# Incomplete Markets and Exchange Rates*

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**Abstract**

We investigate the implications of market incompleteness (both within and across countries) for the co-movement of real exchange rates and relative aggregate consumption, using an analytically tractable two-country, consumption-based asset pricing model with heterogeneous agents. Domestic market incompleteness can recover the observed co-movement as long as periods of exchange rate depreciations coincide with periods of low idiosyncratic risk, making foreign currency returns a poor hedge. International market incompleteness is required to capture the low observed exchange rate volatility, despite the introduction of idiosyncratic risk.

**Keywords:** exchange rates, incomplete markets, heterogeneous agents.

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# 1. Introduction

We revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993) – which describes the cyclicality of exchange rates across countries in terms of co-movement of relative aggregate consumption growth and real exchange rate depreciation – to consider the implications of domestic and international market incompleteness for exchange rate dynamics.

When international financial markets are complete, a large class of representative agent models with power utility admits the following relationship:

$$\left(\frac{C_{t+1}}{C^*_{t+1}}/\frac{C_t}{C^*_t}\right)^s = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where $C_t$ is Home aggregate consumption, $C^*_t$ is Foreign aggregate consumption, $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency, and $s$ is the inverse of the inter-temporal elasticity of substitution. Trade in risk-free Home and Foreign currency denominated bonds implies ex-post redistribution, which must occur entirely through exchange rate movements. For example, following a positive productivity shock in the Home country, the Home currency depreciates leading to higher consumption abroad. Exchange rates thus move to reallocate wealth from Home to Foreign and so are "risky" from the perspective of a Home investor.[1] Put differently, complete markets imply that exchange rate depreciations coincide with periods of low marginal utility growth for Home investors and exchange rates are counter-cyclical. In the data, however, exchange rates usually appreciate when Home consumption rises and marginal utility growth is low, i.e. these are pro-cyclical, constituting the Backus-Smith *puzzle*.

When markets are incomplete internationally, the condition above needs to hold only in expectation and so may fail ex-post. Classical contributions in international

[1]This implication echoes closed economy complete markets models featuring a correlation of -1 between the representative agent stochastic discount factor (SDF) and the market portfolio, see, e.g. Lettau (2002).

macroeconomics such as Corsetti, Dedola and Leduc (2008) and Benigno and Thoenissen (2008) show that non-traded risk, which arises from consumption or production complementarities, enables incomplete market models to generate plausible patterns of international risk sharing. However, these models only allow cross-border trade in a single risk-free asset. Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign risk-free asset imply prices always co-move negatively with consumption – so incomplete market models cannot resolve the puzzle of excessive risk-sharing *regardless* of goods market frictions and other economy specifics. In practice, the number of assets traded across borders is high, so this result has far-reaching implications for both theory and practice.

We consider a generalization of the representative agent, two-country consumption capital-based asset pricing model with heterogeneous consumers within each country, extending Constantinides and Duffie (1996) to an international setting. These heterogeneous consumers trade in fully *integrated* international markets but face uninsurable permanent idiosyncratic income risk, giving rise to a no-trade equilibrium. Hence, domestic markets are incomplete. Within this stylized framework, we analytically derive conditions under which domestic market incompleteness gives rise to procyclicality of exchange rates. Our main result is that, when there is frictionless trade in at least two nominally risk-free bonds, procyclical exchange rates can be obtained if and only if the correlation of idiosyncratic consumption risk with exchange rate depreciation is sufficiently negative, relative to a threshold. The threshold depends on the relative volatility of exchange rates and idiosyncratic risk. This condition implies that the Foreign currency denominated bonds are a poor hedge for the idiosyncratic risk, yielding high returns at a time when domestic investors face low idiosyncratic risk, thus making exchange rates *risky*.

We next specify idiosyncratic risk as linear function of log consumption growth, which is in turn a linear process, and derive a closed-form international incomplete

markets wedge.[2] Two additional results arise in this setting. First, to deliver risky exchange rates with respect to idiosyncratic risk as required above to resolve the Backus-Smith puzzle, idiosyncratic risk must be sufficiently counter-cyclical with respect to aggregate consumption growth. Counter-cyclical risk implies that idiosyncratic consumption uncertainty is high in periods when aggregate consumption growth is low, an assumption which has found support in a large literature on income dynamics, e.g. Storesletten, Telmer and Yaron (2004). When idiosyncratic risk is sufficiently countercyclical there is an implication in terms of observable asset prices. Namely, the model implies that the Sharpe ratio of Home assets will be higher than that of Foreign, which is satisfied for the U.S. Additionally, we find that international market incompleteness is not necessary to generate procyclical exchange rates.

Second, we show that not only is exchange rate volatility larger under internationally complete markets than under incomplete markets for a given cyclicality of idiosyncratic risk, but it increases at a faster rate as risk becomes more counter-cyclical. Taken together, these results imply that while varying the cyclicality of idiosyncratic risk allows us to correct the cyclicality of exchange rates, international market incompleteness is required to correct for the volatility of exchange rates.

We take this model to the data and calibrate SDFs to match Sharpe ratios on equity returns for the U.S., the rest of the world and the cross-country correlation, and take the cyclicality of idiosyncratic risk estimate for the U.S. from the literature. In our quantitative exercise, we are able to match an annualized exchange rate volatility from the data and a slightly negative Backus-Smith covariance with international spanning of 50% for the common factor, illustrating a role for both domestic and international market incompleteness.

[2]Relative to a standard two-country CAPM, we require three additional ingredients to deliver our results: (i) the variance of the idiosyncratic consumption risk is a linear function of domestic consumption growth as in Constantinides and Duffie (1996) and Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016*a*) amongst others, (ii) a common global factor driving consumption growth, and (iii) a Foreign-country specific "noise" factor to discipline the correlation of SDFs across countries to below 1, consistent with the international co-movement of asset prices (Brandt, Cochrane and Santa-Clara, 2006).

We show that the SDFs used in deriving our results can be supported as an equilibrium outcome in a two-country, heterogeneous agent model with multiple goods. We construct an international no-trade equilibrium building on Svensson (1988) but emphasize that, away from the full home bias limit, trade wedges are required for goods markets to clear (as is generally true for a two-country CAPM). We then compare this equilibrium to the allocation of a social planner problem and show that the cyclicality of exchange rates (the Backus-Smith condition) is a sufficient measure for international risk sharing if prices are flexible and the planner faces the same trade wedges as the households.

Finally, we test whether exchange rates are risky with respect to the idiosyncratic risk in the data. In a large class of incomplete market models, Berger, Bocola and Dovis (2023) show that the lack of consumption risk sharing within countries can be summarized by a discount factor wedge which can be measured using panel data on households' consumption shares. Through the lens of our model, idiosyncratic risk can be proxied by the growth in consumption share of households. We develop two tests for our model. First, assuming heterogeneity only in the U.S., we use the discount factor wedge measured by BBD using consumption data to test the condition sufficient for our main results. Second, we construct a proxy of the discount factor wedge using growth of income shares for high-income individuals across seven advanced economies from the Global Repository of Income Dynamics (GRID) (Guvenen, Pistaferri and Violante, 2022), and test a more general condition that allows for heterogeneity across countries.[3] Across both tests, we robustly find support for the key mechanism that the discount factor wedge co-moves negatively with real exchange rate growth, implying exchange rates are indeed risky. Moreover, across a range of measures, we find that the correlation of this wedge with exchange rates is sufficiently negative such that the model is able to reconcile the Backus Smith puzzle.

[3]In the absence of detailed consumption dynamics data, we proxy for the growth in consumption shares of high income individuals with growth in their income shares. While imperfect due to potentially different MPCs across agents, this is nevertheless a useful exercise.

**Related Literature** We build on the work of Lustig and Verdelhan (2019), see also Benigno and Küçük (2012), who show that introducing a second internationally traded bond breaks the ability of canonical international macro models to reconcile the Backus-Smith puzzle, to allow for heterogeneous agents within each country. In doing so, we relate to Ramchand (1999) who considers a two-country model building on Constantinides and Duffie (1996) and derive a modified Cole-Obstfeld balanced trade equilibrium featuring full international risk sharing, and Kocherlakota and Pistaferri (2007) who show that models with domestically incomplete markets impose restrictions on higher order moments of the consumption distribution and exchange rates, assuming international markets are complete. In contrast, we study the implications of idiosyncratic risk for the covariance of aggregate consumption growth with exchange rates and allow for both domestic and internationally incomplete asset markets.[4]

Our approach relates to many important findings literature in international finance. Chernov, Haddad and Itskhoki (2024) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles. Jiang, Krishnamurthy and Lustig (2023) show that Euler equation wedges are necessary to resolve Backus-Smith puzzle.[5] Important precursors have shown that segmentation (Alvarez, Atkeson and Kehoe, 2002; Chien, Lustig and Naknoi, 2020) or intermediation frictions (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021) can resolve various exchange rate puzzles. Our heterogeneous consumers model with integrated markets illustrates that segmentation per se is not required to theoretically reconcile the cyclicality puzzle, and that domestic

[4]Relatedly, Leduc (2002) using a calibrated open economy model featuring idiosyncratic and aggregate risk investigated the role of these risks in generating currency premia. Tessari (2021) extends the asset pricing framework of Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016*b*) to an international finance environment and shows that shocks to common idiosyncratic volatility factor are an important pricing factor for carry trade returns.

[5]Verdelhan (2010), Colacito and Croce (2013), Hassan (2013), Karabarbounis (2014), Farhi and Gabaix (2016), and Kekre and Lenel (2024*b*) investigate the role of habits, long-run risk, country-size, home production, rare disasters, and safety shocks respectively in generating observed correlations between cross country asset returns and exchange rate returns.

market incompleteness offers a complementary micro-founded mechanism for understanding exchange rates. In contemporaneous work, Kekre and Lenel (2024*a*) show that permanent discount factor shocks can resolve the comovement and predictability puzzles. Viewed through the lens of closed economy work on heterogeneous agents in macroeconomics (Nakajima, 2005; Krueger and Lustig, 2010; Werning, 2015; Guerrieri and Lorenzoni, 2017; Berger et al., 2023), domestic incomplete markets provide a microfoundation for discount factor wedges.

More generally, our model with heterogeneous consumers ties to a large literature investigating whether idiosyncratic risk affects macro-finance aggregates (see, e.g. Mankiw, 1986; Weil, 1992; Guvenen, 2009; Heathcote, Storesletten and Violante, 2014; Kaplan, Moll and Violante, 2018; Auclert, Rognlie and Straub, 2023; Di Tella, Hébert and Kurlat, 2024). In the open economy macro literature, for papers with agent heterogeneity, see Ghironi (2006); Kocherlakota and Pistaferri (2007); Farhi and Werning (2016); Fornaro (2018); De Ferra, Mitman and Romei (2020); Auclert, Rognlie, Souchier and Straub (2021); Guo, Ottonello and Perez (2023); Hong (2023); Acharya and Challe (2024) amongst others.

The rest of the paper is organised as follows. Section 2 derives our main conditions on exchange rate cyclicality contrasting the representative agent case to one with idiosyncratic risk. Section 3 derives a closed form incomplete markets wedge and Section 3.1 conducts a quantitative exercise. Section 4 illustrates how the reduced-form conditions we use can be derived as an equilibrium. Section 5 directly evaluates the plausibility of our sufficient condition for reconciling exchange rates using income and consumption data. Section 6 concludes.

## 2. Two-country, Consumption CAPM Models

The starting point of our analysis is the consumption capital based asset pricing model (CAPM) building on Rubinstein (1974); Lucas (1978); Breeden (1979), and

the two-country model in Lucas (1982). Consider a two-country model where $M_{t+1}$ denotes the Home representative household's SDF and $M^*_{t+1}$ denotes the Foreign representative household's SDF. We restrict attention to time-separable utility, e.g. CRRA utility over consumption where $M^{(*)}_{t+1} = \beta \left( \frac{C^{(*)}_{t+1}}{C^{(*)}_t} \right)^{-s}$, as in condition (1).

Home and Foreign households each trade their respective domestic risk-free real bonds with returns $R_t$ and $R^*_t$ respectively frictionlessly (i.e. no borrowing constraints). No-arbitrage pricing implies:

$$\mathbb{E}_t[M_{t+1}] = 1/R_{t+1}, \tag{2}$$

$$\mathbb{E}_t[M^*_{t+1}] = 1/R^*_{t+1} \tag{3}$$

If the Home (Foreign) households also trade the Foreign (home) bond, and the real exchange rate at time $t$ is denoted with $\mathcal{E}_t$, such that an increase in $\mathcal{E}_t$ corresponds to a depreciation, then we additionally obtain the following two Euler conditions:

$$\mathbb{E}_t \left[ M^*_{t+1} \frac{\mathcal{E}_t}{\mathcal{E}_{t+1}} \right] = 1/R_{t+1}, \tag{4}$$

$$\mathbb{E}_t \left[ M_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \right] = 1/R^*_{t+1}. \tag{5}$$

We assume SDFs, allocations and prices are jointly log-normal. To close the model without explicitly specifying goods markets, an exchange rate process is needed, which is consistent with equations (2)–(5) above. This problem reduces to finding an exchange rate process that satisfies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \tag{6}$$

where $x = \log(X)$. Naturally, the process corresponding to complete markets ($\Delta e_{t+1} = m^*_{t+1} - m_{t+1}$) is one candidate. More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (6):

$$\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1} \tag{7}$$

where $\eta_{t+1}$ is an incomplete markets wedge which must satisfy certain conditions

imposed by asset trade. Condition (1) restricts the covariance between relative consumption growth and exchange rate depreciations to be positive. In the data, this covariance is negative, hence the Backus-Smith *puzzle*. Turning to SDFs, this implies the cyclicality of exchange rates $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ is counterfactually positive in the model.

