Risk Markups

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We study optimal policy in an economy where heterogeneous markups arise as compensation for uninsurable persistent idiosyncratic risk and cause misallocation. Entrepreneurs hire labor trading off expected profits against risk. We study the constrained-efficient allocation of a planner who can use a uniform labor tax and time-zero lump-sum transfers. The optimal keep rate equals the product of (1) the aggregate markup and (2) workers' consumption share divided by their Pareto weight. The markup component reflects inefficient risk premia and the consumption-share component reflects inefficient precautionary saving. In the long-run, the precautionary-saving component dominates and the optimal policy is a tax.

1 Introduction

Measured markups are large and heterogeneous across firms and can cause misallocation, i.e., lower TFP.[1] The role for policy to address misallocation depends on the source of the markups. A common view is that measured markups arise because firms have market power. From this perspective, Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023) show that it is optimal to undo the markup distortions by subsidizing production.

In this paper, we study misallocation and optimal policy when measured markups are instead risk premia. Since at least Knight (1921) economists have recognized that profits may reflect risk compensation. Boar, Gorea, and Midrigan (2023) quantify the sources of the dispersion in firm profits and attribute the vast majority of the variation to uninsurable idiosyncratic risk, with some contribution from market power or decreasing returns to scale, and little contribution from collateral constraints.

Our main contribution is to characterize optimal policy in this environment. We obtain a sufficient statistic for the optimal labor tax in terms of (1) the aggregate gross markup and (2) workers' consumption share divided by their Pareto weight. The markup component reflects inefficient risk premia and the consumption-share component reflects inefficient precautionary saving. In the long run it is optimal to tax production, in sharp contrast to the common wisdom derived from the market-power perspective.

Model Overview and Main Results. This paper develops a model to study how uninsurable idiosyncratic entrepreneurial risk affects markups, misallocation, and efficiency. There is a representative worker and a continuum of entrepreneurs. The workers supply labor, which is the only factor of production, and face no risk. Each entrepreneur is endowed with a distinct linear technology to produce the homogeneous consumption good. There are four essential ingredients. First, entrepreneurs persistently differ in the productivity of their technology, which introduces the possibility of misallocation. Second, hiring labor is risky. Entrepreneurs

[1] See Hall (1988), De Loecker and Warzynski (2012), Restuccia and Rogerson (2008), and Hsieh and Klenow (2009).

must hire labor before the realization of their productivity shock. Third, entrepreneurs are risk averse. Fourth, there is incomplete idiosyncratic risk sharing. There are competitive markets for goods, labor, a risk free bond, and firm equity. Motivated by moral hazard arguments, the entrepreneur can sell only up to a fraction $1-\phi$ of the firm value and must retain a ϕ share of the firm equity.

Risk-averse entrepreneurs, risky production, and incomplete risk sharing lead to risk markups. Due to competitive markets, all firms charge the same price for the homogeneous good they sell and pay the same wage to hire labor. But there is a wedge in the first order condition for hiring labor, such that the price of the final good is greater than the unit labor cost. This risk markup wedge is a risk premium. Entrepreneurs hire less labor than they would if they had full insurance. Hiring labor is like investing in a risky asset and they consider the trade off between the expected return and the risk associated with hiring.

Heterogeneous productivity generates heterogeneous risk markups and misallocation. Even though production of the homogeneous good is linear, while more productive entrepreneurs hire more labor, the most productive entrepreneur does not hire all the labor because that would expose him to too much risk. There is a sense in which there are no markups in the competitive equilibrium, since the risk-adjusted marginal cost is equal to the price of the final good for all firms. Nonetheless, the risk markup wedges generate misallocation and is reflected in lower TFP.

In this environment we study the constrained efficient allocation. We study a planner who has access to a time-varying tax on workers' wages (instantly redistributed lump sum) and time-zero lump-sum transfers. The lump-sum transfers allow the planner to achieve any desired redistribution motive, allowing us to focus on whether there are efficiency reasons to use the wage tax. The main result of the paper is that the optimal keep rate (one minus the tax rate) is equal to the product of (1) the aggregate markup and (2) workers' consumption share divided by their Pareto weight.

The markup component reflects inefficient risk premia and the consumption-share com-

ponent reflects inefficient precautionary saving. Even though the risk markup represents compensation for utility-relevant risk taking, it is still inefficient. In isolation, risk premia induce inefficiently low labor demand due to a coordination problem. Individually, as an entrepreneur hires more labor, he is exposed to more risk and consequently has more volatility in his consumption. The planner internalizes that the extra output produced helps all entrepreneurs smooth their consumption, so that consumption is increased while consumption volatility is held constant. To correct this inefficiency in isolation, the planner wants a labor subsidy equal to the aggregate markup.

However, the inefficient risk markup is not the planner's sole concern. Just like in the Aiyagari (1994) model, uninsurable idiosyncratic risk leads to inefficient precautionary saving. Entrepreneurs accumulate too much wealth relative to workers and, since the constrained-efficient planner takes Euler equations as a constraint, the entrepreneurs must receive a larger share of consumption than is warranted by their Pareto weights. For the planner, the share of output above their Pareto weight that goes to entrepreneurs is wasted. A labor tax prevents workers from supplying too much labor to produce this wasted output. To clarify this, we show that a planner who can also tax entrepreneurs' and workers' savings at different rates uses a labor subsidy to undo the aggregate markup and a tax on entrepreneurs' savings to eliminate their precautionary motive.

A key insight of this paper is that the single feature of uninsurable idiosyncratic risk simultaneously generates both inefficient risk markups and inefficient precautionary saving. In the long-run, the precautionary saving effect dominates the risk premia effect and the optimal policy is a tax. Optimal policy when misallocation is caused by uninsurable idiosyncratic risk is the opposite of what would be optimal in many models in which misallocation is caused by monopolistic competition.

1.1 Relationship to Existing Literature

We contribute to the recent literature on risk and misallocation. In David, Schmid, and Zeke (2022), dispersion in the marginal product of capital is generated by heterogeneous risk premia. There is an aggregate shock and different firms have different exogenous loadings on the aggregate risk factor. As in CAPM, the different risk profiles generate different risk premia, which generate deviations from equalizing MPKs, even though the decentralized equilibrium is efficient. Our paper also explores the relationship between risk and misallocation, but in our model risk premia are endogenous, generated by idiosyncratic risk and incomplete markets. This leads to a substantively different perspective on optimal policy. David and Zeke (2024) further explore the implications of aggregate risk and misallocation for optimal monetary policy in a New Keynesian model and David, Ranciere, and Zeke (2023) explores the implications for international diversification and the labor share.

Boar, Gorea, and Midrigan (2023) is a closely related paper that also studies an environment with uninsurable idiosyncratic risk. Theirs is a quantitative paper that explains the source of heterogeneity in firm returns. Their model additionally features decreasing returns to scale production technology (equivalently monopolistic competition) and collateral constraints. They find that uninsurable idiosyncratic risk is by far the most important feature that explains the empirical variation in returns. Motivated by their quantitative results on the importance of uninsurable idiosyncratic risk, we compliment their paper by providing a theoretical analysis of optimal policy in such a world. Since optimal policy is relatively well understood when misallocation is generated by market power or by collateral constraints, we only model uninsurable idiosyncratic risk. This allows us to provide insights from analytical results in a tractable model.

Our model of entrepreneurship is similar to that in Moll (2014) and our analysis of optimal policy is similar to Itskhoki and Moll (2019). Those papers, however, focus on collateral constraints that prevent firms from operating at their optimal scale, which induces capital misallocation. A key insight from their analysis is that the degree of persistence in

the stochastic idiosyncratic productivity process affects the magnitude of steady-state TFP losses from misallocation and the speed of transition towards steady state. Buera and Shin (2013) similarly focus on the effects of collateral constraints on misallocation, although in a model with decreasing returns to scale production and an occupational choice, with a focus on quantitative results. To clearly distinguish the new mechanism of risky production in our paper, we make labor the only factor of production and omit collateral constraints.

Our paper is also similar to Meh and Quadrini (2006) and Angeletos (2007) in that these papers study the effect of risky production, and not collateral constraints, on long-run allocations. Meh and Quadrini (2006) and Angeletos (2007), however, study a model with iid idiosyncratic risk focused on capital accumulation. Di Tella and Hall (2022) study labor risk premiums and investment dynamics in the context of business cycles. These models with iid shocks do not feature heterogeneous types and, thus, no potential for misallocation across types. We study a model without aggregate risk, capital, or capital constraints to focus on the long-run relationship between idiosyncratic risk, markups, and misallocation across persistently heterogeneous entrepreneurs.

Eeckhout and Veldkamp (2023) is another paper exploring how risk affects our interpretation of measured markups. In their paper, firms acquiring data can reduce risk markups by reducing risk (improving forecasting) but can increase markups by inducing investment to capture market share and exploit market power. Dou, Ji, Tian, and Wang (2025) also studies how risk and incomplete markets affects misallocation, with a focus on how misallocation introduces medium-run fluctuations in TFP that affect aggregate growth and asset prices. These are all in contrast to papers on misallocation without an essential role for risk, such as those that generate misallocation with market power, such as Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023), with search frictions as in Menzio (2024), or adjustment costs, as in Asker, Collard-Wexler, and De Loecker (2014).

2 The Model

2.1 The Environment

Time is continuous and infinite, indexed by $t \geq 0$. There is a representative worker and a continuum of entrepreneurs, indexed by $i \in [0,1]$, with preferences

$$U_w(c_w, \ell) = \mathbb{E}\left[\int_0^\infty e^{-\rho_w t}\left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right)dt\right], \tag{1}$$

$$U_e(c_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho_e t}\log c_{it}dt\right]. \tag{2}$$

We focus on the case with entrepreneurs more impatient than workers, $\rho_e > \rho_w > 0$ to obtain a non-degenerate stationary wealth distribution.[2]

Each entrepreneur has a linear production technology that uses labor to produce the final good:

$$dY_{it} = z_{it}\ell_{it}dt + \sigma_y \ell_{it} dB_{yit}, \tag{3}$$

$$dz_{it} = \mu_z(z_{it})dt + \sigma_z(z_{it})dB_{zit}, \tag{4}$$

where B_{yit} and B_{zit} are two Brownian motions specific to the entrepreneur that have correlation λ_{yz}, and $\sigma_y, \sigma_z(z_{it}) > 0$. A central assumption in our setting is that production is risky. At time t when an entrepreneur chooses labor ℓ_{it}, he knows his expected marginal productivity z_{it}, but the final output is exposed to idiosyncratic risk through the term $\sigma_y \ell_{it} dB_{yit}$. An entrepreneur's expected marginal productivity z_{it} evolves stochastically, following a diffusion driven by B_{zit}. We assume z_{it} remains in a bounded interval $[0, \bar{z}]$ and converges to a unique non-degenerate stationary distribution.

[2]The case $\rho_e = \rho_w$ is easy to analyze and doesn't change the main results.

The resource constraints are[3]

$$c_t = \int_0^1 c_{it} di + c_{wt} = \int_0^1 z_{it} \ell_{it} di, \tag{5}$$

$$\ell_t = \int_0^1 \ell_{it} di. \tag{6}$$

2.2 The Competitive Equilibrium

There are competitive markets for the consumption good, labor, a risk-free bond in zero net supply, and entrepreneurs' equity. The consumption good is the numeraire, with price normalized to 1. Markets are incomplete: entrepreneurs must retain a fraction $\phi \in (0, 1)$ of their idiosyncratic risk. Budget constraints are

$$dn_{wt} = (r_t n_{wt} - c_{wt} + w_t \ell_t) dt, \tag{7}$$

$$dn_{it} = (r_t n_{it} - c_{it} + (z_{it} - w_t)\ell_{it}) dt + (\phi \sigma_y) \ell_{it} dB_{yit}. \tag{8}$$

where n_{it} and n_{wt} are agents' bond positions, r_t is the interest rate, and w_t the wage. Entrepreneurs' inability to share idiosyncratic risk is reflected in their budget constraint: they must retain an exposure ϕ to the idiosyncratic risk in their production.[4] Both entrepreneurs and workers face natural borrowing constraints. For entrepreneurs, this means $n_{it} \geq 0$. Workers cannot borrow more than the present value of labor income, $n_{wt} \geq -\int_t^\infty e^{-\int_t^s r_u du} w_s \ell_s ds$. Market clearing of the bond market requires

$$n_{wt} + \int_0^1 n_{it} di = 0. \tag{9}$$

[3] In equation (5) we are assuming a law of large numbers applies. That is, we focus on allocations such that $\int_0^1 \left(\int_0^t \ell_{is} \sigma_y dB_{iys} \right) di = 0$.

[4] Entrepreneurs sell, in a competitive market, a claim to a fraction $(1 - \phi)$ of their profits $dY_{it} - w_t \ell_{it} dt$ in exchange for payment $a_{it} dt$. Since the idiosyncratic risk can be diversified, the market prices this claim so $\mathbb{E}\left[(1 - \phi)(dY_{it} - w_t \ell_{it} dt) - a_{it} dt\right] = 0$. Thus, the entrepreneur's budget constraint is $dn_{it} = (r_t n_{it} - c_{it}) dt + \phi(dY_{it} - w_t \ell_{it} dt) + a_{it} dt = (r_t n_{it} - c_{it} + (z_{it} - w_t)\ell_{it}) dt + (\phi \sigma_y) \ell_{it} dB_{yit}$. Absent quantification of σ_y, it is without loss of generality to consider the $\phi = 1$ case, since ϕ and σ_y always appear together multiplicatively as $\phi \sigma_y$.

The *representative worker's problem* is to choose $(c_{wt} > 0, \ell_t \geq 0)$ to maximize his utility subject to his budget constraint and natural borrowing constraint, taking prices (r, w) and initial bonds n_{w0} given. An *entrepreneur's problem* is to choose $(c_{it} > 0, \ell_{it} \geq 0)$ to maximize his utility subject to his budget constraint and natural borrowing constraint, taking prices (r, w) and initial bonds n_{i0} given. Markets clear if equations 5 and 6 hold (with equation 9 then holding by Walras' Law).

For a given initial distribution of bonds n_{w0} and $n_{i0} > 0$ for $i \in [0, 1]$ satisfying (9), a *Competitive Equilibrium* is an allocation (c_w, ℓ_w) and (c_i, ℓ_i) for $i \in [0, 1]$, and market prices (r, w) such that the representative worker and each entrepreneur solve their problem and markets clear.

