The Distribution of Innovations across Firms *

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Abstract

Highlighting the unaligned incentives problem, this paper models the inventor-firm interplay, and shows the quantitative importance of when thinking about trade in patents, especially its impact on firm dynamics. I establish a theory addressing how innovation transaction market affects inventors' decisions to establish new firms or join incumbents of selected sizes ("which firm creates which innovation"). The model has four main elements: innovative ideas, inventor-firm contracts, synergy between firms and innovations, and innovation transaction market with asymmetric information. The model shows that inventors, particularly those whose outcomes are sensitive to their efforts, tend to align with smaller firms. The model predicts that when the secondary market for innovation is more efficient, small firms specialize in innovating and sell innovations to big firms, who specialize in production. Quantitative analysis shows that at the presence of a noisy signal on innovation quality ($R^2 = 0.4$), though the aggregate growth rate only improves by 0.1 pp, innovations in startups is more than tripled.

Keywords: patent trade, firm dynamics, employee contract, firms' boundaries.

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1 Introduction

Empirical literature shows that, inventors play a crucial role in R&D: an inventor's human capital has a dominant effect on innovation outcomes (Bhaskarabhatla et al., 2021). In contrast, theoretical studies typically model R&D production as a choice made by firms (Akcigit and Kerr, 2018; Acemoglu et al., 2018; Zhang, 2021). Specifically, to produce an innovation, a firm selects its input and converts it into innovations according to a production function, much like producing any other good. This type of model abstracts from the distinct roles played by inventors and firms. Does making this distinction affect the theoretical and quantitative evaluation of R&D policies? What mechanism determines which firm creates which innovation?

This paper aims to address this issue by introducing a model that includes both inventors and firms, and quantitatively investigating the importance of incorporating the inventor-firm interplay in the context of patent trade policies. The literature often focuses on how patent trade policies shape firms' R&D input decision by facilitating specialization (Akcigit et al., 2016; Ma, 2022). I innovate by modeling how these policies influence inventors' incentives to work, the resulting inventor-firm matching (which firm creates what innovation), and inventors' R&D effort. The model predicts that, in equilibrium, an inventor works harder in a startup compared to an incumbent firm. Inventors whose ideas are more sensitive to effort prefer startups.

The model shows that incorporating the inventor-firm interplay plays an important role in policy evaluation. Under a policy which subsidizes patent trade by 5%, startup firms become three times more active in R&D, and the aggregate growth rate increases by 1.4 basis points. Compared to textbook model, incorporating inventor-firm interplay leads to a higher relocation rate but a lower growth rate change.

The starting point is to consider how to capture the interplay between inventors and firms. The paper models this relationship as a contractual one, which determines both inventor-firm matching and R&D outcomes, subject to an incentive problem. Specifically, firms design contracts to incentivize inventors. After reviewing all available contracts, inventors choose which firm to work for and the optimal effort. The interplay centers around the incentive problem. If inventors and firms share aligned interests, it is largely harmless to abstract from the differences between them, as they would make the same decisions regarding R&D. However, empirically, their interests rarely align. A conflict of interest often arises between inventors and firms due to the legal environments of organizations. For instance, in the United States, inventors are typically not directly compensated for their R&D outcomes. Employment contracts often include intellectual property assignment clauses, making em-

ployers the automatic owners of patents. This practice creates an incentive problem between employees (inventors) and employers (firms), which most models abstract from.

There are four key elements in the theory. The first element is innovative ideas. Each idea is owned by one inventor, and idiosyncratic in terms of the sensitivity regarding inventor's effort input[1]. Though innovations improve *firms'* profitability, they are developed by *inventors*, as shown in Bhaskarabhatla et al. (2021), giving rise to the incentive problem between inventors and firms. The incentive problem matters differently across ideas due to heterogeneous effort sensitivity levels.

The second element is the contracts. Firms use contract to hire inventors to develop innovations in house. The contract space is assumed to be a combination of wage and equity [2]. A firm offers a combination of equity and wages to provide incentives, share risk, and split surplus with the inventor; when the firm offers more equity, the inventor is better incentivized but also more exposed to all other risks the firm faces. A given inventor's innovation-related risks contribute to a smaller portion of the equity variance in a larger firm. Specifically, in start-ups, the value of equity depends solely on the inventor's innovation. Thus, start-ups can offer compensation to maximize innovation outcomes merely restricted by the inventor's risk aversion. The inventors are well-incentivized to exert effort. In larger firms, by contrast, much of the equity value depends on stochastic factors unrelated to the innovator's effort (e.g. the success of other product lines). If large firms provided the same amount of incentive as start-ups, which means the large firms provided the same share of equity as start-ups, the inventors would be exposed to lots of unrelated risks and face a highly levered compensation package. The contracts give rise to the equilibrium outcome, that both the equity share and the incentive decrease with the firm size, and an inventor works harder in a smaller firm, which is consistent with empirical findings(Akcigit and Goldschlag, 2023).

The third element is the synergy between innovations and firms. When an innovation complements a firm [3], it is worth more. Otherwise, there are other firms who have synergy with this innovation. The synergy is public information, since it is easy to tell what technology class an innovation is in. An innovation is likely to be more valuable in larger firms, since it has a higher probability of complementing with the firm. (Figueroa and Serrano, 2019) For firms who are not the first-best user of an innovation, they can sell it on a secondary market, where a firm can sell innovations. On the seller side, it is especially useful for small

[1]Idea types are defined by effort sensitivity. It measures the elasticity of the innovation rate with respect to the inventor's effort

[2]See more in Section 2.1.4.

[3]The complementarity includes that a firm can apply the technology to a larger production scope, the innovation is in line with the manager's expertise, or the firm has enough liquidity to meet any potential financial requirement in commercialization. In the appendix, I show that under certain functional form assumptions, financial frictions show up in the model in the same way as complementarity.

firms, since they have a lower probability of complementing with their innovations, and can gain from trade[4] disproportionately

The synergy and the contract together determine the mapping between firms and inventors. On the one hand, the smaller the firm is, the harder the inventor works, hence a higher probability of innovating. On the other hand, conditional on the delivery of an innovation, a larger firm on average extract more value from the innovation. Therefore, the key trade-off is the chances of a successful innovation and the value of the innovation. The former is more important for an inventor with an effort-sensitive idea. As a result, inventors whose ideas are more effort-sensitive work for small firms, whereas inventors with less effort-sensitive ideas work for large firms.

The fourth element is the asymmetric information, the main friction in the secondary market. Specifically, I assume that the firm which owns the innovation has private information about how good the innovation is (Silveira and Wright, 2010; Chiu et al., 2017). The firm can decide either to implement it in-house or to sell it to another firm at a price that is independent of the *privately observed* value. If it does the latter, the gain from trade comes from the difference in synergy—the same innovation is worth more in the buyer firm. The secondary market serves as a reallocation mechanism to make efficient use of innovations. However, due to the asymmetric information, firms cannot fully capture the innovation's full first-best-use value; that is, an innovation's value decreases with the firm size which explains why not everyone goes to startups. This adverse selection feature in innovations is also true in Chatterjee and Rossi-Hansberg (2012), which analyzes the spin-off decisions under asymmetric information. One special case is when the inventor works for a start-up. If the start-up keeps the innovation, the start-up becomes an entrant. Otherwise, it is the model counterpart to a start-up buyout—namely, an incumbent acquires the start-up for its technology.

Combining the four elements, the model demonstrates that large firms in the US, though equipped with better commercialization capabilities, face greater challenges in incentivizing inventors mainly due to legal reasons. However, in countries where invention laws grant all invention rights to inventors rather than firms, these incentive issues disappear. In such cases, my model predicts that large firms would outperform smaller ones. Germany exemplifies this scenario, as its Employee Invention Law (Gesetz über Arbeitnehmererfindungen) mandates that all inventions belong to the inventors. Consequently, in Germany, invention intensity increases with firm size, in contrast to the US where invention intensity tends to decrease as firm size grows; both are consistent with the model's prediction.

[4]This paper focuses on quality-improvement innovations. In the appendix, I consider when innovations are substitutes for existing technologies.

On the quantitative side, I study how the allocation of innovations and the aggregate growth depend on the frictions in the innovation trading market. I calibrate the model using US data moments, including the probability of selling innovations, and the distribution of innovations across firms. To quantify the impact of innovation transactions on "who creates which innovation", I run counterfactual exercises by changing the market conditions in the transaction market, such as adding a quality signal, offering transaction-based subsidy. Qualitatively, when it is easier to sell innovations, inventors move to startups and small firms, and the aggregate growth rate increases. Quantitatively, when potential buyers can observe a signal about the innovation step size, inefficiency in the secondary market is alleviated and more innovations are sold. For example, if the R^2 of the signal is 0.6, the probability of selling increases from 5.3% to 5.9%. Since firms can sell unwanted innovations more easily, smaller firms are more attractive—the share of innovations created in start-ups increases from 0.31% to 5.21%. The growth rate increases from 2.00% to 2.08%. When buyers receive a subsidy purchasing an innovation, it has a similar impact. Conversely, if firms cannot trade innovations, which will happen if patents are invalid, the share of innovations created in start-ups and firms with fewer than 500 employees decreases from 7.9% to 1.4%. By contrast, for firms with more than 100,000 employees, this share increases by more than 10 percentage points. The overall aggregate growth rate in the economy decreases by 0.17%, from 2.00% to 1.83%.

Related Literature My paper relates to a literature exploring the implications of trading knowledge on firm innovations. Examples include Cassiman and Veugelers (2006), Higgins and Rodriguez (2006), Phillips and Zhdanov (2013), Bena and Li (2014), and Liu and Ma (2021), and the impact of the idea market on the economy growth (Eaton and Kortum, 1996; Silveira and Wright, 2010; Chatterjee and Rossi-Hansberg, 2012; Chiu et al., 2017; Cabral, 2018; Cunningham et al., 2021; Perla et al., 2021; Fons-Rosen et al., 2021). Recent papers studying the impact of the secondary market in a general equilibrium model include Akcigit et al. (2016) and Ma (2022). This work and my paper share a common framework, inherited from the endogenous growth literature[5]. The firm production framework is closest to Acemoglu et al. (2018), assuming the intermediate goods producers use innovations to improve qualities, and final goods producers assemble intermediate goods. This paper contributes to the literature by considering the endogenous contracting and matching. I show that the endogenous firm boundaries are important when thinking about the reallocation of inventors across firms in counterfactual scenarios, by mitigating the impact on the aggregate growth rate and amplifies the effect on the innovation distribution.

[5]For example, Romer (1986), Aghion and Howitt (1992), Aghion et al. (2001), Klette and Kortum (2004), Acemoglu et al. (2018), and Akcigit and Kerr (2018)

This model thinking about where inventors work relates to the literature on the boundary of the firm, going back to Coase (1937), important examples of which include Grossman and Hart (1986), Hart and Moore (1990), and Hart and Moore (2008). Closest are Aghion and Tirole (1994) and Schmitz (2005) who analyze the implications of the innovations' ownership. This paper casts these ideas in a quantitative general equilibrium economic setting with heterogeneous inventors and heterogeneous firms. My findings show that the endogenous firm boundaries matter quantitatively for both firm-level results and aggregate growth rate.

This paper is also connected to the discussion on intellectual property right protection. Examples include Boldrin and Levine (2013), Galasso and Schankerman (2015), and Budish et al. (2015). I focus on the general equilibrium implications of tradability. My results are consistent with the evidence derived from two natural experiments by Ma (2022) and Acikalin et al. (2022); namely, decreasing tradability undermines small firms disproportionately, and more innovations are created in large firms.

The rest of the paper is organized as follows: Section 2 describes the model and characterizes the equilibrium. Section 3 describes the calibration. Section 4 reports the quantitative results and counterfactuals, and Section 5 concludes.

2 A Theoretical Model

I build a theoretical model where firms compete to attract inventors for in-house innovation. The goal of this model is to study the innovation distribution among firms and how innovation market affects firms' growth. The model addresses two problems: In the primary market, inventors make strategic decisions regarding the size of the firm they prefer to work for, while in the secondary market, firms engage in transactions to buy and sell innovations among themselves. In the following sections, I first introduce the modeling environment and then describe the equilibrium.

2.1 Environment

Time is continuous. Agents are categorized into four primary groups: households, inventors, intermediate goods producers, and final goods producers. A representative household provides labor and consumes final goods. Inventors use effort to produce innovations in intermediate firms and consume final goods. There is a continuum of intermediate firms, each producing one unique type of product and leveraging innovations to improve product quality. The final good producers assemble intermediate goods to produce final goods.

2.1.1 Preferences

The household is long-lived. It supplies one unit of labor in-elastically. The utility function is

$$U_H = \int_0^\infty e^{-\rho t} log(C_H(t)) dt, \tag{1}$$

where $\rho > 0$ is the discount rate and $C_H(t)$ is the consumption of the household.

An inventor is short-lived and lives for dt time periods. There is a continuum of inventors of measure 1 in every period. An inventor provides effort e_I to produce inventions. She is risk averse and has a mean-variance utility:

$$U_I(c_I, e_I) = \mathbb{E}(c_I) - \frac{var(c_I)}{2\bar{q}} - R(e_I)\bar{q}, \tag{2}$$

where c_I is the consumption, e_I is the effort level, and $R(e_I)\bar{q}$ is the associated cost. $\bar{q}$ (defined below) is the average quality in the economy. The cost is scaled by $\bar{q}$ to keep the problem stable over time. Denote the inventors' aggregate consumption using C_I.

2.1.2 Technology

Firms are owned by household. The final good producers produce final goods using a continuum of intermediate goods $j \in [0, N_F]$ with production technology which is similar to Akcigit and Kerr (2018)[6]:

$$Y(t) = \frac{1}{1-\beta} \int_0^{N_F} q_j^\beta(t) y_j^{1-\beta}(t) dj. \tag{3}$$

In this function, $q_j(t)$ is the quality of the intermediate good j, and $y_j(t)$ is its quantity. I normalize the price of the final good to one in every period. The final good producers are perfectly competitive, taking input prices as given. Henceforth, I will drop the time index t when it does not cause confusion. The final goods are consumed by the household and inventors. The resource constraint of the economy is:

$$Y = C_H + C_I. \tag{4}$$

There is a continuum of measure N_F risk neutral firms producing intermediate goods. Each firm produces one kind of good, using a linear technology using only labor:

$$y_j = \bar{q} l_j, \tag{5}$$

where l_j is the labor input, $\bar{q} = \frac{1}{N_F} \int_0^{N_F} q_j dj$ is the average quality. It means that innovations have positive externality, similar to Romer (1986). The cost is linear in wage w, which firms take as given. The labor market satisfies the constraint:

$$\int_0^{N_F} l_j dj \leq 1. \tag{6}$$

[6]The difference is that in my specification, the final good producers only use intermediate goods, not labor, to assemble the final good. My model yields similar results if the final good producers use labor as well.

The production technologies, together with the market setting on innovation, ensure that a firm's value $V(q_j)$ is linear in quality q_j

$$V(q_j) = \nu q_j, \tag{7}$$

where ν is endogenous. The proof is in Section 2.2.1.

This paper focuses on the balanced growth path. I normalize the variables using the average quality $\bar{q}$. I denote the normalized variables using tilde:

$$\tilde{q_j} \equiv \frac{q_j}{\bar{q}}, \tilde{Q} \equiv \frac{Q}{\bar{q}}, \tilde{V}\left(\tilde{q}\right) \equiv \frac{V\left(q_j\right)}{\bar{q}} = \nu \tilde{q_j}, \tag{8}$$

where $Q \equiv \int_0^{N_F} q_j dj$ is the total technology stock in the economy.

A firm's quality q is affected by both business shocks and innovations. The business shock δ follows a Poisson arrival rate normalized to 1 and is randomly drawn from $\Xi\left(\delta\right)$. When hit by a shock, the firm's quality becomes $q + \delta q$.

