Heterogeneous Mobility in Dynamic Labor Market Equilibrium

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Abstract

We devise a tractable equilibrium model of persistent heterogeneity in labor mobility. The model admits analytical solutions for household values, labor flows, and the distribution of mobility types across markets, a challenge posed by the environment. Equilibrium mobility is ordered: markets shrink or grow in order of mobility type, starting with the most mobile. Mobile types thus sort into better markets while immobile types are "left behind." Equilibrium inequality is, in turn, two-dimensional: gaps emerge across markets not only in labor market outcomes, but also in their mobile-type composition, which is valued by immobile types. Heterogeneity in mobility thus amplifies both between- and within-group inequality. Labor market dynamics exhibit excess persistence, since convergence requires closing both cross-sectional gaps, which generically takes longer. Application of the model to data on local labor market dynamics from the US census provides support for these predictions.

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The mobility of labor across markets plays a central role in the inequality of labor market outcomes in the cross section, and the persistence of these disparities across time. A key stylized fact of labor mobility is the presence of considerable heterogeneity in the propensity to move—notably, it is well-known that the less skilled are less likely to move (Topel 1986; Bound and Holzer 2000). The implications of such heterogeneity for labor market equilibrium are, however, less well-understood.

In this paper, we offer two contributions. First, we develop a tractable framework in which the equilibrium implications of heterogeneous mobility can be inferred analytically. Second, we apply our model to data on local labor market dynamics; we find that it can successfully account for the sorting of types across locations, as well as the large degree of inequality and persistence in employment rates, particularly among the low skilled.

We begin in section 1 with a model of labor market reallocation in the spirit of Lucas and Prescott (1974), specifically its more recent incarnation in Alvarez and Shimer (2011). Labor markets comprised by firms operating a decreasing returns technology face shocks that drive idiosyncratic changes in labor demand. Households choose how much labor to supply, and whether to move across markets subject to a mobility cost.

The key innovation that we explore is to allow households to be *heterogeneous*. Specifically, household types are allowed to differ with respect to their skill, preferences, and mobility costs, and are imperfect substitutes in production. We show that the presence of such household heterogeneity generically implies the presence of heterogeneity in optimal mobility policies.[1]

Solving the model involves an analytical challenge. The presence of heterogeneity in mobility implies households value not only a market's current labor market conditions, but also its *distribution over mobility types*: The presence of more mobile types in a market affects the future path of that market's wages, because more-mobile types are more likely to leave in response to adverse shocks. Mobility decisions both imply, *and are implied by*, the distribution of mobility types, a difficult fixed-point problem.

A central contribution of this paper is to provide a solution to this problem. We show that it is possible to characterize household values, optimal mobility policies, labor flows, the distributions of mobility types across markets and, thereby, steady-state labor market equilibrium in this environment.

[1] We show that households' optimal mobility rules are homogeneous only in a knife-edge case in which the pattern of heterogeneity in skill, preferences, mobility costs, and the relative scarcity of each type aligns in a common ratio across household types.

The key insight is that equilibrium mobility is *ordered*, such that markets facing adverse (favorable) shocks shrink (grow) via population flows in order of mobility type, starting with the most mobile. Formally, optimal mobility decisions are summarized by a sequence of thresholds for market wages and mobility composition at which each household type leaves following adverse idiosyncratic shocks. Consequently, the mobility decisions of less-mobile types become marginal only after more-mobile types have already left a market.

This insight facilitates an analytical characterization of steady-state labor market equilibrium with heterogeneous mobility. As in canonical models of costly reallocation, cross-sectional equilibrium gaps emerge in current labor market conditions (real wages and employment) across markets. A distinctive implication of heterogeneous mobility, however, is that equilibrium gaps also emerge in a further dimension—the mobility-type composition of each market.

We highlight three novel implications that we view as distinctive to the environment. First, a distinctive pattern of equilibrium *sorting* emerges whereby mobile types accumulate in labor markets with better conditions. Intuitively, mobile types act as labor market arbitrageurs: their mobility offsets labor market shocks, even though they comprise only a subset of the market. Formally, the stationary distribution of mobile types over labor market outcomes stochastically dominates its counterpart for immobile types.

Second, echoing the pattern of equilibrium gaps, *inequality* is also two-dimensional. Household values depend on both a labor market's current conditions and its composition. Stationary equilibrium gives rise to a joint distribution over both dimensions. Thus, the model uncovers a new potential source of equilibrium inequality in welfare, one that has its origins in heterogeneity in mobility.

Third, and relatedly, the model naturally gives rise to a notion of *excess persistence*: labor market convergence requires the closing of both equilibrium gaps. For a given gap in current labor market conditions, implied rates of convergence are slower in the presence of heterogeneous mobility, because it also takes time to close composition gaps. Inequality across markets is thus more persistent, and especially so among immobile types.

Thanks to its parsimony and analytical tractability, our model admits straightforward quantification of the general equilibrium impact of heterogeneous mobility. To illustrate, in section 2, we apply our framework to study quantitatively the substantial and persistent inequalities observed across local labor markets (Amior and Manning 2018).

We explore a simple calibration of the model in which, echoing our initial motivation, we map mobility types to skill groups—specifically, nongraduates and graduates. We discipline the model's few parameters using observed average levels and gaps in employment rates by skill, skill composition across locations, together with the overall rate of net population flows in US census data.

We then confront the calibrated model with several untargeted moments in the data. A central implication is that the model requires a much larger mobility cost for less skilled nongraduates to rationalize the data, over four times its counterpart for graduates. Qualitatively, this echoes the findings of Topel (1986) and Bound and Holzer (2000). Quantitatively, we find that the implied population-weighted average mobility cost is considerable, and of a similar order of magnitude to the influential estimates of Kennan and Walker (2011). Furthermore, despite targeting only aggregate net population flows, the model implies net flow contributions by skill that are close to empirical analogues. Taken together, we view the calibrated model as having sensible implications for a range of relevant untargeted outcomes.

In the remainder of the paper, we study the calibrated model's implications for its distinctive novel equilibrium properties: the patterns of sorting, inequality, and persistence in local labor market dynamics.

Sorting takes a particular form in the model: Since skill types are imperfect substitutes (Bound and Johnson 1992; Katz and Murphy 1992), the entry of mobile, high-skilled types into growing locations raises demand for immobile, less-skilled labor, raising nongraduate employment rates. Consequently, mobile graduates are concentrated in markets with higher nongraduate employment rates. Strikingly, this is exactly the tendency in the data. Indeed, the model accounts well for both the magnitude and slope of this sorting pattern.

The calibrated model has particularly rich implications for inequality. To illustrate, we juxtapose our baseline calibration with a counterfactual homogeneous-mobility economy in which low-skilled mobility costs are reduced to equalize mobility across skills. This reveals that heterogeneous mobility contributes significantly both to between- and within-group inequality:[2] The model implies a *mobility premium* in lifetime welfare of as much as 8 percent between mobile graduates and less-mobile nongraduates. More striking, the model implies substantial inequality among the low-skilled, with a standard deviation of log lifetime welfare of 0.2.

[2] Interestingly, the model also clarifies that heterogeneous mobility contributes significantly to misallocation. Indeed, high-skilled types' welfare rises by over 5 percent in the homogeneous-mobility counterfactual.

Finally, we explore two signature predictions of the model for the persistence of local labor market dynamics: that low-skilled types face greater persistence in labor market outcomes; and that heterogeneity in mobility contributes to overall labor market persistence in the aggregate. Both predictions are echoed in the data. Autocorrelations of the log employment rate align well in model and data, both by skill and in the aggregate. Importantly, the model is able to capture the significant persistence of nongraduate employment rates, with a half-life of several decades. This, in turn, contributes to the significant persistence of aggregate employment rates across locations.

Related literature. [To be added.]

1. Model

In this section, we devise a new model of dynamic labor market equilibrium that accommodates the presence of persistent heterogeneity across households. We show that the latter generically gives rise to heterogeneity in the propensity to move across markets. The model is tractable enough to yield analytical solutions for household values, labor flows, the distribution of mobility types over labor market outcomes and, thereby, stationary labor market equilibrium.

1.1 Environment

Time is continuous, and the horizon infinite. There is a continuum of labor markets comprised by firms and households.

Firms. Each market has a unit measure of firms, each of which employs two varieties of labor input. In anticipation of household heterogeneity, we will think of these varieties as a low-skilled labor input, of measure n_L, and a high-skilled labor input, of measure n_H. Firms use these inputs to produce a flow of revenue,

$$y = xn^{\alpha}, \text{ where } n = \left(n_L^{\frac{\varsigma-1}{\varsigma}} + n_H^{\frac{\varsigma-1}{\varsigma}} \right)^{\frac{\varsigma}{\varsigma-1}}. \tag{1}$$

Here, $\alpha \in (0,1)$ captures the presence of decreasing returns to scale in production and/or product market power, and ς denotes the elasticity of substitution between low-

and high-skilled labor input. We will focus on the empirically relevant case in which $\varsigma > 1$ (Bound and Johnson 1992; Katz and Murphy 1992).[3]

The key source of shocks to each market, and thereby of *ex post* heterogeneity and reallocation across markets, is a process of innovations to each market's revenue productivity x; this takes the form of the geometric Brownian motion,

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}z, \tag{2}$$

where $\mathrm{d}z$ is the increment to a Wiener process.

Households. Each market has an endogenous measure of households, each of which is comprised by a unit measure of individuals. A measure ℓ_L of households is low-skilled and the remaining measure ℓ_H is high-skilled. Households differ across three dimensions of heterogeneity: First, as anticipated by the nomenclature, households differ in their *skill*, formalized by their efficiency units of labor input per employed worker, s_i.[4] Second, they differ in their *propensity to supply labor*, formalized by a preference parameter b_i (see (3)). And third, they differ in their (nonpecuniary) *cost of moving*, denoted C_i.

Households obtain a flow utility from consumption c, and employment n, given by

$$u_i(c, n) = c - \frac{1}{b_i^{1/\eta}} \frac{n^{1+1/\eta}}{1 + 1/\eta}. \tag{3}$$

Both firms and households discount the future at common rate r.

Within-market equilibrium. To facilitate our focus on the implications of mobility frictions for between-market equilibrium, within-market equilibrium is thus kept deliberately simple.

Firm optimization implies market demand for labor inputs satisfies

$$n_i^d(w_i; w, n) = \left(\frac{w_i}{w}\right)^{-\varsigma} n, \text{ and } n^d(w, x) = \left(\frac{\alpha x}{w}\right)^{\epsilon}, \text{ where } w \equiv \left(w_L^{1-\varsigma} + w_H^{1-\varsigma}\right)^{\frac{1}{1-\varsigma}}. \tag{4}$$

$\epsilon \equiv 1/(1-\alpha)$ is the elasticity of market demand for total labor input, w_i the market wage per efficiency unit of type-i labor, and w the market wage for total labor input.

[3] The gross complements ($\varsigma \in (0,1)$) and Cobb-Douglas ($\varsigma \to 1$) cases can be solved analogously.

[4] These differences in efficiency units are, of course, isomorphic to differences in skill intensity (since $\varsigma \neq 1$); the latter are thus omitted from the constant-elasticity-of-substitution aggregator in (1).

Household optimization implies that a type-i household supplies $s_i n_i^*(w_i) = b_i s_i^{1+\eta} w_i^{\eta}$ efficiency units of type-i labor, consumes $c^*(w_i) = b_i(s_i w_i)^{1+\eta}$, and enjoys flow indirect utility $u_i^*(w_i) = b_i(s_i w_i)^{1+\eta}/(1+\eta)$. The market supply of labor inputs thus satisfies

$$n_i^s(w_i, m_i) = m_i w_i^{\eta}, \quad \text{and} \quad n^s(w, m) = m w^{\eta}, \quad \text{where} \quad m \equiv \left(m_L^{\frac{\varsigma-1}{\varsigma+\eta}} + m_H^{\frac{\varsigma-1}{\varsigma+\eta}} \right)^{\frac{\varsigma+\eta}{\varsigma-1}}. \tag{5}$$

Here, $m_i \equiv b_i s_i^{1+\eta} \ell_i$ is the market's *effective type-i population*, and m is its *effective total population*. A convenient summary statistic for market type composition will be the *effective population (quasi) shares*,

$$\phi \equiv \left(\frac{m_H}{m}\right)^{\frac{\varsigma-1}{\varsigma+\eta}}, \quad \text{such that } 1-\phi = \left(\frac{m_L}{m}\right)^{\frac{\varsigma-1}{\varsigma+\eta}}. \tag{6}$$

Equilibrium market wages equate demand and supply, both in total and for each type i, given x and m_i (and thus m), yielding the within-market equilibrium outcomes

$$w = \left[\frac{(\alpha x)^{\epsilon}}{m}\right]^{\frac{1}{\epsilon+\eta}}, \quad n = [(\alpha x)^{\eta} m]^{\frac{\epsilon}{\epsilon+\eta}}, \quad w_i = \left(\frac{m_i}{m}\right)^{-\frac{1}{\varsigma+\eta}} w, \quad \text{and} \quad n_i = \left(\frac{m_i}{m}\right)^{\frac{\varsigma}{\varsigma+\eta}} n. \tag{7}$$

Equilibrium market wages and employment rates are interchangeable, since $n_i/m_i = w_i^{\eta}$. Furthermore, at *interior markets*—defined as those with no voluntary mobility, $\mathrm{d}m_i = 0$, and thus no change in population composition, $\mathrm{d}\phi = 0$—market wages and employment rates inherit the stochastic structure of the market revenue productivity shocks in (2),

$$\begin{aligned} \frac{\mathrm{d}w_i}{w_i} &= \frac{\epsilon}{\epsilon+\eta}\left[\left(\mu - \frac{1}{2}\sigma^2 \frac{\eta}{\epsilon+\eta}\right)\mathrm{d}t + \sigma \mathrm{d}z\right] \equiv \mu_w \mathrm{d}t + \sigma_w \mathrm{d}z, \text{ and} \\ \frac{\mathrm{d}(n_i/m_i)}{n_i/m_i} &= \frac{\epsilon\eta}{\epsilon+\eta}\left\{\left[\mu + \frac{1}{2}\sigma^2\left(\frac{\epsilon\eta}{\epsilon+\eta} - 1\right)\right]\mathrm{d}t + \sigma \mathrm{d}z\right\} \equiv \mu_n \mathrm{d}t + \sigma_n \mathrm{d}z. \end{aligned} \tag{8}$$

We will use this property to characterize equilibrium in terms of market wages w_i; but note that an equivalent characterization is available in terms of employment rates n_i/m_i.