2.3 Characterizing the Competitive Equilibrium

Optimality Conditions. The representative worker's optimal allocation is characterized by an Euler equation and the first order condition for labor supply:

$$\frac{dc_{wt}}{c_{wt}} = (r_t - \rho_w)dt, \tag{10}$$

$$c_{wt}^{-1} w_t = \ell_t^{1/\psi}. \tag{11}$$

An entrepreneur's optimal allocation can be characterized by an Euler equation and the first order condition for labor demand:

$$\frac{dc_{it}}{c_{it}} = (r_t - \rho_e + \sigma_{cit}^2)dt + \sigma_{cit} dB_{yit}, \tag{12}$$

$$z_{it} - w_t \leq \sigma_{cit}\phi\sigma_y \quad \text{(with equality if } z_{it} \geq w_t\text{)}, \tag{13}$$

$$\sigma_{cit} = \frac{\ell_{it}}{n_{it}}(\phi\sigma_y) = \frac{\ell_{it}}{c_{it}}(\rho_e \phi\sigma_y). \tag{14}$$

Here we have used the fact that with log preferences $c_{it} = \rho_e n_{it}$ and there are no dynamic hedging motives. His consumption is therefore locally exposed only to the idiosyncratic risk

in his output B_{yt} (and not the idiosyncratic risk in his productivity B_{zt}, although they may be correlated). The last term uses the budget constraint to express the exposure to risk on the entrepreneur's consumption in terms of his labor demand ℓ_{it}, and uses $c_{it} = \rho_e n_{it}$ to substitute out the bonds. Homothetic preferences and linear budget constraints result in labor demand that, when positive, is linear in productivity and in bonds, as can be seen by combining equations (13) and (14). See Appendices A and B for details.

The variable σ_{cit} is the volatility of consumption growth of entrepreneur i and is at the center of our analysis. The σ_{cit}^2 term in the Euler equation captures the precautionary saving motive. The $\sigma_{cit}\phi\sigma_y \geq 0$ term in the labor FOC is a risk premium on labor: it captures the covariation between the marginal utility of the entrepreneur and the fraction ϕ of the marginal product of labor that cannot be shared. Hiring labor is like investing in a risky asset. The demand for labor trades off the expected marginal profit from hiring labor, $z_{it} - w_t$, against increased exposure to risk. Labor demand is increasing in bonds because they provide self-insurance. Entrepreneurs hire less labor compared to the complete markets case. Even though production of the homogeneous good is linear, the most productive entrepreneur does not hire all the labor because that would expose him to too much risk.

The risk premium on labor creates labor wedges at the entrepreneur level that look like markups. For entreprenuers who produce (i.e., those with $z_{it} > w_t$),

$$1 + \pi_{it} := \frac{z_{it}}{w_{it}} = 1 + \frac{\sigma_{cit}\phi\sigma_y}{w_t}. \tag{15}$$

That is the price of the good is higher than the unit labor cost. But risk markups capture the private marginal cost of exposing the entrepreneur to risk. The risk-adjusted marginal product of labor, $z_{it} - \sigma_{cit}\phi\sigma_y$, is equalized to the labor cost w_t for all entrepreneurs who have positive labor demand, $\ell_{it} > 0$. Thus, there is no true markup in the sense that the risk-adjusted marginal cost is equal to the price of the good for all active entrepreneurs. At the aggregate level, however, risk markups show up as a labor wedge generating misallocation

that reduces TFP. Let Z_t be aggregate TFP and $1+\pi_t$ the aggregate labor wedge

$$Z_t := \frac{c_t}{\ell_t} = \int_0^1 z_{it} \frac{\ell_{it}}{\ell_t} di, \tag{16}$$

$$1+\pi_t := \frac{Z_t}{w_t} = \int_0^1 \frac{z_{it}}{w_t} \frac{\ell_{it}}{\ell_t} di. \tag{17}$$

Aggregate productivity Z_t is the labor-weighted average productivity and the aggregate risk markup $1+\pi_t$ is the labor-weighted average labor wedge.

Aggregation and a Separation Property. The environment with log preferences and linear production allows for easy aggregation and a powerful separation result. We can solve for the cross-sectional allocations separately from aggregate employment and output. Because labor demand is linear, when computing equilibrium allocations we do not need to keep track of the joint distribution of bonds n_{it} and productivity z_{it}. It is enough to keep track of the aggregate bond holdings (or equivalently, consumption) of entrepreneurs for each level of productivity. At time t, let η_{it} be the consumption share of entrepreneur i, η_{et} be the consumption share of all entrepreneurs, η_{wt} be the consumption share of the representative worker, and $\eta_t(z)$ be the consumption share of all entrepreneurs with productivity z:

$$\eta_{it} := c_{it}/c_t, \quad \eta_{et} := \int_0^1 \eta_{it} di, \quad \eta_{wt} := c_{wt}/c_t, \quad \eta_t(z) := \int_{\{z_{it}=z\}} \eta_{it}\, di. \tag{18}$$

Then, $\eta_{et} = \int_0^{\bar{z}} \eta_t(z) dz$ and by the resource constraint $\eta_{wt} = 1 - \eta_{et}$.

We can rewrite the entrepreneurs' labor demand as

$$\ell_{it} = \sigma_{cit} \eta_{it} \frac{c_t}{\rho_e \phi \sigma_y}. \tag{19}$$

Combine optimality conditions (13) and (19) with the resource constraints to obtain expressions for the wage and aggregate productivity as a function of the entrepreneur consumption-

share distribution. That is, we implicitly define the functions $w(\eta_t)$ and $Z(\eta_t)$ by

$$\frac{1}{\rho_e \phi \sigma_y} \times \int_w^{\bar{z}} z \eta_t(z) \frac{z-w}{\phi \sigma_y} dz = 1, \tag{20}$$

$$\left(\frac{1}{\rho_e \phi \sigma_y} \times \int_w^{\bar{z}} \eta_t(z) \frac{z-w}{\phi \sigma_y} dz \right)^{-1} = \frac{c_t}{\ell_t} = Z. \tag{21}$$

We also obtain an expression for the exposure to risk for each entrepreneur of productivity type z at each point in time as a function of the entreprenuer's productivity level and the consumption-share distribution. That is, we define the risk exposure function as

$$\sigma_c(z, \eta_t) := \frac{(z - w(\eta_t))^+}{\phi \sigma_y}. \tag{22}$$

We can use the goods resource constraint and the Euler equations for the worker and entrepreneurs to describe the evolution of η_t over time by a Kolmogorov forward equation (KFE) that only depends on η_t itself,

$$\begin{aligned} \partial \eta(z,t) = {} & \eta(z,t) \left[-\mu_z'(z) + \mu_\eta(z, \eta_t) + \sigma_z'^2(z) + \sigma_z(z)\sigma_z''(z) - \left(\sigma_z'(z)\sigma_c(z,\eta_t) + \sigma_z(z)\sigma_c'(z,\eta_t)\right) \lambda_{yz} \right] \\ & + \eta_z'(z,t) \left[-\mu_z(z) + 2\sigma_z(z)\sigma_z'(z) - \sigma_z(z)\sigma_c(z,\eta_t)\lambda_{yz} \right] \\ & + \eta_{zz}''(z,t) \left[\frac{1}{2}\sigma_z^2(z) \right] \end{aligned} \tag{23}$$

where $\mu_z(z), \sigma_z(z), \lambda_{yz}$ are primitives and $\mu_\eta(z, \eta_t) = \bar{\rho}_t - \rho_e + \sigma_c^2(z, \eta_t) - \int \eta_t(z) \sigma_c^2(z, \eta_t) dz$. See Appendix E for details of the KFE derivation.

Constructing the Equilibrium. This separation result allows us to compute an equilibrium in two steps. In step one, start with a given $\eta_{i0} > 0$ such that $\eta_{e0} < 1$ and compute $\eta_0(z)$. Then, solve the KFE forward to obtain the path of $\eta_t(z)$. Then use equations (20)–(22) to solve for the path of w_t, Z_t, and $\sigma_c(z, \eta_t)$. At this point we must verify that there exists a positive w_t that solves equation (20) using η_t at all t.[5]

Due to the separation result, step one does not involve aggregate output c_t, aggregate

[5] If not, no competitive equilibrium exists that is consistent with initial condition n_{i0}, n_{w0}.

employment ℓ_t, or interest rates r_t. In step two, we determine these by using the representative worker's labor supply condition to solve for aggregate employment and the definition of aggregate TFP to compute aggregate output

$$(\overbrace{\eta_{wt} Z_t \ell_t}^{c_{wt}})^{-1} w_t = \ell_t^{1/\psi}, \tag{24}$$

$$c_t = Z_t \ell_t. \tag{25}$$

Once we have the path of ℓ and c we can construct the rest of the equilibrium. The representative worker's consumption is $c_{wt} = \eta_{wt} c_t$, and their Euler equation determines the interest rate

$$r_t = \rho_w + \frac{\dot{c}_{wt}}{c_{wt}}. \tag{26}$$

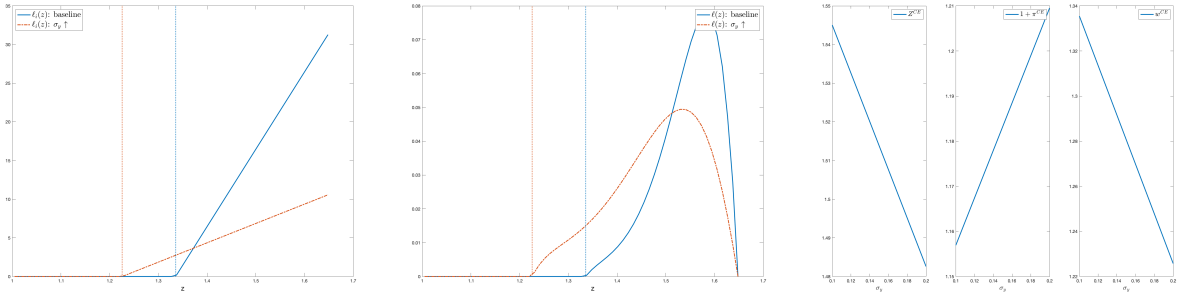
Set $c_{i0} = \eta_{i0} c_0$ and $\sigma_{cit} = \sigma_c(z_{it}, \eta_t)$. We can use entrepreneurs' Euler equation (12) to determine their consumption c_{it} and then their labor demand FOC in equation (14) determines ℓ_{it}. We set the bonds of each entrepreneur to $n_{it} = c_{it}/\rho_e$ and the bonds of the worker to $n_{wt} = -\int_0^1 n_{it} di$.

Proposition 1. *Let $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ be as constructed by the above procedure with associated initial bonds n_{w0} and n_{i0}. Assume that for all t, $w_t, c_{it}, c_{wt} > 0$ and $\lim_{T\to\infty} e^{-\int_0^T r_u du} n_{wT} = 0$. Then $\mathcal{C}$ is a competitive equilibrium for initial bonds n_{w0} and n_{i0}.*

Proof. See Appendix C.

2.4 A Numerical Example: Equilibrium

To illustrate key characteristics of the competitive equilibrium, we explore comparative statics around some roughly chosen baseline parameter values. See Appendix F for details. Figure 1 explores how labor demand, TFP, and the aggregate markup change when hiring labor becomes more risky (increase in σ_y or, equivalently, ϕ). Blue solid lines are from the



(a) Individual Labor Demand (b) Total Type-z Labor (c) TFP, Markup, and Wage

Figure 1: Comparative Static w.r.t. σ_y

baseline calibration and orange dotted lines are from a counterfactual equilibrium with a larger σ_y. The vertical lines mark equilibrium wages.

Figure 1a shows that an entrepreneur's labor demand is positive when productivity is higher than the wage and increases linearly in productivity (holding fixed wealth). When production becomes riskier, entrepreneurs do not scale their firm size with their productivity as aggressively. Figure 1b plots the distribution of total labor hired by all entrepreneurs of type z. There is a steady-state distribution of productivity z, given by the exogenous stationary persistent stochastic process, that is log-normally distributed around a mean of 1. Three forces determine the distribution of labor across productivity types. First, the density of z is declining for $z > 1$. Second, the entrepreneur's optimal labor hiring policy is linearly increasing in productivity. Third, labor demand is a function of the bond distribution.

In this numerical example, there is not much labor hired by low productivity types close to the wage threshold even though there are many entrepreneurs of this type because they each hire little labor. Total labor hired by z-types initially increases in z even though the density of entrepreneurs is decreasing, because each entrepreneur is hiring more workers. Finally, as z further increases, the density of entrepreneurs falls fast enough that the total labor hired by z-type entrepreneurs falls, even as each entrepreneur continues to hire more workers.

Figure 1c shows the effect of increased risk on key macro aggregates. In an economy where

production is riskier, each entrepreneur would hire less labor holding the wage constant. In equilibrium, the wage falls, bringing previously inactive entrepreneurs into production. The end result is that the largest producers reduce labor demand and the smallest producers increase labor demand. The aggregate markup rises, reflecting the increase in risk premia. In this model risk generates misallocation, lowering TFP. Connecting to the misallocation literature, more risk (higher σ_y) or less risk sharing (higher ϕ) could be a potential explanation for differences in cross-country TFP levels and the size distribution of firms. We now turn to analyzing the efficiency properties of the competitive equilibrium.

3 Optimal Policy

Consider a social planner who has access to two policy instruments: (1) a uniform labor tax on the representative worker rebated lump-sum and (2) lump-sum transfers across agents at time $t = 0$. We are interested in studying the efficiency properties of the competitive equilibrium. The $t = 0$ transfers can accomplish any redistribution the social planner may desire, so we can focus on understanding which interventions improve efficiency. We will also consider constraints on the Planner's ability to redistribute at time zero and the role of a tax on entrepreneurs' savings.[6]

In principle we could just study a planner who has access to entrepreneur-specific labor taxes and interest rate taxes. In the absence of a mechanism design approach that delivers the policy instruments needed to implement the planner's constrained optimal allocation, we need to carefully select the tools we give to the planner to avoid studying an uninteresting problem. Motivated by such asymmetric information concerns, we think it may not be feasible for a planner to successfully use entrepreneur-specific taxes without inducing unobserved changes in entrepreneur behavior. For example, a size-dependent tax could induce on-paper mergers that do not change production technologies (such as employing the janitorial staff

[6] A uniform interest rate subsidy is a redundant policy instrument as long as the planner has access to $t = 0$ lump-sum transfers. All agents have the same interest rate in their Euler equations, and a subsidy/tax that affects the return on savings equally for all agents simply serves to redistribute without affecting efficiency.

instead of purchasing janitorial services from another firm). Thus, we think a uniform tax is the most interesting case. However, in order to better understand how to optimally use a uniform labor tax, we will consider a case in which the planner also has access to heterogeneous taxes. Here, we find most insightful analysis when the planner also has a tax on entrepreneur saving, studied in Section 3.5.

