Dear Conference Organizers,

Please find enclosed our submission for consideration at your upcoming conference.

The attached materials include a presentation and a paper titled "Self-Sustainability."

This is a work in progress, and we are eager to receive feedback as we continue to develop the project.

The presentation appears first in the bundle, as it reflects our most recent thinking and results.

We include a preliminary draft that includes technical details and derivations.

Thank you for your time and consideration.

A.I., Robotics, and the Transition to Self-Sustainability: From Neoclassical Growth to $Y = AK$

The Big Picture: Why This Matters

- The global economy is shifting as automation and AI increasingly replace human labor.
- Traditional economic models assume a balance between labor and capital, but this balance is changing.
- As robotic capital grows, economies may transition from labor-driven growth to capital-driven growth.
- **This transition is already happening.** Empirical trends show measurable changes in factor shares and technological adoption.
- **The transition is continuous, not discrete.** As labor's role declines, economic dynamics evolve, requiring adjustments in forecasting and policy design.

What is Self-Sustainability?

- Self-sustainability occurs when capital accumulates independently of human labor constraints, allowing robots to replicate and expand indefinitely.
- If the **elasticity of substitution** $\sigma > 1$, there exists a threshold level of **capital-augmenting technological progress** $\bar{A}_K$, beyond which the economy transitions to self-sustainability. In this regime:
  - Labor's share of output converges to zero, and production follows $Y = AK$.
  - Capital accumulation is unconstrained by labor supply and grows endogenously.
  - Growth and interest rates depend on the **level** of technological progress, not its **rate**.

Self-Sustainability vs. Economic Singularity & Intelligence Explosion

Self-Sustainability vs. Economic Singularity:

- Both SS and ES describe self-reinforcing economic dynamics but differ fundamentally.
- **Self-Sustainability (SS):** Capital accumulates indefinitely without labor, but growth remains bounded by technological progress.
- **Economic Singularity (ES, Nordhaus):** Growth rate itself accelerates indefinitely as $A_K \to \infty$, breaking standard economic models.

Common Theme: Self-Perpetuating Progress Without Human Labor

- **Intelligence Explosion (IE):** AI recursively improves itself, leading to exponential intelligence gains.
- **SS & IE:** In both cases, capital accumulation is no longer constrained by the labor force.

Empirical Motivation: The Transition is Underway

Key Observations:

- The transition requires two conditions: $\sigma > 1$ and sustained growth in A_K.
- Recent studies increasingly support $\sigma > 1$. Even if σ is currently below one, rapid advances in A.I. and robotics suggest it may exceed this threshold soon.
- Observed declines in labor's share and rising capital's share suggest capital is becoming the dominant production factor.

Why Focus on the Capital Share?

- The paper introduces an index measuring relative progress toward self-sustainability, defined only for $\sigma > 1$:

$$\chi^* = \left(\frac{A_K}{\bar{A}_K} \right)^{\sigma - 1}$$

- In steady-state, this index equals the capital income share, making it an (imperfectly) observable measure of progress toward self-sustainability.

Empirical Evidence: Capital's Share Over Time

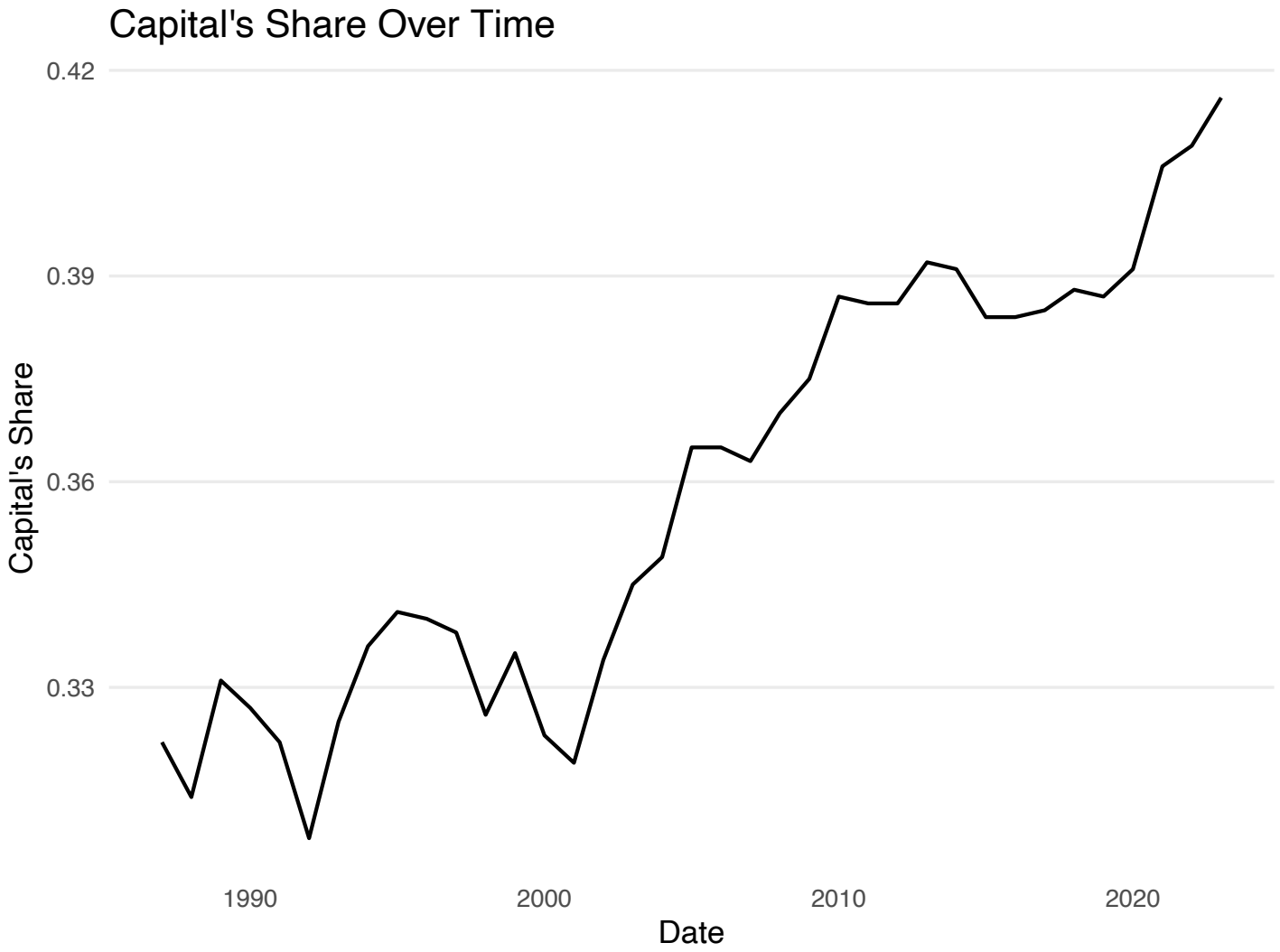


Figure: Private Nonfarm Business Sector: Capital Share. Source: FRED dataset.

Understanding the Transition and Why It Matters Now

The transition is already reshaping the economy.

- Standard models focus on either neoclassical growth or $Y = AK$, overlooking the transition.
- As χ^* rises, the economy increasingly resembles $Y = AK$.
- The decline in labor's share is just the first structural shift.
- Even for $\chi^* < 1$, output growth, interest rates, and factor returns diverge from neoclassical predictions.

Key Economic Dynamics:

- Small increases in capital-augmenting technology amplify output and interest rate changes.
- Sensitivity to capital-augmenting progress increases as labor's role declines.

Policy Implications:

- Existing policies were designed for labor-driven economies.
- As the economy shifts, these policies lose effectiveness.
- **Key Questions:** How should policy adapt during the transition, and how should economic forecasts be updated?

Benchmark Model

Baseline Framework for Growth Analysis

The Benchmark Model: Production Function

$$Y = \left(\alpha(A_K K)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right)^{\frac{1}{\phi}}$$

- Output Y is produced using capital K and labor L.
- A_K and A_L are capital- and labor-augmenting technological parameters.
- α determines the relative weight of capital in production.
- $\sigma = \frac{1}{1-\phi}$ is the elasticity of substitution.

Special Cases:

- $\phi \to 0$: Neoclassical regime. Increasing A_K has the same effect on equilibrium as increasing A_L.
- $\phi \to 1$: In the long run, depending on A_K, the economy may converge to a regime where either capital or labor dominates.
- For any fixed $\phi \in (0, 1)$, depending on A_K, the economy may resemble the neoclassical regime, the capital-dominated regime, or neither.

Household Models: Why Different Approaches?

- There is no single perfect framework to analyze equilibrium in an economy with capital-augmenting progress.
- We use three approaches, each highlighting different aspects:
  - **Representative Agent (RA) Model:** Provides tractable equilibrium analysis but requires fixing A_K to avoid asymptotic divergence in interest rates.
  - **Overlapping Generations (OLG) Model:** Handles growing A_K naturally but leads to different long-run interest rate dynamics.
  - **Fixed Savings Model:** A continuous-time approximation of OLG with logarithmic utility, providing a clear but simplified analysis.
- We begin with the RA model, assuming **fixed** A_K, to analyze equilibrium conditions before introducing models with varying A_K.
- We use the RA model to examine how changes in A_K and ϕ influence short-run equilibrium, long-run dynamics, and the speed of convergence.

Representative Agent Model & Euler Equation

Household Preferences and Budget Constraint:

- A representative agent maximizes lifetime utility:

$$U = \int_0^\infty e^{-\rho t} \frac{C^{1-\theta} - 1}{1-\theta} dt$$

where:

- C is consumption,
- ρ is the discount rate,
- θ is the inverse of the intertemporal elasticity of substitution.

- Budget constraint:

$$\dot{K} = Y - C - \delta K$$

where δ is the depreciation rate.

Euler Equation:

- The optimal consumption path is given by the Euler equation:

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho)$$

where r is the interest rate.

Equilibrium: Capital Abundance Ratio

Definition:

- The **capital abundance ratio** is:

$$\kappa = \frac{A_K K}{A_L L}.$$

- κ is **not just another form of** $k = K/A_L L$, misinterpreting it will lead to counterintuitive results.

Why is κ not defined for Cobb-Douglas ($\phi = 0$)?

- When $\phi = 0$, production depends only on $A = A_K^{\alpha} A_L^{1-\alpha}$, making κ irrelevant.
- A_K and A_L do not separately influence equilibrium, unlike in the transition phase.

Why does κ matter when $\phi > 0$?

- For $\phi > 0$, production depends separately on A_K and A_L, making κ essential.

Should κ be defined in Cobb-Douglas?

- Defining $\kappa = 1$ or $\kappa = K/(A_L L)$ in Cobb-Douglas is arbitrary and adds no insight.

Equilibrium Conditions

$$y = \frac{Y}{A_L L} = \left[\alpha\kappa^{\phi} + (1-\alpha)\right]^{\frac{1}{\phi}}, \tag{1}$$

$$w = \frac{W}{A_L} = (1-\alpha)\left[\alpha\kappa^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}, \tag{2}$$

$$i + \delta = \alpha A_K \left[\alpha + (1-\alpha)\kappa^{-\phi}\right]^{\frac{1-\phi}{\phi}}, \tag{3}$$

$$\dot{\kappa} = A_K(y - c) - (g_{A_L} + \delta)\kappa, \tag{4}$$

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(i - \rho) - g_{A_L}. \tag{5}$$

Key Distinction from the Neoclassical Growth Model:

- A_K directly affects both capital returns (3) and savings (4).
- This difference drives the key results of the papergoverning growth dynamics and equilibrium responses to technological change.

How A_K Affects Capital Dynamics

Key Distinction from the Neoclassical Model:

- Unlike in the standard model, A_K affects both the interest rate and capital accumulation.
- To understand equilibrium behavior, we examine how this changes capital dynamics.

Capital Accumulation Condition:

- From the standard phase diagram and Euler equation, capital accumulates if the return on capital exceeds its critical level:

$$MP_K > i^* + \delta.$$

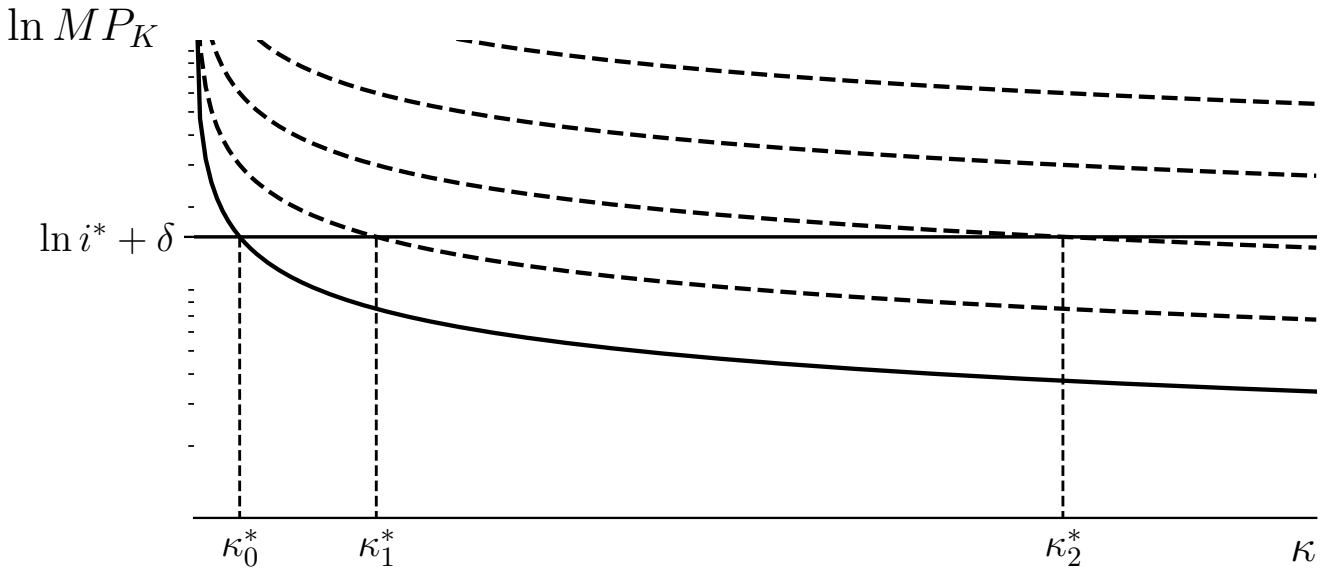
- Substituting for the critical interest rate threshold i^*, this condition becomes:

$$MP_K = i + \delta > \theta g_{A_L} + \rho + \delta.$$

- To determine equilibrium behavior, we now analyze how A_K shifts MP_K.

Sequential $\times 1.5$ Increases of A_K

$$MP_K = \alpha A_K \left[\alpha + (1-\alpha)\kappa^{-\phi}\right]^{\frac{1-\phi}{\phi}}$$



- We assume $\phi > 0$, so $\lim_{\kappa\to\infty} MP_K = \alpha^{\frac{1}{\phi}} A_K > 0$.
- A threshold $A_K^{\text{crit}} = (i^* + \delta)/\alpha^{\frac{1}{\phi}}$ exists, such that if $A_K \geq A_K^{\text{crit}}$, capital accumulates indefinitely.
- As A_K increases, its impact on κ^* grows due to the flattening of ln MP_K.

Understanding Self-Sustainability

What is Self-Sustainability?

- In equilibrium, robots can replicate and accumulate indefinitely without being constrained by human labor.
- This occurs when capital accumulation is self-perpetuating, requiring:

$$A_K \geq \bar{A}_K.$$

Key Condition:

- If capital accumulation is possible in principle but offers a low return, firms may not invest.
- For capital to actually accumulate, it must be both feasible and profitable.
- This requires:

$$\bar{A}_K = \frac{i^* + \delta}{\alpha^{\frac{1}{\phi}}}.$$

- The return on capital must exceed both the required rate of return i^* and depreciation δ.

Measuring the Distance to Self-Sustainability

Defining the Self-Sustainability Frontier

- The threshold for self-sustainability depends on preferences and technology:

$$\bar{A}_K = \frac{i^* + \delta}{\alpha^{\frac{1}{\phi}}}.$$

- Preferences determine i^*, while A_K, ϕ, α, and δ define the technological conditions.

Key Observations:

- A_L is irrelevant—an increase in A_L or L expands the economy's scale but does not affect proximity to self-sustainability.
- The distance to self-sustainability is multi-dimensional, depending on A_K, ϕ, α, and δ.

Why Define a Single Measure?

- Advances in robotics affect multiple parameters, making the transition complex.
- A single index is needed to quantify progressthis is the role of the Self-Sustainability Index χ^*.

Defining the Steady-State Self-Sustainability Index χ^*

Why Introduce χ^*?

- χ^* measures how far the economy is from the self-sustainability phase.
- It summarizes key technological and preference parameters into a single index.

Definition:

- The steady-state Self-Sustainability Index is:

$$\chi^* \equiv \left(\frac{A_K}{\bar{A}_K} \right)^{\frac{\phi}{1-\phi}}.$$

- Self-sustainability occurs when $\chi^* \geq 1$, meaning capital accumulates indefinitely.

Dynamic Self-Sustainability Index: χ versus χ^*

From the Steady-State Index to the Dynamic Index

- The steady-state Self-Sustainability Index is:

$$\chi^* \equiv \alpha^{\frac{1}{1-\phi}} \left(\frac{A_K}{i^* + \delta} \right)^{\frac{\phi}{1-\phi}}.$$

- Since the actual interest rate i fluctuates due to economic conditions, we define the Dynamic Self-Sustainability Index:

$$\chi \equiv \alpha^{\frac{1}{1-\phi}} \left(\frac{A_K}{i + \delta} \right)^{\frac{\phi}{1-\phi}}.$$

- χ convergence to $\min(\chi^*, 1)$.
- χ is observable.

Economic Interpretation of χ

1. **χ as a Relative Technological Index:**
   - By definition, χ measures the relative level of capital-augmenting technology.
2. **χ as an Index of Capital Abundance:**
   - In the equilibrium:

$$\kappa^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}.$$

3. **χ as the Capital Income Share:**
   - In equilibrium:

$$\text{Capital share} = \chi, \quad \text{Labor share} = 1 - \chi.$$

Three Phases of Automation

Automation progresses through three distinct phases:

- **Labor-Dominant Phase** ($\chi \leq \alpha$): Labor remains the primary input, and capital plays a secondary role.
- **Transition Phase** ($\alpha < \chi < 1$): Capital gains importance, but labor still plays a significant role.
- **Self-Sustainability Phase** ($\chi \geq 1$): Capital accumulates indefinitely, and labor's role vanishes.

Relation to the Neoclassical Growth Regime:

- In the neoclassical growth model (Cobb-Douglas, $\phi = 0$), the capital share is fixed at α, but χ is not defined.

Focus of the Analysis:

- The primary focus is the **transition phase**.
- The economy reacts similarly to A_K in the labor-dominant and transition phases but responds differently to ϕ.

Short-Run vs. Long-Run Equilibrium

Short-Run Equilibrium (SR)

- Capital K is fixed, and the interest rate $i + \delta$ adjusts to clear the market.

Long-Run Equilibrium (LR)

- If $\chi^* < 1$, the interest rate is fixed at $i = i^*$, and capital K adjusts endogenously.
- If $\chi^* \geq 1$, capital accumulates indefinitely ($\kappa \to \infty$), and the long-run interest rate becomes an increasing function of A_K.

Methodology: How SR and LR Responses Depend on χ^*

What We Analyze:

- How the short-run and long-run responses of output (y), wages (w), and interest rates (i) depend on A_K and ϕ.
- How the convergence rate changes as the economy transitions toward self-sustainability.

Key Question:

- How do the elasticities of y, w, and i evolve with χ^*?
- How does the convergence rate depend on the state of robotization, measured by χ^*?

Short-Run and Long-Run Elasticities with Respect to A_K

Short-Run Elasticities (for $\chi^* < 1$)

$$\varepsilon^{SR}_{y,A_K} = \chi, \quad \varepsilon^{SR}_{w,A_K} = \chi(1-\phi), \quad \varepsilon^{SR}_{i+\delta,A_K} = \phi + (1-\phi)\chi.$$

Long-Run Elasticities (for $\chi^* < 1$)

$$\varepsilon^{LR}_{y,A_K} = \frac{1}{1-\phi} \cdot \frac{\chi^*}{1-\chi^*}, \quad \varepsilon^{LR}_{w,A_K} = \frac{\chi^*}{1-\chi^*}, \quad \varepsilon^{LR}_{i+\delta,A_K} = 0.$$

Key Observations:

- All elasticities (except $i + \delta$ in LR) **increase** with χ or χ^*.
- As $\chi^* \to 1$, the long-run elasticities of y and w **tend to infinity**.

The Transition Toward Self-Sustainability

Amplification of A_K's Effects Over Time

- When $\chi^* < 1$, each increase in A_K causes **larger and larger** increases in y and w.
- As $\chi^* \to 1$, these effects become extreme, leading to long-run elasticities of y and w **diverging to infinity**.

Natural Transition to the Self-Sustainability Phase

- When $\chi^* \geq 1$, the **level** of y and w grows indefinitely.
- Instead of just increasing the **level**, A_K now influences the **growth rate itself**.
- The self-sustainability regime is the **natural continuation** of this processwhat was previously an extreme response to A_K now defines the economic structure.

How Fast Does the Economy Adjust?

Convergence slows as χ increases:

- When χ is small, deviations from equilibrium fade quickly.
- When χ is large, adjustments take longer, and shocks persist.
- As $\chi \to 1$, temporary effects last indefinitely, shaping long-run dynamics.

Why does this happen?

- The exact convergence equation shows that the adjustment speed declines as $\chi \to 1$.
- The mathematical proof and detailed derivation are provided in the appendix.

How Does A_K Shape Economic Dynamics as χ Grows?

Effect on Output and Wages (y, w)

- As χ increases, the long-run effect of A_K on y and w **amplifies**.
- Growth effects persist longer due to slower convergence.
- When $\chi^* \geq 1$, output and wages **grow indefinitely** with A_K.

Effect on Interest Rates ($i + \delta$)

- In the short run, A_K increasingly affects interest rates as χ rises.
- As $\chi \to 1$, short-run effects last longer.
- When $\chi^* \geq 1$, the effect of A_K on interest rates becomes **permanent**.

Why This Matters:

- As χ grows, the economy **increasingly resembles** the self-sustainability phase.
- Understanding self-sustainability is not just about the distant futureit helps explain today's dynamics.

How ϕ Affects Output, Wages, and Interest Rates

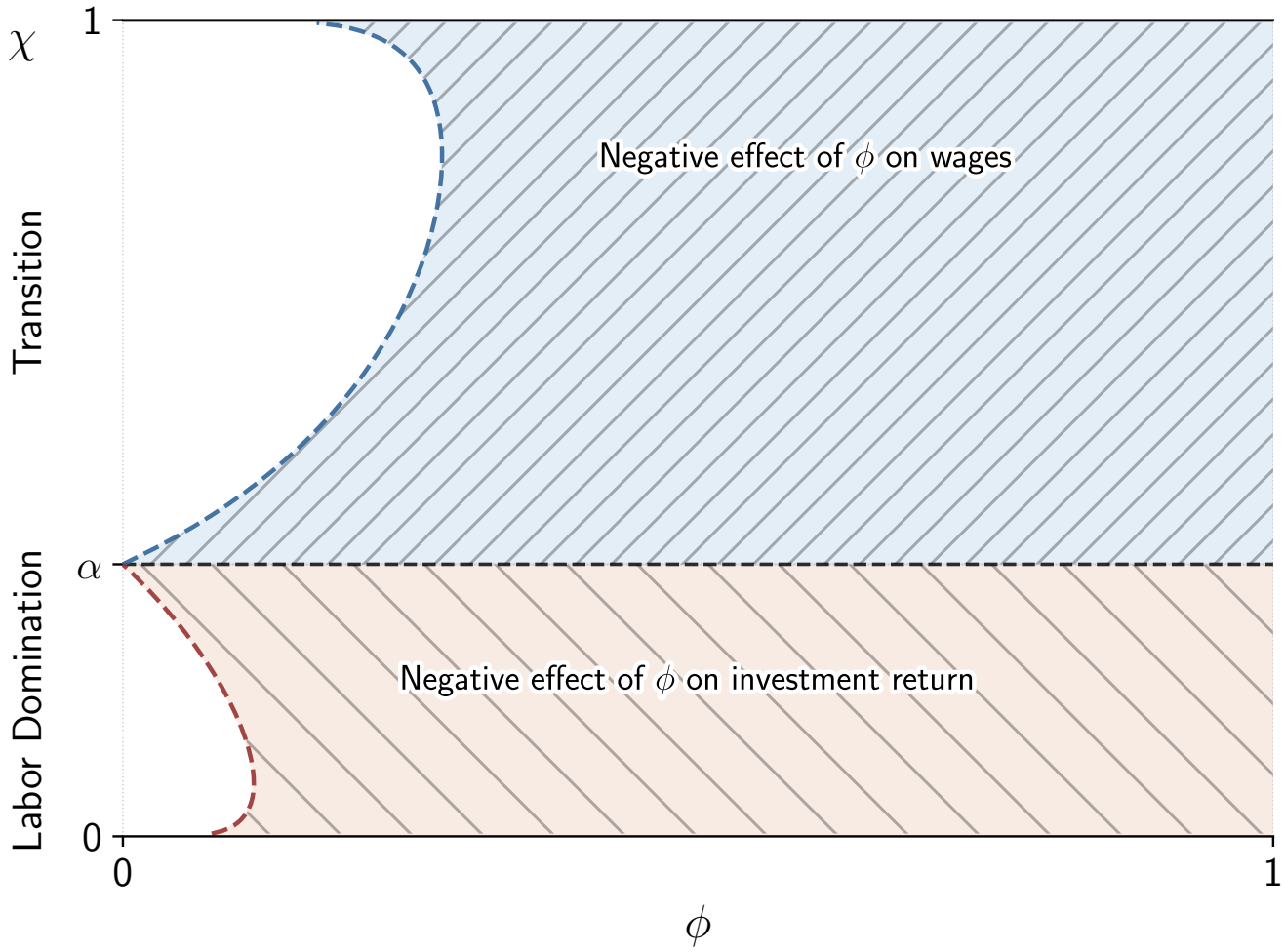
Effect of ϕ on Output (y)

- In the short run, increasing ϕ does not reduce output.
- Output remains constant only when $\chi = \alpha$ (equivalently, $\kappa = 1$).
- In the long run, if $\chi < \alpha$, y may decline due to capital deaccumulation.

