^a \Theta_a(\xi, T) + \Omega \left[\sum_{a=\hat{a}-1}^{T-1} (\beta\rho)^a + (\beta\rho)^T \gamma \right] \\
&= \sum_{a=0}^{\hat{a}-2} (\beta\rho)^a \Theta_a(\xi, T) + (\beta\rho)^{\hat{a}-1}\Omega + \Omega \left[\sum_{a=\hat{a}}^{T-1} (\beta\rho)^a + (\beta\rho)^T \gamma \right] \\
&= \Lambda(\xi, T, \hat{a}) - (\beta\rho)^{\hat{a}-1}\Theta_{\hat{a}-1}(\xi, T) + (\beta\rho)^{\hat{a}-1}\Omega
\end{aligned}$$

Since firms are still financially constrained at age $\hat{a}-1$, we have $\Theta_{\hat{a}-1}(\xi, T) < \Omega$, implying $\Lambda(\xi, T, \hat{a}-1) > \Lambda(\xi, T, \hat{a})$. This means when $\hat{a}$ decreases, long-term debt increases.

Meanwhile, the change in $\hat{a}$ does not affect $\Phi(\xi, T)$ or the value term $\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega$. Therefore, when $\hat{a}$ decreases due to improved enforcement, the numerator $\Lambda(\xi, T, \hat{a})$ increases, while the denominator increases by a smaller proportion, leading to $\frac{\partial \ell_0^L}{\partial \xi} > 0$.

Case 3: Change in both T and $\hat{a}$.

From Proposition 3, we know $T(K)$ increases discretely with ξ. When T increases from T to $T+1$, we have:

$$\Lambda(\xi, T+1, \hat{a}') = \sum_{a=0}^{\hat{a}'-1} (\beta\rho)^a \Theta_a(\xi, T+1) + \Omega \left[\sum_{a=\hat{a}'}^{T} (\beta\rho)^a + (\beta\rho)^{T+1}\gamma \right]$$

Where $\hat{a}'$ is the new ELC duration under the increased maturity. This change has three

effects:

1. The term $\Theta_a(\xi, T+1)$ increases compared to $\Theta_a(\xi, T)$ because the factor $\frac{(\beta\rho)^{T+2-a}}{1-\beta\rho}$ is larger than $\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}$.

2. The sum $\sum_{a=\hat{a}'}^{T}(\beta\rho)^a$ includes an additional term $(\beta\rho)^T$ compared to $\sum_{a=\hat{a}}^{T-1}(\beta\rho)^a$.

3. The terminal payment occurs at $T+1$ instead of T.

All these effects increase $\Lambda(\xi, T+1, \hat{a}')$ compared to $\Lambda(\xi, T, \hat{a})$. Simultaneously, the firm value term changes from $\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\Omega$ to $\frac{(\beta\rho)^{T+2}}{1-\beta\rho}\Omega$, which is smaller since $\beta\rho < 1$.

The short-term debt term $\Phi(\xi, T)$ also changes with T, but the increase in long-term debt capacity dominates because it directly affects the scale of the firm. The combined effect reinforces the positive relationship between enforcement and long-term leverage.

Combining all cases, we have $\frac{\partial \ell_0^L}{\partial \xi} > 0$ for $\xi < \xi^*$.

A.1.8 Part 2: Leverage Throughout the Early Life Cycle

For any age a during the early life cycle ($a \leq \hat{a}$), similar analysis shows that $\frac{\partial \ell_a^L}{\partial \xi} > 0$ for $\xi < \xi^*$.

For short-term leverage $\ell_a^S = \frac{S_a}{V_a + S_a + L_a}$, the relationship $\frac{\partial \ell_a^S}{\partial \xi} < 0$ holds for $\xi < \xi^*$ because as enforcement improves: 1. The term $\frac{1}{1-\xi}$ increases, raising S_a 2. Total assets $(V_a + S_a + L_a)$ increase even more proportionally due to scale effects 3. The duration of constraints shortens (smaller $\hat{a}$)

These effects combine to make short-term debt a smaller fraction of total assets as enforcement improves, leading to $\frac{\partial \ell_a^S}{\partial \xi} < 0$ for $\xi < \xi^*$.

This completes the proof of Proposition 7.

A.2 Perfect Enforcement Benchmark

Under perfect enforcement ($\xi = 1$), the firm chooses initial scale K, employment sequence $\{n_a\}_{a=0}^{\infty}$, short-term debt sequence $\{R_a\}_{a=0}^{\infty}$, and long-term debt coupon structure $\{q_a\}_{a=0}^{\infty}$

to maximize expected present value of dividends subject to the working capital constraint (WCC), the participation constraint of financiers (PCF) and the non-negativity constraint on coupons (NNC):

$$V_0 = \max_{\{K,\{q_a,R_a,n_a\}_{a=0}^{\infty}\}} \sum_{a=0}^{\infty} (\beta\rho)^a \left(z \left[K^{\alpha} \left(\frac{n_a}{w} \right)^{1-\alpha} \right]^{\eta} - (1+r_f)R_a - q_a \right)$$

subject to:

$$w_a n_a \leq R_a \tag{WCC}$$

$$p_k K(1+r_f) \leq \sum_{a=0}^{\infty} (\beta\rho)^a q_a \tag{PCF}$$

$$0 \leq q_a \leq z \left[K^{\alpha} \left(\frac{R_a}{w} \right)^{1-\alpha} \right]^{\eta} - (1+r_f)R_a \tag{NNC}$$

Proposition 9 (Perfect Enforcement Equilibrium) *With no enforcement frictions ($\xi = 1$):*

1. *The scale choice (K) is independent of the maturity T and long-term debt structure $\{q_a\}_{a=0}^{\infty}$.*

2. *Maturity T and long-term debt structure $\{q_a\}_{a=0}^{\infty}$ are indeterminate.*