In intermediate firms, innovations are produced by inventors using effort. Each inventor is born with one innovative idea type θ, that measures the idea's effort sensitivity and is independent of how good an innovation is$(\theta \in (0,1), \theta \sim \Psi(\theta))$. She joins a firm, either an incumbent or a start-up, and exerts unobservable effort e_I to transform the idea into an innovation. Given e_I, an innovation arrives with the instantaneous Poisson flow rate:

$$\lambda_\theta(\theta, e_I) = \mu e_I^\theta. \tag{9}$$

The innovation production function is based on the growth theory literature (Romer, 1990; Klette and Kortum, 2004; Akcigit and Kerr, 2018). The literature usually treats θ as a parameter, whereas here, θ is heterogeneous across innovations. It allows me to consider the mapping between innovations and firms. θ is the effort-elasticity in the innovation production function and I refer to inventors with high θ as effort-sensitive inventors. Additionally, in the literature, a unified firm produces and implements innovations; in my model, however, it is the inventor who creates innovations, and the firm only enjoys the outcome. It is costly to work, and the flow cost of choosing effort e_I is $R\left(e_I\right)\bar{q}$. Thus, without incentives, the inventor chooses $e_I = 0$. Meanwhile, she can always effortlessly make up a useless innovation that does not improve quality. The usefulness is not verifiable.

If the inventor creates a useful innovation, a patent is granted. An innovation's step size z, which is the patent owner's private information, is drawn from a distribution $\Phi\left(z\right)$. By implementing an innovation, a firm's quality increment is Δq, where

$$\Delta q = \begin{cases} \gamma_L z Q & \text{with probability } (1 - h(\tilde{q})) \\ \gamma_H z Q & \text{with probability } h(\tilde{q}), \gamma_H > \gamma_L. \end{cases} \tag{10}$$

The increment is assumed to be scaled up with Q to capture how the total technology stock in the economy affects new innovations. $h(\tilde{q})$ is the probability that a firm has complementarity with the innovation. I assume that $h'(\tilde{q}) > 0$; larger firms are more likely to have

complementarity with an innovation. The complementarity includes that a firm can apply the technology to a larger production scope, the innovation suits its business operation, or the firm has enough liquidity to meet potential requirements in commercialization. γ_L and γ_H capture the different efficiency in applying the technology. When there is complementarity, the firm is the efficient user of the innovation. In this case, the quality increment is $\Delta q = \gamma_H z Q$. Otherwise, the quality improvement is lower. There are always some firms that have complementarity.[7] All firms know whether it complements with the innovation, since it is easy to observe the innovation's technology class.

Because the firm value is linear in its quality, given an innovation step size, its value to any given firm only depends on the complementarity. If the firm is an efficient user, then there is no gain from trade. Otherwise, there is positive gain from trade. All firms try to sell innovations with which they don't have complementarity, and buyers are the firms who complement with the innovation. For each innovation, assume that there are at least two firms have complementarity.

2.1.3 Information Structure

After an inventor is born, the type θ is publicly known. Then the inventor chooses for which firm to work and effort e_I. The effort e_I is unobservable and unverifiable. Hence, contracts cannot be contingent on the effort level.

For each innovation, the step size z is private information and which firms complement with the innovation is public information. This is because though it is very difficult for an outsider to know an innovation's exact value (Chatterjee and Rossi-Hansberg, 2012), it is easy to observe the technology class. For example, if there is a new technology about compressing digital files, all firms in computing know the technology has synergy with their business, yet only the technology owner can tell the pecuniary value. Since the step size is private information and unverifiable, the existence of a useful innovation cannot be proved. Therefore, hiring contracts cannot be contingent on either. Meanwhile, when firms sell and buy innovations, they trade under asymmetric information.

2.1.4 Employment Contracting Problem

In the primary market, a firm hires an inventor by offering an employment contract. Firms compete with each other following Betrand competition. I assume that the employment contract has two dimensions: a constant wage T and a share of equity $a \in [0, 1]$.

[7] Akcigit et al. 2016 and Ma (2022) also consider the technology mismatch, but in a different setting. In their papers, if there is a mismatch, the firm cannot use the innovation while in my model, the firm can still apply it.

One may think that a better tool for providing incentives would be to offer a bonus contingent on the delivery of an innovation, since it would avoid exposing inventors to unrelated shocks. However, it is not feasible. Bonus offers require payoff to be contingent on the delivery of an innovation, which is not achievable due to lack of commitment. On the one hand, a firm cannot commit to pay bonus after an innovation is created, since an inventor cannot prove an innovation is developed by her. On the other hand, an inventor cannot commit to deliver a useful innovation, since the innovation value cannot be proved to an outsider (i.e. the firm knows the actual value, but others don't [8]). Inventors have the incentive to earn bonuses effortlessly by providing useless innovations. Therefore, using a bonus cannot solve the moral hazard problem.

A firm could try to use a detailed contract, and try to directly link the inventor's innovation to the firm's profit increase and use this as a basis for a bonus. I will follow the incomplete contract literature (Grossman and Hart, 1986), who assumes that it is very hard to specify all innovation outcomes in contracts, and, in practice, it is usually not contractible (Acemoglu, 1996; Frésard et al., 2020); I assume that firms can only use equity and wages in the employee contracts, as commonly seen in the real world (Brickley and Hevert, 1991).

The timing is as follows. All firms simultaneously offer firm-inventor-specific contracts to inventors. After viewing all contracts, the inventor chooses her favorite one and joins the firm. If the best contract is offered by $q = 0$, we call it a start-up. Even in this case, the inventor does not bear all the risk nor holding 100% equity. I assume that $q = 0$ still (e.g. venture capital funds) offer contracts with endogenous a.

2.1.5 Secondary Market Setup

Firms can trade unwanted innovations on secondary markets. Sellers are the ones who don't have complementarity with their innovations. Each innovation is unique and not substitutable by others—there is one market for each innovation. Buyers are the firms that have complementarity with the innovation. For each innovation for sale, a firm becomes a potential buyer with a probability proportional to the normalized quality $\tilde{q}$, corresponding

[8] Neither the stock reaction, nor the patent transaction can be used to evaluate innovation value. First of all, public firms only account for less than 13% of all innovative firms in my sample period, which means that most firms do not have access to the stock market, let alone using it to evaluate innovation value. Even for public firms, it is not easy to use stock market return to identify innovation value. It is difficult to implement. One reason is that lots of events, other than innovations, also affect stock prices. Another reason is that, for most public firms, multiple innovations are granted on the same day, which makes it impossible to assign value to each patent separately. Admittedly, for traded patents, their market value can be estimated using the price information. However, most firms do not innovate for the purpose of selling—they innovate to improve their own productivity. That is, many innovations are never sold, and there is no price information available.

to the fact that big firms are more likely to be a buyer (Figueroa and Serrano, 2019). For an innovation, there are at least two potential buyers.

Firms sell and buy under incomplete information because the step size is private information. Firms compete to buy innovations following Bertrand competition. Each buyer offers a price p_z simultaneously, and the seller chooses whether to accept one. It is a one-shot game—neither sellers nor buyers keep track of past trades. The settings lead to a standard lemons market (Akerlof, 1978).

2.1.6 Entry and Exit

There are two types of entry. One is innovation-related, which happens when an inventor chooses to work in $\tilde{q} = 0$ firm, successfully creates an innovation and decides to keep it. Denote the amount of innovation-related entrants with λ_I. The other is the exogenous entry where firms enter due to reasons other than innovations. Denote the amount of exogenous entrants with λ_0. In both cases, upon entry, the firm draws a quality $\tilde{q}$ from a distribution $\tilde{F}_{q0}(\tilde{q})$ and incurs a cost which equals the firm value. This represents the spillover from incumbents to entrants. Assume that $\tilde{f}_{q0}(\tilde{q}/\kappa) = \tilde{f}_q(\tilde{q})/\kappa$. When $\kappa < 1$, entrants are on average smaller. When the new entrant owns an innovation, then its quality q is boosted by Δq because of the innovation.

Firms face an exogenous exit rate τ. I will focus on a balanced growth path such that entry equals exit

$$\tau N_f = \lambda_I + \lambda_0. \tag{11}$$

2.2 Equilibrium

I now characterize the equilibria of the economy in which aggregate variables $(Y, C, R, w, \bar{q})$ grow at the constant rate g, while other variables (such as N_F) are constant.

2.2.1 Production

The final good producer chooses $\{y_j\}_j$ to maximize its profit using the technology described in Section 2.1.2:

$$\max_{\{y_j\}} \frac{1}{1-\beta} \int_0^{N_F} q_j^{\beta} y_j^{1-\beta} dj - \int_0^{N_F} y_j p_j dj. \tag{12}$$

The first-order condition yields the demand function of intermediate firms $p_j = q_j^{\beta} y_j^{-\beta}$. The intermediate goods are produced by corresponding firm $j \in [0, N_F]$ using only labor $y_j = \bar{q} l_j$, where $\bar{q} = \frac{1}{N_F} \int_0^{N_F} q_j dj$ is the average quality, and l_j is the labor input. Intermediate good producers are in monopolistic competition and choose l_j, p_j, y_j to maximize their profit, given

the wage level w:

$$\begin{aligned} &\max_{l_j,p_j,y_j} y_j p_j - w l_j. \\ &s.t.\ y_j = \bar{q} l_j \\ &p_j = q_j^{\beta} y_j^{-\beta} \end{aligned} \tag{13}$$

Therefore, the FOC yields

$$y_j = q_j \left(\frac{\bar{q}\,(1-\beta)}{w} \right)^{\frac{1}{\beta}}, l_j = y_j/\bar{q}, p_j = \frac{w}{\bar{q}\,(1-\beta)}. \tag{14}$$

In each period, the labor market clearing satisfies $\int_0^{N_F} l_j dj = 1$, which gives that $\frac{\int_0^{N_F} q_j \left(\frac{\bar{q}(1-\beta)}{w}\right)^{\frac{1}{\beta}} dj}{\bar{q}} = 1$. The wage w can be solved

$$w = N_F^{\beta}\,(1-\beta)\,\bar{q}. \tag{15}$$

Plug it back into the intermediate firm's problem. Both production y_j and profit π_j are linear in quality

$$y_j = \frac{q_j}{N_F}, \pi_j = \frac{\beta q_j}{N_F^{1-\beta}}. \tag{16}$$

I drop the subscript j from the firm-level variable when it does not cause confusion. In this model, the firm size is linear in its quality and I will use q to denote firm size when it does not cause confusion. Intermediate firms are the ones who hire inventors to innovate. Because of competition, any value from innovations is captured by inventors, and any value from acquisitions is captured by the seller. The discounted value of being a firm of quality q is the same as the net present value under the assumption that the firm never innovates. Thus, the value function of intermediate firm q at time t can be written as

$$V(q,t) = \int_t^{\infty} e^{-(r+\tau)(s-t)} \beta q / N_F^{1-\beta} ds = \nu q. \tag{17}$$

where $\nu = \frac{\beta}{(r+\tau)N_F^{1-\beta}}$. The value function is linear in q and does not depend on time. This result implies that for any firm, the value of a same quality improvement Δq is the same.

The aggregate production is linear in average quality $\bar{q}$. The resource constraint of the economy is $Y = C_H + C_I$. The Euler equation is:

$$g = \frac{\dot{Y}}{Y} = \frac{\dot{C}_H}{C_H} = \frac{\dot{\bar{q}}}{\bar{q}} = r - \rho. \tag{18}$$

2.2.2 Secondary market

The secondary innovation market is where firms buy and sell innovations. For each innovation, there are at least two buyers. Buyers follow Betrand competition and offer prices to the seller. A seller accepts an offer only when it is profitable, which means $\gamma_L z \nu \tilde{Q} \leq \tilde{p}_z$. The left-hand side is the value of the innovation if the seller keeps it, whereas the right-hand side is the value if the seller sells it. The threshold $\hat{z}$ satisfies

$$\gamma_L \hat{z} \nu \tilde{Q} = \tilde{p}_z. \tag{19}$$

Buyers set the price $\tilde{p_z}$ according to the zero-profit condition

$$\int_0^{\hat{z}} \left(\gamma_H \nu z \tilde{Q} - \tilde{p}_z\right) \phi(z) dz = 0, \tag{20}$$

where $\gamma_H \nu z \tilde{Q}$ is the innovation value to any buyer with complementarity; $\tilde{p}_z$ is the price of the innovation; $\hat{z}$ is the step size threshold. If $z \leq \hat{z}$, an innovation is sold.

If a firm is not the efficient user of an innovation, it chooses to sell the innovation if its step size is lower than the threshold $\hat{z}$. Otherwise, the firm uses the innovation to improve its own technology.

For each innovation, the probability of being sold is $(1 - h(\tilde{q}))\,\Phi(\hat{z})$, where $(1 - h(\tilde{q}))$ is the probability that the innovation does not have complementarity with the firm $\tilde{q}$ where the inventor works; $\Phi(\hat{z})$ is the cumulative distribution function of z at $\hat{z}$. The total amount of innovations sold per unit of time is

$$\Phi(\hat{z}) \int_0^1 \lambda_\theta^* \left(1 - h\left(\tilde{q}^*(\theta)\right)\right) \psi(\theta)\, d\theta, \tag{21}$$

where λ_θ^* is the equilibrium innovation arrival rate of type θ. $\tilde{q}^*$ denotes the firm for which inventor θ works in equilibrium. Recall that the probability of buying an innovation is $\lambda_b \tilde{q}$, because the probability of buying an innovation is assumed to be proportional to $\tilde{q}$ and the constant λ_b is what I solve for. The secondary market clearing gives

$$N_F \int \lambda_b \tilde{q} f_q(\tilde{q})\, d\tilde{q} = \Phi(\hat{z}) \int_0^1 \lambda_\theta^* \left(1 - h\left(\tilde{q}^*(\theta)\right)\right) \psi(\theta)\, d\theta, \tag{22}$$

which implies that

$$N_F \lambda_b = \Phi(\hat{z}) \int_0^1 \lambda_\theta^* \left(1 - h\left(\tilde{q}^*(\theta)\right)\right) \psi(\theta)\, d\theta. \tag{23}$$

It pins down the endogenous variable λ_b, and the probability of buying an innovation.

The innovation value for the original firm $\tilde{q}$ satisfies

$$x(z, \tilde{q}) = \begin{cases} \gamma_L \nu \hat{z} \tilde{Q} & \text{no complementarity and } z \leq \hat{z} \\ \gamma_L \nu z \tilde{Q} & \text{no complementarity and } z > \hat{z} \\ \gamma_H \nu z \tilde{Q} & \text{with complementarity.} \end{cases} \tag{24}$$

When there is no complementarity, and the step size z is lower than the threshold $\hat{z}$, the firm sells the innovation on the innovation market at the price $\gamma_L \nu \hat{z} Q$. If the step size z is higher than the threshold, the firm keeps it, and it is worth $\gamma_L \nu z Q$. When there is complementarity, the firm is the efficient user of the innovation. Then the firm also implements it in-house, and its value is $\gamma_H \nu z Q$. In a first-best allocation, all innovations would be sold to the efficient user. The asymmetric information on the secondary market, however, prevents this from happening.

The expected innovation value $\mathbb{E}(x(\tilde{q}))$ can be written as

$$\begin{aligned}\mathbb{E}(x(\tilde{q})) = \gamma_L \left[\Phi(\hat{z})\,\hat{z} + (1-\Phi(\hat{z}))\,\mathbb{E}(z|z>\hat{z})\right](1-h(\tilde{q}))\,\nu Q \\ + \gamma_H \mathbb{E}(z)\,h(\tilde{q})\nu Q.\end{aligned} \tag{25}$$

It increases with the firm size $\tilde{q}$ because $h\tilde{q}$ is increasing in $\tilde{q}$. Therefore, conditional on an innovation being created, its value is higher if it is in a large firm.