Between-market mobility. Households move across markets through two channels. First, households can move, at any time, to a market of their choice, subject to incurring their (nonpecuniary) mobility cost C_i. Second, at rate δ, households receive a mobility shock such that they leave their current market and move to a market drawn randomly from the distribution of population of their type over wages. The latter thus reshuffles households across markets in a population-neutral manner, intuitively capturing a notion of idiosyncratic preferences that generates gross mobility across markets.

Extensions. The preceding environment is amenable to several simple reinterpretations. First, markets can be interpreted several ways; our application in section 2 will view these as geographic locations, but one also could envision markets as industries or occupations.[5] Second, if labor markets are associated with local product markets (as would be natural if one interprets markets as locations), the environment can accommodate the presence of local goods in quasi-fixed supply, such as housing, subject to a simple reinterpretation of the labor demand elasticity ϵ, and of the w_i as *real* market wages (see Appendix A). Third, one can extend the framework to accommodate market amenities in household utilities; all subsequent results then apply subject to a reinterpretation of the w_i as *amenity-adjusted* real market wages.[6] Fourth, if household consumption is hand-to-mouth, and consumption utility exhibits constant relative risk aversion, the results that follow apply subject to an adjustment of the power function in household flow indirect utility.[7]

Analytical challenge. The presence of persistent household heterogeneity gives rise to the following analytical challenge: A household's valuation of a market will, in general, depend not only on (i) market real wages w_i, which determine contemporaneous payoffs, but also: (ii) the *mobility decisions* of other household types which, together with their market *composition*, summarized by the effective population share ϕ, determine the *future evolution* of market real wages; and (iii) the *distribution* of mobility types across markets, which determines the feasible returns to mobility. Household values further imply optimal mobility decisions which, in turn, imply the distribution of mobility types across markets. Equilibrium thus involves a fixed point in both mobility decisions and the distribution of types across markets. In what follows, we devise a method for solving this fixed-point problem analytically and draw out several novel predictions of the resulting equilibrium for labor mobility.

[5] In the latter cases, one could reinterpret households as individuals who supply labor inelastically ($\eta = 0$).

[6] Furthermore, as will become clear, our equilibrium characterization holds whenever local utilities u_i evolve according to a geometric Brownian motion. Thus, one could extend the environment further to allow amenities also to evolve according to a (potentially correlated) geometric Brownian motion.

[7] One also could extend the model to incorporate a within-market "large-firm" search equilibrium (as in, e.g., Elsby and Michaels 2013) if within-market dynamics are relatively short-lived compared to between-market dynamics (our focus). This would allow a reinterpretation of those out of work as being unemployed.

1.2 Homogeneous mobility

A key preliminary input into our solution is a benchmark special case in which a *homogeneous mobility solution* arises, such that optimal mobility decisions are identical across types, and type composition is identical across markets. We will see that such a solution emerges only in a knife-edge case in which the pattern of heterogeneity across skills s_i, preferences b_i, and mobility costs C_i takes a precise form.

Between-market equilibrium in this case can be characterized by two boundaries for market real wages that determine mobility decisions. It will be convenient to express these boundaries in terms of the real wages for a single type—we will use the high-skilled wage w_H for this purpose, such that the homogeneous mobility boundaries are $\underline{w}_H < \overline{w}_H$, common across types i. Given type composition is identical across markets, one can recover L-type wages from $w_L = [\Phi/(1-\Phi)]^{1/(\varsigma-1)} w_H$, where $\Phi/(1-\Phi)$ is the aggregate effective population skill ratio.[8]

At interior markets, $w_H \in (\underline{w}_H, \overline{w}_H)$, there is no voluntary mobility, $\mathrm{d}m_i = 0$, population composition remains unchanged, $\mathrm{d}\phi = 0$, and equilibrium market real wages evolve as in (8). The value of a household of type i, denoted $V_i(w_H)$, therefore satisfies the Bellman equation

$$(r+\delta)V_i(w_H) = \frac{b_i(s_i w_i)^{1+\eta}}{1+\eta} + \mu_w w_H V_i'(w_H) + \frac{1}{2}\sigma_w^2 w_H^2 V_i''(w_H) + \delta \bar{V}_i, \tag{9}$$

for $i \in \{L, H\}$ and $w_H \in (\underline{w}_H, \overline{w}_H)$.

Households at interior markets receive a flow of indirect utility $b_i(s_i w_i)^{1+\eta}/(1+\eta)$, and face capital gains from two sources. First, markets receive idiosyncratic shocks, which are manifested in market wages according to the stochastic law of motion in (8). Application of Ito's lemma gives rise to the terms in the drift μ_w and variance σ_w^2 in (9). Second, at rate δ, the household is reshuffled across markets in a population-neutral manner, realizing an expected payoff denoted $\bar{V}_i$, as in the final term in (9).

At the boundaries $\underline{w}_H$ and $\overline{w}_H$, voluntary mobility occurs. Optimal household mobility requires the boundaries to satisfy value-matching and smooth-pasting conditions,

$$V_i(\underline{w}_H) \equiv V_i(\overline{w}_H) - C_i, \quad \text{and} \quad V_i'(\underline{w}_H) = 0 = V_i'(\overline{w}_H), \quad \text{for } i \in \{L, H\}. \tag{10}$$

(9) and (10) comprise a canonical problem of infinitesimal control (e.g., Abel and Eberly 1996; Alvarez and Shimer 2011).

[8] Specifically, using (5) and (6), $\Phi/(1-\Phi) \equiv (M_H/M_L)^{(\varsigma-1)/(\varsigma+\eta)}$ where $M_i \equiv b_i s_i^{1+\eta} L_i$.

Between-market equilibrium is completed by the aggregate market clearing condition,

$$M = \frac{X}{\int w^{\epsilon+\eta} g(w) \mathrm{d}w}, \tag{11}$$

where $X \equiv \mathbb{E}[(\alpha x)^{\epsilon}]$ reflects aggregate labor demand, and $M \equiv \mathbb{E}[m]$ is aggregate effective population. In each market, supply equals demand, $mw^{\epsilon+\eta} = (\alpha x)^{\epsilon}$. Aggregating over markets and defining $g(w)$ as the density of (effective) population over wages, yields (11).

We will focus on the stationary case in which M and $g(w)$ are unchanged over time, so that X is stationary.[9] Ito's lemma then implies the aggregate stationarity condition

$$\mu_w + \frac{1}{2}\sigma_w^2(\epsilon + \eta - 1) = 0. \tag{12}$$

Proposition 1 characterizes stationary equilibrium under homogeneous mobility.

Proposition 1 *A homogeneous mobility solution exists only if*

$$\mathcal{C}_H \equiv C_H \Phi^{\frac{1+\eta}{\varsigma-1}} / \left(b_H s_H^{1+\eta}\right) = C_L (1-\Phi)^{\frac{1+\eta}{\varsigma-1}} / \left(b_L s_L^{1+\eta}\right) \equiv \mathcal{C}_L, \tag{13}$$

It gives rise to an equilibrium with the following properties:

(i) Household values can be written as $V_H(w_H) = b_H s_H^{1+\eta} \cdot v(w_H)$ *and* $V_L(w_H) = b_L s_L^{1+\eta} [\Phi/(1-\Phi)]^{(1+\eta)/(\varsigma-1)} \cdot v(w_H)$*, where*

$$v(w_H) = \frac{1}{\rho(1+\eta)} \frac{w_H^{1+\eta}}{1+\eta} + \nu_1 w_H^{\gamma_1} + \nu_2 w_H^{\gamma_2} + \frac{\delta \bar{v}}{\rho(0)}, \tag{14}$$

for all $w_H \in (\underline{w}_H, \overline{w}_H)$*,* $\gamma_1 < 0$ *and* $\gamma_2 > 1$ *are roots of the fundamental quadratic*

$$\rho(\gamma) \equiv r + \delta - \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\gamma - \frac{1}{2}\sigma_w^2 \gamma^2, \tag{15}$$

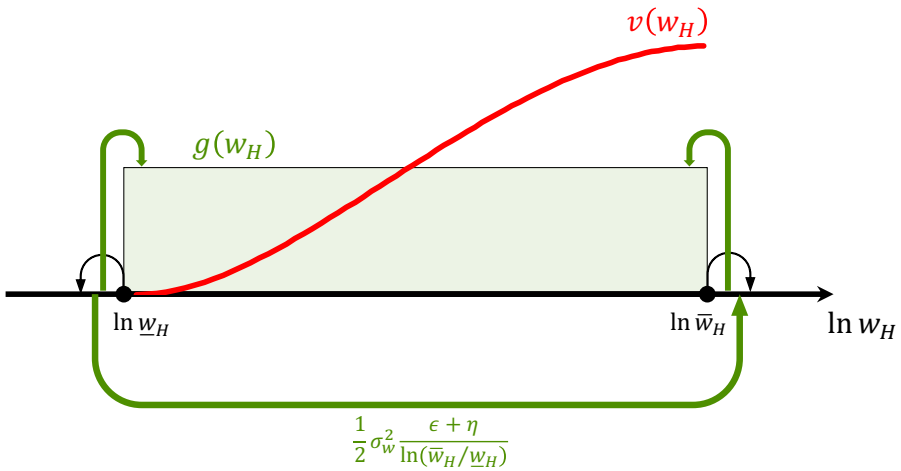
and the coefficients ν_1 *and* ν_2*, and boundaries* $w_{H,l}$ *and* $w_{H,h}$*, satisfy the boundary conditions (10) and the aggregate market clearing condition (11).*

(ii) The distribution of effective population over w_H*,* $G(w_H)$*, and the rate of population outflow at* $\underline{w}_H$*, which is equal to the rate of population inflow at* $\overline{w}_H$*, and denoted* λ*, are*

$$G(w_H) = \frac{\ln(w_H/\underline{w}_H)}{\ln(\overline{w}_H/\underline{w}_H)}, \quad \text{and} \quad \lambda = \frac{1}{2}\sigma_w^2 \frac{\epsilon+\eta}{\ln(\overline{w}_H/\underline{w}_H)}. \tag{16}$$

[9] Analogous results hold for arbitrary μ_w, but with aggregate growth in population or wages.

Figure 1. Homogeneous mobility



Notes. The household value $v(w)$ and effective population density $g(w)$ are depicted over log wages to emphasize the log-uniform nature of $g(w)$. The black arrows are a heuristic depiction of negative (positive) innovations to market wages at the lower (upper) boundaries. The green arrows likewise depict population flows—those moving from $\underline{w}_H$ to $\bar{w}_H$, as well those reflected back to $\underline{w}_H$ and $\bar{w}_H$.

Proposition 1 has the following interpretation. The first term in (14) captures the value of a market with current wage w_H in the absence of population flows; the subsequent two terms in (14) then capture the value of population inflows in favorable future states, and population outflows in adverse future states. Equilibrium population flows then take the following form: At w_l, a density $g(w_l)$ of effective population is subject to shocks to market wages of instantaneous variance σ_w^2. Negative innovations induce population outflows until the market wage is reflected back to w_l, at a rate determined by the sum of the elasticities of local labor demand and supply, $\epsilon + \eta$. Symmetrically, households that leave w_l arrive as population inflows at w_h. Application of the Fokker-Planck (Kolmogorov Forward) equation gives rise to the stationary distribution function of effective population, $G(w_H)$ in (16). The stationarity condition (12) implies that effective population is distributed *uniformly* over log wages in steady state. This, in turn, implies the simple form for population flow λ in (16). Figure 1 illustrates.

Two aspects of Proposition 1 are central to motivating our subsequent analysis. First, under homogeneous mobility, household values in a market are *proportional* across types, where the factor of proportionality reflects differences in skills (s), preferences (b), and relative scarcity (Φ). An implication is that between-group inequality in household values in this case is identical to its counterpart in the *absence* of mobility frictions ($C_i \equiv 0$).

Furthermore, within-group inequality is *identical* across household types. By contrast, we will see that heterogeneous mobility contributes to both forms of inequality.

Second, the presence of persistent heterogeneity in skills, preferences, and mobility costs implies that homogeneous mobility emerges only in a knife-edge case in which the pattern of heterogeneity in b, s, and C exactly offsets the relative scarcity of each type, as in (13). The latter is clearly extreme, motivating study of the more natural case of heterogeneous mobility—our focus in the ensuing sections.

1.3 Heterogeneous mobility

A corollary of Proposition 1 is that a household's *mobility* type is determined by its *effective mobility cost*, $\mathcal{C}_H$ and $\mathcal{C}_L$ in (13). Without loss of generality, we shall adopt the convention that $\mathcal{C}_H < \mathcal{C}_L$, such that H-types are more mobile. We show that the presence of heterogeneous mobility types implies that household values, and thereby mobility decisions, in general depend not only on a market's real wages, but also on its type composition, summarized by the effective population share of the mobile low-$\mathcal{C}_H$ types ϕ, as defined in (6).

We propose an equilibrium in which mobility is *ordered* such that mobile H-types are the first to leave a market following adverse shocks, and the first to enter a market following favorable shocks. Figure 2A illustrates.

Equilibrium is characterized by two sets of boundaries. First, as before, boundaries for the mobile H-type real market wage, $\underline{w}_H < \bar{w}_H$, such that H-types leave markets at $\underline{w}_H$ following adverse shocks, moving to markets at $\bar{w}_H$ that realize favorable shocks. Applying (6) and (7), these H-type wage boundaries, in turn, trace out boundary loci for the wages of the immobile L-type w_L as follows

$$\underline{w}_L(\phi) = \underline{w}_H \cdot \omega(\phi), \text{ and } \bar{w}_L(\phi) = \bar{w}_H \cdot \omega(\phi), \text{ where } \omega(\phi) \equiv \left(\frac{\phi}{1-\phi}\right)^{\frac{1}{\varsigma-1}}. \quad (17)$$

The second set of boundaries applies to the mobile type effective population share, $\underline{\phi} < \bar{\phi}$, such that immobile L-types leave markets at $\underline{\phi}$ with wage $\underline{w}_L(\underline{\phi})$ following adverse shocks, moving to markets at $\bar{\phi}$ with wage $\bar{w}_L(\bar{\phi})$ that realize favorable shocks.