3. *The unconstrained levels of employment, short-term debt, and profits for a firm with scale K and productivity z are:*

$$n_u(K,z) = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w} \right)^{\frac{1}{1-\eta(1-\alpha)}}$$

$$R_u(K,z) = w n_u(K,z)$$

$$\pi_u(K,z) = [1-\eta(1-\alpha)] \frac{1+r_f}{\eta(1-\alpha)} R_u(K,z)$$

4. The NPV of creating a firm with productivity z is:

$$\Pi_u(z) = z^{\frac{1}{1-\eta}} \left(\frac{1-\eta}{1-\beta\rho}\right) \left(\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)}\right)^{\frac{1-\eta(1-\alpha)}{1-\eta}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta}}$$

Without frictions, real choices (K, n) are separable from financial choices $\{q_a\}_{a=0}^{\infty}$. While the path of long-term debt is indeterminate, its face value during the firm's first year is pinned down by K, creating a direct relationship between long-term debt at inception and firm size.

B Appendix: Data and Alternative Empirical Specifications

B.1 Data Description and Alternative Empirical Specifications

Table 2:
Enforcement (Recovery Rates)

Low Enforcement	Recovery Rates
Estonia	38.56
Greece	39.88
Hungary	39.91
France	56.75
Slovenia	59.26
Italy	60.56
High Enforcement	
Portugal	69.87
Spain	73.52
Sweden	76.29
Germany	81.86
Belgium	87.50
Finland	88.87

Source: Doing Business Database, World Bank. Recovery rates are defined as cents on the dollar recovered by secured creditors through judicial reorganization, liquidation or debt enforcement (foreclosure or receivership) proceedings. The calculation takes into account the outcome: whether the business emerges from the proceedings as a going concern or the assets are sold piecemeal.

TABLE 3:
SUMMARY STATISTICS OF FIRM CHARACTERISTICS

	Young Firms					Old Firms				
	Long Leverage	Short Leverage	Age	Employment	N	Long Leverage	Short Leverage	Age	Employment	N
Belgium	0.317	0.230	4.282	59.437	1540	0.236	0.201	30.306	144.514	23821
	(0.253)	(0.210)	(1.452)	(100.686)		(0.209)	(0.191)	(18.568)	(668.328)	
Estonia	0.313	0.184	4.138	7.182	10367	0.250	0.151	14.418	16.901	25893
	(0.219)	(0.162)	(1.462)	(32.934)		(0.205)	(0.146)	(6.244)	(56.393)	
Finland	0.398	0.179	4.118	17.699	9478	0.327	0.152	21.778	32.309	44044
	(0.233)	(0.129)	(1.456)	(76.350)		(0.219)	(0.129)	(13.606)	(169.575)	
France	0.346	0.185	4.225	19.441	25982	0.169	0.136	24.511	73.752	156828
	(0.227)	(0.136)	(1.419)	(115.330)		(0.159)	(0.140)	(14.502)	(809.535)	
Germany	0.390	0.166	4.331	252.496	2272	0.283	0.153	41.419	554.288	31370
	(0.253)	(0.174)	(1.440)	(611.531)		(0.216)	(0.164)	(41.503)	(4866.912)	
Greece	0.262	0.272	4.563	35.892	1310	0.247	0.224	26.122	72.622	22501
	(0.206)	(0.212)	(1.334)	(71.588)		(0.204)	(0.190)	(15.748)	(267.882)	
Hungary	0.291	0.200	4.341	88.100	1849	0.186	0.170	18.961	152.434	15769
	(0.247)	(0.184)	(1.430)	(361.352)		(0.179)	(0.159)	(9.716)	(665.259)	
Italy	0.333	0.332	4.260	22.180	87141	0.257	0.312	23.019	49.384	482327
	(0.238)	(0.250)	(1.407)	(124.894)		(0.204)	(0.238)	(13.752)	(395.162)	
Portugal	0.398	0.215	4.083	14.370	22343	0.307	0.183	22.362	42.157	119223
	(0.247)	(0.196)	(1.504)	(33.348)		(0.216)	(0.167)	(14.338)	(181.730)	
Slovenia	0.378	0.207	3.981	6.832	29742	0.308	0.171	17.324	15.951	112192
	(0.247)	(0.189)	(1.518)	(31.608)		(0.228)	(0.166)	(6.602)	(103.615)	
Spain	0.399	0.182	4.248	18.616	84100	0.313	0.154	20.093	42.156	520001
	(0.256)	(0.180)	(1.432)	(177.656)		(0.241)	(0.161)	(11.194)	(373.392)	
Sweden	0.416	0.168	4.144	7.155	18633	0.359	0.120	21.850	15.560	68167
	(0.239)	(0.131)	(1.396)	(19.554)		(0.231)	(0.121)	(13.578)	(62.302)	

Notes: Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth.

B.2 Alternative Empirical Specification

B.2.1 Early Life Cycle

We estimate the following model where $\omega_{c,s,t}$ denote country-sector-time fixed effect.

$$\begin{aligned} y_{i,c,t} =& \omega_{c,s,t} + \beta_1 AGE_{i,t} + \beta_1^{ELC} AGE_{i,t} \times ELC_{i,c,t} + \beta_2 \ln EMP_{it} + \beta_2^{ELC} \ln EMP_{it} \times ELC_{i,c,t} \\ &+ \beta_3 COL_{i,c,t} + \beta_3^{ELC} COL_{i,c,t} \times ELC_{i,c,t} + \beta_4 PROFIT_{i,t} + \beta_4^{ELC} PROFIT_{i,t} \times ELC_{i,c,t} \\ &+ \beta_5 PROD_{i,t} + \beta_5^{ELC} PROD_{i,t} \times ELC_{i,c,t} + \epsilon_{i,c,t} \end{aligned}$$

This is equivalent to run the regression in two sub-samples of young and old firms, the above specification has the advantage that presents in a convenient way a hypothesis test of the coefficients for young and old firms being different.

