Testing for a Growth Slowdown Using a Model of Punctuated Technological Progress

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Abstract

Disappointing productivity gains in recent decades have raised concerns that the trend rate of TFP improvement may have declined, and this paper examines that possibility. As a framework for analysis, we construct a model with randomly-timed, punctuated technology advances (as from the arrivals of new general purpose technologies), technology-specific capital, and a household sector with 2 agent types. The model generates uneven growth, though there is a stationary long-run average rate. We show that changes in the latter are inherently difficult to establish in practice. We propose an indirect analytic approach, however, based upon duality relations. With the new method, we find little evidence of a recent slowdown — though looking further into the past, arguably there was a decline at the close of the 1920s.

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1. Introduction

Disappointing productivity gains in recent decades have raised concerns that the trend rate of TFP improvement may have declined (Ramey [2020, 2021], Gordon [2019], Jorgenson [2019], and many others), and this paper examines that possibility. As a framework for analysis, we construct an aggregative model with randomly-timed, punctuated technology advances. Conceptualizations of long-term growth often emphasize the importance of revolutionary inventions, and our specification features their role. The model shows that momentary growth rates may be inherently uneven, and it highlights practical difficulties for identifying shifts in their long-run average. Yet, it also enables us to propose an indirect procedure for detecting growth-trend changes — a method that interprets financial data as it relates to the equilibrium behavior of forward-looking market participants. In the end, our applications do not, however, find evidence, at least post 1950, of a secular slowdown in growth.

In the model, growth proceeds in episodes, each initiated by a discrete TFP improvement, which we call a "technology shock." Shock arrivals are Poisson events. The next shock truncates the current episode and starts another. Within an episode, the rate of progress may surge at first, but it asymptotically approaches zero. While the initial spurt will tend to be relatively short-lived, the asymptotic stage can cause lengthy periods of stagnation.

Regardless of short-run fluctuations, the model has a (stochastic) steady-state equilibrium with a constant mean rate of productivity growth. In principle, we could use direct observations of technology shocks to determine whether their underlying stochastic process' parameters may have shifted. That method will tend to be impractical, however, given the wide intervals between shock arrivals and resulting need for a long data set.

For results with a shorter sample, we propose the following indirect approach. Suppose agents within the economy have more information about the pipeline of possible new technologies than external analysts. Major inventions are disruptive. If agents learn, for instance, that future shocks will be weaker and/or less frequent, they may conclude that current investment risks have decreased (see below). Financial markets will then adjust. Since the adjustments can begin as soon as agents' beliefs shift, they can potentially provide a timely alert if the economy's long-run growth trajectory changes.

The idea of punctuated advances has long had adherents. Papers on general purpose technologies (GPTs), for instance, consider abrupt, large-scale technological advances (Helpman and Trajtenberg [1998], Howitt [1998]). Related work by economic historians describes growth as occurring in waves, with major inventions inaugurating periods of accelerated progress (Abramovitz and David [2014], Edquist and Henrekson [2007]).[1] Re-

[1] In general, the idea that seminal inventions are basic to long-run growth seems implicit

cent studies of individual technological revolutions include David [1990], Hobijn and Jovanovic [2001], Mokyr [1990], Jovanovic and Rousseau [2005], Laitner and Stolyarov [2003], Pastor and Veronesi [2009], and Brynjolfsson and McAfee [2011]. In the present paper, each GPT arrival initiates an era of progress, and the procession of such eras determines the economy's steady-state equilibrium.

We make 2 key additional assumptions. The first is that capital is technology specific. That allows punctuated TFP improvements to occur without causing unrealistic abrupt jumps in the GDP. We assume that as the potential advantages of a new technology become evident, the knowledge spreads rapidly. No firm has a particular advantage in adopting the new technology as knowledge is not privately owned.[2] With large technology shocks, the momentary rate of growth of the GDP can swing abruptly. Yet, effects on the level of the GDP are gradual, as they depend upon the economy's accumulation of new-vintage capital.

In contrast, financial-market reactions can be dramatic. As a new GPT arrives, investment in old-vintage capital will cease (recall Solow [1960]) and the market price of existing plant and equipment plummet (due to obsolescence). Given a GPT's wide scope, in fact, a bear market in equities can result.[3] The corresponding sequence of bull and bear markets over time plays a major role in our analysis.

Technology-specific capital may aid in model implementation. In our framework, a GPT's "arrival" corresponds to the time when the new technology's significance is broadly acknowledged. In practice, a seminal invention's main applications may follow its first appearance with a significant lag (David [1990]). While historians' narratives may identify the general time frame of a specific GPT's acceptance, our model may be able to determine the date more precisely from the accompanying bear market. (The bear's depth may help to establish the magnitude of the new technology's TFP contribution as well — see Sect. 1.)

Our second key additional assumption is limited participation (Kocherlakota [1996]). Specifically, we have 2 types of households. According to the Survey of Consumer Finances (SCF), the top 5% of wealth holders own roughly half the U.S. economy's net worth, including a sizable share of the nonresidential capital stock (Kennickell [2009]). We pattern our type-I agents on that group. They finance the business sector's capital.[4] Type-II agents compose the remainder of the population. For simplicity, we assume they are hand-to-mouth consumers.

For tractability, we use a representative agent formulation to describe the sector of type-I households. We think of type-I households as having high earnings and/or inheritances. In practice, they would be prime candidates to leave intentional bequests, and our

in many analyses. For example, Gordon [2012] judges the post-1970 slowdown by asking whether the microprocessor and Internet can measure up to great inventions of the past; Jorgenson [2019] emphasizes the singular role of the IT revolution in recent decades; and, news reports of potential breakthroughs in artificial intelligence, nuclear fusion, and genetic therapies attract a great deal of attention.

[2] For a different treatment, see Laitner and Stolyarov's [2003] use of intangible capital.

[3] Hobijn and Jovanovic [2001] provide direct evidence of this mechanism in the context of the IT revolution, and Albuquerque et al [2015] a roster of such episodes. See Sect. 6.

[4] For simplicity, we omit residential housing from our model.

formulation's infinite time horizon may approximate a dynastic attention span.

The 2-agent specification can capture a potentially significant relation between the distribution of earnings and wealth. We have a production sector with capital and labor inputs. Labor earnings that are imperfectly correlated with asset returns tend to raise an agent's tolerance for portfolio risk. Though top wealth owners in the SCF have above-average wages, their wealth is disproportionately higher still. The same pattern can hold in our analysis as type-II households can account for most of the economy's labor earnings.[5] For a given CRRA, the high relative importance of portfolio returns for our financial-market participants, that is to say, type-I households, can cause them to manifest more risk-averse investment behavior than otherwise.

Another important implication of limited participation pertains to consumption. In practice, aggregate consumption exhibits low volatility. In our model, however, the consumption response of portfolio holders to technology shocks is sizable. Our formulation allows these patterns to be mutually consistent: type-I households are few in number and save a high fraction of their income; consequently, their expenditure behavior need have only a limited effect on aggregate consumption.

Each technology shock simultaneously promotes future growth, causes immediate capital losses in financial markets, and creates exceptional opportunities for new investment. When a shock occurs, the latter 2 effects immediately induce type-I agents to rein in their consumption. Low overall portfolio returns (due to capital losses) thus coincide with a high marginal utility of consumption, fueling an equity premium. A variant of that premium is the basis of our proposed indirect method for detecting growth-trend shifts.

The organization of this paper is as follows. Sections 2-4 describe our model and method for calculating its equilibrium. Sections 5-6 present calibrations and model evaluations. Section 7 uses simulations to illustrate the natural variability of growth and the difficulties that may arise for detecting trend shifts. Section 8 presents our indirect approach for discerning long-run changes, and Section 9 concludes.

Relation to the literature. Our work attempts to bridge two strands of literature: macro-finance general equilibrium analyses where technological change is a source of aggregate risk, and frameworks of past or future growth slowdowns rooted in observations of real activity.

A number of current macroeconomic studies examine the possibility that the supply of promising new ideas may be running out (e.g., Fernald and Jones [2014], Bloom et al [2020]). Part of their argument is that the growth of expenditures on idea production observed during the last century may not be sustainable in the long run. We present a framework in which the absence of idea depletion is a maintained, but testable, hypothesis.

Statistical analyses show that labor productivity is indeed prone to cyclic swings, sometimes lasting for decades (e.g., Fritz, Gries, and Feng [2019], Pagano and Sbracia [2018], Antolin-Diaz, Drechsel, and Petrella [2017]).

[5] Our calibrated examples go further. Among the SCF's top wealth holders, some may be retired or living on inheritances, but others have very high earnings. Assuming an above-average share of the latter own their own business or have a corporate position with an annual bonus dependent upon the company's stock-market performance, we allow a portion of type-I agent earnings to be highly correlated with asset returns.

Recent structural approaches combine models and microeconomic data to examine possible links between growth and impediments to diffusion of information technologies (e.g., Acemoglu, Autor, and Patterson [2023] and Andrews, Criscuolo, and Gal [2016]). Our focus is the long run: rather than studying a particular GPT's attributes, our attention centers on a stochastic steady-state equilibrium in which one GPT displaces another in a sequence of episodes of creative destruction.

One strand of literature — starting with Rietz [1988] and followed by Barro [2006], Gourio [2008] and Gabaix [2012] — studies the financial market implications of major natural disasters and wars. Another, the real business cycle (RBC) literature (Jermann [1998]), emphasizes the role of business-cycle frequency technology shocks in driving risk premia. In contrast to the former, we focus on technology shocks; contrary to the latter, we assume a constant labor supply. Our shocks are more frequent than natural disasters but only about one-third as common as those in RBC models; they always enhance the economy's TFP frontier; the changes that they bring are permanent; and, their (immediate) impact falls upon asset prices rather than GDP, profits, or dividends.

Papers incorporating punctuated change and medium-length cycles include Comin and Gertler [2006] and Garleanu et al [2012], with the latter having vintage capital. Kung and Schmid [2015] formulate an endogenous growth model where persistent changes in productivity growth are echoes of past R&D activity amplified through a positive knowledge externality. Kogan and Papanikolaou [2014] combine high frequency shocks with technology-specific capital, investigating the cross-section of stock returns. Our goals are to contrast the unevenness of momentary growth with the stability of its long-run mean and to develop a methodology for studying the latter.

A number of quantitative macro-finance models incorporate an income distribution and risk-sharing. Guvenen [2009], Chien et al [2011], Danthine and Donaldson [2002], and Kogan, Papanikolaou, and Stoffman [2020] develop heterogeneous-agent models. Our 2-agent formulation is very basic in comparison, more closely resembling, for instance, Campbell and Mankiw [1989]. Our agents play complementary, but quite separate roles: while type-II households provide most of the labor supply and consumption spending, the heavy saving of type-I households finances the economy's capital stock. Our formulation allows portfolio holders to manifest substantial risk aversion without a very high CRRA. It applies an Euler-equation analysis only to the wealthiest segment of the population — households that, in practice, may be the most financially sophisticated and least liquidity constrained (see, for example, Mankiw and Zeldes [1991], Vissing-Jorgensen [2002], and Attanasio et al [2002]). Additionally, it generates a time path of aggregate consumption that differs substantially from that of the consumption of major financial-market participants (Ait-Sahalia, Parker, and Yogo [2004]).

The underlying simplicity of our model may facilitate future extensions. For instance, though the present paper includes only one shock type, we could easily incorporate multiple shock varieties — provided they are distributed independently of one another.

2. Production and Preferences

This section briefly describes our model's aggregate production function, technology-specific capital, and household utility functions.

Production. Technology shock arrivals are Poisson events. The index of the best available technology at time t is J_t, where the frontier TFP level is $[\theta]^{J_t}$, with $\theta > 1$ a parameter. Thus, subsequent technology shocks multiply the frontier TFP level by θ. The hazard rate for shocks is λ.

We have type-I and type-II households. Each household inelastically supplies 1 unit of labor at all times. The effectiveness of the labor of type-I and II households can differ (in practice, type-I agents have a higher wage rate, for example). The number of households of each type is fixed:

$$N_I = 1 \quad \text{and} \quad N_{II} = N\,. \tag{1}$$

We expect $N \gg 1$.

There is 1 output good at each t, its price is always 1, and output is homogeneously divisible into consumption and investment goods. Physical capital is technology specific. At t, $\mathcal{K}_{tj}$ is the quantity of capital specific to technology index j. The time-t production function for physical output $\mathcal{Y}_{jt}$ from labor and vintage $j \leq J_t$ capital is

$$\mathcal{Y}_{jt} = Z \cdot [\theta]^j \cdot [\mathcal{K}_{jt}]^{\alpha} \cdot [L_{I,j,t}]^{\beta\cdot(1-\alpha)} \cdot [L_{II,j,t}]^{(1-\beta)\cdot(1-\alpha)}\,, \quad \alpha\,,\,\beta \in (0,1)\,. \tag{2}$$

Investment is irreversible: once built, a unit of capital cannot be transformed to a different vintage or to consumption. Investment at time t theoretically can include any vintage $j \leq J_t$, with each unit of investment costing 1 dollar. The form of (2) shows, however, that higher TFP levels are unambiguously better. Thus, all new investment at t flows to capital of vintage J_t.

Physical capital depreciates at wear and tear rate $\delta \in (0,1)$. A second source of depreciation works as follows: when a technology shock occurs, the value of existing capital immediately declines through obsolescence. A key outcome is:

Prop. 1: *Let P_{jt} be the time-t price per physical unit of capital of vintage $j \leq J_t$. Then*

$$P_{jt} = [\theta]^{-(J_t - j)/\alpha}\,.$$

Proof: See Laitner and Stolyarov [2003].

In other words, as a new technology arrives, the value of each existing physical unit of capital drops to b times its value the moment before with

$$b \equiv [\theta]^{-1/\alpha}\,. \tag{3}$$

Using Proposition 1, the market value of the time-t capital stock is

$$K_t \equiv \sum_{j \leq J_t} [b]^{J_t - j} \cdot \mathcal{K}_{jt}\,. \tag{4}$$

Let I_t be the current flow of new investment. If there is no technology shock at t, we have

$$\dot{K}_t = I_t - \delta \cdot K_t \,. \tag{5}$$

If, however, a shock occurs at t, at that instant we have

$$J_t = J_{t-0} + 1 \text{ and } K_t = b \cdot K_{t-0} \,. \tag{6}$$

Let Y_t be aggregate output at t. Let L_{Ijt} be the labor allocation from type-I households to vintage j capital, and similarly for $L_{II,j,t}$. Given the labor supplies of (1), competitive behavior gives

$$Y_t = \max_{L_{Ijt}, L_{II,j,t}} \sum_{j \le J_t} \mathcal{Y}_{jt} \tag{7}$$

$$\text{subject to:} \quad \sum_{j \le J_t} L_{Ijt} = 1 \text{ and } \sum_{j \le J_t} L_{II,j,t} = N \,.$$

Fix N. Solow [1960] showed how to construct an aggregate production function $Y_t = F(K_t, J_t)$. We can generalize his methodology to encompass technological change in discrete steps. Let W_t be the wage per unit of type-I household labor and M_t be the marginal product of K_t net of wear and tear depreciation. Then

Prop. 2: *Set $J = J_t$ and, for convenience, set Z to satisfy*

$$Z \cdot [N]^{(1-\beta)\cdot(1-\alpha)} = 1 \,.$$

Let K_t be as in (4). Then

$$\begin{aligned} Y_t = F(K_t, J) \equiv [\theta]^J \cdot [K_t]^\alpha \,, \quad W(K_t, J) = \beta \cdot (1-\alpha) \cdot [\theta]^J \cdot [K_t]^\alpha \,, \\ M_t \equiv M(K_t, J) = \alpha \cdot [\theta]^J \cdot [K_t]^{\alpha-1} - \delta \,. \end{aligned}$$

Proof: See, for example, Laitner and Stolyarov [2003].