3.1 The Planner's Problem

The planner has access to a uniform labor tax τ_t that is rebated lump-sum to the worker. A tax plan is *feasible* if $w_t\tau_t$ is a bounded process. A labor tax rebated to workers distorts the labor margin without affecting any budget constraint in equilibrium. The representative worker's budget constraint is

$$dn_{wt} = (n_{wt}r_t - c_{wt} + (1-\tau_t)w_t\ell_t + T_t)dt, \tag{27}$$

with natural borrowing limit $n_{wt} \geq -\int_t^\infty e^{-\int_t^s r_u du}((1-\tau_s)w_s\ell_s + T_s)ds$. In equilibrium $T_t = \tau_t w_t \ell_t$. The equilibrium condition for labor supply becomes

$$\ell_t^{1/\psi} = c_{wt}^{-1} w_t(1-\tau_t) = (\eta_{wt} Z_t \ell_t)^{-1} w_t(1-\tau_t). \tag{28}$$

Agents' problems are unchanged, except that the representative worker's budget constraint is now given by equation (27) with the associated natural borrowing constraint. A *competitive equilibrium with uniform labor tax and lump-sum transfers* is an allocation (c_w, ℓ, c_i, ℓ_i), prices (r, w), taxes and transfers (τ, T) and initial bonds (n_{i0}, n_{w0}) such that (a) the representative worker and each entrepreneur solve their problem given their initial bonds, (b) markets clear, and (c) $T_t = \tau_t w_t \ell_t$ and $\int n_{i0}di + n_{w0} = 0$. An allocation is *implementable* if there exist prices, feasible taxes and transfers, and initial bonds such that they constitute a *competitive equilibrium with uniform labor tax and lump-sum transfers*. An implementable allocation is

optimal if it maximizes

$$\int_0^\infty e^{-\rho_w t}\tilde{\Gamma}_w \left(\log c_{wt} - \frac{\ell_t^{1+1/\psi}}{1+1/\psi}\right) dt + \int_0^1 \mathbb{E}\left[\int_0^\infty e^{-\rho_e t}\tilde{\Gamma}_e \log c_{it} dt\right] di, \tag{29}$$

where $\tilde{\Gamma}_w > 0$ and $\tilde{\Gamma}_e > 0$ are the Pareto weights on the representative worker and entrepreneurs. For simplicity of exposition, we assume that the planner does not care about entrepreneurs individually (i.e., $\tilde{\Gamma}_i = \tilde{\Gamma}_e \ \forall \ i$). It will be useful to work with the discounted Pareto weights,

$$\Gamma_{et} = e^{-\rho_e t}\tilde{\Gamma}_e, \quad \Gamma_{wt} = e^{-\rho_w t}\tilde{\Gamma}_w. \tag{30}$$

Transforming the Planner's Problem. In principle, the planner's problem above is complicated by needing to choose an optimal transition path for an entire distribution. However, we can exploit a separation property to simplify the problem. This is done in two steps.

First, rewrite the objective function in terms of aggregate labor ℓ_t and the distribution of consumption shares η_{it}:[7]

$$\begin{aligned} &\overbrace{\int_0^\infty (\Gamma_{wt}+\Gamma_{et})\log \ell_t - \Gamma_{wt}\frac{\ell_t^{1+1/\psi}}{1+1/\psi} dt}^{\text{labor}} + \overbrace{\int_0^\infty (\Gamma_{wt}+\Gamma_{et})\log Z_t dt}^{\text{TFP}} \\ &+ \underbrace{\int_0^\infty \Gamma_{wt}\log\eta_{wt} + \Gamma_{et}\log\eta_{et} dt}_{\text{workers vs. entrepreneurs}} + \underbrace{\int_0^\infty \Gamma_{et}\left(\int_0^1 \log\eta_{it} di - \log\eta_{et}\right) dt}_{\text{entrepreneur inequality}}. \end{aligned} \tag{31}$$

The first and second terms of equation (31) concern aggregate labor and TFP. The third term captures the distribution of consumption between workers and entrepreneurs at the aggregate level. The fourth term reflects that the planner dislikes inequality among entrepreneurs (it is weakly negative due to Jensen's inequality).

Second, the choice of aggregate labor can also be separated from the distribution of con-

[7] Write $c_{wt} = \eta_{wt} Z_t \ell_t$ and $c_{it} = \eta_{it} Z_t \ell_t$, apply logs, and use a law of large numbers to eliminate the expectation. Then add and subtract $\int_0^\infty \Gamma_{et}\log\eta_{et} dt$ and regroup terms.

sumption shares in the constraints. Any implementable allocation must satisfy the resource constraints (5) and (6), the Euler equations (10) and (11), and the first order conditions for exposure to risk (14) and labor demand (19). Given these constraints, once the planner chooses the initial distribution η_0, η_t will then follow its KFE and pin down $Z(\eta_t)$, $w(\eta_t)$ and $\sigma_c(z, \eta_t)$ according to equations (20)–(22).

We can therefore recast the planner's problem as choosing an initial distribution η_0 and a path for aggregate labor ℓ to maximize the objective in equation (31) subject to the KFE constraint given by equations (20)-(23). See Appendix D for details on how to construct an optimal allocation from the solution to this problem. A key property of this formulation is that the choices of ℓ and η_0 can be made independently, which is the key to analytical tractability.

3.2 The First-Best Allocation

In the first-best allocation the planner maximizes each term in the objective function. (1) Aggregate labor is set to maximize the first term. (2) Labor is employed only by the highest productivity types. (3) Consumption is split between entrepreneurs and workers according to Pareto weights. (4) There is perfect risk sharing and all entrepreneurs have the same consumption. The first-best allocation is

$$\ell_t^{FB} = \left(\frac{1}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}} \right)^{\frac{\psi}{\psi+1}}, \tag{32}$$

$$Z_t^{FB} = \bar{z}, \tag{33}$$

$$\eta_{et}^{FB} = \frac{\Gamma_{et}}{\Gamma_{wt}+\Gamma_{et}}, \ \eta_{wt}^{FB} = \frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}, \tag{34}$$

$$\eta_{it}^{FB} = \eta_{et}^{FB}. \tag{35}$$

The planner makes workers supply more labor when entrepreneurs' Pareto weight is larger.

Implementation of the First-Best Allocation. With complete markets the competitive equilibrium achieves the first best and the planner implements it with only redistribution at $t = 0$ and no taxes, $\tau_t = 0$.[8] Agents perfectly share risk, $\sigma_{cit} = 0$, so there are no risk markups, $w_t = \bar{z}$, and only the most productive entrepreneurs produce, $Z_t = \bar{z}$. Workers and entrepreneurs' follow their Euler equation without precautionary saving or risk, so $\eta_{it} = \eta_{et}$, and η_{et} follows

$$d\eta_{et} = \eta_{et}(1 - \eta_{et})(\rho_e - \rho_w)dt. \tag{36}$$

This yields $\eta_{et} = \frac{e^{-\rho_e t}\eta_{e0}}{e^{-\rho_w t}\eta_{w0} + e^{-\rho_e t}\eta_{e0}}$. If the planner chooses the initial distribution $\eta_{e0} = \frac{\Gamma_{e0}}{\Gamma_{e0}+\Gamma_{w0}}$, we obtain $\eta_{et} = \frac{\Gamma_{et}}{\Gamma_{et}+\Gamma_{wt}}$. Aggregate labor would then be given by equation (11), which given w_t, Z_t, and η_{wt}, yields the first-best path for aggregate labor.

3.3 The Constrained Optimal Allocation

Since ℓ_t only appears in the labor term, the optimal aggregate labor choice in the constrained planner problem coincides with the first-best, $\ell_t = \ell_t^{FB}$. That is, incomplete markets do not prevent the planner from obtaining the first-best aggregate labor.

In contrast, due to market incompleteness and the limited instruments that the planner has, the choice of the initial wealth distribution η_0 is distorted relative to the first best. The planner takes into account that the initial distribution of wealth η_0 affects both the path of misallocation, Z_t, and risk sharing, σ_{cit}. The choice of η_0 determines the path of η_t according to the KFE in equation (23), and thus the other terms η_{wt}, Z_t, and σ_{cit}^2.[9]

Let's understand how the initial distribution of wealth can affect misallocation and risk sharing. First, TFP is a labor-weighted average of idiosyncratic productivity, and labor hired is increasing in an entrepreneur's wealth. Thus, the planner can increase TFP by allocating

[8] This is an application of the first and second welfare theorems.

[9] If the stationary distribution is independent of initial conditions, then the choice of η_0 has no effect on these terms in the long run. Thus, the choice η_0 is purely about the transition path.

more wealth to higher-productivity entrepreneurs. To the extent that the wealth distribution is somewhat persistent, an initial redistribution can raise TFP along the transition.

Second, entrepreneurs must be exposed to risk if they are to employ labor, and this exposure to risk is higher the lower their consumption share. This is costly for two reasons: (i) entrepreneurs utility is penalized because they are risk averse, and (ii) the path of consumption shares of workers and entrepreneurs departs from the first-best due to entrepreneurs' precautionary saving, i.e., $\eta_{et} \neq \eta_{et}^{FB}$.

This will lead to backloaded consumption for entrepreneurs, in an analogous way to how precautionary saving leads to excessive capital accumulation in an Aiyagari model with incomplete markets. The difference is that here we don't have capital that entrepreneurs can accumulate. Instead, they excessively accumulate claims on the representative worker who is not exposed to uninsurable risk and therefore does not have a precautionary motive.

A Special Case with Only One Productivity Type. To see the second effect of how risk distorts the distribution of consumption more clearly, consider the special case with only one productivity type, $z_{it} = \bar{z}$. In this case, misallocation is not an issue, $Z_t = \bar{z}$. Furthermore, Appendix D shows that the last two terms of the objective function can be rewritten as

$$\underbrace{\int_0^\infty \Gamma_{wt} \log \eta_{wt} + \Gamma_{et} \log \eta_{et} dt}_{\text{workers vs. entrepreneurs}} - \underbrace{\int_0^\infty \Gamma_{et} \frac{1}{\rho_e} \frac{1}{2} \sigma_{ct}^2 dt}_{\text{entrepreneur inequality}} . \tag{37}$$

With complete markets, the planner would choose $\eta_{e0} = \frac{\Gamma_{e0}}{\Gamma_{e0}+\Gamma_{w0}}$, and agents would maximize the two terms on their own without further intervention. In the constrained problem, this is not possible. Equilibrium risk exposure is given by

$$\sigma_c(\eta_{et}) = \frac{\rho_e \phi \sigma_y}{\bar{z} \eta_{et}}. \tag{38}$$

Since the planner uses the labor tax to keep aggregate employment fixed at the first best, a lower wealth share for entrepreneurs η_{et} requires them to take more risk, σ_{ct}. This is costly because entrepreneurs are risk averse (entrepreneur inequality term) and also because their precautionary saving distorts the distribution of consumption (workers-vs.-entrepreneurs term). Entrepreneurs' consumption share η_{et} follows the law of motion

$$d\eta_{et} = \eta_{et}(1-\eta_{et})(\rho_w - \rho_e + \sigma_{ct}^2)dt. \tag{39}$$

Comparing this expression with equation (36), we see that the precautionary saving term, σ_{ct}^2, backloads entrepreneurs' consumption share relative to the path that maximizes the workers-vs.-entrepreneurs term. The result is stark when we consider the long-run. In the first best $\eta_{et} \to 0$, while in the constrained optimal allocation

$$\eta_{et} \to \eta_e^* = \min\left\{\frac{1}{\sqrt{\rho_e - \rho_w}}\frac{\rho_e \phi \sigma_y}{\bar{z}} \ , \ 1\right\}. \tag{40}$$

Even in this case without misallocation, the planner deviates from $\eta_{e0} = \eta_{e0}^{FB}$, trading off the welfare loss from precautionary saving (first term) and exposure to risk (second term).

3.4 Optimal Taxes

While the path for aggregate labor is not distorted relative to the first best, its implementation requires a labor tax. The equilibrium labor supply condition with a labor tax in equation (28) can be expressed as

$$c_{wt}^{-1} \times \underbrace{Z(\eta_t) \times \frac{1}{1+\pi_t(\eta_t)}}_{w_t} \times (1-\tau_t) = \ell_t^{1/\psi}. \tag{41}$$

The planner's first order condition can be expressed as[10]

$$c_{wt}^{-1} \times Z(\eta_t) \times \frac{\eta_{wt}}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}} = \ell_t^{1/\psi}. \tag{42}$$

Comparison of the two equations directly reveals that the tax that supports the planner's allocation is

$$1 - \tau_t = (1 + \pi_t) \times \frac{\eta_{wt}}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}}. \tag{43}$$

Assume that in the long run as $t \to \infty$ the consumption distribution converges to a stationary distribution η^* and the corresponding $\eta_{wt} \to \eta_w^* \in (0, 1)$. Then $\pi(\eta_t)$ and $\eta_w(\eta_t)$ also converge to long-run values. The optimal tax converges to

$$1 - \tau^* = (1 + \pi^*) \times \eta_w^*. \tag{44}$$

We can write the optimal tax in the long run in terms of consumption shares as

$$1 - \tau^* = \frac{c_w^*}{w^* \ell^*}. \tag{45}$$

Entrepreneur wealth must be positive, $n_e^* = \frac{Z^* \ell^* \eta_e^*}{\rho_e} > 0$, so by market clearing $n_w^* < 0$, and thus the budget constraint of the worker implies $c_w^* < w^* \ell^*$. In the long run, workers are paying interest on their debt to entrepreneurs, so workers consume less than their labor income and the optimal labor tax is positive: $\tau^* > 0$.

We can further characterize the optimal path of the tax when $\eta_0 = \eta^*$. This is the tax the planner would choose if the economy started at steady state and he could not redistribute

[10] Isolate $\ell_t^{\frac{1}{\psi}}$ in equation (32), multiply and divide by $\eta_{wt} Z(\eta_t)$, and use $c_{wt} = \eta_{wt} Z(\eta_t) \ell_t$.

at time zero. Equation (43) simplifies to

$$1 - \tau_t = \frac{1 - \tau^*}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}}. \tag{46}$$

When the Planner does not value entrepreneurs welfare ($\Gamma_{e0} = 0$), we have $\tau_t = \tau^*$ for all t. When $\Gamma_{e0} > 0$, τ_t is increasing over time. This could be a subsidy changing to a tax or a small tax growing to be a larger tax. As long as the planner does not care too much about entrepreneurs, if $\Gamma_{e0} \leq \left(\frac{\tau^*}{1-\tau^*}\right)\Gamma_{w0}$, then it is optimal to impose a tax along the entire transition path, i.e., $\tau_t \geq 0$ for all time.

3.5 Understanding the Inefficiency

The fact that the constrained optimal allocation requires a tax on labor implies that the competitive equilibrium is inefficient. The optimal labor tax in equation (43) can be understood in terms of two elements: inefficient risk premium and inefficient precautionary saving. A key insight of this paper is that the single feature of uninsurable idiosyncratic risk generates both inefficient risk markups and inefficient precautionary saving. Furthermore, since the constrained efficient allocation is not the first-best allocation, the single policy instrument of a labor tax cannot simultaneously correct both inefficiencies. As a result, optimal labor policy when misallocation is caused by uninsurable risk is a tax, which is the opposite of an optimal subsidy when misallocation is caused by monopolistic competition (in the Baqaee and Farhi (2020) and Edmond, Midrigan, and Xu (2023) sense). The remainder of the paper clarifies this insight.