Figure 2A traces out an instructive sample path implied by the proposed equilibrium. Consider a market at the initial point $\circ$ in the north-east of each subpanel in Figure 2A. Assume this market subsequently realizes a long, sustained sequence of adverse shocks. Initially, the shocks are not sufficient to induce either mobility type to move; the market's

mobility composition ϕ is unchanged, and both types' equilibrium wages fall horizontally. After some time, the mobile H-type's wage attains its lower boundary $\underline{w}_H$. Thereafter, each subsequent adverse shock is met by the exit of mobile types such that that w_H is reflected back to the boundary $\underline{w}_H$. The market's mobility share therefore falls, $\mathrm{d}\phi < 0$. Mobile types that exit move to any market at the upper boundary $\overline{w}_H$ that realizes a favorable shock, as illustrated by the "watering can" annotation in Figure 2A; these markets' mobility shares therefore rise, $\mathrm{d}\phi > 0$. Because mobility types are imperfect substitutes, immobile-type wages fall (rise) as mobile types exit (enter) the market; thus, w_L falls with ϕ in the left-hand subpanel of Figure 2A. Continued adverse shocks further deplete the mobility share ϕ until it attains its lower boundary $\underline{\phi}$, after which immobile types also begin to exit such that the market is reflected back to the boundary $\underline{\phi}$. Immobile types that exit move to any market at $(\overline{w}_H, \bar{\phi})$ that realizes a favorable shock, as illustrated by the overarching arrow annotation in Figure 2A.

We are now able to characterize an equilibrium of this proposed form. A first implication is that the mobility problem of the H-type remains unchanged relative to the homogeneous case above: Since H-types are the most mobile, they are first to leave (enter) a market facing adverse (favorable) shocks and, conditional on their wage w_H, they otherwise are indifferent to a market's mobility type composition ϕ. Consequently, their household value remains a function solely of their market wage, $V_H(w_H)$, and their mobility problem mirrors (9) and (10), with solution given by result (i) of Proposition 1.

The L-type problem is, however, now very different from the homogeneous case: Since L types are the least mobile, they are last to leave (enter) a market facing adverse (favorable) shocks. Their household value thus will depend not only on a market's wages, but also its type composition. For comparability with the case of homogeneous mobility, it is again convenient to express values in terms of the real wages for a single type, w_H. Thus, we write L-type value as $V_L(w_H, \phi)$.

To see why L-types additionally care about ϕ, return to Figure 2A, and consider a market at the lower wage boundary $\underline{w}_H$. The key is that the exit of mobile H-types following adverse shocks alters the subsequent path of market wages for immobile L-types. The direction and strength of this effect turns on the balance of two forces anticipated in equation (7) above. On the one hand, exit of mobile types lowers m, and thereby the total supply of labor to the market, a force that raises wages w among households that remain. On the other hand, for a given w, exit of mobile types lowers demand for immobile-type labor when types are imperfect substitutes ($\varsigma < \infty$), a force that depresses immobile-type

wages w_L. The key intuition is that, under the proposed equilibrium, a market's type composition ϕ is a sufficient statistic for the *future path* of the immobile type's wage w_L at each boundary; it is for this reason that immobile L-type households also value ϕ.

One might imagine that the presence of two state variables in the immobile type's problem raises a potential analytical challenge: Since L-type households anticipate capital gains from future changes in *both* a market's w_H *and* its type composition ϕ, the Bellman equation for their value $V_L(w_H, \phi)$ will in general be described by a *partial* differential equation, and their optimal mobility decisions take the form of a (w_H, ϕ) *locus*, as opposed to the simple scalar threshold policy in the homogeneous case.

It turns out, however, that the proposed equilibrium implies a useful simplification: market type composition ϕ adjusts only at *discrete points*—the boundaries $\underline{w}_H$ and $\bar{w}_H$. Away from the boundaries, ϕ is *unchanged*. The Bellman equation for the L-type value therefore satisfies a set of *ordinary* differential equations,

$$(r+\delta)V_L(w_H,\phi) = \frac{b_L[s_L w_H \omega(\phi)]^{1+\eta}}{1+\eta} + \mu_w w_H V_{Lw} + \frac{1}{2}\sigma_w^2 w_H^2 V_{Lww} + \delta \bar{V}_L, \tag{18}$$

for all $w_H \in (\underline{w}_H, \bar{w}_H)$ and $\phi \in \left[\underline{\phi}, \bar{\phi}\right]$.

Optimal mobility of L-type households again requires their household value to satisfy value-matching and smooth-pasting conditions. Since, under the proposed equilibrium, L-types flow from markets at $(\underline{w}_H, \underline{\phi})$ to markets at $(\bar{w}_H, \bar{\phi})$, we now have

$$V_L\left(\underline{w}_H, \underline{\phi}\right) \equiv V_L(\bar{w}_H, \bar{\phi}) - C_L, \quad V_{Lw}\left(\underline{w}_H, \underline{\phi}\right) = 0 = V_{Lw}(\bar{w}_H, \bar{\phi}). \tag{19}$$

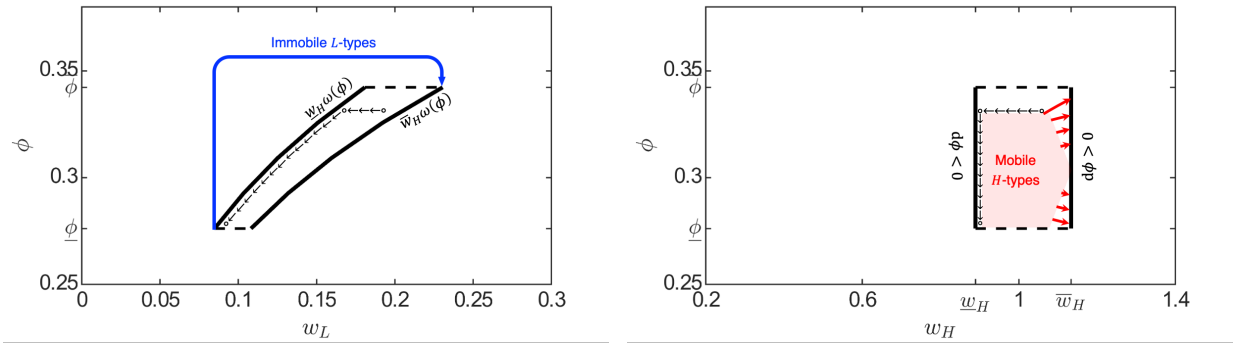
In addition, the L-type household value must evolve continuously at the boundaries $\underline{w}_H$ and $\bar{w}_H$ for all $\phi \in \left(\underline{\phi}, \bar{\phi}\right)$. Recall that adverse (favorable) shocks at the lower (higher) wage boundary are met by outflows (inflows) of H-types such that w_H is reflected back to the boundary. We show in the Appendix that this gives rise to the following additional smooth-pasting conditions for the L-type value,

$$V_{L\phi}(w_H,\phi) = \frac{\varphi'(\phi)}{\varphi(\phi)} w_H V_{Lw}(w_H,\phi), \quad \text{where} \quad \varphi(\phi) \equiv \left[\phi(1-\phi)^{-\frac{\varsigma+\eta}{\epsilon+\eta}}\right]^{\frac{1}{\varsigma-1}}, \tag{20}$$

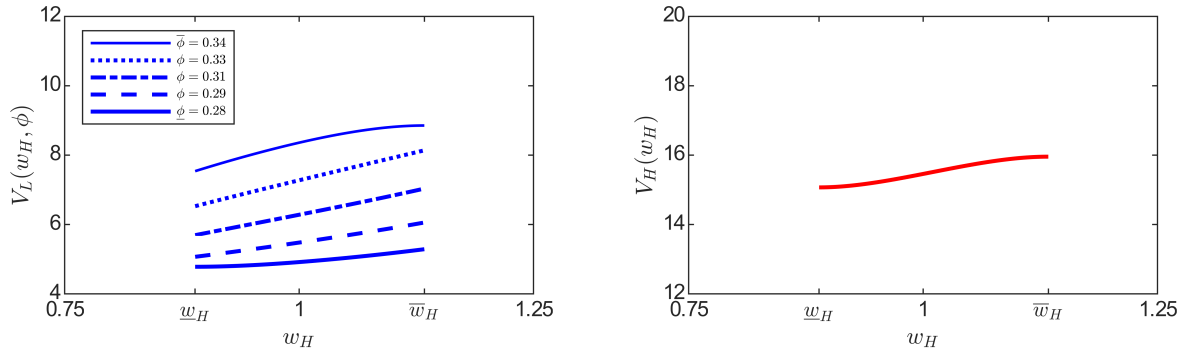
for $w_H \in \{\underline{w}_H, \bar{w}_H\}$ and $\phi \in \left(\underline{\phi}, \bar{\phi}\right)$. Here, $\varphi'(\phi)/\varphi(\phi)$ maps market-level shocks to innovations in composition ϕ at the boundaries $\underline{w}_H$ and $\bar{w}_H$. The transform $\varphi(\phi)$ thus indexes a market's time spent at these boundaries. We will see that $\varphi(\phi)$ plays a central role in the solution.

Figure 2. Heterogeneous mobility

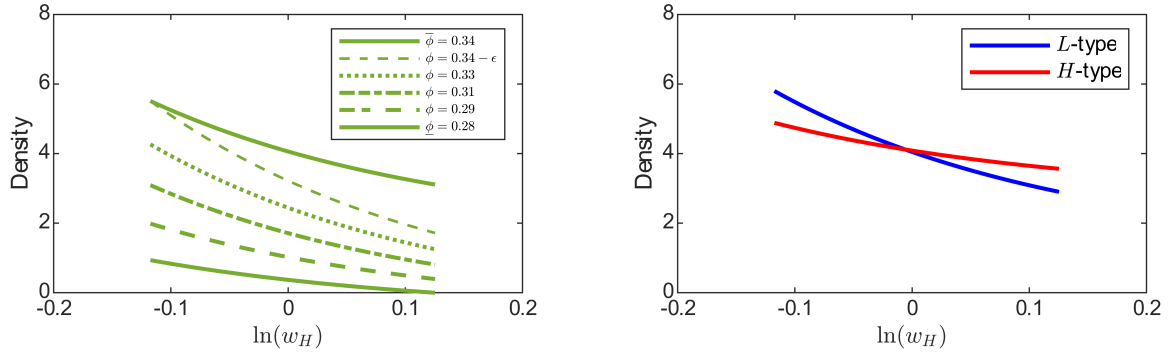
A. Ordered mobility equilibrium



B. Household values



C. Effective population density



Notes. Parameter values are based on the model calibrated as described in Table 1.

Finally, between-market equilibrium is completed by aggregate market clearing conditions, both for total labor input, and for labor input by type $i \in \{L, H\}$,

$$M = \frac{X}{\int\int w^{\epsilon+\eta} g(w,\phi)\mathrm{d}w\mathrm{d}\phi}, \quad \text{and} \quad M_i = \frac{\int\int w^{\varsigma+\eta} g(w,\phi)\mathrm{d}w\mathrm{d}\phi}{\int\int w_i^{\varsigma+\eta} g_i(w_i,\phi)\mathrm{d}w_i\mathrm{d}\phi} M. \tag{21}$$

The former is simply the analogue of (11), where $g(w,\phi)$ denotes the joint density of effective total population m over wages for total labor input w, and type composition ϕ. For the latter, observe from (7) that, in each market, equilibrium real wages by type are such that $m_i w_i^{\varsigma+\eta} = m w^{\varsigma+\eta}$. Aggregating over markets, and defining $M_i \equiv b_i s_i^{1+\eta} L_i$ as the aggregate effective type-i population and $g_i(w_i,\phi)$ the joint density of type-i effective population m_i over type-i wages w_i and ϕ, yields the second equation in (21).

Proposition 2 characterizes stationary equilibrium with heterogeneous mobility.

Proposition 2 *In the heterogeneous mobility case, $\mathcal{C}_L > \mathcal{C}_H$:*

(i) The H-type value is given by $V_H(w_H) = b_H s_H^{1+\eta} \cdot v_H(w_H)$, where

$$v_H(w_H) = \frac{1}{\rho(1+\eta)} \frac{w_H^{1+\eta}}{1+\eta} + \nu_1 w_H^{\gamma_1} + \nu_2 w_H^{\gamma_2} + \frac{\delta \bar{v}_H}{\rho(0)}, \tag{22}$$

for all $w_H \in (\underline{w}_H, \bar{w}_H)$. The constants $\{\nu_1, \nu_2\}$, and the boundary $\underline{w}_H$, satisfy the boundary conditions (10), mirroring result (i) of Proposition 1.

The L-type value is given by $V_L(w_H,\phi) = b_L s_L^{1+\eta} \cdot v_L(w_H,\phi)$, where

$$v_L(w_H,\phi) = \frac{1}{\rho(1+\eta)} \frac{[w_H \omega(\phi)]^{1+\eta}}{1+\eta} + \nu_{L1}(\phi)[w_H \varphi(\phi)]^{\gamma_1} + \nu_{L2}(\phi)[w_H \varphi(\phi)]^{\gamma_2} + \frac{\delta \bar{v}_L}{\rho(0)}, \tag{23}$$

for all $w_H \in (\underline{w}_H, \bar{w}_H)$, and $\phi \in [\underline{\phi}, \bar{\phi}]$, with

$$\nu_{Lj}(\phi) = \psi_j + \nu_j \frac{\gamma_j}{\varsigma - 1} \frac{\epsilon - \varsigma}{\epsilon + \eta} \int \frac{\omega(\phi)^{1+\eta}}{\varphi(\phi)^{\gamma_j}} \frac{\mathrm{d}\phi}{1-\phi}, \text{ for } j \in \{1,2\}. \tag{24}$$

The latter integral can further be expressed as a hypergeometric function (provided in the appendix). The remaining constants $\{\psi_1, \psi_2\}$, and boundaries $\{\bar{w}_H, \underline{\phi}, \bar{\phi}\}$, satisfy the boundary conditions (19) and the aggregate market clearing conditions (21).