Figure 1 illustrates the time paths of K_t and Y_t. Given Prop. 2, if $J = J_t$, the per capita gross income of type-I households — as well as their sector's aggregate gross income — is

$$\eta \cdot [\theta]^J \cdot [K_t]^\alpha \,, \quad \eta \equiv \alpha + \beta \cdot (1-\alpha) \,. \tag{8}$$

Let R_t be the return per dollar's worth of capital, net of depreciation. Let λ be as above. For an investment of \$1 and a short passage of time Δ, we have

$$1 + R_t \cdot \Delta = \begin{cases} 1 + M(K_t, J_t) \cdot \Delta \,, & \text{with prob. } 1 - \lambda \cdot \Delta, \\ b + M(b \cdot K_t, J_t + 1) \cdot \Delta \,, & \text{with prob. } \lambda \cdot \Delta, \end{cases} \tag{9}$$

where a new GPT arrives in the second instance but not the first. At time t, the return on capital in the ensuing instant is a random variable. From (9), the latter's mean is[6]

$$\mathbb{E}_t[R_t] = M(K_t, J_t) - \lambda \cdot (1 - b)\,. \tag{10}$$

Agent Preferences. We assume type-I agents constitute a set of measure 1, take prices as given, and are all alike — behaving, in equilibrium, alike as well. Let C_t be each's time-t consumption flow. Each has isoelastic preferences, $u(C_t) = [C_t]^\gamma/\gamma$, $\gamma < 1$, with subjective discount rate $\rho > 0$.

3. Type-I Agent Behavior

This section describes type-I agents' utility maximization and rational expectations and presents our strategy for characterizing their consumption behavior. (Recall that type-II agents simply consume their income.)

Utility Maximization and Rational Expectations. We model each type-I household with a representative-agent formulation. Call it an RAI specification.

Let $J = J_t$ be the initial "technology index" and society's initial frontier technology be $[\theta]^J$, $\theta > 1$. The period over which a particular technology index prevails is an "era." While index J prevails, all type-I agents anticipate a time path for the aggregate capital stock of

$$\vec{K}(J) \equiv \{K_s\,, s \geq 0 \big| \, J\}\,,$$

where $s \geq 0$ measures time within the era. An RAI solves

$$V\big(A, \vec{K}(J), J\big) = \max_{C_t} \int_0^\infty \lambda \cdot e^{-\lambda \cdot T} \cdot \Big[\int_0^T e^{-\rho \cdot t} \cdot u(C_t)\, dt + e^{-\rho \cdot T} \cdot V\big(b \cdot A_T, \vec{K}(J+1), J+1\big)\Big]\, dT \tag{11a}$$

$$\text{subject to:} \quad \dot{A}_t = W(K_t, J) + M(K_t, J) \cdot A_t - C_t\,, \quad A_0 = A\,, \tag{11b}$$

where $V(.)$ is the value function, A_t the RAI's net worth, and C_t his/her consumption. The wage and marginal product of capital functions, $W(.)$ and $M(.)$, come from Prop. 2, and the RAI evaluates them taking as given values for K_t from $\vec{K}(J)$.

We want rational expectations. Hence, after deriving first-order conditions for (11a-b), giving $(\dot{A}_t\,, \dot{C}_t)$ as a function of (A_t, C_t, K_t), we solve the 2 corresponding differential equations imposing, for all t during the current technology era,

$$K_t = A_t\,. \tag{11c}$$

[6] Note that $\mathbb{E}_t[R_t]$ refers to the expected value of the momentary rate R_t conditional on information up to time t, whereas $\mathbb{E}[R]$ or $\mathbb{E}[R_t]$ refers to the mean of the stationary distribution of R_t.

Having obtained the rational expectations solution, we call the relation between the RAI's current consumption and its net worth its "consumption function," $\Phi(A_t, J) = C_t$. Our attention centers on a long-run equilibrium in which the same consumption function holds for all RAIs and eras and exhibits the homogeneity property

$$\Phi(A, J+1) = x \cdot \Phi(\frac{A}{x}, J) \quad \text{where} \quad x \equiv [\theta]^{\frac{1}{1-\alpha}}. \tag{12}$$

Auxiliary Problems. Given our 2 agent types, we cannot apply the most standard representative-agent solution methodologies of, say, Cooley and Prescott [1995]. Our strategy has 2 parts. The first is, as indicated above, to focus on the RAI's consumption rather than its value function. As (12) suggests, we show we need only construct $\Phi(A, J) = C$ for $J = 0$. Moreover, we show $\Phi(., 0)$ will have a bounded domain and range.

Second, to compute Φ, we develop a sequence of auxiliary problems. The latter are indexed with $m = 0, 1, ...$, where m is the number of future technology shocks allowed. We are interested in the limit as $m \to \infty$. We solve the sequence of auxiliary problems recursively, deriving consumption functions Φ^m. Each problem m is concave, and we derive explicit first-order conditions. After imposing rational expectations as in (11c) above, we solve the conditions numerically. Similarly, we check for convergence $\Phi^m \to \Phi$. Provided uniform convergence obtains, Φ is the basis of our subsequent analysis.

To start the algorithm, set $m = 0$ — so that there are no future GPT arrivals. Let (C_s, A_s) be the consumption and net asset position of our RAI at time s. The RAI solves

$$v^0\big(A, \vec{K}(J), J\big) \equiv \max_{C_s} \int_0^\infty e^{-\rho \cdot s} \cdot u(C_s)\, ds \tag{13a}$$

$$\text{subject to:} \quad \dot{A}_s = W(K_s, J) + M(K_s, J) \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{13b}$$

The K_s for $W(K_s, J)$ and $M(K_s, J)$ comes from $\vec{K}(J)$ — see above — and $W(.)$ and $M(.)$ come from Prop. 2.

Set up a current value Hamiltonian with costate μ_s:

$$\mathcal{H}^0 \equiv u(C_s) + \mu_s \cdot [W(K_s, J) + M(K_s, J) \cdot A_s - C_s]\,. \tag{14}$$

First-order conditions yield

$$\dot{C}_s = \frac{C_s}{1-\gamma} \cdot [M(K_s, J) - \rho]\,, \tag{15a}$$

$$\dot{A}_s = W(K_s, J) + M(K_s, J) \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{15b}$$

Impose rational expectations, i.e., set $\vec{K} = \vec{A}$ as in (11c). We then have

$$\dot{C}_s = \frac{C_s}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^J \cdot [A_s]^{\alpha-1} - (\delta + \rho)\right], \tag{16a}$$

$$\dot{A}_s = \eta \cdot [\theta]^J \cdot [A_s]^\alpha - \delta \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{16b}$$

Figure 2 presents the phase diagram. Suppose it has a stationary point (A^{*0}, C^{*0}) and a saddle path solution to (16a-b), as illustrated. First-order conditions (15a-b) hold by construction on the saddle path, and the conventional transversality condition holds as well. Thus, we have a rational expectation's solution to (13a-b). The height of the saddle path characterizes our $m = 0$ consumption function: $\Phi^0(A, J) = C$ for $A \in (0, A^U]$.

We assume the envelope theorem applies:

$$\frac{\partial v^0\big(A, \vec{K}(J), J\big)}{\partial A}\Big|_{\vec{K}=\vec{A}} = u'\big(\Phi^0(A, J)\big)\,. \tag{17}$$

We often use the shorthand expression $V^0(A, J)$ for the maximized value of the criterion to (13) with rational expectations, though we must keep (17) in mind.

The following Lemma is basic to our analysis.

Lemma 1: *Let $x \equiv [\theta]^{\frac{1}{1-\alpha}}$ as in (12). Then*

$$\Phi^0(A, J+1) = x \cdot \Phi^0(\frac{A}{x}, J)\,, \tag{i}$$

$$V^0(A, J+1) = [x]^{\gamma} \cdot V^0(\frac{A}{x}, J)\,. \tag{ii}$$

Proof: See Appendix.

Proceed to $m = 1$, i.e., the case with 1 GPT arrival in the future. Again, let $\vec{K}(J)$ denote the time path while the current technology index J predominates. The RAI takes $\vec{K}(J)$ as given. Let the era of technology J run from $s = 0$ until the (single) future GPT arrives. Assume that at the latter's arrival, the RAI shifts to problem (13), with its rational expectations solution, consumption function Φ^0, and envelope condition (17).

The criterion for the $m = 1$ problem reflects expected utility. With Poisson shocks, the density for arrival times is exponential. Thus, we have

$$\begin{aligned} &v^1\big(A, \vec{K}(J), J\big) \\ &\quad = \max_{C_t} \int_0^{\infty} \lambda \cdot e^{-\lambda \cdot T} \cdot \Big[\int_0^{T} e^{-\rho \cdot t} \cdot u(C_t)\, dt + e^{-\rho \cdot T} \cdot V^0(b \cdot A_T, J+1) \Big]\, dT. \end{aligned} \tag{18}$$

The right-hand side of (18) sums 2 terms. The first has a double integral. Changing the order of integration, noting that

$$\int_t^{\infty} \lambda \cdot e^{-\lambda \cdot s}\, ds = e^{-\lambda \cdot t}\,,$$

and changing the (dummy) variable of integration in the second term to t, we can rewrite the criterion's integral as

$$\int_0^{\infty} e^{-(\lambda+\rho)\cdot t} \cdot \big[u(C_t) + \lambda \cdot V^0(b \cdot A_t, J+1)\big]\, dt\,. \tag{19}$$

With the constraint remaining as in (13b), the Hamiltonian for the first era of the $m = 1$ problem is

$$\mathcal{H}^1 = u(C_t) + \lambda \cdot V^0(b \cdot A_t, J+1) + \mu_t \cdot [W(K_t, J) + M(K_t, J) \cdot A_t - C_t]\,. \tag{20}$$

Imposing rational expectations (as above) and using (17) and Lemma 1, we have

$$\dot{C}_t = \frac{C_t}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^J \cdot [A_t]^{\alpha-1} + \lambda \cdot b \cdot \frac{u'\big(x \cdot \Phi^0(\frac{b \cdot A_t}{x}, J)\big)}{u'\big(C_t\big)} - (\delta + \lambda + \rho)\right], \tag{21a}$$

$$\text{... same as (16b) ...} \tag{21b}$$

Figure 3 illustrates the phase diagram. The A-isocline is the same as in Figure 2. The C-isocline is upward sloping. Provided we again have a saddle path solution, the transversality condition for the $m = 1$ problem holds. Then define Φ^1 and V^1 analogously to Φ^0 and V^0.

We have

Lemma 2: *If x is as in Lemma 1, then*

$$\Phi^1(A, J+1) = x \cdot \Phi^1(\frac{A}{x}, J)\,, \tag{i}$$

$$V^1(A, J+1) = [x]^\gamma \cdot V^1(\frac{A}{x}, J)\,. \tag{ii}$$

Proof: See Appendix.

Proceeding with $m = 2, 3, ...$, we can see that the proof of Lemma 2 carries over.

In Figures 2-3, notice that although $A = 0$ is not itself a value of interest, if we extend the saddle path to include the origin, Φ^m remains a continuous function. For convenience of exposition, define $\Phi^m(A, J)$ for $A \in [0, A^U]$, setting $\Phi^m(0, J) = 0$.

Numerical Solution. We want to solve our sequence of auxiliary problems numerically, checking for convergence to a limit. Specifically, we determine the saddle path satisfying the analogs (16a-b), for $m = 0$, and (21a-b), for $m = 1, 2, ...$ For each m, set $J = 0$ and compute, and save, the initial era's stationary point (A^{*m}, C^{*m}) and consumption function Φ^m, evaluated on the domain $[0, A^U]$. The $A - isocline$ (recall Figures 2-3) is identical for all $m = 0, 1, ...$ The $C - isocline$ has a positive slope and a fixed intercept for all $m \geq 1$. The stationary points lie at the intersection of the isoclines; so,

$$\bar{C}(A^{*m}) = \Phi^m(A^{*m}, 0) \text{ where } \bar{C}(A) \equiv \eta \cdot [A]^\alpha - \delta \cdot A\,.$$

We are interested in outcomes obeying the following:

Conditions: **(C1)** *In the initial era for each m in our sequence of auxiliary problems, we have a phase diagram, as in Figs 2-3, with an upward sloping saddle path, a single*

positive stationary point (A^{*m}, C^{*m})*, and implied motion toward the stationary point for every initial asset value in* $(0, A^U]$*. We have* $C^U < \infty$ *such that* $\Phi^m(A^U, 0) \leq C^U$ *all* m*. In addition,* **(C2)** *as* $m \to \infty$*, we have uniform convergence* $\Phi^m(A, 0) \to \Phi(A, 0)$ *for* $A \in [0, A^U]$*;* **(C3)** *there is* $A^L > 0$ *such that* $A^L \leq A^{*m} < A^U$ *all* m*;* **(C4)** $A^{*m} \to A^*$ *with* $\bar{C}(A^*) = \Phi(A^*, 0)$ *and* $A < (>) A^*$ *implying* $\bar{C}(A) > (<) \Phi(A, 0)$*; and,* **(C5)** *the sequence* $\partial\Phi^m(A, 0)/\partial A$ *converges uniformly on any compact interval* $[\bar{A}, A^U]$ *with* $\bar{A} > 0$.[7] *Finally,* **(C6)** *the envelope theorem (see (17)) holds each* m*; and,* **(C7)** *if* $\gamma < 0$*, for some* $\xi > 0$ *and all* m *we have* $\Phi^m(A, 0) \geq \xi \cdot A$ *all* $A \in [0, A^U]$.

Given $\Phi^m(A, 0)$, we can derive $\Phi^m(A, J)$ all $J > 0$ from Lemmas 1-2. Provided $(\mathbf{C1} - \mathbf{7})$ hold, our analysis can proceed, centering on $\Phi(A, 0) \equiv \lim_{m\to\infty} \Phi^m(A, 0)$.

4. Steady-State Equilibrium

We are ready to construct our steady-state equilibrium. This section covers, in turn, de-trending key variables, sufficient conditions for the existence of a type-I agent value function, and sufficient conditions for a stationary distribution of per capita wealth.

De-trending. Our basic de-trending formula is an extension of Lemmas 1-2.

Prop. 3A: *Let* $x > 1$ *be as in (12). Assume condition (C2). Then*

$$\Phi(A, J) \equiv \lim_{m\to\infty} \Phi^m(A, J) \text{ exists all } A \in \left[0, [x]^J \cdot a^U\right], \; J \geq 0, \tag{i}$$

$$\Phi(A, J) = x \cdot \Phi(\frac{A}{x}, J - 1) = ... = [x]^J \cdot \Phi(\frac{A}{[x]^J}, 0)\,. \tag{ii}$$

Proof: see Appendix.