The first term on the right of equation (43) is the aggregate risk markup, $1 + \pi_t$. The optimal labor tax has a component that aims to exactly undo the average risk markup with a subsidy, as would be the optimal policy if the wedge was created by monopolistic competition. Here, however, the wedge represents a risk premium that captures a real utility cost of exposing entrepreneurs to more risk. So why is it inefficient? The answer is that

there is an externality associated with risk sharing. To see this, we can write entrepreneur's exposure to risk in the following way:

$$\sigma_{cit} = (\rho_e \phi \sigma_y) \frac{\ell_{it}}{c_{it}} = (\rho_e \phi \sigma_y) \left(\frac{1}{\eta_{it} Z_t(\eta_t)} \right) \frac{\ell_{it}}{\ell_t}. \tag{47}$$

If an entrepreneur increases his labor demand ℓ_{it} individually, he is taking more risk. This is why in the competitive equilibrium entrepreneurs demand a risk premium as compensation. But the planner can implement a deviation that raises all entrepreneurs' ℓ_{it} at the same time and in the same proportion, so that the numerator ℓ_{it} and the denominator ℓ_t both increase proportionally and each entrepreneur's exposure to risk is unchanged. In terms of equilibrium objects, $\sigma_{cit} = (\phi \sigma_y) \frac{\ell_{it}}{n_{it}}$. An individual entrepreneur takes n_{it} as given when choosing ℓ_{it}. If the planner raises aggregate output c_t at time t by raising ℓ_t, then $n_{it} = c_{it}/\rho_e$ raises proportionally; this must occur because $\eta_{it} = c_{it}/c_t$ is predetermined. Since ℓ_{it} is linear in n_{it}, it also raises in proportion. Thus, σ_{cit} is unchanged. In the background, the interest rate increases, according to the Euler equations. Entrepreneurs enter period t with higher n_{it} because the past rate of return on their savings (the interest rate) increases with this deviation. So the risk premium is correctly capturing the private cost of exposing entrepreneurs to risk at the margin, but not the correct social cost.

The second term in equation (43) is

$$\frac{\eta_{wt}}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}} \tag{48}$$

and captures the cumulative effects of inefficient precautionary saving. The numerator is the share of consumption that goes to the representative worker and the denominator is their relative Pareto weight. In the absence of incomplete risk sharing (or in the first best), this ratio would always be one. With incomplete risk sharing, this is not the case due to entrepreneurs' precautionary saving, as shown above.

But even if entrepreneurs' share of consumption is too high, why does this show up as an

inefficiency in the aggregate level of labor? The reason is that, from the perspective of the planner, if he increases labor and output at time t, the share of the extra output that goes to entrepreneurs above their Pareto weights is wasted. So he acts as if the marginal product of labor was not Z_t but rather Z_t times this ratio—as can be seen clearly in the first order condition in equation (42). An equally valid interpretation is that precautionary saving of entrepreneurs makes workers poorer than optimal, so an income effect makes them supply too much labor, and thus a labor tax is needed.

3.5.1 A Tax on Entrepreneurs' Savings

To understand the inefficiency of precautionary saving, consider what the planner would do if he could also tax entrepreneurs' savings. To highlight the role of risk sharing, we return to the simplified environment with only one productivity type, $z_{it} = \bar{z}$. This eliminates the planner's concern for misallocation. The objective function, omitting the independent labor term, reduces to

$$\underbrace{\int_0^\infty \Gamma_{wt} \log \eta_{wt} + \Gamma_{et} \log \eta_{et} dt}_{\text{workers vs. entrepreneurs}} - \underbrace{\int_0^\infty \Gamma_{et} \frac{1}{\rho_e} \frac{1}{2} \sigma_{ct}^2 dt}_{\text{risk penalty}}. \tag{49}$$

Risk exposure is given by

$$\sigma_c(\eta_{et}) = \frac{\rho_e \phi \sigma_y}{\bar{z} \eta_{et}}, \tag{50}$$

and the law of motion of η_{et} is

$$d\eta_{et} = \eta_{et}(1 - \eta_{et})(\rho_w - \rho_e + \sigma_{ct}^2 - \tau_{st})dt. \tag{51}$$

The only difference with equation (36) is the savings tax τ_{st}.

With this new instrument, entrepreneurs' consumption share becomes a control instead of a state variable. The workers-vs.-entrepreneurs term is maximized when consumption shares

equal Pareto weights, but the planner chooses to give entrepreneurs more consumption to reduce their exposure to risk, and thus the risk penalty. The FOC for η_{et} is

$$\Gamma_{et}\frac{1}{\eta_{et}} - \Gamma_{wt}\frac{1}{1-\eta_{et}} + \Gamma_{et}\frac{1}{\rho_e}\sigma_c^2(\eta_{et})\frac{1}{\eta_{et}} = 0.$$

Re-arranging,[11]

$$\eta_{et} = \frac{\Gamma_{et}}{\Gamma_{wt}+\Gamma_{et}} \times \left(1 + (1-\eta_{et})\frac{1}{\rho_e}\sigma_c^2(\eta_{et})\right). \tag{52}$$

From this we deduce that entrepreneurs' consumption share η_{et} falls over time, but is always above their Pareto weight. In the long-run, it converges to zero, $\eta_{et} \to 0$. In contrast, without the tax on savings, $\eta_{et} \to \eta_e^*$ from equation (40).

How is this allocation implemented? The optimal labor tax is still given by equation (43). To implement the optimal path for η_{et}, the saving tax must equate equation (51) with the law of motion of equation (52). After some algebra, we obtain

$$\tau_{st} = \sigma_{ct}^2 + (\rho_w - \rho_e)\left(1 - \frac{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}}{1 + \frac{\Gamma_{et}}{\Gamma_{wt}+\Gamma_{et}}\frac{1}{\eta_{et}}\frac{1}{\rho_e}\sigma_c^2(\eta_{et})\,(2-\eta_{et})}\right). \tag{53}$$

The tax on savings first eliminates precautionary saving, σ_{ct}^2, but then nonetheless distorts consumption paths relative to the first best to account for the effect of η_{et} on risk sharing.

In the long-run entrepreneurs' consumption share η_{et} converges to zero, so their exposure to risk $\sigma_c(\eta_{et})$ goes to infinity, and so do risk markups, $1+\pi_t$, and precautionary saving,

[11] This is a cubic equation for η_{et}, so it has a closed-form expression, but it's not very nice, and this expression is more illuminating.

$\sigma_c^2(\eta_{et})$. In the limit, optimal taxes satisfy[12]

$$\lim_{t\to\infty} \frac{1-\tau_t}{1+\pi_t} = \lim_{t\to\infty} \frac{\eta_{wt}}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}} = 1, \tag{54}$$

$$\lim_{t\to\infty} \frac{\tau_{st}}{\sigma_{ct}^2} = 1. \tag{55}$$

The tax on labor converges to a subsidy (negative tax) to eliminate the aggregate risk markup, while the tax on savings converges to a tax to eliminate the precautionary saving motive. This is in contrast to the case without the tax on savings, where the tax on labor converges to a tax. The difference is precisely the behavior of η_{et}, which encodes the cumulative effect of precautionary saving.

Another special case worth highlighting is the limit as $\Gamma_{e0} \to 0$. From equation (52), we obtain $\eta_{et} \to 0$ for all t. The taxes that implement the optimal allocation satisfy, for all t,

$$\frac{1-\tau_t}{1+\pi_t} = \frac{\eta_{wt}}{\frac{\Gamma_{wt}}{\Gamma_{wt}+\Gamma_{et}}} \to 1, \tag{56}$$

$$\frac{\tau_{st}}{\sigma_{ct}^2} \to 1. \tag{57}$$

The planner uses the labor tax to exactly undo the markup and the tax on savings to exactly undo the precautionary saving, not just in the long run, but along the whole path.

In summary, without a tax on saving, the optimal labor tax deviates from $\tau_t = -\pi_t$ due to the cumulative effects of entrepreneurs' inefficient precautionary saving. If the planner has access to a tax on entrepreneurs' savings, he uses it to eliminate the precautionary motive, but still distorts the path of consumption shares relative to the first best to account for the effect of η_{et} on entrepreneurs' exposure to risk. In the long run (or when he doesn't care about entrepreneurs, $\Gamma_{e0} \to 0$), the planner uses a negative tax (i.e., a subsidy) on labor to exactly offset the risk markup and a tax on saving to exactly offset precautionary saving.

[12]For the saving tax, use equation (52) to replace $\frac{\frac{\Gamma_{et}}{\Gamma_{wt}+\Gamma_{et}}}{\eta_{et}}$ in equation (53), divide by σ_{ct}^2 throughout, and take limits.

3.6 Numerical Examples: Planner Problem

To illustrate key characteristics of the planner's allocation and implementation, we study the effects of different policies in both the long run and on the transition path, using a roughly chosen set of baseline parameter values. See Appendix F for details.

3.6.1 The Long Run

Labor Tax: Long Run				
	Labor tax τ	$\frac{c^\star - c}{c}$	$\frac{\ell^\star - \ell}{\ell}$	Welfare gain
Optimal labor tax	0.09	-0.04	-0.04	+0.22%
Labor subsidy: $\tau = -\pi$	-0.16	+0.07	+0.07	-1.31%

Table 1: The table reports the labor tax, and the change in consumption, labor, and consumption-equivalent welfare induced by the tax. The first row is for the optimal long-run tax. The second row is for a labor subsidy equal to the average markup.

We first analyze how the planner's allocation differs from the zero-tax competitive equilibrium in the long run. In our numerical example, the long-run optimal labor tax is $\tau^* = 1 - (1 + \pi^*)\eta_w^* = 9\%$. The first row of Table 1 compares the long-run difference in consumption, labor, and welfare in planner's allocation to the competitive equilibrium. The labor tax reduces consumption and labor by 4% and increases welfare. According to the planner's objective, the welfare gain from the optimal tax is equivalent to increasing consumption in the zero-tax competitive equilibrium by 0.22% for each person and point in time.

To contrast our findings with those typical of the market-power markup literature, the second row of Table 1 compares the competitive equilibrium to the allocation achieved by a labor subsidy equal to the average markup (i.e., $\tau = -\pi^*$). In the long run, the labor subsidy increases consumption and labor by 7% and reduces welfare by 1.31%.

3.6.2 Transition Dynamics

Results along the transition path depend on the values of the Pareto weights Γ_w and Γ_e. For illustrative purposes, we set $\Gamma_e = 0.04$ and $\Gamma_w = 0.96$, motivated by evidence that approximately 4% of US households are entrepreneurs (Salgado, 2020). The transition differs from the long run because the planner can choose the initial distribution of consumption shares and because the discounted Pareto weights change over time. We first isolate the effect of the time-varying discounted Pareto weights and then study the effect of initial redistribution.

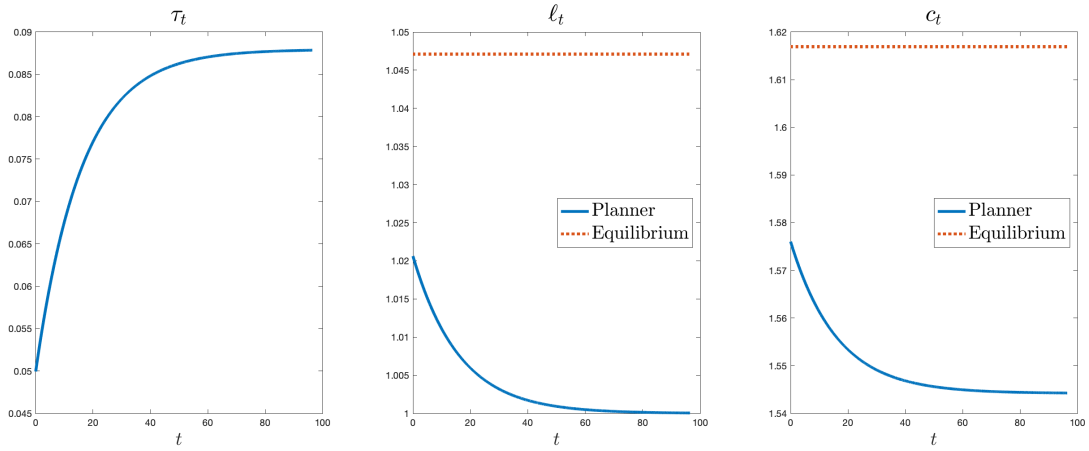


Figure 2: Optimal labor tax τ_t, aggregate labor ℓ_t, and consumption c_t that solve the planner problem. At $t = 0$, the economy is at the steady state of the decentralized equilibrium. From $t = 0$ onward the optimal tax τ_t solves the planner problem.

Fixed Initial Distribution. Figure 2 plots the optimal labor tax, aggregate labor, and consumption that solve the planner problem starting from the steady state of the competitive equilibrium in $t = 0$. Since the distribution of consumption shares and aggregate productivity are identical to the competitive equilibrium, changes in welfare must come from changes in the labor term of the transformed planner's objective:

$$\int_0^\infty (\Gamma_{wt} + \Gamma_{et}) \log \ell_t - \Gamma_{wt} \frac{\ell_t^{1+1/\psi}}{1 + 1/\psi} dt. \tag{58}$$

Labor Tax: Transition Dynamics			
	$\tau_t = 1 - (1+\pi_t)(\eta_{wt})$	$\tau_t = \tau^*$	$\tau_t = -\pi_t$
Welfare gain	0.16%	0.15%	-1.18%

Table 2: This table reports the labor tax and the associated change in consumption-equivalent welfare compared to zero taxes. The first column implements the optimal dynamic tax. The second column implements a constant tax at the optimal long-run tax rate. The third column implements a dynamic labor subsidy equal to the average markup.

The key difference from the long run is that the planner values entrepreneurs' consumption on the transition. This implies that the optimal labor tax at $t = 0$ is less than its long run value, temporarily increasing aggregate labor and consumption. The first column of Table 2 reports a 0.16% welfare gain from the optimal dynamic labor tax τ_t. The second column of shows that the welfare gain from implementing a constant labor tax equal to the long-run optimal rate is 0.15%, very close to that from implementing the optimal dynamic tax. Finally, the third column shows that implementing a dynamic labor subsidy equal to the average markup leads to a 1.18% consumption equivalent welfare loss. Thus, in this simple exercise, long-run results are a good guide to optimal policy on the transition and mistakenly implementing the optimal policy associated with market-power markups is very costly.

Initial Redistribution. We now study a planner who uses labor taxes and can redistribute at time zero. For simplicity, we study a planner who starts with the stationary distribution of wealth and can redistribute between entrepreneurs and workers by changing the consumption share of all entrepreneurs by the same proportion.