2.2.3 Primary market

The primary market is where firms compete to hire inventors using a combination of equity a and wage $\tilde{T}$. The setup yields a principal-agent framework. Firms are risk neutral, and inventors are risk averse. Firms enjoy the innovations produced by inventors, but it is costly for inventors to work and firms cannot monitor the effort. Thus, firms want to split the surplus with the inventor by offering a constant wage; meanwhile, firms need to incentivize the inventor to exert effort by offering equity. Because of the competition, the firm's problem is the same as maximizing the utility it offers to an inventor, subject to the zero profit condition:

$$\begin{aligned}&\max_{a,T,e_I,c_I} U_I\left(a,\tilde{T};\tilde{q},\theta,e_I\right) = \mathbb{E}(\tilde{c_I}) - \frac{1}{2}\text{var}(\tilde{c_I}) - R(e_I)\,dt. \\ &\quad s.t. \lambda_\theta(\theta,e_I)\,\mathbb{E}(x(\tilde{q}))\,dt - \mathbb{E}(\tilde{c_I}) \geq 0 \\ &\qquad e_I = \arg\max\left\{U_I\left(a,\tilde{T};\tilde{q},\theta,e_I\right)\right\} \\ &\qquad \mathbb{E}(\tilde{c_I}) = a\left(\mathbb{E}\left(\tilde{V}_0(\tilde{q})\right) + \lambda_\theta(\theta,e_I)\,\mathbb{E}(x(\tilde{q}))\,dt\right) + \tilde{T} \\ &\qquad \text{var}(\tilde{c_I}) = a^2\left(\lambda_\theta(\theta,e_I)\,\sigma_x^2(\tilde{q}) + \sigma_0^2(\tilde{q})\right)dt\end{aligned} \tag{26}$$

The optimal contract is $\{a^*(\theta,\tilde{q}), T^*(\theta,\tilde{q})\}$ [9]. The first line is the objective function $U_I\left(a,\tilde{T};\tilde{q},\theta,e_I\right)$, representing the inventor θ's utility when working for firm $\tilde{q}$ with effort level e_I; $\left(a,\tilde{T}\right)$ characterize the contract offered by the firm, and $\tilde{c_I}$ is the inventor's consumption level, while $R(e_I)\,dt$ is the cost of effort. The second line is the firm's individual rationality constraint, where $x(\tilde{q})$ is the value of an innovation to a firm $\tilde{q}$. It multiplies by the probability of creating an innovation $\lambda_\theta(\theta,e_I)\,dt$, which gives the expected payoff of hiring an inventor. The cost of hiring is $(\tilde{c_I})$. The firm will only participate if the expected gain is nonnegative. The third line is the inventor's incentive compatibility constraint. The inventor chooses an effort level e_I to maximize the expected utility $U_I\left(a,\tilde{T};\tilde{q},\theta,e_I\right)$ given firm quality $\tilde{q}$ and the wage scheme $\left\{a,\tilde{T}\right\}$. The fourth line describes the inventor's expected consumption. The first part, $a\left(\tilde{V}_0(\tilde{q}) + \lambda_\theta(\theta,e_I)\,\mathbb{E}(x(\tilde{q}))\,dt\right)$, is expected value of the firm equity a, which is the sum of the value without innovations and the expected value of an

[9] The numerical results are in Figure 9.

innovation. The other component, $\tilde{T}$, is the constant transfer from the firm to the inventor. The fifth line shows the variance of the inventor's consumption. The exposure is the equity a. The uncertainty comes from two sources: innovation-related part $\lambda_\theta(\theta, e_I)\sigma_x^2(\tilde{q})$ and the innovation-independent part $\sigma_0^2(\tilde{q})$. In start-ups, the second term is zero. I explain both in detail later.

Given contracts, the inventor with idea θ chooses firm $\tilde{q}^*(\theta) \in [0, \infty)$ and exerts effort e_I to maximize her utility by solving the following problem:

$$\max_{\tilde{q}} U_I\left(a(\tilde{q}, \theta), \tilde{T}(\tilde{q}, \theta), e_I(\tilde{q}, \theta), \tilde{q}, \theta\right). \tag{27}$$

Here $a(\tilde{q}, \theta), \tilde{T}(\tilde{q}, \theta)$ and $e_I(\tilde{q}, \theta)$ are the solutions to the firm's problem in Equation (26). If $\tilde{q}^*(\theta) > 0$, the inventor works in an incumbent firm; $\tilde{q}^*(\theta) = 0$ means the inventor joins a new firm. The equilibrium effort is $e_I^*(\theta)$ and the corresponding arrival rate is $\lambda_\theta^* = \lambda_\theta(\theta, e_I^*(\theta))$.

The firm value is affected by four factors: in-house innovation, purchased innovation, business shocks, and exogenous death. The firm value without in-house innovations $\tilde{V}_0(\tilde{q})$ is

$$\tilde{V}_0(\tilde{q}) = \nu\tilde{q} + \begin{cases} -\nu\tilde{q} & \tau dt \\ \delta\nu\tilde{q} & dt \\ \left(\gamma_H \nu z\tilde{Q} - \tilde{p}_z\right)|_{z\le\hat{z}} & \lambda_b\tilde{q}\Phi(\hat{z})\, dt \\ 0 & 1 - (\tau + 1 + \lambda_b\tilde{q}\Phi(\hat{z}))\, dt. \end{cases} \tag{28}$$

The first line is when the firm dies exogenous. The second line is when hit by a business shock δ that arrives with an arrival rate of 1. The third line is when the firm purchases an innovation. Though the expected gain from purchasing an innovation is zero, the realized gain usually is not zero—it is the gap between the price paid and the firm value improvement. The quality improves by $\gamma_H z\tilde{Q}$, where z is a random draw from the $\Phi(z)$ distribution truncated at $\hat{z}$. The cost of buying an innovation is $\nu\gamma_L\hat{z}\tilde{Q}$. The last line is the firm value otherwise.

The uncertainty in the firm equity value leads to the variance in the inventor's consumption c_I. Because the in-house innovation is independent of other activities, the firm-level risk can be decomposed into two parts: in-house innovation-related risk and the rest. The former one is $\lambda_\theta(\theta, e_I)\sigma_x^2(\tilde{q})dt$, where $\sigma_x^2(\tilde{q})$ is the second moment of the innovation value. The latter one is $\sigma_0^2(\tilde{q})dt$ which contains all shocks unrelated to innovations, including the risk from death, business shocks, and unknown step size in innovation trade. It can be written as

$$\sigma_0^2(\tilde{q}) = \nu^2\left\{\tilde{q}^2\left(\tau + \sigma_\delta^2\right) + \lambda_b\tilde{q}\Phi(\hat{z})\,\mathbb{E}\left[\left(\gamma_H z - \gamma_L\hat{z}\right)^2 | z \le \hat{z}\right]\right\}. \tag{29}$$

This risk increases with the firm size $\tilde{q}$. Therefore, in a larger firm, the innovation-related risk takes a smaller share in the total equity variance.

The firm determines how much equity to offer to the inventor, which depends on the firm size. In start-ups, the equity value depends solely on the inventor's innovation. Thus, start-ups can offer compensation in the form of equity and wages merely restricted by the inventor's risk aversion—the inventor can be well-incentivized to put effort into innovating. In larger firms, by contrast, the inventor's innovation only makes up a small fraction of the equity value; much of the equity value depends on stochastic factors unrelated to the innovator's effort (e.g. success of other product lines). If larger firms were to offer the same incentive as start-ups for the same expected compensation, inventors would be exposed to many unrelated risks and face a highly levered compensation package; this is not attractive to risk averse inventors. Hence, both the optimal equity a and the resulting incentive decrease with firm size. This is consistent with the empirical findings in Akcigit and Goldschlag (2023).

After seeing all contracts, the inventor picks the contract offered by firm $\tilde{q}^*(\theta)$, which gives her the highest utility. Equation (27) can be written as:

$$\begin{aligned}\max_{\tilde{q},e_I} U_I(\tilde{q};\theta,e_I) = \lambda_\theta(\theta,e_I)\,\mathbb{E}(x(\tilde{q}))\,dt - \frac{1}{2}a^2(\tilde{q})\left[\lambda_\theta(\theta,e_I)\,\sigma_x^2(\tilde{q}) + \sigma_0^2(\tilde{q})\right]dt \\ - R(e_I)\,dt. \\ s.t. e_I = \arg\max\{U_I(\tilde{q};\theta,e_I)\}.\end{aligned} \tag{30}$$

The main trade-off an inventor facing is a higher chance to create an innovation versus a better value after an innovation is created, which is represented by the first term of Equation (30): $\lambda_\theta(\theta,e_I)\,\mathbb{E}(x(\tilde{q}))$. The two elements work in opposite directions. A firm with higher quality $\tilde{q}$ on average can implement an innovation more efficiently, which means that the value-added of an innovation $\mathbb{E}(x(\tilde{q}))$ is higher; meanwhile, a smaller firm can offer better incentive and hence higher arrival rate $\lambda_\theta(\theta,e_I)$. The inventor's decision then depends on the relative strength of the two forces, which, in turn, is determined by the inventor type. For an effort-sensitive inventor with high θ, the incentive channel is more important. As a result, she chooses to work for a small firm. Similarly, when an inventor's idea has low effort-elasticity, the value channel is more crucial and she works for a large firm.

Inventor intensity is endogenous. For each inventor, the equilibrium innovation arrival rate generated by herself in the firm $\tilde{q}$ is λ_θ^*. In firm $\tilde{q}$, the innovation arrival rate is affected by both *each innovation's arrival rate* and *the number of inventors per firm*. Denote the arrival rate in firm $\tilde{q}$ by $\lambda_q(\tilde{q})$ and the firm probability distribution function by $f_q(\tilde{q})$. The total amount of innovations in a firm should be the same as the total innovations made by inventors who work for the firm. The inventor market clearing gives

$$N_F\lambda_q(\tilde{q}(\theta))\,f_q(\tilde{q}(\theta))\,d\tilde{q} = \lambda_\theta^*\psi(\theta)d\theta, \tag{31}$$

which pins down the innovation arrival rate per firm.

2.2.4 Entry and Exit

Recall there are two types of entries in this model. One is innovation-related. It happens when an inventor chooses to work in a firm with $q = 0$, successfully gets an innovation, and decides to keep it. The number of firms that enter this way can be written as

$$\lambda_I = \int_{\tilde{q}^*(\theta)=0} \lambda_\theta^* \left[\overbrace{\underbrace{(1-\Phi(\hat{z}))(1-h(0))}_{\text{Non-complementary \& big step size}} + \underbrace{h(0)}_{\text{Complementary}}}^{\text{Innovations kept}} \right] \psi(\theta)\, d\theta. \tag{32}$$

The other type is the exogenous entry, which enters the market without an innovation. The number is denoted by a parameter λ_0.

On the exit front, the exit rate satisfies that the number of firms who exit equals the number that enters.

$$\tau N_F = \lambda_I + \lambda_0. \tag{33}$$

I use (33) to solve for the measure of firms N_F.

2.2.5 Growth rate

The growth rate of the aggregate quality $\bar{q}$ can be written as:

$$g = \frac{\mathbb{E}_t(\bar{q}(t+dt)) - \bar{q}(t)}{\bar{q}(t)dt}. \tag{34}$$

The growth rate is

$$\begin{aligned} g = &-\tau(1-\kappa) + \gamma_H \mathbb{E}(z) \int h(q^*(\theta))\, \lambda_\theta^* \psi(\theta)\, d\theta \\ &+ \left[\gamma_L \mathbb{E}(z) + (\gamma_H - \gamma_L)\, \mathbb{E}(z|z \le \hat{z})\, \Phi(\hat{z})\right] \int \left[1 - h(q^*(\theta))\right] \lambda_\theta^* \psi(\theta)\, d\theta. \end{aligned} \tag{35}$$

It includes three parts. $-\tau(1-\kappa)$ is quality destruction if there were no innovations. For each unit of time, τ fraction of firms with an average quality $\bar{q}$ leave the economy, while the same fraction of firms enter the economy with an average quality κ. The rest reports the quality improvement due to the innovation. The rest of the first line is the quality improvement because of innovations that complement with the firms. The average quality is $\gamma_H \mathbb{E}(z)$. The second line measures the improvement that comes from the innovations that do not complement the initial firm. If there were no secondary market, then the average quality would be $\gamma_L \mathbb{E}(z)$. With the secondary market, some low-step-size innovations are sold to more efficient users, which increases the quality improvement by $(\gamma_H - \gamma_L)\, \mathbb{E}(z|z \le \hat{z})\, F(\hat{z})$. Hence when the innovation is worth more or the arrival rate increases, the growth rate goes up.

2.2.6 Equilibrium

I end this section by summarizing the equilibrium. The in-house R&D expenditure C_I of the economy can be written as

$$C_I = \int \lambda_\theta(\theta)\, \mathbb{E}\left(x\left(q^*(\theta)\right)\right) \psi(\theta)\, d\theta. \tag{36}$$

It captures all transfers made to inventors. The total firm-level R&D expenditure is the sum of R and the spending on purchasing patents on the secondary market

$$X = C_I + \lambda_b p_z. \tag{37}$$

Based on Equation (12), the equilibrium out put level Y is linear in $\bar{q}$

$$Y = \frac{1}{1-\beta} \frac{\bar{q}}{N_F^{1-\beta}}. \tag{38}$$

and the consumption level is

$$C_H = Y - C_I. \tag{39}$$

Definition 1. A balanced growth path of this economy for any combination of t, q, is the mapping between q and θ, the allocation $\left(\left\{y_j^*\right\}_j, Y^*, X^*, C_I^*, C_H^*\right)$, the prices $\left(w^*, \left\{p_j^*\right\}_j, \hat{z}\right)$, the growth rate g^*, the entry rate λ_{eI}^*, and the measure of firms N_F^*, such that (1) for any $j \in [0,1]$, y_j^* and p_j^* satisfy Equation (14); (2) wage w^* satisfies Equation (15); (3) measure of the intermediate producers N_F^* satisfies Equation (33); (4) the mapping is the solution of Equation (27); (5) aggregate creative destruction τ_e satisfies Equation (33); (6) the price offer $\hat{z}$ for inventions satisfies Equation (19); (7) the entry rates λ_{eI}^* satisfy Equation (32); (8) in-house R&D spending C_I^* satisfies Equation (36); (9) total R&D spending X satisfies Equation (37); (10) aggregate output Y^* satisfies Equation (38); (11) aggregate consumption C_H^* satisfies Equation (39); and (12) steady-state growth rate g^* satisfies Equation (35).

3 Calibration

I calibrate the model to the United States. I construct the moments using firm-level information and use patents as a proxy for innovations. Section 3.1 describes the data. Section 3.3 explains how I calibrate the model.

3.1 Data

This paper combines four datasets to calculate moments. The datasets are the US Patent and Trade Office (USPTO), the Center for Research in Security Prices (CRSP), the Merged CRSP-Compustat Database, and the linked CRSP-USPTO data provided by Kogan et al. (2017). In addition, I use the Survey on Business Strategies (ESEE) by Fundacion SEPI

from Spain, which provides firm-level in-house R&D investment and outsourcing R&D expenditure.[10]. The main sample period is from 1982 to 1997. I also calibrate the model to 1998-2010 and the results are reported in the appendix.

3.1.1 US Data Development

I combine the four datasets to obtain firm-level in-house patenting features. I start with the linked CRSP/USPTO dataset. It covers all patents granted to public firms from 1926 to 2010. The dataset provides links between stocks and patents, and an estimated value for each patent. The value is estimated based on the stock market reaction in a three-day window after the announcement of a patent(Kogan et al., 2017). I deflate the value of patents using the Consumer Price Index (CPI) for all urban consumers from Federal Reserve Economic Data.