In textbook treatments of nonstochastic growth, given a rate of Hicks neutral technical change, we transform to a Harrod-neutral measure, divide the capital stock by the latter, and perform most of our analysis on the resulting ratio. Proposition 3A shows the analog applies here: if we transform our Hicks-neutral technology factor θ into a Harrod-neutral factor x (see (12)), we can focus our attention on

$$a_t \equiv A_t/[x]^{J_t}\,, \quad c_t \equiv C_t/[x]^{J_t}\,, \quad k_t \equiv K_t/[x]^{J_t}\,, \quad y_t \equiv Y_t/[x]^{J_t}\,. \tag{22}$$

In particular, note that (12) implies $\theta \cdot [x]^\alpha = x$, recall budget constraint (16b), and define

$$\Gamma(A, J, \Phi) \equiv \eta \cdot [\theta]^J \cdot [A]^\alpha - \delta \cdot A - \Phi(A, J)\,. \tag{23}$$

Then in an era with technology $J = J_t$, using Prop. 3A, part (ii), we have

[7] Suppose (A_t^m, C_t^m) solve our differential equations for stage m. They are continuously differentiable by construction. Accordingly, Φ^m is continuously differentiable in A. We can see $\partial\Phi^m(A_t^m, 0)/\partial A_t^m = \dot{C}_t^m/\dot{A}_t^m$ or, when $A_t^m = A^{*m}$, it equals the slope of the eigenvector associated with the saddle path at the stationary point.

$$
\begin{aligned}
&\dot{A}_t = \Gamma(A_t, J, \Phi)\,, \quad A_0 = A\,,\\
\iff &\frac{\dot{A}_t}{[x]^J} = \frac{\eta\cdot[\theta]^J}{[x]^J}\cdot[A_t]^\alpha - \delta\cdot\frac{A_t}{[x]^J} - \frac{\Phi(A_t,J)}{[x]^J}\,, \quad A_0 = A\,,\\
\iff &\frac{\dot{A}_t}{[x]^J} = \eta\cdot[\frac{A_t}{[x]^J}]^\alpha - \delta\cdot\frac{A_t}{[x]^J} - \frac{[x]^J}{[x]^J}\cdot\Phi(\frac{A_t}{[x]^J},0)\,, \quad A_0/[x]^J = A/[x]^J\,,\\
\iff &\dot{a}_t = \Gamma(a_t, 0, \Phi)\,, \quad a_0 = A/[x]^J\,,
\end{aligned}
\tag{24}
$$

$$
c_t \equiv \frac{C_t}{[x]^J} = \frac{\Phi(A_t,J)}{[x]^J} = \frac{[x]^J\cdot\Phi(a_t,0)}{[x]^J} = \Phi(a_t,0)\,. \tag{25}
$$

Rational expectations implies $K_t = A_t$ all t; so, $k_t = a_t$. And, the algebra for x together with Prop. 2 show

$$
y_t \equiv \frac{Y_t}{[x]^J} = \frac{[\theta]^J\cdot[K_t]^\alpha}{[\theta]^J\cdot[x]^{\alpha\cdot J}} = [k_t]^\alpha\,.
$$

Moreover, we can restrict our attention to a bounded set of values for a_t, as follows. Assume Conditions (C1)-(C7) and, letting A^* be as in (C4), define

$$
a^* \equiv A^* \text{ and } \mathcal{A} \equiv (0, a^*]\,. \tag{26}
$$

In the notation a_t, t refers to calendar time. It is often more convenient, however, to reset the time index to 0 at each GPT arrival. Suppose technology era J lasts for s with $0 \le s < \bar{s}(J)$, with $a_{s,J}$ giving the value of a within the era. By (C5), $\Gamma(a,0,\Phi)$ from (23) is continuously differentiable; thus, we expect to have a solution $a_{s,J} = \Psi(s, a_{0,J})$ from any initial condition $a_{0,J} \in \mathcal{A}$. Condition (C4) implies $\Psi(s, a_{0,J})$ increases in s, asymptotically approaching a^*. So, $\Psi(s, a_{0,J}) \in \mathcal{A}$ each $s \ge 0$. When the era ends, we jump to $a_{0,J+1} = b\cdot\Psi(\bar{s}(J), a_{0,J})/x$. As the latter is clearly within $\mathcal{A}$, we have established

Prop. 3B: *Assume conditions (C1)-(C7). Then*

$$
a_{s,J} \in \mathcal{A} \Longrightarrow a_{s',J'} \in \mathcal{A} \quad \forall s' \ge 0\,,\ J' \ge J\,. \tag{i}
$$

Proof: See the preceding paragraph.

To summarize the notation, for any J and initial condition $a_{0,J}$,

$$
a_{s,J} = \Psi(s, a_{0,J}) \text{ solves (24) all } s \ge 0\,, \text{ and } \lim_{s\to\infty}\Psi(s, a_{0,J}) = a^*\,. \tag{27}
$$

Value Functions. Value functions are an important part of our equilibrium, and this section provides sufficient conditions for the existence of $V(A,J) = \lim_{m\to\infty} V^m(A,J)$. Since this paper's applications do not make explicit use of $V(.)$, the proofs are available on the authors' websites.

We have

Lemma 3: *Let Conditions (C1)-(C7) hold. Then sufficient conditions for* $V^m(A,0)$ *any* $A \in (0, A^U]$ *to be finite all* $m \geq 0$ *are*

$$\lambda \cdot ([x]^\gamma - 1) - \rho < 0\,, \tag{i}$$

$$\lambda \cdot ([b]^\gamma - 1) - \rho < 0\,. \tag{ii}$$

Proof: See authors' website.

Conditions (i)-(ii) together require $\rho > 0$. If $\gamma > 0$, only condition (i) ever binds; if $\gamma < 0$, only condition (ii) matters.

Given Lemma 3 and the consumption function convergence established numerically in (C1-7), we can prove that a bounded $V(A,J)$ exists.

Prop. 3C: *Given the conditions of Lemma 3,* $V(A,J) = \lim_{m\to\infty} V^m(A,J)$ *exists each* $A \in (0, [x]^J \cdot A^U]$*; it is continuously differentiable in* A *with* $\partial V(A,J)/\partial A = u'\big(\Phi(A,J)\big)$*; it is increasing and concave at each saddle path point; and, it has the homogeneity property of Lemmas 1-2, part (ii).*

Proof: See authors' website.

We can verify conditions (i)-(ii) of Lemma 3 for our simulations below.

Saddle Path Growth. Most of the elements of the model's equilibrium are now in place. See Figure 4. In the figure, the a-isocline comes from (16b) and the c-isocline from (21a) — with Φ replacing Φ^0. Our attention centers exclusively on the section of the saddle path with $0 < a_{sJ} \leq a^*$. The saddle path's height is $c_{sJ} = \Phi(a_{sJ}, 0)$. If we start at a point A, we asymptotically converge to (a^*, c^*). If a new GPT arrival interrupts the convergence at B, say, $(\bar{a}_{sJ}, \bar{c}_{sJ})$, we jump to a point C, where the RAI's capital has declined to $b \cdot \bar{a}_{sJ}/x$. At C, convergence along the saddle path toward (a^*, c^*) begins again.

Referring to Fig. 4, characteristics of the equilibrium evident so far are as follows. (i) The set $\mathcal{A} = (0, a^*]$ is invariant — if we start within it, we remain within it. (ii) Within $\mathcal{A}$, our assumption that old-vintage capital cannot be transformed to new-vintage capital or consumption goods holds because net investment is non-negative between GPT arrivals. (iii) Type-I households maximize their expected utility. Specifically, in each era they maximize a Hamiltonian (see (14) and (20)). It is continuously differentiable; nondecreasing in A_t; and, concave in (A_t, C_t). The corresponding transversality condition holds. And, (iv) our derivation of the equations determining the saddle path ensures that agents' expectations are rational.

Steady-State Distribution of Wealth. In a typical Solowesque growth model, after we have detrended K_t, giving k_t, our attention turns to the latter's stationary value. The analog in the present paper is to determine the stationary distribution of equilibrium values for $a_t = k_t$ on its support, $\mathcal{A}$. We call the corresponding probability measure Π (see below). We proceed in 2 stages.

The first stage analyzes a discrete-step, continuous state space Markov process for the de-trended capital stock over technology-era endpoints. Let the era of technology J be

$\big[t(J), \bar{t}(J)\big)$. Note that $t(J+1) = \bar{t}(J)$. Let $\bar{a}_{\bar{t}(J)} \equiv \lim_{t \to \bar{t}(J)} a_t$. Let $\bar{\mathcal{A}}(J+1) \subseteq \mathcal{A}$ be a Borel set and μ Lebesgue measure. Then

$$P\big(\bar{a}_{\bar{t}(J)}, \mathcal{A}(J+1)\big) = \int_{\{s \geq 0 | \Psi(s, b \cdot \bar{a}_{\bar{t}(J)}/x) \in \mathcal{A}(J+1)\}} \lambda \cdot e^{-\lambda \cdot s} \, \mu(ds) \tag{28}$$

defines a Markov transition function from $\bar{a}_{\bar{t}(J)} \in \mathcal{A}$ to points $\bar{a}_{\bar{t}(J+1)} \in \bar{\mathcal{A}}(J+1)$. We have

Prop. 3D: *Suppose conditions (C1)-(C7) hold and*

$$\frac{1}{1-\gamma} - \eta > 0 \, . \tag{i}$$

Then P from (28) has a unique stationary distribution on $\mathcal{A}$, with corresponding stationary probability measure π; $\mathcal{A}$ is an ergodic set; and, $\mathcal{A}$ has no cyclic subsets.

Proof: See Appendix (including Lemma 4 therein).

Our second stage uses Prop. 3D to establish the existence of a unique stationary distribution for a_t, where t is calendar time. We will want to compute moments of the stationary distribution, and we show that the intuitive method for doing so can work. Letting $\bar{s}(J)$ be the length of technology era J as above, we have

Corollary 3D: *Suppose Prop. 3D holds. Then there is a unique stationary probability measure Π for a_t on $\mathcal{A}$. Let $h(.)$ be a measurable function and*

$$\int_{\mathcal{A}} |h(a)| \, \Pi(da) < \infty \, . \tag{i}$$

Then for any $\bar{a}_{\bar{t}(0)} \in \mathcal{A}$, we have

$$\mathbb{E}[h(a)] \equiv \int_{\mathcal{A}} h(a) \, \Pi(da) \overset{\text{P}}{=} \lim_{J \to \infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} h\big(\Psi(s, b \cdot a_{\bar{t}(j-1)}/x)\big) \, ds}{\bar{t}(J) - \bar{t}(0)} \, , \tag{ii}$$

if $\bar{s}(J)$, $J = 1, 2, ...,$ are iid samplings from the exponential distribution with parameter λ.

Proof: See Appendix.

To find the mean of the stationary distribution of a_t, for example, set $h(a) = a$ and simulate a time path for a_t. Cor. 3D shows that the average value of a_t along the simulated path provides a consistent estimate of $\mathbb{E}[a]$.[8]

[8] To calculate $\mathbb{E}[R]$ in the analysis below, we will need to evaluate $I = \int_{\mathcal{A}} [k]^{\alpha-1} \, \Pi(dk)$. As $k \to 0$, $[k]^{\alpha-1}$ is unbounded. That may not, in practice, create a problem for Cor. 3D, cond. (i), because Π will tend to assign a low probability to values of k near 0. Nevertheless, to be sure cond. (i) holds, we can set a small $\epsilon > 0$ and use, in computing I, $[\epsilon, k^*]$ in place of $\mathcal{A}$, and $\Pi(dk)/\Pi([\epsilon, k^*])$ in place of $\Pi(dk)$ — doing likewise for each $\mathbb{E}[.]$ term in the equation we are analyzing. The same issue arises for $\mathbb{E}[r]$ below, where r is the riskfree rate of return, and where a term $[\Phi(b \cdot a, 1)/\Phi(a, 0)]^{\gamma-1}$ arises and may diverge as $a \downarrow 0$. But, the same correction is applicable.

5. Calibrations

Figures 1 and 4 depict qualitative aspects our model's equilibrium. The remainder of this paper uses simulations to provide quantitative results, and this section explains how we calibrate the model's underlying parameters.

Rates of Return. In the preceding sections, we solve for $\Phi(.)$ and the stationary distribution of a_t without the need to specify the structure of the financing of the economy's stock of plant and equipment. Now, following the literature (see below), we assume a fixed share of bonds, υ, and of common-stock, $1-\upsilon$. That enables us to determine separate long-run average rates of return for bonds and stocks, from which the equity premium follows. As discussed in the introduction, we propose using a variant of the latter to detect shifts in the long-run growth rate.

The model has an Euler equation relating the growth rate of type-I agent consumption to the return on (physical) investment (e.g., (16a), (21a), etc.), and we can derive the analog for the return on risk-free bonds. Suppose a risk-free bond issued at t costs \$1, promises an interest rate r_t, and expires at $t+h$, where h is small. Without loss of generality, let the current technology index be 0. The sacrifice of current utility for a type-I household purchasing the bond is $u'(c_t)$. During the bond's lifespan, a new GPT arrives with probability $\lambda \cdot h$. If an arrival occurs, the household's marginal utility jumps to $u'(\tilde{c}_t)$ with $\tilde{c}_t \equiv \Phi(b \cdot a_t, 1) = x \cdot \Phi(b \cdot a_t/x, 0)$. Absent an arrival, the time path of marginal utility is continuously differentiable; so, $u'(c_{t+h}) \approx u'(c_t) + u''(c_t) \cdot \dot{c}_t \cdot h$. Thus,

$$\begin{aligned} u'(c_t) \approx (1-\lambda \cdot h) \cdot \big(1 + (r_t - \rho) \cdot h\big) \cdot \Big(u'(c_t) + (\gamma - 1) \cdot u'(c_t) \cdot \frac{\dot{c}_t}{c_t} \cdot h\Big) \\ + \lambda \cdot h \cdot \big(1 + (r_t - \rho) \cdot h\big) \cdot u'(\tilde{c}_t)\,. \end{aligned}$$

Rearranging terms, letting $h \to 0$, and dropping terms with powers of h greater than 1,

$$r_t = \rho + (1-\gamma) \cdot \frac{\dot{c}_t}{c_t} - \lambda \cdot \Big[\frac{u'(\tilde{c}_t) - u'(c_t)}{u'(c_t)}\Big]\,. \tag{29}$$

Thus, the risk-free rate reflects subjective discounting, within-technology-era consumption growth, and the possible arrival of a new GPT (which would raise the utility-value of the bond's principal).

If R_t^S and R_t are the realized returns per dollar of common stock and physical investment, respectively, then

$$R_t = (1-\upsilon) \cdot R_t^S + \upsilon \cdot r_t\,. \tag{30}$$

The long-run average equity premium is

$$\mathbb{E}[EP] = \mathbb{E}[R^S] - \mathbb{E}[r] = \frac{\mathbb{E}[R] - \upsilon \cdot \mathbb{E}[r]}{1-\upsilon} - \mathbb{E}[r] = \frac{\mathbb{E}[R] - \mathbb{E}[r]}{1-\upsilon}\,, \tag{31}$$

where the second equality uses (30). From Prop. 2, (10), and Cor. 3D, we have

$$\mathbb{E}[R] = \lim_{J\to\infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} \left[\alpha \cdot [\Psi\big(s, b\cdot \bar{a}_{\bar{t}(j-1)}/x\big)]^{\alpha-1} - \delta\right] ds}{\bar{t}(J) - \bar{t}(0)} - \lambda\cdot(1-b)\,. \tag{32}$$

We can derive $r_t = r(a_t)$ from the detrended, $m \to \infty$ version of (21a), (25), and (29); so,

$$\mathbb{E}[r] = \lim_{J\to\infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} r\big(\Psi\big(s, b\cdot \bar{a}_{\bar{t}(j-1)}/x\big)\big)\, ds}{\bar{t}(J) - \bar{t}(0)}\,. \tag{33}$$

Calibration Preliminaries. This subsection derives constants that we use in our calibrations.

As in Laitner and Stolyarov [2003], we assume $b = 0.40$.[9] Following Maddison [1991], we assume a long-run average growth rate of output per worker of $g = 0.0210$. For the share of bond financing, a number of papers use $\upsilon = 0.40$ (e.g., Boldrin et al [1995], Papanikolaou [2011], Barro [2006]) — which we take to be an upper bound.