Combining (2) and (4), with (7) – which implies that the Home bond is internationally traded – yields:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{8}$$

Combining (3) and (5), with (7) – which implies that the Foreign bond is internationally traded – yields:

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m^*_{t+1}, -\eta_{t+1}) \tag{9}$$

These two conditions mirror those studied by Lustig and Verdelhan (2019) and bound the joint dynamics of the incomplete market wedge and the SDFs, carrying strong implications for the macro side of the model. The conditions reflect the return, for a Home and a Foreign investor respectively, from the non-traded component of exchange rates which must compensate for both volatility of returns and the riskiness of the exchange rate.

Replacing the SDFs with the aggregate consumption growth, the incomplete markets wedge is related to:[6]

$$\eta_{t+1} = \log \underbrace{\left( \frac{P_{t+1}}{P_t} \frac{P^*_t}{P^*_{t+1}} \right)}_{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}} - \log \underbrace{\left( \frac{C_t}{C_{t+1}} \frac{C^*_{t+1}}{C^*_t} \right)^s}_{\frac{M_{t+1}}{M^*_{t+1}}}$$

where $P_t$ is the Home price level, $C_t$ is aggregate consumption, $s^{-1}$ is the inter-temporal elasticity of substitution, and terms with asterisks denote the corresponding Foreign objects. The wedge, $\eta$, is often interpreted as the non-traded component of exchange

[6]For a closed form solution of the incomplete markets wedge in a two-country open economy model, see Pavlova and Rigobon (2007).

rate movements or the wealth gap, see e.g. Corsetti, Dedola and Leduc (2023*a*).

## 2.1. Representative Agent and Trade in Risk-free Assets

Having now specified our framework, we illustrate the mechanism through which goods market frictions in incomplete market models can help reconcile the pattern of international risk-sharing.

**Proposition 1** (One Int'l Traded Asset, Representative Agent No-Arbitrage). *When only Foreign bonds are internationally traded such that equations* (2), (3) *and* (5) *hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) \geq var_t(m^*_{t+1} - m_{t+1}) - \log \mathbb{E}_t[e^{\eta_{t+1}}] \tag{10}$$

*where,*

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \tag{11}$$

**Proof.** See Appendix A.2. □

The RHS of condition (10) is equal to the volatility of the exchange rate growth under complete markets and is strictly positive. The condition is satisfied if either the non-traded component $\eta_{t+1}$ leads to relative price fluctuations which are safe from the perspective of a Home investor, as captured by $cov_t(m_{t+1}, \eta_{t+1}) > 0$ or that the volatility of the non-traded component is high. Equation (11) shows that non-traded risk results in relative price fluctuations which are particularly *safe* when the Home SDF is very volatile or when international comovement in SDFs is low relative to the comovement of exchange rates and the Home SDF.[7] In Section 4, we specify a two-country model with multiple goods and investigate the parametrizations consistent with this condition.

The condition (10) provides a general characterization for goods markets mecha-

[7] Brandt et al. (2006) show that SDFs must comove very strongly to explain the relatively low exchange rate volatility in the data.

nisms developed to resolve the Backus-Smith puzzle in models with consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) or global value chain fragmentation (Corsetti, D'Aguanno, Dogan, Lloyd and Sajedi, 2023*b*), amongst others. A limitation of Proposition 1, and the models which satisfy it, is that it may exacerbate other exchange rate puzzles– in particular, that of excess volatility of exchange rates. The RHS of equation (10) is equal to the volatility of exchange rates under complete markets, and models with a low volatility will generally fare better in resolving the cyclicality puzzle, which rationalizes why models such as Corsetti et al. (2008) display very low risk premia (Lustig and Verdelhan, 2019, pp 2241). In Appendix , we simulate an endowment version of Corsetti et al. (2008) and verify Proposition 1 is sufficient, see Figure 2.

Cross-border trade in a second risk-free asset prevents the model from reconciling the Backus-Smith puzzle because this constrains the *safety* of relative price movements due to non-traded risk. From Equation (8), when the non-traded component becomes safe for Home investors, lower returns on their Foreign bond holdings ($\mathbb{E}_t[\eta_{t+1}]$) translate to higher returns on Home bonds for Foreign investors ($\mathbb{E}_t[-\eta_{t+1}]$). This is only consistent with no arbitrage if the non-traded component becomes sufficiently risky for Foreign investors which would restore the Backus-Smith correlation.

**Corollary 1** (Two Int'l Traded Asset, Representative Agent No-Arbitrage).
*As the Home bond also becomes internationally traded without arbitrage,* $cov_t(m_{t+1}, \eta_{t+1}) \rightarrow -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2} var_t(\eta_{t+1})$ *as in equation* (8)*, then condition* (10) *implies* $var_t(\Delta e_{t+1}) \leq 0$.

**Proof.** See Appendix A.2. □

As enough assets become traded and markets approach completeness, $\sigma_\eta \rightarrow 0$. However, even if asset markets are far from complete, trade in just two nominally risk-free bonds is sufficient to ensure $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) > 0$ as we detail in Appendix B.2.

## 2.2. Many (heterogeneous) agents

Assume there is a continuum of consumers $i$ in the Home economy, each facing uninsurable idiosyncratic permanent shocks to their consumption. The financial markets are domestically incomplete with respect to this idiosyncratic risk. For simplicity, we maintain the representative agent assumption in the Foreign economy. We also postpone construction of an equilibrium, with goods market clearing and micro-foundations, and an empirical evaluation to Section 4. Here, we preview our main result.

Each individual consumer $i$ trades frictionlessly in Home and Foreign risk-free bonds, and hence individual Euler equations hold as in (2)-(5), with $M_{t+1}$ replaced by consumer-specific SDF $M^i_{t+1}$. With particular assumptions on agents' consumption processes that allow aggregation as in Constantinides and Duffie (1996), we can construct an aggregate Euler equation for an asset with payoff $\tilde{R}_{t+1}$ summarized by:

$$\mathbb{E}_t \left[ \underbrace{\beta \left( \frac{C_{t+1}}{C_t} \right)^{-s}}_{M_{t+1}} e^{\tilde{y}_{t+1}} \tilde{R}_{t+1} \right] = 1 \tag{12}$$

where $\tilde{y}_{t+1}$ denotes the log of the variance of the idiosyncratic permanent shocks faced by households, which we assume is jointly log-normally distributed with the other variables. We retain $M_{t+1}$ notation to denote aggregate consumption growth based SDF. For the case of the Home bond $\tilde{R}_{t+1} = R_{t+1}$ and for the Foreign bond $\tilde{R}_{t+1} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R^*_{t+1}$. Combining these aggregate Euler equations, along with (3), (4) yields:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) \tag{13}$$

This extended economy admits the exchange rate process described by Equation (7), but with new restrictions imposed by domestic incompleteness on the (international) incomplete markets wedge $\eta_{t+1}$, generalizing Lustig and Verdelhan (2019). Cross-

border trade in Home and Foreign bonds imply:

$$\begin{aligned} \mathbb{E}_t\left[M_{t+1}e^{-\eta_{t+1}}\right] &= \mathbb{E}_t\left[M^i_{t+1}\right], \\ \mathbb{E}_t\left[M^i_{t+1}\frac{M^*_{t+1}}{M_{t+1}}e^{\eta_{t+1}}\right] &= \mathbb{E}_t\left[M^*_{t+1}\right] \end{aligned}$$

The resulting restrictions are given by:

$$\begin{aligned} \mathbb{E}[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) &- cov_t(m_{t+1},\eta_{t+1}) \\ &- \log\mathbb{E}[e^{y_{t+1}}] - cov_t(m_{t+1},y_{t+1}) \end{aligned} \tag{14}$$

$$\begin{aligned} -\mathbb{E}[\eta_{t+1}] = \frac{1}{2}var_t(\eta_{t+1}) &+ cov_t(m^*_{t+1},\eta_{t+1}) \\ &+ \log\mathbb{E}[e^{y_{t+1}}] + cov_t(m^*_{t+1},y_{t+1}) + cov_t(\eta_{t+1},y_{t+1}) \end{aligned} \tag{15}$$

The incomplete markets wedge, or equivalently the non-traded component of exchange rates, must now compensate Home investors for both the aggregate and idiosyncratic volatility, as-well as their risk, i.e. the covariance of valuations and returns.

**Proposition 2** (Two Int'l Traded Assets, Many Agents).
*The two-country model with two internationally traded bonds and a continuum of heterogeneous Home consumers characterized by Equations (3), (4) and (12) evaluated for assets with payoffs* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R^*_t\}$ *delivers* $cov_t(m^*_{t+1} - \log(\int_i e^{\Delta c^i_t}di)^{-s}, \Delta e_{t+1}) < 0$ *if and only if*

$$1 \geq -\rho_{\tilde{y}_{t+1},\Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{y}_{t+1})} \tag{16}$$

where $\rho_{\tilde{y}_{t+1},\Delta e_{t+1}} \equiv \dfrac{cov_t(\tilde{y}_{t+1},\Delta e_{t+1})}{\sigma_t(\Delta e_{t+1})\sigma_t(\tilde{y}_{t+1})}$.

**Proof.** See Appendix A.2. □

Whereas in a representative agent economy, exchange rate risk becomes traded (is spanned) when households trade in both Home and Foreign real bonds across borders, introducing idiosyncratic consumption risk —which co-moves with the exchange rate— recovers a non-traded component and can deliver negative cyclicality of exchange

rates.[8] If variance of the idiosyncratic consumption risk falls with depreciations, that is domestic consumers face lower variance of permanent income shocks at times when foreign bonds yield high returns, Foreign bonds are a poor hedge for domestic consumption. This idiosyncratic risk factor allows a *disconnect* of exchange rates with domestic aggregate consumption growth. When the (negative) comovement of idiosyncratic risk factor with exchange rates is sufficiently large, then the model can accommodate a negative Backus-Smith correlation even with trade in two nominally risk-free bonds. The poor hedging property of Foreign bonds arises in this framework due to a particular comovement of the idiosyncratic risk factor with exchange rates, and is consistent with a solution to *diversification puzzle* offered by Heathcote and Perri (2013).

Note that this reconciliation for Backus Smith puzzle in the aggregate, is sensitive to the number of traded assets. In particular, the necessary requirement to obtain low correlation is relaxed as the corollary below details the case of when only the Foreign bond is internationally traded.

**Corollary 2** (One Int'l Traded Asset, Many Agents).
*When only Foreign bonds are internationally traded and there exist a continuum of heterogeneous Home consumers such that Equations* (3)*, and* (12) *evaluated for assets with returns* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R_t^*\}$ *hold, then* $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$\begin{aligned} cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq \quad & var(m_{t+1}^* - m_{t+1}) \\ & - \log \mathbb{E}_t \left[e^{\tilde{y}_{t+1}}\right] - \ cov_t(\tilde{y}_{t+1}, m_{t+1}^* + \eta_{t+1}) \end{aligned}$$

**Proof.** Follows from the Proof of Proposition 1 and and equations (14) and (15). □

The model requires the non-traded component of exchange rates to be sufficiently safe, generalizing Proposition 1 to an environment with idiosyncratic risk. Indeed, in

[8] Note that we adjust the risk-sharing measure to consider the co-movement of aggregate consumption and exchange rates, in SDF space, consistent with the original definition of the Backus-Smith covariance.

contrast to the case with two internationally traded risk free assets, the condition is relaxed by the variance of the idiosyncratic consumption risk faced by the agents – without relying on the covariance term (i.e. focusing on the case where idiosyncratic and aggregate risk are uncorrelated).

## 2.3. Risky Assets

In practice, many (more than two) assets are traded across borders but few of these are risk-free in real terms, e.g. long-maturity bonds and equity. We extend our main findings to a framework with trading in multiple risky assets but when (nominally) risk-free bonds are not available.

**Proposition 3** (Many Assets, Heterogeneous Consumers No-Arbitrage)
*When Home and Foreign currency risky assets with returns* $\tilde{r}^{(*)}_{t+1}$ *are internationally traded, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$-\rho_{\tilde{y}_{t+1},\Delta e_{t+1}} + \frac{cov_t(\eta_{t+1} + \tilde{y}_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1})}{\sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{y}_{t+1})} \tag{17}$$

**Proof.** See Appendix A.2. □

When only risky assets are traded, exchange rates need not be spanned, allowing further scope for the cyclicality of exchange rates to be negative. In particular, the negative cyclicality is recovered if the incomplete markets wedge internationally or idiosyncratic risk domestically co-vary positively with the return from a risky foreign asset. Notably, the new (second) term in the Proposition is non-zero only if international financial markets are also incomplete.

If, however, we additionally allow for trade in two nominally risk-free assets the new term in Proposition 3 ($cov_t(\eta_{t+1} + \tilde{y}_{t+1}, \tilde{r}^{k(*)}_{t+1} - \tilde{r}^k_{t+1})$) converges to 0, as stated in the corollary below. This implies that Proposition 2 is the necessary and sufficient condition even when there is trade in many assets. Nonetheless, $\rho_{\tilde{y}_{t+1},\Delta e_{t+1}}$ and $\sigma_t(\Delta e_{t+1})$ are themselves changing as the number of traded assets changes.

**Corollary 3 (Many Assets, Many Agents)** *Condition (16) is the necessary and sufficient condition for* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$*, even when there is additional trade in arbitrary risky assets.*

## 3. A Closed-Form Incomplete Markets Wedge

We next specify the law of motion for a common factor that drives aggregate heteroskedastic consumption growth as in Lustig and Verdelhan (2019), which builds on Cox, Ingersoll and Ross (1985, CIR henceforth). Specifying this process allows us to derive the incomplete markets wedge in closed form and tease out the role of international market incompleteness. We introduce two further critical ingredients relative to the Lustig and Verdelhan (2019). First, we allow for countercyclical idiosyncratic risk as in Constantinides and Duffie (1996) and Herskovic et al. (2016*a*) amongst others. Second, using a combination of a common factor and noise, we allow for a correlation of SDFs across countries which is positive but robustly below 1 (Brandt et al., 2006), consistent with the international comovement of asset prices and aggregates.