(ii) The joint density of effective total population over w_H and ϕ satisfies

$$g(w_H,\phi) = \begin{cases} g_0 (1-\bar{\phi})^{-\frac{\varsigma+\eta}{\varsigma-1}} \left[\underline{w}_H^{-(\epsilon+\eta)} - w_H^{-(\epsilon+\eta)} \right] / w_H & \text{if} \quad \phi = \bar{\phi}, \\ g_0 (1-\phi)^{-\frac{\varsigma+\eta}{\varsigma-1}} (\epsilon+\eta) \dfrac{\varphi'(\phi)}{\varphi(\phi)} w_H^{-(\epsilon+\eta)-1} & \text{if} \quad \phi \in (\underline{\phi}, \bar{\phi}), \\ g_0 (1-\underline{\phi})^{-\frac{\varsigma+\eta}{\varsigma-1}} \left[w_H^{-(\epsilon+\eta)} - \bar{w}_H^{-(\epsilon+\eta)} \right] / w_H & \text{if} \quad \phi = \underline{\phi}, \end{cases} \tag{25}$$

for $w_H \in (\underline{w}_H, \bar{w}_H)$, where g_0 is a constant such that $\int\int g(w_H,\phi) dw_H d\phi = 1$. The effective population flow rates by type are

$$\lambda_L = \frac{1}{2}\sigma_w^2(\epsilon+\eta)g_0\left[\underline{w}_H^{-(\epsilon+\eta)} - \overline{w}_H^{-(\epsilon+\eta)}\right],$$
$$\lambda_H = \frac{1}{2}\sigma_w^2(\epsilon+\eta)g_0\left[\left(\frac{\bar{\phi}}{1-\bar{\phi}}\right)^{\frac{\varsigma+\eta}{\varsigma-1}}\underline{w}_H^{-(\epsilon+\eta)} - \left(\frac{\underline{\phi}}{1-\underline{\phi}}\right)^{\frac{\varsigma+\eta}{\varsigma-1}}\overline{w}_H^{-(\epsilon+\eta)}\right]. \quad (26)$$

The key novelty of result (i) is the solution for the immobile L-type household value, in particular its dependence on market type composition ϕ. Most directly, ϕ affects L-types' *current* wages because mobility types are imperfect substitutes, as summarized by the term in $\omega(\phi)$ in (23). But, importantly, ϕ also affects the *future path* of L-type wages; these effects are summarized by the terms in $w_H\varphi(\phi)$ in (23) and their coefficients $v_{L1}(\phi)$ and $v_{L2}(\phi)$ in (24). Observe that, if ϕ were degenerate ($\underline{\phi} = \bar{\phi}$), both $v_H(w_H)$ and $v_L(w_H, \phi)$ would mirror Proposition 1, and differences in value would arise only from differences in the *levels* of wages, rather than their *dynamics*.

The constants ψ_j in (24) capture L-types' valuation of *their own mobility*—that is, leaving markets at $(\underline{w}_H, \underline{\phi})$ and entering markets at $(\overline{w}_H, \bar{\phi})$. Recalling that $\varphi(\phi)$ summarizes a market's time spent at the wage boundaries, $\underline{w}_H$ and $\overline{w}_H$, the ψ_js value the likelihood that a market attains the type-composition boundaries, $\underline{\phi}$ and $\bar{\phi}$, in the future. Accordingly, the ψ_js are determined by the "outer" boundary conditions in (19).

By contrast, the remaining option value coefficients in (24) instead capture L-types' valuation of *H-types' mobility*. Intuitively, they value movements in L-type wages w_L implied by (17) as H-types leave markets at $\underline{w}_H$ and enter markets at $\overline{w}_H$ for $\phi \in (\underline{\phi}, \bar{\phi})$. Accordingly, the solution in (24) is recovered from the "inner" boundary conditions in (20).

Panel B of Figure 2 illustrates the form of the L-type value implied by (23) and (24): $v_L(w_H, \phi)$ "fans out" as type composition ϕ varies. A key implication is that equilibrium inequality in L-type welfare is *two-dimensional* in the presence of heterogeneous mobility, driven not only by disparities in wages w_H across markets (as in the case of homogeneous mobility), but also by disparities in markets' composition ϕ. The extent of this additional dimension of inequality is shaped by the "fanning out" of L-type values in Figure 2B, together with the distribution of population over w_H and ϕ.

The latter is provided in result (ii) of Proposition 2. This completes equilibrium by providing an analytical solution for the stationary effective population density $g(w_H, \phi)$

in (25) which, in turn, can be used to evaluate the market clearing conditions in (21).[10] The density can be understood most straightforwardly with reference to Figure 2C, which plots the cumulative density of total effective population over *log* wages up to a given ϕ, together with marginal densities of the effective population of each type over log wages.

Figure 2C reveals two key properties. First, the marginal density of total effective population over wages no longer exhibits the log-uniform structure implied by the homogeneous mobility solution in Proposition 1. Instead, the total effective population density now declines over log wages. Second, and relatedly, the densities of effective population over wages by type reveal that the distribution of H-types stochastically dominates the distribution of L-types. Intuitively, the equilibration of market wages at the boundaries $\underline{w}_H$ and $\bar{w}_H$ is now achieved by the mobility of just a *subset* of the population—the mobile H-types—who consequently accumulate in higher-wage markets. One interpretation of these mobile H-types is that they are *labor market arbitrageurs*.[11]

1.4 Special cases

Further understanding of the equilibrium characterization in Proposition 2 is aided by the following instructive special cases.

Perfect substitutes ($\varsigma = \infty$). The first is the limit in which types are perfect substitutes. Although not nested by Proposition 2, the form of equilibrium in this case is a simple extension of that in Figure 2A. Now, $w_L = w_H = w$ and $\phi = m_H/m$, and the L-type wage boundaries are independent of ϕ. As a result, declines in ϕ *per se* no longer are sufficient to induce L-types to leave declining markets. Instead, L-types leave after a market exhausts all its mobile types, $\underline{\phi} = 0$, and the wage attains an additional lower wage boundary $\underline{\underline{w}} < \underline{w} < \bar{w}$. Lemma 1 summarizes.

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[10] To mirror its solution for V_L, Proposition 2 reports the density of effective population over w_H and ϕ, denoted (with a slight abuse of notation) by $g(w_H, \phi)$ in (25). It is straightforward to use the latter to recover the relevant densities in (21); for example, $g_L(w_L, \phi) = \left[(1-\phi)^{(\varsigma+\eta)/(\varsigma-1)}/\omega(\phi)\right] g(w_L/\omega(\phi), \phi)$.

[11] A further property of Figure 2C is that the stationary joint density $g(w_H, \phi)$ exhibits "ridges" at $\phi \in \{\underline{\phi}, \bar{\phi}\}$ along which there is an order-of-magnitude jump in density. These arise due to the geometry of the equilibrium. Recalling Figure 2A, equilibrium dynamics are such that markets diffuse across wages w_H for $\phi \in \{\underline{\phi}, \bar{\phi}\}$, but across both w_H and ϕ at the boundaries $\{\bar{w}_H, \phi \in [\underline{\phi}, \bar{\phi})\}$ and $\{\underline{w}_H, \phi \in (\underline{\phi}, \bar{\phi}]\}$. Intuitively, the ridges arise due to "bottlenecks" in the effective population flows at the two corners, $(\bar{w}_H, \bar{\phi})$ and $(\underline{w}_H, \underline{\phi})$.

Lemma 1 *In the perfect substitutes case,* $\varsigma = \infty$*:*

(i) The solution for household values is as in Proposition 1, but with $w_L = w_H = w$*,* $\underline{\phi} = 0$*,* $\underline{w}_L(0) \equiv \underline{\underline{w}} < \underline{w} < \bar{w}$*, and*

$$v_{Lj}(\phi) = v_j + \psi_j(1-\phi)^{-\frac{\gamma_j}{\epsilon+\eta}}, \quad \text{for } j \in \{1,2\}. \tag{27}$$

The remaining constants $\{\psi_1, \psi_2\}$*, and boundaries* $\{\underline{\underline{w}}, \bar{w}, \bar{\phi}\}$*, satisfy (19) and (21).*

(ii) The joint density of effective total population over w *and* ϕ *satisfies*

$$g(w,\phi) = \begin{cases} g_0(1-\bar{\phi})^{-1}\left[\underline{w}^{-(\epsilon+\eta)} - w^{-(\epsilon+\eta)}\right]\dfrac{1}{w} & \text{if} \quad w \in (\underline{w}, \bar{w}), \phi = \bar{\phi}, \\ g_0(1-\phi)^{-2} w^{-(\epsilon+\eta)-1} & \text{if} \quad w \in (\underline{w}, \bar{w}), \phi \in (0, \bar{\phi}), \\ g_0\left[\max\{w, \underline{w}\}^{-(\epsilon+\eta)} - \bar{w}^{-(\epsilon+\eta)}\right]\dfrac{1}{w} & \text{if} \quad w \in (\underline{\underline{w}}, \bar{w}), \phi = 0. \end{cases} \tag{28}$$

A key implication of Lemma 2 is that L-types continue to value market composition ϕ even when it has no direct effect on their wages. The reason is that ϕ remains a sufficient statistic for how resilient a market's future wages will be to adverse shocks—specifically, the likelihood of future wages falling below the intermediate boundary $w_{H,l}$. The perfect substitutes case thus distils one channel through which immobile types value market type composition; accordingly, it gives rise to the simpler solution in Lemma 1. Finally, a further tractable implication of this case is that the effective population density becomes *piecewise* log-uniform on the two intervals, $(\underline{\underline{w}}, \underline{w})$ and $(\underline{w}, \bar{w})$.

Decoupled mobility ($\varsigma = \epsilon$). A second useful special case anticipated by Proposition 2 is when the elasticity of substitution ς equals the elasticity of demand for labor input ϵ. In this case, the L-type household value no longer depends on H-types' mobility.

Intuitively, consider the exit of H-types from markets at $\underline{w}_H$. This imparts two offsetting forces on the wages of L-types that remain. On one hand, it lowers the supply of total labor input, a force that raises wages for all remaining workers, *ceteris paribus*. On the other, because mobility types are imperfect substitutes ($\varsigma < \infty$), exit of H-types lowers the demand for L-type labor, a force that lowers w_L. A symmetric logic applies to the entry of H-types to markets at $\bar{w}_H$. When $\varsigma = \epsilon$, these opposing forces offset exactly, and L-type wages evolve identically with market shocks, *regardless* of H-type mobility.

An implication is that equilibrium household values form a pair of *decoupled* homogeneous mobility solutions in this case, whereby *both types'* optimal mobility rules are intervals for their respective wages whose width reflects their respective mobility.

Indeed, setting $\varsigma = \epsilon$ implies $\varphi(\phi) = \omega(\phi)$, such that the L-type value in (23) now also resembles result (i) of Proposition 1, expressed over L-type wages $w_L = w_H\omega(\phi)$.

1.5 Equilibrium properties

Proposition 2 provides a characterization of dynamic labor market equilibrium in the presence of heterogeneous mobility. Here we highlight distinctive novel features that emerge from this equilibrium. These features, in turn, motivate our quantitative analyses in the sections that follow.

Sorting. A natural intuition is that heterogeneous mobility will imply that mobile types sort into better markets, while immobile types sort into worse markets. This is manifested in the model in the mobile-type population distribution stochastically dominating its immobile-type counterpart in Figure 2C. Although natural, a crucial implication is that the model formalizes a notion of immobile types being *left behind* in equilibrium.

In the sections that follow, we confront these predictions of the model with the pattern of sorting observed in the data and draw out their implications for two further outcomes.

Inequality. First, the model has distinctive novel implications for the origins of inequality in welfare across households. A central message of Proposition 2 is that inequality in household values is two-dimensional, arising not only from cross-sectional gaps in current labor market outcomes, but also from cross-sectional gaps in market type composition ϕ. The magnitude of such gaps in ϕ in turn determine, in a precise sense, the magnitude of welfare inequality associated with immobile types being left behind in worse markets.

Moreover, mobility differentials contribute both to between- and within-group inequality. Between groups, mobile types enjoy a *mobility premium* in average welfare. Within group, cross-sectional differences in household values among immobile types are amplified by the presence of cross-sectional differences in market type composition ϕ.

Persistence. The second key implication we explore is that heterogeneous mobility gives rise to a notion of excess persistence in labor market dynamics, both in the cross section and in the aggregate. In the cross section, labor market outcomes among immobile types will be more persistent, as convergence requires closing equilibrium gaps in both dimensions—i.e., in current labor market outcomes, and in market type composition ϕ. By contrast, convergence among mobile types requires closing only the former gap. Relatedly, in the aggregate, convergence again requires, in part, the closing of both gaps. For a given observed gap in current labor market outcomes, convergence will take longer.

2. Quantitative illustration

We now illustrate the model's quantitative implications in an application to local labor market dynamics, viewing markets in the model as locations. Confronting model predictions with data on local labor market dynamics from the US census, we find support for the distinctive patterns of sorting, inequality and persistence described above. Furthermore, comparison with a counterfactual model in which mobility is homogeneous suggests that heterogeneity in mobility gives rise both to considerable inequality in welfare across local labor markets and considerable persistence of such inequality over time.

2.1 Calibration

To calibrate the model, we begin by considering how model concepts can be mapped to empirical analogues and then use this mapping to discipline the model's parameters.

Mapping model to data. Recall that the model environment described in section 1.1 is isomorphic to several reinterpretations, depending on the definition of markets, the presence of local goods in quasi-fixed supply, and the presence of market amenities. Therefore, we seek a mapping of model to data that accommodates these interpretations.

As we have noted, we view markets in the model as local labor markets in the data. More specifically, we will identify local labor markets with commuting zones (CZs) and use US census microdata throughout to assess the model's quantitative predictions.

Given this definition of markets, a natural local good in quasi-fixed supply is housing. This implies a simple reinterpretation of the elasticity of demand for total labor input ϵ (detailed in Appendix A),

$$\epsilon = 1/\left[1 - \alpha\left(1 - \frac{\beta}{1+\zeta}\right)\right], \tag{29}$$

where β is the housing expenditure share, and ζ is the housing supply elasticity.