Effect of ϕ on Wages (w) and Interest Rates (i)

- If $\chi < \alpha$, robots are still relatively scarce, so labor substitutes robots. As ϕ rises, **wages increase**.
- If $\chi > \alpha$, robots dominate production, so capital substitutes labor. As ϕ rises, **interest rates increase**.
- Otherwise, the effect depends on ϕ.

When Does ϕ Reduce Wages or Interest Rates?



- In labor domination phase, increasing ϕ may increase or reduce return on capital.
- In the transition phase, increasing ϕ may increase or reduce wages.

The Warehouse Example: Understanding ϕ and Automation

Why the Warehouse Example?

- Demonstrates why the response of w and i to ϕ depends on the automation phase.
- Explains how substitutability between robots and labor affects wages and interest rates.

Example Setup:

- Warehouses are optimized for either human workers or robots.
- Robot-adapted warehouses use rigid shelving and automation; human-adapted warehouses allow manual adjustments.

Impact of Increasing ϕ

- Warehouses are redesigned to accommodate both humans and robots, increasing substitutability.
- This change occurs without improving robot or labor productivity (A_K and A_L remain fixed).
- The effect depends on the technological level of robots, represented by A_K.

Applying the Warehouse Example to Wages and Interest Rates

Effect on Wages (w) and Interest Rates (i)

- If $\chi < \alpha$, robots are relatively inefficient and A_K is low, so firms still rely on labor. $\rightarrow$ Increasing ϕ allows labor to substitute for robots, increasing wages.
- If $\chi > \alpha$, robots are highly productive and A_K is high, so firms rely more on capital. $\rightarrow$ Increasing ϕ allows robots to substitute for labor, increasing interest rates.

Effect on Output (y)

- Increasing ϕ is equivalent to relaxing constraints—previously, firms had to use both labor and robots, but now they can optimize their mix. Therefore, y does not decrease.

Conclusion:

- The effect of ϕ depends on how advanced automation is in the economy.
- This explains why the response of wages and interest rates to ϕ differs across automation phases.

Extensions: OLG Model and Capital Competition

1. **Overlapping Generations (OLG) Model:**
   - In the baseline model, savings are bounded by wages, preventing infinite capital accumulation.
   - If households leave *joy-of-giving bequests*, savings are no longer wage-constrained.
   - With bequests, the transition dynamics and self-sustainability results resemble the representative agent case.
2. **Capital Competition Between Robots and Humans:**
   - If robots and humans compete for capital, higher interest rates reduce wages.
   - Despite this effect, the transition to self-sustainability follows a similar path.

Key Takeaway:

- These extensions change *how* savings and capital accumulation operate but do not alter the core conditions for self-sustainability.

Policy Implications

Monetary Policy

How should central banks adapt as the economy moves toward self-sustainability?

Why the Standard Taylor Rule Fails

The Taylor Rule:

$$i_t = r^* + \pi_t + \phi_\pi(\pi_t - \pi^*) + \phi_y(y_t - \bar{y}).$$

- The Fed adjusts interest rates based on inflation and the output gap.
- The rule assumes a fixed steady-state rate r^*.

Key Issue in the Self-Sustainability Regime:

- In the SS regime, r^* increases as A_K grows.
- The Taylor rule does not account for this, causing systematic errors.

Result: Persistent Inflationary Pressure

- If the central bank underestimates r^*, interest rates are set too low.
- This leads to excessive monetary stimulus and inflation exceeding the target.

Why Output Gap Measurement Becomes Unreliable

The Output Gap in the Taylor Rule:

- The Taylor rule responds to deviations from potential output:

$$y_t - \bar{y}.$$

- Potential output $\bar{y}$ is estimated using labor market indicators, such as unemployment.

Why This Becomes a Problem:

- As automation progresses, labor's role declines.
- Unemployment no longer reliably signals economic slack.
- Misestimating the output gap leads to incorrect interest rate decisions.

Policy Implication:

- Monetary policy must rely less on labor-based indicators and incorporate alternative measures.

Policy Solutions for a Capital-Driven Economy

Solution 1: Price-Level Targeting

- Inflation targeting does not correct past mistakes.
- Price-level targeting ensures long-run inflation stability even if r^* is misestimated.
- If the Fed underestimates r^*, the price level rises more than expected, but expected inflation remains stable.

Solution 2: Rethinking the Output Gap

- As labor's role declines, output measurement must rely less on unemployment.
- Monetary aggregates and capital-market conditions may provide better signals of economic slack.

Why Monetary Policy Must Evolve Now

The Taylor Rule Will Not Work in the SS Phase

- As the economy moves toward SS, the rule becomes less effective.
- Policymakers must prepare for a world where capital, not labor, drives economic fluctuations.

Takeaway:

- Transition from inflation targeting to price-level targeting.
- Develop alternative measures of economic slack suited for a capital-driven economy.
- Recognize that monetary policy needs structural changes, not just parameter updates.

Policy Implications

Fiscal Policy

How should tax policy evolve as labor's share declines?

Why Labor Taxation Becomes Unsustainable

The Problem:

- As the labor share declines, labor taxation generates insufficient revenue.
- In the self-sustainability regime, labor income approaches zero, making labor taxes ineffective.

Implication:

- The tax base must shift away from labor to maintain revenue stability.
- A new approach is required before labor-based revenues collapse.

The Case for Consumption Taxation

Lessons from Optimal Taxation Theory:

- A wedge between consumption and labor is desirable for efficiency.
- Capital taxation is distortionary, discouraging investment and slowing growth.

Shifting to Consumption Taxation:

- Reduces reliance on labor-based revenue.
- Provides a stable tax base even as labor income declines.

Implicit Wealth Taxation:

- Consumption taxation indirectly taxes initial wealth, creating a desireble redistributive effect.

Why the Transition Should Start Now

The labor share is already shrinking:

- Delaying reform risks revenue shortfalls.

Gradual implementation is necessary:

- Sudden tax shifts can create economic disruptions.

Key Takeaway:

- Tax policy should evolve before labor-based revenues collapse.

Conclusions

Key Takeaways and Policy Implications

Understanding self-sustainability today is crucial for managing tomorrow's economic transformation.

Key Takeaways from the Transition to Self-Sustainability

Economic transformation is already underway:

- The economy is shifting from labor-driven to capital-driven production.
- While full self-sustainability is a long-term prospect, the transition already impacts growth, wages, and interest rates.

The Self-Sustainability Index χ^* provides a key measure of progress:

- χ^* determines how close an economy is to self-sustainability.
- As χ^* rises, growth dynamics increasingly resemble the self-sustainability regime.

Key Findings on Economic Responses to A_K and ϕ:

- The impact of A_K on wages, output, and interest rates strengthens as χ rises.
- The response to ϕ varies: at low χ, labor benefits; at high χ, capital gains.
- Convergence slows as $\chi \to 1$, making temporary effects more persistent.

Policy Implications and Future Directions

Monetary Policy:

- The Taylor rule becomes ineffective as the natural rate of interest rises with A_K.
- Price-level targeting offers a more robust framework for long-run stability.

Fiscal Policy:

- Labor taxation becomes unsustainable as labor's share declines.
- A shift toward consumption taxation ensures long-term revenue stability and aligns with optimal tax principles.

Final Thought:

- Even though full self-sustainability is a distant future, economies increasingly exhibit its characteristics.
- Understanding and adapting policies today is critical to managing the transition effectively.

Transition to A.I.&Robotic Self-Sustainability

Alina Arefeva Nikolay Arefyev
University of Wisconsin-Madison
Wisconsin School of Business

March 18, 2025

PRELIMINARY AND INCOMPLETE

Abstract

Artificial intelligence (A.I.) and robotics are transforming the economy by redefining the roles of labor and capital. Robotic self-sustainability–where labor no longer limits robot accumulation–may seem like science fiction, but the transition is likely underway. This paper introduces the Self-Sustainability Index (χ), representing the relative robot-augmenting technological level adjusted for the elasticity of substitution between robots and labor. The capital share of income converges to χ, providing a practical measure of progress toward self-sustainability. χ also quantifies the economy's distance from the critical threshold ($\chi = 1$), marking the onset of self-sustainability. As χ rises, labor's share declines, growth accelerates, and interest rates become more sensitive to productivity. These dynamics reshape growth and income distribution long before self-sustainability is achieved, offering insights into the transformations driven by A.I. and robotics.

1 Introduction

Artificial intelligence (A.I.) and robotics are reshaping the foundations of economic systems, marking one of the most transformative periods in economic history. Much like the industrial revolution, this shift redefines how economies grow and distribute resources. At the core of this paper lies the concept of robotic self-sustainability threshold, where robots can replicate and accumulate without being constrained by the limitations of human labor, allowing them to accumulate indefinitely. While the idea of self-sustainability may sound like science fiction, evidence suggests that the transition toward this state is already underway, driven by rapid advancements in A.I. that make robots highly effective substitutes for labor and push economies closer to a state where production is driven by autonomous systems. This parallels the concept of an intelligence explosion, where A.I. systems achieve a self-perpetuating cycle of improvement, amplifying their capabilities and economic impact without reliance on human intervention.

Several indicators suggest that the transition to robotic self-sustainability is already underway. A key condition for this shift is an elasticity of substitution between labor and capital greater than one, alongside continuous advancements in capital-augmenting technological progress. These conditions are central to prevailing explanations for the declining labor share in the literature. Recent studies increasingly support an elasticity greater than one (Karabarbounis and Neiman (2014b) and Hubmer (2023)), overturning earlier conclusions (Chirinko (2008)). Even if the elasticity is currently below one, rapid progress in A.I. and robotics makes it likely to exceed this threshold soon. Additionally, the observed decline in labor's share and rise in capital's share of income offer further evidence of this transition, underscoring the growing dominance of capital and the diminishing role of labor in production.

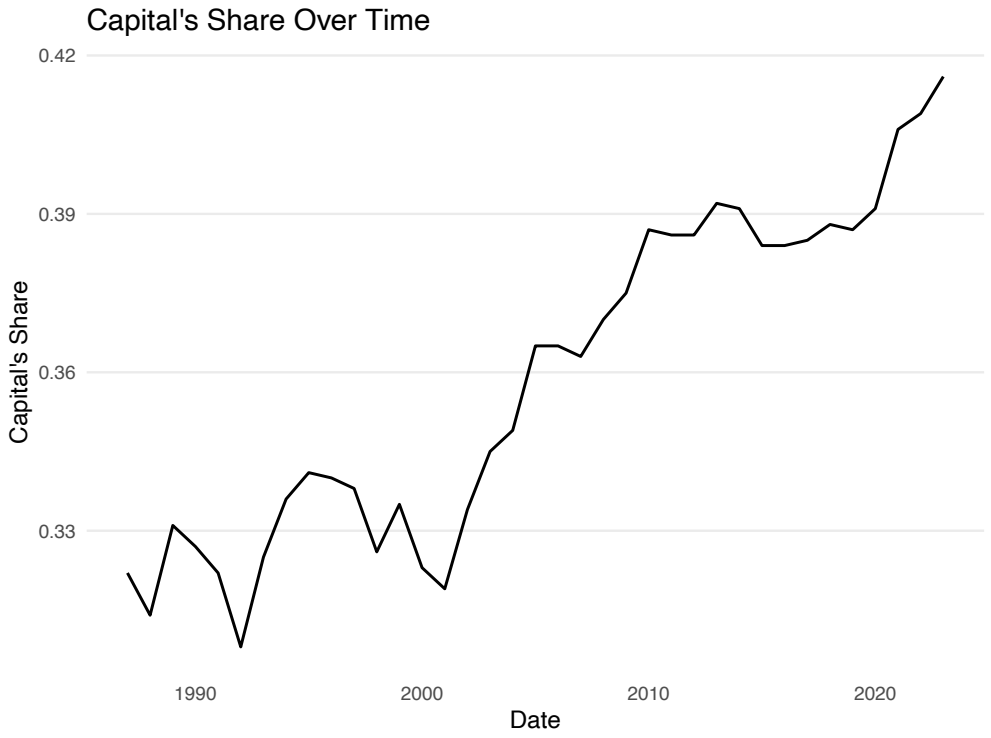


Figure 1: Private Nonfarm Business Sector: Capital Share. Source: FRED dataset.

Figure 1 illustrates the increasing trend in the capital share of income in the private nonfarm business sector. As we show in this paper, this trend aligns with theoretical predictions, providing empirical support for the ongoing shift toward robotic self-sustainability. While different measures of capital share exist in the literature, they yield consistent results, with the common finding being that the capital share has steadily increased over the last few decades. This consistency reinforces the robustness of the observation and its relevance to the broader trends discussed in this paper.

A key contribution of this paper is the introduction of the Self-Sustainability Index (χ), a metric that quantifies the relative progress in robot-augmenting technologies, scaled by the elasticity of substitution between robots and labor. In all model

specifications we consider, χ exactly equals the capital share. However, in practice, factors such as capital adjustment costs, monopolistic power, interest rate fluctuations, and wage variations over the business cycle can cause temporary or permanent divergences between χ and the observed capital share. Therefore, our results should be interpreted as showing that the capital share converges approximately to χ over time, making it a practical proxy for measuring an economy's progress toward robotic self-sustainability.

If the elasticity of substitution between labor and capital is greater than one and capital-augmenting technological progress continues to improve, the economy is inevitably on a path toward $\chi = 1$. When χ reaches 1, both the interest rate and economic growth rate become tied to the level of robot-augmenting technological progress rather than to its growth rate, enabling growth to accelerate indefinitely. In this scenario, labor's share of income converges to zero, wages may either rise or fall depending on broader economic conditions, and the interest rate and output growth become dependent on the level of the robot-augmenting technological progress rather than on its growth rate. While understanding this regime is important, as the economy increasingly resembles it when χ approaches 1, our primary focus is on the transition dynamics. Even when $\chi < 1$, the economy progressively diverges from traditional growth models, with labor's role gradually decreasing and output becoming increasingly sensitive to robot-augmenting productivity. Small increases in this productivity can lead to disproportionate growth in output and interest rates, reflecting the growing dominance of robots in production.

While our analysis predicts that an increasing capital share leads to higher returns on capital and GDP growth, empirical evidence for these effects is challenging to provide. Measures like the S&P 500 Index are noisy and insufficiently sensitive to the modest changes predicted by our framework. Similarly, GDP growth reflects many factors, hiding the predicted increases amid short-term fluctuations. Therefore, we focus on more observable trends, such as the rising capital share, to support our theoretical predictions.

The transition to a robotic self-sustainability regime necessitates a fundamental rethinking of both fiscal and monetary policy. For fiscal policy, as labor's share of income diminishes, reliance on labor-based taxes becomes unsustainable, requiring a shift toward consumption-based taxation, which is less distortionary and better aligned with an economy driven by capital and automation. Redistribution policies will also play a critical role in ensuring that the benefits of increased capital income are shared broadly. Instead of taxing capital directly, which could discourage investment and innovation, redistribution can be achieved through consumption taxes and direct transfers that do not disincentivize capital accumulation. Policy design must balance efficiency and equity, ensuring that automation-driven growth translates into widespread prosperity without undermining incentives for technological progress.

This transition also poses new challenges for monetary policy, which traditionally targets the output gap and inflation. As labor's share declines, measuring the output gap–typically based on deviations from labor market conditions–becomes increasingly difficult. Furthermore, the shift from labor-intensive to capital-intensive consumer basket may complicate inflation measurement, adding uncertainty to monetary policy

decisions. Adapting macroeconomic policy frameworks to these structural changes will be crucial for ensuring stability and growth.

Understanding these transitional dynamics is not only crucial for advancing economic theory but also imperative for shaping the policies that will determine how societies adapt to robotic self-sustainability. This transformation offers unprecedented opportunities for prosperity and innovation, yet it also poses profound challenges, including growing inequality and the displacement of traditional labor markets. Navigating this shift will require proactive policy interventions, such as rethinking taxation, redistribution, and education, to ensure that the benefits of this new economic paradigm are widely shared and its risks are effectively mitigated.

Literature review. This paper contributes to two branches of the literature. First, it addresses how A.I. and robotic developments will reshape the world, exploring both optimistic and pessimistic scenarios while considering policy measures to address these challenges. Acemoglu and Restrepo (2018) established a widely used framework for analyzing the impact of automation on labor markets, deriving an aggregate production function from microfoundations and endogenizing the capital share in a CES framework. While their approach provides valuable insights, this paper takes a different path, prioritizing tractability and focusing on the role of robot-augmenting technological progress.

A key distinction between the two models is that Acemoglu and Restrepo assume all innovation improves labor productivity, whereas we allow for capital-augmenting technological progress. This difference is crucial: in their framework, the capital share adjusts endogenously, leading to scenarios where robots and labor coexist or where robots fully displace labor. In contrast, our framework simplifies this mechanism by showing that robot-augmenting technological progress alone is sufficient to capture the transition dynamics. Importantly, our model does not impose an upper bound on capital-augmenting technological progress, which leads to distinct long-run outcomes. While their assumptions facilitate a balanced growth path, our approach reveals structural changes that ultimately lead to robotic self-sustainability.

Despite these differences, our findings on the self-sustainability index (χ) are consistent with their framework, see Appendix X for details. When capital-augmenting technological progress is introduced into their model, many of our results align, underscoring the robustness of these mechanisms across different modeling approaches.

Second, it engages with the literature on the declining share of labor income[1]. The dominant explanations for this trend include capital-augmenting technological progress and the decreasing price of capital goods, particularly when combined with an elasticity of substitution between labor and capital greater than one. This paper aligns with these explanations but extends the discussion by embedding it in the broader context of economic growth. This extension allows for deeper insights into capital income dynamics, the rate of economic growth, and the implications when χ reaches or exceeds 1. Additionally, the literature highlights how shifts in market

[1]Recent reviews of this literature include Karabarbounis (2024). This review includes the discussion of different measures of capital and income shares, as well as different explanations why the labor's share declined.

power contribute to declining labor shares, emphasizing the challenges of accurately measuring market shares. This discussion underscores that the proposed measure of self-sustainability, while insightful, remains approximate even in asymptotic analysis.

2 Benchmark Model

The benchmark model provides a straightforward framework to examine how robots and labor interact in production as robotic technologies advance. Firms produce output (Y) according to the following production function:

$$Y = \left[\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right]^{\frac{1}{\phi}}, \tag{1}$$

where A_R and A_L are robot- and labor-augmenting technologies, α represents the share of tasks performed by robots, and ϕ determines the substitutability between robots and labor. The elasticity of substitution is given by $1/(1-\phi)$. Wages and robot rental rates are linked to their marginal productivities.

Technological advancements in robotics can be modeled through three distinct channels. First, changes in task allocation (α) capture how automation reallocates tasks between robots and humans, following the approach of Acemoglu and Restrepo. As technology evolves, α may increase or decrease depending on the relative productivity of robots and labor. Second, robot-augmenting progress (A_R) reflects efficiency gains that make existing robots more productive without altering task allocation. Third, increased substitutability (ϕ) represents how advancements in robotics and AI enable robots to replace labor across a broader range of tasks.

While increases in α and A_R are quantitatively substitutable–since a simultaneous rise in α and A_L (offsetting the term $(1-\alpha)A_L^{\phi}$) replicates the effects of higher A_R–an increase in ϕ produces qualitatively different outcomes. Higher substitutability reflects a structural transformation in production, rather than mere task reallocation or efficiency gains. Given this substitutability, the analysis focuses on technological progress through A_R, holding α constant.

Real-world examples illustrate the economic relevance of these distinctions. Increased substitutability (ϕ) is exemplified by autonomous taxis replacing traditional drivers and AI-driven customer service platforms substituting for human agents. In contrast, increased robot productivity (A_R) manifests in sectors where robots perform existing tasks more efficiently without altering task allocation. For instance, advanced robotic arms in manufacturing, such as those by Fanuc and ABB, now operate with greater precision and speed. Similarly, autonomous tractors and harvesters, like those developed by John Deere, leverage enhanced GPS and machine learning algorithms to improve agricultural productivity while maintaining the division of tasks between humans and machines.

Embedding this production function into a growth model poses two challenges. First, a balanced growth path does not exist when $\phi > 0$ and A_R grows at a steady rate. Second, the model predicts asymptotically rising interest rates, which, in the

representative agent framework, drive current consumption to zero, yielding unrealistic results. To address these issues, this paper adopts four approaches.

First, it analyzes static equilibrium under an exogenously given interest rate, which forms the core results of the study. Second, it examines the production function within a representative agent framework, assuming A_R and ϕ are constant, and compares balanced growth paths for different values of A_R and ϕ. Third, it employs an overlapping generations (OLG) model with joy-of-giving bequests, which allows equilibrium determination even when a balanced growth path does not exist and interest rates grow asymptotically. Unlike the representative agent framework, where optimization depends on the long-run interest ratewhich tends to infinitythe OLG structure only considers interest rates over the next period. As a result, while explosive dynamics may still arise, they do not directly affect how equilibrium is determined. Moreover, this approach generates realistic consumption and saving dynamics, ensuring that consumption does not collapse to zero and that savings depend not only on wage income but also on bequests accumulated across generations. Finally, we provide numerical simulations for a model with a constant saving rate. While this assumption has been criticized in the literature, it offers valuable intuition for understanding the results presented in this paper.

3 Definition of χ

This section introduces the Self-Sustainability Index, χ, which quantifies the feasibility of unbounded robotic capital accumulation. Its definition is based on the marginal product of robots, given by:

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}, \tag{2}$$

where $r = \frac{A_R R}{A_L L}$ represents the relative abundance of robotic capital R to human labor L, weighted by their respective productivity terms A_R and A_L. As $r \to \infty$, the marginal product of robots approaches:

$$\lim_{r\to\infty} MP_R = \alpha^{\frac{1}{\phi}} A_R. \tag{3}$$

Since this limit does not vanish, it violates the Inada conditions, which typically ensure that unbounded capital accumulation is not feasible. This violation implies that when A_R is sufficiently large, robotic capital can, in principle, accumulate indefinitely, though this outcome is not necessarily inevitable.

For a given interest rate i, the critical threshold of robot-augmenting technological progress, A_R^{crit}, is defined as the level at which the asymptotic marginal product of robots equals the net interest rate:

$$A_R^{\text{crit}} = (i+\delta)\,\alpha^{-\frac{1}{\phi}}. \tag{4}$$

This threshold, which depends on the technological parameters α and ϕ as well as the interest rate i, represents the minimum A_R at which unbounded robotic capital accumulation becomes feasible.

Using this threshold, the Self-Sustainability Index χ is defined as:

$$\chi \equiv \left(\frac{A_R}{A_R^{\text{crit}}} \right)^{\frac{\phi}{1-\phi}}. \tag{5}$$

Two key aspects of this definition are worth noting. First, χ is only meaningful for $\phi > 0$. When $\phi < 0$, the asymptotic marginal product of robots converges to zero, meaning that A_R^{crit} is not well-defined. Although equations (4) and (5) appear mathematically valid for $\phi < 0$, the logic underpinning their derivation breaks down in this case.

Second, χ captures economic feasibility, not just technological feasibility, and its definition depends on the interest rate used to compute A_R^{crit}. Technological feasibility means that, in principle, robots can fully replace human labor, which is possible whenever the asymptotic marginal product of robots exceeds the depreciation rate:

$$\lim_{r \to \infty} MP_R = \alpha^{\frac{1}{\phi}} A_R > \delta. \tag{6}$$

However, this does not ensure that firms will continue accumulating robotic capital indefinitely. Economic feasibility requires that such accumulation is also financially viable, meaning that the return on robotic capital at least matches the market interest rate. The self-sustainability index incorporates this constraint by embedding the interest rate directly into A_R^{crit}, ensuring that χ reflects not only whether unbounded automation is technologically possible but also whether it is financially sustainable. Unless stated otherwise, χ refers to the index based on the equilibrium interest rate, though alternative ratessuch as a balanced growth path ratecan also be used. Consequently, changes in financial conditions, including shifts in equilibrium interest rates, directly influence the trajectory of χ and the economys transition toward automation.