Figure 3 plots the transition dynamics of this economy in comparison to the economy in which the planner cannot initially redistribute. In the long run, entrepreneurs have a consumption share above 20% while their discounted Pareto weight converges to zero. Even though along the transition Γ_{et} is not zero, it is still well-below η_e^*, and so the planner wants to initially redistribute consumption to the workers. In this example, the planner sets $\eta_{e0} = 0.037$, close to the first-best value of $\Gamma_e = 0.04$. The planner always sets labor to its first best level. After the initial redistribution, workers are relatively rich, and the income

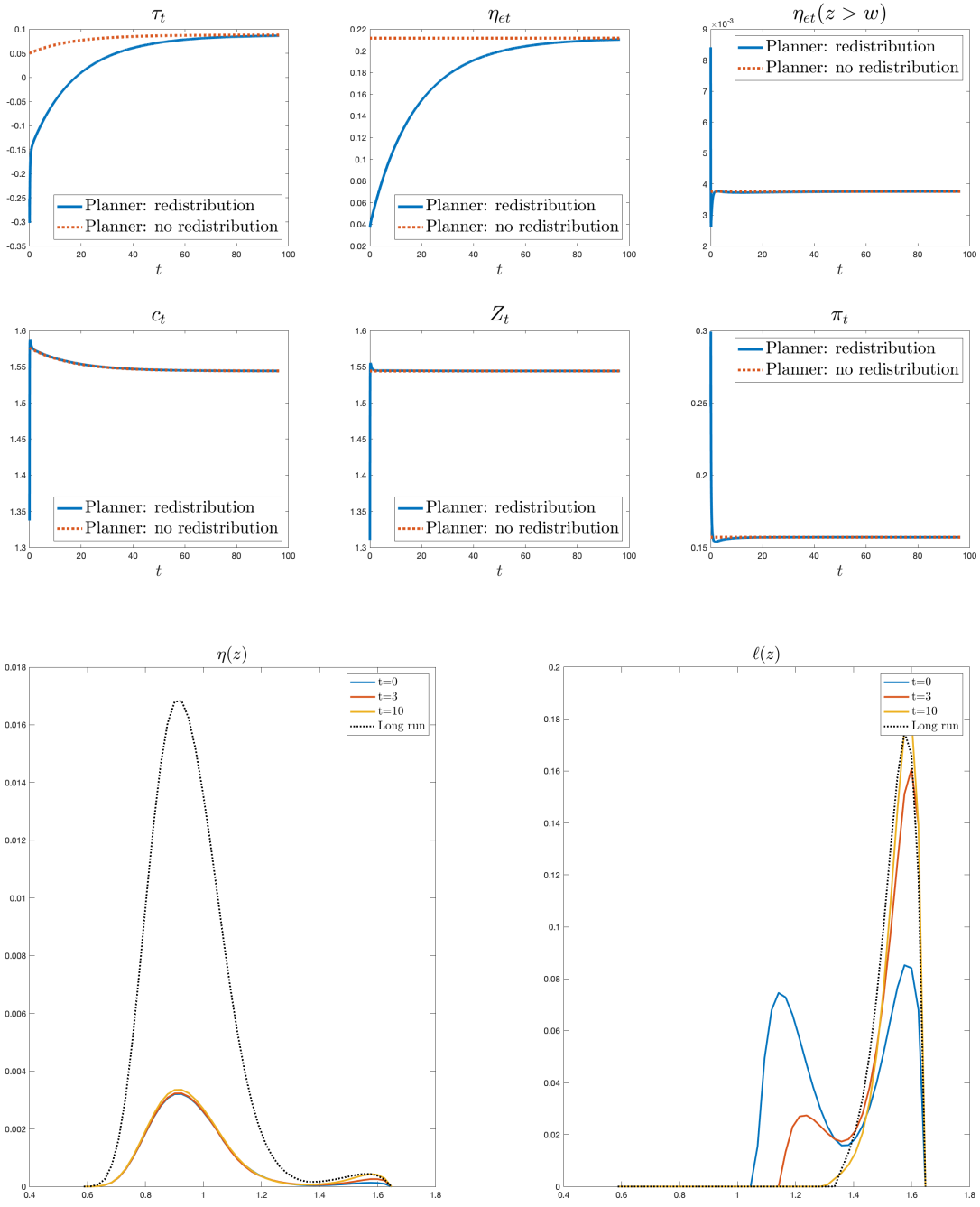


Figure 3: We plot the optimal labor tax τ_t, aggregate labor ℓ_t, consumption c_t, entrepreneurs' consumption share η_{et}, TFP Z_t, and aggregate markup π_t in the planner's allocation. The planner chooses the labor tax and how to redistribute wealth between entrepreneurs and workers at $t = 0$, starting from the steady state of the competitive equilibrium. We also plot the distribution of consumption shares among entrepreneurs $\eta(z)$ and the total labor hired by type-z entrepreneurs $\ell(z)$ at different times along the transition.

effect makes them want to work less, so, the planner needs to subsidize labor in the short run. The lower level of net worth for entrepreneurs makes it riskier for them to hire these workers, so they demand larger risk premia, causing markups to rise and TFP to fall. As the entrepreneurs accumulate wealth over time, the labor subsidy becomes a tax along the transition.

TFP and the aggregate markup converge quickly to their long-run values, even though the consumption share of entrepreneurs is much slower to converge. We also plot the distribution of consumption shares among entrepreneurs η_{zt} and the total labor hired by type-z entrepreneurs $\ell_t(z)$ at different times along the transition. The key insight is that TFP and risk premia are driven by the wealth shares of entrepreneurs who have high-enough productivity to produce, while the majority of entrepreneurs wealth is held by lower-productivity individuals. Figure 3 shows that the right tail of the wealth distribution converges quickly, as high-productivity entrepreneurs earn large profits from large risk premia. This rapid accumulation of wealth in the right tail of the distribution increases the self-insurance of high-productivity entrepreneurs and leads them to hire more workers, raising the wage and pushing lower-productivity entrepreneurs to exit. In this numerical example, the majority of entrepreneurs are rentiers, low productivity and not actively producing, living off the profits of days past and waiting to become profitable again in the future.

Table 3 reports that the planner achieves a consumption-equivalent welfare gain of 5.5% and decomposes this gain into the four components of the planner's objective. In line with the separation result, the welfare gain from the labor component is identical to the case in which the planner cannot redistribute. However, the overall welfare gain is larger when the planner can redistribute, almost entirely due to improvements in the worker-vs.-entrepreneur consumption share component. There is also a decrease in welfare from the increased inequality among entrepreneurs caused by the increase in risk.

Labor Tax and Redistribution: Transition Dynamics				
	Welfare Gain Decomposition			
Total	Labor	TFP	Workers vs. Entrepreneurs	Entrepreneurs inequality
+5.50%	+0.16%	+ 0.01%	+6.79%	- 1.37%

Table 3: The table reports changes in welfare between the planner solution and the competitive equilibrium. We consider the case in which the planner can redistribute wealth between entrepreneurs and workers at $t = 0$ and implement the optimal labor tax or subsidy τ_t for all t. The first column reports the consumption-equivalent welfare gain. The other four columns decompose changes in welfare according to components of the transformed planner's problem, reflecting changes in labor, aggregate TFP, inequality between workers and entrepreneurs, and inequality within entrepreneurs.

4 Conclusion

This paper develops a theoretical framework to study how uninsurable idiosyncratic risk affects markups, misallocation, and efficiency. Hiring labor is like investing in a risky asset, so entrepreneurs trade off the expected profits against the uninsurable risk and demand a risk premium. This risk premium creates labor wedges at the entrepreneur level that look like measured markups, since the price of the good is above the marginal cost of labor. However, the risk-adjusted marginal product is equalized for all active entrepreneurs. Thus, there is no true markup in the sense that the risk adjusted marginal cost is equal to the price of the good for all active entrepreneurs. At the aggregate level, however, risk markups show up as a labor wedge and generates misallocation that reduces TFP.

We show how a planner who cannot complete markets, but can use taxes, subsidies, and transfers, can improve welfare. While the constrained planner can achieve the first-best level of labor, the distribution of consumption between entrepreneurs and workers and among entrepreneurs is distorted. The optimal labor tax can be understood in terms of an inefficient risk premium that pushes for a labor subsidy (negative tax) and an inefficient precautionary saving motive that pushes for a labor tax. In the long-run, the precautionary inefficiency dominates and the optimal policy is a tax, which is the opposite of the optimal subsidy typically found in the literature with alternative microfoundations for measured markups.

According to our simple numerical exercise, there is a modest welfare gain from imple-

menting the optimal dynamic labor tax, almost the same gain from implenting a constant labor tax at the long-run optimal level, large welfare losses from implemnting a labor subsidy equal to the aggregate markup, and a large welfare gain from redistributing from entreprenuers to workers. Since policy recommendations differ qualitatively, and there is a suggestion of large quantitative differences, further research to identify the relative importance of uninsurable risk and monopoly power in generating misallocation is highly valuable.

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A Entrepreneur Optimality

The entrepreneurs' problem is to chose adapted processes $(c_i > 0, \ell_i \geq 0)$ to maximize utility

$$U(c_i) = \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \log c_{it} dt\right] \tag{A.1}$$

subject to the dynamic budget constraint

$$dn_{it} = (r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t))dt + \ell_{it}\sigma_y \phi dB_{yit} \tag{A.2}$$

with initial bonds n_{i0} given and the natural borrowing limit $n_{it} \geq 0$.

Any plan (c_i, ℓ_i) satisfying the dynamic budget constraint satisfies the intertemporal budget constraint

$$\mathbb{E}\left[\int_0^\infty \xi_{it} c_{it} dt\right] \leq n_{i0}, \tag{A.3}$$

where ξ_{it} is the solution to

$$d\xi_{it}/\xi_{it} = -r_t dt - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} dB_{yit}, \tag{A.4}$$

with $\xi_{i0} = 1$. To see this, use Itô's lemma to compute

$$\begin{aligned} d(\xi_{it} n_{it}) &= \xi_{it}(r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t) - r_t n_{it} - \frac{(z_{it} - w_t)^+}{\phi\sigma_y}\ell_{it}\phi\sigma_y)dt + \xi_{it}\left(\ell_{it}\phi\sigma_y - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} n_{it}\right) dB_{yt}, \\ &= \xi_{it}(-c_{it} + \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right))dt + \xi_{it}\left(\ell_{it}\phi\sigma_y - \frac{(z_{it} - w_t)^+}{\phi\sigma_y} n_{it}\right) dB_{yt}. \end{aligned}$$

Integrate and take expectations to obtain

$$\mathbb{E}\left[\xi_{i\tau_j} n_{i\tau_j}\right] = \xi_{i0} n_{i0} + \mathbb{E}\left[\int_0^{\tau_j} \xi_{it}(-c_{it} + \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right))dt\right],$$

where $\{\tau_j\}$ is an increasing sequence of stopping times with $\lim_{j\to\infty} \tau_j = \infty$ a.s., such that

$\int_0^{\tau_j} \xi_{it} \left(\ell_{it} \phi \sigma_y - \frac{(z_{it}-w_t)^+}{\phi \sigma_y} n_{it} \right) dB_{yt}$ is a martingale. Reorganizing we get

$$\mathbb{E}\left[\int_0^{\tau_j} \xi_{it} (c_{it} - \ell_{it} \left((z_{it} - w_t) - (z_{it} - w_t)^+ \right)) dt \right] + \mathbb{E}\left[\xi_{i\tau_j} n_{i\tau_j} \right] = \xi_{i0} n_{i0}.$$

Notice that $\xi_{it}(c_{it} - \ell_{it}\left((z_{it} - w_t) - (z_{it} - w_t)^+\right)) \geq \xi_{it} c_{it} \geq 0$. Take limits as $j \to \infty$ using the monotone convergence theorem and recall $\xi_{it} n_{it} \geq 0$ for any plan that satisfies the natural borrowing constraint. We obtain

$$\mathbb{E}\left[\int_0^{\infty} \xi_{it} c_{it} dt \right] \leq n_{i0}.$$

We will therefore consider a relaxed problem: choose consumption c_i to maximize utility $U(c_i)$ subject to the intertemporal budget constraint (A.3) and then later check that the proposed solution satisfies the dynamic budget constraint. We have a first-order condition

$$e^{-\rho_e t} c_{it}^{-1} = \lambda \xi_{it}, \tag{A.5}$$

where the value of $\lambda > 0$ ensures the constraint holds with equality:

$$\begin{aligned} \mathbb{E}\left[\int_0^{\infty} \xi_{it} c_{it} dt \right] = \mathbb{E}\left[\int_0^{\infty} e^{-\rho_e t} \lambda^{-1} dt \right] = \frac{1}{\rho_e \lambda} = n_{i0}, \\ \implies \lambda = \frac{1}{\rho_e} \frac{1}{n_{i0}}. \end{aligned}$$

These conditions are sufficient for optimality of the relaxed problem. To see this, consider any alternative plan $\tilde{c}_i$ satisfying the intertemporal budget constraint. Use the gradient

inequality for concave functions to note that $\log \tilde{c}_{it} \leq \log c_{it} + c_{it}^{-1}(\tilde{c}_{it} - c_{it})$. Then,

$$\begin{aligned} U(\tilde{c}_i) &= \mathbb{E}\left[\int_0^\infty e^{-\rho_e t} \log \tilde{c}_{it}\ dt\right] \\ &\leq \mathbb{E}\left[\int_0^\infty e^{-\rho_e t}\left(\log c_{it} + c_{it}^{-1}(\tilde{c}_{it} - c_{it}\right) dt\right] \\ &= U(c_i) + \mathbb{E}\left[\int_0^\infty \lambda \xi_{it}(\tilde{c}_{it} - c_{it}) dt\right] \leq U(c_i). \end{aligned}$$

The final inequality holds because $\mathbb{E}\left[\int_0^\infty \xi_{it}\tilde{c}_{it}\right] \leq n_{i0}$ since $\tilde{c}$ satisfies the intertemporal budget constraint in equation (A.3) while $\mathbb{E}\left[\int_0^\infty \xi_{it} c_{it}\right] = n_{i0}$ by construction.

We now want to derive the stochastic processes that these sufficient conditions imply. From the first-order condition (A.5), we obtain that $c_{it} > 0$ is an Itô process,

$$dc_{it}/c_{it} = \mu_{cit} dt + \sigma_{cit} dB_{yit}.$$

Using Itô's lemma we obtain

$$d(e^{-\rho_e t} c_{it}^{-1}) = d(\lambda \xi_{it}) \tag{A.6}$$

$$e^{-\rho_e t} c_{it}^{-1}\left(-\rho_e - \mu_{cit} + \sigma_{cit}^2\right) dt - e^{-\rho_e t} c_{it}^{-1} \sigma_{cit} dB_{yit} = -\lambda \xi_{it} r_t dt - \lambda \xi_{it} \frac{(z_{it} - w_t)^+}{\phi \sigma_y} dB_{yit}. \tag{A.7}$$

Matching terms we get the Euler equation and the first-order condition for labor,

$$\mu_{cit} = r_t - \rho_e + \sigma_{cit}^2, \tag{A.8}$$

$$\sigma_{cit} = \frac{(z_{it} - w_t)^+}{\phi \sigma_y}. \tag{A.9}$$

And any plan $c_i > 0$ satisfying these conditions and $c_{i0} = \lambda^{-1} = \rho_e n_{i0}$ satisfies FOC (A.5) and the intertemporal budget constraint with equality and is therefore optimal in the relaxed problem.

It only remains to show that our candidate plan c_i can indeed be implemented as part of

a plan (c_i, ℓ_i) that satisfies the dynamic budget constraint. We set

$$\ell_{it} = \frac{\sigma_{cit} c_{it}}{\phi \sigma_y \rho_e} \geq 0, \tag{A.10}$$

We now define the process for bonds $n_{it} = c_{it}/\rho_e > 0$ which satisfies the initial condition by construction. Using the optimality conditions we compute

$$\begin{aligned} dn_{it}/n_{it} &= \left(r_t - \rho_e + \sigma_{cit}^2\right) dt + \sigma_{cit} dB_{yit}, \\ &= \left(r_t - c_{it}/n_{it} + \frac{\ell_{it}}{n_{it}} \phi \sigma_y \frac{(z_{it} - w_t)^+}{\phi \sigma_y}\right) dt + \frac{\ell_{it}}{n_{it}} \sigma_y \phi dB_{yit}, \\ &= \left(r_t - c_{it}/n_{it} + \frac{\ell_{it}}{n_{it}} (z_{it} - w_t)\right) dt + \frac{\ell_{it}}{n_{it}} \phi \sigma_y dB_{yit}, \end{aligned}$$

where the last step uses that $\ell_{it} = 0$ if $z_{it} < w_t$ by construction in equation (A.10).