TABLE 4:
LEVERAGE PATTERNS OF YOUNG FIRMS

	Short Leverage	Long Leverage
Young firms (Age 0-10)		
Age	-0.0009***	-0.0013***
	(0.0000)	(0.0000)
Age× ELC	0.0022***	-0.0152***
	(0.0003)	(0.0007)
Size	0.0013***	-0.0129***
	(0.0004)	(0.0003)
Size× ELC	0.0013***	-0.0023***
	(0.0005)	(0.0005)
Collateral	-0.1502***	0.2484***
	(0.0044)	(0.0036)
Collateral× ELC	0.0065***	0.0447***
	(0.0025)	(0.0036)
Profitability	-0.2455***	-0.2014***
	(0.0072)	(0.0050)
Profitability× ELC	0.0224***	-0.0867***
	(0.0062)	(0.0062)
Productivity	0.0078***	-0.0156***
	(0.0006)	(0.0004)
Productivity× ELC	0.0001	0.0096***
	(0.0002)	(0.0004)
Observations	1,647,120	1,647,120
Adjusted R^2	0.2340	0.2158

Notes: ELC (Early Life Cycle) is a dummy variable that takes the value 1 for firms that are 7 years old or younger. Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. We consider firms for wich we have at least 3 observations and have positive debt.

$$
\begin{aligned}
y_{i,c,t} =& \omega_i + \beta_1 AGE_{i,t} + \beta_1^{ELC} AGE_{i,t} \times ELC_{i,c,t} + \beta_2 \ln EMP_{it} + \beta_2^{ELC} \ln EMP_{it} \times ELC_{i,c,t} \\
&+ \beta_3 COL_{i,c,t} + \beta_3^{ELC} COL_{i,c,t} \times ELC_{i,c,t} + \beta_4 PROFIT_{i,t} + \beta_4^{ELC} PROFIT_{i,t} \times ELC_{i,c,t} \\
&+ \beta_5 PROD_{i,t} + \beta_5^{ELC} PROD_{i,t} \times ELC_{i,c,t} + \epsilon_{i,c,t}
\end{aligned}
$$

Table 5:
Leverage Patterns of Young Firms

	Short Leverage	Long Leverage
Age	-0.0029***	-0.0044***
	(0.0002)	(0.0002)
Age× ELC	0.0041***	-0.0216***
	(0.0004)	(0.0008)
Size	0.0157***	-0.0112***
	(0.0008)	(0.0010)
Size× ELC	-0.0014***	-0.0023***
	(0.0004)	(0.0005)
Collateral	-0.0982***	0.2909***
	(0.0045)	(0.0051)
Collateral× ELC	-0.0128***	0.0546***
	(0.0021)	(0.0031)
Profitability	-0.2033***	-0.1948***
	(0.0062)	(0.0058)
Profitability× ELC	0.0612***	0.0032
	(0.0057)	(0.0064)
Productivity	0.0116***	-0.0115***
	(0.0007)	(0.0008)
Productivity× ELC	-0.0009***	0.0104***
	(0.0002)	(0.0005)
Observations	1,562,247	1,562,247
Adjusted R^2	0.7474	0.7325

Notes: ELC (Early Life Cycle) is a dummy variable that takes the value 1 for firms that are 7 years old or younger. Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include firm fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. We consider firms for wich we have at least 3 observations and have positive debt.

B.2.2 ELC and Financial Development

Table 6:
Leverage Patterns of Young Firms and Financial Development

	Short Leverage	Long Leverage
Age	0.00835***	-0.02580***
	(0.00101)	(0.00119)
FD × Age	-0.00006***	0.00008***
	(0.00001)	(0.00001)
Size	0.00454***	-0.01665***
	(0.00039)	(0.00042)
Collateral	-0.15075***	0.28785***
	(0.00177)	(0.00232)
Profitability	-0.23712***	-0.29815***
	(0.00358)	(0.00457)
Productivity	0.00771***	-0.01513***
	(0.00059)	(0.00062)
Observations	233,921	233,921
Adjusted R^2	0.21105	0.22628
All firms		
Age	-0.00087***	0.00048***
	(0.00004)	(0.00005)
FD × Age	0.00000	-0.00002***
	(0.00000)	(0.00000)
Size	0.00117***	-0.01233***
	(0.00013)	(0.00014)
Collateral	-0.14960***	0.24804***
	(0.00067)	(0.00088)
Profitability	-0.24450***	-0.20091***
	(0.00413)	(0.00394)
Productivity	0.00782***	-0.01379***
	(0.00024)	(0.00025)
Observations	1,411,686	1,411,686
Adjusted R^2	0.23708	0.20030

Notes: Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. FD is defined as Private Credit over GDP.

TABLE 7:
LEVERAGE PATTERNS OF YOUNG FIRMS AND FINANCIAL DEVELOPMENT

	Short Leverage	Long Leverage
Age	0.00382***	-0.02534***
	(0.00100)	(0.00116)
FD × Age	-0.00001	-0.00005***
	(0.00001)	(0.00001)
Size	0.01315***	-0.00893***
	(0.00128)	(0.00145)
Collateral	-0.09834***	0.33916***
	(0.00470)	(0.00620)
Profitability	-0.16750***	-0.23535***
	(0.00414)	(0.00551)
Productivity	0.01235***	-0.00934***
	(0.00121)	(0.00133)
Observations	191,970	191,970
Adjusted R^2	0.73422	0.73430
All firms		
Age	-0.00289***	-0.00443***
	(0.00005)	(0.00006)
FD × Age	-0.00000	0.00000***
	(0.00000)	(0.00000)
Size	0.01554***	-0.01294***
	(0.00049)	(0.00055)
Collateral	-0.10063***	0.29029***
	(0.00164)	(0.00212)
Profitability	-0.19556***	-0.18184***
	(0.00519)	(0.00554)
Productivity	0.01103***	-0.00967***
	(0.00052)	(0.00057)
Observations	1,333,407	1,333,407
Adjusted R^2	0.75993	0.74110

Notes: Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. FD is defined as Private Credit over GDP.