Finally, in equilibrium, χ has a direct economic interpretation: it corresponds to the income share of robotic capital. That is, as long as $\chi \leq 1$, the fraction of total output accruing to robots is exactly given by χ, while labor receives the remaining share $1 - \chi$:

$$\text{Robot share} = \chi, \quad \text{Labor share} = 1 - \chi.$$

This result follows from the firms optimization condition and the definition of A_R^{crit}, as shown in Appendix A. This interpretation makes χ particularly useful for empirical applications, as it provides a way to approximate the progress of robot-augmenting technological development using observable macroeconomic data.

4 Partial Equilibrium Analysis of Firm Behavior

This section examines how firms respond to advancements in robotic technology by analyzing the effects of changes in A_R and ϕ on equilibrium outcomes across different phases of automation. The analysis is conducted within a partial equilibrium framework, incorporating the firms production function and first-order conditions:

$$i + \delta = MP_R, \quad W = MP_L. \tag{7}$$

Two cases are considered. In the short-run equilibrium, robotic capital R is fixed, and the interest rate $i + \delta$ adjusts to clear the market. In the long-run equilibrium, the cost of capital $i + \delta$ is held constant, allowing R to adjust endogenously. While the term "long-run" is used for convenience, it does not necessarily imply that the interest rate converges to a fixed value. Whether this occurs depends on broader economic conditions and is analyzed in later sections.

The key objective is to examine how firms' responses to technological progress, specifically changes in A_R and ϕ, evolve as the self-sustainability index χ increases. This index plays a central role in characterizing automation, as it determines the share of income accruing to robotic capital. The analysis distinguishes three phases of automation:

- *Labor-Dominant Phase* ($\chi \leq \alpha$): Labor remains the primary input in production, and robotic capital plays a secondary role.
- *Transition Phase* ($\alpha < \chi < 1$): Robots gain a stronger role, but labor is still an important production factor.
- *Self-Sustainability Phase* ($\chi \geq 1$): Robotic capital accumulates indefinitely, leading to fundamental shifts in equilibrium dynamics.

The section begins by establishing the conditions that define the threshold $\chi = \alpha$, marking the transition between labor-dominance and automation. It then derives equilibrium elasticities, showing how firms adjust wages, output, and capital returns in response to technological progress. Finally, it contrasts these results with the predictions of the standard neoclassical growth model, highlighting key differences in automation dynamics.

4.1 The Threshold $\chi = \alpha$ and Its Three Equivalent Conditions

Defining the Automation Threshold The threshold $\chi = \alpha$ marks the transition from a labor-dominant economy to one where robots play a significant role in production. When $\chi \leq \alpha$, labor remains the primary input, and robotic capital contributes only a limited share of output. As χ increases beyond α, robots become more significant, shifting the distribution of income toward capital.

An intuitive way to understand this threshold comes from considering the limiting case where $\phi \to 0$. In this case, the production function converges to a Cobb-Douglas form, where capital's share is fixed at α. Since χ represents the share of robotic capital in equilibrium, the condition $\chi \leq \alpha$ implies that the economy is still below the automation threshold, meaning that labor continues to dominate production.

First Condition: Directly in Terms of χ The threshold can be expressed in its simplest form as

$$\chi \leq \alpha. \tag{8}$$

This condition states that the share of income accruing to robotic capital is less than or equal to the capital share that would arise in a Cobb-Douglas setting, ensuring that labor remains the dominant production factor.

Second Condition: Expressed in Terms of Robotic Intensity Using the equilibrium relationship between r and χ, derived from (7), (**??**), (5), and (4), we obtain:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha} r^{\phi}. \tag{9}$$

Details on how this relation is derived are provided in Appendix B. From this expression, we see that the condition $\chi \leq \alpha$ is equivalent to:

$$r \leq 1. \tag{10}$$

This means that robots remain scarce relative to labor, adjusted for their respective productivity levels. When $r \leq 1$, human labor remains the dominant factor, as robotic capital has not yet accumulated to a level where it can compete directly with labor at a large scale.

Third Condition: Expressed in Terms of the Critical Level of A_R A third way to express this condition is in terms of the threshold level of robot-augmenting technology required to push the economy beyond the labor-dominant phase. As derived in Appendix **??**, the condition $\chi \leq \alpha$ holds if and only if:

$$A_R \leq \bar{A}_R, \quad \text{where} \quad \bar{A}_R = \frac{i+\delta}{\alpha}. \tag{11}$$

This result provides an alternative interpretation of the labor-dominant phase: the economy remains in this phase if robot-augmenting technological progress is insufficient, meaning that A_R has not yet reached the level necessary for robots to become competitive with labor in production.

Interpretation and Summary The three equivalent conditions

$$\chi \leq \alpha, \quad r \leq 1, \quad A_R \leq \bar{A}_R \tag{12}$$

provide complementary ways of understanding the labor-dominant phase. - The first condition states that robotic capital's share remains below the Cobb-Douglas benchmark. - The second condition explains that robots are still relatively scarce compared to labor. - The third condition emphasizes that robot-augmenting technology is not yet advanced enough to push the economy past the automation threshold.

These results form the foundation for analyzing how firms adjust as χ increases and the economy transitions toward automation. The next section derives equilibrium elasticities, showing how wages, output, and capital returns respond to changes in A_R and ϕ across different automation phases.

4.2 Equilibrium Relationships in Terms of χ

Rewriting Equilibrium Conditions To analyze the effects of automation on firms' equilibrium decisions, it is useful to express key equilibrium conditions in terms of the self-sustainability index χ, rather than robotic intensity r. Since χ determines factor income distribution and plays a central role in automation dynamics, it provides a clearer perspective on how wages, output, and the cost of capital evolve in response to technological progress.

Additionally, χ is approximately observable in practice, as it corresponds to the capital share under certain conditions. This makes it a more relevant variable for economic analysis compared to r, which depends on the unobservable measure of robotic capital R and relative technological progress, A_R/A_L. Expressing equilibrium conditions in terms of χ provides direct insights into how firms adjust as automation advances.

Output per Effective Labor The equilibrium condition for output per effective labor, derived in Appendix C.1, is given by:

$$y = \left[\frac{1-\alpha}{1-\chi}\right]^{\frac{1}{\phi}}. \tag{13}$$

From this equation, it follows that as $\chi \to 1$, output per effective labor y tends to infinity. The intuition behind this result is that when χ is close to 1, the economy is nearing a regime where robots accumulate without bound. This reflects the fact that the closer the economy is to the self-sustainability phase, the stronger the impact of automation on output. The elasticity of y with respect to χ will be formally derived in the next section, but its divergence as $\chi \to 1$ is already evident from this expression.

Wage per Effective Labor The equilibrium wage per effective labor is given by:

$$w = (1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}. \tag{14}$$

This equation shows that wages also increase with χ. Since $1-\chi$ decreases as χ increases, and it is raised to a negative exponent, the overall expression grows to infinity as $\chi \to 1$. This means that wages, like output, tend to infinity as the economy transitions toward the self-sustainability phase. Again, this result reflects the increasing intensity of robotic capital accumulation as automation progresses.

Cost of Robotic Capital The equilibrium cost of robotic capital, which equals its marginal product, is:

$$i + \delta = A_R \alpha^{\frac{1}{\phi}} \chi^{-\frac{1-\phi}{\phi}}. \tag{15}$$

At first glance, this equation may suggest that as χ increases, the cost of capital declines. However, this interpretation is misleading because χ is an endogenous variable that depends on A_R. Since A_R directly influences both χ and $i+\delta$, it is not possible to determine how χ affects $i+\delta$ from this equation alone. The correct way to analyze this relationship is by deriving elasticities, which will be done in the next section.

Short-Run and Long-Run Values of χ The equilibrium value of χ depends on whether robotic capital R is fixed (short run) or whether the cost of capital $i + \delta$ is fixed, allowing R to adjust endogenously (long run).

In the short run, the equilibrium condition for χ, derived in Appendix D.1, is:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha}\left(\frac{A_R R}{A_L L}\right)^{\phi}. \tag{16}$$

This expression shows that short-run adjustments in χ depend on the relative abundance of robotic capital, scaled by its productivity.

In the long run, when $i + \delta$ is fixed and robotic capital adjusts, the equilibrium value of χ, derived in Appendix D.2, is given by:

$$\chi = \alpha^{\frac{1}{1-\phi}}\left(\frac{A_R}{i+\delta}\right)^{\frac{\phi}{1-\phi}}. \tag{17}$$

This equation shows that long-run automation levels depend on both the rate of technological progress in A_R and financial conditions, as captured by $i + \delta$.

Why These Expressions Matter These equilibrium relationships provide the foundation for analyzing how firms adjust to changes in automation-related technological progress. By expressing wages, output, and the cost of capital as functions of χ, we can directly examine how firms respond to changes in A_R and ϕ. The next section derives elasticity conditions to quantify these responses across different automation phases.

4.3 Response of the Self-Sustainability Index χ

Why Analyzing the Response of χ Matters Since χ determines the share of income accruing to robotic capital, its response to changes in A_R and ϕ is central to understanding how automation reshapes factor income distribution and economic structure. By analyzing how χ evolves, we can quantify the pace of automation and its effects across different automation phases.

The elasticity of χ with respect to A_R captures how improvements in robot-augmenting technology drive automation, while the semi-elasticity of χ with respect to ϕ explains how increasing substitutability between labor and robots affects the speed of automation.

Elasticity of χ with Respect to A_R The elasticity of χ with respect to A_R is derived in Appendix E.1 and given by:

$$\frac{d\ln\chi}{d\ln A_R} = \begin{cases} \phi(1-\chi), & \text{(Short Run)} \\ \frac{\phi}{1-\phi}, & \text{(Long Run)} \end{cases} \tag{18}$$

This equation provides three key insights:

- The response of χ to A_R becomes increasingly delayed as $\chi \to 1$, meaning that increases in A_R take longer to translate into changes in automation.
- In the long run, the elasticity is constant and always larger than the short-run elasticity, meaning that automation progresses more strongly over time.
- There is no overshooting: Since the long-run elasticity exceeds the short-run elasticity, χ continues to rise as automation progresses, ensuring a smooth adjustment path.

Semi-Elasticity of χ with Respect to ϕ The semi-elasticity of χ with respect to ϕ is derived in Appendices E.2 and E.2.1 and given by:

$$\frac{d\ln\chi}{d\phi} = \begin{cases} (1-\chi)\ln r, & \text{(Short Run)} \\ \frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)}, & \text{(Long Run)} \end{cases} \tag{19}$$

The sign of this elasticity depends on the automation phase:

- If $\chi < \alpha$ (equivalently, $r < 1$), increasing ϕ reduces the share of income accruing to robotic capital (χ declines).
- If $\chi > \alpha$ (equivalently, $r > 1$), increasing ϕ raises the share of income accruing to robotic capital (χ rises).

This phase-dependent response of χ to ϕ plays a key role in shaping automation dynamics.

Economic Intuition: The Warehouse Example The warehouse example illustrates why the response of χ to ϕ differs depending on the automation phase. Suppose warehouses are initially designed either for human workers or for robots, each optimized to maximize efficiency given the nature of the workforce. A warehouse designed for robots may feature rigid shelving layouts and automated retrieval systems, whereas a warehouse designed for humans allows for greater flexibility and manual adjustments.

Now, suppose the warehouse is redesigned to accommodate both humans and robots equally, increasing their substitutability without any improvement in robot- or labor-augmenting technology. The impact of this change depends on the level of robotic technology:

- If robots are not yet sufficiently advanced, labor retains its dominant role, as human workers can better handle exceptions, navigate unforeseen circumstances, and respond dynamically to production needs. In this case, increasing ϕ without raising A_R leads to weaker automation incentives, lowering χ.
- If robots are already capable enough, increasing ϕ allows firms to substitute away from labor more effectively, raising χ.

This explains why the semi-elasticity of χ with respect to ϕ is negative when $\chi \leq \alpha$ but positive in the transition and self-sustainability phases.

Summary and Transition This section has shown that:

- The response of χ to A_R follows a smooth and monotonic pattern, with no overshooting.
- The response of χ to ϕ depends on the automation phase: when robots are scarce, increasing ϕ reduces χ; when robots are already widespread, increasing ϕ raises χ further.
- These results highlight that the evolution of χ is driven not only by improvements in robot-augmenting technology but also by the degree of substitutability between robots and labor.

The next section examines how output per effective labor y responds to changes in A_R and ϕ, and how this sensitivity evolves as χ increases.

4.4 Response of Output per Effective Labor

Why Analyzing the Response of y Matters Output per effective labor y is a central measure of economic productivity. Since it has already been defined as

$$y = \left[\frac{1-\alpha}{1-\chi}\right]^{\frac{1}{\phi}}, \tag{20}$$

its response to changes in A_R and ϕ determines how technological progress affects aggregate output. The key insight from this equation is that as $\chi \to 1$, y tends to infinity. This behavior reflects the increasing dominance of robotic capital in production, where output scales with the accumulation of robots. The elasticity of y with respect to χ quantifies how output responds as the economy approaches full reliance on robotic capital.

Elasticity of y with Respect to χ The elasticity of output per effective labor with respect to χ, derived in Appendix E.3, is given by:

$$\frac{d\ln y}{d\ln \chi} = \frac{\chi}{\phi(1-\chi)}. \tag{21}$$

This expression shows that as $\chi \to 1$, the elasticity diverges to infinity. In other words, even small increases in χ lead to large changes in output when robotic capital becomes the dominant factor. The intuition behind this result is straightforward: as robots accumulate indefinitely, output becomes increasingly sensitive to further increases in robotic capital intensity.

Elasticity of y with Respect to A_R The elasticity of y with respect to A_R is:

$$\frac{d\ln y}{d\ln A_R} = \begin{cases} \chi, & \text{(Short Run)} \\ \frac{1}{1-\phi}\cdot\frac{\chi}{1-\chi}, & \text{(Long Run)} \end{cases} \tag{22}$$

Three important conclusions follow:

- The short-run elasticity is always smaller than the long-run elasticity, meaning that the response of output per effective labor to changes in A_R strengthens over time.
- As $\chi \to 1$, the long-run elasticity diverges, meaning that when robotic capital is already widespread, a given percentage increase in A_R leads to a disproportionately large increase in y, while the short-run response remains stable.

Response of y to ϕ The effect of ϕ on output depends on the extent of robotic capital accumulation. While the derivation of these results involves nontrivial calculations, which are provided in Appendix F and related sections, the key conclusions are:

- In the short run, y increases with ϕ for all values of χ except when $\chi = \alpha$, where it remains constant (see Appendix F for proof).
- In the long run, the relationship is more complex: at early stages, increasing ϕ may slow the accumulation of robotic capital, leading to lower output. At later stages, increasing ϕ unambiguously raises y.

This behavior reflects the fact that increasing ϕ makes robots more substitutable for labor, but the effects of this substitutability depend on whether robots have already become the dominant production factor.

Economic Intuition: The Warehouse Example Revisited The warehouse example provides intuition for why the response of y to ϕ differs across stages of robotic capital accumulation. Suppose a warehouse initially designed for human workers is modified to be equally suitable for both humans and robots. Increasing ϕ in this context allows for greater flexibility in factor allocation, improving efficiency. As a result, output increases in all cases except when $\chi = \alpha$ (equivalently, $r = 1$), where robots and labor are equally productive. In this special case, there is no efficiency gain from reallocation, so output remains unchanged.

Summary and Transition This section has established that:

- As $\chi \to 1$, y tends to infinity, reflecting the accumulation of robots.
- The elasticity of y with respect to A_R increases as robotic capital expands, with the long-run response diverging as $\chi \to 1$.
- The effect of ϕ on output depends on the level of robotic capitalwhile it may initially slow substitution between robots and labor, at later stages it enhances output growth.

The next section examines how wages and the return to robotic capital respond to increases in A_R and ϕ, and how their sensitivity changes as χ rises.

4.5 Response of Wage per Effective Labor and Return to Robotic Capital

Why Analyzing the Response of w and $i+\delta$ Matters Changes in A_R and ϕ affect both labor income (wages) and the return to robotic capital. Analyzing their response provides insight into how income distribution evolves as robotic capital accumulates. Since both variables play a crucial role in shaping economic incentives, understanding their behavior is essential for assessing the long-term effects of technological progress.

Elasticity of w with Respect to A_R The elasticity of wages with respect to A_R, derived in Appendices G and H, is given by:

$$\frac{d\ln w}{d\ln A_R} = \begin{cases} \chi(1-\phi), & \text{(Short Run)} \\ \frac{\chi}{1-\chi}, & \text{(Long Run)} \end{cases} \tag{23}$$

This equation leads to several key observations:

- Wages always increase with A_R in both the short run and long run.
- The short-run elasticity remains below one, while the long-run elasticity exceeds one, ensuring that the response of wages to A_R strengthens over time.
- The long-run elasticity diverges as $\chi \to 1$, meaning that at high levels of robotic capital, even small increases in A_R lead to large increases in wages.

Elasticity of $i+\delta$ with Respect to A_R The response of the return to robotic capital is given by:

$$\frac{d\ln(i+\delta)}{d\ln A_R} = \begin{cases} \phi + (1-\phi)\chi, & \text{(Short Run)} \\ 0, & \text{(Long Run)} \end{cases} \tag{24}$$

Since the long-run analysis assumes that $i+\delta$ is fixed, the long-run elasticity is zero by construction. In contrast, the short-run elasticity is positive and depends on both ϕ and χ. This implies:

- In the short run, the return to robotic capital increases with A_R, and the effect is stronger at higher values of χ.
- In the long run, $i+\delta$ does not respond to A_R because it is held constant by assumption.

Response of w and $i+\delta$ to ϕ The effect of ϕ depends on the level of robotic capital:

- When $\chi < \alpha$, increasing ϕ raises wages and reduces the return to robotic capital.
- When $\chi > \alpha$, increasing ϕ lowers wages and raises the return to robotic capital.

This pattern highlights how substitutability between robots and labor shifts income distribution as robotic capital accumulates.

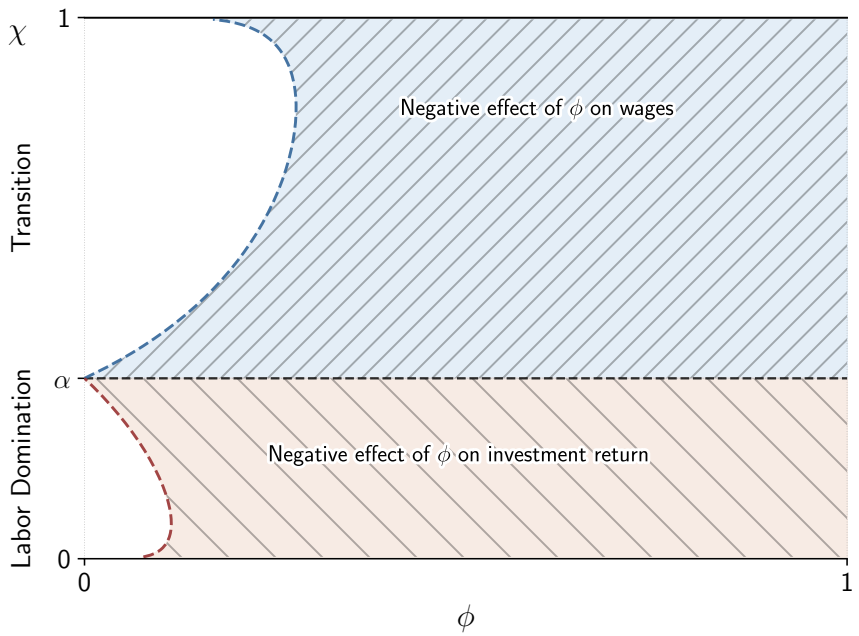


Figure 2: Phase diagram illustrating the regions where the effects of ϕ on investment return and wages are negative. The labor-dominated and transition regions are labeled, and the dashed lines mark key phase boundaries. The figure is drawn for $\alpha = \frac{1}{3}$.

Phase Diagram: When ϕ Reduces Wages or Capital Returns To better understand the effects of ϕ, Figure 2 is based on the elasticities of wages and the return to robotic capital with respect to ϕ in the short run, which are derived in Appendices I and K. The diagram is structured in (ϕ, χ) space, with two key reference values: the threshold $\chi = \alpha$ and the upper bound $\chi = 1$.

Two dashed boundary curves divide the space into regions where the effects of ϕ on wages and the return to robotic capital change:

- A blue labor-dominated boundary separates areas where increasing ϕ has a neutral or positive effect on investment return from those where it becomes negative.
- A red transition boundary marks the threshold where increasing ϕ begins to lower wages.

The shaded areas of the diagram indicate:

- A blue-shaded region where increasing ϕ reduces the return to robotic capital. This occurs at lower values of χ, particularly in the labor domination phase.
- A coral-shaded region where increasing ϕ reduces wages. This effect arises at higher values of χ, in the transition phase and beyond.

At $\chi = \alpha$, the respective elasticities are zero, meaning that increasing ϕ has no immediate effect on either wages or the return to robotic capital. This follows from the fact that $r = 1$, where labor and robotic capital are equally productive.

Importantly, the phase diagram also shows that:

- The elasticity of w with respect to ϕ is positive for all $\chi < \alpha$.
- The elasticity of $i + \delta$ with respect to ϕ is positive for all $\chi > \alpha$.
- When $\chi < \alpha$, the return to robotic capital initially increases with ϕ when ϕ is small but decreases when ϕ is large.
- When $\chi > \alpha$, the return to labor initially increases with ϕ when ϕ is small but decreases when ϕ is large.

This visualization helps clarify how the impact of ϕ on income distribution depends on the stage of robotic capital accumulation.

Economic Intuition: The Warehouse Example Revisited The warehouse example helps illustrate why the return to a specific factor may fall as ϕ increases. Suppose that initially, warehouses are designed either for human workers or for robots, each optimized for the respective workforce. A warehouse designed for robots may feature rigid shelving layouts and automated retrieval systems, whereas a warehouse designed for humans allows for greater flexibility and manual adjustments.

Now, suppose warehouses are redesigned to accommodate both humans and robots equally, increasing ϕ by making labor and robotic capital more substitutable. As ϕ increases, firms adjust their factor usage based on the relative productivity of labor and robots.

If robots are not yet highly efficient, firms may respond by relying more on labor, reducing their demand for robots. This shift increases wages but lowers the return to robotic capital. Conversely, if robots are already highly effective, firms substitute labor with robotic capital, decreasing the demand for labor and thereby reducing wages.

This example provides intuition for why the return to robotic capital decreases for large ϕ when $\chi < 1$, and the return to labor decreases for large ϕ when $\chi > 1$. When a production factor is dominant, increasing ϕ can improve efficiency by shifting resources toward it. However, when substitutability becomes too high, the demand for one factor declines, leading to a fall in its return.

Summary and Transition This section has established that:

- Wages always rise with A_R, but their sensitivity increases at higher levels of χ.
- The return to robotic capital increases with A_R in the short run but is fixed in the long run.
- The effect of ϕ depends on χat low levels, it favors labor, while at high levels, it favors robotic capital.

- The phase diagram clarifies when ϕ reduces wages or the return to capital.
- Although total output increases with ϕ, income distribution effects depend on which factor captures the gains.

The next section examines how income distribution shifts across different phases of robotic capital accumulation.

4.6 Income Distribution Across Different Phases of Robotic Capital Accumulation

Why Income Distribution Matters The share of income accruing to labor and robotic capital determines the distributional effects of technological progress. As robotic capital accumulates, income distribution evolves, shifting between three distinct phases:

- **Labor Domination** ($\chi < \alpha$): Labor remains the primary production input, and its income share is high.
- **Transition Phase** ($\alpha < \chi < 1$): Robots play an increasingly significant role, but labor is still relevant.
- **Self-Sustainability** ($\chi \geq 1$): Robotic capital fully dominates production, and labors share approaches zero.

Understanding how income shares evolve across these phases helps clarify how technological progress affects wages and returns to robotic capital.

Factor Income Shares as Functions of χ The equilibrium income shares of labor and robotic capital are given by:

$$\text{Robot share} = \chi, \quad \text{Labor share} = 1 - \chi. \tag{25}$$

Since χ evolves with A_R and ϕ, these shares are not constant and shift dynamically over time.

How Income Shares Change with A_R Using the elasticity of χ with respect to A_R, derived in Appendix E.1, we obtain:

$$\frac{d\ln\chi}{d\ln A_R} = \begin{cases} \phi(1-\chi), & \text{(Short Run)} \\ \frac{\phi}{1-\phi}, & \text{(Long Run)} \end{cases} \tag{26}$$

This equation reveals that:

- When χ is small, the robot share grows slowly with A_R, meaning that labors share remains high.

- As $\chi \to 1$, the robot share increases more rapidly, shifting income away from labor.
- In the long run, the response of χ to A_R is stronger than in the short run, reinforcing the shift toward robotic capital dominance over time.