More generally, the presence of nontradeable goods further underscores that the w_is in the model properly should be interpreted as *real consumer* market wages. The latter are notoriously difficult to measure across locations. Recall from equation (8), however, that an equivalent characterization of the model is available in terms of employment rates, which are much more straightforward to measure. Our baseline analysis therefore uses observed employment rates in US census data to discipline model parameters.

It is also natural that local labor markets offer heterogeneous amenities to households. For this reason, we also will examine the robustness of our results to using adjusted employment rates that are purged of variation associated with observable amenities.

Finally, we map low-skilled and high-skilled types in the model respectively to nongraduates and graduates in the data. This, in turn, is consistent with our convention of aligning the high-skilled with mobile types in the model, as empirical mobility rates among graduates are much larger than among nongraduates (Bound and Holzer 2000). Accordingly, we map the effective population quasi-share ϕ to the graduate-nongraduate population ratio in the data.

The loglinear structure of the model also admits a further simplification: Exchanging w_i with $\ln(n_i/\ell_i)$ and ϕ with $\ln(\ell_H/\ell_L)$ in Figure 2A transforms the model such that the low-skilled mobility boundaries are linear with slope $1 + (\varsigma/\eta)$. This will inform our calibration strategy, to which we now turn.

Calibration strategy. Table 1 summarizes our approach. We begin by externally calibrating several parameters directly.

We set the discount rate $r = 0.05$ to yield an annual real interest rate of 5 percent.

We calibrate the elasticity of demand for labor input ϵ using the transform in (29). We use standard values from the literature: We set the housing expenditure share $\beta = 0.33$ (Bureau of Labor Statistics 2019), the housing supply elasticity $\zeta = 1.75$ (Saiz 2010), and the revenue elasticity $\alpha = 5/6$ to reflect an elasticity of local product demand of 6 (Costinot and Rodriguez-Clare 2014). Together, these imply $\epsilon = 3.75$.

We set the labor supply elasticity η to 0.28, the extensive margin Frisch elasticity reported in the meta-analysis of Chetty et al. (2011).

We set the elasticity of substitution between high- and low-skilled workers, ς, to 1.41 to mirror the estimates of Katz and Murphy (1992).

We then calibrate remaining parameters using simple moments from cross-sectional microdata from the 1990 US census. In what follows, we provide intuition for which moments are most informative for which parameters.

We first normalize the aggregate population $L = L_H + L_L = 1$, and set the high-skilled population share $L_H = 0.192$ to replicate the graduate population share in the 1990 census.

We then normalize the efficiency units of the high skilled $s_H = 1$, and calibrate the efficiency units of the low skilled s_L such that the average log graduate wage premium in the model is equal to 0.497, the average observed in 1990 census data.

Table 1. Parameters and targeted moments of calibrated model (annual frequency)

Parameter		Value	Reason / Moment	Model	Target
A. Externally calibrated					
r	Discount rate	0.05	Annual real interest rate	0.05	0.05
ϵ	n^d elasticity (total input)	3.75	See text	3.75	3.75
η	n^s elasticity	0.28	Chetty et al. (2011)	0.28	0.28
ς	Elasticity of substitution	1.41	Katz and Murphy (1992)	1.41	1.41
L	Total population	1	Normalization	—	—
L_H	H-type population	0.192	Graduate population share	0.192	0.192
s_H	H-type efficiency units	1	Normalization	—	—
δ	Gross mobility rate	0.03	Gross population churn rate	0.03	0.03
B. Internally calibrated					
s_L	L-type efficiency units	4.34	Avg graduate wage premium	0.497	0.497
X	Agg labor demand scaler	0.0003	Avg graduate wage (normalized)	1	1
σ_w	Std dev of w innovations	0.042	Net population flow rate	0.005	0.005
C_H	H-type mobility cost	0.89	Log graduate e-pop gap	0.068	0.068
C_L	L-type mobility cost	4.07	Log population ratio gap	1.28	1.28
b_H	H-type labor supply pref	0.86	Avg graduate e-pop rate	0.861	0.861
b_L	L-type labor supply pref	0.77	Avg nongraduate e-pop rate	0.672	0.672

Notes. The rationale and source for each targeted moment are explained in the main text.

We choose the scale of aggregate labor demand X such that the population-weighted average high-skilled wage in the model $\mathbb{E}_g[w_H]$ is normalized to one. All flow variables are thus expressed in units of average high-skilled wages.

We then set the gross mobility rate δ to the observed rate of gross population churn, defined as the excess of the gross population flow rate over the net population flow rate.[12] The census records place of residence five years previously, allowing measurement of population flows over a five-year interval. The rate of gross population churn across CZs over the preceding five years in the 1990 census was 14.9 percent; we thus set $\delta = 0.0298$.

[12] The gross population flow rate is defined as the population-weighted average of the population inflow rate plus the population outflow rate, divided by two. The net population flow rate is defined as the population-weighted average of the absolute value of net population flows, divided by two.

The standard deviation of shocks, σ_w in (8), is calibrated to replicate observed net population flows. The net population flow rate over the preceding five years in the 1990 census was 2.58 percent; we thus target an annual rate of 0.517 percent.

Central ingredients to the model are the mobility costs by type, C_H and C_L. Recalling the determination of the mobility boundaries, and the mapping from model to data described above, we calibrate C_H to replicate observed gaps in the log employment rate among graduates, and C_L to replicate observed gaps in the log graduate-nongraduate population ratio, across locations.

An attractive feature of this latter step is that it requires only simple measures of *gaps* in employment rates by skill, and population skill ratios. We measure these gaps using observed differentials between the 95$^{\text{th}}$ and 5$^{\text{th}}$ percentiles of population across CZs in 1990 census data. These are, respectively, 0.068 for the log graduate employment rate, and 1.28 for the log graduate-nongraduate population ratio.

To complete the calibration, we calibrate the preference parameters b_H and b_L to match the aggregate employment rates among graduates and nongraduates: respectively, 0.861 and 0.672 in the 1990 census.

2.2 Properties of the calibrated model

A key message of Table 1 is that the target moments imply a low-skilled mobility cost that is much larger than its high-skilled analogue, $C_L \gg C_H$; indeed, over four times larger. Viewed through the model, the data support the presence of considerable heterogeneity in mobility frictions, a central premise of this paper.

In the sections that follow, we describe several important corollaries of this result. Before doing so, however, we highlight that the calibrated model also has reasonable implications for a range of relevant untargeted outcomes.

First, recalling that mean high-skilled wages are normalized to one in the calibration, it is possible to convert the mobility costs in Table 1 to dollar equivalents and compare them with the influential estimates of Kennan and Walker (2011). They report average mobility costs of \$312,000 in 2010 dollars (see their Table IV). In our calibrated model, C_L corresponds to around \$267,000, while C_H comes to around \$58,000, in 2010 dollars. Together, these imply a population-weighted average mobility cost of around \$227,000.[13]

[13] The average hourly wage among graduates in the 2010 census was \$31.48 (in 2010 dollars), after removing top and bottom percentiles. Scaling up based on a 40-hour workweek implies an annual wage of a little more than \$65,000. Applying the latter to the entries for C_L and C_H in Table 1 yields the stated numbers.

Although somewhat smaller than Kennan and Walker's estimate (perhaps because their data capture interstate moves, while ours capture commuting zones), the shared message—based on different methods and data—is that mobility costs are considerable.

Second, although the model is calibrated to match the *total* net population flow rate, a natural implication of heterogeneous mobility is that more-mobile subgroups should account for a disproportionate share of total net population flows. Consistent with this, the high skilled account for 36.3 percent of net population flows in the model, despite comprising only 19.2 percent of the population. Analogous outcomes in the data deliver a similar message: there, graduates account for 27.2 percent of net population flows (with a population share of 19.2 percent).

Third, recall that the calibration targets empirical gaps in graduate employment rates and graduate-nongraduate population ratios, but not in nongraduate employment rates. A key implication of the model is that there should be considerable variation in the latter. It turns out that the model fares quite well on this dimension: It implies a log low-skilled employment rate gap of 0.280; in the data, the 95-5 percentile differential for the log nongraduate employment rate is 0.222. These, in turn, are substantially larger than their high-skilled/graduate analogue: 0.068 in model and data.

Finally, the implied standard deviation of shocks to labor productivity in a CZ, σ_w, comes out at 0.042 at an annual frequency. This too seems reasonable: much larger than the standard deviation of annual aggregate labor productivity growth, and much smaller than estimates of the cross-sectional standard deviation of productivity innovations.[14]

We thus view the calibrated model as having sensible implications for a range of relevant untargeted outcomes. This, in turn, motivates us to study its implications for the distinctive novel equilibrium properties of the model noted earlier: the patterns of sorting, inequality, and persistence in local labor market dynamics.

2.3 Sorting

Consider first the model's implications for the sorting of worker types across markets. Recall from Figure 2C that, in equilibrium, more-mobile high-skilled types accumulate disproportionately in better markets. Here, we confront this key implication of the calibrated model with empirical analogues in the data.

[14] For example, on one hand, the standard deviation of postwar annual aggregate labor productivity growth in the Bureau of Labor Statistics' Productivity and Costs data is 0.015. On the other, Decker at al. (2020) report estimates of the cross-sectional standard deviation of TFP innovations of about 0.3.

Table 2. Untargeted outcomes

Outcome		**Model**	**Prior estimates / Data**
Mobility costs ($2010)	Low skilled	$266,577	—
	High skilled	$57,969	—
	Pop-weighted average	$226,524	$312,000
Net population flows	High-skilled share	36.3%	27.2%
Nongrad e-pop rate	Log gap (Data: 95-5)	0.280	0.222

Notes. Model outcomes implied by baseline calibration in Table 1. Mobility costs converted to 2010 dollars by scaling by average graduate wages in 2010 census data. Prior estimate of average mobility cost is from Kennan and Walker (2011, Table IV). The high-skilled share of net population flows and the 95-5 percentile differential of the log nongraduate employment-population rate are calculated from 1990 census data.

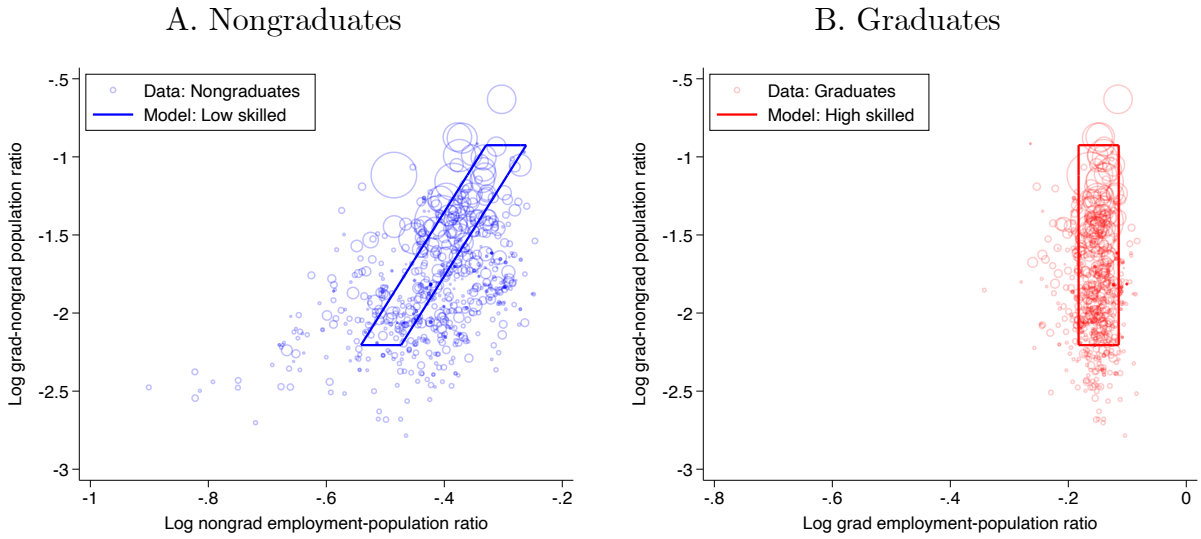
The model provides a natural barometer for the extent of such equilibrium sorting: Combining the mobility boundaries in Figure 2A with our mapping of model outcomes to empirical counterparts yields an equilibrium correspondence between the log skill ratio, $\ln(\ell_H/\ell_L)$, and its log employment rates for each type, $\ln(n_i/\ell_i)$. The latter, in turn, can be confronted with the observed relation between these outcomes across CZs in the data.

Figure 3 illustrates. As designed, our mapping of model to data is such that the implied equilibrium mobility boundaries form a rectangle for the mobile, high-skilled type, and a parallelogram for the immobile, low-skilled type in the model.

Strikingly, the data echo the pattern of sorting implied by the calibrated model. Employment rates among graduates are clustered at high levels in a manner unrelated to a location's composition. And, importantly, there is a clear tendency for nongraduate employment rates to be positively associated with a CZ's skill composition in the data. Recall that the latter is a signature implication of heterogeneous mobility in the model: there, the mobility boundaries for the immobile, low-skilled type take the form of thresholds on market composition. More striking still, the implied slope of the mobility boundaries in the model is empirically reasonable.

Of course, there is also some slippage between model and data—most saliently, the variation in nongraduate employment rates in the data is greater than that implied by the model. But, given the parsimony of the model, and given that these outcomes were not targeted by the calibration in Table 1, we view the patterns in Figure 3 as encouraging.

Figure 3. Sorting: Model versus data



Notes. The scatterplots depict outcomes across 722 commuting zones (CZs) using 1990 census data, where the size of each "bubble" is proportional to the CZ's population. The bold lines depict the calibrated model's implications for the mobility boundaries, transformed to log employment rate-log population ratio space.

Figure 3 also motivates our next exercise: Viewed through the model, the variation in *both* labor market outcomes *and* a market's type composition will give rise to equilibrium inequality in welfare—both between types and within immobile, low-skilled types.

2.4 Inequality

An informal sense of the implications of heterogeneous mobility for inequality can be gleaned by returning to Figure 2, which depicts key equilibrium outcomes in the calibrated model. Panel B reveals that gaps in market composition ϕ are just as—if not more—important for immobile, low-skilled type welfare as gaps in current labor market conditions. This, in turn, also contributes to the implied inequality in welfare between immobile and mobile types. The magnitude of these effects turns on the interaction between the values in Figure 2B and the joint distribution of population in Figure 2C. We now quantify formally the contribution of heterogeneous mobility to inequality in the calibrated model.