Suppose that a common factor $z_t$ drives aggregate consumption growth globally

$$z_{t+1} = \rho z_t + u_{t+1}$$

Aggregate consumption growth in the Home country is given by $\Delta c_{t+1} = \sqrt{z_t} u_{t+1}$ while consumption growth abroad is given by $\Delta c^*_{t+1} = \xi^* \sqrt{z_t} u_{t+1} + \sigma_\nu u^\nu_{t+1}$, such that $\xi^* > 1$ corresponds to an environment where the shock $u_t$ has a relatively larger effect on the Foreign aggregate consumption growth and $u^\nu_{t+1}$ is Foreign consumption specific shock. Both $u_{t+1}$ and $u^\nu_{t+1}$ are mean zero normal i.i.d innovations with standard deviations $\sigma$ and $\sigma_\nu/\xi$ respectively. Define the (aggregate) Home and Foreign SDFs defined on aggregate consumption are given by:

$$-m_{t+1} = \beta + \chi z_t + \gamma \sqrt{z_t} u_{t+1}, \tag{18}$$

$$-m^*_{t+1} = \beta^* + \chi^* z_t + \gamma^* (\sqrt{z_t} u_{t+1} + \sigma_\nu u^\nu_{t+1}) \tag{19}$$

where we define $\gamma^*$ as the product of Foreign risk aversion and $\xi^*$.[9] The volatility of aggregate consumption is given by $z_t$ and $var_t(m_{t+1}) = \gamma^2 z_t$, $var_t(m^*_{t+1}) = \gamma^{*2}(z_t + \sigma^2_\nu)$. In the presence of uninsurable idiosyncratic risk, the relevant pricing kernel is instead characterised by (12) and additionally depends on the scaled log of the variance of the idiosyncratic permanent shocks ($\tilde{y}_{t+1}$), for which we specify a linear process:

$$\tilde{y}_{t+1} = \tilde{\phi}\sqrt{z_t}u_{t+1}, \tag{20}$$

where $\tilde{\phi} = \phi\frac{\gamma(\gamma+1)}{2}$ captures the cyclicality of idiosyncratic risk. We normalize the foreign cyclicality parameter $\phi^*$ to 0 such that it is captured by the parameter $\gamma^*$.

**Proposition 4 (Equilibrium Incomplete Markets Wedge)**

*In the model with heterogeneous consumers, satisfying (3), (4), (12) evaluated for* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R^*_t\}$ *and (7), the incomplete markets wedge is given by:*[10]

$$\begin{aligned}\eta_{t+1} = -\frac{1}{2}(\gamma - \tilde{\phi} + \gamma^*)\sqrt{(\gamma - \tilde{\phi} - \gamma^*)^2 - \lambda}\ z_t + \frac{1}{2}\gamma^*\sqrt{\gamma^{*2} - \lambda^\nu}\ \sigma^2_\nu \\ -\sqrt{(\gamma - \tilde{\phi} - \gamma^*)^2 - \lambda}\ \sqrt{z_t}u_{t+1} + \sqrt{\gamma^{*2} - \lambda^\nu}\ \sigma_\nu u^\nu_{t+1} \\ -\sqrt{\lambda - \kappa}\ \sqrt{z_t}\epsilon_{t+1} + \sqrt{\lambda^\nu - \kappa^\nu}\ \sigma_\nu \epsilon^\nu_{t+1},\end{aligned} \tag{21}$$

*where* $\lambda \geq \kappa, \lambda^\nu \geq \kappa^\nu$ *are parameters governing cross border spanning and*

$$\kappa = (\gamma - \tilde{\phi} - \gamma^*)^2 - (\gamma - \tilde{\phi} - \gamma^*)\sqrt{(\gamma - \tilde{\phi} - \gamma^*)^2 - \lambda} \geq 0, \tag{22}$$

$$\kappa^\nu = \gamma^{*2} - \gamma^*\sqrt{\gamma^{*2} - \lambda^\nu} \geq 0 \tag{23}$$

*and* $var_t(\Delta e_{t+1}) = \kappa z_t + \kappa^\nu \sigma^2_\nu$.

[9] If $\chi = \chi^* = 0$, this corresponds to power utility over consumption as in the canonical representative agent Lucas model. Allowing $\chi > 0$ is a reduced form way to capture a connection between the conditional mean and variance of SDFs which may arise due to habit formation or recursive preferences, see e.g. Hassan, Mertens and Wang (2024).

[10] Square roots refer to the positive root only unles we include the $\pm$ ahead. For the $z_t$ factor, while the negative root is a potential solution, conditional on $\phi < 0$, it violates $\lambda > \kappa$ which we require for a real solution.

**Proof.** See Appendix A.2. □

It is also useful to define the benchmark of internationally complete markets where $\eta_{t+1}^{CM} = -\tilde{y}_{t+1} = -\tilde{\phi}\sqrt{z_t}u_{t+1}$, i.e. when exchange rates move to reflect differences in the as-if representative agent SDFs, such that $\lambda_{CM} = \kappa_{CM} = (\gamma - \phi - \gamma^*)^2$ and $\lambda_{CM}^{\nu} = \kappa_{CM}^{\nu} = \gamma^{*2}\sigma_{\nu}^2$. To capture incomplete spanning, we parametrize

$$\lambda = \alpha(\gamma - \tilde{\phi} - \gamma^*)^2 \tag{24}$$

for $\alpha \in (0, 1)$. It can be verified that $\alpha \to 1$ coincides with the complete markets benchmark and lower levels of spanning imply increasing levels of unspanned risk. We similarly define $\lambda^{\nu} = \alpha^{\nu}\gamma^{*2}$.[11] Armed with a closed form expression for the international incomplete markets wedge (21), we can revisit the condition which determines the cyclicality of exchange rates.

**Proposition 5 (Exchange rate cyclicality in Cox et al. (1985) model)**

*In the model with heterogeneous consumers, satisfying (3),(4), (12), evaluated for* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R_t^*\}$ *and (7), in the limit* $\sigma_{\nu} \to 0$, $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$(\gamma - \gamma^*)[(\gamma - \gamma^*) - \tilde{\phi} + \beta_1] \leq 0.$$

*Suppose further that* $\gamma - \gamma^* < 0$ *then this is satisfied when* $var_t(\hat{m}_{t+1}) > var_t(\hat{m}_{t+1}^*)$.

**Proof.** See Appendix A.2. □

Suppose that $\beta_1 \to 0$. Proposition 5 states that the Backus-Smith covariance is negative, allowing for counter-cyclical idiosyncratic risk $\phi < 0$, as long as $\gamma - \gamma^* < 0$. Allowing spanning to be linear in risk (24), we leverage the Hansen and Jagannathan (1991) bounds to argue the condition is satisfied if equity returns in the U.S. are

[11] As discussed in Section 2.3, additional cross-border trade in risky assets which span $u, u^{\nu}$ will impose $\alpha \to 1$, $\alpha^{\nu} \to 1$.

higher than abroad– supported by the data and discussed in the calibration below.[12] The limit as $\sigma_\nu \to 0$ is taken for simplicity, as this limit allows us to express the threshold on $\tilde{\phi}$ for the Backus-Smith condition, to be orthogonal to the degree of market incompleteness. Away from this limit, the degree of market completeness does matter for the threshold (i.e. $\lambda$ and $\lambda^\nu$ would feature). Once more, contrasting this to the case $\tilde{\phi} \to 0$ which corresponds to the representative agent limit (Lustig and Verdelhan, 2019), Proposition 5 can never be satisfied, see proof for details.

**The Role for Market Incompleteness** The cyclicality and volatility of exchange rates are intricately connected. It turns out that if SDFs are sufficiently volatile to explain (e.g.) stock prices, models with complete markets will struggle to reconcile the low exchange rate volatility observed in the data.[13] While introducing idiosyncratic risk can recover pro-cyclical exchange rates consistent with low risk-sharing, in the next Proposition we show that this is at the cost of further exacerbating exchange rate volatility.

**Proposition 6 (Idiosyncratic Risk and Exchange Rate Volatility)** *If income risk is counter-cyclical* ($\phi < 0$)*, exchange rate volatility* (22) *is always higher under international complete markets* $\kappa^{CM} > \kappa^{IM}$ and $\dfrac{d\kappa^{CM}}{d|\phi|} > \dfrac{d\kappa^{IM}}{d|\phi|} > 0$.

**Proof.** Consider (22) for $\phi < 0$ and allow $\lambda > \kappa$ and denote this as $\kappa^{IM}$. Imposing complete markets $\lambda^{CM} = (\gamma - \gamma^* - \phi)^2$ and defining this as $\kappa^{CM}$ confirms the result. □

Taken together, our results imply that while varying the cyclicality of idiosyncratic risk (which requires domestic market incompleteness) allows us to correct the cyclicality of exchange rates, international market incompleteness is critical to correct the volatility of exchange rates, both in the common factor and the noise. As we next illustrate,

[12] The variance condition in Proposition 5 is robust to not normalizing $\phi^*$ such that the condition reduces to

$$(\gamma - \gamma^* + \tilde{\phi}^*)[(\gamma - \tilde{\phi} - \gamma^* + \tilde{\phi}^*)(1 + \sqrt{1-\alpha})] \leq 0$$

A simple case arises when $\gamma = \gamma^*$. Then, the condition reduces to $\phi < \phi^* < 0$.

[13] Consistent with Brandt et al. (2006), only in the limit where we shut down $\sigma_\nu$ such that SDFs are perfectly correlated, can internationally complete markets be consistent with observed exchange rate volatility.

enriching models in both directions allows more flexibility to match the data – a low exchange rate volatility and a cyclicality that is close to zero but negative in SDF space.

## 3.1. Quantitative Exercise

**Calibration** We build on the calibration of Backus et al. (2001), Lustig and Verdelhan (2019) and calibrate $z_t$ using the process for U.S. interest rates. Starting from: $r_t = [\chi - \frac{1}{2}(\gamma - \phi)^2]z_t$, we choose $\chi = -1 + \frac{1}{2}(\gamma - \phi)^2$ to ensure counter-cyclical interest rates needed to match the Fama puzzle (Verdelhan, 2010). Specifically, we use $\theta = 0.0044$ so $\mathbb{E}_t[z_{t+1}] = \theta$, $\sigma = 0.0084$ and $\rho = 0.976$ to be directly comparable to the literature, and select $\alpha$ to match average interest rates. We additionally target a correlation of SDFs across countries of 0.8, the upper limit in Brandt et al. (2006). This implies $\rho_{m,m^*} = 0.8 = \theta/\sqrt{\theta}\sqrt{\theta + \sigma_\nu^2}$, from which we back out $\sigma_\nu^2 = 0.002475$.

To calibrate the asymmetry in $\gamma - \phi$ and $\gamma^* - \phi^*$, we use Sharpe ratios on equity from the U.S. and abroad. We use a Sharpe ratio of 0.5 for the U.S. based on (S&P 500) and we take $\phi = 5$ from the data for the U.S. following (Acharya, Challe and Dogra, 2023). We back out $\gamma \approx 1.1722$ using $(\gamma - \tilde{\phi})\sqrt{\theta} = 0.5$. Since we have less reliable data on income risk for the rest of the world, we estimate $\gamma^* - \tilde{\phi}^*$ jointly, repeating the exercise with a Sharpe ratio of 0.4, which results in $\gamma^* - \phi^* = 4.8242$.[14]

Figure 1 illustrates the main results, plotting the relationship between idiosyncratic risk cyclicality, the Backus-Smith covariance and exchange rate volatility for three levels of market completeness: complete markets, very little spanning and an intermediate level of spanning that seems to best fit the data. The left panel shows that for sufficiently countercyclical – and empirically relevant– idiosyncratic risk $\phi < 0$, the Backus-Smith covariance is negative. Interestingly, higher levels of spanning for the

[14]These Sharpe ratios are comparable to the literature and slightly higher than the historical averages in (Jordà, Schularick, Taylor and Ward, 2018), who however also document a higher relative ratio for the U.S.

common factor (higher values of $\alpha$) lead to more negative values for the covariance. In contrast, if the noise shock was spanned $\alpha^{\nu} = 1$, lower values of $\phi$ are required to generate a negative covariance.

The right panel illustrates exchange rate volatility under complete markets is too high when we realistically calibrate to an imperfect correlation of SDFs across countries, consistent with the literature. We show the problem is exacerbated further as $\phi$ becomes increasingly negative– resulting in extremely high volatility under internationally complete markets. However, any level of volatility in the shaded area can be attained by varying $\phi, \alpha$ and $\alpha^{\nu}$ and we show as an example that $\phi = -5, \alpha = 0.5$ (countercylical idiosyncratic risk and a moderate amount of spanning of the common factor) fits well with plausible levels of $var(\Delta e_{t+1})$ which are around $0.11^2$.

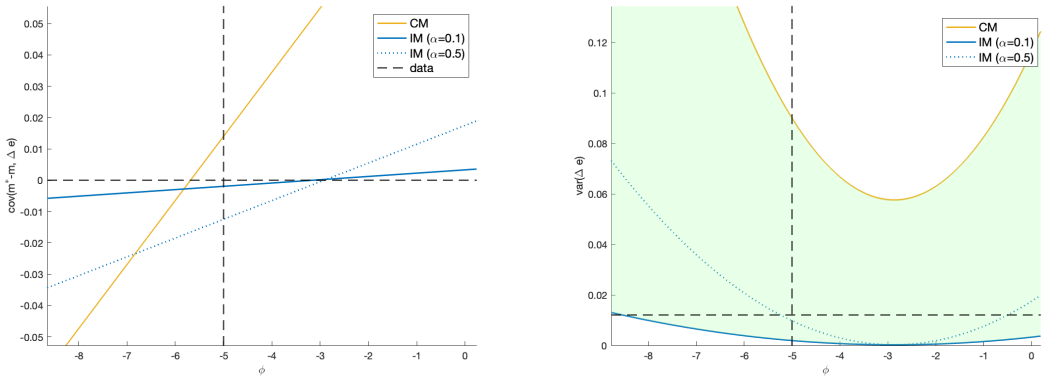
We discuss two special cases, the representative agent limit ($\phi = 0$) and the limit with no noise shocks ($\sigma_{\nu} = 0$). The representative agent limit ($\phi = 0$) would deliver a counter-factually positive covariance (in fact a correlation of 1), although exchange rate volatility can be plausible under incomplete markets. Absent the noise shock, the correlation of SDFs across countries would be perfect - a counterfactual prediction. However, the model more easily delivers a negative covariance, as discussed above. Moreover, international market incompleteness becomes less necessary for moderating exchange rate volatility when the cross-country SDF correlation is perfect. Figure 4 in Appendix D illustrates comparative statics with respect to how much $u^{\nu}$ is spanned (i.e. varying $\alpha^{\nu}$ parameter).