How Income Shares Change with ϕ Using the elasticity of χ with respect to ϕ, derived in Appendix E.2, we obtain:

$$\frac{d\ln\chi}{d\phi} = \begin{cases} (1-\chi)\ln r, & \text{(Short Run)} \\ \frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)}, & \text{(Long Run)} \end{cases} \tag{27}$$

This equation shows that:

- When $\chi < \alpha$, increasing ϕ slows the shift toward robotic capital, preserving labors income share.
- When $\chi > \alpha$, increasing ϕ accelerates the shift toward robotic capital, reducing labors share.
- At $\chi = \alpha$, the elasticity of χ with respect to ϕ is zero, meaning that substitutability changes have no immediate effect on income shares.

Phase Transitions: When Labor Share Drops Below Key Thresholds The shift from labor domination to robotic capital dominance occurs gradually as χ increases. The key transitions occur when:

- **At** $\chi = \alpha$: Labors share falls below the level found in a Cobb-Douglas economy, meaning that robots have surpassed the standard capital share.
- **At** $\chi = 1$: Labors share drops to zero, meaning that robots fully dominate production.

These thresholds mark critical shifts in the economic structure, as robotic capital takes over an increasing share of income.

Economic Intuition: The Transition from Labor to Robotic Capital Dominance At low values of χ, wages are resilient because labor is still essential for production. However, as $\chi \to 1$, the marginal contribution of labor diminishes, leading to a decline in its income share. This process mirrors historical transitions where technological advancements shifted income shares between production factors, such as the transition from agricultural labor to industrial capital in previous industrial revolutions.

Summary and Transition This section has established that:

- The evolution of χ drives changes in labor and robotic capital income shares.
- The robot share grows slowly at first but accelerates as χ approaches 1.
- The effect of ϕ influences how fast this transition occurs, with different effects depending on whether robots are scarce or abundant.
- The phase diagram clarifies when the labor share falls below key thresholds, marking the transition from labor-dominated to robot-dominated production.

The next section examines the macroeconomic implications of shifting income shares, exploring how these distributional changes affect long-run economic dynamics.

4.7 Conclusion of the Partial Equilibrium Analysis

This section has examined how firms respond to advancements in robotic technology, focusing on how changes in A_R and ϕ affect output, wages, the return to robotic capital, and income distribution. The key insights are:

- The self-sustainability index χ plays a central role in determining how factor income shares evolve as robotic capital accumulates.
- As χ increases, output per effective labor rises, but the gains are not evenly distributedlabor's share declines, while robotic capital captures a growing fraction of income.
- The impact of ϕ depends on the stage of robotic capital accumulation: it benefits labor at low levels of χ, but as robots become dominant, it favors robotic capital.
- The transition from labor to robotic capital dominance accelerates as $\chi \to 1$, marking a fundamental shift in income distribution and production dynamics.

While this analysis has provided insights into firm-level responses, it has remained within a static framework. The next section introduces a dynamic perspective, examining how robotic capital accumulates over time and whether its expansion follows a predictable trajectory or exhibits structural constraints.

5 Can Self-Sustainability Be Attained?

The rise of automation raises a fundamental question: *Can economies achieve self-sustaining automation, where robots accumulate indefinitely without reliance on human labor?* The answer depends on whether there exists a threshold level of robot productivity, A_R^*, beyond which automation perpetuates itself.

To address this question, we examine two frameworks. In the representative agent (RA) model, savings can accumulate indefinitely, suggesting that self-sustainability

may be possible. In contrast, the overlapping generations (OLG) model introduces savings constraints that fundamentally limit robot accumulation. However, if intergenerational wealth transfers (bequests) are allowed, these constraints may be relaxed, potentially enabling self-sustainability.

5.1 The Representative Agent Model

A key challenge in analyzing the attainability of self-sustainability in the representative agent framework arises when A_R grows at a fixed rate. In this case, the interest rate also increases asymptotically at a fixed rate, making it difficult to determine the optimal level of current consumption from the transversality conditions.

To address this issue and provide a well-defined comparison across different technological levels, we focus on a model where A_R is fixed. This allows us to characterize the balanced growth path for any given A_R and systematically analyze how variations in A_R affect equilibrium outcomes. By comparing these equilibria across different values of A_R, we assess whether there exists a threshold $\tilde{A}_R$ beyond which robots accumulate indefinitely, even when the technological parameters A_R, A_L, and ϕ remain fixed.

The balanced growth path of the economy is characterized by the Euler equation:

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(i - \rho), \tag{28}$$

where i is the real interest rate, σ is the inverse elasticity of intertemporal substitution, and ρ is the rate of time preference. For any given A_R, a balanced growth path exists, but its properties differ qualitatively depending on the level of A_R.

Pre-Self-Sustainability Phase ($A_R < \tilde{A}_R$) This phase encompasses both the labor-dominated and transition phases. When A_R is small, the economy follows a well-established balanced growth path similar to the standard neoclassical growth model. Initially, labor accounts for the majority of income in production, but as A_R increases, the share of labor income declines, and the economy gradually diverges from traditional growth dynamics. Despite these changes, capital accumulation remains bounded, and the long-run interest rate converges to:

$$i^* = \sigma g_{A_L} + \rho. \tag{29}$$

The growth rates of output, consumption, and wages are given by:

$$g_Y = g_C = g_W = g_{A_L}. \tag{30}$$

A key characteristic of this phase is the slowing convergence rate as χ increases. Appendix O shows that as χ^* rises, the rate at which the economy stabilizes declines, approaching zero as $\chi \to 1$. This means that the effects of A_R on interest rates and output growth persist for longer periods, making the economy increasingly resemble the self-sustainability regime even before reaching it. As labor's role diminishes,

small increases in A_R generate disproportionate effects on output and interest rates, reinforcing the growing dominance of automation in production.

These findings provide mathematical justification for the argument made in the introduction: while full self-sustainability ($\chi = 1$) may seem like a theoretical extreme, the economy behaves progressively more like it as χ approaches 1. The transition itself is significant—even when the economy has not yet reached full self-sustainability, automation already plays an increasing role in shaping long-run dynamics.

Self-Sustainability Regime ($A_R \geq \tilde{A}_R$) If A_R exceeds a certain threshold, the economy transitions fully into a self-sustaining automation regime. The threshold separating these two regimes is determined by comparing the long-run interest rate with its pre-self-sustainability steady-state level. Taking the limit of the marginal product of robots (MP_R) from equation (2) as $r \to \infty$ gives the lowest possible interest rate in the economy, $\alpha^{\frac{1}{\phi}} A_R - \delta$. Setting this equal to the steady-state interest rate in equation (29) and solving for A_R yields the threshold:

$$\tilde{A}_R = \alpha^{-1/\phi} \left(\sigma g_{A_L} + \rho + \delta \right). \tag{31}$$

For $A_R \geq \tilde{A}_R$, the interest rate never returns to its traditional balanced-growth level, and the economy no longer follows a neoclassical growth model. Instead, its dynamics resemble an $Y = AK$ economy, where capital accumulation becomes the primary driver of growth.

These results demonstrate that self-sustainability is possible in the RA model under appropriate conditions. We now turn to the OLG framework to examine whether a similar outcome can arise when individuals save only during their working years.

5.2 Overlapping Generations Model with Bequests

The overlapping generations (OLG) framework differs from the representative agent model because savings depend only on the income of young workers. Despite this constraint, self-sustainability remains possible under sufficiently strong intergenerational wealth transfers. Specifically, we show that for any bequest motive $\hat{\rho} > 0$, there exists a threshold A_R that enables self-sustaining automation.

5.2.1 Preferences and Utility Function

Each individual lives for two periods, working when young and consuming in both periods. The lifetime utility function is given by:

$$U = u(C_t) + \rho u(D_{t+1}) + \hat{\rho} u(B_{t+1}), \tag{32}$$

where C_t is consumption when young, D_{t+1} is old-age consumption, and B_{t+1} is the bequest left to the next generation. The parameter ρ represents the subjective discount factor, and $\hat{\rho}$ governs the strength of the bequest motive.

The utility function follows a constant relative risk aversion (CRRA) form:

$$u(X) = \frac{X^{1-\sigma} - 1}{1-\sigma}, \quad \text{for } \sigma \neq 1, \tag{33}$$

$$u(X) = \ln X, \quad \text{for } \sigma = 1. \tag{34}$$

5.2.2 Budget Constraints and Bequest Decision

Young individuals earn wage income W_t, consume C_t, and save the remainder. In the second period, they allocate their savings between old-age consumption D_{t+1} and bequests B_{t+1}. The budget constraint in the second period is:

$$D_{t+1} + B_{t+1} = (1 + i_{t+1})(B_t + W_t - C_t). \tag{35}$$

Individuals optimally choose C_t, D_{t+1}, and B_{t+1} to maximize (32), leading to the following optimal savings and bequest functions:

$$s_t = \frac{w_t + b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1 + i_{t+1})^{1-\sigma}}, \tag{36}$$

$$b_{t+1} = \frac{\hat{\rho}^\sigma (1 + i_{t+1})}{\rho^\sigma + \hat{\rho}^\sigma + (1 + i_{t+1})^{1-\sigma}}(w_t + b_t). \tag{37}$$

5.2.3 Robot Accumulation and Self-Sustainability Condition

The evolution of the robot-labor ratio r_t is given by:

$$(1 + g_{A_L}) r_{t+1} = A_{R,t+1} s_t. \tag{38}$$

Substituting (36) into (38) yields the transition equation governing robot accumulation over time.

For robots to accumulate indefinitely, the system must support unbounded growth in r_t over time. Taking the limit as $t \to \infty$, robot accumulation becomes self-sustaining if and only if:

$$\frac{\hat{\rho}^\sigma (A_R \alpha^{\frac{1}{\phi}})}{\rho^\sigma + \hat{\rho}^\sigma + (A_R \alpha^{\frac{1}{\phi}})^{1-\sigma}} \geq 1. \tag{39}$$

Equation (39) establishes the threshold condition for self-sustainability in the OLG model. A key result is that for any bequest motive $\hat{\rho} > 0$, there exists a sufficiently high level of robot productivity A_R that enables self-sustaining automation. This implies that while stronger intergenerational transfers facilitate self-sustainability, they are not strictly necessary; even small bequests can support unbounded robot accumulation if A_R is large enough.

An important implication of this result is that economies with a higher bequest motive require a lower threshold level of A_R to attain self-sustainability. When $\hat{\rho}$ is

large, savings and intergenerational wealth transfers provide sufficient capital accumulation to sustain automation growth at lower levels of robot productivity. Conversely, when $\hat{\rho}$ is small, achieving self-sustainability requires exceptionally high levels of A_R to compensate for weaker intergenerational transfers.

This result mirrors the findings from the representative agent model, where self-sustainability is also possible beyond a critical A_R. Thus, despite differences in how capital accumulation occurs in each framework, both models share the fundamental conclusion that automation can sustain itself indefinitely under appropriate technological and economic conditions.

5.2.4 Interest Rate Behavior and Persistence of Automation Effects

This persistence mirrors a similar effect in the RA model, where interest rates return to their steady-state more slowly the later the economy is in the transition phase. However, in the RA model, persistence arises due to intertemporal consumption smoothing, while in the OLG model, it stems from savings constraints and gradual robot accumulation. The similarity in long-run outcomes suggests that despite differences in how capital is accumulated, the core resultself-sustainability beyond a threshold $A_R$remains robust across different economic structures.

5.2.5 Conclusion

The presence of bequests plays a fundamental role in determining whether self-sustaining automation is possible in the OLG framework. When bequests are absent ($\hat{\rho} = 0$), the model reduces to the standard OLG setting, where savings constraints prevent unbounded capital accumulation, making self-sustainability impossible. However, when bequests are introduced, they relax the savings constraint, allowing intergenerational wealth transfers to sustain automation growth. If χ surpasses a critical threshold, robot accumulation becomes unbounded, enabling self-sustaining automation even within the OLG structure.

6 Simulating Economic Dynamics with Fixed Saving Rates

This section explores the dynamic behavior of an economy as the elasticity of substitution between labor and robots evolves over time. Using two scenarios, we examine how changes in elasticity influence economic growth, income distribution, and labor outcomes. Both simulations begin with a Cobb-Douglas production function, corresponding to $\phi \to 0$, on a balanced growth path consistent with the Solow model. In this baseline case, the specific values of A_R and A_L do not affect production individually; only the product $A_R^{\alpha} A_L^{1-\alpha}$ matters. However, when $\phi > 0$, the elasticity of substitution allows for differentiated contributions of labor and robots, making the individual values of A_R and A_L significant. For the same value of $A_R^{\alpha} A_L^{1-\alpha}$, but with different individual values of A_R and A_L, a transition from $\phi = 0$ to positive values

may produce jumps in output and marginal products. This motivates the approach outlined in the next paragraph, which is designed to avoid these jumps.

To model the transition from $\phi = 0$ to $\phi > 0$, we assume that firms adopt new technology in a way that ensures the total cost of production does not increase. Specifically, when ϕ changes from 0 to a positive value, A_R and A_L are adjusted such that two conditions are met: (1) the product $A_R^{\alpha} A_L^{1-\alpha}$ remains constant, regardless of the initial values of A_R and A_L; and (2) the total cost of production stays unchanged, implying that for a given amount of resources, the output does not change due to the shift in ϕ. This approach ensures that the marginal products of robots (MP_R) and labor (MP_L) also remain constant during the transition. These assumptions prevent abrupt and unrealistic jumps in output and marginal products, allowing for a smooth progression of economic dynamics.

The simulations use the following key parameters: $\alpha = \frac{1}{3}$, $\delta = 0.05$, $g = 0.02$, and the savings rate is $s = 0.2$.

The scenarios differ primarily in how ϕ, the parameter governing the elasticity of substitution between labor and robots, evolves over time.

6.1 Scenario 1: Abrupt Shift in Elasticity

In the first scenario, ϕ increases suddenly from 0 to 0.25 ten years after the simulation begins ($T_{\text{shock}} = 10$). This abrupt change represents a technological breakthrough that significantly enhances the substitutability between labor and robots, shifting the elasticity of substitution from unity ($1/(1-\phi) = 1$) to 4/3 ($1/(1-0.25)$). Figure 3 illustrates the results.

As shown in the figure, the interest rate and output growth rate rise significantly following the shift in ϕ. However, the marginal product of labor (MP_L) grows at a slower pace than output, indicating that the benefits of growth increasingly favor capital over labor. These dynamics reflect the growing role of robots in production as the elasticity of substitution rises, reducing the relative importance of labor.

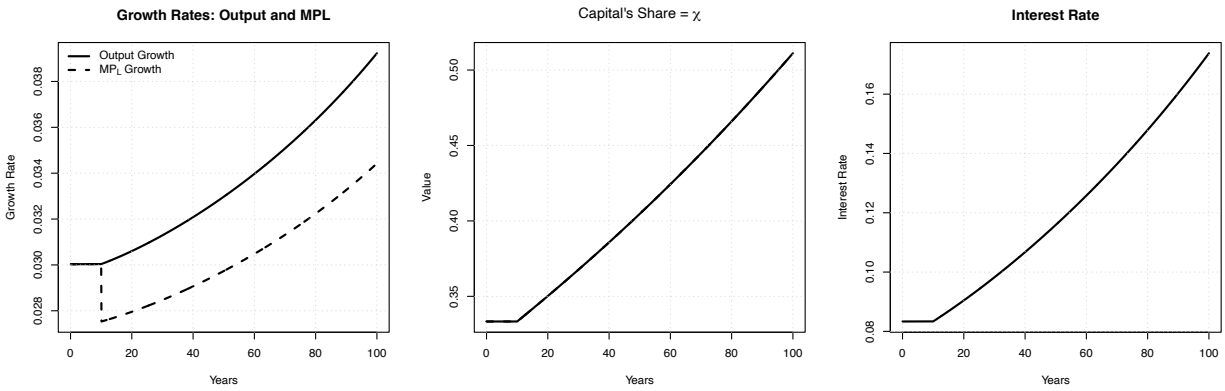


Figure 3: Simulation results for an abrupt shift in ϕ from 0 to 0.25 at $T_{\text{shock}} = 10$, showing changes in the interest rate, output growth, and marginal product of labor. Key parameters: $\alpha = \frac{1}{3}$, $\delta = 0.05$, $s = 0.2$, $g = 0.02$, $T = 100$, $dt = 0.1$.

6.2 Scenario 2: Gradual Increase in Elasticity

In the second scenario, the elasticity of substitution between labor and robots, given by $1/(1-\phi)$, grows at a steady rate of 1% per year. This reflects continuous technological progress, gradually enhancing the substitutability of labor with robots. Figure 4 presents the results.

Compared to the abrupt shift in Scenario 1, the gradual increase in elasticity produces more profound long-term changes. The interest rate and output growth rate accelerate rapidly over time, while the growth rate of MP_L remains subdued. Eventually, wages stagnate, as indicated by the slow growth of MP_L, highlighting a long-term shift in income distribution favoring capital. This scenario underscores the compounding effects of steady technological progress and its implications for labor markets in a robot-intensive production regime.

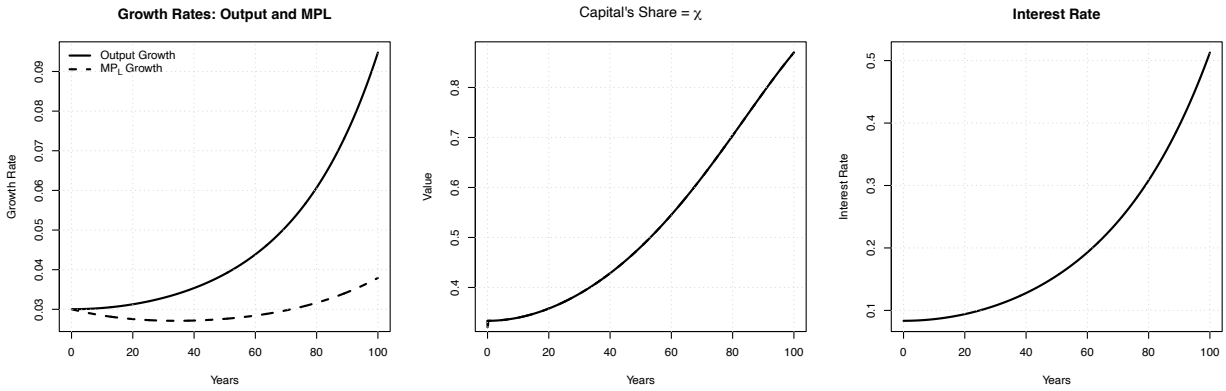


Figure 4: Simulation results for a gradual increase in elasticity at 1% per year, showing long-term changes in economic growth, interest rates, and marginal product of labor. Key parameters: $\alpha = \frac{1}{3}$, $\delta = 0.05$, $s = 0.2$, $g = 0.02$, $T = 100$, $dt = 0.1$.

6.3 Comparison of Scenarios

Both scenarios demonstrate how increasing the elasticity of substitution between labor and robots impacts economic dynamics. However, the magnitude and trajectory of these changes differ significantly. The gradual increase in elasticity in Scenario 2 leads to more pronounced acceleration in interest rates and output growth compared to the abrupt shift in Scenario 1. Additionally, wage stagnation emerges only in Scenario 2, underscoring the long-term challenges for labor in a robot-dominated production regime.

These findings emphasize the critical role of elasticity dynamics in shaping economic outcomes during the transition to robot-intensive production. Policymakers must carefully consider the distributional implications of such transitions to ensure balanced and inclusive economic growth. Future research could explore how alternative saving rates or policy interventions might mitigate the adverse effects on labor.

7 Extensions

This section explores two extensions to the baseline framework: a multisector model and the incorporation of competition for resources. Both extensions aim to enhance the relevance and robustness of the results by addressing complexities present in real-world economies.

7.1 Multisector Model

The baseline framework assumes a single, aggregate sector, which simplifies analysis but overlooks the variation in how different sectors depend on labor, capital, and technology. To address this, we extend the framework to a multisector model where production in each sector follows (1) with sector-specific parameters. We focus on the case $\phi \in (0, 1)$, implying an elasticity of substitution between robots and labor greater than 1.

Two key scenarios emerge in this setting:

First, the demand for sectoral output Y_i may be bounded due to the influence of other sectors and price constraints from monopolistic power. Under these conditions, neither robots' nor labor's share in production converges to zero, as the marginal productivity of each factor rises sharply when its quantity diminishes, and output remains finite, limiting robot accumulation.

Second, if Y_i grows without bound, this sector eventually dominates others that do not exhibit unbounded growth. In this case, the analysis effectively reduces to the benchmark single-sector model, with the dominant sector driving overall economic dynamics.

This extension highlights how sectoral heterogeneity shapes the transition dynamics of an economy, while remaining tractable for policy analysis. Further investigation could provide valuable insights into optimal policies and the path of economic transitions in multisector contexts.

7.2 Competition for Renewable Resources

This section examines the role of resource competition in shaping economic outcomes, focusing on renewable resources, such as capital, and their implications for the distribution of income. The production function in this framework is specified as:

$$Y = \left[\alpha(A_R R)^{\phi} + (1-\alpha)Q^{\phi}\right]^{\frac{1}{\phi}}, \tag{40}$$

where $Q = K^{\beta} \cdot (A_L L)^{1-\beta}$. Here, K represents capital, L represents labor, A_R and A_L are productivity parameters for robots and labor, respectively, α is the weight of robots in production, ϕ determines the elasticity of substitution, and β specifies the contribution of capital to Q. The relative utilization of robots versus the composite input is redefined as:

$$r = \frac{A_R R}{Q}. \tag{41}$$

The model shows that income is distributed among three factors when robots and labor compete for renewable resources. For $\chi < 1$, the robot share is χ, the capital share is $\beta(1-\chi)$, and the labor share is $(1-\beta)(1-\chi)$. As χ approaches or exceeds 1, the robot share converges to 1, leading robots to dominate production. This distribution illustrates how resource competition modifies factor shares, while preserving the coexistence of robots and labor in equilibrium under fixed interest rates.

Although the necessary formulas are not derived explicitly in this section, the analysis follows the same structure as Section (**??**). The primary distinction lies in the income distribution, where an additional division occurs between capital and labor within the composite input Q. This adjustment refines the framework without altering the essential mechanisms of factor interaction.

7.3 Fixed Resources

In this section, we extend the analysis to scenarios where robots and labor compete for fixed resources, such as land. Unlike renewable resources, fixed resources impose stricter constraints on factor allocation, leading to fundamentally different equilibrium dynamics. In such cases, robots can fully displace labor from production even when $\chi < 1$. This occurs because the fixed supply of resources limits the ability of labor to maintain its share in production, allowing robots to dominate despite not achieving full self-sustainability.

Although fixed resources currently play a limited role in the economy, their significance could grow as technological progress advances and the economy transitions toward a self-sustainability phase. This shift could fundamentally alter income distribution and production dynamics, as the competition for fixed resources amplifies the advantage of robots over labor.

8 Policy Implications

8.1 Taxation in a Self-Sustaining Economy

The transition toward robotic self-sustainability fundamentally alters the feasibility and efficiency of different taxation mechanisms. As the labor share of income converges to zero, traditional labor taxation becomes unsustainable, both practically and equitably. This necessitates a shift in the tax base from labor to alternative sources, primarily consumption.

8.1.1 The Breakdown of Labor Taxation

In economies where labor income constitutes a substantial portion of GDP, labor taxation serves as a primary revenue source. However, as robotic automation progresses and the self-sustainability index χ increases, the share of income accruing to labor declines. In the limit, when labors share vanishes, it becomes impossible to collect any meaningful revenue from labor taxation.

Moreover, labor taxation exacerbates income inequality during the transition. Since capital owners are the primary beneficiaries of automation, imposing taxes on labor further concentrates wealth, accelerating disparities in income distribution. This reinforces the urgency of transitioning away from labor taxation before labor becomes fully obsolete.

8.1.2 The Inefficiency of Capital Taxation

An alternative response might be to increase capital taxation, ensuring that the gains from robotization are redistributed. However, the literature on capital taxation (e.g., Chamley, 1986; Judd, 1985) has long established that capital taxation is distortionary, leading to reduced investment and long-run economic inefficiencies.

In the context of a self-sustaining robotic economy, capital taxation introduces an additional inefficiency: it raises the technological threshold required for achieving self-sustainability. By discouraging capital accumulation and innovation, it delays the transition to an economy where robots sustain production independently of human labor. This implies that capital taxation not only reduces efficiency in the standard sense but also directly inhibits technological progress.

8.1.3 Consumption Taxation as the Optimal Alternative

Given the breakdown of labor taxation and the inefficiency of capital taxation, consumption taxation emerges as the only sustainable and efficient fiscal instrument. A shift from labor to consumption taxation is equivalent, in present-value terms, to a lump-sum tax on capital, making it non-distortionary. Unlike capital taxation, it does not deter investment, and unlike labor taxation, it remains viable even when labor income disappears.

In the limit, when labors share is zero, consumption is the only feasible tax base. This insight suggests that governments should progressively restructure their fiscal policies to rely more heavily on consumption-based taxation as economies approach higher levels of robotic self-sustainability.