Rearranging we obtain the dynamic budget constraint,

$$dn_{it} = (r_t n_{it} - c_{it} + \ell_{it}(z_{it} - w_t))dt + \ell_{it} \phi \sigma_y dB_{yit}.$$

Lemma 1. *Assume adapted processes $(c_i > 0, \ell_i \geq 0)$ satisfy* (A.8)-(A.10) *with $c_{i0} = \rho_e n_{i0}$. Then the plan (c_i, ℓ_i) and the associated process $n_{it} = c_{it}/\rho_e$ satisfies the dynamic budget constraint subject to initial bonds n_{i0} and the natural borrowing limit and achieves the maximum utility of the relaxed intertemporal problem. It is therefore an optimal solution of the entrepreneur's problem.*

B Worker Optimality

The worker's problem is to choose a non-negative processes (c_w, ℓ) to maximize utility $U_w(c_w, \ell)$ subject to the dynamic budget constraint

$$dn_{wt} = (n_{wt} r_t - c_{wt} + w_t \ell_t)dt,$$

with initial bonds n_{w0} given and natural borrowing limit $n_{wt} \geq -\int_t^\infty e^{-\int_t^s r_u du} w_s \ell_s ds$.

This is a standard problem. First, for any plan (c_w, ℓ) satisfying the dynamic budget constraint and natural borrowing limit we satisfy the intertemporal budget constraint

$$\int_0^\infty e^{-\int_0^t r_u du}(c_{wt} - w_t \ell_t) dt \leq n_{w0}. \tag{B.1}$$

To see this compute

$$d(e^{-\int_0^t r_u du} n_{wt}) = e^{-\int_0^t r_u du}\left(-n_{wt} r_t + n_{wt} r_t - c_{wt} + w_t \ell_t\right) dt.$$

Integrating we obtain

$$\int_t^T e^{-\int_t^s r_u du}(c_{ws} - w_s \ell_s) ds + e^{-\int_t^T r_u du} n_{wT} = n_{w0}.$$

Use the natural borrowing limit to obtain

$$\int_t^T e^{-\int_t^s r_u du} c_{ws} ds - \int_t^\infty e^{-\int_t^\infty r_u du} w_s \ell_s ds \leq n_{w0},$$

and taking the limit $T \to \infty$ we obtain the intertemporal budget constraint. We can therefore consider the relaxed problem of maximizing utility subject to the intertemporal budget constraint.

The first-order conditions of the relaxed problem are

$$e^{-\rho_w t} c_{wt}^{-1} = \lambda e^{-\int_0^t r_u du} \tag{B.2}$$

$$e^{-\rho_w t} \ell_t^{1/\psi} = \lambda e^{-\int_0^t r_u du} w_t \iff \ell_t^{1/\psi} = c_{wt}^{-1} w_t, \tag{B.3}$$

and λ is set so the intertemporal budget constraint (B.1) holds with equality. These conditions are also sufficient for optimality.

From (B.2), taking time-derivatives, we obtain the Euler equation

$$dc_{wt}/c_{wt} = (r_t - \rho_w)dt. \tag{B.4}$$

A plan (c_w, ℓ) satisfying the first-order conditions (B.4) and (B.3) and the intertemporal budget constraint with equality, achieves the optimum of the relaxed problem.

It remains to show this plan can be implemented with the dynamic budget constraint. Set

$$n_{wt} = \int_t^\infty e^{-\int_t^s r_u du}(c_{ws} - w_s\ell_s)ds, \tag{B.5}$$

which satisfies the initial condition by construction. Taking derivatives with respect to time, we obtain the dynamic budget constraint

$$dn_{wt} = (n_{wt}r_t - c_{wt} + w_t\ell_t)dt.$$

Since $c_w \geq 0$, the natural borrowing limit is automatically satisfied.

Lemma 2. *Assume adapted processes* $(c_w > 0, \ell \geq 0)$ *satisfy* (B.4) *and* (B.3) *and the intertemporal budget constraint with equality.Then the plan* (c_w, ℓ) *and the associated process* n_w *given by* (B.5) *satisfies the dynamic budget constraint subject to initial bonds* n_{w0} *and the natural borrowing limit and achieves the maximum utility of the relaxed intertemporal problem. It is therefore an optimal solution of the worker's problem.*

C Proof of Proposition 1

We want to show that the prices and allocations $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ constructed as described in Section 2.3 constitute a competitive equilibrium. To do so we need to show that allocations and prices are in their permissible domain, that markets clear, and that agents are solving their problem taking prices as given.

C.1 Market Clearing

First, since $w_t, c_{it}, c_{wt} > 0$, we have that $\ell_{it} = \sigma_c(z_{it}, \eta_t)\frac{c_{it}}{\rho_e \phi \sigma_y} \geq 0$ and $\ell_t = c_{wt}^{-1} w_t > 0$. Furthermore, by construction $\int_0^1 n_{it} di + n_{wt} = 0$.

To show that markets clear we can use that the KFE is derived from the Euler equations of entrepreneurs and the worker, yielding that

$$\frac{\int_{\{z_{it}=z\}} c_{it} di}{c_{wt}} = \frac{\eta_t(z)}{\eta_{wt}}.$$

Integrating across z and using $\eta_{et} = 1 - \eta_{wt}$ and $c_{wt} = (1 - \eta_{et}) Z_t \ell_t$, we obtain

$$\int_0^1 c_{it} di = \eta_{et} \frac{c_{wt}}{1 - \eta_{et}} = \eta_{et} Z_t \ell_t,$$

and therefore

$$\int_0^1 c_{it} di + c_{wt} = Z_t \ell_t.$$

Equations (20) and (21) then ensure that market clearing (equations 5 and 6) hold.

C.2 Optimality

It remains to show that the allocations (c_i, ℓ_i) and (c_w, ℓ) satisfy the dynamic budget constraints, natural borrowing constraints, and are optimal given prices (r, w).

Since we constructed the entrepreneurs' allocations (c_i, ℓ_i) to satisfy the first-order conditions and $c_{i0} = \rho_e n_{i0}$, Lemma 1 in Appendix A ensures they satisfy budget constraints and are optimal.

The allocation satisfies the first order conditions for the worker. So given Lemma 2 in Appendix B, we only need to show that it satisfies the worker's intertemporal budget

constraint with equality. We defined $n_{wt} = -\int_0^1 n_{it} di$. We therefore have that

$$\begin{aligned}
e^{-\int_t^T r_u du} n_{wT} &= -\int_0^1 e^{-\int_t^T r_u du} n_{it} di, \\
&= \int_0^1 \left(-n_{it} + \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds - \int_t^T e^{-\int_t^s r_u du} \ell_{is} \phi \sigma_y dB_{iys} \right) di, \\
&= \int_0^1 -n_{it} di + \int_0^1 \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds di - \int_0^1 \int_t^T e^{-\int_t^s r_u du} \ell_{is} \phi \sigma_y dB_{iys} di \\
&= \left(n_{wt} + \int_0^1 \int_t^T e^{-\int_t^s r_u du} (c_{is} - (z_{is} - w_s)\ell_{is}) ds di \right).
\end{aligned}$$

where the last term is eliminated imposing a law of large numbers.[13] We can switch the order of integration using Tonelli to obtain,

$$e^{-\int_t^T r_u du} n_{wT} = n_{wt} + \int_t^T e^{-\int_t^s r_u du} \left(\int_0^1 c_{is} di - \int_0^1 z_{is} \ell_{is} + w_s \int_0^1 \ell_{is} di \right) ds.$$

Use the resource constraints,

$$e^{-\int_t^T r_u du} n_{wT} = n_{wt} + \int_t^T e^{-\int_t^s r_u du} (-c_{ws} + w_s \ell_s) ds.$$

Use that $e^{-\int_t^T r_u du} n_{wT} = \left(\frac{e^{-\rho_w t} c_{wt}^{-1}}{c_{w0}^{-1}} \right)^{-1} e^{-\int_0^T r_u du} n_{wT}$ and take the limit $T \to \infty$. Since $\lim_{T\to\infty} e^{-\int_0^T r_u du} n_{wT} = 0$, then

$$\int_t^\infty e^{-\int_t^s r_u du} (c_{ws} - w_s \ell_s) ds = n_{wt}, \tag{C.1}$$

as desired.

C.3 Conclusion

Thus, initial bonds n_{w0} and n_{i0} and allocations and prices $\mathcal{C} = (c_w, \ell, c_i, \ell_i, r, w)$ that satisfy the proposition assumptions and are constructed by the detailed procedure are a *Competitive*

[13] This is implied by the LLN that we imposed when writing the goods resource constraint in equation (5) because $e^{-\int_t^s r_u du}$ as a function of s is deterministic and continuous on $[0, T]$.

Equilibrium.

D Constructing the Optimal Allocation

Starting with η_0 and a path for ℓ from the Planner optimization problem, we can construct the allocation following the same steps we used to construct the competitive equilibrium using $w(\eta_t)$ and $Z(\eta_t)$. Now, however, we do not need to use the worker's first-order condition for labor supply to determine labor supply ℓ_t. Instead, we know ℓ_t and we need to find the labor tax τ_t and the initial bond distribution, n_{w0} and n_{i0}, that implement the allocation.

As in the competitive equilibrium, starting from η_0, solve the KFE forward to obtain the path of η_t, $w(\eta_t)$, and $Z(\eta_t)$. Then set the labor tax according to equation (28). We then set the representative worker's consumption to $c_{wt} = (1 - \eta_{et})Z_t\ell_t$, and use his Euler equation (10) to determine the interest rate. Set $c_{i0} = \eta_{i0}c_0$ and use entrepreneurs' Euler equation (12) and $\sigma_{cit} = \sigma_c(z_{it}, \eta_t)$ to pin down their consumption c_{it}, and set $\ell_{it} = \sigma_c(z_{it}, \eta_t)\frac{c_{it}}{\rho_e \phi \sigma_y}$. The tax rebate is $T_t = \tau_t w_t \ell_t$ and the bonds are $n_{it} = c_{it}/\rho_e$ and $n_{wt} = -\int_0^1 n_{it} di$.

Proposition 2. *Let $\mathcal{S} = (c_w, \ell, c_i, \ell_i)$ be an allocation built as described above, with associated prices (r, w), taxes (τ, T) and initial bonds (n_{i0}, n_{w0}). Assume that τ is feasible, that $w_t(1 - \tau_t), c_{it}, c_{wt} > 0$ for all t and $\lim_{T\to\infty} e^{-\int_0^T r_u du} n_{wT} = 0$. Then $\mathcal{S}$ is an optimal allocation.*

Proof. The allocation maximizes the social planner's transformed objective (31). To show it is an optimal allocation we only need to show that it is implementable. An argument analogous to the one in Proposition 1 shows that $(c_w, \ell, c_i, \ell_i, r, w, \tau, T, n_w, n_i)$ is a competitive equilibrium with tax and lump-sum transfers, so $\mathcal{S}$ is implementable. Since $w_t(1-\tau_t), c_{it}, c_{wt} > 0$, we have $\ell_{it} \geq 0$ and $\ell_t > 0$, and the first order conditions for the workers and entrepreneurs are satisfied and $n_{wt} + \int_0^1 n_{it} di = 0$ by construction. The same argument as in the proof of Proposition 1 shows that the resource constraints hold and that the dynamic budget constraint of entrepreneurs hold with the natural borrowing limit and the intertemporal budget constraints hold with equality, which imply that the plan (c_i, ℓ_i) is optimal for entrepreneur

i. Since $T_t = \tau_t w_t \ell_t$, the budget constraint of the worker is the same as in the case without taxes, so the same argument as in the proof of Proposition 1 shows that the workers dynamic budget constraint and natural borrowing limit hold, and the intertemporal budget constraint holds with equality, which implies the plan (c_w, ℓ) is optimal for the worker. We conclude that $(c_w, \ell, c_i, \ell_i, r, w, \tau, T, n_w, n_i)$ is a competitive equilibrium with tax and lump-sum transfers, so $\mathcal{S}$ is implementable. □

D.1 Transforming the Planner's Objective

We want to write the last term of the objective function in a way that is more useful for numerical computation,

$$\int_0^\infty \Gamma_{et} \left(\int_0^1 \log \eta_{it} di - \log \eta_{et} \right) dt.$$

Use the Euler equations to write

$$\begin{aligned} c_{wt} &= c_{w0} \exp \left(\int_0^t r - \rho_w dt \right), \\ c_{it} &= c_{i0} \exp \left(\int_0^t r - \rho_e + \sigma_{cit}^2 - \frac{1}{2}\sigma_{cit}^2 dt + \int_0^t \sigma_{cit} dB_{yit} \right). \end{aligned}$$

Putting them together, we can express the consumption of entrepreneurs in terms of workers' consumption,

$$c_{it} = c_{i0} \times \frac{c_{wt}}{c_{w0}} \times \exp \left(\int_0^t \rho_w - \rho_e + \frac{1}{2}\sigma_{cit}^2 dt + \int_0^t \sigma_{cit} dB_{yit} \right).$$

Divide and multiply by c_0 on the rights-hand-side, and then divide by c_t on both sides,

$$\frac{c_{it}}{c_t} = \frac{c_{i0}}{c_0} \times \frac{c_{wt}}{c_t} \times \frac{c_0}{c_{w0}} \times \exp \left(\int_0^t \rho_w - \rho_e + \frac{1}{2}\sigma_{cit}^2 dt + \int_0^t \sigma_{cit} dB_{yit} \right).$$

Now take logs,

$$\log \eta_{it} = \log \eta_{wt} + (\log \eta_{i0} - \log \eta_{w0}) + \int_0^t \rho_w - \rho_e + \frac{1}{2}\sigma_{cit}^2 dt + \int_0^t \sigma_{cit} dB_{yit}.$$

Integrate over i, and use a LLN and the fact that σ_{cit} is bounded because τ is feasible,

$$\int_0^1 \log \eta_{it} di = \log \eta_{wt} + \left(\int_0^1 \log \eta_{i0} di - \log \eta_{w0} \right) + \int_0^t \left(\rho_w - \rho_e + \int_0^1 \frac{1}{2}\sigma_{cit}^2 di \right) dt.$$

Now subtract $\log \eta_{et}$, multiply by Γ_{et}, and integrate over time,

$$\int_0^\infty \Gamma_{et} \left(\int_0^1 \log \eta_{it} di - \log \eta_{et} \right) dt =$$
$$\int_0^\infty \Gamma_{et} \left(\log \eta_{wt} - \log \eta_{et} + \left(\int_0^1 \log \eta_{i0} di - \log \eta_{w0} \right) + \int_0^t \left(\rho_w - \rho_e + \int_0^1 \frac{1}{2}\sigma_{cis}^2 di \right) ds \right) dt.$$

Finally, change the order of integration for the last term

$$\int_0^\infty \Gamma_{et} \int_0^t \left(\rho_w - \rho_e + \int_0^1 \frac{1}{2}\sigma_{cis}^2 di \right) ds dt = \int_0^\infty \Gamma_{et} \frac{1}{\rho_e} \left(\rho_w - \rho_e + \frac{1}{2} \int_0^1 \sigma_{cit}^2 di \right) dt.$$