## 4. Equilibrium in a Two-Country Two-Good Economy

Finally, we specify a parsimonious two-country, two-good model with incomplete domestic markets where the aggregate Euler equation (12) described above can be supported as an equilibrium outcome. There is a continuum of heterogeneous consumers $i$ in the Home economy trading in *fully integrated* asset markets.

**Figure 1**

**(a)** *Backus-Smith Covariance* **(b)** *Exchange Rate Volatility*




*Solid yellow line reflects complete markets equilibrium, solid blue line corresponds to incomplete markets with spanning summarised by* $\alpha = 0.1, \alpha^{\nu} = 0.01$. *The dotted blue line is an example which seems to fit the data well corresponding to* $\alpha = 0.5$. *Dashed lines reflect the zero-line in the left panel,* $(\Delta e)$ *from the data in the right panel.*

Each agent $i$ derives per-period utility from consumption:

$$u(C_t^i) = \beta(C_{t-1})\frac{{C_t^i}^{1-s}}{1-s} \tag{25}$$

where $\beta(C_{t-1}) = \omega C_{t-1}^{-u}$, with $\omega \in (0,1)$, is the discount factor used as a stationarity-inducing device following Bodenstein (2011), $C_t$ is the aggregate consumption bundle at Home, and the individual $i$'s consumption bundle is given by:[15]

$$C_t^i = \left[\alpha^{\frac{1}{\phi}} {C_{H,t}^i}^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} {C_{F,t}^i}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{26}$$

where $\phi$ is the trade elasticity and $\alpha$ is the measure of home-bias. The corresponding price index is given by:

$$P_t = \left[\alpha P_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha) P_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{27}$$

$C^*$ and $P^*$ are defined symmetrically. The real exchange rate is given by $\mathcal{E} = P^*/P$.

[15]The endogenous discount factor is not important for our results but is a device to induce a stationary equilibrium in this model.

Individual $i$'s budget constraint at Home is given by:

$$C_t^i - I_t^i \leq R_t B_{t-1}^i - B_t + \mathcal{E}_t R_t^* B_t^{i*} - \mathcal{E}_t B_{t-1}^{i*} - \int_{j\in J} X_{t-1}^{i,j} + \int_{j\in J} \tilde{R}^j X_t^{i,j} \tag{28}$$

where $I_t^i$ is the income drawn by agent $i$ and we define $\frac{P_{H,t}}{P_t} I_{H,t}^i + \frac{P_{F,t}}{P_t} I_{F,t}^i = I_t^i$, $B_t^{i(*)}$ is the position in Home (Foreign) risk-free bonds and $R_t^{(*)}$ is the corresponding returns and $X^{i,j}$ denote risky assets indexed by $j$ held by agent $i$. Foreign agents face an analogous maximization, but we assume there is no heterogeneity in the Foreign country for expositional purpose. Finally, goods market clearing requires:

$$\int_i C_{H,t}^i di + C_{H,t}^* = \int_i I_{H,t}^i di + I_{H,t}^*; \quad \int_i C_{F,t}^i di + C_{F,t}^* = \int_i I_{F,t}^i di + I_{F,t}^* \tag{29}$$

where the aggregate endowment processes are given by $I_{H,t}^{(*)} = \rho I_{H,t-1}^{(*)} + (1-\rho) I_H^{(*)} + \epsilon_{H,t}^{(*)}$, $I_{F,t}^{(*)} = \rho I_{F,t-1}^{(*)} + (1-\rho) I_F^{(*)} + \epsilon_{F,t}^{(*)}$ and $\epsilon_{H,t}^{(*)}$ and $\epsilon_{F,t}^{(*)}$ are iid $N(0, \sigma_\epsilon)$ endowment shocks.

No-arbitrage pricing requires that the Euler equations for household $i$ investing in an arbitrary security with return $\tilde{R}_{t+1}$:

$$\mathbb{E}_t \left[ \beta \left( \frac{C_{t+1}^i}{C_t^i} \right)^{-s} \tilde{R}_{t+1} \right] = 1 \tag{30}$$

We allow individual consumption draws to be related to the aggregate consumption draw and follows a heteroskedastic log-normal process $\Delta c_{t+1}^i = \log\left( \frac{\delta_{t+1}^i}{\delta_t^i} \frac{C_{t+1}}{C_t} \right) \sim \mathcal{N}(\mu_{c_t^i}, \sigma_{i,t}^2)$ where $\frac{C_{t+1}}{C_t} \equiv \exp(\Delta c_{t+1})$ denotes aggregate consumption growth and is drawn from a lognormal distribution given by $\Delta c_{t+1}^{(*)} \sim \mathcal{N}\left( \mu_{C_t^{(*)}}, \sigma_{C_t^{(*)}}^2 \right)$, where $\mu_{C_t^{(*)}}$ and $\sigma_{C_t^{(*)}}^2$ are conditional mean and variance of this process, and the asterisks denote the corresponding aggregate consumption process for the Foreign economy. The individual consumption exposures satisfy the law of large numbers and are themselves log-normally distributed, i.e. $\delta_{t+1}^i / \delta_t^i$ satisfies $\int_i \delta_t^i \, di = 1$, $\log\left( \frac{\delta_{t+1}^i}{\delta_t^i} \right) \sim \mathcal{N}(\mu_{\delta_t}, \sigma_{\delta_t}^2)$, $\forall t$.[16,17]

[16] We repeatedly use the result that if $X \sim \mathcal{N}(\mu_X, \sigma_X)$, $e^X \sim \log\mathcal{N}(\mu_X, \sigma_X)$ and if $Y \sim \mathcal{N}(\mu_Y, \sigma_Y)$, then $Z = XY \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$. From this it follows that $e^Z \sim \log\mathcal{N}(\mu_X + \mu_Y, \sigma_X + \sigma_Y + 2\sigma_{XY})$.

[17] Together, Equations (4) and (4) thus imply $\mu_{c_t^i} = \mu_t^\delta + \mu_{c_t^i}$ and $\sigma_{i,t}^2 = \sigma_{\delta,t}^2 + \sigma_{C,t}^2 + 2\sigma_{\delta_t, C_t}$, where $\sigma_{\delta,t}^2 \equiv var_t\left( \log\left( \frac{\delta_{t+1}^i}{\delta_t^i} \right) \right)$, and $\sigma_{\delta_t, C_t} \equiv cov_t\left( \log\left( \frac{\delta_{t+1}^i}{\delta_t^i} \right), \log\left( \frac{C_{t+1}}{C_t} \right) \right)$.

Substituting in (4) and assuming independence of the consumption share growth process, the above Euler equation can be rewritten as:

$$\mathbb{E}_t\left[\beta\hat{\mathbb{E}}_t\left[\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\right]\left(\frac{C_{t+1}}{C_t}\right)^{-s}\tilde{R}_{t+1}\right]=1 \tag{31}$$

where $\hat{\mathbb{E}}_t$ is a cross-sectional expectation operator and we denote $\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}=\hat{\mathbb{E}}_t\left[\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\right]$ corresponding to the consumption wedge in Berger et al. (2023). To arrive at the aggregate Euler equation in Section 2.2, we adopt the following process for $\delta^i_t$:

$$\frac{\delta^i_{t+1}}{\delta^i_t}=\exp(\xi^i_{t+1}\sqrt{y}_{t+1}-y_{t+1}/2) \tag{32}$$

where $\xi^i_{t+1}$ are the uninsurable idiosyncratic shocks which are distributed as standard normal for all $i$ and $t$, independently distributed from the aggregate state $y_{t+1}$.[18] This process satisfies (4) implying $\log(\delta^i_{t+1}/\delta^i_t)\sim\mathcal{N}(-\frac{y_{t+1}}{2},y_{t+1})$.

The individual Euler equations can be aggregated to condition (12) used in Section (2.2), using the consumption share process (32), as in Constantinides and Duffie (1996). Two key results follow. First, the aggregate Euler equality fails, i.e. Equation (12) features a risk premium over aggregate consumption growth. Second, as a result of the specification of $\delta^i$, the Euler for each individual household is independent of $i$, indicating all households agree on a common valuation. Assuming individual endowments are given by $I^i_t=\delta^i_tC_t+D_t$, where $D_t$ denotes the aggregate dividend in the economy, conditional on a process for aggregate consumption $C_t$, standard arguments from Constantinides and Duffie (1996) imply $C^i_t=I^i_t$ is an equilibrium in this economy for all $i,t$.

[18] The law of large numbers follows from properties of the normal distribution for $z^i$. Treating $y_{t+1}$ as a constant, and using the moment generating function $M_z(h)=e^{hz}$ for $h\in\mathbb{R}$, $\mathbb{E}[e^{z^i\sqrt{y}-y/2}]=M_z(x)e^{-y/2}=e^0=1$.

## 4.1. International Equilibrium

We construct an equilibrium where each agent $i \in [0,1]$ chooses not to trade in financial assets with agents in their own or the foreign country.[19] We first establish that if the Home country is in financial autarky, the individual SDF $m^i_{t+1}$ is independent of $i$, so starting from an equilibrium where every household is the same, the no-trade allocation constitutes an equilibrium within each country. Then, taking any foreign endowment process as given, we select a process for aggregate domestic endowments such that agents would optimally choose not to trade across borders even when assets are available. More generally, the endowment processes we consider can be interpreted as the wealth of consumers after all gains from trade have been exhausted.

The adjusted risk-sharing or Backus-Smith condition is then given by:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s} e^{\frac{s(s+1)}{2}y_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\right] \tag{33}$$

Using (33), replacing $C_t$ with $Y_t = \int_i Y^i_t$ and $C^*_t$ with $Y^*_t = \int_{i^*} Y^i_t{}^*$ implies equalization of the autarky rates across countries resulting in no trade (Svensson, 1988).

## 4.2. Goods Market Clearing and Risk Sharing

The equilibrium we construct does not always lie on the static Pareto frontier (goods markets do not clear), but can be supported with static goods market wedges.[20] While this may sound undesirable, such wedges are non-zero in the data and play a significant role in impeding risk sharing (Fitzgerald, 2012; Bodenstein, Cuba-Borda, Gornemann and Presno, 2024). Moreover, as illustrated in Section 2, and consistent with Lustig and Verdelhan (2019), such goods markets frictions alone are unable to explain the

[19] In general, heterogeneous agent models with endogenous portfolio choice are few in the literature. We favor our approach for its analytical tractability in characterizing the exchange rates.

[20] This is true of any consumption based asset pricing model. A contribution of our paper is that we specify the wedges needed to support equilibrium and discuss their impact on international risk-sharing.

Backus Smith puzzle with trade in many assets.

Given a level of aggregate consumption $C_t$, households optimally choose their consumption bundle $\{c^i_{H,t}, c^i_{F,t}\}$.[21] The proposition below provides the sufficient condition for goods market clearing in our model.

**Proposition 7 (Goods Market Clearing)**

*Given processes for* $\{Y_t, Y_{H,t}, Y_{F,t}, \tau_t\}$ *and the Pareto Frontier* $\{c_H(C_t), c_F(C_t)\}$*, goods markets are efficient and clear if and only if:*

$$\left[ Y_t^{\frac{\phi-1}{\phi}} - \alpha^{\frac{1}{\phi}} c_H(Y_t)^{\frac{\phi-1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \left( \hat{b} Y_{H,t} + (1-\hat{b}) c_H(Y_t) \right) = c_H(Y_t) Y_{F,t} (1-\alpha)^{\frac{1}{\phi-1}} \quad (35)$$

*where* $\hat{b} = (\frac{\alpha}{1-\alpha})^2 (1+\tau_t)^\phi$ *and* $\tau_t > -1$*. At the limit of full home bias* $\alpha \to 1$ $\tau_t = 0$ *satisfies the condition.*

**Proof.** Consider the static problem (34). Aggregating (36) over all agents $i$, combining with market clearing (29) and the (first order homogeneous) aggregator (26) delivers the result. □

Consider (33). Taking $\hat{Y}^*_t = \int_{i^*} Y^{i\,*}_t$ as given, the process $\hat{Y}_t$ ensures a no-trade equilibrium. However, we cannot simultaneously choose $Y_{H,t}$, $Y_{F,t}$ and $\hat{Y}_t$ to satisfy (35), implying the need for wedges. However, a zero wedge is consistent with equilibrium in the limit of full home bias– where there is no scope for risk sharing in a no trade equilibrium.[22]

[21] The optimal consumption bundle satisfies:

$$C^*(C_t) = \max_{\{c^i_{H,t}, c^i_{F,t}\}} \Big\{ g^*(c^{i\,*}_H, c^{i\,*}_F) \quad \text{s.t.} \quad \int_i c^i_H di + \int_{i^*} c^{i\,*}_H di^* = Y_H, \quad (34)$$
$$\int_i c^i_F (1+\tau_t) di + \int_{i^*} c^{i*}_F di^* = Y_F \text{ and } g(c^i_H, c^i_F) \geq C^i_t \Big\},$$

where we define $g(c_H, c_F)$ by (26). See Costinot, Lorenzoni and Werning (2014) for a friction-less representative agent framework, and Lloyd and Marin (2024) for a treatment with static wedges. Our generalization to many agents follows from the first order homogeneity of $g(\cdot)$.

[22] Note that full home bias is close to what is often estimated in the literature for the U.S., e.g. Colacito and Croce (2013) estimate home bias of 90%.