Maddison's g reflects the expansion of GDP over a long time period of years. Suppose we start at time 0 and end at t, with the frontier TFP indices being $J_0 = 0$ and $J_t = J$, and the corresponding values of k being k_0 and k_t. Algebraically,

$$\theta\cdot[x]^\alpha = x \quad \text{and} \quad \ln(x) = \frac{\alpha}{1-\alpha}\cdot\ln(1/b)\,, \tag{34}$$

with the second equality using Prop. 1. Recall that the population and labor force are fixed. Using Prop. 2 and the first part of (34), we have

$$e^{g\cdot t} = \frac{GDP_t}{GDP_0} = \frac{[\theta]^J\cdot[[x]^J\cdot k_t]^\alpha}{[k_0]^\alpha} = \frac{[x]^J\cdot[k_t]^\alpha}{[k_0]^\alpha} \Longleftrightarrow g = \lambda\cdot\ln(x) + \epsilon_1(t) + \epsilon_2(t)\,,$$

$$\epsilon_1(t) \equiv [\frac{J_t}{t} - \lambda]\cdot\ln(x) \quad \text{and} \quad \epsilon_2(t) \equiv \frac{\alpha\cdot[\ln(k_t) - \ln(k_0)]}{t}\,.$$

Consider any sequence $t_n \to \infty$. As in the proof of Cor. 3D, we have $\text{plim}_n\, \epsilon_1(t_n) = 0$. Cor. 3D shows that if we allow ourselves to adjust any t_n higher when necessary, we can assume $k_{t_n} \geq \mathbb{E}[k] > 0$ all n.[10] Hence, $\lim_{n\to\infty} \epsilon_2(t_n) = 0$. Given the second part of (34), we thus set

$$g = \lambda\cdot\frac{\alpha}{1-\alpha}\cdot\ln(1/b)\,. \tag{35}$$

[9] Greenwood, Hurcowitz, and Krusell [1997] derive the average rate of decline in the relative price of investment goods 1954-1990 commensurate with an assessment of vintage-based quality gains. They attribute such gains only to equipment, but their average applied to all plant and equipment, over the average episode length derived below, yields an alternative estimate $b = 0.41$.

[10] In practice, we can pick GDP_0 and GDP_t above each midway between stock-market bears, for instance.

The parameter β is labor's share for type-I households. In Kennickell [2009], the 2007 SCF seems representative of 1989-2007. The income share of the top 5% of wealth holders was about 33%. Let $\mathcal{Y}$ measure 2007 GDP (excluding services of residential housing, investment in residential housing, and depreciation on the same). Let $\mathcal{D}$ measure depreciation (excluding residential housing). Then $\mathcal{Y} = \$12,308.3\ bil/yr$ and $\mathcal{D} = \$1458.3\ bil/yr$, and we derive β from

$$\frac{1}{3} \cdot [\mathcal{Y} - \mathcal{D}] = [\alpha \cdot \mathcal{Y} - \mathcal{D}] + \beta \cdot (1 - \alpha) \cdot \mathcal{Y}\,. \tag{36}$$

Following Laitner and Stolyarov [2003], we set the overall annual depreciation rate for capital from NIPA data, $d = 0.075$. We assume d includes both wear and tear depreciation, say, at rate δ, and resale-price declines from obsolescence. Suppose d reflects year-to-year depreciation at different times for a wide selection of capital goods of different ages. Then the total depreciation per dollar's worth of capital from t to $t + h$ obeys

$$d \cdot h = \delta \cdot h + \begin{cases} 0\,, & \text{with prob. } (1 - \lambda) \cdot h\,, \\ 1 - b\,, & \text{with prob. } \lambda \cdot h\,. \end{cases}$$

Assuming the NIPA estimated d from the expected value of the right-hand side, we have

$$d = \delta + \lambda \cdot (1 - b)\,. \tag{37}$$

Stock-Market Data. Data from Table 1 provides the final inputs for our calibrations.

Table 1. Mean Annual Rates from U.S. Financial Market Data

Row	Time Span	Bull Market Years	No Excluded Years Within Time Span			Bull-Market Years Only
			Riskless Rate of Interest	Stock-Market Return	Equity Premium	Equity Premium
	Long Intervals					
1	1916-2020	... [a]	0.92%/yr	6.46%/yr	5.53%/yr	9.10%/yr
2	1929-2020	... [b]	0.90	6.03	5.13	8.44
	Individual Technology Eras					
3	1916-1928	1921-1928	1.05	9.44	8.39	15.95
4	1929-1972	1933-1972	0.67	5.99	5.32	8.71
5	1973-1999	1975-1999	2.51	7.59	5.08	7.84
6	2000-2020	2003-2020	-0.68	4.14	4.81	8.66

Source: See Appendix 2.

(a) Union from this column, rows 3-6, below.

(b) Union from this column, rows 4-6, below.

According to our model, the stock market has episodes with a new GPT causing a short, severe bear market, followed by a bull.[11] In our terminology, each bear-then-bull episode corresponds to a "technology era." For the 20th century, we think of the following eras: development of a practical farm tractor about the end of WWI, which greatly reduced agricultural labor input (Dahlstrom [2022], Cox and Alm [1992]); electrification of factories and development of methods of mass production for automobiles and home appliances by the close of the 1920s (Landes [1969]); the microprocessor about 1970 (Laitner and Stolyarov [2003], Hobijn and Jovanovic [2001]); and, the Internet about 2000 (Pastor and Veronesi [2009]). As noted in the Introduction, a bear market in stocks can help to pin down the date at which the economy accepts a seminal invention's viability. We use the cycles summarized, and labeled according to their characteristic technology, in Albuquerque et al. [2015, tab. 2]. We choose the following 4 to correspond to our technology eras: 1916-1928, 1929-1972, 1973-1999, and 2000-2020. Although the last has not yet ended, it has almost reached the average length of the others — and we include it to bring the analysis up to date.[12]

Parameter Values. Table 2 presents our calibrations. Each GPT arrival causes a sharp negative spike in the equity premium; hence, the number of arrivals in a finite interval can dramatically affect our estimate of $\mathbb{E}[EP]$. In that light, we limit our analysis to sets of complete technology episodes. Table 2, row 1, uses all 4 eras from Table 1. As we argue below that the first episode may have been anomalous, Table 2, row 2, excludes it.

Table 2. Parameter Values for Simulations Matching Data Averages from Table 1, Rows 1-2

Data Range:	ρ	γ	λ	b	υ	α	β	η	δ
1916-2020	0.0155	0.258	0.0357	0.400	0.400	0.391	0.0350	0.412	0.0538
1929-2020	0.0157	0.273	0.0359	0.400	0.350	0.390	0.0371	0.412	0.0537

Source: See text.

Table 2, row 1, is based on Table 1, row 1. Set $b = 0.40$ and $\upsilon = 0.40$, as above. Eq. (9) shows that during bull markets, $R_t = M(K_t, J_t)$. Eq. (10) shows that conditional of information just prior to t, $\mathbb{E}_t[R_t] = M(K_t, J_t) - \lambda \cdot (1 - b)$. Hence, (31) shows the long-run difference between the bull-market and the unrestricted equity premium is

$$\mathbb{E}[BMEP] - \mathbb{E}[EP] = \frac{\lambda \cdot (1 - b)}{1 - \upsilon} \,. \tag{38}$$

[11] Note: this paper's "bull markets" consist of all times outside of bear markets.

[12] Years before 1916 are outside the scope of the present paper. Other authors have noted that technological change was slower in earlier parts of the industrial revolution (e.g., Landes [1969]); the US stock market was less mature (e.g., Smith [2003]); and, the equity premium tended to be much lower (e.g., Siegel and Thaler [1997], Siegel [1998]).

Given the data in Table 1, cols. 6-7, and our calibrations for b and υ, eq. (38) determines λ. Then Maddison's g and (35) determine α; (36) sets β; (8) sets η; and (37) sets δ. To complete the calibration, we adjust (ρ, γ) to jointly match the risk-free rate, $\mathbb{E}[r]$, and the unrestricted equity premium, $\mathbb{E}[EP]$.

For Table 2, row 2, we use Table 1, row 2. Adjusting υ downward slightly, we repeat the steps above.

As stated, we worry below that the economy may not have attained equilibrium 1916-1929. Thus, our simulations emphasize Table 2, row 2. In any case, Table 2, rows 1-2, are quite similar to one another.

Computations. We simulate the model on a desktop computer, working in Fortran and making extensive use of the IMSL subroutine library. Given parameter values, the execution time for determining $\Phi(a, 0)$ and the economy's stationary moments is 1-2 seconds.

6. Model Overview

Before turning to analysis of the long-run growth rate, we highlight evidence supportive of our key assumptions and basic simulation outcomes.

Direct Evidence. We can note previous analyses consistent with our specification.

One of the most thoroughly studied examples of a GPT arrival is the microprocessor revolution of the early 1970s. Hobijn and Jovanovic [2001] provide detailed evidence of large-scale changes and technology-specific capital. They link the 1972-74 bear market to investor recognition of the significance of the microprocessor. Separating firms into those with old versus new-vintage capital, they document divergences after 1972, as follows: the value of firms with old-vintage capital fell by more than 50 percent, never fully recovering. Firms with new-vintage capital, on the other hand, accounted for a subsequent tripling of the market capitalization-to-GDP ratio. Notably, the vintage effect was most pronounced in the IT-intensive sectors of the economy (see Hobijn and Jovanovic's Fig. 10).

Pastor and Veronesi's [2009] analysis of the advent of the Internet has a related nature to our framework as well (though in contrast to our treatment, they emphasize phases in agent recognition of the significance of a new technology).

Landes' [1969] view of technological change emphasizes waves of growth and renewal. He writes:

> From 1870 on ... British industry had exhausted the gains implicit in the original cluster of innovations that had constituted the Industrial Revolution.... Not until a series of major advances opened new areas of investment around the turn of the century was this deceleration reversed. These [later] years saw the lusty childhood, if not the birth, of electrical power and motors; organic chemistry and synthetics; the internal-combustion engine and automotive devices; precision manufacture and assembly-line production.... (pp. 234-5)

As in our steady-state, a GPT's arrival opens investment options that raise national output. Diminishing returns eventually set in and growth slows. But, as the next GPT follows, gains from investment rise again and productivity accelerates.

Stock Market. Our model generates low-frequency financial-market "cycles."

Tables 1-2 in Albuquerque et al. [2015], for example, segment US stock market data 1869-2013 into bear-then-bull episodes. Their narrative identifies 5 bull markets associated with major technology diffusions, suggesting a hazard similar to our financial-variable-based calibration in (38). Their turning points show that bear markets are relatively short and bulls longer, with the equity premium positive in bulls but negative in bears. Our model's cycles have analogous asymmetries.

Another empirical stock-market pattern is that after plunging during bears, price-earnings (PE) ratios tend to work their way back up in bulls (e.g., Shiller [2000, Fig. 1.2]). Our analysis suggests a potential explanation, as follows.

In the model, the value of existing capital plummets when a new GPT arrives. Because the GDP is not immediately affected and capital's share is invariant, the PE ratio falls. Thereafter, a_t converges toward a^* (recall Fig. 4). The price of efficiency units of capital is constant as long as the current episode lasts; hence, the PE ratio rises as the the law of diminishing returns causes its denominator to fall.

A simulation can illustrate. Using detrended variables, GDP_t is $[a_t]^\alpha$. As described, our α may encompass some human capital. Suppose the share of income for physical capital is $\bar{\alpha} = 0.33$ and for human capital $\alpha - \bar{\alpha}$. An equity investment of $1 - \upsilon$ dollars buys control of 1 unit of physical capital. If $r_t = r(a_t)$ is the riskless rate of return, let

$$PE_t = \frac{1 - \upsilon}{\bar{\alpha} \cdot [a_t]^{\alpha - 1} - \delta - \upsilon \cdot r(a_t)} \,. \tag{39}$$

Then a 1000-episode simulation with parameters from Table 2, row 2, yields an average PE ratio of 5.99 at the start of bull markets and 20.25 at the end.[13]

Economywide versus Type-I Agent Consumption. Ait-Sahalia et al. [2004] construct a measure of luxury expenditures which, they argue, provides a proxy for stockholder consumption. They find that the growth rate of luxury expenditures has a standard deviation roughly an order of magnitude larger than that of aggregate consumption and is strongly positively correlated with stock returns. Both outcomes are consistent with our mechanism for the equity premium — which generates large declines in type-I agent consumption during bear markets and recoveries during bulls. The timing of consumption changes in Ait-Sahalia et al's Fig 1B seems consistent as well. Their data covers 1961-2001. The largest changes in luxury consumption are dips, and they find the sharpest dips in the early to mid 1970s, the period Hobijn and Jovanovic analyze, and 2000-2001, a period which Pastor and Veronesi study.

In our simulations with Table 2 parameter values, the wealth of type-I agents falls 60% as a technology shock occurs, and the average drop in their annual consumption is 66.7%. Because of type-I agents' small share of total consumption, however, the maximum shock-induced annual decline in aggregate consumption is only 9.6%. Viewed in a different way, 40% of our model's shocks cause aggregate consumption to drop 5% or more. As the hazard for GPT arrivals is 0.036, the implied average number per century of shock-induced declines in aggregate consumption exceeding 5% is thus 1.4. By way of comparison, Mehra and Prescott [1988] propose benchmark upper bounds for credible annual changes

[13] If a bull market reached $a_t = a^*$, the model's PE-ratio would climb to 35.25.

in aggregate consumption (based on a 100-year data sample), and their figures display a maximal drop of 8.8% and an annual decline greater than 5% in 4 cases.

Equity Premium. We are able to use isoelastic preferences, which simplifies our analysis and may facilitate future extensions of it. What is more, our calibrated CRRAs are moderate (Mankiw and Zeldes [1991, p.105], Mehra and Prescott [1988, p.135]).[14]

7. Short-Run Unevenness

Punctuated technological change makes our model's growth episodic. We now illustrate how uneven that growth can be — and the difficulties we are likely to face if we try to establish, from direct evidence, that a shift in the growth trend has occurred.

Uneven Growth. Cor. 3D shows how to characterize the stationary distribution for k_t. Table 3 presents an illustration based upon a 1000-era (about 28,000-year) simulation with parameters from Table 2, row 2. It summarizes information on the deciles of the stationary distribution. For ease of interpretation, the size of the capital stock is characterized in terms of the capital-to-GDP ratio, $k_t/y_t = k_t/[k_t]^{\alpha}$ (recall Prop. 2).[15] Since the population and labor supply are constant, both k_t and y_t can be thought of as either constant-dollars per capita or per worker.

Table 3. Stationary Distribution for k_t: Capital-to-Output Ratio (k_t/y_t) = 4.01 at Mean k_t

Decile for k_t	Within Decile Range k_t/y_t	Fraction Time in Decile	Range Productivity Growth Rate	Average Productivity Growth Rate
1	0.00-1.38	2.42%	9.11%/yr - ...	12.96%/yr
2	1.38-2.11	7.72	5.01-9.11	6.50
3	2.11-2.70	11.38	3.28-5.01	4.05
4	2.70-3.22	10.72	2.28-3.28	2.75
5	3.22-3.69	10.09	1.61-2.28	1.93
6	3.69-4.12	9.75	1.13-1.61	1.36
7	4.12-4.53	9.77	0.75-1.13	0.93
8	4.53-4.91	10.29	0.45-0.75	0.59
9	4.91-5.28	11.28	0.21-0.45	0.32
10	5.28-5.66	16.59	0.00-0.21	0.09

Source: Model simulation for 1000 technology eras; parameters from Table 2, row 2.

[14] It seems fair to say, however, that in the absence of additions to our model or refinements, we would have difficulty matching significantly higher equity premium values than those of Table 1, rows 1-2.

[15] Although the mean capital-to-output ratio, 4.01, may seem high, our K_t includes human capital (recall Sect. 5). It can differ further from the NIPA concept because of our treatment of depreciation from obsolescence — recall (5) and (6).

Within any technology era J, the time-t productivity growth rate is

$$\dot{Y}_t/Y_t = \dot{y}_t/y_t = \alpha \cdot \dot{k}_t/k_t \,,$$

where y_t is detrended but $Y_t = [x]^J \cdot y_t$ is not. During each technology era, k_t increases, asymptotically approaching k^*. The last means the momentary growth rate decreases to 0 as time spent within an era lengthens. In fact, in Table 3 $\dot{y}_t/y_t$ monotonically declines throughout the stationary distribution's support.