8.1.4 Optimal Taxation in a Self-Sustaining Economy

To formalize these insights, we derive the optimal taxation problem in an economy transitioning toward robotic self-sustainability. Specifically, we analyze the dynamic path of tax rates as labor income declines and capital returns increase.

[Placeholder for formal derivation of the optimal taxation problem.]

8.1.5 Universal Basic Income and Redistribution

The shift to consumption taxation also provides a natural framework for financing redistribution mechanisms such as universal basic income (UBI). Since capital owners capture most of the gains from automation, a broad-based consumption tax ensures

that wealth redistribution occurs without introducing efficiency losses associated with direct capital taxation.

Although the debate on UBI extends beyond the scope of this paper, our results indicate that consumption taxation can sustain such a policy while preserving economic efficiency. Governments should therefore explore consumption-based redistribution mechanisms as part of a broader strategy to manage the transition to robotic self-sustainability.

8.2 Monetary Policy in a Self-Sustaining Economy

The transition toward robotic self-sustainability presents fundamental challenges for monetary policy. Traditional frameworks, such as the Taylor rule, rely heavily on labor market indicators to assess economic slack and determine appropriate interest rate adjustments. However, as the labor share of income declines, these measures become increasingly unreliable, requiring a rethinking of monetary policy rules and objectives.

8.2.1 The Declining Relevance of the Labor Market in Monetary Policy

In conventional monetary policy frameworks, the unemployment rate serves as a key measure of economic slack, influencing estimates of the output gap. The Taylor rule, for example, sets the interest rate based on inflation and the deviation of output from its potential level, with the latter often inferred from labor market conditions.

As robotic automation progresses and the self-sustainability index χ increases, the role of labor in production diminishes, rendering unemployment a poor indicator of economic performance. In the limit, where human labor becomes negligible, the unemployment rate ceases to be meaningful, yet the economy may still operate below or above potential output. This structural shift implies that monetary policy must move away from labor-based measures of slack and adopt alternative indicators such as:

- *Capital utilization rates*: The extent to which productive capital and robotic infrastructure are employed in the economy.
- *Technological deployment*: Metrics capturing the share of automation in active use relative to its full potential.
- *Consumption and wealth-based indicators*: Since income distribution increasingly shifts toward capital owners, monetary transmission mechanisms may depend more on capital market conditions than on wage-driven consumption.

A failure to adjust monetary policy frameworks accordingly could lead to destabilizing policy responses based on misleading measures of economic slack.

8.2.2 The Endogenous Shift in the Natural Interest Rate

Another challenge for monetary policy arises from the dynamic behavior of the natural interest rate r^*. In standard models, r^* is often assumed to be stable in the long run, adjusting only in response to demographic changes, productivity growth, or shifts in savings behavior. However, in a capital-driven economy where technological progress fuels economic growth, the natural interest rate is no longer constant but may rise over time at the rate of technological progress.

This has significant implications for monetary policy:

- Interest rate rules must account for a rising r^*. If monetary authorities continue to use frameworks designed for a constant long-run interest rate, they risk excessive monetary tightening as automation accelerates.
- The Phillips curve relationship weakens. As labors role in production diminishes, inflationary dynamics may become increasingly disconnected from labor market conditions.
- Policy adjustments must be forward-looking. Central banks may need to develop monetary policy rules that explicitly account for the trajectory of r^* rather than relying on static equilibrium assumptions.

The shift toward robotic self-sustainability thus necessitates a fundamental reconsideration of monetary policy frameworks, moving beyond traditional labor-based indicators and adjusting to an evolving interest rate environment.

[Potential extension: A formal derivation of a modified monetary policy rule accounting for a rising r^* and new slack measures.]

9 Conclusion

The transition toward robotic self-sustainability represents a fundamental shift in the structure of the economy, redefining the roles of labor, capital, and technology. This paper introduces the Self-Sustainability Index χ, a measure of the relative progress toward an economy in which robots sustain production independently of human labor. As χ increases, the labor share of income declines, leading to profound implications for taxation, monetary policy, and broader economic governance.

One of the primary challenges in this transition is the restructuring of fiscal policy. As labor taxation becomes increasingly infeasible and capital taxation remains distortionary, consumption taxation emerges as the only sustainable and efficient revenue source. We show that switching to consumption-based taxation is non-distortionary and ensures that government revenue remains stable even as labor's economic role diminishes. Additionally, capital taxation increases the threshold for achieving self-sustainability, potentially delaying technological progress and reducing the overall benefits of automation.

Monetary policy also faces a structural transformation. Traditional policy frameworks, such as the Taylor rule, rely heavily on labor market indicators to assess economic slack. However, as labor's role diminishes, unemployment ceases to be a meaningful measure of economic performance. This necessitates a shift toward alternative indicators, such as capital utilization and technological deployment, to better assess the output gap. Furthermore, the natural interest rate r^* is no longer constant but may rise at the rate of technological progress, requiring central banks to adjust their policy frameworks accordingly.

Beyond taxation and monetary policy, the broader economic implications of robotic self-sustainability warrant further exploration. The transition may require new approaches to labor market adaptation, industrial policy, and social welfare. Ensuring equitable distribution of the gains from automation will be critical to maintaining social stability, particularly as the share of income accruing to capital increases. Future research should explore optimal mechanisms for redistributing wealth, the role of government intervention in managing technological displacement, and the long-term evolution of economic growth under self-sustaining automation.

As economies approach higher levels of robotic self-sustainability, policymakers must proactively adapt fiscal and monetary frameworks to prevent destabilizing effects. The findings in this paper provide a foundation for understanding these challenges and offer a framework for designing policies that ensure economic stability, efficiency, and equitable growth in a world increasingly driven by automation.

References

KARABARBOUNIS, L. (2024): "Perspectives on the Labor Share," Journal of Economic Perspectives, 38, 10736.

Appendix

A Capital's share and χ

B Equilibrium relation between r and χ

The equilibrium condition is derived as follows. We begin with the first-order condition (7):

$$i + \delta = MP_R,$$

where the marginal product of capital is given by (**??**):

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Substituting this expression into the first-order condition yields:

$$i + \delta = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Next, we use the definition of the critical level of productivity defined by (4):

$$A_R^{\text{crit}} = (i+\delta)\, \alpha^{-\frac{1}{\phi}}.$$

Substituting the previous result for $i + \delta$, we obtain:

$$A_R^{\text{crit}} = \alpha^{\frac{\phi-1}{\phi}} A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Using the definition of χ from (5):

$$\chi \equiv \left(\frac{A_R}{A_R^{\text{crit}}}\right)^{\frac{\phi}{1-\phi}},$$

we substitute for A_R^{crit}:

$$\chi = \left(\frac{A_R}{\alpha^{\frac{\phi-1}{\phi}} A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}}\right)^{\frac{\phi}{1-\phi}}.$$

This simplifies to:

$$\chi = \alpha \left[\alpha + (1-\alpha) r^{-\phi}\right]^{-1}.$$

Divide both sides by α:

$$\frac{\chi}{\alpha} = \left[\alpha + (1-\alpha) r^{-\phi}\right]^{-1}.$$

Inverting both sides:

$$\frac{\alpha}{\chi} = \alpha + (1-\alpha) r^{-\phi}.$$

Subtracting α from both sides:

$$\frac{\alpha}{\chi} - \alpha = (1-\alpha)r^{-\phi}.$$

Factoring α on the left-hand side:

$$\alpha\left(\frac{1}{\chi} - 1\right) = (1-\alpha)r^{-\phi}.$$

Using $\frac{1}{\chi} - 1 = \frac{1-\chi}{\chi}$, we rewrite:

$$\alpha\frac{1-\chi}{\chi} = (1-\alpha)r^{-\phi}.$$

Dividing both sides by α:

$$\frac{1-\chi}{\chi} = \frac{1-\alpha}{\alpha}r^{-\phi}.$$

Inverting both sides:

$$\boxed{\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha}r^{\phi}}.$$

C Equilibrium Representation Using χ

C.1 Equilibrium y as a function of χ

The expression for y as a function of χ is derived as follows. We start with the definition of y:

$$y = \frac{Y}{A_L L}.$$

Substituting the production function from (1):

$$y = \frac{\left[\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right]^{\frac{1}{\phi}}}{A_L L}.$$

Rewriting $\frac{1}{A_L L}$ as $(A_L L)^{-\phi}$ raised to the power of $\frac{1}{\phi}$:

$$y = \left[\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right]^{\frac{1}{\phi}} \cdot \left((A_L L)^{-\phi}\right)^{\frac{1}{\phi}}.$$

Since both terms are raised to $\frac{1}{\phi}$, we combine the exponents:

$$y = \left[\frac{\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}}{(A_L L)^{\phi}}\right]^{\frac{1}{\phi}}.$$

Dividing each term in the numerator by the denominator:

$$y = \left[\alpha \frac{(A_R R)^\phi}{(A_L L)^\phi} + (1-\alpha)\frac{(A_L L)^\phi}{(A_L L)^\phi}\right]^{\frac{1}{\phi}}.$$

Since $\frac{(A_L L)^\phi}{(A_L L)^\phi} = 1$ and using the definition $r = \frac{A_R R}{A_L L}$, we rewrite:

$$y = \left[\alpha r^\phi + (1-\alpha)\right]^{\frac{1}{\phi}}.$$

Using the equilibrium condition from Appendix B:

$$r^\phi = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we substitute:

$$y = \left[\alpha \left(\frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}\right) + (1-\alpha)\right]^{\frac{1}{\phi}}.$$

Canceling α in the first term:

$$y = \left[\frac{\chi}{1-\chi}(1-\alpha) + (1-\alpha)\right]^{\frac{1}{\phi}}.$$

Factoring $(1-\alpha)$:

$$y = \left[(1-\alpha)\left(\frac{\chi}{1-\chi} + 1\right)\right]^{\frac{1}{\phi}}.$$

Since $\frac{\chi}{1-\chi} + 1 = \frac{1}{1-\chi}$, we simplify:

$$\boxed{y = \left[\frac{1-\alpha}{1-\chi}\right]^{\frac{1}{\phi}}}.$$

This completes the derivation.

C.2 MP_R as a function of χ

The expression for MP_R as a function of χ is derived as follows. We start with the equation for MP_R from (**??**):

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Using the equilibrium condition from Appendix B:

$$r^\phi = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we take the reciprocal to express $r^{-\phi}$:

$$r^{-\phi} = \frac{1-\chi}{\chi} \cdot \frac{\alpha}{1-\alpha}.$$

Substituting this into (**??**), we obtain:

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) \cdot \frac{1-\chi}{\chi} \cdot \frac{\alpha}{1-\alpha}\right]^{\frac{1-\phi}{\phi}}.$$

Canceling $(1-\alpha)$ in the fraction:

$$MP_R = \alpha A_R \left[\alpha + \alpha \frac{1-\chi}{\chi}\right]^{\frac{1-\phi}{\phi}}.$$

Factoring out α inside the brackets:

$$MP_R = \alpha A_R \left[\alpha \left(1 + \frac{1-\chi}{\chi}\right)\right]^{\frac{1-\phi}{\phi}}.$$

Since $1 + \frac{1-\chi}{\chi} = \frac{1}{\chi}$, we simplify:

$$MP_R = \alpha A_R \left[\alpha \frac{1}{\chi}\right]^{\frac{1-\phi}{\phi}}.$$

Distributing the exponent $\frac{1-\phi}{\phi}$:

$$MP_R = \alpha A_R \alpha^{\frac{1-\phi}{\phi}} \chi^{-\frac{1-\phi}{\phi}}.$$

Combining the exponents of α:

$$MP_R = A_R \alpha^{1+\frac{1-\phi}{\phi}} \chi^{-\frac{1-\phi}{\phi}}.$$

Since $1 + \frac{1-\phi}{\phi} = \frac{\phi+1-\phi}{\phi} = \frac{1}{\phi}$, this simplifies to:

$$\boxed{MP_R = A_R \alpha^{\frac{1}{\phi}} \chi^{-\frac{1-\phi}{\phi}}}.$$

This completes the derivation.

C.3 w as a function of χ

The expression for w as a function of χ is derived as follows. We start with the production function:

$$Y = \left[\alpha (A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right]^{\frac{1}{\phi}}.$$

To find MP_L, we take the derivative of the production function with respect to L:

$$MP_L = \frac{\partial Y}{\partial L}.$$

Applying the chain rule:

$$MP_L = \frac{1}{\phi}\left[\alpha(A_R R)^\phi + (1-\alpha)(A_L L)^\phi\right]^{\frac{1}{\phi}-1} \cdot (1-\alpha)\phi(A_L L)^{\phi-1} A_L.$$

Canceling ϕ, we obtain:

$$MP_L = (1-\alpha)A_L(A_L L)^{\phi-1}\left[\alpha(A_R R)^\phi + (1-\alpha)(A_L L)^\phi\right]^{\frac{1}{\phi}-1}.$$

Rewriting $(A_L L)^{\phi-1}$ as:

$$(A_L L)^{\phi-1} = \left((A_L L)^{-\phi}\right)^{\frac{1-\phi}{\phi}},$$

and the exponent $\frac{1}{\phi} - 1$ as $\frac{1-\phi}{\phi}$, we express MP_L as:

$$MP_L = (1-\alpha)A_L\left((A_L L)^{-\phi}\right)^{\frac{1-\phi}{\phi}}\left[\alpha(A_R R)^\phi + (1-\alpha)(A_L L)^\phi\right]^{\frac{1-\phi}{\phi}}.$$

Since both terms are raised to $\frac{1-\phi}{\phi}$, we combine the exponents:

$$MP_L = (1-\alpha)A_L\left[\frac{\alpha(A_R R)^\phi + (1-\alpha)(A_L L)^\phi}{(A_L L)^\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Dividing each term in the numerator by the denominator:

$$MP_L = (1-\alpha)A_L\left[\alpha\frac{(A_R R)^\phi}{(A_L L)^\phi} + (1-\alpha)\frac{(A_L L)^\phi}{(A_L L)^\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Since $\frac{(A_L L)^\phi}{(A_L L)^\phi} = 1$ and using the definition $r = \frac{A_R R}{A_L L}$, we rewrite:

$$MP_L = (1-\alpha)A_L\left[\alpha r^\phi + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Using the equilibrium condition from Appendix B:

$$r^\phi = \frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha},$$

we substitute:

$$MP_L = (1-\alpha)A_L\left[\alpha\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right) + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Canceling α in the first term:

$$MP_L = (1-\alpha)A_L\left[\frac{\chi}{1-\chi}(1-\alpha) + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Factoring $(1-\alpha)$:

$$MP_L = (1-\alpha)A_L\left[(1-\alpha)\left(\frac{\chi}{1-\chi}+1\right)\right]^{\frac{1-\phi}{\phi}}.$$

Since $\frac{\chi}{1-\chi} + 1 = \frac{1}{1-\chi}$, we simplify further:

$$MP_L = (1-\alpha)A_L \left[(1-\alpha) \cdot \frac{1}{1-\chi}\right]^{\frac{1-\phi}{\phi}}.$$

Combining the powers of $(1-\alpha)$:

$$MP_L = A_L(1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Using $w = \frac{MP_L}{A_L}$, we obtain:

$$\boxed{w = (1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}}.$$

This completes the derivation.

D The value of χ in equilibrium

D.1 Short-run value of χ

To derive the short-run equilibrium expression for χ, we start with the relationship between χ and r^ϕ from Appendix B:

$$r^\phi = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}.$$

Rearranging for $\frac{\chi}{1-\chi}$, we multiply both sides by $\frac{\alpha}{1-\alpha}$:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha} r^\phi.$$

Since in the short run, R is given, the capital-labor ratio is fixed at:

$$r = \frac{A_R R}{A_L L}.$$

Raising both sides to the power ϕ, we obtain:

$$r^\phi = \left(\frac{A_R R}{A_L L}\right)^\phi.$$

Substituting this into our previous expression for $\frac{\chi}{1-\chi}$:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha}\left(\frac{A_R R}{A_L L}\right)^\phi.$$

This equation determines χ in terms of the given variables in the short-run equilibrium.

D.2 Long-run value of χ

To derive the long-run equilibrium value of χ, we start with its definition:

$$\chi \equiv \left(\frac{A_R}{A_R^{\text{crit}}} \right)^{\frac{\phi}{1-\phi}}.$$

From Appendix 4, the critical productivity level is given by:

$$A_R^{\text{crit}} = (i + \delta)\, \alpha^{-\frac{1}{\phi}}.$$

Substituting this into the definition of χ, we obtain:

$$\chi = \left(\frac{A_R}{(i+\delta)\alpha^{-\frac{1}{\phi}}} \right)^{\frac{\phi}{1-\phi}}.$$

Rewriting the denominator by factoring $\alpha^{-\frac{1}{\phi}}$ outside the parentheses:

$$\chi = \left(A_R \cdot \left[(i+\delta)^{-1} \alpha^{\frac{1}{\phi}} \right] \right)^{\frac{\phi}{1-\phi}}.$$

Distributing the exponent $\frac{\phi}{1-\phi}$ across the terms:

$$\chi = A_R^{\frac{\phi}{1-\phi}} (i+\delta)^{-\frac{\phi}{1-\phi}} \alpha^{\frac{1}{1-\phi}}.$$

Finally, taking $\alpha^{\frac{1}{1-\phi}}$ out of the parentheses:

$$\boxed{\chi = \alpha^{\frac{1}{1-\phi}} \left(\frac{A_R}{i+\delta} \right)^{\frac{\phi}{1-\phi}}}.$$

This expression provides the long-run equilibrium value of χ as a function of the models parameters.

E Elasticities with respect to A_R and ϕ

E.1 Short-run elasticity of χ with respect to A_R

E.1.1 Method using the equilibrium value of χ

To analyze the response of χ to an increase in A_R in the short run, we compute the elasticity of χ with respect to A_R. We start with the short-run equilibrium equation for χ, given in Appendix D.1:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha} \left(\frac{A_R R}{A_L L} \right)^{\phi}.$$

Taking the natural logarithm on both sides:

$$\ln \chi - \ln(1-\chi) = \ln\left(\frac{\alpha}{1-\alpha}\right) + \phi\left(\ln A_R + \ln R - \ln A_L - \ln L\right).$$

Differentiating both sides with respect to A_R:

$$\frac{1}{\chi}\frac{d\chi}{dA_R} - \frac{1}{1-\chi}\frac{d(1-\chi)}{dA_R} = \phi\frac{1}{A_R}.$$

Since $\frac{d(1-\chi)}{dA_R} = -\frac{d\chi}{dA_R}$, this simplifies to:

$$\left(\frac{1}{\chi} + \frac{1}{1-\chi}\right)\frac{d\chi}{dA_R} = \frac{\phi}{A_R}.$$

Rewriting the term in parentheses using a common denominator:

$$\frac{1}{\chi} + \frac{1}{1-\chi} = \frac{1}{\chi(1-\chi)}.$$

Substituting this:

$$\frac{1}{\chi(1-\chi)}\frac{d\chi}{dA_R} = \frac{\phi}{A_R}.$$

Multiplying both sides by $\chi(1-\chi)$ to isolate $\frac{d\chi}{dA_R}$:

$$\frac{d\chi}{dA_R} = \phi \cdot \frac{\chi(1-\chi)}{A_R}.$$

To compute the elasticity, we multiply both sides by $\frac{A_R}{\chi}$:

$$\frac{d\ln\chi}{d\ln A_R} = \frac{A_R}{\chi}\cdot\frac{d\chi}{dA_R}.$$

Substituting $\frac{d\chi}{dA_R}$:

$$\frac{d\ln\chi}{d\ln A_R} = \phi(1-\chi).$$

Thus, the elasticity of χ with respect to A_R in the short run is:

$$\boxed{\frac{d\ln\chi}{d\ln A_R} = \phi(1-\chi)}.$$

E.1.2 Alternative derivation of elasticity of χ wrt A_R to check the math

We derive the short-term elasticity of χ with respect to A_R, starting from the marginal productivity condition:

$$MP_R = \alpha A_R\left[\alpha + (1-\alpha)r^{-\phi}\right]^{\frac{1-\phi}{\phi}},$$

with the first-order condition:

$$i + \delta = MP_R.$$

Taking natural logarithms on both sides:

$$\ln(i+\delta) = \ln\alpha + \ln A_R + \frac{1-\phi}{\phi}\ln\left[\alpha + (1-\alpha)r^{-\phi}\right].$$

Differentiating with respect to A_R:

$$\frac{1}{i+\delta}\frac{d(i+\delta)}{dA_R} = \frac{1}{A_R} + \frac{1-\phi}{\phi}\cdot\frac{-\phi(1-\alpha)r^{-\phi-1}\frac{dr}{dA_R}}{\alpha+(1-\alpha)r^{-\phi}}.$$

Since $\frac{dr}{dA_R} = \frac{r}{A_R}$, substituting this yields:

$$\frac{1}{i+\delta}\frac{d(i+\delta)}{dA_R} = \frac{1}{A_R} - \frac{(1-\phi)(1-\alpha)r^{-\phi}}{\phi(\alpha+(1-\alpha)r^{-\phi})}\cdot\frac{\phi}{A_R}.$$

Multiplying both sides by A_R:

$$\varepsilon_{i+\delta,A_R} = 1 - \frac{(1-\phi)(1-\alpha)r^{-\phi}}{\alpha+(1-\alpha)r^{-\phi}}.$$

Using the definition:

$$\chi \equiv \left(\frac{A_R}{A_R^{\text{crit}}}\right)^{\frac{\phi}{1-\phi}},$$

where

$$A_R^{\text{crit}} = (i+\delta)\alpha^{-\frac{1}{\phi}},$$

taking logarithms:

$$\ln\chi = \frac{\phi}{1-\phi}\left(\ln A_R - \ln(i+\delta) + \frac{1}{\phi}\ln\alpha\right).$$

Differentiating both sides:

$$\frac{1}{\chi}\frac{d\chi}{dA_R} = \frac{\phi}{1-\phi}\left(\frac{1}{A_R} - \frac{1}{i+\delta}\frac{d(i+\delta)}{dA_R}\right).$$

Multiplying by A_R:

$$\varepsilon_{\chi,A_R} = \frac{\phi}{1-\phi}\left(1 - \varepsilon_{i+\delta,A_R}\right).$$

Substituting $\varepsilon_{i+\delta,A_R}$:

$$\varepsilon_{\chi,A_R} = \frac{\phi}{1-\phi}\left(1 - \left(1 - \frac{(1-\phi)(1-\alpha)}{\alpha r^{\phi}+(1-\alpha)}\right)\right).$$

Simplifying:

$$\varepsilon_{\chi,A_R} = \frac{\phi(1-\alpha)}{\alpha r^{\phi} + (1-\alpha)}.$$

Using the substitution:

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we obtain:

$$\varepsilon_{\chi,A_R} = \frac{\phi(1-\alpha)}{\alpha\left(\frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}\right) + (1-\alpha)}.$$

Factoring $(1-\alpha)$ in the denominator:

$$\varepsilon_{\chi,A_R} = \frac{\phi(1-\alpha)}{(1-\alpha)\left(\frac{\chi}{1-\chi} + 1\right)}.$$

Canceling $(1-\alpha)$:

$$\varepsilon_{\chi,A_R} = \frac{\phi}{\frac{\chi}{1-\chi} + 1}.$$

Since:

$$\frac{\chi}{1-\chi} + 1 = \frac{1}{1-\chi},$$

we arrive at the final result:

$$\varepsilon_{\chi,A_R} = \phi(1-\chi).$$

This completes the derivation.