## 4.3. Planning Problem, Backus-Smith and Risk-Sharing

In this section we show that the cyclicality of exchange rates with respect to aggregate consumption growth is informative for risk-sharing and we define exactly what this means. Consider the following planning problem where we assume i) equal weights for each agent within each country and ii) the planner potentially faces costs to allocate good F to Home households. As a result of (ii), if costs are positive, the allocation is constrained efficient. The problem is given as follows:

$$\max_{\{c^i_{H,t},c^i_{F,t},c^{i*}_{H,t},c^{i*}_{F,t}\}} \mu \int_i \frac{1}{1-s}(g(c^i_{H,t},c^i_{F,t}))^{1-s}di + (1-\mu)\int_{i^*}\frac{1}{1-s}(g^*(c^{i*}_{H,t},c^{i*}_{F,t}))^{1-s}di^*,$$
$$\text{s.t.} \int_i c^i_{H,t}di + \int_{i^*} c^{i\ *}_{H,t}di^* = Y_H,\ \int_i c^i_{F,t}di(1+\tau^{plan}_t) + \int_{i^*} c^{i*}_{F,t}di^* = Y_{F,t}$$

where $g(\cdot)$ is characterized by (26).

There are three associated efficiency conditions. The first reveals that the planner wishes to equate marginal utility within countries for every agent, e.g. $g_H(c^i_{H,t},c^i_{F,t})^{-s} = \lambda_{H,t}$ where $\lambda_{H,t}$ is the multiplier on the H good market clearing which is invariant to $i$. Therefore, we drop $i$ superscripts from the following efficiency conditions.

The second condition relates relative goods consumption across the two countries:

$$\frac{c^i_{H,t}}{c^i_{F,t}} = \left(\frac{\alpha}{1-\alpha}\right)^2 (1+\tau^{plan}_t)^{\phi}\ \frac{c^{i*}_{H,t}}{c^{i*}_{F,t}}, \tag{36}$$

which coincides with the decentralized allocation.

The final condition requires:[23]

$$\frac{C_t}{C^*_t}^{-\sigma} = \frac{1-\mu}{\mu}\frac{1}{\mathcal{E}_t} \tag{38}$$

[23]This equation follows because:

$$\frac{C_t}{C^*_t}^{-\sigma} = \frac{1-\mu}{\mu}\frac{g^*_{H,t}}{g_{H,t}} = \frac{1-\mu}{\mu}\frac{g^*_{F,t}}{g_{F,t}}(1+\tau_t) \tag{37}$$

and $\dfrac{g^*_{F,t}}{g_{F,t}} = \dfrac{\frac{dC^*_t}{dc^*_{F,t}}}{\frac{dC_t}{dc_{F,t}}} = \dfrac{\frac{dC^*_t}{dC_t}}{\frac{dc^*_{F,t}}{dc_{F,t}}} = \dfrac{\frac{dC^*_t}{dC_t}}{\frac{dY_{F,t}-c_{F,t}(1+\tau_t)}{dc_{F,t}}} = -\dfrac{dC^*_t}{dC_t}(1+\tau_t)^{-1}$

where we have derived the real exchange rate as the ratio of multipliers as follows $\frac{dC_t^*}{dC_t} = \frac{1}{\mathcal{E}_t}$, see Costinot et al. (2014); Lloyd and Marin (2024).[24] Evaluating this condition at $t$ and $t+1$ yields (1) – our benchmark for full risk-sharing. In general, comparing the planner conditions with the decentralized allocation reveals two impediments to risk-sharing, $\tau_t$ and $y_{t+1}$. If $\tau_t = \tau_t^{plan}$, then idiosyncratic risk would be the only reason why risk sharing is low.

## 5. Empirical Evidence

In this section, we investigate whether exchange rates are indeed risky with respect to idiosyncratic risk in the data. To take the model to the data, we generalize the sufficient condition (16) to condition (39) below by relaxing the model along three dimensions. (i) we consider a general form of wedges consistent with equation (31), (ii) we allow for domestic market incompleteness both domestically and abroad, (iii) we allow for mean zero financial shocks which only affecting returns on foreign positions, consistent with e.g. Itskhoki and Mukhin (2021). Then the model implies $cov_t(m_{t+1} - m_{t+1}^*, \Delta e_{t+1}) < 0$ with trade in two risk-free assets if and only if:

$$1 \geq -\frac{cov_t\left(\log\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s} - \log\left(\frac{\delta_{t+1}^*}{\delta_t^*}\right)^{-s}, \Delta e_{t+1}\right)}{\sigma_t\left(\log\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s} - \log\left(\frac{\delta_{t+1}^*}{\delta_t^*}\right)^{-s}\right)\sigma_t(\Delta e_{t+1})} \geq \frac{(1-u)\sigma_t(\Delta e_{t+1})}{\sigma_t\left(\log\left(\frac{\delta_{t+1}}{\delta_t^i}\right)^{-s} - \log\left(\frac{\delta_{t+1}^*}{\delta_t^*}\right)^{-s}\right)} \quad (39)$$

where $u \in [0,1]$ is the share of exchange rate volatility stemming from financial shocks $var_t(u_{t+1}) = u \times var_t(\Delta e_{t+1})$. The derivation of this general model is in Appendix E.1.

An advantage of expressing the condition in terms of the consumption shares wedge (instead of cross-sectional income volatility) is the ability to measure this wedge

[24] Away from the flex price limit, Aguiar, Itskhoki and Muhkin (2024) argue that, due to pricing to market frictions, the real exchange rate is inefficient and does not correspond to the ratio of multipliers, and therefore the Backus-Smith condition is not informative for risk-sharing.

reliably in the data. Whereas our baseline model assumes all agents are unconstrained, Berger et al. (2023) show that the growth in consumption shares of unconstrained agents is a reliable proxy for discount factor wedge generated by a large class of incomplete markets models.

We test (39) in two ways: (i) assume only the US has heterogeneity ($\delta_t^* = 1$ for all $t$) and use the measured consumption wedge in Berger et al. (2023), (ii) construct a proxy of this wedge using growth of income shares for high-income individuals for each country in a panel of seven advanced economies from the Global Repository of Income Dynamics (GRID) (Guvenen et al., 2022).[25]

**Data description** We construct real exchange rates from annual nominal exchange rates and consumer price indices using the Macro-History database (Jordà, Schularick and Taylor, 2017).[26] The average standard deviations of real exchange rate growth is approximately 0.11. For our first test, we use the annual discount factor wedge series constructed by Berger et al. (2023) using growth in consumption-shares of financially unconstrained households directly from the Consumption Expenditure Survey panel data spanning 1992–2017.[27] Second, we construct the bilateral wedge $\log\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s} - \log\left(\frac{\delta^{i*}_{t+1}}{\delta^{i*}_t}\right)^{-s}$ using growth in income shares for a high income group over a balanced sample of 1998–2015 in seven countries (Canada, Denmark, France, Germany, Italy, Norway, and Sweden) against the United States.[28] To capture unconstrained

[25] An important caveat with the second approach is that we only have data on income dynamics, not consumption. In the absence of detailed consumption dynamics data, we proxy growth in consumption shares of high income individuals with growth in their income shares. However, it is understood that the pass through from income to consumption is not perfect (e.g. Blundell, Pistaferri and Preston, 2008; Kaplan and Violante, 2010).

[26] Australia, Belgium, Canada, Denmark, France, Finland, Germany, Ireland, Italy, Japan, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, and the United Kingdom. The constructed bilateral real exchange rate growth variable denotes depreciation of the U.S. consumption basket against the foreign country basket.

[27] Berger et al. (2023) consider high income/low net worth household group to be a representative group of financially unconstrained households who would save for precautionary reasons.

[28] We use residual log earnings for Male in age groups 25–55 in our calculation of income shares. GRID provides these residuals from "regressing log earnings on age dummies". We show results are robust to using data on log earnings, or using data for all genders.

agents, we consider the top 0.01 percentile group in each country's income distribution as our benchmark group, and show robustness with the top 0.1 percentile, and top 1 percent groups in the Appendix. The country-specific income-based wedge constructed for the top 0.01 percentile income group in GRID is similar to the discount factor wedge constructed by Berger et al. (2023) in its volatility and cyclicality with respect to each country's output gap.[29] Finally, to construct the volatility of the wedge, we need to calibrate the inverse of the elasticity of intertemporal substitution. Following Best, Cloyne, Ilzetzki and Kleven (2020), we set $s = 10$.

**Empirical Results** We report correlations for each bilateral pair against the required threshold in the Appendix Tables 1 and 4. In the data, the average correlation of the Berger et al. (2023) wedge and the income-share based bilateral wedge with real exchange rate growth is $-0.25$ and $-0.30$ respectively. The standard deviation of the Berger et al. (2023) wedge is 0.42, while the average standard deviation of the bilateral wedge is 0.30. Substituting these moments into (39), the threshold correlations when $u = 0$ (i.e. no financial shocks) required are $-0.26$ and $-0.39$ respectively.

The two main takeaways are (i) the sign of the correlation of exchange rates with the wedge is broadly negative, indicating that periods of high idiosyncratic risk correspond to low returns on foreign currency denominated assets. This is consistent with our mechanism that exchange rates need to be *risky* for domestic market incompleteness to have a shot at exchange rate cyclicality. (ii) Quantitatively, for the consumption based wedge, the magnitude of this correlation (0.25) is close to the threshold required (0.26). For the income based (bilateral) wedge, the correlation is below the threshold. This is consistent with findings in the literature that 55-75% of exchange rate movements are uncorrelated to any domestic asset returns (Chernov et al., 2024). Allowing for $u > 0$

[29] We report the summary statistics of these wedges along with the correlation with country's output gap in the Appendix. See Tables 2 and 3. The constructed wedges from all three groups are counter-cyclical with respect to the respective country's real GDP growth as well as a measure of output gap obtained with a quadratic trend. The wedge for the Top 5 percent group is procyclical and so we do not use it in our analysis.

can lower the thresholds but as long as $u \leq 1$, the correlation must be negative.

## 6. Conclusion

A classical strand of the literature in international macroeconomics has focused on formulating goods-market mechanisms which generate a negative relationship between consumption growth and depreciation– the opposite sign to that implied by the Backus-Smith condition– as long as financial markets are incomplete. We show that any model which achieves this resolution must rely on a non-traded component to relative prices which is "safe" from a domestic investor perspective. However, Lustig and Verdelhan (2019) determine that any two-country model with a representative agent and frictionless trade in Home and Foreign currency denominated risk-free bonds recovers the exchange rate cyclicality implied by complete markets.

We propose a generalization of the model beyond the representative agent no-arbitrage benchmark to a model of heterogeneous consumers trading in integrated markets but facing uninsurable idiosyncratic risk. Introducing domestic heterogeneity generates a non-traded component to exchange rate movements, resulting in low risk-sharing even with cross-border trade in both Home and Foreign bonds. Building on Lucas (1982) and Cox et al. (1985), we derive the incomplete markets wedge in closed form and show that the cyclicality of within-country idiosyncratic risk is critical to reconcile the Backus-Smith correlation. At the same time, internationally incomplete markets are necessary to prevent unrealistically high exchange rate volatility from a model that can reconcile volatile asset prices domestically. Using estimates of idiosyncratic income risk from the data, we find that its correlation with exchange rates is negative, consistent with our mechanism. However, idiosyncratic risk is not always volatile enough, indicating a role for financial shocks as in Itskhoki and Mukhin (2021).

Our paper highlights the need to extend a workhorse international macroeconomic

model to allow not only for heterogeneous agents and specifically account for portfolio choice with both idiosyncratic and aggregate risk. This is a challenging endeavour that goes beyond most of the existing literature, see e.g. Auclert, Rognlie, Straub and Ťapák (2024), which either abstracts from heterogeneity (Benigno and Küçük, 2012; Heathcote and Perri, 2013) or delegates the portfolio choice problem to a mutual fund (Auclert et al., 2021).

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# A. Appendix

## A.1. Additional Derivations for Section 2.

To find the admissible set of processes, consider the log expansions of the above conditions, assuming joint log normality:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) = -r_{t+1}, \quad (40)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) = -r^*_{t+1} \quad (41)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \quad (42)$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \quad (43)$$

where lower case levels denote logs, e.g. $\log(M_{t+1}) = m_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (40) and (43), and (41) and (42) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(m_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \quad (44)$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(m^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \quad (45)$$

## A.2. Proofs to Propositions

**Proof to Proposition 1** The Backus-Smith condition is related to the covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ which can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \quad (46)$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \quad (47)$$

Imposing (9) (international trade in the Foreign asset), but not (8) (international trade in the Home asset) as is done in Lustig and Verdelhan (2019), assuming $\mathbb{E}_t[\eta_{t+1}] = 0$, and rearranging yields the result. □

**Proof to Corollary 1** The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}, \eta_{t+1}) - 2cov_t(m_{t+1}, \eta_{t+1})$$

Imposing (8) and (9):

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1})$$

Taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \to$ (10) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

**Proof to Proposition 2.** Taking a log expansion of (12) for the case of the Home households investing in the Home bond ($\tilde{R}_t = R_t$) can be written as:

$$-r_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + var_t(\tilde{y}_{t+1}) + 2cov_t(m_{t+1}, \tilde{y}_{t+1})] \tag{48}$$

Turning to the Euler for a Home household $i$ investing in Foreign bonds ($\tilde{R}_t = R^*_t \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$):

$$\begin{aligned} -r^*_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + var_t(\tilde{y}_{t+1}) + 2cov_t(m_{t+1}, \tilde{y}_{t+1})] + \\ \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1} + \tilde{y}_{t+1}, \Delta e_{t+1}) \end{aligned} \tag{49}$$

Combining with (3)-(4), and taking log expansions:

$$var_t(\eta_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - var_t(\Delta e_{t+1}) - 2cov(\tilde{y}_{t+1}, \Delta e_{t+1}) \tag{50}$$

Then, using $e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{51}$$

In turn, combining (14)-(15) yields

$$0 = var(\eta_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{y}_{t+1})$$

Combining by substituting $cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1})$ in (51) yields:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1}) - cov_t(\Delta e_{t+1}, \tilde{y}_{t+1})$$

Using (50), this implies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) + cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}).$$

Then, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ requires $-cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) > var_t(\Delta e_{t+1})$. Finally, the Cauchy-Schwarz identity implies $|cov_t(\tilde{y}_{t+1}, \Delta e_{t+1})| \leq \sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})$. Dividing through by $\sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})$ yields the result.