As the Introduction suggests, our model implies that intermittent periods of slow growth are likely to occur, and Table 3 bears that out. Although the long-run average growth rate is $g = 2.10\%$/yr, the mean of the stationary distribution of k_t falls in the 6th decile, where $\dot{y}_t/y_t = 1.13$-1.61%/yr. A linearization of our Euler and budget equations yields a half-life for movement along our phase diagram's saddle path of 14-15 years. A single half-life would carry us from the bottom of decile 6 to the interior of decile 10, where productivity grow is miniscule. Our shocks are Poisson; so, the average waiting time until the next GPT's arrival has an exponential distribution. The mean wait time is $1/\lambda$, which in our case is about 28 yrs. For a 90% chance of a new GPT arrival, the wait time is roughly 64 yrs. The chance of a long period of relative stagnation in the latter part of almost any episode is therefore high.

The last 2 rows of Table 3 show productivity growth rates under 0.5%/yr prevail about 30% of the time, and rates under 1.1%/yr about half the time. We also simulated for 1250 consecutive 20-year intervals, computing the average growth rate for each. Creating bins for 20-year average productivity rates 0-0.5%/yr, 0.5-1.0%/yr, etc., the fullest bin is the first, with 264 intervals. There are 580 20-yr intervals with an average growth rate 0-1.5%/yr.

Conversely, growth tends to be fast in the aftermath of a shock. Suppose k is near k^* when the next GPT arrives. At the arrival, our $b = 0.40$ then causes k to plunge to decile 3 in Table 3. In decile 3, the growth rate is 3.28-5.01%/yr. (Nonetheless, a single half-life will carry the economy all the way from decile 3 to decile 7.)

The spurt of growth can be stronger if several shocks follow in rapid succession. Suppose that at the end of technology era J we have $k_{\bar{t}(J)}$. Then technology $J+1$ begins with $k_{t(J+1)} = (b/x) \cdot k_{\bar{t}(J)}$. If a second shock arrives very quickly, it takes hold with $k_{t(J+2)} \approx (b/x)^2 \cdot k_{\bar{t}(J)}$. As $b < 1$ and $x > 1$, the second shock drives us further into the stationary distribution's support. The effect on output can then be stronger and longer lived.[16]

Detecting Changes in the Average Growth Rate. Provided the factor-share parameter α is constant, (35) shows that λ and b determine g. To emphasize the latter's endogeneity in our analysis, henceforth we write $g_t \equiv \dot{y}_t/y_t$ and refer to the long-run av-

[16] Put another way, the TFP gain from a new GPT achieves its full expression when k_t has a chance to rise all the way to the stationary level k^*. When, however, a new shock interrupts the diffusion of its predecessor, the truncated portion of the predecessor's TFP gain is not lost to the economy. Rather, the residual is manifested jointly with the gains to future GPT arrivals.

erage g_t as $\mathbb{E}[g]$. A fall in λ lengthens the time between seminal inventions, reducing $\mathbb{E}[g]$; an increase in b lessens the TFP gain from each revolutionary change, again reducing $\mathbb{E}[g]$.

Though we can look for changes in λ and b by examining data on the magnitude of shocks and their spacing (recall the Introduction), the generally small magnitude of our hazard is likely to lead to difficulties in practice. As noted above, the average duration of a technology era is about 28 years in our calibrations. A single unusually long era could be due to random chance. If we want to establish with some assurance that the parameter λ has declined, we will need to observe a number of eras and average their lengths. (If b were a random variable and its mean was our chief concern, the same problem would arise.)

Table 4. Duration of Single-Sided Test of Null, $\lambda = 0.0359$, and Power Against the Specific Alternatives $\lambda^{ALT} = 0.5 \times \lambda$ & $0.75 \times \lambda$

Reject Null, 10% Significance Level, If Number GPT Arrivals During Test is $< i$, where $i =$	Approximate Test Length	Power Against λ^{ALT}, where	
		$\lambda^{ALT} = 0.5 \times \lambda$	$\lambda^{ALT} = 0.75 \times \lambda$
1	64 yrs	32%	18%
3	149	50	24
9	364	79	36
18	661	94	50
54	1782	100	80

Source: See text.

Let the null hypothesis be that λ remains unchanged and the alternative be that λ has declined. Consider Table 4. Start at time 0. The density for the arrival time, say, t, of the next GPT is $\lambda \cdot e^{-\lambda \cdot t}$. The delay in the next GPT's arrival implying that we can reject the null with a 10% chance of a type-1 error is about 64 years (Table 4, row 1).

In the preceding test, even if the hazard has dropped by half (which (35) shows would correspond to a halving of $\mathbb{E}[g]$), the chance of a type-II error is about 68%. To increase the power of the test, we could monitor a larger sample of GPT arrivals, switching our analysis to the gamma distribution. Suppose λ has declined by 50%. Then Table 4 shows that to reject the null at the 10% significance level while holding the probability of a type-II error to 50%, we need almost 150 years of observations.

In sum, the model, by nature, generates uneven growth. Long periods of slow progress will not be surprising. If the periodicity and/or magnitude of technology shocks does shift, detecting the change and establishing its statistical significance from measures of annual productivity, or direct observations of the shocks themselves, may take many decades — or centuries.

8. An Indirect Approach

Table 4 warns that detecting a change in trend growth by analyzing momentary rates g_t may require an extremely long time series of observations. The current section presents an alternative, indirect approach that is capable of yielding faster results.

Our idea as follows. Suppose households in the economy have access to granular information that is unavailable to external analysts. If the households receive news that the parameter λ has fallen and/or b risen, they will anticipate a reduction in the frequency and/or severity of future bear markets — and a slowdown in long-term growth. The former should reduce the equity premium, an adjustment that can take place immediately. We propose to use changes in the equity premium as early alerts of long-run changes in growth.

Assuming the economy is in equilibrium, the measured equity premium for a given period reflects a combination of what agents think the stochastic process for technology shocks is, which determines their demand for assets, and how many GPT arrivals have occurred during the interval in question, with each shock contributing a negative spike, lowering the realized average return on equities. The magnitude of what we call the bull-market equity premium (BMEP) — the equity premium computed for bull-market years alone — excludes the spikes. Notice that spikes have large magnitudes but occur rarely; therefore, unless we are studying a very long interval, an atypically large (small) number of spike realizations will make our estimate of the stationary equity premium too low (high). We focus on the BMEP to avoid this source of noise.

Our theoretical analysis gives us conditions for a steady-state equilibrium in which the long-run average bull-market equity premium, $\mathbb{E}[BMEP]$, is constant. Since the momentary premium $BMEP_t$ is observable, we will be interested in a regression formulation

$$BMEP_t = \mathbb{E}[BMEP] + \epsilon_t \,, \tag{40}$$

where the error ϵ_t has mean 0 by construction. Given Poisson shocks and rational expectations, according to the model systematic deviations between $BMEP_t$ and $\mathbb{E}[BMEP]$ should depend on k_t. The latter dependence seems slight in our calibrated simulations, however. In Table 3, for instance, $BMEP_t$ monotonically declines as k_t rises, but does so very slowly: it is 8.71%/yr in the first decile of the stationary distribution of k_t, 8.64 in the second, ..., and 8.32 in the tenth. In our analysis, we proceed, therefore, without regressors reflecting k_t.[17]

Table 5 presents simulation results showing the effect of a decrease in λ, or increase in b, on $\mathbb{E}[BMEP]$. For a fixed α, (35) implies that such parameter changes reduce $\mathbb{E}[g]$ — as the table's first column illustrates. Table 5's remaining columns confirm our conjecture above that such changes also cause a reduction in $\mathbb{E}[BMEP]$. In fact, the changes in $\mathbb{E}[g]$ and $\mathbb{E}[BMEP]$ move together virtually 1-for-1 in all rows. Our approach uses a version of (40) to search for changes in $\mathbb{E}[BMEP]$, relating the latter, through Table 5, to $\mathbb{E}[g]$.

Data. The remainder of this section uses the data of Appendix 3. Where designated, however, we drop the recession years 1938-39 and 2007-09 as outliers.

T-tests. We test for a decline in $\mathbb{E}[BMEP]$ as follows.

Think of a step downward that occurs during the time interval $[t_1, t_2]$. Suppose y_i , $i = 1, ..., m$, are observations of $BMEP_t$ for $t < t_1$ and x_j , $j = 1, ..., n$, are the same for $t > t_1$. Assume y_i are iid $\mathcal{N}(\mu_y, \sigma)$ and x_j are iid $\mathcal{N}(\mu_x, \sigma)$. Our null hypothesis, H_0, is that

[17] None of our preliminary empirical tests show evidence of autocorrelation for the error in (40), a possibility that we disregard in the rest of our analysis.

Table 5. Sensitivity of the Steady-State Equilibrium to Changes in λ and b

New Parameter Value	Growth Rate $\mathbb{E}[g]$ (Old Value 2.10%/yr)	Equity Premium $\mathbb{E}[EP]$ (Old Value 5.53%/yr)	Bull-Market Equity Premium $\mathbb{E}[BMEP]$ (Old Value 9.10%/yr)
Parameter Adjusted: λ. Old Value: $\lambda_0 = 0.0357$. New Value: λ_1.			
$\lambda_1 = 0.9 \times \lambda_0$	1.89%/yr	4.96%/yr	8.18%/yr
$\lambda_1 = 0.8 \times \lambda_0$	1.68	4.39	7.25
$\lambda_1 = 0.7 \times \lambda_0$	1.47	3.83	6.33
$\lambda_1 = 0.6 \times \lambda_0$	1.26	3.27	5.41
$\lambda_1 = 0.5 \times \lambda_0$	1.05	2.71	4.50
Parameter Adjusted: b. Old Value: $b_0 = 0.40$. New Value: b_1.			
$b_1 = 1.1 \times b_0$	1.88%/yr	4.45%/yr	7.78%/yr
$b_1 = 1.2 \times b_0$	1.68	3.57	6.67
$b_1 = 1.3 \times b_0$	1.50	2.86	5.72
$b_1 = 1.4 \times b_0$	1.33	2.28	4.89
$b_1 = 1.5 \times b_0$	1.17	1.80	4.18

Source: Simulations as in Table 3 run for various values of λ and b. See text.

the economy has attained equilibrium and $\mu_y = \mu_x$. We form a statistic (e.g., Hogg and Craig [1978])

$$\tilde{T} = \frac{\sqrt{\frac{m \cdot n}{m+n}} \cdot (\bar{y} - \bar{x})}{\sqrt{\frac{\sum_1^m (y_i - \bar{y})^2 + \sum_1^n (x_j - \bar{x})^2}{m+n-2}}}, \quad \bar{y} \equiv \sum_1^m y_i/m\,, \ \bar{x} \equiv \sum_1^n x_j/n\,. \tag{41}$$

Let

$$\Lambda \equiv \sqrt{\frac{m \cdot n}{m+n}} \cdot \frac{\mu_y - \mu_x}{\sigma}\,. \tag{42}$$

Under H_0, $\Lambda = 0$ and $\tilde{T}$ has a central t distribution with $m + n - 2$ degrees freedom. Call the distribution function $D_{0,m+n-2}(.)$. Our alternative hypothesis, H_1, has the growth rate falling during the interval $[t_1, t_2]$, keeping its lower level for $t > t_1$. Under H_1, $\mu_y - \mu_x > 0$; so, $\Lambda > 0$. Then $\tilde{T}$ has noncentral t distribution, with distribution function $D_{\Lambda,m+n-2}(.)$ and noncentrality parameter Λ.

Table 6 provides comparisons to Table 4. We employ a single-tailed test, testing the null at the 10% significance level. Table 6 presents power calculations: if $h_{0.10}$ is the critical value for testing H_0, it solves $D_{0,n+m-2}(h_{0.10}) = 0.90$; the test's power against H_1 is $1 - D_{\Lambda,n+m-1}(h_{0.10})$. For conformity with Table 4, we set $\mu_y - \mu_x = 0.5 \cdot 0.0844$ in (42). In the denominator of (42), we set $\sigma = \hat{\sigma}_{1916-2020}$, the sample standard deviation for

Table 6. Power for Test of H_0 Against H_1 [a]

$m \setminus n$	$n = 10$	20	30	40	50
$m = 10$	0.2759	0.3148	0.3334	0.3444	0.3516
20	..[b]	0.3825	0.4197	0.4433	0.4596
30	..	..	0.4707	0.5045	0.5285
40	..	..	..	0.5462	0.5763
50	..	..	..	..	0.6113

Source: See text.

(a) $\sigma = \hat{\sigma}_{1916-2020}$ with data from Appendix 3, outliers and bear markets removed. $\mu_y - \mu_x = 0.0844/2$. See text.

(b) Table's lower triangle omitted — all cell entries symmetric in m , n.

Appendix 3's data. We consider various numbers of blocks of 10 observations before and after $[t_1, t_2]$. (The table entries do not depend on the magnitude of $t_2 - t_1$.)

A comparison of Tables 4 and 6 reveals the potential gains from our approach. In the former, we needed 64 years to complete a test with a power of 32%. In Table 6, on the other hand, if $[t_1, t_2]$ covers 5 years and we have 20 years of background data prior to t_1, we can attain roughly the same power with 15 years of data post t_1. In Table 4, we reach of power of 50% with 149 years of data following a reduction in $\mathbb{E}[g]$, whereas in Table 6 with 40 years of background data, we need 35 years of new observations.

Supremum Estimators. In practice, if a growth slowdown occurs, we would like to learn its date. Moreover, if several changes in $\mathbb{E}[g]$ occur, we need to establish the date of the first to know when to begin analysis of the second. We now turn, therefore, to a supremum estimator (e.g., Andrews [1993], Hansen [2000]).

The 1920s (i.e., the "roaring twenties") are widely perceived to have been a period of energetic change. We first test for evidence of a decline in $\mathbb{E}[g]$ in that time frame. Then we consider the possibility of a recent change.

For our sup estimator, we set $t_2 - t_1 = 4$. Let the sample be $[t_0, t_3]$, with bear-market years and outliers omitted. We calculate (41) for a sequence of values of t_1 — say, $t_1 > t_0 + 10$ to $t_1 + 4 \leq t_3 - 10$ — choosing the highest resulting $\tilde{T}$. As all calculations in a given sequence have the same $m + n$, the $\sqrt{m \cdot n/(m+n)}$ term of (41) implies our test will have its highest power when $m = n$. We require $m, n \geq 5$ or 10 to keep our critical values (see below) from getting extremely high.

We first compute the critical value for testing H_0. We draw independent random $\mathcal{N}(0,1)$ realizations for $t = t_0, ..., t_3$; compute (41) for each interval of change, $[t_1, t_2]$; and, set T^* equal to the highest $\tilde{T}$ (over the sequence of change intervals) and t^* to the corresponding t_1. Repeating for $k = 1, ..., 1000$ such vectors and ordering T_k^* from lowest to highest, T_{900}^* gives the critical value $h_{0.10}$. We then calculate the sup $\tilde{T}$ for our empirical vector $\vec{BMEP}_t$, and compare it with $h_{0.10}$ to test H_0.