Table 3 summarizes. It juxtaposes differences in the mean and dispersion of household values by type in the baseline calibration with a homogeneous-mobility counterfactual. The latter recalibrates the low-skilled mobility cost C_L such that both types are equally mobile in equilibrium, per the homogeneous mobility solution in Proposition 1. All other targets remain as in Table 1.

Table 3. Between- and within-group inequality implied by the model

Outcome	**Baseline**	**Counterfactual** ($\mathcal{C}_L^{\text{new}} = \mathcal{C}_H$)	**Difference**
A. Between-group inequality			
$\ln \mathbb{E}[V_L]$	1.849	1.983	0.134
$\ln \mathbb{E}[V_H]$	2.738	2.792	*Misallocation loss*: 0.054
Difference	0.890	0.809	*Mobility premium*: 0.080
B. Within-group inequality			
Std dev $\ln V_L$	0.206	0.019	0.187
Std dev $\ln V_H$	0.020	0.019	0.001

Notes. "Baseline" outcomes refer to those implied by the baseline calibration in Table 1. "Counterfactual" outcomes are those implied by a homogeneous-mobility counterfactual in which $\mathcal{C}_L$ is recalibrated such that L-types are as mobile as H-types in the baseline model.

Panel A of Table 3 begins by considering the contribution of heterogeneity in mobility to between-group inequality—that is, differences in average household values across types. The first row reveals that the low-skilled type is considerably better off, on average, under the homogeneous-mobility counterfactual—to the tune of more than 13 log points.

The subsequent rows reveal that this average welfare gain for the L-types originates from two economic forces. The first is that mean welfare in fact rises for both types in the counterfactual. High-skilled households' average value rises by more than 5 log points. Intuitively, increased low-skilled mobility reduces the equilibrium *misallocation* of labor in the counterfactual economy, benefitting both skill types on average.

But this improvement in misallocation explains only part of the decrease in between-group inequality in the homogeneous-mobility counterfactual. The remainder is accounted for by the second economic force in the model, the presence of a *mobility premium* in average welfare. Intuitively, this summarizes the loss in average low-skilled value due to their tendency to accumulate disproportionately in less auspicious markets. Under the calibrated model, this mobility premium turns out to be nontrivial, at 8 log points.

Panel B then draws out the implications of heterogeneous mobility for within-group inequality. These turn out to be even starker: The standard deviation of log low-skilled household values in the baseline calibrated model is 0.206; low-skilled households one

standard deviation up in the distribution of values are $e^{0.206} - 1 = 23$ percent better off. By contrast, within-high-skilled inequality implied by the baseline calibration is an order of magnitude smaller, with a standard deviation of log values of 0.02. This, in turn, is almost the same as within-group inequality in the homogeneous-mobility counterfactual. The implication, then, is that heterogeneity in mobility gives rise to considerable within-group inequality in welfare among less-mobile, low-skilled types.

Observe that none of the implications of Table 3—for misallocation, the mobility premium, or within-group inequality—would arise in the absence of heterogeneity in mobility. For this reason, we see these as distinctive, novel implications of our model.

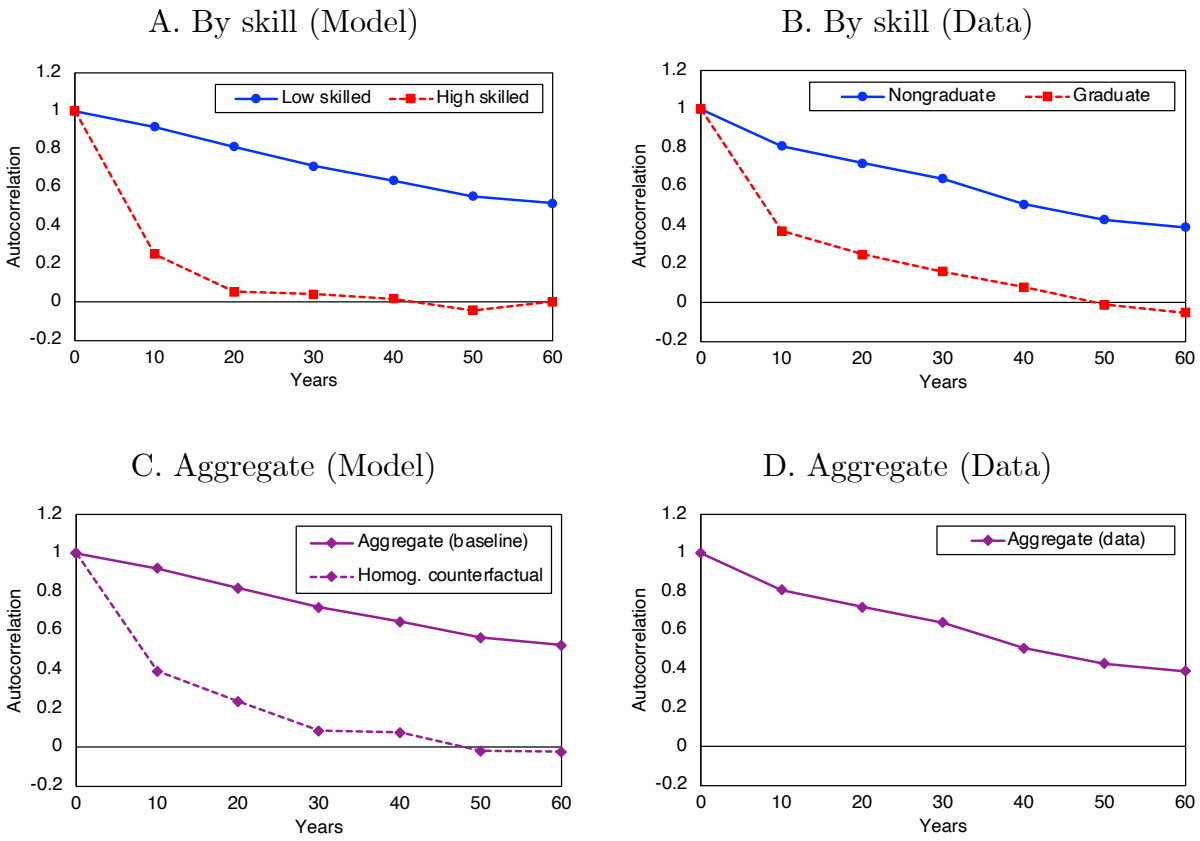
2.5 Persistence

A final corollary of heterogeneous mobility that we emphasize is its signature implications for labor market *dynamics*. We highlight two key predictions: First, that we should observe differentials in labor market dynamics whereby less-mobile, low-skilled types experience greater persistence in their labor market outcomes. Second, that heterogeneity in mobility should contribute to overall labor market persistence in the aggregate.

The intuition for these predictions can again be gleaned by returning to Figure 2. Consider first the high-skilled mobility boundaries in Panel A. Observe that the dynamics of real consumer wages w_H exhibit a memoryless property whereby, upon attaining either boundary, the history of shocks becomes irrelevant for the subsequent dynamics of w_H. And, more generally, high-skilled labor market conditions will inherit this property. However, the same logic applies to low-skilled labor dynamics only once gaps in *both* w_H *and* market composition ϕ have been closed; low-skilled labor market dynamics will thus be more sclerotic relative to the high skilled. Furthermore, this force naturally will spill over to aggregate labor market dynamics. We now examine this logic quantitatively.

Figure 4 presents a simple diagnostic for these outcomes—the autocorrelation function of the log employment rate. Panels A and B depict these, respectively, by skill in the calibrated model, and for graduates and nongraduates in the data (reported by Amior and Manning 2018). These reveal very large differentials in the persistence of local labor market dynamics by skill, in both model and data: Among high-skilled graduates, the half-life of employment rates is less than a decade; among low-skilled nongraduates the half-life is on the order of *several decades*. Although these differences are somewhat larger in the model, they are broadly consistent with their empirical analogues.

Figure 4. Autocorrelation function of log employment rates: Model versus data



Notes. Model outcomes implied by baseline calibration in Table 1. The homogeneous-mobility counterfactual recalibrates C_L such that L-types are as mobile as H-types in the baseline model. Data are taken from Amior and Manning (2018, Table 1).

Panels C and D of Figure 4 then reveal that these forces leave their imprint on the dynamics of aggregate employment rates. Model and data again align well, with both exhibiting extremely persistent aggregate local labor market dynamics. Panel C provides a perspective on the contribution of heterogeneous mobility to this outcome in the model. Echoing our earlier analysis, we again juxtapose the results from the baseline calibration with the homogeneous-mobility counterfactual described above. Once the low skilled are rendered as mobile as the high skilled in the model, the implied persistence of local labor market dynamics shrinks significantly.

Thus, not only can the model broadly capture the magnitude of local labor market persistence, both across groups, and in the aggregate, it also implies that a significant contributor to both outcomes is the presence of considerable heterogeneity in mobility. We view this as lending credence both to the central premise of this paper, and the importance of its implications.

3. Extensions and robustness

[To be added.]

4. Summary and discussion

[To be added.]

Appendix

A. Local goods (housing) in quasi-fixed supply

In this appendix, we show how our environment can be extended to accommodate the presence of local goods in quasi-fixed supply, such as housing, subject to a simple reinterpretation of the labor demand elasticity ϵ.

Suppose that household utility in (3) instead takes the form

$$u_i(c, n) = c^{1-\beta}\hbar^{\beta} - \frac{1}{b_i^{1/\eta}}\frac{n^{1+1/\eta}}{1+1/\eta}, \tag{30}$$

where $\hbar$ is housing and $\beta \in (0,1)$ is the housing expenditure share. Household optimization implies that a type-i household supplies $s_i n_i^*(\omega_i) = B^{\eta} b_i s_i^{1+\eta} \omega_i^{\eta}$ efficiency units of type-i labor, and enjoys flow indirect utility $u_i^*(\omega_i) = b_i (B s_i \omega_i)^{1+\eta}/(1+\eta)$, where $\omega_i \equiv w_i/p^{\beta}$ is the real wage for type-i labor input, p is the local housing price, and $B \equiv \beta^{\beta}(1-\beta)^{1-\beta}$.

The market supply of labor inputs thus satisfies (5), but with w_i and w replaced by the real wages ω_i and $\omega \equiv w/p^{\beta}$, and with the effective type-i population now given by $m_i \equiv B^{\eta} b_i s_i^{1+\eta} \ell_i$. Market labor demand remains as in (4). Within-market equilibrium real wages and employment, for a given housing price p, are thus

$$\omega = \left[\left(\frac{\alpha x}{p^{\beta}}\right)^{\epsilon}\frac{1}{m}\right]^{\frac{1}{\epsilon+\eta}}, \quad n = \left[\left(\frac{\alpha x}{p^{\beta}}\right)^{\eta} m\right]^{\frac{\epsilon}{\epsilon+\eta}}, \quad \omega_i = \left(\frac{m_i}{m}\right)^{-\frac{1}{\varsigma+\eta}}\omega, \text{ and } n_i = \left(\frac{m_i}{m}\right)^{\frac{\varsigma}{\varsigma+\eta}} n. \tag{31}$$

Market housing demand can be written as $h^d(p) = \beta m \omega^{1+\eta}/p^{1-\beta}$. If housing supply is isoelastic, $h^s(p) \propto p^{\zeta}$, then the equilibrium housing price, for a given real wage ω, satisfies $p \propto (m\omega^{1+\eta})^{1/(1-\beta+\zeta)}$. The full equilibrium real wage thus satisfies

$$\omega \propto \left(\frac{x^{\epsilon}}{m^{1+\frac{\beta\epsilon}{1-\beta+\zeta}}}\right)^{\frac{1}{\epsilon+\eta+(1+\eta)\frac{\beta\epsilon}{1-\beta+\zeta}}} \equiv \left(\frac{\tilde{x}^{\tilde{\epsilon}}}{m}\right)^{\frac{1}{\tilde{\epsilon}+\eta}}, \text{ such that } \tilde{\epsilon} = \epsilon\left(1+\beta\frac{\epsilon-1}{1+\zeta}\right)^{-1}. \tag{32}$$

Thus, the extended model with quasi-fixed housing is isomorphic to the baseline model, with x and ϵ exchanged with $\tilde{x}$ and $\tilde{\epsilon}$. Observe that the presence of quasi-fixed housing, $\beta/(1+\zeta) > 0$, lowers the effective elasticity of market demand for labor input ($\tilde{\epsilon} < \epsilon$).

B. Proofs

Proof of Proposition 1. (i) A solution of the form $V_H(w_H) = b_H s_H^{1+\eta} \cdot v(w_H)$ and $V_L(w_H) = b_L s_L^{1+\eta}[\Phi/(1-\Phi)]^{(1+\eta)/(\varsigma-1)} \cdot v(w_H)$ satisfies the Bellman equation in (9) if

$$(r+\delta)v(w_H) = \frac{w_H^{1+\eta}}{1+\eta} + \mu_w w_H v'(w_H) + \frac{1}{2}\sigma_w^2 w_H^2 v''(w_H) + \delta\bar{v}. \tag{33}$$

The latter has general solution as in (14) and (15). The constants, ν_1 and ν_2, and the boundaries, $\underline{w}_H$ and $\bar{w}_H$, must satisfy (14), and value-matching and smooth-pasting conditions implied by (10).

The smooth-pasting conditions imply $v'(\underline{w}_H) = 0 = v'(\bar{w}_H)$, and

$$\nu_1 = -\frac{1}{\gamma_1}\frac{\underline{w}_H^{1+\eta-\gamma_1}}{\rho(1+\eta)}\vartheta(\mathcal{G}), \qquad \nu_2 = -\frac{1}{\gamma_2}\frac{\underline{w}_H^{1+\eta-\gamma_2}}{\rho(1+\eta)}[1-\vartheta(\mathcal{G})], \tag{34}$$

where $\mathcal{G} \equiv \bar{w}_H/\underline{w}_H$ and $\vartheta(\mathcal{G}) \equiv (\mathcal{G}^{\gamma_2} - \mathcal{G}^{1+\eta})/(\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1})$.