We obtain the final expression,

$$\int_0^\infty \Gamma_{et} \left(\int_0^1 \log \eta_{it} di - \log \eta_{et} \right) dt = \tag{D.1}$$
$$\int_0^\infty \Gamma_{et} \left(\log \eta_{wt} - \log \eta_{et} + \left(\int_0^1 \log \eta_{i0} di - \log \eta_{w0} \right) + \frac{1}{\rho_e} \left(\rho_w - \rho_e + \frac{1}{2} \int_0^1 \sigma_{cit}^2 di \right) \right) dt.$$

Special Case of $z_{it} = \bar{z}$. In the special case with only one type, $z_{it} = \bar{z}$, the expression simplifies. Start with

$$c_{it} = c_{i0} \exp \left(\int_0^t r - \rho_e + \sigma_{ct}^2 - \frac{1}{2}\sigma_{ct}^2 dt + \int_0^t \sigma_{ct} dB_{yit} \right).$$

Integrate across i using the LLN and σ_{ct} bounded, to obtain

$$\int_0^1 c_{it}di = \int_0^1 c_{i0}di \times \exp\left(\int_0^t r - \rho_e + \sigma_{ct}^2\right).$$

Plug back into the expression for c_{it},

$$c_{it} = \frac{c_{i0}}{\int_0^1 c_{i0}di}\int_0^1 c_{it}di \exp\left(\int_0^t -\frac{1}{2}\sigma_{ct}^2 dt + \int_0^t \sigma_{ct}dB_{yit}\right).$$

Divide by c_t, and use $\eta_{it} = c_{it}/c_t$ and $\eta_{et} = \int_0^1 c_{it}di/c_t$,

$$\eta_{it} = \frac{c_{i0}}{\int_0^1 c_{i0}di}\eta_{et} \exp\left(\int_0^t -\frac{1}{2}\sigma_{ct}^2 dt + \int_0^t \sigma_{ct}dB_{yit}\right).$$

Given that we have equal weights for entrepreneurs and they all have the same productivity, we can without loss of generality take $c_{i0} = \int_0^1 c_{i0}di$. Take logs and integrate across i, again using a LLN to eliminate the last term,

$$\int_0^1 \log \eta_{it}di - \log \eta_{et} = \int_0^t -\frac{1}{2}\sigma_{ct}^2 dt.$$

Now multiply by Γ_{et} and integrate over t, and switch the order of integration to obtain

$$\int_0^\infty \Gamma_{et}\left(\int_0^1 \log \eta_{it}di - \log \eta_{et}\right)dt = -\int_0^\infty \Gamma_{et}\frac{1}{\rho_e}\frac{1}{2}\sigma_{ct}^2 dt. \tag{D.2}$$

E Derivation of KFE for $\eta_t(z)$

E.1 Derive process for η_{it}

Keep track of (z_i, η_i) where $\eta_{it} = c_{it}/c_t$. We need to compute the growth rate of aggregate consumption c_t:

$$c_{wt}\mu_{cwt} + \int c_{it}\mu_{cit}di = \mu_{ct}c_t$$

$$\eta_{wt}\mu_{cwt} + \int \eta_t(z)\mu_{ct}(z)dz = \mu_{ct}$$

(using the fact that growth rates depend only on z type).

Now use the Euler equations to rewrite this as

$$\eta_{wt}(r_t - \rho_w) + \int \eta_t(z)\left(r_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}\right)dz = \mu_{ct}$$

$$\left(\underbrace{\eta_{wt} + \int \eta_t(z)dz}_{=1}\right)r_t - \underbrace{\left(\eta_{wt}\rho_w + \rho_e\int \eta_t(z)dz\right)}_{=\bar{\rho}_t} + \int \eta_t(z)\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz = \mu_{ct}$$

$$\mu_{ct} = r_t - \bar{\rho}_t + \int \eta_t(z)\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz.$$

So now we can compute the law of motion of $\eta_{it} = c_{it}/c_t$:

$$d(c_{it}/c_t) = \left(\frac{c_{it}}{c_t}\right)(\mu_{cit} - \mu_{ct})\,dt + \left(\frac{c_{it}}{c_t}\right)\sigma_{cit}dB_{it}$$

$$d(c_{it}/c_t) = \left(\frac{c_{it}}{c_t}\right)\underbrace{\left(\bar{\rho}_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2} - \int \eta_t(z)\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz\right)}_{\mu_\eta(z,t)}dt$$

$$+ \left(\frac{c_{it}}{c_t}\right)\underbrace{\frac{(z_t - w_t)^+}{\phi\sigma_y}}_{\sigma_{ct}(z)}dB_{yt},$$

or, more compactly,

$$d\eta_{it} = \eta_{it}\mu_\eta(z,t)dt + \eta_{it}\sigma_{ct}(z)dB_{yt}.$$

E.2 Derive KFE

Let $D(\eta, z, t)$ be the joint distribution of η, z. $D(\eta, z, t)$ is the measure of entrepreneurs with $z_i = z$ and $\eta_i = \eta$. This distribution has a convenient mapping into $\eta(z,t)$ as

$$\eta(z,t) = \int D(\eta, z, t)\eta d\eta$$

Let $X = (z_i, \eta_i)$

$$dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_z(z) & 0 \\ 0 & \sigma_c(z,\eta_t)\eta \end{bmatrix} \begin{bmatrix} dB_{zt} \\ dB_{yt} \end{bmatrix}.$$

In order to derive a KFE for the joint distribution $D(\cdot,\cdot)$, we re-write dX_t as a function of a standard multivariate Wiener process with independent component. Consider two independent Wiener processes W_{1t}, W_{2t}:

$$\begin{aligned} B_{zt} &= W_{1t} \\ B_{yt} &= \lambda_{yz} W_{1t} + \sqrt{1-\lambda_{yz}^2} W_{2t}. \end{aligned}$$

Therefore, we have

$$dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_z(z) & 0 \\ \sigma_c(z,\eta_t)\eta\lambda_{yz} & \sigma_c(z,\eta_t)\eta\sqrt{1-\lambda_{yz}^2} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

To simplify the notation, we can write more generally

$$dX_t = \begin{bmatrix} \mu_z(z) \\ \mu_\eta(z,t)\eta \end{bmatrix} dt + \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix}.$$

We derive the KFE for $D(\eta, z, t)$, starting from the first order terms[14]

$$\begin{aligned} \partial D(\eta,z,t) &= -\frac{\partial}{\partial z}\left[\mu_z(z)D(\eta,z,t)\right] - \frac{\partial}{\partial \eta}\left[\mu_\eta(z,t)\eta D(\eta,z,t)\right] + \text{ 2nd order terms} \\ &= -\left[\mu_z'(z)D(\eta,z,t) + \mu_z(z)D_z'(\eta,z,t)\right] \\ &\quad -\left[\mu_\eta(z)D(\eta,z,t) + \mu_\eta(z)\eta D_\eta'(\eta,z,t)\right] + \text{ 2nd order terms.} \end{aligned}$$

[14]Since $d\eta_{it}/\eta_{it} = dc_{it}/c_{it}$, we have $d\eta_i = \mu_c\eta_i dt + \sigma_c\eta_i dB_t$.

Now, we define the second order terms. The variance covariance matrix is

$$\Lambda = \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix} \begin{bmatrix} \sigma_{z1}(z) & \sigma_{z2}(z) \\ \sigma_{c1}(z,t)\eta & \sigma_{c2}(z,t)\eta \end{bmatrix}'$$

$$\Lambda = \begin{bmatrix} \sigma_{z1}^2(z) + \sigma_{z2}^2(z) & \eta\sigma_{z1}(z)\sigma_{c1}(z) + \eta\sigma_{z2}(z)\sigma_{c2}(z) \\ \eta\sigma_{z1}(z)\sigma_{c1}(z) + \eta\sigma_{z2}(z)\sigma_{c2}(z) & \sigma_{c1}^2(z)\eta^2 + \sigma_{c2}^2(z)\eta^2 \end{bmatrix},$$

$$\text{2nd order terms} = \frac{1}{2}\frac{\partial^2}{\partial z^2}[\Lambda_{11}D(\eta,z,t)] + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}[\Lambda_{22}D(\eta,z,t)] + \frac{\partial^2}{\partial \eta z}[\Lambda_{12}D(\eta,z,t)].$$

Further evaluating the matrix by elements, the first term is:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial z^2}[\Lambda_{11}D(\eta,z,t)] &= \frac{1}{2}\frac{\partial^2}{\partial z^2}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D(\eta,z,t)] \\
&= \frac{1}{2}\frac{\partial}{\partial z}[2(\sigma_{z1}(z)\sigma'_{z1}(z) + \sigma_{z2}(z)\sigma'_{z2}(z))D(\eta,z,t) + (\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D_z(\eta,z,t)] \\
&= \frac{\partial}{\partial z}[(\sigma_{z1}(z)\sigma'_{z1}(z) + \sigma_{z2}(z)\sigma'_{z2}(z))D(\eta,z,t)] \\
&\quad + \frac{1}{2}\frac{\partial}{\partial z}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D_z(\eta,z,t)] \\
&= [\left(\sum_j \sigma'_{zj}(z)^2 + \sigma_{zj}(z)\sigma''_{zj}(z)\right)D(\eta,z,t)] + [\left(\sum_j \sigma_{zj}(z)\sigma'_{zj}(z)\right)D_z(\eta,z,t)] \\
&\quad + [\left(\sum_j \sigma_{zj}(z)\sigma'_{zj}(z)\right)D_z(\eta,z,t)] + \frac{1}{2}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D_{zz}(\eta,z,t)] \\
&= [\left(\sum_j \sigma'_{zj}(z)^2 + \sigma_{zj}(z)\sigma''_{zj}(z)\right)D(\eta,z,t)] + 2[\left(\sum_j \sigma_{zj}(z)\sigma'_{zj}(z)\right)D_z(\eta,z,t)] \\
&\quad + \frac{1}{2}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D_{zz}(\eta,z,t)].
\end{aligned}$$

The second term is:

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial\eta^2}[\Lambda_{22}D(\eta,z,t)] =& \frac{1}{2}\frac{\partial^2}{\partial\eta^2}[(\sigma_{c1}^2(z)\eta^2+\sigma_{c2}^2(z)\eta^2)D(\eta,z,t)] \\
=& (\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\frac{1}{2}\frac{\partial^2}{\partial\eta^2}[\eta^2 D(\eta,z,t)] \\
=& (\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\frac{1}{2}\frac{\partial}{\partial\eta}[2\eta D(\eta,z,t)+\eta^2 D_\eta(\eta,z,t)] \\
=& (\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[\frac{\partial}{\partial\eta}[\eta D(\eta,z,t)]+\frac{1}{2}\frac{\partial}{\partial\eta}[\eta^2 D_\eta(\eta,z,t)]\right] \\
=& (\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[D(\eta,z,t)+\eta D_\eta(\eta,z,t)]+\eta D_\eta(\eta,z,t)+\frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right] \\
=& (\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[D(\eta,z,t)+2\eta D_\eta(\eta,z,t)]+\frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right].
\end{aligned}$$

The third term is:

$$\begin{aligned}
\frac{\partial^2}{\partial\eta\partial z}[\Lambda_{12}D(\eta,z,t)] =& \frac{\partial^2}{\partial\eta\partial z}[[\eta\sigma_{z1}(z)\sigma_{c1}(z)+\eta\sigma_{z2}(z)\sigma_{c2}(z)]D(\eta,z,t)] \\
=& \frac{\partial^2}{\partial\eta\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))\eta D(\eta,z,t)] \\
=& \frac{\partial}{\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))D(\eta,z,t)]+\frac{\partial}{\partial z}[\sum_{j=1}^{2}(\sigma_{zj}(z)\sigma_{cj}(z))\eta D_\eta(\eta,z,t)] \\
=& \sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]D(\eta,z,t)+\sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z)D_z(\eta,z,t)] \\
&+\sum_{j=1}^{2}\left[\sigma'_{zj}(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma'_{cj}(z)\right]\eta D_\eta(\eta,z,t) \\
&+\sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z))\eta D_{\eta z}(\eta,z,t)].
\end{aligned}$$

Combining all first and second order terms yields

$$\begin{aligned}
\partial D(\eta,z,t) = &- \left[\mu_z'(z)D(\eta,z,t) + \mu_z(z)D_z'(\eta,z,t)\right] - \left[\mu_\eta(z)D(\eta,z,t) + \mu_\eta(z)\eta D_\eta'(\eta,z,t)\right] \\
&+ \left(\sum_j \sigma_{zj}'(z)^2 + \sigma_{zj}(z)\sigma_{zj}''(z)\right) D(\eta,z,t) + 2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right) D_z(\eta,z,t) \\
&\quad + \frac{1}{2}[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))D_{zz}(\eta,z,t)] \\
&+(\sigma_{c1}^2(z) + \sigma_{c2}^2(z))\left[[D(\eta,z,t) + 2\eta D_\eta(\eta,z,t)] + \frac{1}{2}[\eta^2 D_{\eta\eta}(\eta,z,t)]\right] \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right] D(\eta,z,t) + \sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z)D_z(\eta,z,t)] \\
&\quad + \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta D_\eta(\eta,z,t) + \sum_{j=1}^{2}[\sigma_{zj}(z)\sigma_{cj}(z))\eta D_{\eta z}(\eta,z,t)]\,.
\end{aligned}$$

To obtain $\eta(z,t)$, use that

$$\begin{aligned}
\eta(z,t) &= \int \eta D(\eta,z,t)d\eta \\
\partial_t\eta(z,t) &= \int \eta \partial D(\eta,z,t)d\eta \\
\eta_z(z,t) &= \int \eta D_z(\eta,z,t)d\eta \\
\eta_{zz}(z,t) &= \int \eta D_{zz}(\eta,z,t)d\eta.
\end{aligned}$$

Then,

$$\begin{aligned}\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right]-\left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)\int\eta^2D_\eta'(\eta,z,t)d\eta\right]\\&+\left(\sum_j\sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j\sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t)\\&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)]\\&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2\int\eta^2D_\eta(\eta,z,t)d\eta]+\frac{1}{2}[\int\eta^3D_{\eta\eta}(\eta,z,t)d\eta]\right]\\&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right]\\&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\int\eta^2D_\eta(\eta,z,t)d\eta+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z))\int\eta^2D_{\eta z}(\eta,z,t)d\eta\right].\end{aligned}$$

Conjecture that $\eta^2D(z,\eta,t)|_0^\infty=\lim_{\eta\to\infty}\eta^2D(z,\eta,t)-\lim_{\eta\to0}\eta^2D(z,\eta,t)=0-0=0$. Then, using integration by parts,

$$\int\eta^2D_\eta(z,\eta,t)d\eta=\eta^2D(z,\eta,t)|_0^\infty-2\int_0^\infty\eta D(z,\eta,t)d\eta=-2\eta(z,t).$$