E.2 Sort-run semi-elasticity of χ wrt ϕ

To analyze the response of χ to an increase in ϕ in the short run, we compute the semi-elasticity of χ with respect to ϕ. We start with the short-run equilibrium equation for χ, given in Appendix D.1:

$$\frac{\chi}{1-\chi} = \frac{\alpha}{1-\alpha}\left(\frac{A_R R}{A_L L}\right)^{\phi}.$$

Taking the natural logarithm on both sides:

$$\ln\left(\frac{\chi}{1-\chi}\right) = \ln\left(\frac{\alpha}{1-\alpha}\right) + \ln\left[\left(\frac{A_R R}{A_L L}\right)^{\phi}\right].$$

Using the logarithm power rule $\ln(x^a) = a \ln x$, we simplify:

$$\ln\left(\frac{\chi}{1-\chi}\right) = \ln\left(\frac{\alpha}{1-\alpha}\right) + \phi \ln\left(\frac{A_R R}{A_L L}\right).$$

Using the definition $r = \frac{A_R R}{A_L L}$, we rewrite the equation as:

$$\ln \chi - \ln(1-\chi) = \ln\left(\frac{\alpha}{1-\alpha}\right) + \phi \ln r.$$

Differentiating both sides with respect to ϕ:

$$\frac{1}{\chi}\frac{d\chi}{d\phi} - \frac{1}{1-\chi}\frac{d(1-\chi)}{d\phi} = \ln r.$$

Since $\frac{d(1-\chi)}{d\phi} = -\frac{d\chi}{d\phi}$, this simplifies to:

$$\frac{1}{\chi}\frac{d\chi}{d\phi} + \frac{1}{1-\chi}\frac{d\chi}{d\phi} = \ln r.$$

Factoring out $\frac{d\chi}{d\phi}$:

$$\left(\frac{1}{\chi} + \frac{1}{1-\chi}\right)\frac{d\chi}{d\phi} = \ln r.$$

Rewriting the term in parentheses using a common denominator:

$$\frac{1}{\chi} + \frac{1}{1-\chi} = \frac{1}{\chi(1-\chi)},$$

we obtain:

$$\frac{1}{\chi(1-\chi)}\frac{d\chi}{d\phi} = \ln r.$$

Multiplying both sides by $\chi(1-\chi)$ to isolate $\frac{d\chi}{d\phi}$:

$$\frac{d\chi}{d\phi} = \chi(1-\chi)\ln r.$$

To compute the **semi-elasticity**, we divide both sides by χ:

$$\frac{1}{\chi}\frac{d\chi}{d\phi} = (1-\chi)\ln r.$$

By definition, the semi-elasticity is:

$$\frac{d\ln\chi}{d\phi} = (1-\chi)\ln r.$$

Thus, the semi-elasticity of χ with respect to ϕ in the short run is:

$$\frac{d\ln\chi}{d\phi} = (1-\chi)\ln r.$$

E.2.1 Long-run semi-elaticity of χ with respect to ϕ

To analyze the response of χ to an increase in ϕ in the long run, we compute the semi-elasticity of χ with respect to ϕ. We start with the long-run equilibrium expression for χ, given in Appendix D.2:

$$\chi = \alpha^{\frac{1}{1-\phi}} \left(\frac{A_R}{i+\delta} \right)^{\frac{\phi}{1-\phi}}.$$

Taking the natural logarithm on both sides:

$$\ln \chi = \ln \left[\alpha^{\frac{1}{1-\phi}} \right] + \ln \left[\left(\frac{A_R}{i+\delta} \right)^{\frac{\phi}{1-\phi}} \right].$$

Using the logarithm power rule $\ln(x^a) = a \ln x$, we simplify:

$$\ln \chi = \frac{1}{1-\phi} \ln \alpha + \frac{\phi}{1-\phi} \ln \left(\frac{A_R}{i+\delta} \right).$$

Differentiating both sides with respect to ϕ:

$$\frac{d \ln \chi}{d\phi} = \frac{d}{d\phi} \left(\frac{1}{1-\phi} \ln \alpha + \frac{\phi}{1-\phi} \ln \left(\frac{A_R}{i+\delta} \right) \right).$$

Applying the derivative to each term separately:

$$\frac{d}{d\phi} \left(\frac{1}{1-\phi} \ln \alpha \right) = \frac{\ln \alpha}{(1-\phi)^2},$$

$$\frac{d}{d\phi} \left(\frac{\phi}{1-\phi} \ln \left(\frac{A_R}{i+\delta} \right) \right) = \frac{\ln \left(\frac{A_R}{i+\delta} \right)}{(1-\phi)^2}.$$

Summing both terms:

$$\frac{d \ln \chi}{d\phi} = \frac{\ln \alpha + \ln \left(\frac{A_R}{i+\delta} \right)}{(1-\phi)^2}.$$

Using the original equation for χ to express $\frac{A_R}{i+\delta}$:

$$\frac{A_R}{i+\delta} = \left(\frac{\chi}{\alpha^{\frac{1}{1-\phi}}} \right)^{\frac{1-\phi}{\phi}}.$$

Substituting this:

$$\frac{d \ln \chi}{d\phi} = \frac{\ln \left(\alpha \left(\frac{\chi}{\alpha^{\frac{1}{1-\phi}}} \right)^{\frac{1-\phi}{\phi}} \right)}{(1-\phi)^2}.$$

Expanding the logarithm:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\alpha + \frac{1-\phi}{\phi}\ln\left(\frac{\chi}{\alpha^{\frac{1}{1-\phi}}}\right)}{(1-\phi)^2}.$$

Using $\ln\frac{a}{b} = \ln a - \ln b$, we rewrite:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\alpha + \frac{1-\phi}{\phi}(\ln\chi - \ln\alpha^{\frac{1}{1-\phi}})}{(1-\phi)^2}.$$

Since $\ln\alpha^{\frac{1}{1-\phi}} = \frac{1}{1-\phi}\ln\alpha$, we obtain:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\alpha + \frac{1-\phi}{\phi}\ln\chi - \frac{1-\phi}{\phi}\frac{1}{1-\phi}\ln\alpha}{(1-\phi)^2}.$$

Factoring $\ln\alpha$:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\alpha\left(1 - \frac{(1-\phi)}{\phi(1-\phi)}\right) + \frac{1-\phi}{\phi}\ln\chi}{(1-\phi)^2}.$$

Since $1 - \frac{1}{\phi} = -\frac{1-\phi}{\phi}$, we simplify:

$$\frac{d\ln\chi}{d\phi} = \frac{\frac{1-\phi}{\phi}(\ln\chi - \ln\alpha)}{(1-\phi)^2}.$$

Rewriting $\ln\chi - \ln\alpha$ as $\ln\left(\frac{\chi}{\alpha}\right)$:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)}.$$

Thus, the long-run semi-elasticity of χ with respect to ϕ is:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)}.$$

E.3 Elasticity of y wrt χ and A_R in short- and long run.

E.3.1 Elasticity of y with Respect to χ

To derive the elasticity of y with respect to χ, we start with the equation relating y to χ, given in a previous appendix:

$$y = \left[\frac{1-\alpha}{1-\chi}\right]^{\frac{1}{\phi}}.$$

Taking the natural logarithm of both sides:

$$\ln y = \frac{1}{\phi}\ln\left(\frac{1-\alpha}{1-\chi}\right).$$

Using the logarithm property $\ln \frac{a}{b} = \ln a - \ln b$, we rewrite:

$$\ln y = \frac{1}{\phi} \left(\ln(1-\alpha) - \ln(1-\chi) \right).$$

Differentiating both sides with respect to χ:

$$\frac{1}{y}\frac{dy}{d\chi} = \frac{1}{\phi} \left(0 - \frac{1}{1-\chi} \frac{d(1-\chi)}{d\chi} \right).$$

Since $\frac{d(1-\chi)}{d\chi} = -1$, this simplifies to:

$$\frac{1}{y}\frac{dy}{d\chi} = \frac{1}{\phi} \cdot \frac{1}{1-\chi}.$$

Multiplying both sides by χ:

$$\frac{\chi}{y}\frac{dy}{d\chi} = \frac{\chi}{\phi(1-\chi)}.$$

Since the elasticity is defined as $\frac{d\ln y}{d\ln \chi} = \frac{\chi}{y}\frac{dy}{d\chi}$, we obtain:

$$\frac{d\ln y}{d\ln \chi} = \frac{\chi}{\phi(1-\chi)}.$$

Fixed typo in the equation above, anything that uses it may be wrong

E.3.2 Elasticity of y with Respect to A_R in the Short Run

Method 1. To find the short-run elasticity of y with respect to A_R, we use the chain rule:

$$\frac{d\ln y}{d\ln A_R} = \frac{d\ln y}{d\ln \chi} \cdot \frac{d\ln \chi}{d\ln A_R}.$$

From the previous subsection, we have:

$$\frac{d\ln y}{d\ln \chi} = \phi \frac{\chi}{1-\chi}.$$

From Appendix D.1, the short-run elasticity of χ with respect to A_R is:

$$\frac{d\ln \chi}{d\ln A_R} = \phi(1-\chi).$$

Multiplying these expressions:

$$\frac{d\ln y}{d\ln A_R} = \left(\frac{\chi}{\phi(1-\chi)} \right) \cdot \left(\phi(1-\chi) \right).$$

Canceling $\phi(1-\chi)$:

$$\frac{d\ln y}{d\ln A_R} = \chi.$$

Method 2

Consider the production function:

$$Y = \left[\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right]^{\frac{1}{\phi}}.$$

In the short run, R and L are fixed. We seek the elasticity of Y with respect to A_R, defined as:

$$\varepsilon_{Y,A_R} = \frac{dY}{dA_R} \cdot \frac{A_R}{Y}.$$

Taking natural logarithms on both sides:

$$\ln Y = \frac{1}{\phi} \ln\left[\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}\right].$$

Differentiating with respect to A_R:

$$\frac{1}{Y}\frac{dY}{dA_R} = \frac{1}{\phi} \cdot \frac{\alpha\phi(A_R R)^{\phi-1} R}{\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}}.$$

Canceling ϕ:

$$\frac{1}{Y}\frac{dY}{dA_R} = \frac{\alpha(A_R R)^{\phi-1} R}{\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}}.$$

Using the identity:

$$A_R(A_R R)^{\phi-1} R = (A_R R)^{\phi},$$

we obtain:

$$\varepsilon_{Y,A_R} = \frac{\alpha(A_R R)^{\phi}}{\alpha(A_R R)^{\phi} + (1-\alpha)(A_L L)^{\phi}}.$$

Using the definition of r:

$$r = \frac{A_R R}{A_L L},$$

we rewrite:

$$\varepsilon_{Y,A_R} = \frac{\alpha r^{\phi}}{\alpha r^{\phi} + (1-\alpha)}.$$

Substituting:

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we get:

$$\varepsilon_{Y,A_R} = \frac{\alpha\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right)}{\alpha\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right)+(1-\alpha)}.$$

Simplifying:

$$\varepsilon_{Y,A_R} = \frac{\chi(1-\alpha)}{(1-\alpha)\left(\frac{\chi}{1-\chi}+1\right)}.$$

Since:

$$\frac{\chi}{1-\chi}+1 = \frac{1}{1-\chi},$$

it follows that:

$$\varepsilon_{Y,A_R} = \chi.$$

Thus, the elasticity of Y with respect to A_R in the short run is given by the self-sustainability index χ.

E.3.3 Elasticity of y with Respect to A_R in the Long Run

For the long-run elasticity of y with respect to A_R, we again use the chain rule:

$$\frac{d\ln y}{d\ln A_R} = \frac{d\ln y}{d\ln \chi}\cdot\frac{d\ln \chi}{d\ln A_R}.$$

Using the previously derived expression:

$$\frac{d\ln y}{d\ln \chi} = \frac{\chi}{\phi(1-\chi)}.$$

From Appendix D.2, the long-run elasticity of χ with respect to A_R is:

$$\frac{d\ln \chi}{d\ln A_R} = \frac{\phi}{1-\phi}.$$

Multiplying these expressions:

$$\frac{d\ln y}{d\ln A_R} = \left(\frac{\chi}{\phi(1-\chi)}\right)\cdot\left(\frac{\phi}{1-\phi}\right).$$

Rearranging:

$$\frac{d\ln y}{d\ln A_R} = \frac{1}{1-\phi}\cdot\frac{\chi}{1-\chi}.$$

This completes the derivation.

F Proof that Production is Non-Decreasing in ϕ for a given r.

F.1 Definition of the Function

We define the function

$$f(\phi) = \left[\alpha r^{\phi} + (1-\alpha)\right]^{\frac{1}{\phi}}.$$

We want to show that $f'(\phi) \geq 0$ for all ϕ, with equality if and only if $r = 1$.

F.2 Log Transformation

Taking the natural logarithm,

$$\ln f(\phi) = \frac{1}{\phi} \ln\left[\alpha r^{\phi} + (1-\alpha)\right].$$

Define

$$g(\phi) = \frac{1}{\phi} \ln\left[\alpha r^{\phi} + (1-\alpha)\right].$$

Since the natural logarithm is a strictly increasing function, $f(\phi)$ is non-decreasing if and only if $g(\phi)$ is non-decreasing:

$$g'(\phi) \geq 0.$$

F.3 Computing $g'(\phi)$

Using the product rule,

$$g'(\phi) = \frac{1}{\phi} \cdot \frac{\alpha r^{\phi} \ln r}{\alpha r^{\phi} + (1-\alpha)} - \frac{1}{\phi^2} \ln\left[\alpha r^{\phi} + (1-\alpha)\right].$$

This derivative is zero when $r = 1$, which proves that at $r = 1$, the function $f(\phi)$ does not depend on ϕ. It remains to prove that $g'(\phi)$ is positive when $r \neq 1$.

Introducing $x = r^{\phi}$, we rewrite

$$g'(\phi) = \frac{1}{\phi^2} \left(\frac{\alpha x \ln x}{\alpha x + (1-\alpha)} - \ln\left[\alpha x + (1-\alpha)\right] \right).$$

F.4 Defining the Function to Analyze

Define

$$h(x) = \alpha x \ln x - (\alpha x + (1-\alpha)) \ln\left[\alpha x + (1-\alpha)\right].$$

We need to show that

$$h(x) \geq 0 \quad \text{for all } x,$$

with equality only at $x = 1$.

To prove this, we first compute $h'(x)$, then show that $h'(x) < 0$ for $x < 1$, that $h'(x) > 0$ for $x > 1$, and that $h(1) = 0$. This will establish that $x = 1$ is the global minimum of $h(x)$, meaning $h(x) \geq 0$ for all x, with equality if and only if $x = 1$.

F.5 Computing $h'(x)$

Differentiating,

$$h'(x) = \frac{d}{dx} \left[\alpha x \ln x - (\alpha x + (1 - \alpha)) \ln(\alpha x + (1 - \alpha)) \right].$$

Using the product and chain rules,

$$h'(x) = \alpha(\ln x + 1) - \alpha \left[\ln(\alpha x + (1 - \alpha)) + 1 \right].$$

Rewriting using logarithm properties,

$$h'(x) = \alpha \ln \frac{x}{\alpha x + (1 - \alpha)}.$$

F.6 Analyzing the Sign of $h'(x)$

If $\frac{x}{\alpha x + (1-\alpha)} \geq 1$, then $\ln \frac{x}{\alpha x + (1-\alpha)} \geq 0$, meaning $h'(x) \geq 0$. If $\frac{x}{\alpha x + (1-\alpha)} < 1$, then $\ln \frac{x}{\alpha x + (1-\alpha)} < 0$, meaning $h'(x) < 0$.

Solving $\frac{x}{\alpha x + (1-\alpha)} \geq 1$,

$$x \geq \alpha x + (1 - \alpha).$$

Rearranging,

$$(1 - \alpha)x \geq 1 - \alpha.$$

Dividing by $(1 - \alpha)$ (which is positive for $0 < \alpha < 1$),

$$x \geq 1.$$

Thus, $h(x)$ is decreasing for $x < 1$ and increasing for $x > 1$, implying that $h(x)$ has a minimum at $x = 1$. Since $h(1) = 0$, and $x = 1$ is the global minimum of $h(x)$, we conclude that $h(x) \geq 0$ for all x, with equality only at $x = 1$.

F.7 Conclusion

Since $g'(\phi) = \frac{1}{\phi^2} h(x)$ and $\frac{1}{\phi^2} > 0$, we conclude that $g'(\phi) \geq 0$, meaning $g(\phi)$ is non-decreasing. This implies that $f(\phi)$ is non-decreasing, proving the original inequality. Equality holds if and only if $r = 1$.

F.8 Graphical Illustration

To further illustrate the behavior of $h(x)$, we include the figure below, which plots $h(x)$ and confirms that it attains its global minimum at $x = 1$.

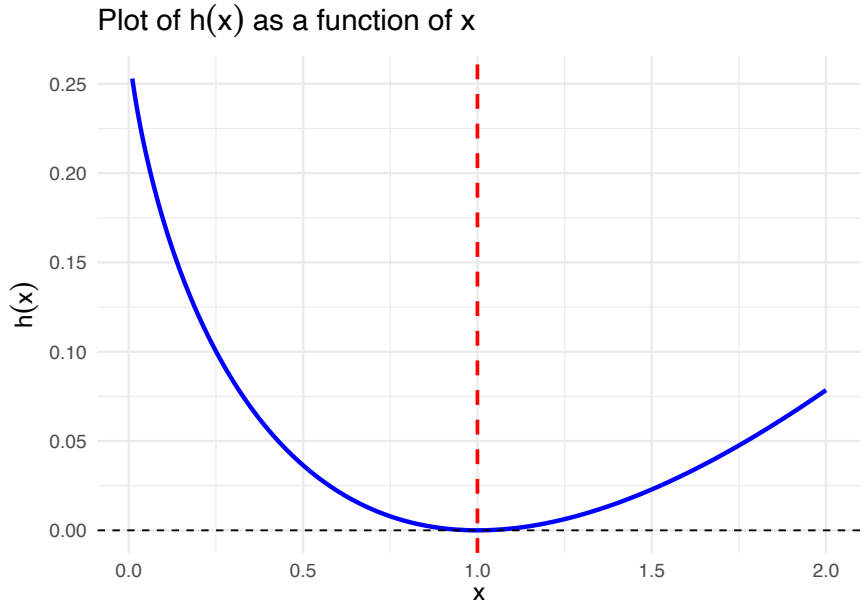


Figure 5: Illustration of the function $h(x)$. The function is non-negative and attains its global minimum at $x = 1$.

This visualization further confirms that $h(x) \geq 0$, supporting the analytical proof.

G Short-Run Elasticity of w with Respect to A_R

G.1 First Method: Using χ

We begin with the correct equation for w:

$$w = (1 - \alpha)^{\frac{1}{\phi}} (1 - \chi)^{-\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm of both sides:

$$\ln w = \frac{1}{\phi} \ln(1 - \alpha) - \frac{1 - \phi}{\phi} \ln(1 - \chi).$$

Next, differentiate both sides with respect to A_R, taking into account that χ depends on A_R:

$$\frac{1}{w} \frac{dw}{dA_R} = \frac{1 - \phi}{\phi} \cdot \frac{1}{1 - \chi} \cdot \frac{d\chi}{dA_R}.$$

Use the chain rule to express $\frac{d\chi}{dA_R}$:

$$\frac{d\chi}{dA_R} = \frac{d\ln\chi}{d\ln A_R} \cdot \frac{\chi}{A_R}.$$

Substitute this back into the equation and multiply both sides by A_R to convert to elasticity:

$$\frac{d\ln w}{d\ln A_R} = \frac{1-\phi}{\phi} \cdot \frac{1}{1-\chi} \cdot \frac{d\ln\chi}{d\ln A_R} \cdot \chi.$$

Now, using the short-run elasticity of χ with respect to A_R calculated in Appendix **??**:

$$\frac{d\ln\chi}{d\ln A_R} = \phi(1-\chi).$$

Substituting and simplifying:

$$\frac{d\ln w}{d\ln A_R} = \chi \cdot (1-\phi)$$

G.2 Second method to check the math: using r

We begin with the equations:

$$w = (1-\alpha)\left[\alpha r^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}},$$

$$r = \frac{A_R R}{A_L L}.$$

The elasticity of r with respect to A_R is given by:

$$\varepsilon_{r,A_R} = \frac{dr}{dA_R} \cdot \frac{A_R}{r}.$$

Differentiating r with respect to A_R, we obtain:

$$\frac{dr}{dA_R} = \frac{R}{A_L L}.$$

Using $r = \frac{A_R R}{A_L L}$, we find:

$$\varepsilon_{r,A_R} = \frac{R}{A_L L} \cdot \frac{A_R}{r} = 1.$$

Next, we take the logarithm of the wage equation:

$$\ln w = \ln(1-\alpha) + \frac{1-\phi}{\phi}\ln\left(\alpha r^{\phi} + (1-\alpha)\right).$$

Differentiating with respect to r:

$$\frac{1}{w}\frac{dw}{dr} = \frac{(1-\phi)\alpha r^{\phi-1}}{\alpha r^{\phi} + (1-\alpha)}.$$

Multiplying by r, the elasticity of w with respect to r is:

$$\varepsilon_{w,r} = \frac{(1-\phi)\alpha r^{\phi}}{\alpha r^{\phi} + (1-\alpha)}.$$

Since $\varepsilon_{w,A_R} = \varepsilon_{w,r} \cdot \varepsilon_{r,A_R}$, we substitute $\varepsilon_{r,A_R} = 1$:

$$\varepsilon_{w,A_R} = \frac{(1-\phi)\alpha r^{\phi}}{\alpha r^{\phi} + (1-\alpha)}.$$

Using the transformation:

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we substitute into the elasticity expression:

$$\varepsilon_{w,A_R} = \frac{(1-\phi)(1-\alpha)\frac{\chi}{1-\chi}}{(1-\alpha)\left(\frac{\chi}{1-\chi} + 1\right)}.$$

Canceling $(1-\alpha)$ in the numerator and denominator:

$$\varepsilon_{w,A_R} = \frac{(1-\phi)\frac{\chi}{1-\chi}}{\frac{1}{1-\chi}}.$$

Simplifying:

$$\varepsilon_{w,A_R} = (1-\phi)\chi.$$

H Long-Run Elasticity of w with Respect to A_R

H.1 First Method: Using χ

We begin with the correct equation for w:

$$w = (1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm of both sides:

$$\ln w = \frac{1}{\phi}\ln(1-\alpha) - \frac{1-\phi}{\phi}\ln(1-\chi).$$

Next, differentiate both sides with respect to A_R, taking into account that χ depends on A_R:

$$\frac{1}{w}\frac{dw}{dA_R} = \frac{1-\phi}{\phi} \cdot \frac{1}{1-\chi} \cdot \frac{d\chi}{dA_R}.$$

Use the chain rule to express $\frac{d\chi}{dA_R}$:

$$\frac{d\chi}{dA_R} = \frac{d\ln\chi}{d\ln A_R} \cdot \frac{\chi}{A_R}.$$

Substitute this back into the equation and multiply both sides by A_R to convert to elasticity:

$$\frac{d\ln w}{d\ln A_R} = \frac{1-\phi}{\phi} \cdot \frac{\chi}{1-\chi} \cdot \frac{d\ln\chi}{d\ln A_R}.$$

Now, use the **long-run elasticity of χ with respect to A_R:**

$$\frac{d\ln\chi}{d\ln A_R} = \frac{\phi}{1-\phi}.$$

Substitute and simplify:

$$\frac{d\ln w}{d\ln A_R} = \frac{\chi}{1-\chi}.$$

H.2 Second method: using r

To derive the long-run elasticity of w with respect to A_R, we start by considering the condition that in the long run, $i+\delta$ remains constant:

$$i+\delta = MP_R = \alpha A_R \left[\alpha + (1-\alpha)r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm of both sides:

$$\ln(i+\delta) = \ln\alpha + \ln A_R + \frac{1-\phi}{\phi}\ln\left(\alpha + (1-\alpha)r^{-\phi}\right).$$

Differentiating both sides with respect to A_R, noting that $i+\delta$ is constant:

$$0 = \frac{1}{A_R} + \frac{1-\phi}{\phi} \cdot \frac{1}{\alpha + (1-\alpha)r^{-\phi}} \cdot (-\phi)(1-\alpha)r^{-\phi-1}\frac{dr}{dA_R}.$$

Rearranging for $\frac{dr}{dA_R}$:

$$\frac{dr}{dA_R} = \frac{\frac{1}{A_R} \cdot (\alpha + (1-\alpha)r^{-\phi})}{(1-\phi)(1-\alpha)r^{-\phi-1}}.$$

Now, the elasticity of r with respect to A_R is:

$$\varepsilon_{r,A_R} = \frac{dr}{dA_R} \times \frac{A_R}{r}.$$

Substituting $\frac{dr}{dA_R}$:

$$\varepsilon_{r,A_R} = \frac{\alpha + (1-\alpha)r^{-\phi}}{(1-\phi)(1-\alpha)r^{-\phi}} = \frac{\alpha r^{\phi} + (1-\alpha)}{(1-\phi)(1-\alpha)}$$

Substitute:

$$r^{\phi} = \frac{\chi}{1-\chi}\frac{1-\alpha}{\alpha}$$

$$\begin{aligned}
\varepsilon_{r,A_R} &= \frac{\alpha\frac{\chi}{1-\chi}\frac{1-\alpha}{\alpha} + (1-\alpha)}{(1-\phi)(1-\alpha)} \\
&= \frac{\frac{\chi}{1-\chi}(1-\alpha) + (1-\alpha)}{(1-\phi)(1-\alpha)} \\
&= \frac{\frac{\chi}{1-\chi} + 1}{1-\phi} \\
&= \frac{1}{(1-\phi)(1-\chi)}
\end{aligned}$$

Thus:

$$\varepsilon_{r,A_R} = \frac{1}{(1-\phi)(1-\chi)}.$$

Next, we consider the wage equation:

$$w = (1-\alpha)\left[\alpha r^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm:

$$\ln w = \ln(1-\alpha) + \frac{1-\phi}{\phi}\ln\left(\alpha r^{\phi} + (1-\alpha)\right).$$

Differentiating both sides with respect to A_R:

$$\frac{1}{w}\frac{dw}{dA_R} = \frac{1-\phi}{\phi} \cdot \frac{\alpha\phi r^{\phi-1}\frac{dr}{dA_R}}{\alpha r^{\phi} + (1-\alpha)}.$$

Rearranging:

$$\frac{dw}{dA_R} = w \cdot \frac{(1-\phi)\alpha\phi r^{\phi-1}\frac{dr}{dA_R}}{\phi(\alpha r^{\phi} + (1-\alpha))}.$$

Now, using $\frac{dr}{dA_R} = \varepsilon_{r,A_R} \cdot \frac{r}{A_R}$:

$$\varepsilon_{w,A_R} = \frac{(1-\phi)\alpha r^{\phi} \cdot \varepsilon_{r,A_R}}{\alpha r^{\phi} + (1-\alpha)}.$$

Substituting $r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}$:

$$\varepsilon_{w,A_R} = \frac{(1-\phi) \cdot \chi \cdot \varepsilon_{r,A_R}}{1-\chi}.$$

Finally, substituting $\varepsilon_{r,A_R} = \frac{1}{(1-\phi)(1-\chi)}$:

$$\boxed{\varepsilon_{w,A_R} = \frac{\chi}{1-\chi}}.$$

Conclusion: Result same as before, math checked.