Finally, we draw 3 new sets of 1000 random vectors as above and simulate a reduction

Table 7. Supremum Estimation Results: Single-Tailed Tests with $t_2 - t_1 = 4$

Row	Domain of Analysis [a]	Domain for Change Intervals	Value sup $\tilde{T}$	Change Interval for sup $\tilde{T}$	Critical Value 10% (5%) Test
1	1916-2020	1926-2014	2.5010	[1939,1943]	2.3922 (2.7394)
2	1916-2020	1935-2010	2.5010	[1939,1943]	2.2545 (2.6346)
3	1916-1980	1935-1968	2.3799	[1957,1961]	2.1695 (2.5409)
4	1950-2020	1960-2010	1.4846	[1960,1964]	2.1491 (2.5184)
5	1960-2020	1970-2010	-1.3337	[1976,1980]	2.0955 (2.4977)
Addendum 1: Other Significant $\tilde{T}$ Values and Their Change Intervals; Rows as above					
1, 4-5	na				
2	2.2741 [1940,1944]				
3	2.3144 [1939,1943], 2.2075 [1951,1955], 2.2105 [1952,1956], 2.2540 [1954,1958], 2.3562 [1955,1959], 2.3626 [1956,1960], 2.2028 [1959,1963], 2.1894 [1960,1964]				
Addendum 2: Power (%) to Detect a Decline $(\mu_y - \mu_x)/\hat{\sigma} = 0.5 \cdot 0.0844/\hat{\sigma}$ that occurs (i) Start Year + 11, (ii) Midyear, or (iii) End Year - 10;[b] Rows as above					
1	(i) 20%, (ii) 54%, (iii) 22%				
2	(i) 31%, (ii) 58%, (iii) 32%				
3	(i) 25%, (ii) 39%, (iii) 24%				
4	(i) 30%, (ii) 47%, (iii) 30%				
5	(i) 28%, (ii) 39%, (iii) 29%				

Source: See text.

(a) Sample excludes bear markets and outliers. See text.

(b) We use $\mu_y - \mu_x = 0.0844/2$ to match Table 4, and we set $\hat{\sigma}$ from (43).

in $\mathbb{E}[g]$. The reduction is $0.5 \cdot 0.0844/\hat{\sigma}$ with the numerator corresponding to the change in Tables 4 and 6, and

$$\hat{\sigma} = \big(\sum_{1}^{m} (y_i - \bar{y})^2 + \sum_{1}^{n} (x_j - \bar{x})^2\big)/(m + n - 2)\,, \tag{43}$$

with the m, n, and $[t_1, t_2]$ from our sup $\tilde{T}$ estimation with $B\vec{M}EP_t$ above. For the first set of 1000 random vectors, we make the downward adjustment for all $t \geq t_0 + 11$; for the second, we do so for all $t \geq$ the sample midyear; and, for the third set, for all $t \geq t_3 - 10$. Performing a sup estimation on each (adjusted) random vector, we derive power assessments.

Table 7 presents results.

Results 1920s. Table 7, rows 1-2, reject H_0 at the 10% significant level. We chose a starting point for the search of dates near the sample's boundary in row 1 in order to be able to spot a shift during the 1920s. In both rows, however, the sup estimator points to a reduction in the economy's technological dynamism 1939-43.

Row 3 lowers t_3 to 1980. When we truncate the sample, somewhat surprisingly we

find $t^* = 1957$. A graph of the $\vec{BMEP}_t$ data (see Appendix 3) provides an explanation. Namely, the data reaches a nadir 1960-80 and then edges higher. As we truncate the sample at 1980, $\bar{x}$ thus falls relative to $\bar{y}$. The disposition of observations to $\bar{y}$ and $\bar{x}$ then changes. (We also find that $\tilde{T}(t_1)$ exceeds $h_{0.10}$ for a number of t_1 values at which it was marginally below the critical level in row 2.)

The exclusion of outliers does not change the test verdicts. When, however, we include 1937-38 and 2007-09, we find $t^* = 1937$ in rows 1-3 — 1937 being a recession year with huge capital losses in the stock market. In fact, $t = 1937$ becomes the exclusive significant outcome for rows 1-3.

One interpretation for the rejection of H_0 in the early 20th century would be that λ fell and/or b rose, that is to say, that growth did slow down. A second would be that the economy had not yet attained a steady-state equilibrium during the interval 1919-1929 — so that presuming it did thereafter, we reject the hypothesis of a lack of change 1916-2020. Arguably financial variables rather than momentary growth rates seem askew in the 1920s (see, for example, Maddison [1991, Table 3.13]). As explained in Section 5, there had been a substantial increase in the pace of technological progress relative to the previous century. Moreover, we identify 2 GPT arrivals in the first 15 years of our sample but only 2 more over the next 90 — making it easy to believe that agents over-estimated λ in the 1920s. Finally, a modern stock market was only beginning to emerge — perhaps, again, suggesting a rational expectations equilibrium had not had time to be established.

Results After 1950. Our primary interest is the possibility of a slowdown of growth in the interval 1950-2020.

In Table 7, rows 4-5 show no sign of a significant slowdown for $t_0 = 1950$ or 1960. The t_0 values complement the $[t_1, t_2]$ findings from rows 1-3. Raising t_0 still further does not provide a significant $\sup \tilde{T}$. As noted, $BMEP_t$ tends to drift up, rather than down, in later years; hence, a higher t_0 often leads to $\bar{y} < \bar{x}$ — as is already evident in row 5.

In general, the power assessments at the bottom of Table 7 offer a pleasant surprise: near the sample boundary, they resemble Table 4, row 1; at the sample's center, however, they are often reminiscent of Table 4, row 2.

9. Conclusion

This paper studies ways of detecting changes in the long-run dynamism of the economy. For our analysis, we present a model in which new general purpose technologies are the fundamental driver of progress, their arrivals are Poisson events, and agents have rational expectations.

The model is highly aggregative and, in its present form, includes only 1 type of shock. Nonetheless, we show that it can illustrate (i) the likely volatility of the momentary growth rate even when its long-run average is stationary; (ii) the difficulty of establishing a shift in the long-run growth rate from direct observations; and, (iii) the value of monitoring financial variables for clues of market-participant beliefs about the future.

We ultimately focus our attention on what we call the bull-market equity premium, arguing that if agents within the economy begin to predict a slowdown in long-term growth, $\mathbb{E}[BMEP]$ should decrease. When we examine US data 1916-2020, we find a possible signal for such a decline after the 1920s, but nothing for the period 1950-2020.

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APPENDIX 1: Proofs

Proof Lemma 1: Noting that algebra shows $\theta \cdot [x]^{\alpha-1} = 1$, we have

$$\begin{aligned}
\dot{A}_s &= \eta \cdot [\theta]^{J+1} \cdot [A_s]^{\alpha} - \delta \cdot A_s - C_s && \textit{from eq } 16b \\
\iff \frac{\dot{A}_s}{x} &= \frac{1}{x} \cdot \eta \cdot \frac{[\theta]^{J+1} \cdot [A_s]^{\alpha}}{\theta \cdot [x]^{\alpha-1}} - \delta \frac{A_s}{x} - \frac{C_s}{x} && \textit{division by } x \\
\iff \frac{\dot{A}_s}{x} &= \eta \cdot [\theta]^{J} \cdot [\frac{A_s}{x}]^{\alpha} - \delta \cdot \frac{A_s}{x} - \frac{C_s}{x} \,. && (A1)
\end{aligned}$$

And,

$$\begin{aligned}
\dot{C}_s &= \frac{C_s}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^{J+1} \cdot [A_s]^{\alpha-1} - (\delta + \rho)\right] && \textit{from eq } 16a \\
\iff \frac{\dot{C}_s}{x} &= \frac{\frac{C_s}{x}}{1-\gamma} \cdot \left[\alpha \cdot \frac{[\theta]^{J+1} \cdot [A_s]^{\alpha-1}}{\theta \cdot [x]^{\alpha-1}} - (\delta + \rho)\right] && \textit{division by } x \\
\iff \frac{\dot{C}_s}{x} &= \frac{\frac{C_s}{x}}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^{J} \cdot [\frac{A_s}{x}]^{\alpha-1} - (\delta + \rho)\right] . && (A2)
\end{aligned}$$

The saddle path in the phase diagram for (A_s, C_s) determines $C_s = \Phi^0(A_s, J+1)$. If (16a-b) hold for $(A_s, C_s)\,,\ s \geq 0$, then (A1)-(A2) show they continue to hold if we shift the technology level from $J+1$ to J and divide both A_s and C_s by x. Hence, result (i) holds.

We have

$$\begin{aligned}
\dot{A}_s &= \eta \cdot [\theta]^{J+1} \cdot [A_s]^{\alpha} - \delta \cdot A_s - \Phi^0(A_s, J+1)\,,\ A_0 = A\,, && \textit{from } 16b \\
\iff \dot{A}_s &= \eta \cdot \frac{[\theta]^{J+1} \cdot [A_s]^{\alpha}}{\theta \cdot [x]^{\alpha-1}} - \delta \cdot A_s - x \cdot \Phi^0(\frac{A_s}{x}, J)\,,\ A_0 = A\,, && \textit{from result } i \\
\iff \frac{\dot{A}_s}{x} &= \eta \cdot [\theta]^{J} \cdot \frac{x}{x} \cdot [\frac{A_s}{x}]^{\alpha} - \delta \cdot \frac{A_s}{x} - \frac{x \cdot \Phi^0(\frac{A_s}{x}, J)}{x}\,,\ \frac{A_0}{x} = \frac{A}{x} && \textit{division by } x \\
\iff \frac{\dot{A}_s}{x} &= \eta \cdot [\theta]^{J} \cdot [\frac{A_s}{x}]^{\alpha} - \delta \cdot \frac{A_s}{x} - \Phi^0(\frac{A_s}{x}, J)\,,\ \frac{A_0}{x} = \frac{A}{x}\,. && (A3)
\end{aligned}$$

Let $A_s = \Psi^0(s, J+1, A)\,,\ s \geq 0\,,$ solve the first line of (A3). Then comparing the first and last lines of (A3), we can see that

$$\Psi^0(s, J, \frac{A}{x}) = \frac{\Psi^0(s, J+1, A)}{x}\,. \qquad (A4)$$

Using the notation $V^0(A, J) \equiv v^0(A, \vec{K}(J), J)|_{\vec{K}=\vec{J}}$, we then have

$$
\begin{aligned}
V^0(A, J+1) &= \int_0^\infty e^{-\rho \cdot s} \cdot u\Big(\Phi^0\big(\Psi^0(s, J+1, A), J+1\big)\Big)\, ds \\
&= \int_0^\infty e^{-\rho \cdot s} \cdot u\Big(\Phi^0\big(x \cdot \Psi^0(s, J, \frac{A}{x}), J+1\big)\Big)\, ds \qquad \textit{from A4} \\
&= \int_0^\infty e^{-\rho \cdot s} \cdot u\Big(x \cdot \Phi^0\big(\frac{x \cdot \Psi^0(s, J, \frac{A}{x})}{x}, J\big)\Big)\, ds \quad \textit{from result i} \\
&= [x]^\gamma \cdot \int_0^\infty e^{-\rho \cdot s} \cdot u\Big(\Phi^0\big(\Psi^0(s, J, \frac{A}{x}), J\big)\Big)\, ds \qquad \textit{algebra} \\
&= [x]^\gamma \cdot V^0(\frac{A}{x}, J)\,, \qquad (A5)
\end{aligned}
$$

completing the proof. ∎

Proof Lemma 2: The proofs of (A1) and (A3) remain as before. For (A2), we add additional right-hand side terms to the [.] in the first line, which progress in the remaining 2 lines, as follows:

$$
\begin{aligned}
&\lambda \cdot b \cdot \frac{u'\big(x \cdot \Phi^1(\frac{b \cdot A_t}{x}, J+1)\big)}{u'(C_t)} - \lambda \\
=&\lambda \cdot b \cdot \frac{u'\big(\Phi^1(\frac{b \cdot A_t}{x}, J+1)\big)}{u'\big(\frac{C_t}{x}\big)} - \lambda \\
=&\lambda \cdot b \cdot \frac{u'\big(x \cdot \Phi^1(\frac{b \cdot A_t / x}{x}, J)\big)}{u'\big(\frac{C_t}{x}\big)} - \lambda\,.
\end{aligned}
$$

For (A5), we replace $e^{-\rho \cdot s}$ with $e^{-(\lambda+\rho) \cdot s}$ and again add an additional right-hand side term that progresses in the remaining 2 lines, as follows:

$$
\begin{aligned}
&\int_0^\infty e^{-(\lambda+\rho) \cdot s} \cdot \lambda \cdot V^0\big(b \cdot \Psi^1(s, J+1, A), J+2\big)\, ds \\
&\quad = \int_0^\infty e^{-(\lambda+\rho) \cdot s} \cdot \lambda \cdot V^0\big(b \cdot x \cdot \Psi^1(s, J, \frac{A}{x}), J+2\big), ds \qquad \textit{from A4} \\
&\quad = [x]^\gamma \cdot \int_0^\infty e^{-(\lambda+\rho) \cdot s} \cdot \lambda \cdot V^0\big(b \cdot \Psi^1(s, J, \frac{A}{x}), J+1\big)\, ds\,, \quad \textit{from Lem. 1, part ii}
\end{aligned}
$$

completing the proof. ∎

Proof of Prop. 3A: Lemmas 1-2 and their extension for $m \geq 2$ imply

$$\Phi^m(A, J) = x \cdot \Phi^m(\frac{A}{x}, J-1) = ... = [x]^J \cdot \Phi^m(\frac{A}{[x]^J}, 0)\,, \tag{A6}$$

establishing result (ii). We have

$$\begin{aligned} [x]^J \cdot \Phi(\frac{A}{[x]^J}, 0) &= [x]^J \cdot \lim_{m\to\infty} \Phi^m(\frac{A}{[x]^J}, 0) && \textit{from cond. C2} \\ &= \lim_{m\to\infty} \Phi^m(A, J)\,, && \textit{from A6} \end{aligned}$$

showing $\Phi(A, J) \equiv \lim_{m\to\infty} \Phi^m(A, J)$ exists. That establishes result (i). ∎

Lemma 4 and Proof

Lemma 4: *Suppose we have a Solow model:*

$$\dot{k_t} = \sigma \cdot Z \cdot [k_t]^\alpha - \delta \cdot k_t\,, \quad k_0 \textit{ given}\,,$$

where $\sigma \in (0,1)$, $\alpha \in (0,1)$, and $k_0 > 0$. Let the solution to the differential equation be $k_t = \psi(t, k_0)$. The stationary solution is

$$k^* = [\frac{\sigma \cdot Z}{\delta}]^{\frac{1}{1-\alpha}}\,.$$

Then
(i) For any $\bar{k} \in (0, k^*)$, we have a finite $T = T(\bar{k})$ such that $\psi(t, k_0) \geq \bar{k}$ all $t \geq T(\bar{k})$,
(ii) $T(\bar{k}) \to 0$ as $\bar{k} \to 0$.