B.3 Alternative Specification: No Firm Controls

$$y_{i,t} = \eta_i + \sum_{a \in A} \beta_a \mathcal{D}^a_{i,t} + \epsilon_{i,t}$$

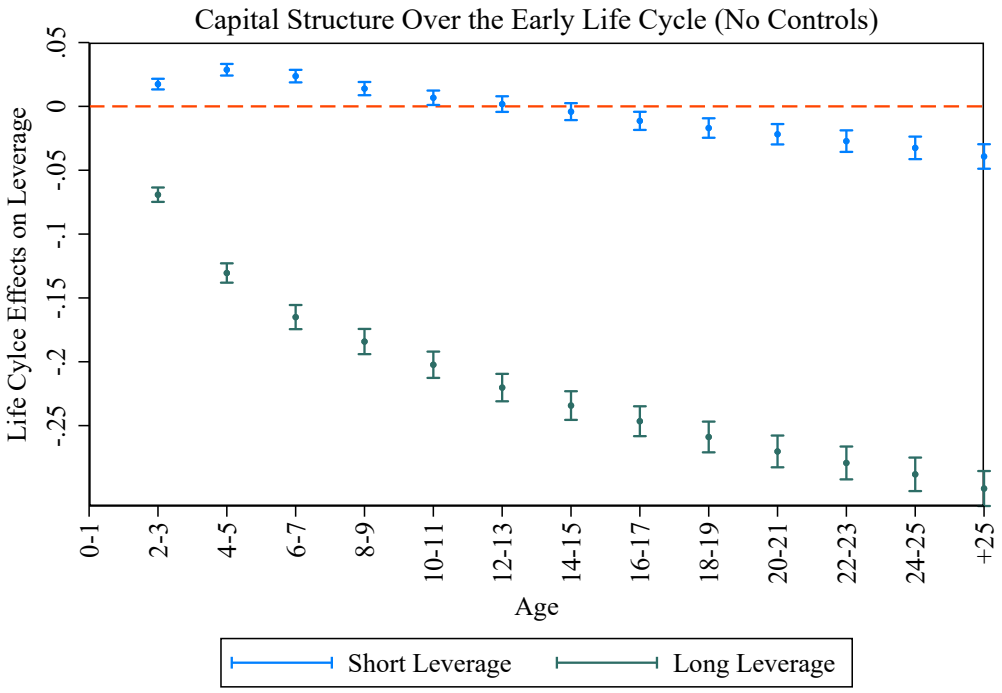


FIGURE 5: LIFE CYCLE PATTERNS OF CAPITAL STRUCTURE OF FIRMS FOR WHICH WE HAVE MORE THAN 4 OBSERVATIONS. CONFIDENCE INTERVALS ARE AT 90% AND STANDARD ERRORS ARE CLUSTERED AT COUNTRY-INDUSTRY-TIME LEVEL.

We run the regression over the sample of firms for which we have at least 3 observations.

B.4 Calibration

We calibrate the model to the US economy assuming that $\xi = 1$.

Employment size of firms:

$$n^u(K,z) = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$K^u = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

$$\left[K^u\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} = z^{\frac{\eta\alpha}{[1-\eta(1-\alpha)][1-\eta]}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}}$$

$$
\begin{aligned}
\frac{K}{n^u(K,z)} &= \frac{K}{K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
&= \frac{K^{\frac{1-\eta}{1-\eta(1-\alpha)}}}{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
\frac{K^u}{n^u(K^u,z)} &= \frac{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]}{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
&= \frac{(1+r_f)w}{\eta(1-\alpha)} \frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}
\end{aligned}
$$

If we target firms of 10 yrs old:

- w is $\frac{\text{Wage bill}}{\text{Employment}}$
- $\rho = 0.85$ (Average exit rate of US economy is roughly 10% according to Business Dynamics Statistics.)
- β to match a risk free rate of less than a year $r_f = \frac{1-\sqrt{\beta}}{\sqrt{\beta}}$
- $\eta = 0.95$.
- $\frac{K}{n^u(K,z)}$ is the average capital to employment ratio of firms
- $\phi = 0.4$ is in the range of reasonable values according to Kermani and Ma (2022).
- We can get $p_k = \frac{n^u(K,z)}{K} \frac{(1+r_f)w}{\eta(1-\alpha)} \frac{\eta\alpha}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}$

We will calibrate p_k to be the price of capital in US, we have in the data:

$$\frac{p_k K}{n}$$

B.5 Capital Labor Ratios US

Code	Sector	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
01-99	Total	0.38	0.38	0.38	0.37	0.37	0.37	0.37	0.37	0.37	0.37
01-03	Agriculture, hunting, forestry and fishing [A]	0.43	0.43	0.43	0.44	0.46	0.44	0.45	0.45	0.45	0.45
05-09	Mining and quarrying [B]	3.01	2.77	2.62	2.65	2.64	2.97	3.54	3.43	3.20	3.21
10-33	Manufacturing [C]	0.29	0.29	0.29	0.29	0.29	0.29	0.30	0.30	0.30	0.30
35-39	Electricity, gas (...) and remediation activities [D-E]	4.06	4.10	4.15	4.19	4.21	4.22	4.29	4.29	4.27	4.30
41-43	Construction [F]	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
45-47	Wholesale and retail trade, repair of motor vehicles [G]	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
49-53	Transportation and storage [H]	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.23	0.23
55-56	Accommodation and food service activities [I]	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
58-63	Information and communication [J]	0.45	0.45	0.45	0.46	0.47	0.47	0.48	0.50	0.51	0.52
69-75	Professional, scientific and technical activities [M]	0.11	0.11	0.11	0.11	0.11	0.10	0.10	0.11	0.11	0.11
77-82	Administrative and support service activities [N]	0.07	0.07	0.07	0.07	0.07	0.08	0.08	0.08	0.08	0.09
87-88	Residential care and social work activities [QB]	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
90-93	Arts, entertainment and recreation [R]	0.30	0.30	0.29	0.29	0.28	0.29	0.28	0.29	0.29	0.29
45-99	Total services	0.36	0.36	0.36	0.35	0.35	0.35	0.35	0.35	0.35	0.35

Table 8: United States Capital-Labor ratio based on OECD STAN number of employees and the volume of net capital stock.