Note also that we construct the correct measure of the Backus-Smith correlation, in SDF space, when there are heterogeneous agents. If log quantities are jointly normally distributed it follows that the log of the Backus-Smith covariance in this economy is equal to the covariance of the logs:[30]

$$\log\left(cov_t\left(\int_i e^{\Delta c^i_{t+1}} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(\log \underbrace{\int_i e^{\Delta c^i_{t+1}} di}_{C_{t+1}}, \Delta e_{t+1}\right) \implies$$

$$\log\left(cov_t\left(\int_i M^i_{t+1} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(m_{t+1}, \Delta e_{t+1}\right) \tag{52}$$

□

**Proof to Proposition 3 and Corollary 3** Combining (74)-(77) in Section B.1 yields (84). Using (84), $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if $var_t(\Delta e_{t+1}) + cov_t(\tilde{y}, \Delta e_{t+1}) - cov_t(\eta_{t+1} - \tilde{y}_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) < 0$. Setting $\tilde{r}^{(*)}_{t+1} = r^{(*)}_t$ and comparing again with (74)-(77) yields additional restrictions imposed by trade in risk-free assets. □

**Proof to Proposition 4** The individual Euler equation can be expressed as:

$$\mathbb{E}_t\left[\beta\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-\gamma}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-\gamma}\tilde{R}_{t+1}\right] = 1, \tag{53}$$

for any (risky) asset with return $\tilde{R}_{t+1}$, so that e.g. foreign nominally risk-free bonds are captured by $(\mathcal{E}_{t+1}/\mathcal{E}_t)R_{t+1}$. Using

$$\frac{\delta^i_{t+1}}{\delta^i_t} = e^{(z^i_{t+1}\sqrt{y}_{t+1} - y_{t+1}/2)} \tag{54}$$

[30] In general $cov_t(X_{t+1}, Y_{t+1}) = \mathbb{E}_t[X_{t+1}]\mathbb{E}_t[Y_{t+1}]\left(e^{cov_t(U,V)} - 1\right)$ when $U, V \sim \mathcal{N}$.

we show:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}\right]=\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}e^{\frac{y_{t+1}(\gamma+\gamma^2)}{2}}\tilde{R}_{t+1}\right]=1, \tag{55}$$

where $y_t$ is the cross-sectional volatility of consumption. We define

$$\Delta e_{t+1}=m_{t+1}^*-\hat{m}_{t+1}+\eta_{t+1}$$

and assume a process for the incomplete markets wedge:

$$\eta_{t+1}=\beta_0 z_t+\beta_0^\nu z_t^\nu+\beta_1\sqrt{z_t}u_{t+1}+\beta_1^\nu\sigma^\nu u_{t+1}^\nu+\beta_2\sqrt{z_t}\epsilon_{t+1}+\beta_2^\nu\sigma^\nu\epsilon_{t+1}^\nu \tag{56}$$

where $u^{(\nu)}$ are spanned innovations and $\epsilon^{(\nu)}$ are unspanned. We proceed to determine the coefficients of the IM wedge consistent with no arbitrage. Using (50) and denoting $var_t(\Delta e_{t+1})=\kappa z_t+\kappa^\nu\sigma_\nu^2$:

$$\underbrace{(\beta_1^2+\beta_2^2)z_t+(\beta_1^{\nu 2}+\beta_2^{\nu 2})\sigma_\nu^2}_{var_t(\eta_{t+1})}=\underbrace{(\gamma-\phi-\gamma^*)^2z_t+\gamma^{*2}\sigma_\nu^2}_{var(m_{t+1}^*-\hat{m}_{t+1})}-\underbrace{\left(\kappa z_t+\kappa^\nu\sigma_\nu^2\right)}_{var_t(\Delta e_{t+1})} \tag{57}$$

which yields:

$$\beta_1=\pm\sqrt{(\gamma-\phi-\gamma^*)^2-\lambda},\quad \beta_1^\nu=\pm\sqrt{\gamma^{*2}-\lambda^\nu} \tag{58}$$

$$\beta_2=\pm\sqrt{\lambda-\kappa},\quad \beta_2^\nu=\pm\sqrt{\lambda^\nu-\kappa^\nu} \tag{59}$$

where for real solutions, the following restrictions are required:

$$(\gamma-\phi-\gamma^*)^2\geq\lambda\geq\kappa, \tag{60}$$

$$\gamma^{*2}\geq\lambda^\nu\geq\kappa^\nu \tag{61}$$

Using (14)-(15) imply:

$$\beta_0=\frac{1}{2}\left\{(\gamma-\phi-\gamma^*)^2-\kappa\right\}-(\gamma-\phi)\beta_1, \tag{62}$$

$$\beta_0^\nu=\frac{1}{2}\left\{\gamma^{*2}-\kappa^\nu\right\}, \tag{63}$$

and

$$\beta_0+\frac{1}{2}\left\{(\gamma-\phi-\gamma^*)^2-\kappa\right\}-\gamma^*\beta_1=0, \tag{64}$$

$$\beta_0^\nu+\frac{1}{2}\left\{\gamma^{*2}-\kappa^\nu\right\}-\gamma^*\beta_1^\nu=0, \tag{65}$$

respectively. Adding (63) and (65) for $z_t$ and $\sigma_\nu^2$ respectively, this can be rewritten as:

$$\beta_0 = \frac{1}{2}(\gamma - \phi + \gamma^*)\sqrt{(\gamma - \phi - \gamma^*)^2 - \lambda}, \tag{66}$$

$$\beta_0^\nu = \frac{1}{2}\gamma^*\sqrt{\gamma^{*2} - \lambda^\nu} \tag{67}$$

Subbing $\beta_0$ and $\beta_0^\nu$ back into (63) and (65) respectively:

$$\kappa = (\gamma - \phi - \gamma^*)^2 + (\gamma - \phi - \gamma^*)\beta_1 \geq 0, \tag{68}$$

$$\kappa^\nu = \gamma^{*2} - \gamma^*\beta_1^\nu \geq 0 \tag{69}$$

This completes the characterization of the $\eta$ process. □

**Proof to Proposition 5** First consider:

$$cov_t(m_{t+1}^* - \hat{m}_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) =$$
$$cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) - cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) \tag{70}$$

Then, the Backus-Smith covariance is negative if and only if:

$$var_t(\Delta e_{t+1}) + cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) \leq 0 \tag{71}$$

Using Proposition 4:

$$\kappa z_t + \kappa^\nu \sigma_\nu^2 + \tilde{\phi}(\gamma - \tilde{\phi} - \gamma^* + \beta_1)z_t \leq 0 \tag{72}$$

and taking a limit $\sigma_\nu \to 0$:

$$\kappa + \tilde{\phi}(\gamma - \tilde{\phi} - \gamma^* + \beta_1) \leq 0 \tag{73}$$

Immediately, we can see that the representative agent limit $\tilde{\phi} \to 0$ leads to a contradiction since $\kappa \geq 0$ is required for $var_t(\Delta e_{t+1})$. Continuing for the case $|\tilde{\phi}| \geq 0$,

substituting $\kappa$:

$$\begin{aligned}
&(\gamma - \tilde{\phi} - \gamma^*)^2 + (\gamma - \tilde{\phi} - \gamma^*)\beta_1 + \tilde{\phi}(\gamma - \tilde{\phi} - \gamma^* + \beta_1) \\
&= (\gamma - \gamma^*)^2 + \tilde{\phi}^2 - 2\tilde{\phi}(\gamma - \gamma^*) + (\gamma - \gamma^*)\beta_1 + \tilde{\phi}(\gamma - \gamma^*) - \tilde{\phi}^2 \\
&= (\gamma - \gamma^*)^2 - \tilde{\phi}(\gamma - \gamma^*) + (\gamma - \gamma^*)\beta_1 \\
&= (\gamma - \gamma^*)[(\gamma - \gamma^*) - \tilde{\phi} + \beta_1]
\end{aligned}$$

so $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ if:

$$(\gamma - \gamma^*)[(\gamma - \gamma^*) - \tilde{\phi} + \beta_1] \leq 0$$

Next, suppose $\gamma^* > \gamma$ and $\lambda = \alpha(\gamma - \phi - \gamma^*)^2$ then:

$$(\gamma - \gamma^* - \tilde{\phi})(1 + \sqrt{1 - \alpha}) \geq 0 \leftrightarrow$$

$$\gamma - \gamma^* - \tilde{\phi} \geq 0$$

To express this in terms of volatility (still in the limit $\sigma_\nu \to 0$), consider that :

$$var_t(\hat{m}^{(*)}_{t+1}) = (\gamma^{(*)} - \phi^{(*)})^2 \mathbb{E}[z_t]$$

with $\phi^*$ normalized to zero. Both $\gamma - \phi > 0$ and $\gamma^* - \phi^* > 0$ therefore $var(\hat{m}_{t+1}) > var(m^{(*)}_{t+1}) \implies \gamma - \phi > \gamma^*$ (with normalization) delivering the final result. $\square$

# B. ONLINE APPENDIX

## B.1. Trade in Risky Assets

If instead of allowing for trade in both risk-free assets, we allow for trade in Home and Foreign risky assets, then equations (2)–(5) are replaced by:

$$\mathbb{E}_t[M_{t+1}e^{\tilde{y}_{t+1}}\tilde{R}_{t+1}] = 1, \tag{74}$$

$$\mathbb{E}_t[M_{t+1}e^{\tilde{y}_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\tilde{R}^*_{t+1}] = 1, \tag{75}$$

$$\mathbb{E}_t[M^*_{t+1}\tilde{R}^*_{t+1}] = 1, \tag{76}$$

$$\mathbb{E}_t[M^*_{t+1}\left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1}\tilde{R}_{t+1}] = 1, \tag{77}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (74)- (77) hold. Assuming joint log normality, the above Euler equations imply:

$$\mathbb{E}_t[m_{t+1}+\tilde{y}_{t+1}] + \frac{1}{2}var_t(m_{t+1}+\tilde{y}_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}_{t+1}) = 0, \tag{78}$$

$$\mathbb{E}_t[m_{t+1}+\tilde{y}_{t+1}] + \frac{1}{2}var_t(m_{t+1}+\tilde{y}_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \cdots$$
$$cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}^*_{t+1}) + cov_t(m_{t+1}+\tilde{y}_{t+1},\Delta e_{t+1}) + cov_t(\Delta e_{t+1},\tilde{r}^*_{t+1}) = 0, \tag{79}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + cov_t(m^*_{t+1},\tilde{r}^*_{t+1}) = 0, \tag{80}$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1},\tilde{r}_{t+1}) + cov_t(m^*_{t+1},-\Delta e_{t+1}) + cov_t(-\Delta e_{t+1},\tilde{r}_{t+1}) = 0 \tag{81}$$

Combining (78) and (79):

$$\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}^*_{t+1}] + \frac{1}{2}var_t(\tilde{r}^*_{t+1}) - \mathbb{E}_t[\tilde{r}_{t+1}] - \frac{1}{2}var_t(\tilde{r}_{t+1}) + \cdots$$
$$cov_t(m_{t+1}+\tilde{y}_{t+1},\Delta e_{t+1}) + cov_t(\Delta e_{t+1},\tilde{r}^*_{t+1}) + cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}^*_{t+1}-\tilde{r}_{t+1}) = 0 \tag{82}$$

Combining (80) and (81):

$$-\mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + \mathbb{E}_t[\tilde{r}_{t+1}] + \frac{1}{2}var_t(\tilde{r}_{t+1}) - \mathbb{E}_t[\tilde{r}^*_{t+1}] - \frac{1}{2}var_t(\tilde{r}^*_{t+1}) + \cdots$$
$$cov_t(m^*_{t+1}, -\Delta e_{t+1}) + cov_t(-\Delta e_{t+1}, \tilde{r}_{t+1}) - cov_t(m^*_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) = 0 \quad (83)$$

Together, the above conditions yield:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1} - \tilde{y}_{t+1}, \Delta e_{t+1}) - cov_t(\eta_{t+1} - \tilde{y}_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) \quad (84)$$

## B.2. Degree of Market Completeness and Risk-Sharing

To illustrate that two nominally risk-free asssets suffice to span exchange rates, regardless of market completeness, we turn to a framework in the tradition of Lucas (1978). Agents have time-separable, CRRA preferences over an exogenous consumption stream. In particular:

$$\Delta c_{t+1} = \sum_{k=1}^{N} g_{y_{k,t+1}}, \quad (85)$$
$$m_{t+1} = -s\Delta c_{t+1}, \quad (86)$$

where $g_{y_{k,t+1}} = y_{k,t+1} - y_{k,t} \sim i.i.d\ \mathcal{N}(\mu_{y_k}, \sigma_{y_k})$ denotes the growth rate of $k$-th productive unit that comprises the consumption good. Corresponding variables for the Foreign economy are denoted with an asterisk. Start with the case of $N = 1$ productive units, discussed in (Lustig and Verdelhan, 2019, Sec III.A). Frictionless international trade in Home and Foreign risk-free bonds (8)- (9) and additional trade in a Home and a Foreign risky asset (a claim on $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$ respectively) will imply that the incomplete markets wedge $\eta_{t+1}$ is orthogonal to $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$, and it then follows that the only equilibrium is $\eta_{t+1} = 0$– i.e. markets are complete. When $N > 1$, additional risky claims need to traded to complete the market. However, for any $N$, frictionless international trade in just the Home and the Foreign real bonds ensures risk-sharing consistent with complete markets (Corollary 1). Relatedly, Chernov et al. (2024) show that incompleteness alone cannot match exchange rate puzzles, unless there is some lack of integration in financial markets (i.e. only Foreign bonds traded

as in Corollary 2).