Next, I match this dataset with patent information. I download the supplemental information, including application date and granted date, from USPTO.[11] I clean the data using the following steps. First, I only keep patents with nonmissing citation observations. Second, I omit any observations where the patent granted date is before the patent application date, or the patent is granted more than ten years after the initial application.[12] This gives me a dataset with all in-house patents owned by public firms.

Then, I link it with firm-level data. I download firm-level employment data from Compustat and stock data from CRSP. I deflate the market capitalization using CPI. I clean the firms using three criteria: (1) positive employment, (2) located in the United States, and (3) obtained at least one patent during the sample period. My final sample is the universe of patenting US public firms. There are 3,175 unique firms in the sample. For comparison, Akcigit and Kerr (2018) uses census data and reports 23,927 firms in total. The discrepancy is because my sample includes only public firms, while their sample contains private firms as well.

3.1.2 ESEE

I use Survey on Business Strategies (ESEE) to evaluate the model implications on untargeted moments (in-house versus purchased innovations). ESEE is a panel survey of manufacturing firms in Spain conducted by Fundacion SEPI. The firms in the survey are selected through

[10]Spain has a similar employee invention policy as the US: firms can ask employees to give up ownership of all on-job innovation. Thus, the firm innovation decision in Spain is comparable with US.

[11]For patents granted after 1976, high-quality details are available at PatentsView, a patent data visualization and analysis platform supported by USPTO.

[12]On average, it takes 2.4 years (29 months) for a patent to be granted.

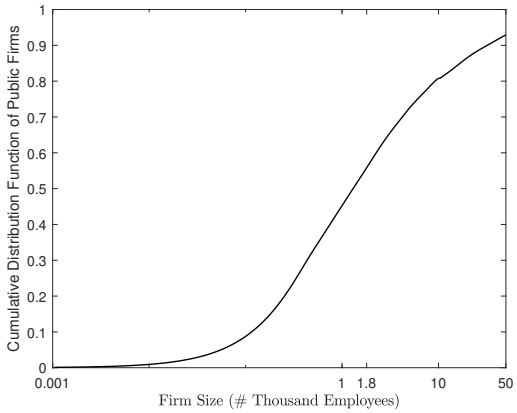


Figure 1: Public Innovative Firm Size Distribution

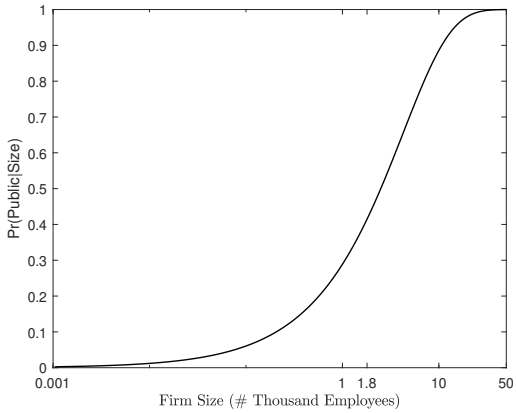


Figure 2: Probability of Being Public Conditional on Firm Size

stratified, proportional, and systematic sampling with a random seed. It reports firms' R&D activities, including annual expenses on in-house innovations and external innovation-related activities. Although Spain is admittedly different from the United States, the employee invention law in the two countries are similar regarding who owns the rights of employee inventions. Therefore, the innovation activities in the two countries are comparable. I use this dataset as a qualitative reference for the in-house versus external expenses comparison.

3.2 Moment Estimation

3.2.1 A Statistical Model

The dataset described in the previous section only includes information for public firms. The problem with using public firm data is that the dataset includes large firms disproportionately, which means that the distribution is not representative—particularly it lacks input from small firms. I assume that being a public firm is a random sampling weighted by the firm size.

Because firm size is linear in quality q in the model, I use q to denote the firm size. The public firm size follows distribution $g(q)$ (Figure 1) while the population distribution of all innovative firms is $f_q(q)$. I obtain $g(q)$ from the public firm data and need to estimate $f_q(q)$. Assume that for a firm size q, the probability of being public is $p(q)$, which only depends on firm size q. I estimate $p(q)$ parametrically using the generalized method of moments (GMM). $g(q)$ and $f_q(q)$ satisfy

$$f_q(q)\, p(q) = g(q)\, \mathbb{E}(p(q)), \tag{40}$$

where $\mathbb{E}(p(q))$ is the share of public firms among all patenting firms:

$$\mathbb{E}(p(q)) = \int f_q(q)\, p(q)\, dq. \tag{41}$$

Table 1: Model Fit for Key Moments of the Statistical Model

Moments	Data	Model
Share of firms with fewer than 17 employees	0.25	0.25
Share of firms with fewer than 70 employees	0.50	0.45
Share of firms with fewer than 370 employees	0.75	0.76
Average firm size	1,805	1,450

There are 23,927 patenting firms in the economy among which 3,175 are public. Therefore, $\mathbb{E}(p(q)) = 0.13$. I assume that $p(q)$ takes the form of the cumulative distribution function of a shifted gamma distribution: $p(q) = \Gamma(q + q_0, \Gamma_a, \Gamma_b)$. Then I use a generalized method of moments (GMM) to estimate the parameters $(q_0, \Gamma_a, \Gamma_b)$.

There are five moments in the data. First, because $f_q(q)$ is probability distribution function, by definition, it satisfies $\int_0^\infty f_q(q)\,dq = 1$, which gives the first condition that $p(q)$ must meet:

$$\int_0^\infty \frac{g(q)}{p(q)} dq = \frac{1}{\mathbb{E}(p(q))}. \tag{42}$$

Second, the average employee size of all patenting firms is 1,805 (Akcigit and Kerr, 2018). Set the unit of q to be 1,000: $\mathbb{E}(q) = 1.805$. It yields the moment below:

$$\int_0^\infty q \frac{g(q)}{p(q)} \mathbb{E}(p(q))\,dq = 1.805. \tag{43}$$

The rest of the moments are the three quartiles. The n^{th} quartile of the population q_n satisfies

$$\int_0^{q_n} f_q(q)\,dq = \frac{n}{4}, n = 1, 2, 3. \tag{44}$$

The quartiles of size of all patenting firms are approximately 17, 70, and 370 employees (Akcigit and Kerr, 2018). Therefore, it gives three moment conditions:

$$\int_0^{q_n} \frac{g(q)}{p(q)} \mathbb{E}(p(q))\,dq = \frac{n}{100}, n = 1, 2, 3. \tag{45}$$

The five moments described in Equation (42), (45), and (43) overidentified the parameters. I set up the estimation so that the parameters must satisfy the first moment and assign uniform weights to the other four. The estimated parameters are:$q_0 = 6.90 \times 10^{-5}$, $\Gamma_a = 0.70$, and $\Gamma_b = 6.0645$. The estimated probability of being public is in Figure 2.

Combining the size distribution of public firms in Figure1 and the probability of being public in Figure 2, Figure 3 reports the estimated distribution of firm sizes. The estimated moments of the firm size distribution are shown in Table 1.

3.2.2 Innovation Distribution

I estimate the patent distribution across firms as an empirical counterpart of the innovation distribution, which is used as moments in the quantitative analysis. Because the public firm

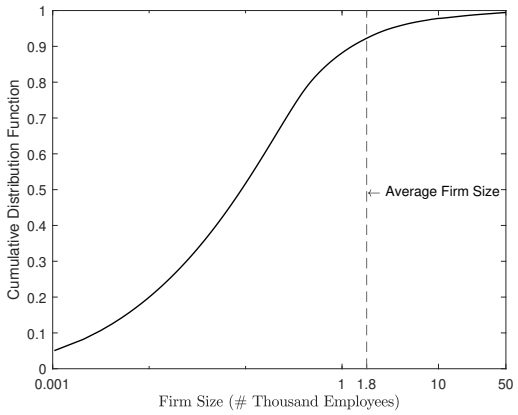


Figure 3: Estimated Firm Size Distribution of Innovative Firms

Notes: This is estimated using the public firm data and the statistical model.

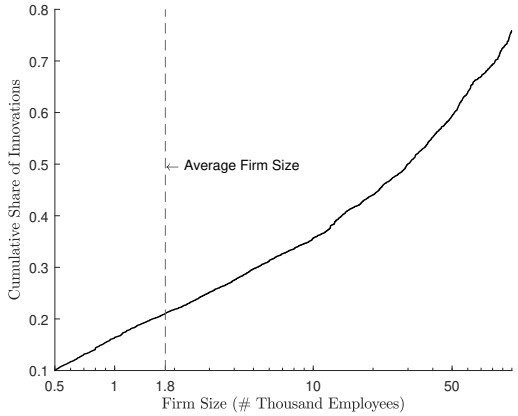


Figure 4: Estimated Innovation Distributions across Innovative Firms (with more than 500 employees)

Notes: The cumulative share of innovations in firms with fewer than 500 employees is set to 10% according to Figueroa and Serrano (2019).

Table 2: List of Parameters Used in the Model

Parameter	Description	Identification
β	Quality share in final goods	External calibration
ρ	Discount rate	External calibration
τ	Exit rate	External calibration
σ_δ^2	Business shock arrival rate	External calibration
λ_0	Exogenous entry	Indirect inference
κ	Avg quality of exogenous entrants	Indirect inference
μ	R&D arrival rate multiplier	Indirect inference
η	Curvature of Pr(complementarity)	Indirect inference
k_1	Pr(complementarity|emp>500)	Indirect inference
k_2	Pr(complementarity|emp>500) -Pr(complementarity|start-up)	Indirect inference
m	Scale para of step size dist	Indirect inference
α	Shape para of step size dist	Indirect inference
β_a, β_b	Shape para of inventor dist	Indirect inference
γ_L	Scale of R&D without complementarity	Normalized to 1
γ_H	Scale of R&D value with complementarity	Indirect inference

data have only limited the observations of firms with a few employees, one concern is the sample of small public firms suffers from selection bias. That is, the reason why a small firm goes public is correlated with its innovations. Therefore, I only use the statistical model to estimate the share of patents held by firms with more than 500 employees and assume that the share of patents held by firms with fewer than 500 employees is 10%, as suggested by the literature (Figueroa and Serrano, 2019). The estimated cumulative share of patents is reported in Figure 4, for example, the 60th percentile is about 50,000 employees per firm.

3.3 Calibration

In this section I calibrate the model. To solve the model, I use the following functional form assumptions. The cost function is quadratic: $R(e_I) = \frac{e_I^2}{2}$.

The probability of having complementarity with an innovation is

$$h(\tilde{q}) = k_1 - k_2 \exp(-\eta \tilde{q}), k_1 \geq k_2 > 0, \eta > 0. \tag{46}$$

The step size distribution of an innovation follows Pareto distribution:

$$\Phi(z) = 1 - \frac{m^\alpha}{z^{\alpha+1}}. \tag{47}$$

The type distribution of inventors follows beta distribution:

$$\Psi(\theta) = \mathcal{B}(\theta; \beta_a, \beta_b). \tag{48}$$

The business shock δ is a random draw from a truncated normal distribution:

$$\Xi(\delta) \left(\delta \in [-1, 1], \mathbb{E}(\delta) = 0, \text{var}(\delta) = \sigma_\delta^2\right) \tag{49}$$

The model has 16 parameters, which are listed in Table 2. The parameters (γ_L, γ_H, m) cannot be identified separately, so I normalize γ_L to 1.

I calibrate $(\rho, \beta, \tau, \sigma_\delta^2)$ externally. The discount rate ρ is set to 2%. I use β, the quality share in the production function, to target the firm profitability, defined as $\frac{\pi_j}{y_j p_j}$, which is 10.9% for the sample period between 1982 to 1997 (Akcigit and Kerr, 2018). In the model, the profitability equals β. Thus, I set $\beta = 10.9\%$. The firm exit rate τ is 6.5%, the average annual entry rate of innovative firms. The entry rate is estimated as follows. I obtain the annual entry rate for the whole economy from Decker et al. (2016) which is about 11.6%. To estimate the entry rate for innovative firms, I use two other measures from the same paper: the share of employment at young firms for the whole economy (14.3%) and the information industry (8.0%). I calculate the ratio between the two and use this ratio as a proxy for the ratio between the economy-wide entry rate, and the innovative firm entry rate is the same as the employment share because the information industry constitutes a large portion of innovation firms. Then combining the ratio and the economy-wide entry rate, I estimate the entry rate for innovative firms as 6.5%. Meanwhile, σ_δ^2, the variance of business shock δ is related to the firm growth rate volatility. The median standard deviation of firm growth

Table 3: External Calibrated Parameters

ρ	β	τ	σ_δ^2
0.02	0.109	0.065	0.14

rates within 10 years is about 16% to 20% for all firms (Comin and Philippon, 2005). Using public patenting firms, the results are similar. I set σ_δ to 14%. The external calibrated parameter values are listed in Table 3

There are 11 remaining parameters to be estimated. I identify these parameters using a simulated method of moments approach in the spirit of Lentz and Mortensen (2008), which is widely used in the growth literature to match firm-level evidence (Lentz and Mortensen, 2016; Acemoglu et al., 2018; Akcigit and Kerr, 2018). I calculate the model implied moments based on the model and compare them to the data-generated moment to minimize

$$min \sum_{i=1}^{11} \left(\frac{model\,(i) - data\,(i)}{data\,(i)} \right)^2. \tag{50}$$

Though every parameter affects every moment, below shows what parameter is especially important for matching each moment.

The probability of selling an innovation—The secondary market is an essential feature of the model. To characterize the secondary market. I match three moments related to the probability of selling an innovation: the probability of a big firm[13] selling its innovation, the start-up buyout rate, and the average probability of a firm selling its innovation. In the model, the probability of selling is directly affected by the probability of having complementarity. The first moment is 5.1% (Figueroa and Serrano, 2019) and governed by parameter k_1. As shown in Figure 5, when k_1 increases (decreases), it shifts up (down) the probability of having complementarity $h(\tilde{q})$. The startup buyout rate is governed by parameter k_2 (9.4%, from Gao et al. (2013)). Figure 6 confirms that $k2$ mainly affects the $h(q)$ for low q. The average probability of a firm selling it innovation is matched to α, the shape factor of the invention step size distribution. When α is large, then the uncertainty in the inventor's income stream is dominated by operational reasons that are unrelated to her own innovating effort. As α approaches 2, the innovation-related uncertainty goes up, which affects small to medium-small-sized firms disproportionately. Thus, I adjust α to match the average probability of a firm selling a patent. In the data, the probability is about 5.4% (Figueroa and Serrano, 2019).

Innovation distribution across firms—I obtain the innovation distribution across firms using the public firm data, patent data from USPTO, and a statistical estimation in Section 3.2.2. I target four data points from the distribution to pin down four parameters: η, μ, β_a,

[13]Following USPTO, a big firm is defined as a firm with more than 500 employees.