	Local Currency	2017 PPP $	U.S. dollars	2017 PPP $	U.S. dollars
Belgium	48201	53938	63901	1.33	1.57
Estonia	9828	16762	13029	0.41	0.32
Finland	36528	37341	48426	0.92	1.19
France	30804	34347	40837	0.84	1.00
Germany	34584	40598	45849	1.00	1.13
Greece	21588	27558	28620	0.68	0.70
Hungary	2692080	18835	12946	0.46	0.32
Italy	27432	33529	36367	0.82	0.89
Portugal	9372	13397	12425	0.33	0.31
Slovenia	17724	25390	23497	0.62	0.58
Spain	22066	27572	29253	0.68	0.72
Sweden	33650	3681	4669	0.09	0.11
United States	40668	40668	40668	1.00	1.00

Table 9: Annualized wages and relative wages compared to US for the year 2010. *Source: International Labor Organization.*

We get a price of capital for US in dollar using

$$p_k = \frac{n^u(K,z)}{K}\frac{(1+r_f)w}{\eta(1-\alpha)}\frac{\eta\alpha}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}$$

Denote this p_k^{US} in dollars, we take the wages in dollars w^{US} and for each country convert

them to local currency and divide them by the price of consumption (private consumption deflator) of that country.

TABLE 10: ALTERNATIVE CALIBRATIONS OF PRICES FOR THE COUNTRIES IN THE SAMPLE

Country	Local Currency	2017 PPP $	U.S. dollars	(2017 PPP $)	(U.S. dollars)	Deflator	pk	Relative p_k	p_k/w	Relative p_k/w
Belgium	48200.88	53937.60	63900.72	1.33	1.57	0.89	491372.38	0.89	9.11	0.67
Estonia	9828.00	16762.08	13029.12	0.41	0.32	0.59	322392.28	0.59	19.23	1.42
Finland	36528.00	37340.88	48425.76	0.92	1.19	0.98	537884.14	0.98	14.40	1.07
France	30804.00	34347.24	40837.32	0.84	1.00	0.90	493131.41	0.90	14.36	1.06
Germany	34584.00	40597.80	45848.64	1.00	1.13	0.85	468403.48	0.85	11.54	0.85
Greece	21588.00	27558.00	28619.52	0.68	0.70	0.78	430736.93	0.78	15.63	1.16
Hungary	2692080.00	18834.96	12946.20	0.46	0.32	142.93	78590608.63	142.93	4172.59	308.61
Italy	27432.00	33528.84	36367.08	0.82	0.89	0.82	449869.28	0.82	13.42	0.99
Portugal	9372.00	13396.56	12424.56	0.33	0.31	0.70	384668.28	0.70	28.71	2.12
Slovenia	17724.00	25390.20	23496.96	0.62	0.58	0.70	383833.62	0.70	15.12	1.12
Spain	22065.84	27572.40	29253.00	0.68	0.72	0.80	440041.15	0.80	15.96	1.18
Sweden	33649.80	3681.00	4668.72	0.09	0.11	9.14	5026481.20	9.14	1365.52	101.00
United States	40668.24	40668.24	40668.24	1.00	1.00	1.00	549854.01	1.00	13.52	1.00

Let K, N denote the aggregate factors of production used in the economy, N_f the number of firms, then once we consider firms as a reproducible factor we can recover an homogenized aggregate production function. We can write GDP in our model economy as:

$$Y = \left(K^{\alpha} N^{1-\alpha}\right)^{\eta} \left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)^{1-\eta} N_f^{1-\eta}$$

We get then a transformation of the average firm productivity $\left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$ that can be calculated:

$$\left[\frac{Y}{(K^{\alpha} N^{1-\alpha})^{\eta}} \right]^{\frac{1}{1-\eta}} \frac{1}{N_f} = \left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$$

Using US GDP in 2010, the number of establishments in 2010 from BDS, capital stock and number of employees in USA we get a value for $\left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$. Assuming that all firms have the same productivity then: $\bar{z} N_f = \sum_i z_i^{\frac{1}{1-\eta}} = z_i^{\frac{1}{1-\eta}} N_f$ and $\bar{z}^{1-\eta} = z_i$. In our calibration: $z_i^{US} = 170089$

C Appendix: Measurement of Financial Frictions

This appendix provides detailed derivations of the relationships between funding gaps and outcome distortions in our model.

C.1 Outcome Distortions as Functions of Funding Gaps

We define the total distortion in variable x as $\Lambda^x = \sum_{a=0}^{\infty} \Lambda_a^x$, where the period-specific distortions are:

$$\Lambda_a^n = \rho^a[n^u(K, z, 1, p_k, \alpha) - n_a(K, z, \xi, p_k, \alpha)]$$
$$\Lambda_a^\pi = (\beta\rho)^a[\pi^u(K, z, 1, p_k, \alpha) - \pi_a(K, z, \xi, p_k, \alpha)]$$
$$\Lambda_a^V = (\beta\rho)^a[V^u(K, z, 1, p_k, \alpha) - V_a(K, z, \xi, p_k, \alpha)]$$