## C. Representative Agent Limit

To speak to classical contributions in the literature, consider the representative agent limit ($I^i = I$ for all $i$), such that $C_t^i = C_t, B_t^i = B_t, B_t^{i*} = B_t^*, X_t^{i,j} = X_t^j$. Lemma 1 describes the autarky limit for prices and allocations, attained at the limit of zero liquidity (Corsetti et al., 2008) and full home-bias limit (Itskhoki and Mukhin, 2021, 2023).

**Lemma 1** (Autarky Limit).

*In the autarky limit $\alpha \to 1$, $B, B^* \to 0$, the model is summarized by the following equations:*

$$\begin{aligned} m_{t+1} &= -s g_{y_{H,t+1}}, \\ m_{t+1}^* &= -s g_{y_{F,t+1}}, \\ \Delta e_{t+1} &= \frac{1}{1-2(1-\phi)} (g_{y_{H,t+1}} - g_{y_{F,t+1}}), \\ \eta_{t+1} &= \left[ \frac{1}{1-2(1-\phi)} - s \right] (g_{y_{H,t+1}} - g_{y_{F,t+1}}) \end{aligned}$$

*where $g_{y_{t+1}} = y_{t+1} - y_t$. Then, $m_{t+1}, m_{t+1}^*, \eta_{t+1}$ and $\Delta e_{t+1}$ are jointly normally distributed.*

Our approximation technique relies on taking the autarky limit for real quantities. There are two reasons why this limit is attractive. First, we prove the joint log normality of SDFs and the exchange rate at the full home-bias limit as $\alpha \to 1$, whereas this is not the case for $\alpha < 1$. Second, the allocations and prices are invariant to the number of assets traded.[31] Then, whereas we calculate the covariance in the financial autarky case by direct computation, we back out the implied covariance at the autarky limit, denoted $cov_t^{\to FA}(m_{t+1}^* - m_{t+1}, \Delta e_{t+1})$, between SDFs and depreciations when

[31] While the invariance and the log-normality properties also hold at the complete markets allocation or Cole and Obstfeld (1991) limit, these cannot by construction break the Backus-Smith condition.

there is trade in assets by imposing $cov_t(m_{t+1}^{(*)}, \eta_{t+1})$ consistent with (8)-(9).

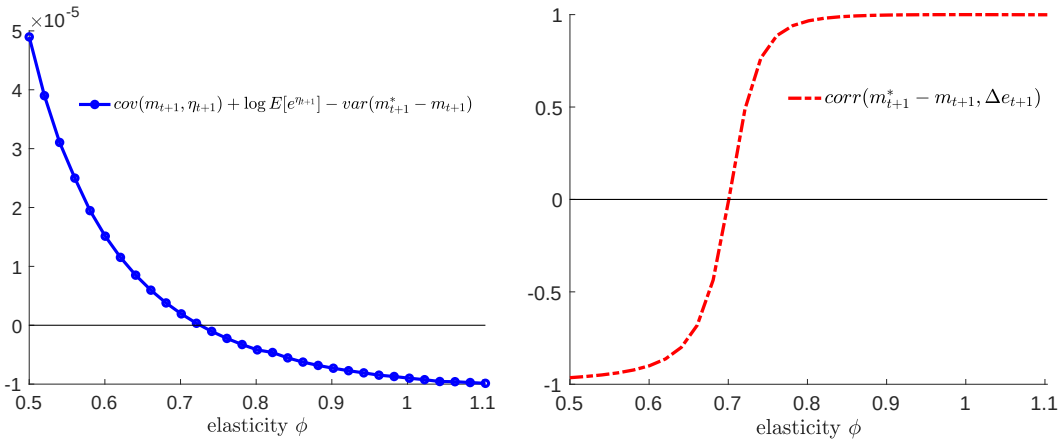
**Proposition** (Representative agent, Autarky Limit).
*The two-country model at the autarky limit can deliver* $cov_t^{\to FA}(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) \leq 0$ *conditional on shocks to* $y_{H,t}$ *if* $\frac{-s}{1-2(1-\phi)} \leq 0$ *when at most one asset is traded internationally. With two risk-free assets (and potentially also risky assets) are internationally traded, the model requires* $\phi \to \infty$ *which implies* $var_t(\Delta e_{t+1}) \to 0$.

When the trade elasticity is sufficiently low, the non-traded component of exchange rates is risky ($cov(m_{t+1}, \eta_{t+1})) < 0$ and following an increase in Home productivity, demand for Home goods rises so much, that prices must adjust to constrain Foreign consumption of the Home good for markets to clear, as in Corsetti et al. (2008). Under autarky the ability of the model to match risk sharing depends only on $\phi$ given that $s > 0$. When there is trade in a risk-free asset, condition (10) additionally depends on the inter-temporal elasticity of substitution but this can never have an effect on whether the inequality is satisfied. With trade in two nominally risk-free bonds, a zero covariance between relative SDFs and the exchange rate arises in the limit where $var_t(\Delta e_{t+1})$ approaches zero, but a negative covariance can never be achieved.

Figure 2a illustrates Proposition 1 away from the autarky limit, i.e. $\alpha < 1$. Specifically, condition (10) is satisfied for values over which the plotted line is positive. Figure 2b shows that the sign on the simulated Backus-Smith correlation tracks condition (10). The correlation is negative for low values of $\phi$, positive for higher values and intersects 0 at approximately the same value for $\phi$ as Panel (a), even though, away from the autarky limit, the SDFs and exchange rates in the equilibrium model are not exactly log-normal.

**Figure 2**

**(a)** *Prop. 1 Statistic* **(b)** *Backus-Smith Correlation*




*Calibration*: $\omega = 0.96, u = 0.005, s = 1, \phi \in [0.5, 1.1], \alpha = 0.6, \alpha^* = 1 - \alpha, \rho = 0.96, Y_H = 1$. Unconditional moments calculated from second-order simulation with one million draws.

## D. Additional results for Section 3

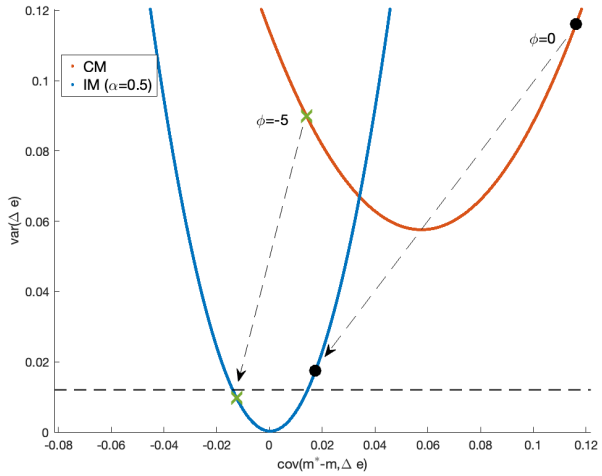
Figure 3 show the tight link between volatility and cyclicality. The complete and incomplete market loci are constructed by considering different values for $\phi$, keeping spanning constant. On the other hand, the dots and crosses illustrate how the outcomes vary with spanning, keeping $\phi$ constant. The case of $\phi = 0$ corresponds to the representative agent limit), studied in Chernov et al. (2024). When idiosyncratic risk is not priced, incomplete international spanning can at best deliver a zero covariance and only as exchange rate volatility approaches zero.

Figure 4 presents the comparative statics with respect to $\alpha^\nu$, i.e. how spanned the $\nu-$ shock is. The left panel illustrates that incomplete spanning is critical to allow idiosyncratic risk to result in a negative aggregate Backus-Smith covariance. To see this, notice that:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = \kappa z_t + \kappa^\nu \sigma_n u^2 + \tilde{\phi}(\gamma - \tilde{\phi} - \gamma^* + \beta_1) z_t \leq 0$$

Given that $\kappa^\nu$ is minimized at $\alpha^\nu = 0$, as spanning rises, then $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ becomes more positive. More specifically, assuming $\alpha^\nu \to 0$, the curve pivots around

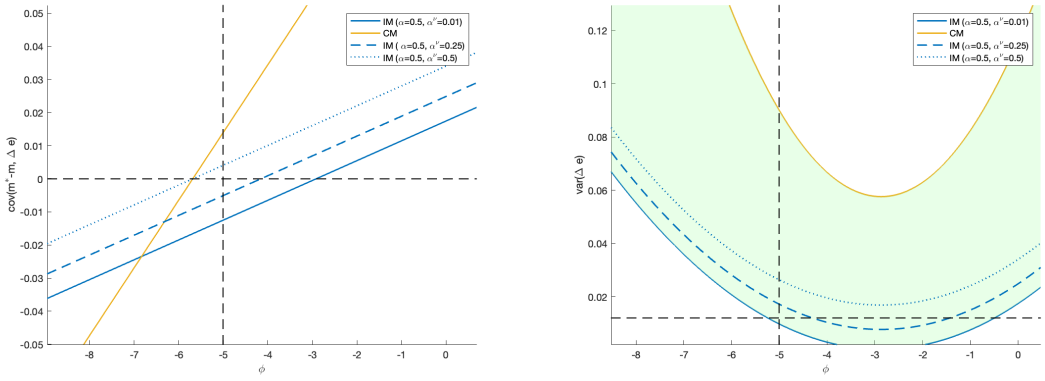
**Figure 3:** *Uninsurable Idiosyncratic Risk vs. Incomplete International Spanning*




*Loci of admissible variance and cyclicality of exchange rates in equilibrium. Each point in the lines denotes a different value of idiosyncratic risk cyclicality $\phi$. Dashed arrows show how the equilibria vary with international spanning $\alpha$ holding $\phi$ at values of* 0 *and* $-5$ *respectively. Dashed horizontal black line denotes the observed exchange rate volatility. The representative agent limit is obtained at* $\phi = 0$.

**Figure 4**

**(a)** *Backus-Smith Covariance* **(b)** *Exchange Rate Volatility*




the same point. As $\alpha$ rises, the line becomes steeper and the covariance more negative at e.g. $\phi = -5$. Instead, as $\alpha^{\nu}$ rises, the curve shifts to the left towards the complete

markets results.

# E. Empirical Results and Robustness

## E.1. Generalization of Proposition 4

Consider the following aggregate Euler equations capturing within-country idiosyncratic risk according to (31) and allowing for financial shocks $u_{t+1}$:

$$\mathbb{E}_t\left[M_{t+1}\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}\right]R_{t+1}=1, \tag{87}$$

$$\mathbb{E}_t\left[M^*_{t+1}\left(\frac{\delta^*_{t+1}}{\delta^*_t}\right)^{-s}\right]R^*_{t+1}=1, \tag{88}$$

$$\mathbb{E}_t\left[M^*_{t+1}\left(\frac{\delta^*_{t+1}}{\delta^*_t}\right)\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right]R_{t+1}=\mathbb{E}_t[e^{u_{t+1}}], \tag{89}$$

$$\mathbb{E}_t\left[M_{t+1}\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right]R^*_{t+1}=\mathbb{E}_t[e^{u_{t+1}}] \tag{90}$$

where $\mathbb{E}_t[u_{t+1}]=0$. Taking logs, we derive:

$$\begin{aligned}&var_t(\Delta e_{t+1})-var_t(u_{t+1})+cov_t(m_{t+1}-m^*_{t+1},\Delta e_{t+1})\\&+cov_t\left(\log\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}-\log\left(\frac{\delta^*_{t+1}}{\delta^*_t}\right)^{-s},\Delta e_{t+1}\right)=0\end{aligned} \tag{91}$$

Substituting $var_t(u_{t+1})=uvar_t(\Delta e_{t+1})$ and assuming $u\in[0,1]$, rearrranging:

$$\begin{aligned}var_t(\Delta e_{t+1})(1-u)+cov_t&\left(\log\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}-\log\left(\frac{\delta^*_{t+1}}{\delta^*_t}\right)^{-s},\Delta e_{t+1}\right)\\&=cov_t(m^*_{t+1}-m_{t+1},\Delta e_{t+1})\end{aligned} \tag{92}$$

Then, $cov_t(m^*_{t+1}-m_{t+1},\Delta e_{t+1})<0$ requires:

$$-cov_t\left(\log\left(\frac{\delta_{t+1}}{\delta_t}\right)^{-s}-\log\left(\frac{\delta^*_{t+1}}{\delta^*_t}\right)^{-s},\Delta e_{t+1}\right)>var_t(\Delta e_{t+1})(1-u)$$

Applying the Cauchy-Schwarz identity and dividing by standard deviations yields the general condition (39).

## E.2. Berger et al. (2023) wedge

We use the discount factor wedge $\beta$ constructed by Berger et al. (2023) to check whether the observed real exchange rate of the following 17 advanced economies with the US satisfies the inequality restriction discussed in inequality (39). Sample covers annual data over 1992–2017.

BBD define the discount factor wedge as " the conditional expectation, across realizations of the idiosyncratic state, of the change in the consumption share between two periods raised to a power of -$\sigma$", where $\sigma$ in their notation is the inverse of the intertemporal elasticity of substitution. They measure it from Consumption Expenditure Survey as follows: for each household $i$, define the consumption share $\varphi_{it} = \frac{c_{it}}{C_t}$. Group households with similar characteristics at date $t$. For each group $g \in G$ groups, compute the discount factor statistic as average across $N_g$ individuals in each group $g$:

$$\beta_{g,t+1} \equiv \frac{1}{N_g} \sum_{i=1}^{N_g} \left( \frac{\varphi_{it+1}}{\varphi_{it}} \right)^{-\sigma} \tag{93}$$

They use the discount factor wedge computed from high income/low net worth individuals as their model consistent wedge. We use this wedge from their measurement in testing the inequality (39) assuming domestic incomplete markets only within the US ($\delta_t^* = 1$ for all $t$).