Substitute this into the KFE to obtain

$$\begin{aligned}\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t)+\mu_z(z)\eta_z'(\eta,z,t)\right]-\left[\mu_\eta(z)\eta(z,t)+\mu_\eta(z)(-2\eta(z,t))\right]\\&+\left(\sum_j\sigma_{zj}'(z)^2+\sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t)+2\left(\sum_j\sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t)\\&+\frac{1}{2}[(\sigma_{z1}^2(z)+\sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)]\\&+(\sigma_{c1}^2(z)+\sigma_{c2}^2(z))\left[[\eta(z,t)+2(-2\eta(z,t))]+\frac{1}{2}[\int\eta^3D_{\eta\eta}(\eta,z,t)d\eta]\right]\\&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t)+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right]\\&+\sum_{j=1}^2\left[\sigma_{zj}'(z)\sigma_{cj}(z)+\sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z,t))+\sum_{j=1}^2\left[\sigma_{zj}(z)\sigma_{cj}(z))\int\eta^2D_{\eta z}(\eta,z,t)d\eta\right].\end{aligned}$$

Conjecture that $\eta^3 D_\eta(z,\eta,t)|_0^\infty = \lim_{\eta\to\infty}\eta^3 D_\eta(z,\eta,t) - \lim_{\eta\to 0}\eta^3 D_\eta(z,\eta,t) = 0 - 0 = 0$. Then, using integration by parts,

$$\int \eta^3 D_{\eta\eta}(\eta,z,t)d\eta = \eta^3 D_\eta(z,\eta,t)\big|_0^\infty - 3\int \eta^2 D_\eta(\eta,z,t)d\eta = 6\eta(z,t).$$

Again, substitute this into the KFE to obtain

$$\begin{aligned}
\partial\eta(z,t) =& -\left[\mu_z'(z)\eta(z,t) + \mu_z(z)\eta_z'(\eta,z,t)\right] - \left[\mu_\eta(z)\eta(z,t) + \mu_\eta(z)(-2\eta(z,t))\right] \\
&+ \left(\sum_j \sigma_{zj}'(z)^2 + \sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t) + 2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t) \\
&+ \frac{1}{2}\left[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)\right] \\
&+ (\sigma_{c1}^2(z) + \sigma_{c2}^2(z))\left[\left[\eta(z,t) + 2(-2\eta(z,t))\right] + \frac{1}{2}[6\eta(z,t)]\right] \\
&+ \sum_{j=1}^2 \left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t) + \sum_{j=1}^2 \left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right] \\
&+ \sum_{j=1}^2 \left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z,t)) + \sum_{j=1}^2 \left[\sigma_{zj}(z)\sigma_{cj}(z)\right)\int \eta^2 D_{\eta z}(\eta,z,t)d\eta\Bigg].
\end{aligned}$$

A similar argument can be applied to obtain

$$\int \eta^2 D_{\eta z}(\eta,z,t)d\eta = -2\eta_z'(z,t),$$

which implies that

$$\begin{aligned}
\partial\eta(z,t) = &- \left[\mu_z'(z)\eta(z,t) + \mu_z(z)\eta_z'(\eta,z,t)\right] - \left[\mu_\eta(z)\eta(z,t) + \mu_\eta(z)(-2\eta(z,t))\right] \\
&+ \left(\sum_j \sigma_{zj}'(z)^2 + \sigma_{zj}(z)\sigma_{zj}''(z)\right)\eta(z,t) + 2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right)\eta_z'(z,t) \\
&+ \frac{1}{2}\left[(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))\eta_{zz}''(\eta,z,t)\right] \\
&+ (\sigma_{c1}^2(z) + \sigma_{c2}^2(z))\left[\left[\eta(z,t) + 2(-2\eta(z,t))\right] + \frac{1}{2}\left[6\eta(z,t)\right]\right] \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right]\eta(z,t) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z)\eta_z'(z,t)\right] \\
&+ \sum_{j=1}^{2}\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right](-2\eta(z,t)) + \sum_{j=1}^{2}\left[\sigma_{zj}(z)\sigma_{cj}(z))(-2\eta_z'(z,t))\right].
\end{aligned}$$

Grouping terms yields

$$\begin{aligned}
\partial\eta(z,t) = &\ \eta(z,t)\left[-\mu_z'(z) + \mu_\eta(z) + \left(\sum_j \sigma_{zj}'(z)^2 + \sigma_{zj}(z)\sigma_{zj}''(z)\right) - \sum_j\left[\sigma_{zj}'(z)\sigma_{cj}(z) + \sigma_{zj}(z)\sigma_{cj}'(z)\right]\right] \\
&+ \eta_z'(z,t)\left[-\mu_z(z) + 2\left(\sum_j \sigma_{zj}(z)\sigma_{zj}'(z)\right) - \sum_j \sigma_{zj}(z)\sigma_{cj}(z)\right] \\
&+ \eta_{zz}''(\eta,z,t)\left[\frac{1}{2}(\sigma_{z1}^2(z) + \sigma_{z2}^2(z))\right].
\end{aligned}$$

In our model

$$\begin{aligned}
\sigma_{z1}(z) &= \sigma_z(z) \\
\sigma_{z2}(z) &= 0 \\
\sigma_{c1}(z) &= \sigma_c(z,\eta_t)\lambda_{yz} \\
\sigma_{c2}(z) &= \sigma_c(z,\eta_t)\sqrt{1-\lambda_{yz}^2}.
\end{aligned}$$

Therefore, the KFE is

$$\begin{aligned}\partial\eta(z,t) = \eta(z,t)\left[-\mu_z'(z) + \mu_\eta(z,t) + \sigma_z'(z)^2 + \sigma_z(z)\sigma_z''(z) - \left(\sigma_z'(z)\sigma_c(z,\eta_t) + \sigma_z(z)\sigma_c'(z,t)\right)\lambda_{yz}\right] \\ + \eta_z'(z,t)\left[-\mu_z(z) + 2\sigma_z(z)\sigma_z'(z) - \sigma_z(z)\sigma_c(z,\eta_t)\lambda_{yz}\right] \\ + \eta_{zz}''(\eta,z,t)\left[\frac{1}{2}\sigma_z^2(z)\right]\end{aligned}$$

where $\mu_z(z), \sigma_z(z), \lambda_{yz}$ are primitives, and $\sigma_c(z,\eta_t), \mu_\eta(z,t)$ are equilibrium objects given by:

$$\begin{aligned}\sigma_c(z,\eta_t) &= \frac{(z_t - w_t)^+}{\phi\sigma_y} \\ \mu_\eta(z,t) &= \bar{\rho}_t - \rho_e + \frac{(z_t - w_t)^2}{\phi^2\sigma_y^2} - \int \eta_{zt}\frac{(z_t - w_t)^2}{\phi^2\sigma_y^2}dz.\end{aligned}$$

F Numerical Example Details

Parameter values are reported in Table F.1. We set the workers discount factor ρ equal to 0.035, and entrepreneurs discount factor ρ^e equal to 0.097 as in Di Tella and Hall (2022).[15] We set the Frisch elasticity ψ equal to one, a conventional value in the literature that lies between micro and macro estimates Chetty, Guren, Manoli, and Weber (2011).

Finally, we specify parameters for the stochastic process driving fluctuations in firms' output and productivity. The law of motions of entrepreneurs' net worth and productivity are characterized by

$$\begin{aligned}dn_{it} &= \left(r_t n_{it} - c_{it} + (z_{it} - w_t)\,\ell_{it}\right)dt + (\phi\sigma_y)\,\ell_{it}dB_{yit} \\ dz_{it} &= \mu_z\left(z_{it}\right)dt + \sigma_z\left(z_{it}\right)dB_{zit}\end{aligned}$$

where B_{yit} and B_{zit} are two Brownian motions specific to the entrepreneur that have correlation $\lambda_{yz}, \sigma_y > 0$, and $\sigma_z\left(z_{it}\right) > 0$.

[15] These parameter values imply that in steady state the annual interest rate is 3.5% and the consumption share of entrepreneurs is 20%.

Parameter	Description	Value
ψ	Frisch Elasticity	1
ρ	Household's discount factor	0.035
ρ_e	Entrepreneurs' discount factor	0.097
θ	Speed of mean reversion of z_{it}	0.3
σ_y	Volatility of innovations to output, B_{yt}	0.1
$\sigma_{\log z}$	Volatility of innovations to $\log(z)$, B_{zt}	0.1
λ_{yz}	Correlation of B_{yt}, B_{zt}	1
ϕ	Exposure to idiosyncratic risk	1

Table F.1: Parameter values for the numerical illustration.

In order to solve the model numerically, we impose more structure on the stochastic process for firms' productivity. We assume that $\log z$ follows a standard Ornstein-Uhlenbeck process according to equation (F.1).

$$d\log z_{it} = \theta\left(\log \bar{z} - \log z_{it}\right)dt + \sigma_{\log z} dB_{zt} \tag{F.1}$$

Moreover, we set $\log(\bar{z}) = -\frac{1}{\theta}\frac{1}{2}\sigma^2_{\log z}$, so that the implied stochastic process for z takes a simple form:

$$\begin{aligned} dz_{it} &= z_{it}\left(-\theta \log\left(z_{it}\right) + \theta\log(\bar{z}) + \frac{1}{2}\sigma^2_{\log z}\right)dt + z_{it}\sigma_{\log z} dB_{zt} \\ &= -\theta z_{it}\log\left(z_{it}\right)dt + z_{it}\sigma_{\log z} dB_{zt} \end{aligned}$$

Therefore, the stochastic process driving fluctuations in firms' output and productivity is disciplined by a set of parameters $(\sigma_y, \sigma_{\log z}, \lambda_{yz}, \theta)$.

We set $\theta = 0.3$ and $\sigma_{logz} = 0.1$, meaning that the annual autocorrelation of firm productivity is equal to 0.7, and the standard deviation of productivity shocks is equal to 0.1 as in Khan and Thomas (2013); Bloom, Guvenen, and Salgado (2023). Finally, we assume that innovations to firms' productivity at time t proportionally affect output at time t, so that $\sigma_y = \sigma_{\log z}$ and $\lambda_{yz} = 1$. In the baseline calibration, we set the exposure to idiosyncratic risk $\phi = 1$.

F.1 Numerical algorithm

Competitive Equilibrium. We use the separation result to compute the competitive equilibrium in two steps. First, we use the KFE and the good market-clearing condition to solve for the distribution of consumption shares $\eta_t(z)$ and for the wage w_t. Then, it is easy to compute aggregate labor and consumption using the labor market-clearing condition and the first-order condition for labor supply.

In order to solve for the distribution of consumption shares, we construct a grid for productivity $\mathbf{z}$ that takes values between $\underline{z}$ and $\bar{z}$. We denote by η the vector obtained by evaluating $\eta(z)$ at $\mathbf{z}$. Let $\mathbf{D_f}$ be the forward operator and $\mathbf{D_b}$ be the backward operator, defined as:

$$\mathbf{D_f} = \begin{pmatrix} -1/dz & 1/dz & 0 & \dots & 0 \\ 0 & -1/dz & 1/dz & 0 & \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & & -1/dz & 1/dz \\ 0 & \dots & & 0 & -1 \end{pmatrix}$$

$$\mathbf{D_b} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1/dz & 1/dz & 0 & & \\ \vdots & -1/dz & 1/dz & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \dots & & -1/dz & 1/dz \end{pmatrix}$$

For first derivatives, we use the operator $\mathbf{D^1}$ defined as:

$$\mathbf{D^1} = \mathbf{1}(z > \bar{z})\mathbf{D_b} + \mathbf{1}(z < \bar{z})\mathbf{D_f}.$$

For second derivatives, we use the operator $\mathbf{D^2}$ defined as:

$$\mathbf{D^2} = \mathbf{D_f D_b}.$$

To handle the finite domain, we follow Moll (2014) to ensure that the support of the distribution is bounded. We assume two reflecting barriers at the bounds $\underline{z}$ and $\bar{z}$.

We initialize the algorithm with a value for the initial distribution of consumption shares $\boldsymbol{\eta_0}$. Then, at each step of the algorithm:

1. We use the good market clearing condition to find the wage that solves:

$$\left(\int_{w,}^{\infty} \frac{\eta_{zt}(z)}{\sigma_y(z)} z \frac{z - w_t}{\sigma_y(z)} dz\right) \frac{1}{\rho^e} = 1$$

2. We update the distribution $\boldsymbol{\eta_1}$ using the Runge-Kutta method

3. In order to minimize numerical errors when we iterate forward, we use our equation for $d\eta_e$ to normalize the distribution of consumption shares at each step, such that:

$$\boldsymbol{\eta_1}^{norm} = \boldsymbol{\eta_1} \times \frac{\sum \boldsymbol{\eta_0} + d\eta_e}{\sum \boldsymbol{\eta_1}}$$

4. We update the distribution by setting $\boldsymbol{\eta_0} = \boldsymbol{\eta_1}^{norm}$ and iterate to convergence

We simulate the algorithm until the economy converges to a steady state, that is when $\boldsymbol{\eta_1} \approx \boldsymbol{\eta_0}$.

Once we have solved for the entire path of the distribution of consumption shares and wages, we can easily compute aggregate labor, consumption, and productivity as

$$\begin{aligned} Z_t &= \left(\tfrac{1}{\rho^e} \int_{w_t}^{\bar{z}} \tfrac{z - w_t}{\sigma_y(z)} \tfrac{\eta_{zt}(z)}{\sigma_y(z)} dz\right)^{-1} && \text{(Aggregate TFP)} \\ \ell_t^{1/\psi} &= (\eta_{wt} Z_t \ell_t)^{-1} w_t && \text{(Aggregate labor)} \\ c_t &= \ell_t Z_t && \text{(Aggregate consumption)} \end{aligned}$$

Planner's Problem. The numerical algorithm that we use to solve the planner's problem is very similar to the one for the competitive equilibrium.

Once we have solved for the entire path of the distribution of consumption shares and wages, as in the competitive equilibrium, we can easily find the optimal labor tax as a function of consumption shares, markups, and Pareto weights. Then, given the optimal labor tax and consumption shares, we solve for aggregate labor and consumption.

In order to assess welfare gains, we need to evaluate the objective function of the social planner:

$$\int_0^\infty (\Gamma_{wt} + \Gamma_{et}) \log \ell_t - \Gamma_{wt} \frac{\ell_t^{1+1/\psi}}{1+1/\psi} dt + \int_0^\infty (\Gamma_{wt} + \Gamma_{et}) \log Z_t dt$$
$$+ \int_0^\infty \Gamma_{et} \left(\log \eta_{wt} - \log \eta_{et} + \left(\int_0^1 \log \eta_{i0} di - \log \eta_{w0} \right) + \frac{1}{\rho_e} \left(\rho_w - \rho_e + \frac{1}{2} \int_0^1 \sigma_{cit}^2 di \right) \right) dt \tag{F.2}$$

In principle, evaluating equation (F.2) poses some challenges, as it depends on the distribution of consumption shares across entrepreneurs at time $t = 0$ because of the term η_{i0}. This is challenging because one should know the distribution of net worth across entrepreneurs with the same level of productivity z. In practice, we always consider situations in which the planner cannot alter the initial distribution of wealth shares across entrepreneurs at $t = 0$. Therefore, the term η_{i0} will always be the same in the competitive equilibrium and under the planner's allocation, so that one does not need to evaluate η_{i0} to assess welfare gains.

Finally, once we evaluate changes in welfare $\Delta Welfare$ between the planner allocation and the competitive equilibrium, we can easily evaluate welfare gains in consumption equivalent terms by solving the following equation for x:

$$\Delta Welfare = \int_0^\infty \Gamma_{wt} \log(1+x) + \int_0^\infty \Gamma_{et} \log(1+x).$$