Proof: We derive a closed form solution for $\psi(.)$ and then establish (i)-(ii).
Step 1. Set $f_t = k_t \cdot e^{\delta \cdot t}$. Then

$$k_t = f_t \cdot e^{-\delta \cdot t}\,. \tag{A7}$$

From the Solow differential equation and (A7), we have

$$\begin{aligned} \dot{f}_t \cdot e^{-\delta\cdot t} - \delta \cdot f_t \cdot e^{-\delta\cdot t} &= \sigma \cdot Z \cdot [f_t]^\alpha \cdot e^{-\alpha\cdot\delta\cdot t} - \delta \cdot f_t \cdot e^{-\delta\cdot t} \\ \Longrightarrow \dot{f}_t &= \sigma \cdot Z \cdot [f_t]^\alpha \cdot e^{\delta\cdot(1-\alpha)\cdot t}\,. \end{aligned} \tag{A8}$$

Step 2. For positive constants L and B, set $F_t = [f_t/L]^{1/B}$. Then

$$f_t = L \cdot [F_t]^B\,. \tag{A9}$$

So, (A8) implies

$$L \cdot B \cdot [F_t]^{B-1} \cdot \dot{F}_t = \sigma \cdot Z \cdot [L]^\alpha \cdot [F_t]^{\alpha\cdot B} \cdot e^{\delta\cdot(1-\alpha)\cdot t}\,. \tag{A10}$$

Let

$$B \equiv \frac{1}{1-\alpha} \quad \text{and} \quad L \equiv [\sigma \cdot Z \cdot (1-\alpha)]^{\frac{1}{1-\alpha}} . \tag{A11}$$

Given B from (A11), we can cancel $[F_t]$ from both sides of (A10) — provided $F_t > 0$. Given L from (A11), we then have

$$\dot{F}_t = e^{\delta \cdot (1-\alpha) \cdot t} . \tag{A12}$$

Step 3: Applying the fundamental theorem of calculus to (A12):

$$F_t = F_0 + \int_0^t e^{\delta \cdot (1-\alpha) \cdot s} \, ds = F_0 + \frac{1}{\delta \cdot (1-\alpha)} \cdot [e^{\delta \cdot (1-\alpha) \cdot t} - 1] . \tag{A13}$$

Using (A7) and (A9), (A13) implies

$$\begin{aligned} [\frac{k_t \cdot e^{\delta \cdot t}}{L}]^{1/B} &= [\frac{k_0}{L}]^{1/B} + \frac{e^{\delta \cdot (1-\alpha) \cdot t} - 1}{\delta \cdot (1-\alpha)} \\ \iff [k_t \cdot e^{\delta \cdot t}]^{1-\alpha} &= [k_0]^{1-\alpha} + \frac{\sigma \cdot Z}{\delta} \cdot [e^{\delta \cdot (1-\alpha) \cdot t} - 1] \qquad \textit{from A11} \\ \iff [k_t]^{1-\alpha} &= [k_0]^{1-\alpha} \cdot e^{-\delta \cdot (1-\alpha) \cdot t} + [k^*]^{1-\alpha} \cdot [1 - e^{-\delta \cdot (1-\alpha) \cdot t}] . \end{aligned}$$

Thus, $[\psi(t, k_0)]^{1-\alpha} \geq [k^*]^{1-\alpha} \cdot [1 - e^{-\delta \cdot (1-\alpha) \cdot t}]$ any $k_0 > 0$. So,

$$\psi(t, k_0) \geq k^* \cdot [1 - e^{-\delta \cdot (1-\alpha) \cdot t}]^{\frac{1}{1-\alpha}} \quad \text{any} \quad k_0 > 0 . \tag{A14}$$

Step 4: Fix any $\bar{k} \in (0, k^*)$. We can see there is a unique $T(\bar{k}) > 0$ such that

$$\bar{k} = k^* \cdot [1 - e^{-\delta \cdot (1-\alpha) \cdot T(\bar{k})}]^{\frac{1}{1-\alpha}} . \tag{A15}$$

For $k_0 > 0$, $k_t = \psi(t, k_0)$ converges monotonically to k^* as t rises. Hence, for $k_0 \geq \bar{k}$, $\psi(t, k_0) \geq \bar{k}$ all $t \geq 0$. For $k_0 \in (0, \bar{k}]$ and $t \geq T(\bar{k})$, using (A14) and then (A15), we have

$$\psi(t, k_0) \geq \psi\big(T(\bar{k}), k_0\big) \geq k^* \cdot [1 - e^{-\delta \cdot (1-\alpha) \cdot T(\bar{k})}]^{\frac{1}{1-\alpha}} = \bar{k} , \tag{A16}$$

establishing result (i). From (A15), $\bar{k} \to 0 \iff T(\bar{k}) \to 0$, establishing result (ii). ∎

Proof of Prop. 3D: Letting P be the transition function of (28) and $\mathcal{A} \equiv (0, a^*]$ the invariant set (see (26)) for P, we establish the following version of the hypothesis of Doeblin (Doob [1953]): Let $\mu(.)$ be Lebesgue measure. We can find $\epsilon > 0$ such that if $\mu(\mathcal{Z}) < \epsilon$ for measurable set $\mathcal{Z}$ implies $P(a, \mathcal{Z}) \leq 1 - \epsilon$ all $a \in \mathcal{A}$.

We work with detrended asset levels. Without loss of generality, fix current technology $J = 0$ and the starting time for the current era to $t = 0$. The current era begins with asset level a_0; the previous era ended with asset level a_{00}; and, a_0 , $a_{00} \in \mathcal{A}$.

Step 1. Consider the upper bound, a^*, of $\mathcal{A}$. Note that $P(a^*, [b \cdot a^*/x, a^*]) = 1$ because a_s increases during an era. Because a_s asymptotically converges to a^* but never reaches it, $P(a^*, [a^*]) = 0$. The continuity of $P(a^*, [\Delta, a^*])$ with respect to Δ implies we can choose $\Delta_1 \in (b \cdot a^*/x, a^*)$ with $P(a^*, [\Delta_1, a^*]) = 1/6$. The monotonicity of a_s in its era then implies

$$a_{00} \leq a^* \Longrightarrow P(a_{00}, [\Delta_1, a^*]) \leq P(a^*, [\Delta_1, a^*]) = 1/6\,. \tag{A17}$$

Step 2. Consider the left bound, 0, for $\mathcal{A}$. The RAI's average propensity to consume is $APC = C_s/(\eta \cdot Y_s)$. Prop. 2 shows $Y_s = [a_s]^\alpha$. Use the notation $\widehat{C}_s \equiv \dot{C}_s/C_s$. Setting $m = \infty$ in (21a-b), we have

$$\widehat{APC} = \hat{C}_s - \alpha \cdot \hat{a}_s \geq \frac{1}{1-\gamma} \cdot \left[[\alpha \cdot [a_s]^{\alpha-1} - (\delta + \lambda + \rho)] - \alpha \cdot [\eta \cdot [a_s]^{\alpha-1} - \delta] \right]. \tag{A18}$$

Given this proposition's condition (i), (A18) implies there is $\tilde{a}$ with our model's $APC(a)$ increasing all $a \in (0, \tilde{a}]$. So, $APS(a) = 1 - APC(a)$ is decreasing on the same domain. In other words,

$$APS(a) \geq APS(\tilde{a}) \quad \forall a \in (0, \tilde{a}]\,. \tag{A19}$$

Turn to Lem. 4. Use the notation $a_s^S = k_s^S$ for the Lemma's Solow model $k_s^S = \psi(s, k_0^S)$ and k^{S*} for its stationary value. For the Lemma's Solow model, fix $\sigma = APS(\tilde{a})$ and $Z = \eta$. Let P^S be the version of transition function (28) with the Lemma's ψ in place of Ψ. Lem. 4 shows that for each $\Delta \in (0, k^{S*})$, we have $T(\Delta) > 0$ such that $\psi(s, a_0^S) \geq \Delta$ all $s \geq T(\Delta)$ and all $a_0^S \in \mathcal{A}$. *Ceteris paribus*, a larger APS leads to faster growth in a_s; hence,

$$P\big(a_{00}, [0, \min\{\tilde{a}, \Delta\}]\big) \leq P^S\big(a_{00}, [0, \min\{\tilde{a}, \Delta\}]\big) \leq \int_0^{T(\Delta)} \lambda \cdot e^{-\lambda \cdot s}\, ds\,.$$

Recalling Lem. 4, part (ii), choose Δ so small that the last term in the preceding expression is $\leq 1/6$. Set $\Delta_2 \equiv \min\{\tilde{a}, \Delta\}$. Then

$$P\big(a_{00}, [0, \Delta_2]\big) \leq 1/6 \quad \forall a_{00} \in \mathcal{A}\,. \tag{A20}$$

Step 3. Consider the interval $[\Delta_2, \Delta_1]$. In our model's phase diagram, $\dot{a}_s$ from differential equation (24) equals the vertical distance at a_s between the A-isocline and the saddle path. (C4) ensures the distance is positive for $a \in (0, a^*)$. Since the isocline and the saddle path

are continuous, $\dot{a}_s$ takes a minimum value, say, $\dot{a}^{min} > 0$, on $[\Delta_2, \Delta_1]$. Note that $\lambda \cdot e^{-\lambda \cdot s}$ is decreasing in s. Hence for any $\mathcal{Z} \subset [\Delta_2, \Delta_1]$, we have

$$P(a_{00}, \mathcal{Z}) \leq \int_0^{\mu(\mathcal{Z})/\dot{a}^{min}} \lambda \cdot e^{-\lambda \cdot s}\, ds \quad \forall a_{00} \in \mathcal{A}\,. \tag{A21}$$

Choose $\epsilon_3 > 0$ so small that $\mu(\mathcal{Z}) \leq \epsilon_3$ implies

$$\int_0^{\mu(\mathcal{Z})/\dot{a}^{min}} \lambda \cdot e^{-\lambda \cdot s}\, ds \leq 1/6\,. \tag{A22}$$

Step 4. Set $\epsilon \equiv \min\{\epsilon_3, 0.5\}$. Consider $\mathcal{Z}$ with $\mu(\mathcal{Z}) \leq \epsilon$. Let

$$\mathcal{Z}_1 \equiv \mathcal{Z} \cap [0, \Delta_2] \quad \mathcal{Z}_2 \equiv \mathcal{Z} \cap [\Delta_2, \Delta_1] \quad \mathcal{Z}_3 \equiv \mathcal{Z} \cap [\Delta_1, a^*]\,.$$

Then

$$\begin{aligned} P(a_{00}, \mathcal{Z}) &= P(a_{00}, \mathcal{Z}_1) + P(a_{00}, \mathcal{Z}_2) + P(a_{00}, \mathcal{Z}_3) \\ &\leq P(a_{00}, [0, \Delta_2]) + P(a_{00}, \mathcal{Z}_2) + P(a_{00}, [\Delta_1, a^*]) \\ &\leq 1/6 + P(a_{00}, \mathcal{Z}_2) + 1/6 \qquad \textit{from A17 and A20} \\ &\leq 1/6 + 1/6 + 1/6 \qquad \textit{from A21 - 22} \\ &= 1/2 \leq 1 - \epsilon \quad \forall a_{00} \in \mathcal{A}\,. \end{aligned}$$

Thus, the hypothesis of Doeblin holds. Hence, we have a stationary distribution for a_{00} on $\mathcal{A}$ (Doob [1953, thm. 5.7]).

Step 5. In our notation for the solution of (24), $a_s = \Psi(s, a_0)$. The continuity and monotonicity of the latter in s and the fact that $\lim_{s\to\infty} \Psi(s, a_0) = a^*$, any $a_0 \in \mathcal{A}$, show $\mathcal{A}$ is an ergodic set. Hence, the stationary distribution is unique. Furthermore, the monotonicity of $\Psi(s, a_0)$ in s rules out cyclic subsets. ∎

Proof of Corollary 3D: We work exclusively with detrended values of a_t. Using the notation $a_{s,J}$ for technology era J, Prop. 3B shows $a_{s,0} \in \mathcal{A}$ implies $a_{s',J} \in \mathcal{A}$ all $J \geq 0$. Choose any $a_{\bar{t}(0)} \in \mathcal{A}$. Without loss of generality, let $J = 1$.

Step 1. Let $h(a) = [a]^n$ any $n = 0, 1, ...$ Let

$$f(a, T) \equiv \int_0^T h\big(\Psi(s, b \cdot a/x)\big)\, ds \quad \text{and} \quad \bar{f}(a) \equiv \int_0^\infty \lambda \cdot e^{-\lambda \cdot T} \cdot f(a, T)\, dT\,. \tag{A23}$$

Then $\bar{f}(a_{\bar{t}(0)})$ gives the average of the cumulated, over technology era $J = 1$, value of $h(a_{s,1})$, conditional on $a_{\bar{t}(0)}$, with $a_{0,J} = b \cdot a_{\bar{t}(0)}/x$ and $s \in \big[0, \bar{s}(J)\big)$. Thus, Prop. 3D shows

$$\int_{\mathcal{A}} \bar{f}(a_{\bar{t}(0)})\, \pi(da_{\bar{t}(0)})$$

gives the unconditional value of $h(a_t)$ cumulated over technology era $J = 1$. The same outcome applies for any J. With Poisson GPT arrivals, the average technology-era length is always $1/\lambda$. Therefore, the unconditional mean $\mathbb{E}[h(a_t)]$ obeys

$$(1/\lambda) \cdot \mathbb{E}[h(a_t)] = \int_{\mathcal{A}} \bar{f}(a)\, \pi(da)\,.$$

So,

$$\mathbb{E}[h(a_t)] = \lambda \cdot \int_{\mathcal{A}} \bar{f}(a)\, \pi(da)\,. \tag{A24}$$

Using (A24), we can determine all noncentral moments $\mathbb{E}\big[[a_t]^n\big]$. Because $\mathcal{A}$ is bounded, the moments uniquely determine the stationary distribution for a_t (e.g., Harris [1966, ch. 4]).

Step 2. Now let $h(a)\,,\ a \in \mathcal{A}\,,$ be any measurable function obeying the corollary's condition (i).

We have

$$\int_{\mathcal{A}} \bar{f}(a_{\bar{t}(0)})\, \pi(da_{\bar{t}(0)}) \overset{\text{a.e.}}{=} \lim_{J\to\infty} \frac{1}{J} \cdot \sum_{j=0}^{J-1} \bar{f}(a_{\bar{t}(j)}) \overset{\text{a.e.}}{=} \lim_{J\to\infty} \frac{1}{J} \cdot \sum_{j=0}^{J-1} f\big(a_{\bar{t}(j)}, \bar{s}(j+1)\big)\,, \tag{A25}$$

where the first equality comes from Doob [1953, thm. 6.2] and the second from the methods of Stokey and Lucas [1989, thms. 14.4-14.5]. Familiar analysis of the exponential distribution yields $\lambda = \text{plim}_{\mathrm{J}}\, \mathrm{J}/[\bar{\mathrm{t}}(\mathrm{J}) - \bar{\mathrm{t}}(0)]$. Because $\mathcal{A}$ is bounded, convergence a.e. implies convergence in probability. Furthermore, $\text{plim}_{\mathrm{J}}\, \mathrm{X_J} = \mathrm{X}$ and $\text{plim}_{\mathrm{J}}\, \mathrm{Y_J} = \mathrm{Y}$ implies $\text{plim}_{\mathrm{J}}\, (\mathrm{X_J} \cdot \mathrm{Y_J}) = \mathrm{X} \cdot \mathrm{Y}$. Hence,

$$\begin{aligned}
\lambda \cdot \int_{\mathcal{A}} \bar{f}(a_{\bar{t}(0)})\, \pi(da_{\bar{t}(0)}) &= \lambda \cdot \text{plim}_{\mathrm{J}}\, \frac{1}{\mathrm{J}} \cdot \sum_{\mathrm{j}=0}^{\mathrm{J}-1} \mathrm{f}\big(\mathrm{a}_{\bar{\mathrm{t}}(\mathrm{j})}, \bar{\mathrm{s}}(\mathrm{j}+1)\big) && \textit{from A25} \\
&= \text{plim}_{\mathrm{J}}\, \frac{\mathrm{J}}{\bar{\mathrm{t}}(\mathrm{J}) - \bar{\mathrm{t}}(0)} \cdot \text{plim}_{\mathrm{J}}\, \frac{1}{\mathrm{J}} \cdot \sum_{\mathrm{j}=0}^{\mathrm{J}-1} \mathrm{f}\big(\mathrm{a}_{\bar{\mathrm{t}}(\mathrm{j})}, \bar{\mathrm{s}}(\mathrm{j}+1)\big) \\
&= \text{plim}_{\mathrm{J}}\, \frac{\mathrm{J}}{\bar{\mathrm{t}}(\mathrm{J}) - \bar{\mathrm{t}}(0)} \cdot \frac{1}{\mathrm{J}} \cdot \sum_{\mathrm{j}=0}^{\mathrm{J}-1} \mathrm{f}\big(\mathrm{a}_{\bar{\mathrm{t}}(\mathrm{j})}, \bar{\mathrm{s}}(\mathrm{j}+1)\big) \\
&= \text{plim}_{\mathrm{J}}\, \frac{1}{\bar{\mathrm{t}}(\mathrm{J}) - \bar{\mathrm{t}}(0)} \cdot \sum_{\mathrm{j}=0}^{\mathrm{J}-1} \mathrm{f}\big(\mathrm{a}_{\bar{\mathrm{t}}(\mathrm{j})}, \bar{\mathrm{s}}(\mathrm{j}+1)\big)\,, && (A26)
\end{aligned}$$

where the last term is, see (A23), the right-hand side of the corollary's eq (ii). ∎

APPENDIX 2: Annual Financial Data

Our annual financial data comes from Shiller [1989], updated to 2020.