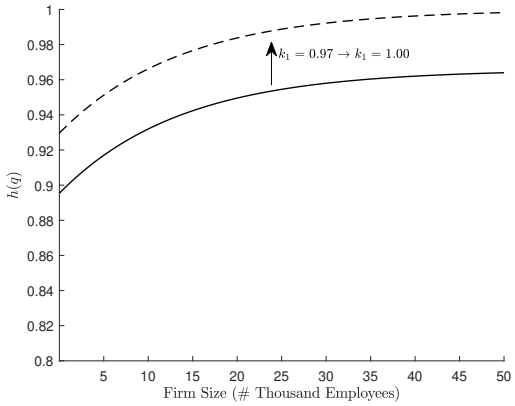


Figure 5: The Impact of k_1 on $h(q)$

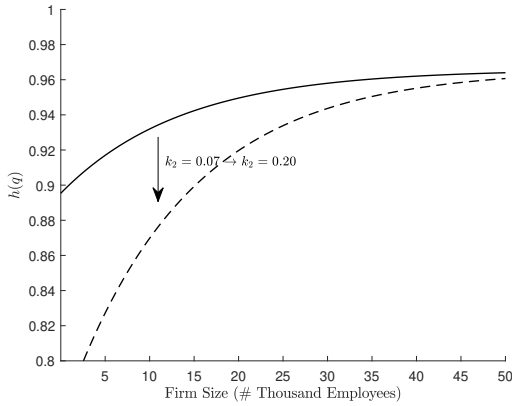


Figure 6: The Impact of k_2 on $h(q)$

and β_b. First, η governs the curvature of the complementarity probability. For firms large enough, because they all offer sufficiently low equity shares, η governs the firm-inventor matching. When η increases, it acts as if those firms are larger by the same factor regarding the complementarity probability. The impact is shown in Figure 7. η mainly affects the slope of the cumulative share of innovations by firm size. For example, if η increases from 0.13 to 0.26, then the 60th percentile of the size of all patenting firms shrinks by half. So I match it to the 60th percentile of size (employment size=51,250, which is $28.47\bar{q}$). The second parameter μ, the R&D arrival rate multiplier disciplines the share of innovations held by firms with fewer than 2,000 employees. μ affects both the value of innovating and the innovation-related uncertainty linearly. It affects the innovation value in all firms uniformly; it acts as a normalization. However, it has a disproportional impact on the equity risk across firms, since it only affects the innovation-related risk. As a result, in the relevant parameter space, μ mainly governs the share of innovation held by small- to medium-sized firms. I match it to the share of innovations held by firms with fewer than 2,000 employees (22%). The last two parameters β_a and β_b determine the inventor type distribution $\Psi(\theta)$. β_a affects the amount of low effort-sensitive inventors whereas β_b affects the amount of high effort-sensitive inventors. Therefore, I use the cumulative share of innovations held by firms with fewer than 5,000 employees (30%) and 500 employees (10%) to match β_a and β_b, respectively.

Average value of an innovation in firms with more than 500 employees—In the data, I calculate the average value of a patent in firms with more than 500 employees using patent value estimated by the stock market reaction (Kogan et al., 2017). I first keep all firms with more than 500 employees. Then I calculate the average patent value per firm and weigh it by the statistical model developed in Section 3.2.2. The average value per patent is about 3.6% of the average market capitalization. In the model, given the measure of firms N_F, the size factor m of the step size distribution determines the average value per innovation. It

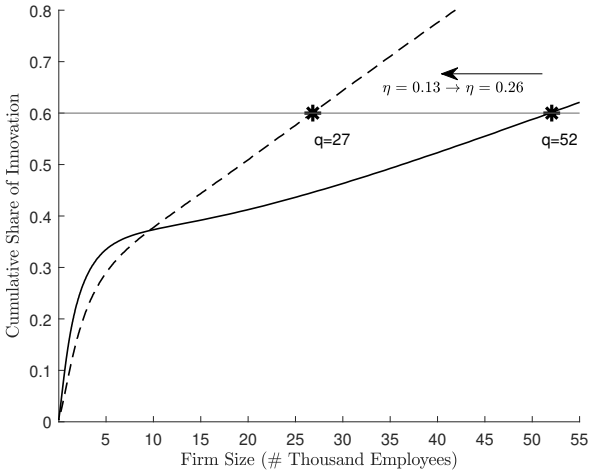


Figure 7: The Impact of η on Innovation Distribution

pins down the value of m.

The aggregate growth rate and the average growth rate—The aggregate growth rate measures the growth rate of the average quality whereas the average growth is the unweighted average of firm-level growth rates. When the aggregate growth rate is lower, it means that firms with low q grow faster. The amount of exogenous firm entry λ_0, from Equation (33), determines the measure of firms N_F. Conditional on the innovation value parameter m, λ_0 impacts the aggregate growth rate through N_F, for innovation value is scaled by the total technology stock ($Q = \bar{q}N_F$). Hence, I set λ_0 so that the aggregate growth rate g is 2%. In the model, the average growth rate is mainly affected by κ, the average quality of exogenous entrants. When κgoes up, holding the aggregate growth rate constant, the measure of firms N_F needs to adjust. Therefore, κ affects the innovations per firm and consequentially firm-level growth rate. The average productivity growth rate, based on Equation (14), can be written as:

$$\mathbb{E}\left(\frac{\dot{q_j}}{q_j}\right) = g + \mathbb{E}\left(\frac{\dot{l_j}}{l_j}\right), \tag{51}$$

where g is the aggregate growth rate, and $\mathbb{E}\left(\frac{\dot{l_j}}{l_j}\right)$ is the average employment growth in the model. In the data, the average employment growth per firm (7.4%(Akcigit and Kerr, 2018)) are driven by two forces: the productivity change and the working population growth (2.1%, US Bureau of Labor Statistics). Nevertheless, the model does not have population changes. I calibrate $\mathbb{E}\left(\frac{\dot{l_j}}{l_j}\right)$ to match the average employment growth driven by productivity, which is 5.3%. Adding the 2% aggregate growth rate, the average productivity growth is 7.3%.

Firm growth versus firm size regression—γ_H measures the value of having complementarity with an invention. When γ_H goes up, having complementarity is more valuable, which affects the inventor's trade-off; that is, higher γ_H means big firms are more attractive, and

Table 4: Indirect Inference Calibrated Parameters

λ_0	k_2	m	κ	μ	η
0.035	0.070	0.0147	0.014	1.87	0.13
k_1	α	β_a	β_b	γ_H	
0.97	2.0013	0.081	1.19	1.07	

Table 5: Model Fit for Key Moments—Targeted Moments

Moment	Data	Model	Parameter it informs
Profitability	0.109	0.109	β
Discount rate	0.020	0.020	ρ
Entry rate	0.065	0.065	τ
Firm growth volatility	0.17	0.18	σ_δ^2
Aggregate growth rate	0.02	0.020	λ_0
Average patent value	0.036	0.036	m
Startup buyout rate	0.094	0.094	k_2
Pr(Sell| big) [1]	0.051	0.050	k_1
Pr(Sell)	0.054	0.053	α
Growth-size relation β_g [2]	-0.031	-0.037	γ_H
% Inno, $emp < 500$ [3]	0.10	0.079	β_b
% Inno, $emp < 2,000$	0.22	0.23	μ
% Inno, $emp < 5,000$	0.30	0.33	β_a
60th pctl $\tilde{q}$ weighted by R&D	28.47	28.92	η
Average growth rate	0.073	0.072	κ

[1] A firm with more than 500 employees is defined as A "Big firm", according to USPTO.
[2] β_g is the coefficient of the growth-size regression.
[3] The % Inno is the cumulative density function of innovations created in firms with less than certain employment. For example, "% Inno, emp< 500" means the share of innovations that are invented in a firm with fewer than 500 employees.

hence grow faster. It is related to the growth rate versus firm size regression (Akcigit and Kerr, 2018):

$$g_{ft} = \eta + \beta_g ln\left(qft\right) + \epsilon_{ft}.$$

The empirical coefficient from the literature is $\beta_g = -0.035$.

The results are shown in Table 4

4 Results

The calibration moments are reported in Table 5. Overall, my model matches closely the targeted moments. Next, I discuss model features in more detail.

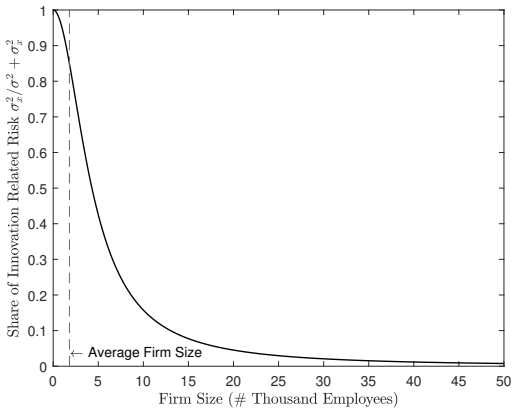


Figure 8: The Innovation-Related Risks across Firms

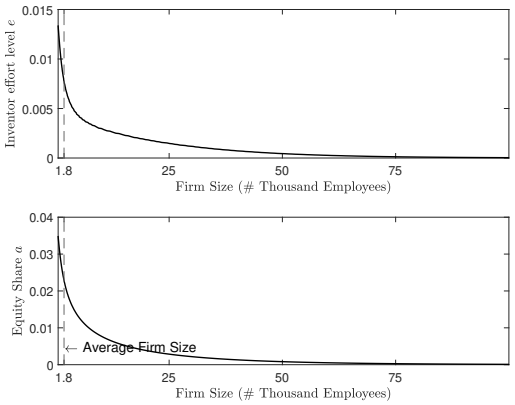


Figure 9: The Equilibrium Effort and the Optimal Contract

Notes: The figures report the actual effort level and the optimal contract for each equilibrium firm-inventor matching.

4.1 Optimal Contracts

Figure 8 shows what share of total risk is the innovation-related risk for firms of different sizes. As a result, it is more difficult for large firms to incentivize inventors without exposing them to unrelated risks. This is one optimal contract for the inventor each firm actually hires in equilibrium. I plot the contracts across firms in Figure 9. The top panel plots the resulting effort level e for the inventor works for a firm $\tilde{q}$. The bottom panel shows that, in equilibrium, the equity $a(\tilde{q})$ decreases with the firm size. This is consistent with the empirical findings in Akcigit and Goldschlag (2023).

4.2 The Distribution of Inventors across Firms

Figure 10 shows that an inventor with a more effort-sensitive idea chooses a smaller firm, receives more equity and chooses a higher effort level. Namely, inventors with low θ are less effort-sensitive and work for big firms; instead of working for an incumbent firm, inventors with θ higher than a threshold $\bar{\theta}$ join a start-up.

When choosing where to work, an inventor faces the trade-off between chances and value. The chance to successfully innovate decreases with firm sizes, because, as shown in Figure 9, smaller firms are better at incentivizing inventors to exert more effort. The value of an innovation increases with firm size, for larger firms are more likely to implement the innovation efficiently. When an inventor's idea is more effort sensitive, the incentive channel weighs more. Therefore, a more effort-sensitive inventor chooses to work for a smaller-size firm.

Figure 11 shows the inventor intensity (inventor-sales ratio) by firm size. It is consistent

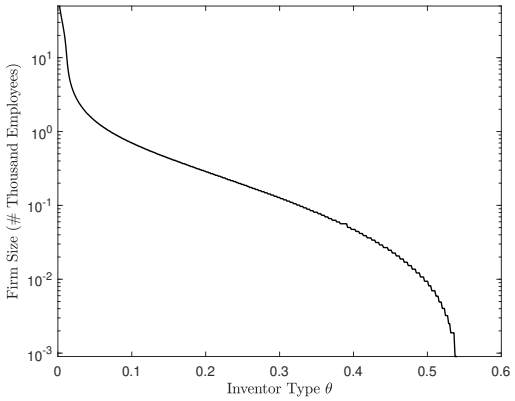


Figure 10: The Relationship between Inventors and Firms

Notes: Inventors with $\theta > 0.5$ work for start-ups with $q = 0$ are not shown in this figure, because the x-axis is in log.

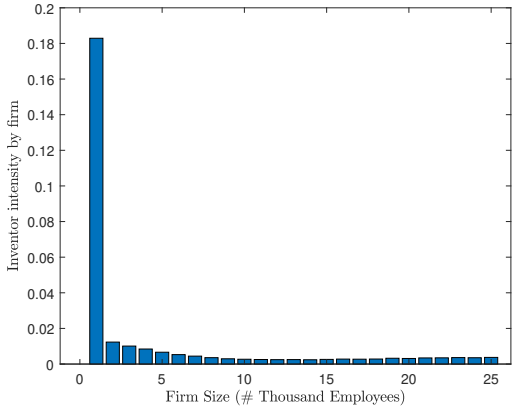


Figure 11: Inventor intensity by firm sales

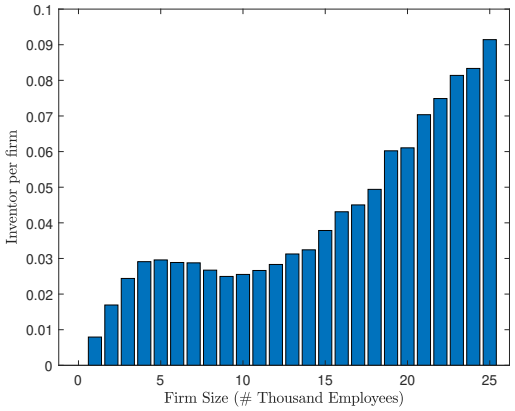


Figure 12: Inventor per firm

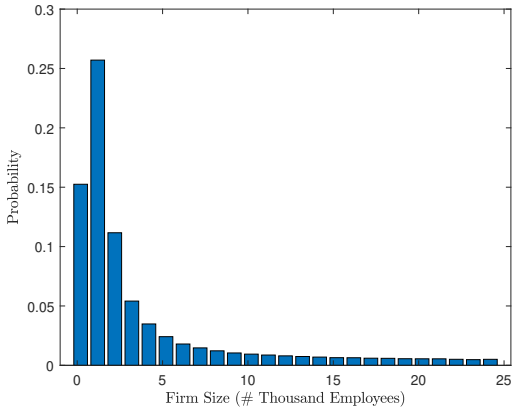


Figure 13: The Firm Size Distribution

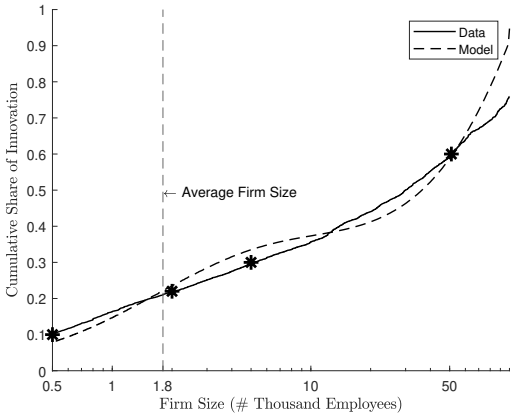


Figure 14: Cumulative Share of Innovations

Notes: The solid line plots the cumulative share for firms with more than 500 employees, the same as Figure 4. The stars represents the data points I used to calibrate the model.

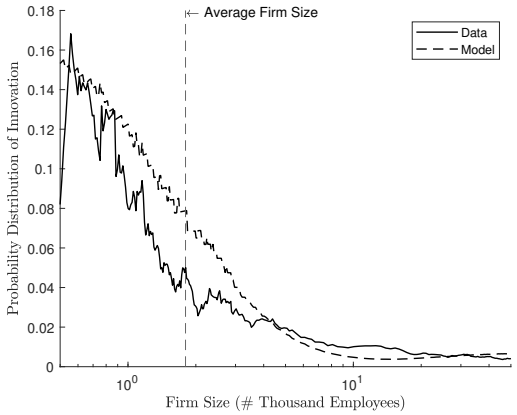


Figure 15: Probability Density of Innovation

Notes: This figure shows the probability distribution function of innovations for firms with more than 500 employees both in data and in model.

with the data and decreases with firm sizes. Figure 12 shows the inventor distribution by firm size, which is increasing, meaning that bigger firms have more inventors. The discrepancy between two figures is driven by the firm size distribution (13): the distribution density decreases with firm size.

4.3 Untargeted Moments

I next compare the model with untargeted features in the data.

Cumulative shares of inventions by firm sizes—I use four moments from the innovation distribution to calibrate the model; the rest is used as untargeted moments. The comparison is shown in Figure 14, and some moments are reported in Table 6. In general, the model is able to match the overall pattern of the cumulative shares. It overestimates when the firm size is medium-small to medium and large and underestimates when the size is around medium to medium-large. This is mainly because of the assumption that θ follows a beta distribution. Figure 15 shows the probability distribution function of Figure 14. The model follows a similar pattern as the data. *Growth rate gap between 90th percentile and 50th percentile*—Decker et al. (2016) shows that in high-tech industries (defined by Heckler (2005)) the growth rate difference between 90th percentile and 50th percentile is about 31%. In the model, I simulate 5 million firms for one year to find the annual growth rate for each firm. Then I only keep firms that have innovations within this year. The growth rate of firms that die within this time interval is defined as -1, following the literature. I rank the firms according to their growth rate and find the growth rate gap between the 90th percentile and 50th percentile.