The value-matching condition in (10) implies $v(\underline{w}_H) \equiv v(\bar{w}_H) - \mathcal{C}$, where $\mathcal{C} \equiv \mathcal{C}_H/\left[\Phi^{(1+\eta)/(\varsigma-1)}\right] = \mathcal{C}_L/\left[(1-\Phi)^{(1+\eta)/(\varsigma-1)}\right]$ in (13). This implies

$$\frac{\underline{w}_H^{1+\eta}}{1+\eta}\varphi(\mathcal{G}) = \frac{\bar{w}_H^{1+\eta}}{1+\eta}\varphi(\mathcal{G}^{-1}) - \mathcal{C}, \tag{35}$$

where

$$\varphi(\mathcal{G}) \equiv \frac{1}{\rho(1+\eta)}\left[1 - (1+\eta)\frac{\vartheta(\mathcal{G})}{\gamma_1} - (1+\eta)\frac{1-\vartheta(\mathcal{G})}{\gamma_2}\right]. \tag{36}$$

The aggregate market clearing condition (11) provides a final equation in the boundaries and completes the solution.

To verify that the latter homogeneous solution arises *only if* (13) holds, suppose the contrary, and observe that the value-matching condition in (10) would then be violated for common boundaries $\underline{w}_H$ and $\bar{w}_H$.

(ii) Denote the logarithm of the H-type wage $\mathit{w}_H \equiv \ln w_H$. In the inaction region, where $\mathit{w}_H \in (\underline{\mathit{w}}_H, \bar{\mathit{w}}_H)$ and $\mathrm{d}\ln m = 0$, w_H evolves according to

$$\mathrm{d}\mathit{w}_H = \frac{\epsilon}{\epsilon+\eta}\mathrm{d}\ln x = \frac{\epsilon}{\epsilon+\eta}\left[\left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}z\right] \equiv \mu_{\mathit{w}}\mathrm{d}t + \sigma_{\mathit{w}}\mathrm{d}z. \tag{37}$$

To derive the effective population distribution and flows, we use the fact that the latter can be approximated by a discrete-time, discrete-state process:

$$w_{Ht+\mathrm{d}t} = \begin{cases} w_{Ht} + \Delta & \text{with probability} \quad p, \\ w_{Ht} - \Delta & \text{with probability} \quad q, \end{cases} \tag{38}$$

where $\Delta = \sigma_w \sqrt{\mathrm{d}t}$, $p = \frac{1}{2}\left(1 + \frac{\mu_w}{\sigma_w}\sqrt{\mathrm{d}t}\right)$, and $q = \frac{1}{2}\left(1 - \frac{\mu_w}{\sigma_w}\sqrt{\mathrm{d}t}\right)$.

Consider a household at $\underline{w}_H$. With probability q, their market crosses the lower boundary and sheds a fraction $(\epsilon + \eta)\Delta$ of effective population such that it returns to $\underline{w}_H$. Denoting the stationary density of effective population over log wages w_H by $g(w_H)$, the rate of outflow of total effective population from markets at $\underline{w}_H$ is given by

$$\lambda = \frac{1}{\mathrm{d}t} q(\epsilon + \eta)\Delta[g(\underline{w}_H)\Delta] \to \frac{1}{2}\sigma_w^2(\epsilon + \eta)g(\underline{w}_H), \text{ as } \mathrm{d}t \to 0. \tag{39}$$

By symmetry, the rate of inflow of total effective population at $\overline{w}_H$ is given by $(\sigma_w^2/2)(\epsilon + \eta)g(\overline{w}_H)$.

The Fokker-Planck (Kolmogorov Forward) equation for the density $g(w_H)$ is

$$-\mu_w g'(w_H) + \frac{1}{2}\sigma_w^2 g''(w_H) = 0. \tag{40}$$

To determine boundary conditions for the latter, consider the flows in and out of $g(\underline{w}_H)$. An inflow $qg(\underline{w}_H + \Delta)$ comes from markets at $\underline{w}_H + \Delta$ that receive a negative shock, which occurs with probability q. An outflow $pg(\underline{w}_H)$ comes from markets at $\underline{w}_H$ that receive a positive shock, which occurs with probability p. A second outflow $q(\epsilon + \eta)\Delta g(\underline{w}_H)$ comes from the fraction $(\epsilon + \eta)\Delta$ of the effective population at $\underline{w}_H$ that leaves these markets upon realization of a negative shock: A market at $\underline{w}_H$ receives a negative innovation, $\mathrm{d}w_H = -\Delta$, with probability q; it sheds effective population such that the equilibrium wage in the market returns to $\underline{w}_H$, so that $\mathrm{d}\ln m = \epsilon \mathrm{d}\ln x = -(\epsilon + \eta)\Delta$. Expanding $g(\underline{w}_H + \Delta)$, using the definitions of p, q and Δ, and collecting terms in $\sqrt{\mathrm{d}t}$ yields the net flow *rate*

$$\frac{1}{\mathrm{d}t}\left(\left\{\left[\mu_w + (\epsilon + \eta)\frac{1}{2}\sigma_w^2\right]g(\underline{w}_H) - \frac{1}{2}\sigma_w^2 g'(\underline{w}_H)\right\}\sqrt{\mathrm{d}t} + o\left(\sqrt{\mathrm{d}t}\right)\right). \tag{41}$$

As $\mathrm{d}t \to 0$, the latter explodes, unless the following boundary condition holds

$$\left[\mu_w + (\epsilon + \eta)\frac{1}{2}\sigma_w^2\right]g(\underline{w}_H) - \frac{1}{2}\sigma_w^2 g'(\underline{w}_H) = 0. \tag{42}$$

Repeating the same steps implies that the analogous boundary condition for the steady-state density at the higher boundary, $g(\overline{w}_H)$, takes the same form

$$\left[\mu_w + (\epsilon+\eta)\frac{1}{2}\sigma_w^2\right] g(\overline{w}_H) - \frac{1}{2}\sigma_w^2 g'(\overline{w}_H) = 0. \tag{43}$$

Finally, the aggregate stationarity condition (12) implies $\mu_w + (\epsilon+\eta)(\sigma_w^2/2) = 0$, such that (40), and the boundaries (42) and (43), simplify to

$$(\epsilon+\eta)g'(w_H) + g''(w_H) = 0, \tag{44}$$

subject to $g'(\underline{w}_H) = 0 = g'(\overline{w}_H)$. It is straightforward to confirm that the uniform solution $g(w_H) = 1/(\overline{w}_H - \underline{w}_H)$ satisfies the latter.

Mapping back from logarithms to levels yields the results for λ and $G(w_H)$ in (16).

Derivation of the smooth-pasting conditions in (20). Consider first the lower wage boundary, $\underline{w}_H$. Recalling (37), here adverse innovations to w are now met by outflows of mobile types that reflect market wages back to the lower boundary. Using the market wages for the mobile type from (7), evaluating at the lower wage boundary $\underline{w}_H$, noting that m_L is fixed, and totally differentiating implies

$$\mathrm{d}w_H = \frac{1}{\varsigma+\eta}\mathrm{d}\ln m_H + \left(\frac{1}{\epsilon+\eta} - \frac{1}{\varsigma+\eta}\right)\mathrm{d}\ln m. \tag{45}$$

The definition of the effective total population m in (5) implies that $\mathrm{d}\ln m = \phi \mathrm{d}\ln m_H$. Likewise, the definition of the effective population quasi shares in (6) implies that $\mathrm{d}\phi = [(\varsigma-1)/(\varsigma+\eta)]\phi(1-\phi)\mathrm{d}\ln m_H$. Using the discrete-time, discrete-state approximation in (38), the implied downward step in ϕ following a negative innovation is therefore

$$\Delta_\phi = \frac{(\epsilon+\eta)(\varsigma-1)\phi(1-\phi)}{\epsilon+\eta-(\epsilon-\varsigma)\phi}\Delta = \frac{\Delta}{\varphi'(\phi)/\varphi(\phi)}, \tag{46}$$

where $\varphi(\phi)$ is defined as in (20). Continuity of the immobile-type value at $\underline{w}_H$ requires

$$p\underline{w}_H V_{Lw}(\underline{w}_H, \phi)\Delta - q V_{L\phi}(\underline{w}_H, \phi)\Delta_\phi = 0. \tag{47}$$

With probability p, a positive innovation Δ to $w \equiv \ln w_H$ is realized; with probability q a negative innovation $-\Delta_\phi$ to ϕ is realized. Since Δ (and therefore Δ_ϕ) is of order $\sqrt{\mathrm{d}t}$, the terms in (47) must cancel (otherwise, the immobile-type value will explode at the boundary $\underline{w}_H$ as $\mathrm{d}t \to 0$). Using the definitions of p, q, Δ and Δ_ϕ, and taking $\mathrm{d}t \to 0$, yields the stated smooth-pasting condition

$$V_{L\phi}(\underline{w}_H, \phi) = \frac{\varphi'(\phi)}{\varphi(\phi)}\underline{w}_H V_{Lw}(\underline{w}_H, \phi). \tag{48}$$

An analogous derivation yields the smooth-pasting condition at $\overline{w}_H$.

Proof of Proposition 2. (i) The H-type household value satisfies the Bellman equation in (9) for $i = H$. A solution of the form $V_H(w_H) = b_H s_H^{1+\eta} \cdot v_H(w_H)$ satisfies (9), such that

$$(r+\delta)v_H(w_H) = \frac{w_H^{1+\eta}}{1+\eta} + \mu_w w_H v_H'(w_H) + \frac{1}{2}\sigma_w^2 w_H^2 v_H''(w_H) + \delta\bar{v}_H, \tag{49}$$

subject to the boundary conditions

$$v_H(\underline{w}_H) \equiv v_H(\overline{w}_H) - \left[C_H/\left(b_H s_H^{1+\eta}\right)\right], \quad \text{and} \quad v_H'(\underline{w}_H) = 0 = v_H'(\overline{w}_H). \tag{50}$$

The problem thus mirrors result (i) in Proposition 1, *mutatis mutandis*, with coefficients ν_1 and ν_2 as in (34), and a solution for $\underline{w}_H$, for any given $\overline{w}_H$, analogous to (35).

Turning to the L-type household value, a solution of the form $V_L(w_H, \phi) = b_L s_L^{1+\eta} \cdot v_L(w_H, \phi)$ satisfies the Bellman equation in (18), such that

$$(r+\delta)v_L(w_H,\phi) = \frac{[w_H\omega(\phi)]^{1+\eta}}{1+\eta} + \mu_w w_H v_{Lw} + \frac{1}{2}\sigma_w^2 w_H^2 v_{Lww} + \delta\bar{v}_L. \tag{51}$$

The general solution to the latter is

$$v_L(w_H,\phi) = \frac{1}{\rho(1+\eta)}\frac{[w_H\omega(\phi)]^{1+\eta}}{1+\eta} + v_{L1}(\phi)w_H^{\gamma_1} + v_{L2}(\phi)w_H^{\gamma_2} + \frac{\delta\bar{v}_2}{\rho(0)}, \tag{52}$$

where the functions $v_{L1}(\phi)$ and $v_{L2}(\phi)$ are to be determined from the boundary conditions (19) and (20). The latter implies

$$\begin{aligned}\left[\frac{\varphi'(\phi)}{\varphi(\phi)} - \frac{\omega'(\phi)}{\omega(\phi)}\right]\frac{[w_H\omega(\phi)]^{1+\eta}}{\rho(1+\eta)} + \left[\gamma_1\frac{\varphi'(\phi)}{\varphi(\phi)}v_{L1}(\phi) - v_{L1}'(\phi)\right]w_H^{\gamma_1} \\ + \left[\gamma_2\frac{\varphi'(\phi)}{\varphi(\phi)}v_{L2}(\phi) - v_{L2}'(\phi)\right]w_H^{\gamma_2} = 0,\end{aligned} \tag{53}$$

for $w_H \in \{\underline{w}_H, \overline{w}_H\}$ and for all $\phi \in \left(\underline{\phi}, \bar{\phi}\right)$.

(53) is a system of first-order ordinary differential equations in $\nu_{L1}(\phi)$ and $\nu_{L2}(\phi)$. The homogeneous solution is

$$v_{Lj}^h(\phi) = \psi_j\varphi(\phi)^{\gamma_j} = \psi_j\left[\phi(1-\phi)^{-\frac{\varsigma+\eta}{\epsilon+\eta}}\right]^{\frac{\gamma_j}{\varsigma-1}}, \quad \text{for } j \in \{1,2\}, \tag{54}$$

where the ψ_j are constants to be determined. Using the method of variation of parameters, one can construct the particular solution $v_{Lj}^p(\phi) = A_j(\phi)v_{Lj}^h(\phi)$, where the $A_j(\phi)$ are functions to be determined. Substituting into (53) and solving yields

$$\left[\frac{\varphi'(\phi)}{\varphi(\phi)}-\frac{\omega'(\phi)}{\omega(\phi)}\right]\frac{[w_H\omega(\phi)]^{1+\eta}}{\rho(1+\eta)}-\left[A_1'(\phi)v_{L1}^h(\phi)w_H^{\gamma_1}+A_2'(\phi)v_{L2}^h(\phi)w_H^{\gamma_2}\right]=0, \tag{55}$$

for $w_H \in \{\underline{w}_H, \overline{w}_H\}$. Recalling (17) and (20), and solving yields

$$A_j'(\phi)=\nu_j\frac{\gamma_j(\epsilon-\varsigma)}{(\epsilon+\eta)(\varsigma-1)}\omega(\phi)^{1+\eta}\frac{1}{(1-\phi)v_{Lj}^h(\phi)}, \quad \text{for } j\in\{1,2\}, \tag{56}$$

where the ν_js are the coefficients from the mobile H-type solution (which further correspond to those in Proposition 1). Integrating yields the particular solution

$$v_{Lj}^p(\phi)=\nu_j\frac{\gamma_j(\epsilon-\varsigma)}{(\epsilon+\eta)(\varsigma-1)}\int^{\phi}\omega(\tilde{\phi})^{1+\eta}\frac{v_{Lj}^h(\phi)}{v_{Lj}^h(\tilde{\phi})}\frac{\mathrm{d}\tilde{\phi}}{1-\tilde{\phi}}, \quad \text{for } j\in\{1,2\}. \tag{57}$$

Combining, yields the stated solution. Furthermore, recalling the homogeneous solution in (54), it is possible to evaluate the latter integral (using, e.g., Mathematica) to write

$$v_{Lj}^p(\phi)=\nu_j\frac{\gamma_j(\epsilon-\varsigma)}{(\epsilon+\eta)\left(\varsigma+\eta-\gamma_j\right)}\phi^{\frac{\varsigma+\eta}{\varsigma-1}}(1-\phi)^{-\frac{1+\eta}{\varsigma-1}}\mathcal{F}_j(\phi), \tag{58}$$

where

$$\mathcal{F}_j(\phi)\equiv\ {}_2F_1\left(1,1-\frac{\gamma_j}{\epsilon+\eta}\frac{\epsilon-\varsigma}{\varsigma-1},1+\frac{\varsigma+\eta-\gamma_j}{\varsigma-1},\phi\right), \tag{59}$$

and ${}_2F_1$ denotes the hypergeometric function.