C.1.1 Employment Distortion

The employment distortion directly follows from the working capital constraint that labor needs to be paid in advance:

$$\Lambda^n = \sum_{a=0}^{\infty} \rho^a[n^u(K, z, 1, p_k, \alpha) - n_a(K, z, \xi, p_k, \alpha)] \tag{40}$$
$$= \sum_{a=0}^{\infty} \rho^a \left[\frac{S^u(K, z, 1, p_k, \alpha)}{w} - \frac{S_a(K, z, \xi, p_k, \alpha)}{w}\right] \tag{41}$$
$$= \frac{1}{w} \sum_{a=0}^{\infty} \rho^a[S^u(K, z, 1, p_k, \alpha) - S_a(K, z, \xi, p_k, \alpha)] \tag{42}$$
$$= \frac{1}{w} \frac{1}{1-\rho} \Gamma^{S,a} \tag{43}$$

C.1.2 Investment Distortion

The investment distortion equals the long-term funding gap:

$$\Lambda^K = p_k[K(z,1,p_k,\alpha) - K(z,\xi,p_k,\alpha)] = \Gamma^L \tag{44}$$

C.1.3 Profit Distortion

The profit distortion is more complex and involves both the observable funding gap and the target distortion:

$$\Lambda^\pi = \sum_{a=0}^{\infty}(\beta\rho)^a[\pi^u - \pi_a] \tag{45}$$

$$= \sum_{a=0}^{\hat{a}-1}(\beta\rho)^a[\pi^u(1,K(1),\cdot) - \pi_a(\xi,K(\xi),\cdot)] + \frac{(\beta\rho)^{\hat{a}}}{1-\beta\rho}[\pi^u(1,K(1),\cdot) - \pi_a(1,K(\xi),\cdot)] \tag{46}$$

For the second term, we exploit the homogeneity of our production function:

$$\pi^u(K) = [1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \tag{47}$$

Therefore:

$$[\pi^u(1,K(1),\cdot) - \pi_a(1,K(\xi),\cdot)] = [1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}}\left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \tag{48}$$

$$\times\left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] \tag{49}$$

This can be expressed in terms of the short-term debt target distortion:

$$[\pi^u(1, K(1), \cdot) - \pi_a(1, K(\xi), \cdot)] = [1 - \eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega^S(K(\xi, \cdot)) \tag{50}$$

For the first term, we analyze $[\pi^u(1, K(\xi)) - \pi_a(\xi, K(\xi))]$:

$$[\pi^u(1, K(\xi)) - \pi_a(\xi, K(\xi))] = zK(\xi)^{\eta\alpha}\left(\frac{S(1, K(\xi), \cdot)}{w}\right)^{\eta(1-\alpha)} - (1+r_f)S(1, K(\xi), \cdot) \tag{51}$$

$$- zK(\xi)^{\eta\alpha}\left(\frac{S_a(\xi, K(\xi), \cdot)}{w}\right)^{\eta(1-\alpha)} + (1+r_f)S_a(\xi, K(\xi), \cdot) \tag{52}$$

$$= zK(\xi)^{\eta\alpha}\left(\frac{S(1, K(\xi), \cdot)}{w}\right)^{\eta(1-\alpha)}\left[1 - \left(\frac{S_a(\xi, K(\xi), \cdot)}{S(1, K(\xi), \cdot)}\right)^{\eta(1-\alpha)}\right] \tag{53}$$

$$- (1+r_f)[S(1, K(\xi), \cdot) - S_a(\xi, K(\xi), \cdot)] \tag{54}$$

Recognizing that $[S(1, K(\xi), \cdot) - S_a(\xi, K(\xi), \cdot)] = \tilde{\Gamma}^{S,a}(K(\xi, \cdot))$:

$$[\pi^u(1, K(\xi)) - \pi_a(\xi, K(\xi))] = zK(\xi)^{\eta\alpha}\left(\frac{S(1, K(\xi), \cdot)}{w}\right)^{\eta(1-\alpha)}\left[1 - \left(1 - \frac{\tilde{\Gamma}^{S,a}(K(\xi, \cdot))}{S(1, K(\xi), \cdot)}\right)^{\eta(1-\alpha)}\right] \tag{55}$$

$$- (1+r_f)\tilde{\Gamma}^{S,a}(K(\xi, \cdot)) \tag{56}$$

Defining $y^u(\xi, \cdot) = zK(\xi)^{\eta\alpha}\left(\frac{S(1,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)}$ as the revenue of a firm at the end of the ELC, we can express the total profit distortion as:

$$\Lambda^{\pi} = \sum_{a=0}^{\hat{a}-1} (\beta\rho)^a \left[y^u(\xi,\cdot) \left(1 - \left(1 - \frac{\tilde{\Gamma}^{S,a}(K(\xi,\cdot))}{S(1,K(\xi,\cdot),\cdot)} \right)^{\eta(1-\alpha)} \right) - (1+r_f)\tilde{\Gamma}^{S,a}(K(\xi,\cdot)) \right] \quad (57)$$

$$+ \frac{[1-\eta(1-\alpha)]}{1-\beta\rho}(1+r_f)\frac{1}{\eta(1-\alpha)}\Omega^S(K(\xi,\cdot)) \quad (58)$$

C.1.4 Firm Value Distortion

Building on the profit distortion, the firm value distortion is:

$$\Lambda^V = \Lambda^{\pi} + \frac{(\beta\rho)^{\hat{a}}}{1-\beta\rho}\beta(1-\rho)\Gamma^L \quad (59)$$

The second term represents the distortion in firm value due to the scale distortion, which persists beyond the ELC.

C.2 Detailed Decomposition of the Short-Term Funding Gap

The observable short-term funding gap can be expressed as:

$$\tilde{\Gamma}^S(K(\xi,\cdot)) = S^u(K(\xi,\alpha,z),\alpha,z) \left[1 - \left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi} \right]^{\frac{1}{\eta(1-\alpha)}} \left\{ \omega_0 + K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z) \right\} \right] \quad (60)$$

where $\omega_0 = [1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$.