Table 6: Model Fit for Untargeted Moments

Moment	Data	Model
Growth gap: 90th pctl - 50th pctl [1]	0.31	0.38
% Inno, $q < 20$	0.44	0.41
% Inno, $q < 50$	0.59	0.59
% Inno, $q < 100$	0.76	0.90

[1] The difference between the 90th percentile of the firm-level growth rate and the median growth rate.

Table 7: Model Fit for Untargeted Moments Using Spanish Data[1]

DV	$\frac{\text{R\&D}}{\text{Sales}}$	$\frac{\text{Internal}}{\text{Sales}}$	$\frac{\text{R\&D}}{\mathbb{E}(\text{Sales})}$	$\frac{\text{Internal}}{\mathbb{E}(\text{Sales})}$
1990	-0.009	-0.006	0.016	0.012
1994	-0.001	-0.001	0.023	0.013
Model	-0.025	-0.025	0.0030	0.0020

[1] The data are from ESEE of the SEPI Foundation, Spain. I use the Spanish dataset because it shares a similar employee invention law as the US, and provides detailed information on R&D investment.

The comparison of untargeted moments is listed in Table 6. Overall I am able to match the data; specifically, I match the 90th–50th growth rate gap and cumulative share held by medium- to medium-large-size firms closely.

In-house versus outsourcing choice—In addition, I report the comparison between in-house versus outsourcing choice both in the data and in the model in Table 7. In the model, a firm can both innovate in-house and purchase innovations from others. The data moments are obtained from ESEE in 1990 and 1994. The first wave of ESEE is in 1990. I run four specifications for 1990 and 1994 according to:

$$y_{ft} = \text{constant}_t + \text{coeff}_t \times \ln(\text{Sales}_{ft}) + \epsilon_{ft}, t = 1990, 1994, \tag{52}$$

where y_{ft} is the dependent variable of firm f and time t. Four dependent variables are (1) total R&D expenditure to sales ratio, (2) internal R&D expenditure to sales ratio, (3) total R&D expenditure normalized by average sales, and (4) internal R&D expenditure normalized by average sales. The first two measure the innovation intensity, and the rest measure the absolute level of innovation expenditure. The full regression results are in the appendix. Both R&D and internal R&D expenditure intensity decreases with firm size and the absolute levels increase with firm size. Though the data are from Spain and the model is calibrated to the US the model can match at least the direction of the coefficients, which is reassuring because it captures some of the trade-offs between in-house and outsourcing.

4.4 Growth Decomposition

I now use the model to document the sources of growth. The growth rate in Equation (35) can be written as

$$g = \overbrace{-\tau\left(1-\kappa\right)}^{\text{Destruction}} + \overbrace{\left(\gamma_H-\gamma_L\right)\Phi\left(\hat{z}\right)\mathbb{E}\left(z|z\leq\hat{z}\right)\int\left[1-h\left(q^*\left(\theta\right)\right)\right]\lambda_\theta^*\psi\left(\theta\right)d\theta}^{\text{Secondary market}} + \overbrace{\mathbb{E}\left(z\right)\int\left\{\gamma_H h\left(q^*\left(\theta\right)\right)+\gamma_L\left(1-h\left(q^*\left(\theta\right)\right)\right)\right\}\lambda_\theta^*\psi\left(\theta\right)d\theta}^{\text{Primary market}}. \tag{53}$$

It depends on three forces: (1) destruction and replacement with an entrant subtracted the innovation effect, (2) firms purchasing innovation on the secondary market, and (3) firms hiring inventors from the primary market to innovate in-house.

My model estimates that the average quality decreases by 6.41 percentage points due to the destruction and replacement process. For exogenous entries, this is because new entrants start at a quality on average lower than the firms who exit due to exogenous destruction.

Average quality increases by 8.23 percentage points because of innovation in-house. On average about 5% of all innovations are sold (Figueroa and Serrano, 2019), and the secondary market adds 0.18 percentage points to the aggregate growth rate. In Section 4.5 I analyze the effects of changes on the secondary market.

4.5 Counterfactual

In this section, I consider counterfactuals about the innovation tradability on the secondary market. The analysis shows that tradability affects both firm-level innovation allocations and the aggregate growth rate. I start with an extreme case when the secondary market no longer exists in Section 4.5.3. Section 4.5.1 analyzes the case if there is a signal about the invention step size. Section 4.5.2 studies the case when innovation transactions are subsidized. Online appendix shows the results if there is an invention transaction tax.

4.5.1 Signal

In the benchmark model, the secondary market is frictional due to information asymmetry. This section consider a publicly available signal χ associated with each innovation z, which alleviates the information asymmetry in the secondary market. The signal satisfies

$$\chi = \varepsilon z, \tag{54}$$

Table 8: Counterfactuals[1]

Moment	(1) Benchmark	(2) Signal	(3) Signal	(4) Subsidy	(5) Subsidy	(6) No Market
		$\alpha_\varepsilon = 1.5$ $R^2 = 0.4$	$\alpha_\varepsilon = 2.5$ $R^2 = 0.6$	5.0%	7.0%	
Growth rate	2	0.025	0.079	0.014	0.032	−0.168
% Inno in startups	0.31	0.814	4.899	0.836	1.485	−0.311
% Inno, emp< 500	7.64	−1.008	−4.778	−1.079	−2.066	−6.224
% Inno, emp$\in [0.5k, 20k)$	33.2	3.666	8.026	3.715	5.312	−8.897
% Inno, emp$\in [20k, 0.1m)$	49.29	6.096	1.42	6.095	4.836	4.41
% Inno, emp$\geq 0.1m$	9.57	−9.567	−9.567	−9.567	−9.567	11.022
Pr(Sell)	5.31	0.383	0.629	0.172	0.213	−5.309

[1] The table reports the **percentage points difference** with respect to the benchmark. For example, the growth rate change is calculated according to $(g_{\text{counter}} - g) \times 100$. The first column reports the benchmark case in percentage points. The second and third columns analyze cases in which there is a noisy signal when firms buy and sell innovations. The noise ϵ follows a Pareto distribution with scale parameter 1 and shape parameter α_ϵ. The fourth and fifth columns report the results when there is a one-time innovation transaction subsidy. The subsidy rates are 5% and 7% of the transaction price.The last column shows a case where firms cannot trade innovations at all.

where ε is noise. Assume that ε follows a Pareto distribution, where $f_\varepsilon(\varepsilon) = \frac{\alpha_\varepsilon}{\varepsilon^{\alpha_\varepsilon+1}}, \alpha_\varepsilon > 0$. Buyers use this information to update the innovation step size distribution, which yields:

$$f_{z|\chi}(z|\chi) = \begin{cases} \frac{-\tilde{\alpha} z^{-\tilde{\alpha}-1}}{\chi^{-\tilde{\alpha}} - m^{-\tilde{\alpha}}} & \text{if} z \leq \chi \\ 0 & \text{otherwise,} \end{cases} \tag{55}$$

where $\tilde{\alpha} = \alpha - \alpha_\varepsilon$.

Buyers now use this information to determine their bid price on the secondary market. As before, the buyer chooses a price $\tilde{p}_z(\chi)$ according to the zero profit condition.

$$\int_0^{\max(\hat{z},\chi)} \left[\gamma_H \nu z \tilde{Q} - \tilde{p}_z(\chi)\right] f_{z|\chi}(z|\chi)\, dz = 0. \tag{56}$$

The innovations are sold if they are worth less than the price to the seller. Equivalently, all innovations with step size lower than the threshold $\hat{z}$ are sold, where $\gamma_L \hat{z} \nu \tilde{Q} = p_z(\chi)$. The solution $\hat{z}$ satisfies

$$\frac{\gamma_H}{\tilde{\alpha}-1}\left(m^{-\tilde{\alpha}+1} - \hat{z}^{-\tilde{\alpha}+1}\right) = \frac{\hat{z}}{\tilde{\alpha}}\left(m^{-\tilde{\alpha}} - \hat{z}^{-\tilde{\alpha}}\right), \tag{57}$$

if $\hat{z} \leq \chi$. In this case, the bid price is independent of the signal. Otherwise, it means that the signal shows the step size is low enough, and the solution $\hat{z}$ satisfies

$$\hat{z} = \frac{\tilde{\alpha}(\gamma_H)\left[m\chi^{\tilde{\alpha}} - \chi m^{\tilde{\alpha}}\right]}{(\tilde{\alpha}-1)\left[\chi^{\tilde{\alpha}} - m^{\tilde{\alpha}}\right]}. \tag{58}$$

Using the updated expression for $\hat{z}$, the firm's problem and the inventor's problem are both the same as in the benchmark model.

To measure the informativeness of a signal, I regress $\ln z$ on the signal $\ln \chi$ and use R^2

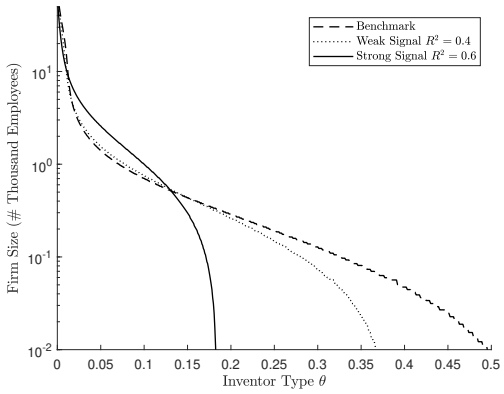


Figure 16: The Mapping between Inventors and Firms

as the information's accuracy. In the regression, the R^2 is

$$R^2 = \frac{\text{Var}(\ln z)}{\text{Var}(\ln \chi)} = \frac{\alpha_\varepsilon^2}{\alpha^2 + \alpha_\varepsilon^2}, \tag{59}$$

which increases with the shape parameter α_ε^2 of the signal distribution. It implies that when the signal is more accurate, R^2 is higher, and consequentially, we can better infer z from the signal.

I consider two different signal levels: $\alpha_\varepsilon = 1.5, 2.5$, and the corresponding R^2 are 0.4 and 0.6, which means the signal informativeness level is low and high, respectively. The results are reported in the second and third columns of Table 8.

The first row reports the percentage change in the aggregate growth rate. When $R^2 = 0.4$, the growth rate increases by 0.03 percentage points while it increases by 0.08 percentage points when $R^2 = 0.6$. Namely, when the R^2 of the signal is 0.6, the growth rate increases from 2.00% to 2.08%. This is because that more information means less severe adverse selection and thus it is easier for firms to sell their innovations. Therefore, the economy is more efficient and the growth rate is higher.

The second to the sixth rows of Table 8 report the changes in the innovation distribution across firms. The signal has heterogeneous effects on inventors. The signal has a significant impact on the share in start-ups. Quantitatively, the share of innovations created in start-ups is more than triple, from 0.31% to 1.13%, with a signal as noisy as $R^2 = 0.4$. When the signal is more informative, with $R^2 = 0.6$, the share of innovations in start-ups increases by 4.90 percentage points. Meanwhile, the share of innovations created in medium-small (with 500–20,000 employees) and medium-large firms (with 20,000–100,000 employees) also increases with the signal strength; the share of innovations in both small firms (with fewer than 500 employees) and large firms (with more than 100,000 employees) decreases with the signal strength.

The heterogeneity is because the signal effect is two-fold on the inventor's choice. Figure

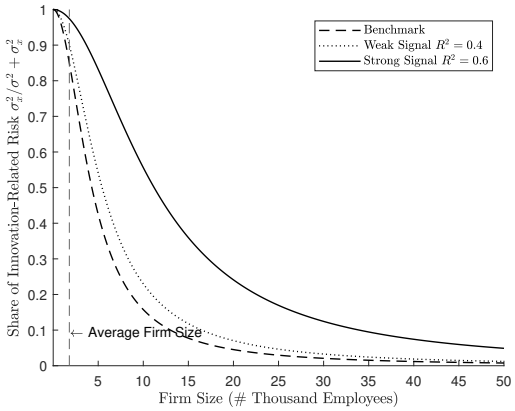


Figure 17: The Innovation-Related Risk Share Increases with the Signal

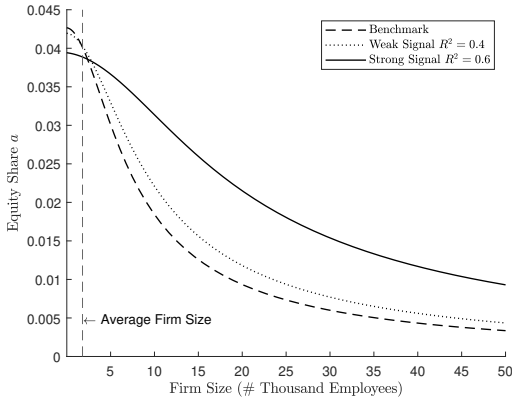


Figure 18: The Optimal Contract Changes with the Signal

16 reports the change in the mapping between inventors and firms. For inventors with the effort sensitivity level $\theta > 0.15$ and $\theta < 0.015$, more precise information leads to them working for smaller firms. On the contrary, for inventors with the effort sensitivity level $0.015 < \theta < 0.15$, the more informative the signal is, the bigger firm they choose. This is because the signal precision affects the inventor's decision in two ways. First, because firms can sell unmatched innovations more easily, the expected value of an innovation decreases more slowly with the firm size. Namely, for each innovation, there is a smaller value gap of being commercialized in different sizes. Second, innovations contribute more to firm's equity, because of the adding variance to the innovation selling prices. The innovative risk takes a larger share in a firm's total uncertainty profile, as shown in Figure 17. The increasing innovative uncertainty has two impacts: the firm equity variance is higher, and more uncertainty in the firm equity are related to the inventor's own choice. The former channel restricts the start-ups' and small firms' ability to offer equity since there is more demand for risk-sharing. The latter channel, on the contrary, enables medium- to large-sized firms to offer more equity by lowering the relative background noise level. Figure 18 confirms the mechanism. The share of equity offered by start-ups drops from 4.3% to 3.9% when there is a strong signal ($R^2 = 0.60$). And the slope is flatter in the signal's informativeness, especially at the left end. Therefore, the capacity to incentivize inventors drops more slowly with the firm size, especially for small firms.

The relative strength of the two forces determines the influence of the signal. For inventors with $\theta > 0.15$ and $\theta < 0.015$, the increasing innovation value is more important. Hence, inventors work for smaller firms. For the rest, the incentive improvement plays a more critical role, and inventors shift to bigger firms. As a result, more innovations are created in start-ups and medium-sized firms.

4.5.2 Subsidy

This section considers when the transactions on the secondary market are subsidized, which, similar to the public signal, alleviates the friction in the secondary market. It is a fixed transfer Γ per transaction, paid from lump sum taxes. The buyer now pays $(p_z - \Gamma)$ to purchase an innovation. The rest of the settings are the same as in the benchmark model. The buyer's zero-profit condition is

$$\int_0^{\hat{z}} \left(\gamma_H \nu z Q - p_z + \Gamma\right) \phi(z) dz = 0, \tag{60}$$

where $\hat{z} = \frac{p_z}{\gamma_L \nu Q}$ is the step size threshold; a firm agrees to sell an innovation when it does not have complementarity and the step size is lower than $\hat{z}$. Define $\psi \equiv \frac{p_z}{\gamma_L \nu Q}$. The zero-profit condition can be re-written as

$$\int_0^{\hat{z}} \left(\gamma_H z - \gamma_L \hat{z} + \gamma_L \psi\right) \phi(z) dz = 0. \tag{61}$$

With the subsidy, it is less costly to buy innovations. Therefore, $\hat{z}$ is higher than in the benchmark case, and a firm is more likely to sell an innovation.[14]

I evaluate two cases where the subsidy levels are 5% and 7% of the transaction price, respectively. The third and fourth column of Table 8 reports the results. The result is similar to the effect of adding a signal. It improves innovation tradability, which has two impacts on firm-level outcomes. It also has heterogeneous effect on inventors. There are more innovations in both start-ups and medium-small- to medium-large-sized firms and fewer innovations in small and big firms. The second to the sixth rows of Table 8 report the changes in the distribution. When there is a 5% subsidy, quantitatively, the share of innovations in start-ups increases by 0.84 percentage points. When the subsidy is 5%, the share in start-ups increases by 1.49 percentage points. Meanwhile, the share of innovations created in medium-small (with 500–20,000 employees) and medium-large firms (with 20,000–100,000 employees) also increases with the subsidy; the share of innovations in both small firms (with fewer than 500 employees) and large firms (with more than 100,000 employees) decreases with it. The seventh row confirms that it is more likely for a firm to sell its innovation.