Table 1 provides the negative of the correlation of the real exchange growth with the U.S. discount factor wedge and the threshold in the inequality (39). The final column lists the standard deviation of the real exchange rate growth for each country against the U.S. The standard deviation of the discount factor wedge is 0.42 (constructed with intertemporal elasticity of substitution $s^{-1} = 0.1$). Threshold for the inequality restriction is defined by $\frac{\sigma(\Delta e)}{\sigma(\beta)}$.

With the exception of Japan, the correlation of the exchange rate with the discount factor wedge is negative. Hence, exchange rates are largely risky with respect to the discount factor wedge. The magnitude of the correlation is broadly consistent with

**Table 1:** *Empirical Moments: Real Exchange Rate Growth and Berger et al. (2023) wedge*

| iso | $-\text{Corr}(\beta,\Delta e)$ | Threshold | $\sigma(\Delta e)$ |
|---|---|---|---|
| AUS | 0.22 | 0.32 | 0.13 |
| BEL | 0.25 | 0.25 | 0.10 |
| CAN | 0.37 | 0.23 | 0.10 |
| CHE | 0.20 | 0.22 | 0.09 |
| DEU | 0.24 | 0.25 | 0.10 |
| DNK | 0.26 | 0.24 | 0.10 |
| ESP | 0.30 | 0.27 | 0.11 |
| FIN | 0.24 | 0.28 | 0.12 |
| FRA | 0.25 | 0.25 | 0.10 |
| GBR | 0.25 | 0.26 | 0.11 |
| IRL | 0.34 | 0.21 | 0.09 |
| ITA | 0.25 | 0.27 | 0.11 |
| JPN | -0.07 | 0.26 | 0.11 |
| NLD | 0.24 | 0.25 | 0.11 |
| NOR | 0.30 | 0.29 | 0.12 |
| PRT | 0.26 | 0.26 | 0.11 |
| SWE | 0.33 | 0.31 | 0.13 |
| AVERAGE | 0.25 | 0.26 | 0.11 |

*Notes:* Table lists the negative of the correlation between the US discount factor wedge ($\beta$) the bilateral real exchange rate growth ($\Delta e$), the threshold for exchange rate cyclicality described in equation 39 assuming domestic incomplete markets only within the US ($\delta_t^* = 1$ for all $t$), and the standard deviation of the real exchange rate growth for pair of seventeen advanced economies. Real exchange rate growth is constructed from Jordà et al. (2017) database. US discount factor wedge is constructed by Berger et al. (2023) from Consumer Expenditure Survey in the US. Sample: 1992–2017 (annual). See text for details.

satisfying the inequality restriction for nine of the seventeen bilateral pairs in the sample.

## E.3. Bilateral Wedge from GRID

We measure bilateral wedges using micro statistics from the Global Repository of Income Dynamics (Guvenen et al., 2022, GRID henceforth). We focus on sample of advanced economies for which we can obtain the longest panel to construct the bilateral wedge against the U.S. The annual data sample covers Canada, Denmark,

France, Germany, Italy, Norway, Sweden, the UK, and the US spanning 1998–2015.

Variables used in our wedge construction are log earnings and residual log earnings for top 0.01 percentile, top 0.1 percentile, top 1 percent, and top 5 percent of each country's log earnings distribution in year $t$, and the mean log earnings and residual log earnings for that year. GRID constructs residual log earnings as "the residuals after regressing log earnings on age dummies for each year $t$". For baseline results, we use the residual log earnings data for Male. Results are robust to using (i) log earnings data for Male (i.e. unresidualized data), (ii) residual log earnings data for all gender, and (ii) log earnings data for all gender.

Denote with $\overline{\log I_{gt}}$ the average (residual) log earnings for the group $g$, and $\overline{\log I_t}$ as the average (residual) log earnings in the country at time $t$. Then the income share at date $t$ for a group $g$ is defined as $\varphi^I_{gt} \equiv \exp\left(\overline{\log I_{gt}} - \overline{\log I_t}\right)$. The discount factor wedge for a group $g$ is then constructed as:[32]

$$\beta^I_{gt} = \left(\frac{\varphi^I_{gt+1}}{\varphi^I_{gt}}\right)^{-s} \tag{94}$$

where $s$ is the inverse of the intertemporal elasticity of substitution. We de-mean each country's wedge for (i) partially addressing measurement error as in Berger et al. (2023), and (ii) comparison across countries.[33] The bilateral wedge is then given by the difference in the de-meaned country specific wedges for group $g$:

$$\Delta\beta_{gt} \equiv \beta^I_{gt} - \beta^{*I}_{gt} \tag{95}$$

[32]The data in GRID is averages at the percentile level. This proxy construction is thus different from Berger et al. (2023) in that we are taking average income shares to construct a wedge, whereas Berger et al. (2023) construct average of the individual wedges.

[33]Results are robust to using the non-deamened wedges.

### E.3.1 Summary Statistics

**Table 2:** *Summary Statistics for Country-specific Wedges by Percentile Groups*

| | Top 0.01 percentile | | Top 0.1 percentile | | Top 1 percent | |
|---|---|---|---|---|---|---|
| iso | $\sigma(\beta^I_{gt})$ | $\text{Corr}(\beta^I_{gt}, \hat{Y}_t)$ | $\sigma(\beta^I_{gt})$ | $\text{Corr}(\beta^I_{gt}, \hat{Y}_t)$ | $\sigma(\beta^I_{gt})$ | $\text{Corr}(\beta^I_{gt}, \hat{Y}_t)$ |
| CAN | 0.032 | -0.437 | 0.030 | -0.434 | 0.017 | -0.486 |
| DEU | 0.020 | -0.538 | 0.016 | -0.548 | 0.008 | -0.341 |
| DNK | 0.031 | -0.510 | 0.030 | -0.376 | 0.018 | -0.106 |
| FRA | 0.031 | -0.435 | 0.016 | -0.688 | 0.010 | -0.531 |
| ITA | 0.009 | -0.093 | 0.009 | 0.029 | 0.011 | 0.005 |
| NOR | 0.044 | -0.490 | 0.028 | -0.468 | 0.012 | -0.353 |
| SWE | 0.043 | -0.335 | 0.035 | -0.302 | 0.018 | -0.125 |
| USA | 0.022 | -0.609 | 0.021 | -0.701 | 0.009 | -0.656 |
| AVERAGE | 0.029 | -0.431 | 0.023 | -0.436 | 0.013 | -0.324 |

*Notes:* Table lists the standard deviation of the country specific wedges constructed for different groups $\beta^I_{gt}$, and the correlation of these wedges with the respective country's output gap $\hat{Y}_t$. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. Output gap is proxied with real GDP growth. Real GDP growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

**Table 3:** *Standard Deviation of Bilateral Exchange Rates and Wedges*

| | | $\sigma(\Delta\beta_{gt})$ | | |
|---|---|---|---|---|
| iso | $\sigma(\Delta e_t)$ | Top 0.01% | Top 0.1% | Top 1% |
| CAN | 0.11 | 0.02 | 0.02 | 0.01 |
| DEU | 0.11 | 0.02 | 0.02 | 0.01 |
| DNK | 0.10 | 0.04 | 0.03 | 0.02 |
| FRA | 0.11 | 0.04 | 0.02 | 0.01 |
| ITA | 0.11 | 0.03 | 0.03 | 0.02 |
| NOR | 0.13 | 0.04 | 0.02 | 0.01 |
| SWE | 0.13 | 0.03 | 0.02 | 0.01 |
| AVERAGE | 0.12 | 0.03 | 0.02 | 0.01 |

*Notes:* Table lists the standard deviation of the bilateral real exchange rate ($\Delta e$) and the bilateral discount factor wedges constructed for different groups $\beta^I_{gt}$. Wedges are constructed using residual log earnings data for Male ages between 25–55 in GRID. Real exchange rate growth is constructed from Jordà et al. (2017) database. Sample: 1998–2015 (annual). See text for details.

### E.3.2 Results from residual log income based shares

**Table 4:** *Correlation and Thresholds by Percentile Groups*

| iso | Top 0.01 percentile $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | Top 0.1 percentile $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | Top 1 percent $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. |
|---|---|---|---|---|---|---|
| CAN | 0.34 | 0.44 | 0.32 | 0.52 | 0.23 | 0.84 |
| DEU | 0.18 | 0.47 | -0.10 | 0.67 | -0.33 | 1.26 |
| DNK | 0.25 | 0.29 | 0.00 | 0.31 | 0.04 | 0.53 |
| FRA | 0.37 | 0.29 | -0.03 | 0.54 | 0.01 | 0.92 |
| ITA | -0.07 | 0.41 | -0.22 | 0.44 | -0.10 | 0.71 |
| NOR | 0.46 | 0.35 | 0.12 | 0.61 | 0.33 | 1.23 |
| SWE | 0.55 | 0.48 | 0.57 | 0.62 | 0.43 | 1.11 |
| AVERAGE | 0.30 | 0.39 | 0.10 | 0.53 | 0.09 | 0.94 |

*Notes:* Table lists the negative of the correlation between the bilateral discount factor wedge ($\Delta\beta_{gt}$) and the bilateral real exchange rate growth ($\Delta e$), and the threshold for exchange rate cyclicality described in equation 39. Real exchange rate growth is constructed from Jordà et al. (2017) database. Bilateral discount factor wedge is constructed using residual log earnings data for Male ages between 25–55 in GRID. Sample: 1998–2015 (annual). See text for details.

### E.3.3 Results from log income based shares

**Table 5:** *Correlation and Thresholds by Percentile Groups*

| iso | Top 0.01 percentile $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | Top 0.1 percentile $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | Top 1 percent $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. |
|---|---|---|---|---|---|---|
| CAN | 0.35 | 0.45 | 0.35 | 0.55 | 0.33 | 0.80 |
| DEU | 0.22 | 0.54 | -0.11 | 0.67 | -0.31 | 1.24 |
| DNK | 0.23 | 0.29 | 0.01 | 0.32 | 0.09 | 0.50 |
| FRA | 0.32 | 0.35 | -0.12 | 0.61 | -0.02 | 1.02 |
| ITA | -0.05 | 0.45 | -0.23 | 0.44 | -0.11 | 0.66 |
| NOR | 0.46 | 0.37 | 0.06 | 0.67 | 0.47 | 1.18 |
| SWE | 0.51 | 0.51 | 0.59 | 0.69 | 0.33 | 1.05 |
| AVERAGE | 0.29 | 0.42 | 0.08 | 0.56 | 0.11 | 0.92 |

*Notes:* Table lists the negative of the correlation between the bilateral discount factor wedge ($\Delta\beta_{gt}$) and the bilateral real exchange rate growth ($\Delta e$), and the threshold for exchange rate cyclicality described in equation 39. Real exchange rate growth is constructed from Jordà et al. (2017) database. Bilateral discount factor wedge is constructed using log earnings data for Male ages between 25–55 in GRID. Sample: 1998–2015 (annual). See text for details.

### E.3.4 Results from residual log income based shares and all genders

**Table 6:** *Correlation and Thresholds by Percentile Groups*

| | Top 0.01 percentile | | Top 0.1 percentile | | Top 1 percent | |
|---|---|---|---|---|---|---|
| iso | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. |
| CAN | 0.41 | 0.47 | 0.32 | 0.59 | 0.33 | 1.04 |
| DEU | 0.33 | 0.58 | -0.18 | 0.80 | -0.41 | 1.35 |
| DNK | 0.25 | 0.29 | 0.02 | 0.37 | 0.05 | 0.68 |
| FRA | 0.20 | 0.29 | 0.12 | 0.53 | 0.10 | 0.98 |
| ITA | -0.01 | 0.35 | -0.22 | 0.50 | -0.12 | 0.91 |
| NOR | 0.31 | 0.35 | 0.19 | 0.76 | 0.34 | 1.35 |
| SWE | 0.56 | 0.47 | 0.50 | 0.70 | 0.41 | 1.29 |
| AVERAGE | 0.29 | 0.40 | 0.11 | 0.61 | 0.10 | 1.09 |

*Notes:* Table lists the negative of the correlation between the bilateral discount factor wedge ($\Delta\beta_{gt}$) and the bilateral real exchange rate growth ($\Delta e$), and the threshold for exchange rate cyclicality described in equation 39. Real exchange rate growth is constructed from Jordà et al. (2017) database. Bilateral discount factor wedge is constructed using residual log earnings data for all genders ages between 25–55 in GRID. Sample: 1998–2015 (annual). See text for details.

### E.3.5 Results from log income based shares and all genders

**Table 7:** *Correlation and Thresholds by Percentile Groups*

| | Top 0.01 percentile | | Top 0.1 percentile | | Top 1 percent | |
|---|---|---|---|---|---|---|
| iso | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. | $-\text{Corr}(\Delta\beta_{gt}, \Delta e)$ | Thresh. |
| CAN | 0.35 | 0.51 | 0.29 | 0.63 | 0.38 | 1.06 |
| DEU | 0.25 | 0.60 | -0.14 | 0.87 | -0.37 | 1.34 |
| DNK | 0.24 | 0.32 | 0.04 | 0.38 | 0.12 | 0.67 |
| FRA | 0.16 | 0.35 | 0.01 | 0.70 | 0.04 | 1.06 |
| ITA | -0.03 | 0.45 | -0.24 | 0.60 | 0.06 | 1.06 |
| NOR | 0.39 | 0.38 | 0.27 | 0.84 | 0.51 | 1.35 |
| SWE | 0.52 | 0.52 | 0.51 | 0.74 | 0.41 | 1.39 |
| AVERAGE | 0.27 | 0.45 | 0.11 | 0.68 | 0.16 | 1.13 |

*Notes:* Table lists the negative of the correlation between the bilateral discount factor wedge ($\Delta\beta_{gt}$) and the bilateral real exchange rate growth ($\Delta e$), and the threshold for exchange rate cyclicality described in equation 39. Real exchange rate growth is constructed from Jordà et al. (2017) database. Bilateral discount factor wedge is constructed using residual log earnings data for all genders ages between 25–55 in GRID. Sample: 1998–2015 (annual). See text for details.