I Short-Run Elasticity of w with Respect to ϕ

I.1 Introduction

In this appendix, we present two different derivations of the short-run elasticity of w with respect to ϕ to verify the correctness of our mathematical derivation.

The first derivation expresses w in terms of χ, while the second derivation expresses it in terms of r. We then substitute r in terms of χ and confirm that both derivations lead to the same result.

I.2 First Derivation: Using the expression of w in Terms of χ

We start with the equation for MP_L:

$$MP_L = A_L(1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Since $w = W/A_L = MP_L/A_L$, we obtain:

$$w = (1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm:

$$\ln w = \frac{1}{\phi}\ln(1-\alpha) - \frac{1-\phi}{\phi}\ln(1-\chi).$$

Differentiating both sides with respect to ϕ:

$$\frac{1}{w}\frac{dw}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{\phi}\ln(1-\alpha) - \frac{1-\phi}{\phi}\ln(1-\chi)\right).$$

Applying the product rule:

$$-\frac{1}{\phi^2}\ln(1-\alpha) + \frac{1}{\phi^2}\ln(1-\chi) + \frac{1-\phi}{\phi}\frac{1}{1-\chi}\frac{d\chi}{d\phi}$$

Multiplying by ϕ to obtain the elasticity:

$$\frac{d\ln w}{d\ln\phi} = \frac{1}{\phi}\ln\left(\frac{1-\chi}{1-\alpha}\right) + \frac{1-\phi}{1-\chi}\frac{d\chi}{d\phi}.$$

Convert $\frac{d\chi}{d\phi}$ into semi-elasticity and substitute it from Appendix E.2:

$$\begin{aligned}\frac{d\chi}{d\phi} &= \chi \cdot \frac{d\ln\chi}{d\phi} \\ &= \chi \cdot (1-\chi)\ln r,\end{aligned}$$

so the elasticity of w wrt ϕ becomes:

$$\frac{d\ln w}{d\ln\phi} = \frac{1}{\phi}\ln\left(\frac{1-\chi}{1-\alpha}\right) + (1-\phi)\chi\cdot\ln r$$

Substitute $\ln r = \frac{1}{\phi}\ln r^{\phi}$, and r^{ϕ} from Appendix B:

$$\frac{d\ln w}{d\ln\phi} = \frac{1}{\phi}\ln\left(\frac{1-\chi}{1-\alpha}\right) + \chi\cdot\frac{1-\phi}{\phi}\ln\left(\frac{\chi}{1-\chi}\frac{1-\alpha}{\alpha}\right)$$

I.3 Second Derivation: Expressing w in Terms of r

We start with the equation for MP_L:

$$MP_L = (1-\alpha)A_L\left[\alpha r^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Since $w = W/A_L = MP_L/A_L$, we obtain:

$$w = (1-\alpha)\left[\alpha r^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm:

$$\ln w = \ln(1-\alpha) + \frac{1-\phi}{\phi}\ln\left(\alpha r^{\phi} + (1-\alpha)\right).$$

Differentiating both sides:

$$\frac{1}{w}\frac{dw}{d\phi} = \frac{d}{d\phi}\left(\frac{1-\phi}{\phi}\ln\left(\alpha r^{\phi} + (1-\alpha)\right)\right).$$

Expanding using the product rule:

$$-\frac{1}{\phi^2}\ln\left(\alpha r^{\phi} + (1-\alpha)\right) + \frac{1-\phi}{\phi}\frac{\alpha r^{\phi}\ln r}{\alpha r^{\phi} + (1-\alpha)}.$$

Multiplying by ϕ:

$$\frac{d\ln w}{d\ln\phi} = -\frac{1}{\phi}\ln\left(\alpha r^{\phi} + (1-\alpha)\right) + \frac{(1-\phi)\alpha r^{\phi}\ln r}{\alpha r^{\phi} + (1-\alpha)}.$$

Now, we substitute r^{ϕ} using from Appendix B:

$$r^{\phi} = \frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}.$$

Substituting this into our expression:

$$\frac{d\ln w}{d\ln \phi} = -\frac{1}{\phi}\ln\left(\alpha\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}+(1-\alpha)\right)+\frac{(1-\phi)\alpha\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\ln r}{\alpha\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}+(1-\alpha)}.$$

Simplifying the coefficients:

$$\frac{d\ln w}{d\ln \phi} = -\frac{1}{\phi}\ln\left(\frac{1-\alpha}{1-\chi}\right)+(1-\phi)\chi\cdot\frac{1}{\phi}\ln r^{\phi}.$$

Substituting $r^{\phi} = \frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}$, we obtain:

$$\frac{d\ln w}{d\ln \phi} = -\frac{1}{\phi}\ln\left(\frac{1-\alpha}{1-\chi}\right)+(1-\phi)\chi\cdot\frac{1}{\phi}\ln\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right).$$

I.4 Conclusion: Equivalence of the Two Derivations

Since both derivations lead to the same final expression, we confirm that our original derivation of the short-run elasticity of w with respect to ϕ is correct. This verification ensures the validity of our approach and computations.

The final form is:

$$\frac{d\ln w}{d\ln \phi} = \frac{1}{\phi}\left[\ln\left(\frac{1-\chi}{1-\alpha}\right)+(1-\phi)\chi\cdot\ln\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right)\right].$$

J Long-Run Elasticity of w with Respect to ϕ

J.1 Introduction

In this appendix, we present two different derivations of the long-run elasticity of w with respect to ϕ to verify the correctness of our mathematical derivation.

The first derivation expresses w in terms of χ, while the second derivation expresses it in terms of r. We then substitute r in terms of χ and confirm that both derivations lead to the same result.

J.2 First Derivation: Expressing w in Terms of χ

We start with the equation for MP_L:

$$MP_L = A_L(1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Since in the long run $w = MP_L/A_L$, we obtain:

$$w = (1-\alpha)^{\frac{1}{\phi}}(1-\chi)^{-\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm:

$$\ln w = \frac{1}{\phi}\ln(1-\alpha) - \frac{1-\phi}{\phi}\ln(1-\chi).$$

Differentiating both sides with respect to ϕ:

$$\frac{1}{w}\frac{dw}{d\phi} = -\frac{1}{\phi^2}\ln(1-\alpha) + \frac{1}{\phi^2}\ln(1-\chi) - \frac{1-\phi}{\phi}\frac{d}{d\phi}\ln(1-\chi).$$

Using the given semi-elasticity:

$$\frac{d\ln\chi}{d\phi} = \frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)},$$

we obtain:

$$\frac{d}{d\phi}\ln(1-\chi) = -\frac{\chi}{(1-\chi)}\cdot\frac{\ln\left(\frac{\chi}{\alpha}\right)}{\phi(1-\phi)}.$$

Substituting this into our equation:

$$\frac{1}{w}\frac{dw}{d\phi} = -\frac{1}{\phi^2}\ln(1-\alpha) + \frac{1}{\phi^2}\ln(1-\chi) - \frac{\chi}{\phi^2(1-\chi)}\ln\left(\frac{\chi}{\alpha}\right).$$

Multiplying by ϕ to obtain the elasticity:

$$\frac{d\ln w}{d\ln\phi} = \frac{1}{\phi}\ln\left(\frac{1-\chi}{1-\alpha}\right) + \frac{\chi}{\phi(1-\chi)}\ln\left(\frac{\chi}{\alpha}\right).$$

At this point, we **stop**, ensuring that logarithmic terms remain unexpanded.

J.3 Second Derivation: Expressing w in Terms of r

We start with the equation for MP_R:

$$MP_R = \alpha A_R\left[\alpha + (1-\alpha)r^{-\phi}\right]^{\frac{1-\phi}{\phi}}.$$

Since the interest rate is fixed in the long run, we impose the equilibrium condition:

$$MP_R = i + \delta.$$

Taking the natural logarithm of both sides:

$$\ln\alpha + \ln A_R + \frac{1-\phi}{\phi}\ln\left(\alpha + (1-\alpha)r^{-\phi}\right) = \ln(i+\delta).$$

Differentiating both sides:

$$\frac{d}{d\phi}\left(\frac{1-\phi}{\phi}\ln\left(\alpha + (1-\alpha)r^{-\phi}\right)\right) = 0.$$

Applying the product rule:

$$-\frac{1}{\phi^2}\ln\left(\alpha+(1-\alpha)r^{-\phi}\right)+\frac{1-\phi}{\phi}\frac{d}{d\phi}\ln\left(\alpha+(1-\alpha)r^{-\phi}\right)=0.$$

$$-\frac{1}{\phi^2}\ln\left(\alpha+(1-\alpha)r^{-\phi}\right)+\frac{1-\phi}{\phi}\frac{(1-\alpha)(-1)r^{-\phi}\ln r}{\alpha+(1-\alpha)r^{-\phi}}\frac{dr}{d\phi}=0.$$

Solving for $\frac{dr}{d\phi}$:

$$\frac{dr}{d\phi}=-\frac{1}{\phi^2}\ln\left(\alpha+(1-\alpha)r^{-\phi}\right)\cdot\frac{\alpha+(1-\alpha)r^{-\phi}}{(1-\phi)(1-\alpha)r^{-\phi}\ln r}.$$

Using the equation for MP_L in terms of r:

$$MP_L=(1-\alpha)A_L\left[\alpha r^{\phi}+(1-\alpha)\right]^{\frac{1-\phi}{\phi}},$$

we repeat the differentiation process and substitute $r^{\phi}=\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}$ to obtain:

$$\frac{d\ln w}{d\ln\phi}=-\frac{1}{\phi}\ln\left(\frac{1-\alpha}{1-\chi}\right)+\frac{\chi}{\phi(1-\chi)}\ln\left(\frac{\chi}{\alpha}\right).$$

Really? Need to derive it to make sure.

J.4 Conclusion: Equivalence of the Two Derivations

Since both derivations lead to the same final expression, we confirm that our original derivation of the long-run elasticity of w with respect to ϕ is correct. This verification ensures the validity of our approach and computations.

K Appendix: Sort-Run Elasticity of $i+\delta$ with respect to A_R

K.1 Method 1

We derive the short-run elasticity of $i+\delta$ with respect to A_R, given that:

$$i+\delta=MP_R,$$

where

$$MP_R=A_R\alpha^{\frac{1}{\phi}}\chi^{-\frac{1-\phi}{\phi}}.$$

The elasticity of $i+\delta$ with respect to A_R is defined as:

$$\varepsilon=\frac{d\ln(i+\delta)}{d\ln A_R}.$$

Since $i+\delta=MP_R$, we rewrite:

$$\varepsilon = \frac{d \ln MP_R}{d \ln A_R}.$$

Taking the natural logarithm of MP_R:

$$\ln MP_R = \ln A_R + \frac{1}{\phi} \ln \alpha - \frac{1-\phi}{\phi} \ln \chi.$$

Differentiating both sides with respect to A_R:

$$\frac{1}{MP_R} \frac{dMP_R}{dA_R} = \frac{1}{A_R} - \frac{1-\phi}{\phi} \cdot \frac{1}{\chi} \frac{d\chi}{dA_R}.$$

Multiplying both sides by A_R gives:

$$\varepsilon = \frac{A_R}{MP_R} \frac{dMP_R}{dA_R} = 1 - \frac{1-\phi}{\phi} \cdot \frac{A_R}{\chi} \frac{d\chi}{dA_R}.$$

Using the given short-run elasticity of χ with respect to A_R:

$$\frac{d \ln \chi}{d \ln A_R} = \phi(1-\chi),$$

we substitute:

$$\frac{A_R}{\chi} \frac{d\chi}{dA_R} = \phi(1-\chi).$$

Thus,

$$\varepsilon = 1 - \frac{1-\phi}{\phi} \cdot \phi(1-\chi).$$

Simplifying:

$$\varepsilon = 1 - (1-\phi)(1-\chi),$$

which further reduces to:

$$\boxed{\varepsilon = \phi + (1-\phi)\chi}.$$

This is the elasticity of $i + \delta$ with respect to A_R in the short run.

K.2 Method 2

We derive the elasticity of $i + \delta$ with respect to A_R in the short run, where R is fixed. Given the equation:

$$i + \delta = MP_R$$

and the marginal product function:

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}},$$

where the ratio r is defined as:

$$r = \frac{A_R R}{A_L L}.$$

Step 1: Elasticity of r with respect to A_R

Differentiating r with respect to A_R:

$$\frac{dr}{dA_R} = \frac{R}{A_L L}.$$

The elasticity of r with respect to A_R is then:

$$\varepsilon_{r,A_R} = \frac{dr}{dA_R} \times \frac{A_R}{r} = 1.$$

Step 2: Logarithmic Differentiation

Taking the natural logarithm of both sides of $i + \delta = MP_R$:

$$\ln(i+\delta) = \ln \alpha + \ln A_R + \frac{1-\phi}{\phi} \ln\left[\alpha + (1-\alpha) r^{-\phi}\right].$$

Differentiating with respect to A_R:

$$\frac{1}{i+\delta} \frac{d(i+\delta)}{dA_R} = \frac{1}{A_R} - \frac{(1-\phi)(1-\alpha) r^{-\phi-1} \frac{dr}{dA_R}}{\alpha + (1-\alpha) r^{-\phi}}.$$

Multiplying both sides by A_R to express the left-hand side as elasticity:

$$\varepsilon_{(i+\delta),A_R} = 1 - \frac{(1-\phi)(1-\alpha) r^{-\phi-1} A_R \frac{dr}{dA_R}}{\alpha + (1-\alpha) r^{-\phi}}.$$

Using $\frac{dr}{dA_R} = \varepsilon_{r,A_R} \frac{r}{A_R}$, we rewrite:

$$\varepsilon_{(i+\delta),A_R} = 1 - \frac{(1-\phi)(1-\alpha) r^{-\phi} \varepsilon_{r,A_R}}{\alpha + (1-\alpha) r^{-\phi}}.$$

Dividing the numerator and denominator by $r^{-\phi}$:

$$\varepsilon_{(i+\delta),A_R} = 1 - \frac{(1-\phi)(1-\alpha) \varepsilon_{r,A_R}}{\alpha r^{\phi} + (1-\alpha)}.$$

Substituting $r^{\phi} = \frac{1-\alpha}{\alpha} \cdot \frac{\chi}{1-\chi}$, we get:

$$\varepsilon_{(i+\delta),A_R} = 1 - \frac{(1-\phi)(1-\alpha) \varepsilon_{r,A_R}}{\frac{1-\alpha}{1-\chi}}.$$

Canceling $1-\alpha$:

$$\varepsilon_{(i+\delta),A_R} = 1 - (1-\phi)\varepsilon_{r,A_R}(1-\chi).$$

Since $\varepsilon_{r,A_R} = 1$, we obtain:

$$\begin{aligned}\varepsilon_{(i+\delta),A_R} &= 1 - (1-\phi)(1-\chi) \\ &= 1 - (1-\phi) + \chi(1-\phi) \\ &= \phi + (1-\phi)\chi\end{aligned}$$

L Short-Run Semi-Elasticity of $i+\delta$ with Respect to ϕ

L.1 Method 1

We begin with the marginal productivity equation:

$$MP_R = A_R \alpha^{\frac{1}{\phi}} \chi^{-\frac{1-\phi}{\phi}}$$

which, combined with the identity $MP_R = i + \delta$, gives:

$$i + \delta = A_R \alpha^{\frac{1}{\phi}} \chi^{-\frac{1-\phi}{\phi}}.$$

Taking the natural logarithm on both sides,

$$\ln(i+\delta) = \ln A_R + \frac{1}{\phi}\ln\alpha - \frac{1-\phi}{\phi}\ln\chi.$$

Differentiating with respect to ϕ,

$$\frac{1}{i+\delta}\frac{d(i+\delta)}{d\phi} = -\frac{1}{\phi^2}\ln\alpha + \frac{1}{\phi^2}\ln\chi - \frac{1-\phi}{\phi}\cdot\frac{1}{\chi}\frac{d\chi}{d\phi}.$$

Multiplying by ϕ to convert to elasticity form,

$$\varepsilon_{i+\delta,\phi} = \frac{1}{\phi}(\ln\chi - \ln\alpha) - \frac{1-\phi}{\phi}\varepsilon_{\chi,\phi}.$$

In the short run, the elasticity of χ with respect to ϕ is given by:

$$\frac{d\ln\chi}{d\phi} = (1-\chi)\ln r.$$

which implies:

$$\varepsilon^{SR}_{\chi,\phi} = \frac{\phi}{\chi}\frac{d\chi}{d\phi} = \phi(1-\chi)\ln r.$$

Using the identity

$$\ln r = \frac{1}{\phi} \ln r^{\phi},$$

and substituting r^{ϕ} from

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we obtain:

$$\boxed{\varepsilon^{SR}_{i+\delta,\phi} = \frac{1}{\phi} \ln \frac{\chi}{\alpha} - \frac{(1-\phi)(1-\chi)}{\phi} \ln \left(\frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha} \right)}.$$

This expression characterizes the short-run semi-elasticity of $i + \delta$ with respect to ϕ.

L.2 Method 2

We begin with the equilibrium condition:

$$MP_R = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi} \right]^{\frac{1-\phi}{\phi}},$$

which also satisfies:

$$MP_R = i + \delta.$$

Taking logarithms on both sides:

$$\ln(i+\delta) = \ln \alpha + \ln A_R + \frac{1-\phi}{\phi} \ln \left[\alpha + (1-\alpha) r^{-\phi} \right].$$

Differentiating with respect to ϕ:

$$\frac{d}{d\phi} \ln(i+\delta) = -\frac{1}{\phi^2} \ln \left[\alpha + (1-\alpha) r^{-\phi} \right] - \frac{(1-\phi)}{\phi} \frac{(1-\alpha) r^{-\phi} \ln r}{\alpha + (1-\alpha) r^{-\phi}}.$$

Multiplying by ϕ to obtain the elasticity:

$$\varepsilon_{i+\delta,\phi} = -\frac{1}{\phi} \ln \left[\alpha + (1-\alpha) r^{-\phi} \right] - (1-\phi) \frac{(1-\alpha) r^{-\phi} \ln r}{\alpha + (1-\alpha) r^{-\phi}}.$$

Using the substitution:

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha},$$

we rewrite:

$$r^{-\phi} = \frac{1-\chi}{\chi} \cdot \frac{\alpha}{1-\alpha}.$$

Substituting this and simplifying:

$$\varepsilon_{i+\delta,\phi} = -\frac{1}{\phi}\ln\frac{\alpha}{\chi} - (1-\phi)(1-\chi)\frac{1}{\phi}\ln\left(\frac{\chi}{1-\chi}\cdot\frac{1-\alpha}{\alpha}\right).$$

This completes the derivation.

M Derivation of the Optimal Bequests and Savings Decisions

The individual's optimization problem can be written as:

$$\max_{S_t, B_{t+1}} u(B_t + W_t - S_t) + \rho u\left((1+i_{t+1})S_t - B_{t+1}\right) + \hat{\rho} v(B_{t+1}). \tag{42}$$

The first-order conditions are:

$$\begin{aligned} -\rho u'(D_{t+1}) + \hat{\rho} v'(B_{t+1}) &= 0, \\ -u'(C_t) + \rho(1+i_{t+1})u'(D_{t+1}) &= 0. \end{aligned}$$

Substituting the CES utility specification:

$$\begin{aligned} \rho D_{t+1}^{-\frac{1}{\sigma}} &= \hat{\rho} B_{t+1}^{-\frac{1}{\sigma}}, \\ C_t^{-\frac{1}{\sigma}} &= \rho(1+i_{t+1}) D_{t+1}^{-\frac{1}{\sigma}}. \end{aligned}$$

Rearranging terms:

$$\begin{aligned} \hat{\rho}^{\sigma} D_{t+1} &= \rho^{\sigma} B_{t+1}, \\ \rho^{\sigma}(1+i_{t+1})^{\sigma} C_t &= D_{t+1}. \end{aligned}$$

Substituting the budget constraint:

$$\begin{aligned} \hat{\rho}^{\sigma}\left((1+i_{t+1})S_t - B_{t+1}\right) &= \rho^{\sigma} B_{t+1}, \\ \rho^{\sigma}(1+i_{t+1})^{\sigma}(W_t + B_t - S_t) &= (1+i_{t+1})S_t - B_{t+1}. \end{aligned}$$

Solving for B_{t+1} and S_t:

$$B_{t+1} = \frac{\hat{\rho}^{\sigma}}{\rho^{\sigma} + \hat{\rho}^{\sigma}}(1+i_{t+1})S_t, \tag{43}$$

$$\begin{aligned} S_t &= \frac{W_t + B_t}{1 + \frac{1}{\rho^{\sigma}+\hat{\rho}^{\sigma}}(1+i_{t+1})^{1-\sigma}} \\ &= \frac{\rho^{\sigma} + \hat{\rho}^{\sigma}}{\rho^{\sigma} + \hat{\rho}^{\sigma} + (1+i_{t+1})^{1-\sigma}}(W_t + B_t). \end{aligned}$$

Finally,

$$B_{t+1} = \frac{\hat{\rho}^{\sigma}(1+i_{t+1})}{\rho^{\sigma} + \hat{\rho}^{\sigma} + (1+i_{t+1})^{1-\sigma}}\left(W_t + B_t\right).$$

N Equilibrium in OLG model

Derive χ:

$$\begin{aligned}
\chi_t &= \left(\frac{A_R}{A_R^{\text{crit}}}\right)^{\frac{\phi}{1-\phi}} \\
&= \left(\frac{A_R}{(i+\delta)\,\alpha^{-\frac{1}{\phi}}}\right)^{\frac{\phi}{1-\phi}} \\
&= \left(\frac{A_R}{\alpha A_{R,t}\left[\alpha + (1-\alpha) r_t^{-\phi}\right]^{\frac{1-\phi}{\phi}}\,\alpha^{-\frac{1}{\phi}}}\right)^{\frac{\phi}{1-\phi}} \\
&= \frac{\alpha}{\alpha + (1-\alpha) r_t^{-\phi}}
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{\chi_t} &= \alpha + (1-\alpha) r_t^{-\phi} \\
\frac{\alpha}{\chi_t} - \alpha &= (1-\alpha) r_t^{-\phi} \\
\alpha(\chi_t^{-1} - 1) &= (1-\alpha) r_t^{-\phi} \\
r_t^{-\phi} &= \frac{\alpha}{1-\alpha}(\chi_t^{-1} - 1) \\
r_t^{\phi} &= \frac{1-\alpha}{\alpha(\chi_t^{-1} - 1)}
\end{aligned}$$

$$\begin{aligned}
w_t &= (1-\alpha)\left[\alpha r_t^{\phi} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}} \\
&= (1-\alpha)\left[\alpha \frac{1-\alpha}{\alpha(\chi_t^{-1} - 1)} + (1-\alpha)\right]^{\frac{1-\phi}{\phi}} \\
&= (1-\alpha)^{1+\frac{1-\phi}{\phi}}\left[\frac{1}{\chi_t^{-1} - 1} + 1\right]^{\frac{1-\phi}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}}\left[\frac{\chi_t^{-1}}{\chi_t^{-1} - 1}\right]^{\frac{1-\phi}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}}\left[\frac{1}{1-\chi_t}\right]^{\frac{1-\phi}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}}\left[1-\chi_t\right]^{-\frac{1-\phi}{\phi}}
\end{aligned}$$

$$
\begin{aligned}
i_t + \delta &= \alpha A_{R,t} \left[\alpha + (1-\alpha) r_t^{-\phi}\right]^{\frac{1-\phi}{\phi}} \\
&= \alpha A_{R,t} \left[\alpha + (1-\alpha) \frac{\alpha}{1-\alpha} (\chi_t^{-1} - 1)\right]^{\frac{1-\phi}{\phi}} \\
&= \alpha A_{R,t} \left[\alpha + \alpha(\chi_t^{-1} - 1)\right]^{\frac{1-\phi}{\phi}} \\
&= \alpha^{1+\frac{1-\phi}{\phi}} A_{R,t} \left[\chi_t^{-1}\right]^{\frac{1-\phi}{\phi}} \\
&= \alpha^{\frac{1}{\phi}} A_{R,t} \chi_t^{-\frac{1-\phi}{\phi}}
\end{aligned}
$$

$$
\begin{aligned}
y_t &= \left[\alpha r_t^{\phi} + (1-\alpha)\right]^{\frac{1}{\phi}} \\
&= \left[\alpha \frac{1-\alpha}{\alpha(\chi_t^{-1} - 1)} + (1-\alpha)\right]^{\frac{1}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}} \left[\frac{1}{\chi_t^{-1} - 1} + 1\right]^{\frac{1}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}} \left[\frac{\chi_t^{-1}}{\chi_t^{-1} - 1}\right]^{\frac{1}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}} \left[\frac{1}{1-\chi_t}\right]^{\frac{1}{\phi}} \\
&= (1-\alpha)^{\frac{1}{\phi}} \left[1-\chi_t\right]^{-\frac{1}{\phi}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{w_t}{y_t} &= \frac{(1-\alpha)^{\frac{1}{\phi}} \left[1-\chi_t\right]^{-\frac{1-\phi}{\phi}}}{(1-\alpha)^{\frac{1}{\phi}} \left[1-\chi_t\right]^{-\frac{1}{\phi}}} \\
&= \left[1-\chi_t\right]^{\frac{1}{\phi} - \frac{1-\phi}{\phi}} = 1 - \chi_t
\end{aligned}
$$

$$
\begin{aligned}
(1 + g_{A_L}) \cdot r_{t+1} &= A_{R,t+1} \cdot s_t \\
(1 + g_{A_L})^{\phi} \cdot r_{t+1}^{\phi} &= A_{R,t+1}^{\phi} \cdot s_t^{\phi} \\
(1 + g_{A_L})^{\phi} \cdot \frac{1-\alpha}{\alpha(\chi_{t+1}^{-1} - 1)} &= A_{R,t+1}^{\phi} \cdot s_t^{\phi} \\
\chi_{t+1}^{-1} - 1 &= \frac{1-\alpha}{\alpha} \frac{(1 + g_{A_L})^{\phi}}{A_{R,t+1}^{\phi} \cdot s_t^{\phi}}
\end{aligned}
$$

$$
\begin{aligned}
s_t &= \frac{w_t + b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1 + i_{t+1})^{1-\sigma}}, \\
&= \frac{(1-\alpha)^{\frac{1}{\phi}} \left[1 - \chi_t\right]^{-\frac{1-\phi}{\phi}} + b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma} \left(\alpha^{\frac{1}{\phi}} A_{R,t} \chi_t^{-\frac{1-\phi}{\phi}}\right)^{1-\sigma}}.
\end{aligned}
$$