4.5.3 Shut Down the Secondary Market

Consider an extreme case where there is no secondary market and firms cannot buy or sell innovations. This happens when an innovation is not protected by a patent. It hurts small firms disproportionately because they are the ones who rely more on the secondary market. The results are reported in the last column of Table 8. The benchmark results are shown

[14] Adding a transaction tax has the opposite effect of a subsidy. The counterfactual results of a transaction tax are reported in Appendix

in the first column for comparison. Inventors shift to bigger firms. The share of innovations created in start-ups drops by 0.31 percentage points, which is 100%—there are no innovations in start-ups anymore; start-ups cannot attract any inventors. The share of innovations in small firms also decreases dramatically, by 6.22 percentage points. This exercise speaks to one of the effects of intellectual property rights protection: without protection, firms cannot trade patents and the secondary market does not exist. The model implications are consistent with the empirical observations in Acikalin et al. (2022), which shows that when facing sudden patent invalidation, small firms lose disproportionately. In the contrast to the well-known "small firms innovate" idea, without the secondary market, innovations are created in big firms. For example, quantitatively, the share of innovations created in large firms (with more than 100,000 employees) increases by more than 11.02 percentage points—from 9.57% to 20.59%. The impact of shutting down the secondary market goes beyond firm-level distribution shift—it also has an aggregate implication. The aggregate growth rate decreases by about 0.17 percentage points. The growth rate drops from 2.00% to 1.83%.

Next, I explore what moments in the data are responsible for the quantitative finding—specifically, how much the growth rate drop would change if the observed data were different. To do this, I consider changes in two groups of data moments: the probability of selling an innovation and the share of innovations in small firms. After changing the moments, I recalibrate the model and re-run the counterfactual. The results are shown in Table 9. If firms are, in fact, more likely to sell an innovation than the data, the growth rate drop will be larger. For example, consider the case where the parameter $k_1 = 0.99$, which means instead of 5.3%, on average 3.6% of innovations are sold if there is a secondary market. Shutting down the secondary market will result in a drop that is 6 base points lower than before—the growth rate decreases by 0.10 percentage points, compared with the 0.16 percentage points in the baseline calibration. If in contrast, more innovations are created in small firms than observed, then the results will be bigger.

For example, consider a case in which on average 15% of innovations are sold, and no innovations are created in firms with more than 100,000 employees. Shutting down the secondary market leads to an additional 0.5 percentage points drop in the growth rate—now the growth rate would be 1.5% if there were no secondary market. This is because the impact of tradibility is mainly about innovation reallocation between firms. Both a high probability of selling and a small share of innovations in big firms means that the reallocation has a strong implication for efficiency improvement in which case shutting down the secondary market has a more salient effect.

Table 9: The Secondary Market's Impact Depends on the Probability of Selling an Innovation [1]

		(1) Baseline	(2) Change Sell Only	(3) Sell+Distribution
Parameters	k_1	0.97	0.99	0.85
	η	0.13	0.27	0.29
Moments	Pr(sell)	5.3%	3.6%	15.1%
	% Inno, emp$\geq 50k$	58.6%	55.2%	90.4%
Counterfactual	Δ g(No Market)	0.17%	0.10%	0.51%

[1] The table reports the counterfactual aggregate growth rate drop when shutting down the secondary market. The first column reports the baseline counterfactual as in Table 8. The second column shows when holding other moments constant, and the probability of selling decreases. In the third column, there are fewer innovations in big firms and more trade between firms.

Table 10: Distinguishing inventors and firms[1]

No Sell	(1) Benchmark Results	(2) Baseline Counterfactual	(3) Alternative Counterfactual
Growth rate	2.00	-0.17	-1.00
% Inno in startups	0.31	-0.31	0.08
% Inno, emp<500	7.64	-6.22	-7.44
% Inno, emp$\in[500, 20k)$	33.20	-8.90	-24.65
% Inno, emp$\in[20k, 100k)$	49.29	4.41	-10.54
% Inno, emp$\geq 100k$	9.57	11.02	42.54

[1] The table reports the **percentage points difference** with respect to the benchmark. The first column reports the benchmark case in percentage points. The second column reports the counterfactual results from the baseline model. The third column is the counterfactual results from the alternative model

4.6 Distinguishing inventors and firms

This section explores to what extent distinguishing inventors and firms affect the quantitative results. For comparison, following similar setup in section 2, I build an alternative model. The model is a textbook macroeconomic model with innovation[15]. There are three major differences between the alternative model and the benchmark model. In the alternative model: (1) there is no "inventor"; (2) firms produce innovation follows a R&D production function subject to firm-specific cost; (3) the measure of potential startups is constant and given by a parameter. The model is calibrated using the same data moments as in section 3. Then, I use the model to run same counterfactual exercises. Taking the shutting down the secondary market as an example[16], the comparison between two models is reported in

[15] More details can be found in the appendix

[16] Full results are reported in the appendix

Table 10. Comparing with the alternative case, both the aggregate effects and the relocation effects are less significant. This is because in the baseline model, both contract terms and inventor-firm mapping are endogenous. After shutting down the secondary market, firms can adjust contracts and inventors relocate between firms, which mitigates the impact. Another difference is that the alternative counterfactual predicts that share of innovation in startups will slightly increase, while the baseline model predicts that it will drop. This is because in the baseline model, the measure of potential startups is constant. Though in the alternative model, when shutting down the secondary market, in terms of absolute numbers, startups create fewer innovations. However, fewer innovations in the startup also leads to less entry, and fewer firm as an equilibrium result. Therefore, there are also fewer innovations in incumbent firms. As a general equilibrium result, after shutting down the secondary market, the share of innovations created by startups slightly increases.

5 Conclusion

This paper explores why some inventions are invented inside a large firm while others are in a small firm or start-up. The model characterizes the contractual relationship between two sets of heterogeneous agents: inventors and firms. Heterogeneous firms hire inventors to innovate in-house, and then trade non-complementary inventions on a secondary market. How much equity a firm offers depends on its size. Large firms find it difficult to give inventors high-power incentives without exposing them to unrelated risks. Therefore, both the incentive to exert effort and the optimal equity share decrease with firm size. A key trade-off that an inventor faces when choosing firms is between value and opportunities. On average, innovation value is higher in bigger firms, since they are more likely to have synergy with the innovation; meanwhile, the chance to innovate is higher in small firms, for they are better at incentivizing inventors using equity. The tradability of innovations on the secondary market is important to this trade-off. The model suggests that inventors work for small firms if their ideas are more sensitive to effort whereas inventors with ideas less sensitive to effort work for big firms.

This model offers a framework to think about the boundaries of firms quantitatively. The counterfactual exercise shows that, if we shut down the innovation market, the growth rate would decrease by 0.17 percentage points, and the share of innovations created in start-ups and firms with fewer than 500 employees will drop from 7.9% to 1.4%.

It would be useful to extend and generalize the analysis in several directions. One possible direction is to consider how individuals can decide whether to become an inventor or a worker. Another direction is to incorporate financial frictions in the model and to think

about when an inventor becomes an entrepreneur.

In addition, the contracting problem points to an understudied cost associated with moving to multi-unit firms. When multiple firms merge together, though the stock market can provide a forward-looking estimate of the total firm value, the cost is having an individual evaluation.

References

Acemoglu, D. (1996): "A Microfoundation for Social Increasing Returns in Human Capital Accumulation," *The Quarterly Journal of Economics*, 111, 779–804.

Acemoglu, D., U. Akcigit, H. Alp, N. Bloom, and W. Kerr (2018): "Innovation, Reallocation, and Growth," *American Economic Review*, 108, 3450–3491.

Acikalin, U., T. Caskurlu, G. Hoberg, and G. M. Phillips (2022): "Intellectual Property Protection Lost and Competition: An Examination Using Machine Learning," Working Paper 30671, National Bureau of Economic Research.

Aghion, P., C. Harris, P. Howitt, and J. Vickers (2001): "Competition, Imitation and Growth with Step-by-Step Innovation," *The Review of Economic Studies*, 68, 467–492.

Aghion, P. and P. Howitt (1992): "A Model of Growth through Creative Destruction," *Econometrica*, 60, 323–351.

Aghion, P. and J. Tirole (1994): "The Management of Innovation," *The Quarterly Journal of Economics*, 109, 1185–1209.

Akcigit, U., M. A. Celik, and J. Greenwood (2016): "Buy, Keep, or Sell: Economic Growth and the Market for Ideas," *Econometrica*, 84, 943–984.

Akcigit, U. and N. Goldschlag (2023): "Where have all the" creative talents" gone? Employment dynamics of us inventors," Working Paper 31085, National Bureau of Economic Research.

Akcigit, U. and W. R. Kerr (2018): "Growth through Heterogeneous Innovations," *Journal of Political Economy*, 126, 1374–1443.

Akerlof, G. A. (1978): "The Market for "Lemons": Quality Uncertainty and the Market Mechanism," in *Uncertainty in Economics*, Elsevier, 235–251.

Bena, J. and K. Li (2014): "Corporate Innovations and Mergers and Acquisitions," *The Journal of Finance*, 69, 1923–1960.

Bhaskarabhatla, A., L. Cabral, D. Hegde, and T. Peeters (2021): "Are inventors or firms the engines of innovation?" *Management Science*, 67, 3899–3920.

Boldrin, M. and D. K. Levine (2013): "The Case Against Patents," *Journal of Economic Perspectives*, 27, 3–22.

Brickley, J. A. and K. T. Hevert (1991): "Direct employee stock ownership: An empirical investigation," *Financial Management*, 70–84.

Budish, E., B. N. Roin, and H. Williams (2015): "Do Firms Underinvest in Long-term Research? Evidence from Cancer Clinical Trials," *American Economic Review*, 105, 2044–2085.

Cabral, L. (2018): "Standing on the Shoulders of Dwarfs: Dominant Firms and Innovation Incentives," No. DP13115, CEPR Discussion Paper.

Cassiman, B. and R. Veugelers (2006): "In Search of Complementarity in Innovation Strategy: Internal R&d and External Knowledge Acquisition," *Management Science*, 52, 68–82.

Chatterjee, S. and E. Rossi-Hansberg (2012): "Spinoffs and the Market for Ideas," *International Economic Review*, 53, 53–93.

Chiu, J., C. Meh, and R. Wright (2017): "Innovation and Growth with Financial, and Other, Frictions," *International Economic Review*, 58, 95–125.

Coase, R. H. (1937): "The Nature of the Firm," *Economica*, 4, 386–405.

Comin, D. and T. Philippon (2005): "The Rise in Firm-level Volatility: Causes and Consequences," *Nber Macroeconomics Annual*, 20, 167–201.

Cunningham, C., F. Ederer, and S. Ma (2021): "Killer Acquisitions," *Journal of Political Economy*, 129, 649–702.

Decker, R. A., J. Haltiwanger, R. S. Jarmin, and J. Miranda (2016): "Where Has All the Skewness Gone? the Decline in High-growth (young) Firms in the Us," *European Economic Review*, 86, 4–23.

Eaton, J. and S. Kortum (1996): "Trade in Ideas Patenting and Productivity in the Oecd," *Journal of International Economics*, 40, 251–278.

Figueroa, N. and C. J. Serrano (2019): "Patent Trading Flows of Small and Large Firms," *Research Policy*, 48, 1601–1616.

Fons-Rosen, C., P. Roldan-Blanco, and T. Schmitz (2021): "The Aggregate Effects of Acquisitions on Innovation and Economic Growth," Working Paper 3785485, SSRN.

Frésard, L., G. Hoberg, and G. M. Phillips (2020): "Innovation Activities and Integration through Vertical Acquisitions," *The Review of Financial Studies*, 33, 2937–2976.

Galasso, A. and M. Schankerman (2015): "Patents and Cumulative Innovation: Causal Evidence from the Courts," *The Quarterly Journal of Economics*, 130, 317–369.

Gao, X., J. R. Ritter, and Z. Zhu (2013): "Where Have All the IPOs Gone?" *Journal of Financial and Quantitative Analysis*, 48, 1663–1692.

Grossman, S. J. and O. D. Hart (1986): "The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Integration," *Journal of Political Economy*, 94, 691–719.

Hall, B. H., A. B. Jaffe, and M. Trajtenberg (2001): "The Nber Patent Citation Data File: Lessons, Insights and Methodological Tools," .

Hart, O. and J. Moore (1990): "Property Rights and the Nature of the Firm," *Journal of Political Economy*, 98, 1119–1158.

——— (2008): "Contracts As Reference Points," *The Quarterly Journal of Economics*, 123, 1–48.

Heckler, D. E. (2005): "High-technology Employment: A Naics-based Update," *Monthly Lab. Rev.*, 128, 57.

Higgins, M. J. and D. Rodriguez (2006): "The Outsourcing of R&D through Acquisitions in the Pharmaceutical Industry," *Journal of Financial Economics*, 80, 351–383.

Klette, T. J. and S. Kortum (2004): "Innovating Firms and Aggregate Innovation," *Journal of Political Economy*, 112, 986–1018.

Kogan, L., D. Papanikolaou, A. Seru, and N. Stoffman (2017): "Technological Innovation, Resource Allocation, and Growth," *The Quarterly Journal of Economics*, 132, 665–712.

Lentz, R. and D. T. Mortensen (2008): "An Empirical Model of Growth through Product Innovation," *Econometrica*, 76, 1317–1373.

——— (2016): "Optimal Growth through Product Innovation," *Review of Economic Dynamics*, 19, 4–19.

Liu, E. and S. Ma (2021): "Innovation Networks and R&D Allocation," Working Paper 29607, National Bureau of Economic Research.

Ma, Y. (2022): "Specialization in a Knowledge Economy," Working Paper 4052990, SSRN.

Perla, J., C. Tonetti, and M. E. Waugh (2021): "Equilibrium Technology Diffusion, Trade, and Growth," *American Economic Review*, 111, 73–128.

Phillips, G. M. and A. Zhdanov (2013): "R&d and the Incentives from Merger and Acquisition Activity," *The Review of Financial Studies*, 26, 34–78.

Romer, P. M. (1986): "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94, 1002–1037.

——— (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71–S102.

Schmitz, P. W. (2005): "Allocating Control in Agency Problems with Limited Liability and Sequential Hidden Actions," *Rand Journal of Economics*, 318–336.

Silveira, R. and R. Wright (2010): "Search and the Market for Ideas," *Journal of Economic Theory*, 145, 1550–1573.

Zhang, Y. (2021): "Quantitative Assessment on Frictions in Technology Market," Working Paper 122170, MPRA.

Appendix A Algorithm

I use value function iteration to solve the model. First, I define a set of discrete points for normalized firm sizes $\{\tilde{q}_0, \tilde{q}_1, ..., \tilde{q}_N\}$ and for idea types $\{\theta_0, \theta_1, \theta_2, ...\theta_M\}$. The algorithm employs a computational loop with the following steps

1. Guess the measure of the firm, the innovation without complementarity, the death rate and the growth rate $(N_F, \lambda_b, \tau, g)$.