Finally, the coefficients ψ_1 and ψ_2, and boundaries $\overline{w}_H$, $\underline{\phi}$, and $\bar{\phi}$ satisfy the value-matching and smooth-pasting conditions in (19), and the aggregate market clearing conditions in (21), completing the solution.

(ii) To derive the effective population distribution, we return to the discrete-time, discrete-state approximation in (38), and extend it to accommodate movements in ϕ. Specifically, we first use this approximation to solve for a joint density $g(w_H, \phi)$ of total effective population over $w_H \equiv \ln w_H$ and $\phi(\phi) \equiv \ln \varphi(\phi)$, where $\varphi(\phi)$ is defined in (20); we then transform back to a solution for $g(w_H, \phi)$.

Away from the boundaries, w_H evolves as in (37) and (38) with step size $\Delta = \sigma_w\sqrt{\mathrm{d}t}$. At the boundaries, (46) then implies a step size for ϕ of $\Delta_\phi = [\varphi'(\phi)/\varphi(\phi)]\cdot\Delta_\phi = \Delta$. These transforms thus conveniently partition the state-space grid into *squares* of size Δ.

After imposing the aggregate stationarity condition in (12), the stationary density $g(w_H, \phi)$ satisfies the Fokker-Planck (Kolmogorov Forward) equation,

$$(\epsilon+\eta)g_w(w_H,\varphi)+g_{ww}(w_H,\varphi)=0, \tag{60}$$

for $w_H \in (\underline{w}_H, \overline{w}_H)$, $\phi \in [\underline{\phi}, \overline{\phi}]$. We propose a solution of the form

$$g(w_H, \phi) = \begin{cases} \overline{g}(w_H) & \text{if} \quad \phi = \overline{\phi}, \\ \widetilde{g}(w_H, \phi) & \text{if} \quad \phi \in (\underline{\phi}, \overline{\phi}), \\ \underline{g}(w_H) & \text{if} \quad \phi = \underline{\phi}. \end{cases} \tag{61}$$

Consider first $\widetilde{g}(w_H, \phi)$. Its general solution is

$$\widetilde{g}(w_H, \phi) = \tilde{B}_1(\phi) + \tilde{B}_2(\phi) \exp[-(\epsilon + \eta) w_H], \tag{62}$$

for some functions $\tilde{B}_1(\phi)$ and $\tilde{B}_2(\phi)$, to be determined. From the derivation of (46) above, one can derive the log change in total effective population m induced by adverse (favorable) shocks at $\underline{w}_H$ ($\overline{w}_H$) as

$$\Delta_{\ln m} = \Gamma(\phi)\Delta, \quad \text{where} \quad \Gamma(\phi) \equiv \frac{(\epsilon + \eta)(\varsigma + \eta)\phi^{-1}(\phi)}{\epsilon + \eta - (\epsilon - \varsigma)\phi^{-1}(\phi)}, \tag{63}$$

It is straightforward to verify that the transform $\phi(\phi)$ defined above is invertible (since $\varsigma > 1$). The flows into (left-hand side) and out of (right-hand side) $\widetilde{g}(\underline{w}_H, \phi)$ are thus

$$q_l \widetilde{g}(\underline{w}_H + \Delta, \phi)\Delta^2 + q_l[1 - \Gamma(\phi)\Delta]\widetilde{g}(\underline{w}_H, \phi + \Delta)\Delta^2 = \widetilde{g}(\underline{w}_H, \phi)\Delta^2. \tag{64}$$

Dividing by Δ^2, expanding, keeping terms of order $\sqrt{dt}$, and noting that the aggregate stationarity condition (12) implies $\mu_w = -(\sigma_w^2/2)(\epsilon + \eta)$, yields

$$[\epsilon + \eta - \Gamma(\phi)]\widetilde{g}(\underline{w}_H, \phi) + \widetilde{g}_w(\underline{w}_H, \phi) + \widetilde{g}_\phi(\underline{w}_H, \phi) = 0. \tag{65}$$

Applying a symmetric logic to the flows into and out of $\widetilde{g}(\overline{w}_H, \phi)$ yields

$$[\epsilon + \eta - \Gamma(\phi)]\widetilde{g}(\overline{w}_H, \phi) + \widetilde{g}_w(\overline{w}_H, \phi) + \widetilde{g}_\varphi(\overline{w}_H, \phi) = 0. \tag{66}$$

Applying these boundary conditions to the general solution for $\widetilde{g}(w_H, \phi)$ then implies

$$\begin{aligned} \tilde{B}_1(\phi) &= \tilde{B}_1 \exp[-(\epsilon + \eta)\phi] \exp\left[\int \Gamma(\phi) d\phi\right] = \tilde{B}_1 \exp[-(\epsilon + \eta)\phi] \, [1 - \phi^{-1}(\phi)]^{-\frac{\varsigma + \eta}{\varsigma - 1}}, \\ \tilde{B}_2(\phi) &= \tilde{B}_2 \exp\left[\int \Gamma(\phi) d\phi\right] = \tilde{B}_2 [1 - \phi^{-1}(\phi)]^{-\frac{\varsigma + \eta}{\varsigma - 1}}, \end{aligned} \tag{67}$$

for some constants $\tilde{B}_1$ and $\tilde{B}_2$. Thus, we have the solution

$$\widetilde{g}(w_H, \phi) = [1 - \phi^{-1}(\phi)]^{-\frac{\varsigma + \eta}{\varsigma - 1}} \{\tilde{B}_1 \exp[-(\epsilon + \eta)\phi] + \tilde{B}_2 \exp[-(\epsilon + \eta) w_H]\}. \tag{68}$$

Now consider $\overline{g}(w_H)$. Its general solution is

$$\overline{g}(w_H) = \bar{B}_1 + \bar{B}_2 \exp[-(\epsilon + \eta) w_H], \tag{69}$$

for some constants $\bar{B}_1$ and $\bar{B}_2$, to be determined. Since $(\underline{w}_H, \bar{\phi})$ is an absorbing boundary, it must be that $\bar{g}(\underline{w}_H) = 0$. The flows into (left-hand side) and out of (right-hand side) $\bar{g}(\bar{w}_H)$ are

$$p\bar{g}(\bar{w}_H - \Delta)\Delta + p[1 + \Gamma(\phi)\Delta]\tilde{g}(\bar{w}_H, \bar{\phi} - \Delta)\Delta^2 + p(\epsilon + \eta)\Delta\bar{g}(\bar{w}_H)\Delta = q\bar{g}(\bar{w}_H)\Delta. \quad (70)$$

Expanding, keeping terms of order $\sqrt{\mathrm{d}t}$, and imposing the aggregate stationarity condition, $\mu_w = -(\sigma_w^2/2)(\epsilon + \eta)$, yields

$$\bar{g}_w(\bar{w}_H) = \tilde{g}(\bar{w}_H, \bar{\phi}). \quad (71)$$

Applying the boundary conditions to the general solution for $\bar{g}(w_H)$ then yields

$$\bar{g}(w_H) = \frac{1}{\epsilon + \eta}\tilde{g}(\bar{w}_H, \bar{\phi})\{\exp[(\epsilon + \eta)(\bar{w}_H - \underline{w}_H)] - \exp[(\epsilon + \eta)(\bar{w}_H - w_H)]\}. \quad (72)$$

Finally, consider $\underline{g}(w_H)$. Its general solution is

$$\underline{g}(w_H) = \underline{B}_1 + \underline{B}_2 \exp[-(\epsilon + \eta)w_H], \quad (73)$$

for some constants $\underline{B}_1$ and $\underline{B}_2$, to be determined. Since $(\bar{w}_H, \underline{\phi})$ is an absorbing boundary, $\underline{g}(\bar{w}_H) = 0$. The flows into (left-hand side) and out of (right-hand side) $\underline{g}(\underline{w}_H)$ are

$$q\underline{g}(\underline{w}_H + \Delta)\Delta + q[1 - \Gamma(\phi)\Delta]\tilde{g}(\underline{w}_H, \underline{\phi} + \Delta)\Delta^2 = p\underline{g}(\underline{w}_H)\Delta + q(\epsilon + \eta)\Delta\underline{g}(\underline{w}_H)\Delta. \quad (74)$$

Expanding, keeping terms of order $\sqrt{\mathrm{d}t}$, and imposing the aggregate stationarity condition, $\mu_w = -(\sigma_w^2/2)(\epsilon + \eta)$, yields

$$\underline{g}_w(\underline{w}_H) = -\tilde{g}(\underline{w}_H, \underline{\phi}). \quad (75)$$

Applying the boundary conditions to the general solution for $\underline{g}(w_H)$ then yields

$$\underline{g}(w_H) = \frac{1}{\epsilon + \eta}\tilde{g}(\underline{w}_H, \underline{\phi})\{\exp[-(\epsilon + \eta)(w_H - \underline{w}_H)] - \exp[-(\epsilon + \eta)(\bar{w}_H - \underline{w}_H)]\}. \quad (76)$$

It remains to pin down the constants, $\tilde{B}_1$ and $\tilde{B}_2$. Setting the effective population outflow of L-types from $(\underline{w}_H, \underline{\phi})$ equal to the effective population inflow of L-types to $(\bar{w}_H, \bar{\phi})$, and recalling that $m_L/m = (1 - \phi)^{(\varsigma+\eta)/(\varsigma-1)} = [1 - \phi^{-1}(\phi)]^{(\varsigma+\eta)/(\varsigma-1)}$, we have

$$\frac{1}{2}\sigma_w^2(\epsilon + \eta)(1 - \underline{\phi})^{\frac{\varsigma+\eta}{\varsigma-1}}\underline{g}(\underline{w}_H) = \frac{1}{2}\sigma_w^2(\epsilon + \eta)(1 - \bar{\phi})^{\frac{\varsigma+\eta}{\varsigma-1}}\bar{g}(\bar{w}_H). \quad (77)$$

Using the solutions for $\underline{g}(w_H)$, $\bar{g}(w_H)$, and $\tilde{g}(w_H, \phi)$ implies $\tilde{B}_1 = 0$. The remaining constant $\tilde{B}_2$ then follows from the law of total probability, $\int\int g(w_H, \phi)\mathrm{d}w_H\mathrm{d}\phi = 1$.

To transform from $(\underline{w}_H, \phi)$ space to (w_H, ϕ) space, observe that $\tilde{g}(w_H, \phi)\mathrm{d}w_H\mathrm{d}\phi = \tilde{\underline{g}}(\underline{w}_H, \phi)\mathrm{d}\underline{w}_H\mathrm{d}\phi$ and, likewise, $\bar{g}(w_H)\mathrm{d}w_H = \bar{\underline{g}}(\underline{w}_H)\mathrm{d}\underline{w}_H$ and $\underline{g}(w_H)\mathrm{d}w_H = \underline{\underline{g}}(\underline{w}_H)\mathrm{d}\underline{w}_H$. These give rise to the stated solution (25), with $g_0 \equiv \tilde{B}_2/(\epsilon+\eta)$.

Finally, the effective population flows are as follows. First, recalling (77), the effective L-type population flow rate is

$$\begin{aligned}\lambda_L &= \frac{1}{2}\sigma_w^2(\epsilon+\eta)g_0\{\exp[-(\epsilon+\eta)\underline{w}_H] - \exp[-(\epsilon+\eta)\bar{w}_H]\} \\ &= \frac{1}{2}\sigma_w^2(\epsilon+\eta)g_0\left[\underline{w}_H^{-(\epsilon+\eta)} - \bar{w}_H^{-(\epsilon+\eta)}\right].\end{aligned} \tag{78}$$

Second, consider the effective H-type population flows. A market at $\underline{\underline{w}}_H$ with $\phi \in (\underline{\phi}, \bar{\phi})$ crosses $\underline{\underline{w}}_H$ with probability q and, recalling (63), sheds a fraction $\Gamma(\phi)\Delta$ of its effective *total* population such that it returns to $\underline{\underline{w}}_H$. Equivalently, the market sheds a fraction $\Gamma(\phi)\Delta/\phi$ of its *H-type* population. In addition, a market at $(\underline{\underline{w}}_H, \underline{\phi})$ crosses $\underline{\underline{w}}_H$ with probability q and sheds a fraction $(\epsilon+\eta)\Delta$ of its effective *total* population, proportionately by type. Recalling that $m_H/m = \phi^{(\varsigma+\eta)/(\varsigma-1)}$, we have

$$\begin{aligned}\lambda_H &= \frac{1}{2}\sigma_w^2\left[\int_{\underline{\phi}}^{\bar{\phi}} \phi^{\frac{\varsigma+\eta}{\varsigma-1}-1}\Gamma(\phi)\tilde{\underline{g}}(\underline{\underline{w}}_H, \phi)\mathrm{d}\phi + (\epsilon+\eta)\underline{\phi}^{\frac{\varsigma+\eta}{\varsigma-1}}\underline{\underline{g}}(\underline{\underline{w}}_H)\right] \\ &= \frac{1}{2}\sigma_w^2(\epsilon+\eta)g_0\left[\left(\frac{\bar{\phi}}{1-\bar{\phi}}\right)^{\frac{\varsigma+\eta}{\varsigma-1}}\underline{w}_H^{-(\epsilon+\eta)} - \left(\frac{\underline{\phi}}{1-\underline{\phi}}\right)^{\frac{\varsigma+\eta}{\varsigma-1}}\bar{w}_H^{-(\epsilon+\eta)}\right].\end{aligned} \tag{79}$$

Proof of Lemma 1. [To be added.]

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