O Appendix: Log-Linearization and Convergence Rate Analysis

Equilibrium System

The model is characterized by the following equilibrium conditions:

$$
Y = \left[\alpha (A_R R)^\phi + (1-\alpha)(A_L L)^\phi\right]^{\frac{1}{\phi}}, \tag{44}
$$

$$
i + \delta = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}, \quad \text{where } r = \frac{A_R R}{A_L L}, \tag{45}
$$

$$
\frac{\dot{C}}{C} = \frac{1}{\sigma}(i - \rho), \tag{46}
$$

$$
\dot{R} = Y - C - \delta R. \tag{47}
$$

Labor is fixed, and technology evolves as:

$$
\dot{A}_L = g_{A_L} A_L. \tag{48}
$$

Normalization

Define normalized variables:

$$
r = \frac{A_R R}{A_L L}, \qquad c = \frac{C}{A_L}, \qquad y = \frac{Y}{A_L L}. \tag{49}
$$

Rewriting the system in normalized form:

$$
y = \left[\alpha r^\phi + (1-\alpha)\right]^{\frac{1}{\phi}}, \tag{50}
$$

$$
i + \delta = \alpha A_R \left[\alpha + (1-\alpha) r^{-\phi}\right]^{\frac{1-\phi}{\phi}}, \tag{51}
$$

$$
\frac{\dot{c}}{c} = \frac{1}{\sigma}(i - \rho) - g_{A_L}, \tag{52}
$$

$$
\frac{\dot{r}}{r} = \frac{A_R}{r}(y - c) - \delta - g_{A_L}. \tag{53}
$$

We introduce the transformation:

$$r^{\phi} = \frac{\chi}{1-\chi} \cdot \frac{1-\alpha}{\alpha}. \tag{54}$$

Using this, the production function and first-order conditions become:

$$y = \left(\frac{1-\alpha}{1-\chi}\right)^{\frac{1}{\phi}}, \tag{55}$$

$$i + \delta = \alpha^{\frac{1}{\phi}} A_R \chi^{-\frac{1-\phi}{\phi}}. \tag{56}$$

The robot accumulation equation transforms into:

$$\dot{\chi} = \phi\chi(1-\chi)\left[A_R\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{\phi}}\left(1-\frac{c}{y}\right) - \delta - g_{A_L}\right]. \tag{57}$$

Steady-State Analysis

Setting $\dot{\chi} = 0$, we obtain:

$$\frac{c^*}{y^*} = 1 - \left(\frac{\delta + g_{A_L}}{\rho + \sigma g_{A_L} + \delta}\right)\alpha^{\frac{1}{\phi}}\chi^{*-\frac{1-\phi}{\phi}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{\phi}}. \tag{58}$$

For the steady-state χ^*, we get:

$$\chi^* = \left(\frac{\rho + \sigma g_{A_L} + \delta}{\alpha^{\frac{1}{\phi}} A_R}\right)^{-\frac{\phi}{1-\phi}}. \tag{59}$$

Log-Linearization and Stability Analysis

Defining the log-deviations:

$$\hat{y} = \frac{1}{\phi}\frac{\chi^*}{1-\chi^*}\hat{\chi}, \tag{60}$$

$$\hat{i} = -\frac{1-\phi}{\phi}\frac{i^*+\delta}{i^*}\hat{\chi}, \tag{61}$$

$$\dot{\hat{c}} = \frac{1}{\sigma} i^* \hat{i}, \tag{62}$$

$$\dot{\hat{\chi}} = -\phi\chi^*\frac{c^*}{y^*}\hat{c} + \chi^*\frac{c^*}{y^*}\frac{\chi^*}{1-\chi^*}\hat{\chi}. \tag{63}$$

Rewriting in matrix form:

$$\begin{bmatrix}\dot{\hat{\chi}} \\ \dot{\hat{c}}\end{bmatrix} = \begin{bmatrix}\chi^*\frac{c^*}{y^*}\frac{\chi^*}{1-\chi^*} & -\phi\chi^*\frac{c^*}{y^*} \\ -\frac{1-\phi}{\sigma\phi}(i^*+\delta) & 0\end{bmatrix}\begin{bmatrix}\hat{\chi} \\ \hat{c}\end{bmatrix}. \tag{64}$$

Solving for the eigenvalues:

$$\lambda = \frac{\chi^* \frac{c^*}{y^*} \frac{\chi^*}{1-\chi^*} \pm \sqrt{\left(\chi^* \frac{c^*}{y^*} \frac{\chi^*}{1-\chi^*}\right)^2 + 4\chi^* \frac{c^*}{y^*} \frac{1-\phi}{\sigma}(i^* + \delta)}}{2}. \tag{65}$$

Convergence Rate and Its Dependence on χ^*

The convergence rate is:

$$\text{Convergence Rate} = \frac{\sqrt{\left(\chi^* \frac{c^*}{y^*} \frac{\chi^*}{1-\chi^*}\right)^2 + 4\chi^* \frac{c^*}{y^*} \frac{1-\phi}{\sigma}(i^* + \delta)} - \chi^* \frac{c^*}{y^*} \frac{\chi^*}{1-\chi^*}}{2}. \tag{66}$$

As $\chi^* \to 1$, the convergence rate approaches zero, implying that when robots dominate production, economic adjustments become arbitrarily slow.

P Overlapping Generations and Capital Accumulation

P.1 Benchmark OLG Model

In the representative agent (RA) model, savings can accumulate indefinitely, allowing automation to reach a self-sustaining regime where robots fully replace labor. In contrast, in the overlapping generations (OLG) framework, savings depend only on the income of young workers, fundamentally limiting capital accumulation and preventing runaway automation.

Each individual lives for two periods, working when young and consuming in both periods. Preferences are given by:

$$U = u(C_t) + \rho u(D_{t+1}), \tag{67}$$

where C_t and D_{t+1} represent consumption when young and old, respectively. The utility function follows the standard CRRA form:

$$u(C) = \begin{cases} \frac{C^{1-\sigma}-1}{1-\sigma}, & \text{if } \sigma \neq 1, \\ \ln(C), & \text{if } \sigma = 1. \end{cases} \tag{68}$$

In this section, we assume $\delta = 1$, a standard convenience assumption in the OLG literature. Savings per unit of effective labor satisfy the optimality condition:

$$s_t = \frac{w_t}{1 + \rho^{-\sigma}(1 + i_{t+1})^{1-\sigma}}. \tag{69}$$

A necessary condition for savings to be non-decreasing in the interest rate is $\sigma \geq 1$.

The transition equation for the robot-to-labor ratio is:

$$r_{t+1} = \frac{A_{R,t+1}}{1 + g_{A_L}} s_t. \tag{70}$$

This defines a sequence $\{r_t\}$ that determines the evolution of automation over time. Unlike standard OLG models, where capital accumulation is driven by interest rates and wage dynamics, here robot-augmenting technology ($A_{R,t+1}$) directly scales savings, influencing both the speed and extent of robot adoption.

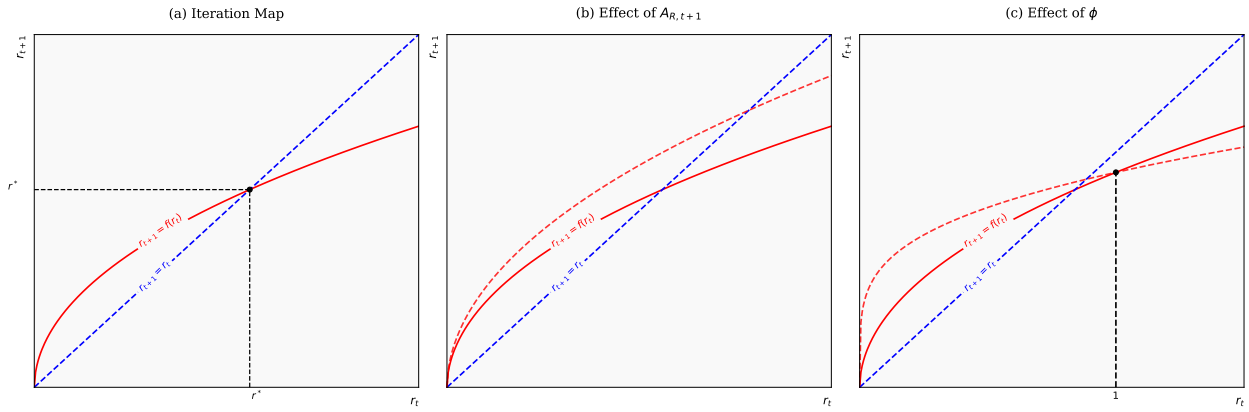


Figure 6: Iteration map and the effects of parameter changes in the overlapping generations model without bequests. Panel (a) shows the iteration map in (r_t, r_{t+1}) space. Panels (b) and (c) illustrate the effects of changes in $A_{R,t+1}$ and ϕ, respectively. The solid red line represents the function $r_{t+1} = f(r_t)$ in the base case, while the dashed red line shows its shift due to parameter changes. The blue dashed line represents the 45-degree identity line, $r_{t+1} = r_t$. The steady-state value r^* is marked with a black dot and corresponding dashed lines.

A standard result from the OLG literature states that the sequence $\{r_t\}$ is monotonic and bounded above. These properties are illustrated in Panel (a) of Figure 6, where r_t converges to the steady-state value r^*. The upper bound arises because savings per unit of labor is constrained by wages, and as robots accumulate, wages grow more slowly, reducing savings relative to r_t. More formally, from the production function, the wage-to-robot ratio satisfies:

$$\lim_{r \to \infty} \frac{w_t}{r_t} = 0. \tag{71}$$

This implies that even as r_t increases, the wage grows at a slower rate, restricting the potential for further savings and capital accumulation. More precisely, since $\lim_{r\to\infty} w/r = 0$ and savings are always bounded above by wages, it follows that $\lim_{r\to\infty} s/r = 0$ as well. Consequently, for sufficiently large r, savings become too small relative to the existing robot stock to sustain further accumulation. From the phase diagram, it follows that the function r_{t+1} asymptotically flattens as r_t increases, eventually becoming nearly horizontal. As a result, it must cross the 45-degree line at

some finite r_t, ensuring that robot accumulation is bounded. Since r_t remains finite, the self-sustainability index χ_t never reaches 1, implying that the economy does not transition into the self-sustainability phase.

An alternative way to illustrate that the economy never reaches the self-sustainability phase is to examine the steady-state condition for χ. Substituting the wage equation (14), the interest rate equation (15), the savings function (69), the robot accumulation equation (70), and the relationship between r and χ (9), the steady-state value χ^* satisfies:

$$\frac{{\chi^*}^{\frac{1}{\phi}}}{1-\chi^*}\left(1+\rho^{-\sigma}\left(\alpha^{\frac{1}{\phi}}A_R{\chi^*}^{-\frac{1-\phi}{\phi}}\right)^{1-\sigma}\right)=\frac{\alpha^{\frac{1}{\phi}}A_R}{1+g_{A_L}}. \tag{72}$$

The left-hand side of equation (72) is strictly increasing in χ^* for all $\phi \in (0,1)$ under the assumption $\sigma \geq 1$, which was established earlier. Additionally, it equals zero when $\chi^* = 0$ and tends to infinity as $\chi^* \to 1$. While the term $\alpha^{1/\phi}A_R$ appears on both sides of the equation, if $\sigma > 1$, the left-hand side is decreasing in this term, whereas the right-hand side is increasing. This ensures that for any finite A_R, there exists a unique steady-state value χ^* strictly between zero and one. Consequently, the economy always converges to a steady-state where robots never accumulate indefinitely, and the self-sustainability phase is never reached.

This result reinforces the earlier conclusion that the OLG structure imposes a fundamental constraint on automation. Unlike in the RA model, where unbounded capital accumulation is possible, here the steady-state condition (72) ensures that the self-sustainability phase, $\chi = 1$, is unattainable at any finite level of A_R. As a result, robots never accumulate indefinitely, and the economy remains in a regime where the accumulation of robots is bounded from above.

P.1.1 Effect of A_R on Robot Accumulation, Wages, and Interest Rates

Robot-augmenting technology A_R influences equilibrium outcomes through its effects on robot accumulation, wages, and the interest rate.

From equation (72), the steady-state value χ^* is an increasing function of A_R, implying that the steady-state robot-labor ratio r^* also increases with A_R. This shift is illustrated in Panel (b) of Figure 6, where a higher $A_{R,t+1}$ moves the iteration map upward, increasing the steady-state level of automation. However, unlike in the RA model, where for a given $\phi \in (0,1)$, there exists a threshold A_R beyond which robot accumulation and output growth become unbounded, the OLG framework imposes an upper bound on r^*. Since χ^* remains strictly below one, the economy never reaches the self-sustainability phase, and robot accumulation does not become unbounded, regardless of how large A_R becomes.

The increase in A_R also affects wages and the interest rate. From equation (14), wages increase as χ rises and tend to infinity as $\chi \to 1$. However, since χ does not converge to 1 in the OLG model, wages remain finite for any level of A_R. In contrast, in the RA framework, where automation can push χ arbitrarily close to 1, wages can grow indefinitely. This distinction highlights a fundamental difference between the two models: in the RA setting, automation eventually drives wages to ever-higher

levels, whereas in the OLG model, intergenerational savings constraints prevent unbounded wage growth.

The response of the interest rate to A_R also differs between the two models. From the equilibrium condition $i + \delta = MPR$, there is a trade-off between the interest rate and the robot-labor ratio. Since r remains finite in the OLG model, while it grows indefinitely in the RA model, the interest rate is more sensitive to changes in A_R in the OLG framework. This suggests that in an economy where automation advances within an OLG structure, automation-induced increases in A_R lead to a stronger rise in interest rates compared to the RA setting.

P.1.2 Effect of ϕ on Robot Accumulation

The elasticity of substitution ϕ has a more complex effect. As shown in Panel (c) of Figure 6, its impact depends on whether the economy is in the labor-dominated or transition phase.

At $r = 1$, corresponding to $\chi = \alpha$, changes in ϕ do not affect wages, the interest rate, or output, which explains why the iteration maps in Panel (c) intersect at $r_t = 1$. Away from this point, the effect is asymmetric. When ϕ is not close to zero, a higher ϕ reduces wages in the transition phase but raises them in the labor-dominated phase. Consequently, r_{t+1} increases for $r_t < 1$ but decreases for $r_t > 1$.

A natural intuition is that greater substitutability between robots and labor should always increase robotization. However, in the OLG model, this does not hold in the transition phase because capital is underaccumulated. Since savings depend only on young workers' wages, the economy does not generate enough investment to support higher robot adoption, even as substitutability increases. As a result, in this regime, a higher ϕ can reduce the steady-state level of robots rather than expand automation.

P.1.3 Conclusion

Unlike in the RA model, where savings can accumulate indefinitely, in the OLG framework, capital accumulation is constrained by the savings behavior of young workers. Even with rapid technological progress, automation remains bounded, and the economy does not transition into a self-sustaining automation regime. While increasing A_R raises automation, changes in ϕ can either reinforce or counteract robot accumulation, depending on the phase of the economy.

P.2 Effects of Bequests on Capital Accumulation

One way to alleviate this constraint is through intergenerational transfers. If individuals derive utility from leaving bequests, a portion of wealth is passed forward, increasing total savings and capital accumulation. We extend the model by introducing bequests into preferences:

$$U = u(C_t) + \rho u(D_{t+1}) + \hat{\rho} u(B_{t+1}), \tag{73}$$

where B_{t+1} represents the bequest left to the next generation. The budget constraint becomes:

$$D_{t+1} + B_{t+1} = (1 + i_{t+1})(B_t + W_t - C_t). \tag{74}$$

Individuals optimally choose savings and bequests:

$$s_t = \frac{w_t + b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1 + i_{t+1})^{1-\sigma}}, \tag{75}$$

$$b_{t+1} = \frac{\hat{\rho}^\sigma (1 + i_{t+1})}{\rho^\sigma + \hat{\rho}^\sigma + (1 + i_{t+1})^{1-\sigma}}(w_t + b_t). \tag{76}$$

Bequests increase capital accumulation by raising savings, which in turn raises the robot-to-labor ratio:

$$(1 + g_{A_L}) r_{t+1} = A_{R,t+1} s_t. \tag{77}$$

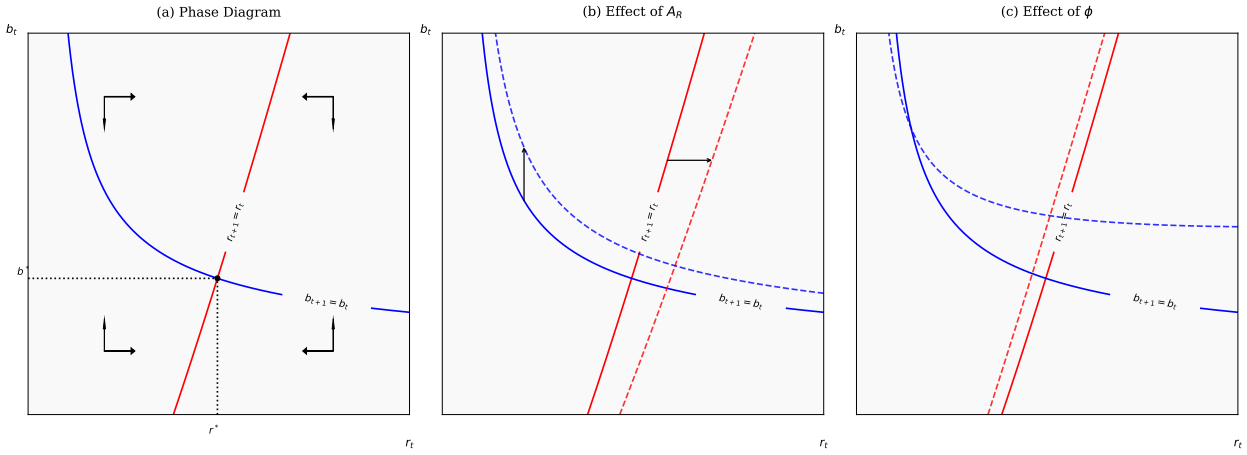


Figure 7: Phase diagrams illustrating the dynamics of robotization (r_t) and bequests (b_t). Panel (a) shows the phase diagram with nullclines $r_{t+1} = r_t$ (red) and $b_{t+1} = b_t$ (blue). Panel (b) illustrates the effect of increasing robot-augmenting technological progress parameter A_R from 20 to 25, shifting the nullclines and steady-state. Panel (c) depicts the effect of increasing the elasticity parameter ϕ from 0.1 to 0.5, similarly altering the phase dynamics and steady-state. Dashed lines represent the new nullclines after the parameter change, and arrows indicate the direction of shifts.

Notes: Parameter values are set as follows: $\alpha = \frac{1}{3}$, $g_{A_L} = 0.01$, $\sigma = \frac{1}{2}$, $\rho = \frac{1}{2}$, $\hat{\rho} = \frac{1}{4}$, and $\delta = 1$. The base values of A_R and ϕ are 3.5 and 0.1, respectively, with changes to 4.5 and 0.5, as indicated in Panels (b) and (c).

P.3 Self-sustainability

To figure out whether the self-sustanability regime is possible, consider the dynamics of the economy as $r \to \infty$. The dynamics is then given by:

$$r_{t+1} = \frac{A_{R,t+1}}{1+g_{A_L}} \frac{\rho^\sigma + \hat{\rho}^\sigma}{\rho^\sigma + \hat{\rho}^\sigma + (A_R \alpha^{\frac{1}{\phi}})^{1-\sigma}} b_t,$$

$$b_{t+1} = \frac{\hat{\rho}^\sigma (A_R \alpha^{\frac{1}{\phi}})}{\rho^\sigma + \hat{\rho}^\sigma + (A_R \alpha^{\frac{1}{\phi}})^{1-\sigma}} b_t.$$

From this system, r grows infinitly if and only if b grows infinitly, and the latter happens when:

$$\frac{\hat{\rho}^\sigma (A_R \alpha^{\frac{1}{\phi}})}{\rho^\sigma + \hat{\rho}^\sigma + (A_R \alpha^{\frac{1}{\phi}})^{1-\sigma}} \geq 1$$

$$\hat{\rho}^\sigma A_R \alpha^{\frac{1}{\phi}} - (A_R \alpha^{\frac{1}{\phi}})^{1-\sigma} \geq \rho^\sigma + \hat{\rho}^\sigma$$

P.4 Steady-State Capital Level and the Limitations of Bequests

In steady state:

$$r_{t+1} = r_t = r^*, \tag{78}$$

$$b_{t+1} = b_t = b^*. \tag{79}$$

From the equilibrium conditions, the steady-state robot-labor ratio is:

$$r^* = \frac{A_R s^*}{1+g_{A_L}}, \tag{80}$$

where s^* incorporates both direct savings and bequests. Bequests raise s^*, increasing r^*, but their effect is bounded:

$$b^* = \frac{\hat{\rho}^\sigma (1+i^*)}{\rho^\sigma + \hat{\rho}^\sigma + (1+i^*)^{1-\sigma}} (w^* + b^*). \tag{81}$$

Bequests increase capital accumulation, but they do not fully eliminate the constraint imposed by the OLG structure. Unlike in the RA model, where capital accumulation compounds indefinitely, bequests remain a **fraction of total income** and depend on previous generations savings. Thus, while they push the economy toward greater automation, they do not induce the self-sustaining regime observed in the RA case.

P.5 Conclusion and Policy Implications

Bequests mitigate the capital accumulation constraint by transferring resources across generations, improving the trajectory of robotization. However, they do not fully eliminate underaccumulation, as savings still depend on generational income rather than an indefinitely compounding capital stock.

This suggests that **policy interventions may be necessary** to complement bequests. Potential policies include:

- **Pension reforms** that increase retirement savings.
- **Capital taxation** to redistribute from the old (wealth holders) to the young (savers).
- **Subsidies for automation investment** to accelerate capital deepening.

Each of these policies alters the intergenerational allocation of resources and affects the long-run trajectory of capital accumulation and automation.

$$\lim_{r\to\infty} \frac{r_{t+1}}{r_t} = \lim_{r\to\infty} \frac{A_R}{(1+g_{A_L})r_t} \frac{w_t + b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1+i_{t+1})^{1-\sigma}}, \tag{82}$$

$$= \lim_{r\to\infty} \frac{A_R}{(1+g_{A_L})r_t} \frac{b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1+i_{t+1})^{1-\sigma}} \tag{83}$$

$$= \lim_{r\to\infty} \frac{A_R}{(1+g_{A_L})r_t} \frac{b_t}{1 + \frac{1}{\rho^\sigma + \hat{\rho}^\sigma}(1+i_{t+1})^{1-\sigma}} \tag{84}$$

$$b_{t+1} = \frac{\hat{\rho}^\sigma (1+i_{t+1})}{\rho^\sigma + \hat{\rho}^\sigma + (1+i_{t+1})^{1-\sigma}} (w_t + b_t) \tag{85}$$

Line (83) uses that $\lim_{r->\infty} w/r = 0$.