2. Based on the Equation (7), solve for the corresponding ν.

3. For each inventor firm pair $(\theta, \tilde{q})$, use Equation (26) to solve for the stock share offered a and the utility level.

4. For each inventor θ, find the firm $\tilde{q}$ that offers the highest utility level according to Equation (27).

5. Update the guess on the measure of the firm, the innovation without complementarity, the death rate, and the growth rate $(N_F, \lambda_b, \tau, g)$ using Equation 31, 23, 33, and 35.

6. Repeat steps 2 to 5 until convergence.

Appendix B Growth Rate

The growth rate of the aggregate quality $\bar{q}$ by definition is:

$$g = \frac{\mathbb{E}_t\left(\bar{q}(t+dt)\right) - \bar{q}(t)}{\bar{q}(t)dt}. \tag{62}$$

The expected average quality at $t+dt$ includes two parts: the quality of incumbents and the quality of new firms. $\mathbb{E}_t\left(\bar{q}(t+\Delta t)\right)$ can be written as

$$\begin{aligned}\mathbb{E}_t\left(\bar{q}(t+dt)\right) = &\overbrace{\int \mathbb{E}q\left(t+dt\right) f_q\left(q,t\right) dq}^{\text{Incuments}} + \overbrace{\tau\kappa\bar{q}dt}^{\text{Entry}} \\ &+ \overbrace{\left[\gamma_L\lambda_{I,L}\mathbb{E}\left(z|z>\hat{z}\right) + \gamma_H\lambda_{I,H}\mathbb{E}\left(z\right)\right]\bar{q}dt}^{\text{Innovative Entry}},\end{aligned} \tag{63}$$

where $\lambda_{I,L} = \frac{(1-h(0))(1-\Phi(\hat{z}))}{(1-h(0))(1-\Phi(\hat{z}))+h(0)}\lambda_I$, $\lambda_{I,H} = \frac{h(0))}{(1-h(0))(1-\Phi(\hat{z}))+h(0)}\lambda_I$,.

Rearrange the equation to get the growth rate:

$$\begin{aligned}g = &-\tau\left(1-\kappa\right) \\ &+ \lambda_b N_F\left[\gamma_L\frac{\mathbb{E}\left(z\right)}{F\left(\hat{z}\right)} + \gamma_H\mathbb{E}\left(z|z\leq\hat{z}\right)\right] \\ &+ \gamma_H\mathbb{E}\left(z\right)\int h\left(q^*\left(\theta\right)\right)\lambda_\theta^*\psi\left(\theta\right)d\theta.\end{aligned} \tag{64}$$

Appendix C Counterfactuals

C.1 Counterfactual: Tax

Now I add a sales tax to the secondary market. It is a fixed cost Γ per transaction, which means that the buyer now pays $(p_z + \Gamma)$ to purchase an innovation. The rest of the settings are the same as in the benchmark model. The buyer's zero-profit condition is

$$\int_0^{\hat{z}}\left(\gamma_H\nu zQ - p_z - \Gamma\right)\phi(z)dz = 0, \tag{65}$$

Table C1: Counterfactual on Secondary Market Tax: 1% of the Transaction Price[1]

Moment	Benchmark	Tax
Growth rate	2.00	−0.002
% Inno in startups	0.31	−0.079
% Inno, emp< 500	7.64	−0.074
% Inno, emp∈ $[500, 20k)$	33.20	−0.652
% Inno, emp∈ $[20k, 100k)$	49.29	+0.803
% Inno, emp≥ $100k$	9.57	+0.001
Pr(Sell)	5.31	−0.045

[1] The table reports the **percentage points difference** with respect to the benchmark. For example, the growth rate change is calculated according to $(g_{\text{counter}} - g) \times 100$. The second column reports the results when there is a one-time innovation transaction tax. The tax rate is as low as 1% of the transaction price.

where $\hat{z} = \frac{p_z}{\gamma_L \nu Q}$ is the step size threshold; a firm agrees to sell an innovation when it does not have complementarity and the step size is lower than $\hat{z}$. Define $\psi \equiv \frac{p_z}{\gamma_L \nu Q}$. The zero-profit condition can be re-written as

$$\int_0^{\hat{z}} (\gamma_H z - \gamma_L \hat{z} - \gamma_L \psi)\, \phi(z) dz = 0. \tag{66}$$

With the tax, it is more costly to buy innovations. Therefore, $\hat{z}$ is lower than in the benchmark case, and a firm is less likely to sell an innovation.

I evaluate a case where the tax level is 1% of the transaction price. The fourth column of Table C1 reports the results. The tax mainly affects the innovation distribution. The tax, opposite to a signal, adds friction to the secondary market. As a result, firms sell a smaller portion of innovations. There is a larger value gap between innovations implemented by a big firm and those implemented by a small firm, which makes big firms more attractive. Similar to the signal case, the tax also affects firms' risk composition—fewer risks come from innovations. In this case, the second force is weaker than the first for all inventors. All innovations shift to larger firms. Quantitatively, the proportion of innovations in start-ups slides by 0.079 percentage points, which is a 25% drop. The share in small firms (with fewer than 500 employees) and medium-small firms (with 500–20,000 employees) drop by 0.074 percentage points and 0.652 percentage points, respectively. The medium-large firms take a share 0.803 percentage points higher than before. For firms with more than 100,000 employees, the share increases slightly, by about 0.001 percentage points. The tax has a mild impact on the aggregate growth rate. This is because the effect is mitigated by endogenous contracts.

C.2 Alternative Model

I build an alternative model without distinguishing inventors and firms. The model is described in this section. Time is continuous. Agents are categorized into three primary groups: households, intermediate goods producers, and final goods producers. Both the representative household and final goods producers are the same as in section 2. I will only focus on the intermediate firms. The intermediate firms hire labor to produce output as in section 2. Meanwhile, they also conduct R&D by choosing input e_I. Given the input level, an innovation arrives with the instantaneous Poisson flow rate:

$$\lambda_a(q) = \mu e_I^{\frac{1}{2}} \tag{67}$$

The variable cost of effort follows:

$$C_{a1}(q, e_I) = c_{mg}(q) e_I \tag{68}$$

where c_{mg} is a firm-specific R&D management cost. A firm chooses its innovation input level to maximize its profit:

$$\max_{e_I} \mathbb{E}(x(q)) \lambda_a(e_I) - C_{a1}(q, e_I) - C_{a2}(q) \tag{69}$$

where $\mathbb{E}(x(q))$ is the expected value of innovation defined in section 2. C_{a2} is a firm specific fixed cost. By combining the two cost, it ensures that the alternative model can generate the same intermediate firm value function and R&D allocation in the baseline calibration as the main model.

In the alternative model, the measure of startup-specific R&D N_N determines the potential entry. Startups produce innovation in the same way as incumbent firms. The exit rate τ is exogenous and homogeneous as the baseline model. In equilibrium, exit equals entry:

$$N_N \lambda_a(q = 0) = N_F \tau \tag{70}$$

where pins down the measure of startup specific R&D.

Appendix D Extensions

D.1 A More Recent Sample Period

In this section, I calibrate the model to the sample period from 1998 to 2010 to cover more recent observations. The firm-level data are from the enter for Research in Security Prices (CRSP) and the Merged CRSP-Compustat Database. I apply the statistical model derived in Section 3.1 on the public firm data to estimate the economy-wide moments. I use patent data for innovations. The patent data are from Patent Examination Research Dataset and Patent Assignment Dataset (PAD), both provided by the US Patent and Trademark Office

Table C2: Directly Calibrated Parameters Given Indirect Inference Results (A More Recent Sample Period)

λ_0	k_2	m
0.021	0.119	0.007

Table C3: Indirect Inference Calibrated Parameters (A More Recent Sample Period)

κ	μ	η	k_1	α	β_a	β_b	γ_H
0.006	1.821	0.213	0.937	2.001	0.203	1.953	0.937

(USPTO).[17] I link firm-level data and the patent data using the linked CRSP-USPTO data provided by Kogan et al. (2017).

I calibrate the model in the same way as explained in section 3. The external calibrated parameters are also the same as in Table 3. Other parameters are reported in Table C3 and Table C2. The targeted moments are reported in Table C4. Table C5 shows the counterfactual results. The patterns are the same as the baseline sample period.

D.2 Financial Friction

The model implicitly incorporates the idea of financial frictions in the complementarity. The financial friction is an additive part of the complementarity under certain assumptions. One example is described below.

When an inventor finishes working, the firm pays the wage and purchases the stock from the inventor. The total spending is $W(\tilde{q})$. Firms can borrow from outside at the interest rate r to finance the spending under a collateral constraint

$$W(\tilde{q}) \leq \varepsilon(\tilde{q}) V(\tilde{q}), \tag{71}$$

where $\varepsilon(\tilde{q})$ is a random variable ($\varepsilon > 0$, $f_\varepsilon(\varepsilon)$). If the constraint binds, the firm cannot fully develop the business potential of the innovation. The firm can sell the innovation on the secondary market, where the buyer can further commercialize the innovation. In this case, the effect of financial frictions shows up exactly as the complementarity in the benchmark model.

[17] I estimate the probability of selling an innovation using PAD for two sample periods. Then scale all probability related moments using the ratio. This is because, rather than using all patents observations, Figueroa and Serrano (2019) keeps only **corporate patents**, which take about 75% in the whole patent sample. But "corporate patents" are not marked in PAD. Hence, to ensure the samples are consistent, I scale all patent trade-related moments using the same ratio.

Table C4: Model Fit for Key Targeted Moments (A More Recent Sample Period)

Moment	Data	Model
Profitability	0.11	0.11
Discount rate	0.02	0.02
Entry rate	0.07	0.07
Firm growth volatility	0.17	0.17
Aggregate growth rate	0.02	0.02
Average patent value	0.03	0.03
start-up buyout rate	0.16	0.16
Pr(Sell|Big)[1]	0.09	0.09
Pr(Sell)	0.09	0.09
Growth-size relation β_g[2]	-0.04	-0.04
% Inno, emp<500[3]	0.07	0.07
% Inno, emp$<2,000$	0.24	0.24
% Inno, emp$<5,000$	0.34	0.35
60th pctl $\tilde{q}$ weighted by R&D	19.91	16.16
Average growth rate	0.07	0.05

[1] Pr(Sell|Big) measures the probability to sell an innovation given it is invented by a big firm. A "Big firm" is defined as a firm with more than 500 employees, according to USPTO.

[2] β_g is the coefficient of the growth-size regression.

[3] The % Inno is the cumulative density function of innovations created in firms with less than certain employment. For example, "% Inno, emp<500" means the share of innovations, among all innovations created in this period, that are invented in a firm with fewer than 500 employees.

Table C5: Counterfactuals (A More Recent Sample Period)

	(1)	(2)	(3)	(4)	(5)	(6)
Moment	Benchmark	No Market	Signal	Signal	Subsidy	Tax
			$\alpha_\epsilon = 1$ $R^2 = 0.2$	$\alpha_\epsilon = 2.5$ $R^2 = 0.6$	1%	1%
Growth rate	2.003	-0.266	0.031	0.122	0.003	-0.004
% Inno in startups	0.24	-0.240	0.32	4.27	0.098	-0.075
% Inno, $q < 0.5$	6.43	-5.680	-0.410	-4.490	-0.10	-0.076
% Inno, $q \in [0.5, 20)$	43.646	-11.185	1.864	7.484	0.582	-0.708
% Inno, $q \in [20, 100)$	49.685	17.101	-1.775	-7.265	-0.585	0.858
% Inno, $q \geq 100$	0	0	0	0	0	0
Pr(Sell)	2.451	-9.090	0.39	0.7	0.069	-0.079

[1] The table reports the **percentage points difference** with respect to the benchmark. For example, the growth rate change is calculated according to $(g_{\text{counter}} - g) \times 100$. The first column reports the benchmark case in percentage points. The second and third columns analyze a case in which there is a noisy signal when firms trade innovations on the secondary market. The noise ϵ follows a Pareto distribution with scale parameter 1 and shape parameter α_ϵ. The fourth (fifth) column reports the results when there is a one-time innovation transaction subsidy(tax). The tax rate is as low as 1% transaction value. The sixth column shows a case where firms cannot trade innovations at all.

D.3 Innovations as Substitutions

The benchmark model assumes that each innovation can improve a firm's quality. Cunningham et al. (2021) suggests that some innovations may be substitutions of existing technology. The model can incorporate this by adding one assumption—firms are exposed to random random negative quality shocks, which are proportional to the aggregate new innovation arrival rate.

D.4 Use Stock Options Instead of Equity

Using the stock options mainly affect the constant transfer of a contract; stock options capture the equity variance and protected by a lower bound. The same logic still applies: in a bigger firm, the innovation contributes to a smaller share of the equity value. Therefore, the only difference is the functional form of the variance associated with the incentive. Essentially, the mechanism behind the incentive problem does not change. The qualitative results still hold while the quantitative results may be different.

D.5 Patent Originality Distribution

One possible mapping of effort-sensitivity in the real life is the patent originality, as defined by Hall et al. (2001):

$$\text{Originality}_j = 1 - \sum_{i}^{n_j} s_{ij}^2$$

where s_{ij} denotes the percentage of citations made by patent j that belong to patent class i. Higher Originality_j means the patent j relies on technologies in many different fields, and hence it is more novel. Figure D1 plots the average originality by firm size in 1997. It decreases with firm size, which is consistent with the model implications.

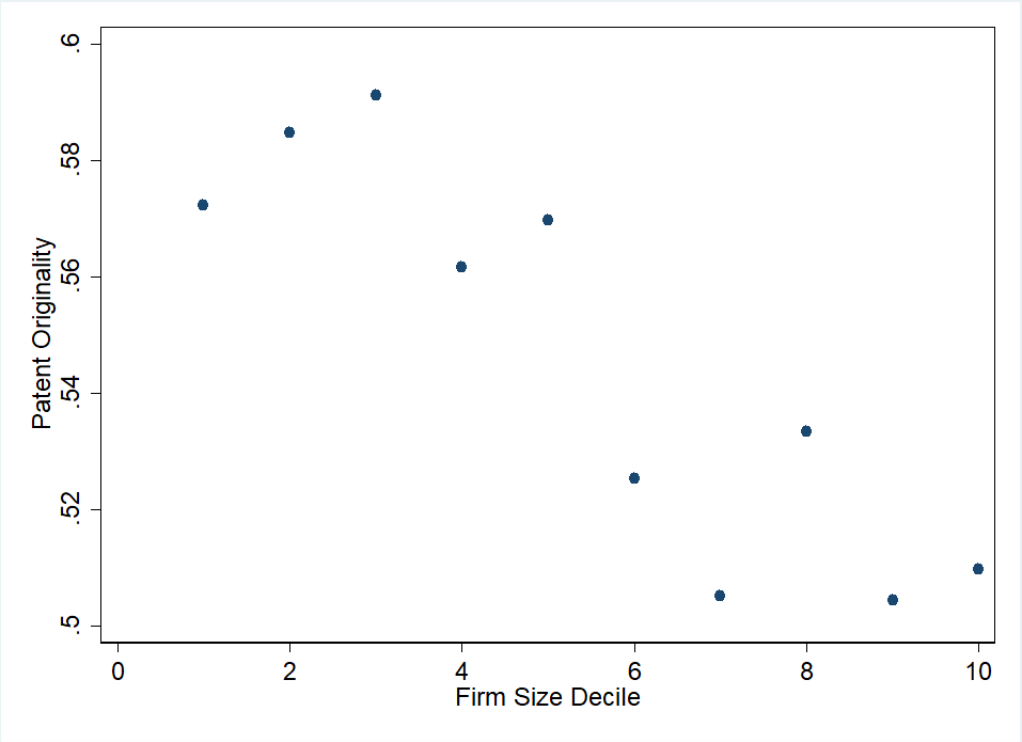


Figure D1: The Average Originality By Firm Size in 1997