1. If $rnom_t$ is the riskfree nominal interest rate in decimal form and cpi_t the consumer price index, our annualized continuous-time real riskfree rate is

$$r_t = \ln\big(\frac{cpi_t * (1.0 + rnom_t)}{cpi_{t+1}}\big)\,.$$

2. If P_t^S is the real stock price and Div_t the real dividend, our annualized real rate of return on equities is

$$R_t^S = \ln\big(\frac{P_{t+1}^S + Div_t}{P_t^S}\big)\,.$$

3. Our equity premium is $EP_t = R_t^S - r_t$. And, $\mathbb{E}[EP_t]$ is an arithmetic average.

FIGURES FOR TEXT

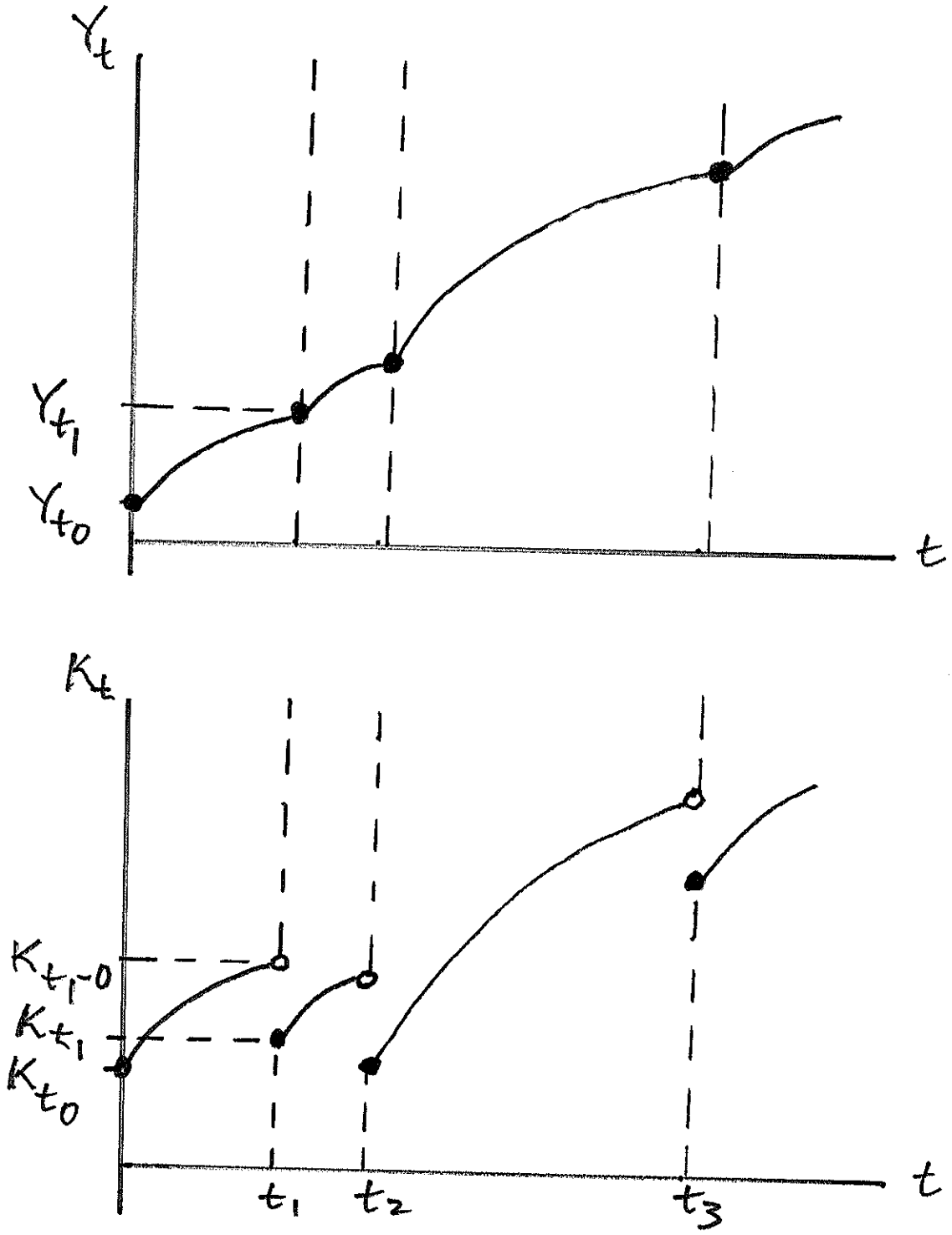


Figure 1: Time paths for K_t and Y_t, with GPT arrivals at $t_0, t_1, ...$

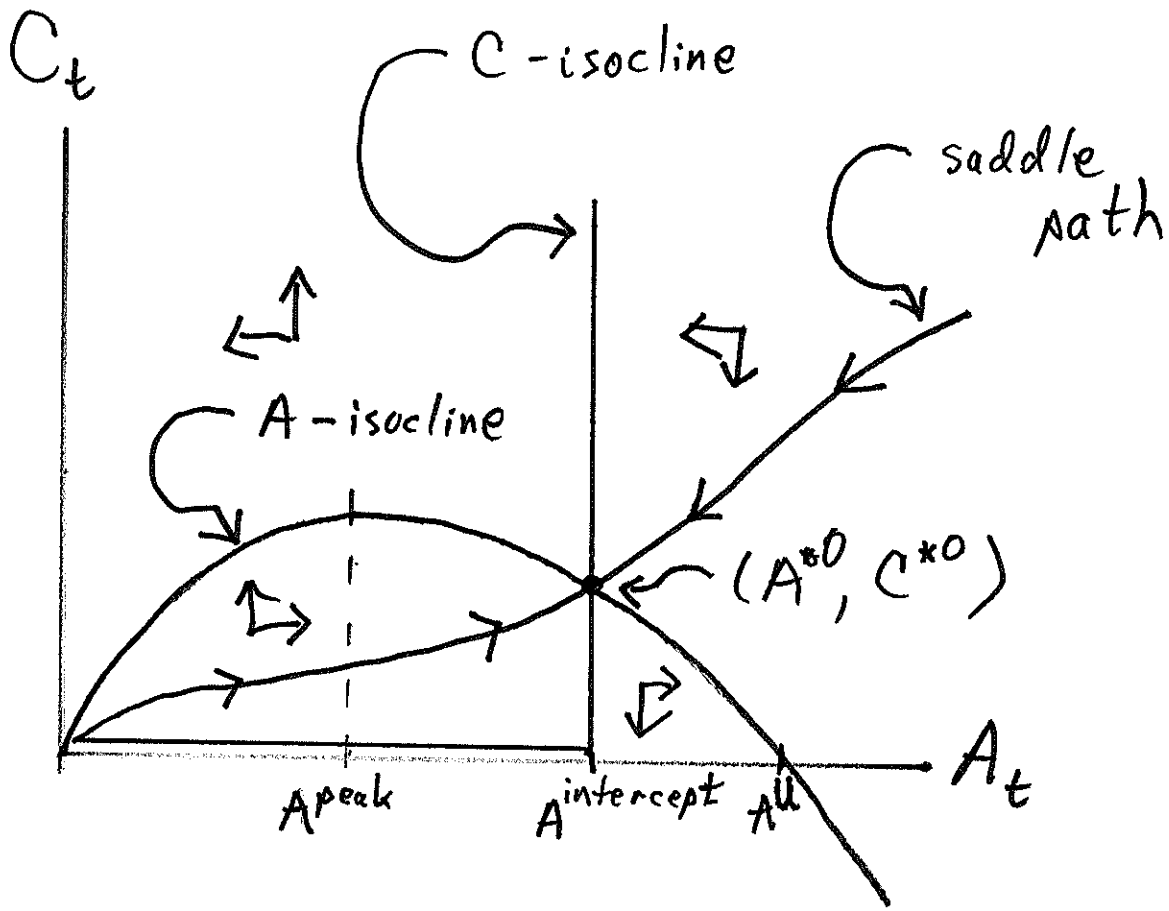


Figure 2: The $m = 0$ case. Let $J = 0$. We have $A^{Peak} = [\alpha \cdot \eta/\delta]^{1/(1-\alpha)}$ and $A^{Intercept} = [\alpha/(\delta + \rho)]^{1/(1-\alpha)}$.

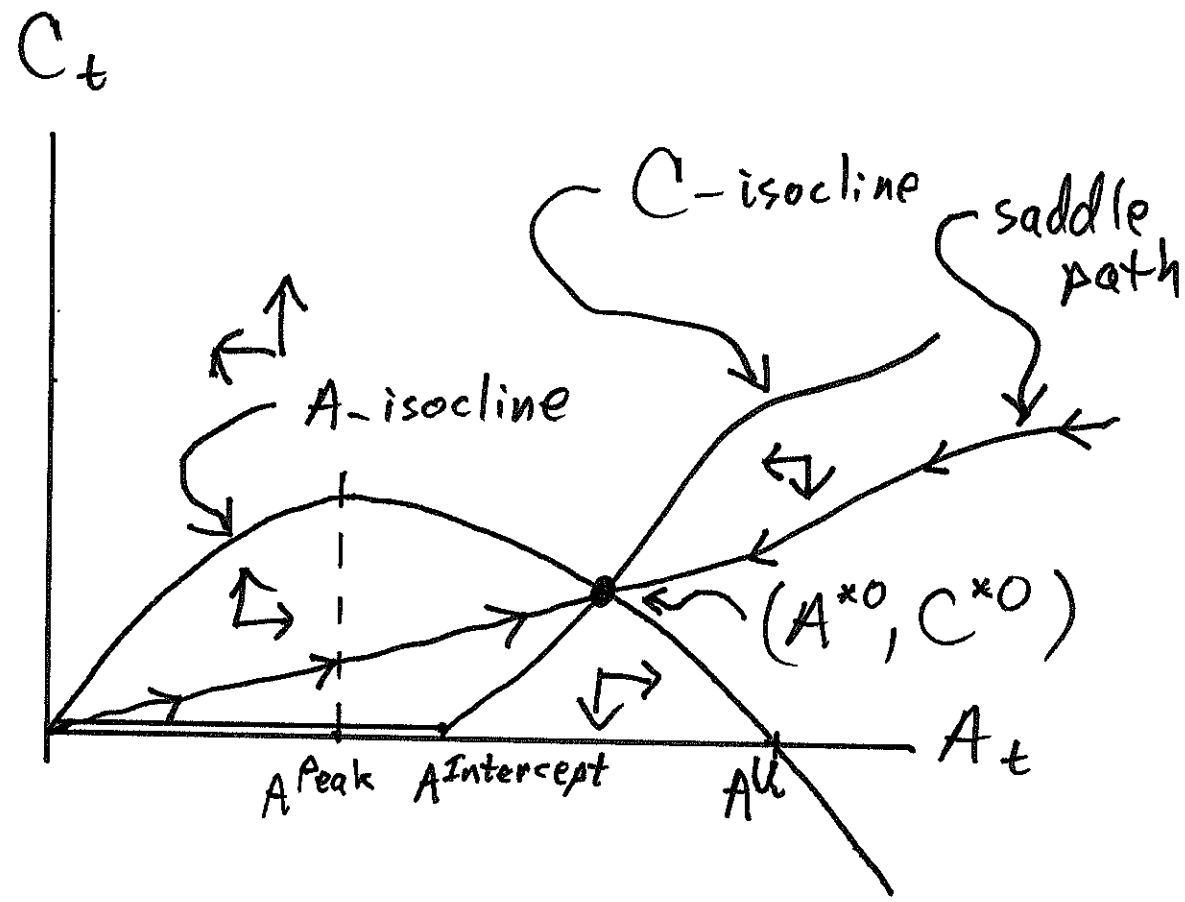


Figure 3: The $m = 1$ case. Let $J = 0$. The A-isocline as in Fig. 2. Now $A^{Intercept} = [\alpha/(\delta + \rho + \lambda)]^{1/(1-\alpha)}$.

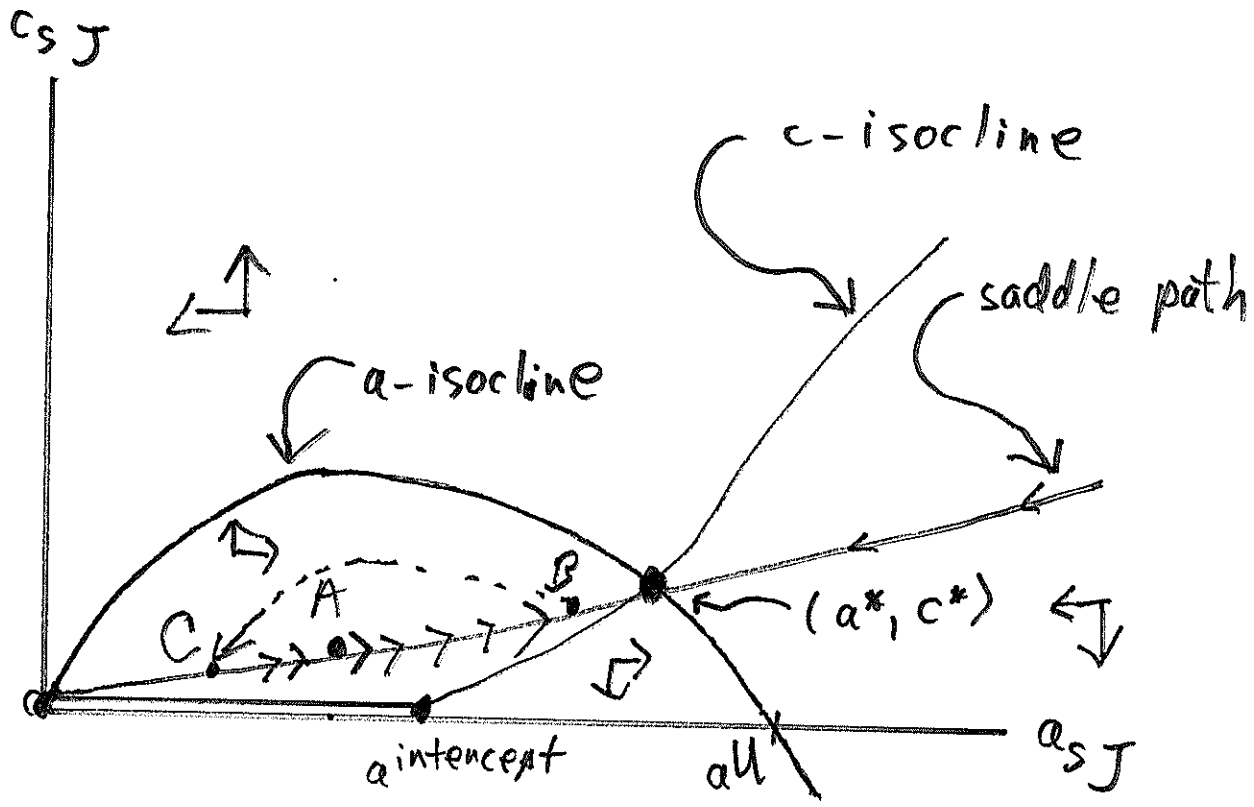


Figure 4: The $m = \infty$ case. $J \geq 0$. Detrended variables a_{sJ} and c_{sJ}. The a-isocline as in Figs. 2-3; $a^{Intercept}$ as in Fig. 3. Points A-C — see text.