The non-monotonicity of the observable funding gap with respect to enforcement quality ξ results from four competing effects:

C.2.1 Direct Effect on Borrowing Constraint

Increasing ξ directly reduces the tightness of the borrowing constraint for a given scale $K(\xi, \cdot)$:

$$-\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}S^u(K(\xi,\cdot),\cdot)<0 \tag{61}$$

This negative effect reduces the funding gap as enforcement improves.

C.2.2 Indirect Effect on Target Funds

Increasing ξ raises $K(\xi, \cdot)$ and consequently increases the target funds:

$$\frac{\partial S^u(K(\xi,\cdot),\cdot)}{\partial K(\xi,\alpha,z)}\frac{\partial K}{\partial \xi}\left[1-\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\right]>0 \tag{62}$$

This positive effect increases the funding gap as enforcement improves.

C.2.3 Effect on Debt Maturity and ELC Duration

Increasing ξ raises debt maturity T and potentially ELC duration $\hat{a}(\xi)$ due to increased long-term debt:

$$\frac{\partial \hat{a}(\xi)}{\partial \xi}\ln(\frac{1}{\beta\rho})(\beta\rho)^{\hat{a}(\xi)-a}\frac{1}{1-\beta\rho}\frac{1}{1-\xi}\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}} \tag{63}$$

$$\times\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}S^u(K(\xi,\cdot),\cdot)\geq 0 \tag{64}$$

This non-negative effect can increase the funding gap as enforcement improves.

C.2.4 Indirect Effect on Borrowing Constraint through Value

Increasing ξ raises $K(\xi, \cdot)$ and reduces the borrowing constraint due to a higher firm value:

$$- \frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]} S^u(K(\xi,\alpha,z),\alpha,z) \left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \tag{65}$$

$$\times K(\xi,\cdot)^{\frac{1-\eta-\eta(1-\alpha)[1-\eta(1-\alpha)]}{\eta(1-\alpha)[1-\eta(1-\alpha)]}} \omega_1(z) < 0 \tag{66}$$

This negative effect reduces the funding gap as enforcement improves.

The interplay between these four effects explains why the observable funding gap is not monotonic in enforcement quality. The relative strength of these effects depends on the capital intensity α of the industry, creating different patterns across industries.

C.3 Inferring Target Distortion from Observables

Using a benchmark economy ($\xi = 1$), the unobservable target distortion can be inferred from any of the following three moments:

$$\Omega^S(K(\xi,\cdot)) = \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \tag{67}$$

$$= \left[\left[\frac{L_0(1)}{p_k(1+r_f)}\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - \left[\frac{L_0(\xi)}{p_k(1+r_f)}\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \tag{68}$$

$$= w[n^u(1,K(1),\alpha,z) - n^u(1,K(\xi),\alpha,z)] \tag{69}$$

While initial scale K and long-term debt L_0 are difficult to measure accurately for young

firms, employment levels at the end of the ELC provide a practical alternative for empirical work.

C.4 Proof of Outcome Gaps as Functions of Funding Gaps

We derive the relationship between funding gaps and outcome distortions in our model.

C.4.1 Employment and Investment Distortions

The employment and investment distortions follow directly from the definitions:

$$\begin{aligned}\Lambda^n &= \frac{1}{w}\frac{1}{1-\rho}[S(K(1,\cdot),z,1,p_k,\alpha) - S_a(K(\xi,\cdot),z,\xi,p_k,\alpha)] \\ &= \frac{1}{w}\frac{1}{1-\rho}\Gamma^S \end{aligned} \tag{70}$$

$$\begin{aligned}\Lambda^K &= p_k[K(z,1,p_k,\alpha) - K(z,\xi,p_k,\alpha)] \\ &= \Gamma^L \end{aligned} \tag{71}$$

C.4.2 Profit Distortion

The profit distortion is more complex:

$$\begin{aligned}\Lambda^\pi &= \sum_{a=0}^{\infty}(\beta\rho)^a[\pi^u - \pi_a] \\ &= \sum_{a=0}^{\hat{a}-1}(\beta\rho)^a[\pi^u(1,K(1),\cdot) - \pi_a(\xi,K(\xi),\cdot)] + \frac{(\beta\rho)^{\hat{a}}}{1-\beta\rho}[\pi^u(1,K(1),\cdot) - \pi_a(1,K(\xi),\cdot)] \end{aligned} \tag{72}$$

For the second term, we exploit the homogeneity of our production function to express unconstrained profits as:

$$\pi^u(K) = [1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \tag{73}$$

Therefore:

$$\begin{aligned}[\pi^u(1,K(1),\cdot) - \pi_a(1,K(\xi),\cdot)] = [1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}} \left(\frac{\eta(1-\alpha)}{(1+r_f)w}\right)^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \\ \times \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right]\end{aligned} \tag{74}$$

We can express this in terms of the short-term debt target distortion:

$$[\pi^u(1,K(1),\cdot) - \pi_a(1,K(\xi),\cdot)] = [1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega^S(K(\xi,\cdot)) \tag{75}$$

For the first term in the profit distortion, we analyze $[\pi^u(1,K(\xi)) - \pi_a(\xi,K(\xi))]$:

$$\begin{aligned}[\pi^u(1,K(\xi)) - \pi_a(\xi,K(\xi))] &= zK(\xi)^{\eta\alpha}\left(\frac{S(1,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)} - (1+r_f)S(1,K(\xi),\cdot) \\ &\quad - zK(\xi)^{\eta\alpha}\left(\frac{S_a(\xi,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)} + (1+r_f)S_a(\xi,K(\xi),\cdot) \\ &= zK(\xi)^{\eta\alpha}\left(\frac{S(1,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left(\frac{S_a(\xi,K(\xi),\cdot)}{S(1,K(\xi),\cdot)}\right)^{\eta(1-\alpha)}\right] \\ &\quad - (1+r_f)[S(1,K(\xi),\cdot) - S_a(\xi,K(\xi),\cdot)]\end{aligned} \tag{76}$$

Recognizing that $[S(1,K(\xi),\cdot) - S_a(\xi,K(\xi),\cdot)] = \tilde{\Gamma}^{S,a}(K(\xi,\cdot))$, we have:

$$[\pi^u(1,K(\xi)) - \pi_a(\xi,K(\xi))] = zK(\xi)^{\eta\alpha}\left(\frac{S(1,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left(1-\frac{\tilde{\Gamma}^{S,a}(K(\xi,\cdot))}{S(1,K(\xi),\cdot)}\right)^{\eta(1-\alpha)}\right]$$
$$-(1+r_f)\tilde{\Gamma}^{S,a}(K(\xi,\cdot)) \tag{77}$$

Defining $y^u(\xi,\cdot) = zK(\xi)^{\eta\alpha}\left(\frac{S(1,K(\xi),\cdot)}{w}\right)^{\eta(1-\alpha)}$ as the revenue of a firm at the end of the ELC, we can express the total profit distortion as:

$$\Lambda^\pi = \sum_{a=0}^{\hat{a}-1}(\beta\rho)^a\left[y^u(\xi,\cdot)\left(1-\left(1-\frac{\tilde{\Gamma}^{S,a}(K(\xi,\cdot))}{S(1,K(\xi,\cdot),\cdot)}\right)^{\eta(1-\alpha)}\right)-(1+r_f)\tilde{\Gamma}^{S,a}(K(\xi,\cdot))\right]$$
$$+\frac{[1-\eta(1-\alpha)]}{1-\beta\rho}(1+r_f)\frac{1}{\eta(1-\alpha)}\Omega^S(K(\xi,\cdot)) \tag{78}$$

C.4.3 Firm Value Distortion

Building on the profit distortion, the firm value distortion incorporates the effect on firm value:

$$\Lambda^V = \Lambda^\pi + \frac{(\beta\rho)^{\hat{a}}}{1-\beta\rho}\beta(1-\rho)\Gamma^L \tag{79}$$

The second term represents the distortion in firm value due to the scale distortion, which persists beyond the ELC.

C.4.4 Short-Term Funding Gap Decomposition

We can decompose the short-term funding gap as:

$$\Gamma^{S,a} = S(K(1,\cdot),z,1,p_k,\alpha) - S_a(K(\xi,\cdot),z,\xi,p_k,\alpha) \tag{80}$$

$$= S(K(1,\cdot),z,1,p_k,\alpha) - S(K(\xi,\cdot),z,1,p_k,\alpha) \tag{81}$$

$$+ S(K(\xi,\cdot),z,1,p_k,\alpha) - S_a(K(\xi,\cdot),z,\xi,p_k,\alpha) \tag{82}$$

$$= \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \tag{83}$$

$$+ S_a(K(\xi,\cdot),z,1,p_k,\alpha) - S_a(K(\xi,\cdot),z,\xi,p_k,\alpha) \tag{84}$$

$$= \Omega^S(K(\xi,\cdot)) + \tilde{\Gamma}^{S,a}(K(\xi,\cdot)) \tag{85}$$

The third line follows from the property that firms converge to be unconstrained when they mature: $S(K(\xi,\cdot),z,1,p_k,\alpha) = S_a(K(\xi,\cdot),z,\xi,p_k,\alpha)$ for $a \geq \hat{a}(\xi)$.

The observable short-term funding gap can be written as:

$$\tilde{\Gamma}^S(K(\xi,\cdot)) = S^u(K(\xi,\alpha,z),\alpha,z)\left[1 - \left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\right] \tag{86}$$

where $\omega_0 = [1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$.

C.4.5 Effects of Changes in Enforcement Quality

Increasing the level of development (ξ) induces four effects:

1. A direct effect reducing the tightness of the borrowing constraint for a given scale

$K(\xi,\cdot)$:

$$-\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}S^u(K(\xi,\cdot),\cdot)<0 \tag{87}$$

2. An indirect effect that increases $K(\xi,\cdot)$ and the target funds:

$$\frac{\partial S^u(K(\xi,\cdot),\cdot)}{\partial K(\xi,\alpha,z)}\frac{\partial K}{\partial \xi}\left[1-\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\right]>0 \tag{88}$$

3. An increase in debt maturity T and potentially in $\hat{a}(\xi)$ due to increased long-term debt:

$$\frac{\partial \hat{a}(\xi)}{\partial \xi}\ln(\frac{1}{\beta\rho})(\beta\rho)^{\hat{a}(\xi)-a}\frac{1}{1-\beta\rho}\frac{1}{1-\xi}\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}S^u \tag{89}$$

4. An indirect effect that increases $K(\xi,\cdot)$ and reduces the borrowing constraint due to higher firm value:

$$-\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}S^u(K(\xi,\alpha,z),\alpha,z)\left[\frac{(\beta\rho)^{\hat{a}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}K(\xi,\cdot)^{\frac{1-\eta-\eta(1-\alpha)[1-\eta(1-\alpha)]}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)<0 \tag{90}$$

These competing effects explain why the relationship between enforcement quality and the observable funding gap is non-monotonic, particularly in high capital intensity industries.

C.5 Alternative Expressions for Target Distortion

Using a benchmark economy ($\xi = 1$), the unobservable target distortion can be inferred from initial scale K, initial long-term debt (L_0), or mature size (n^u):

$$\Omega^S(K(\xi,\cdot)) = \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \tag{91}$$

$$= \left[\left(\frac{L_0(1)}{p_k(1+r_f)}\right)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - \left(\frac{L_0(\xi)}{p_k(1+r_f)}\right)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \tag{92}$$

$$= w[n^u(1,K(1),\alpha,z) - n^u(1,K(\xi),\alpha,z)] \tag{93}$$

While data at firm inception is difficult to obtain, employment at the end of a firm's early life cycle provides a viable alternative for measuring the severity of financial constraints.